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Matrices and Representations over Rings of Analytic Functions and other one dimensional Rings

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1 Introduction

Suppose Ω is a noncompact Riemann surface (e.g. a domain in the complex plane). Let R denote the ring of holomorphic functions on Ω. If A and B are $n \times n$ matrices over R, they are said to be pointwise similar on Ω if $A(z)$ and $B(z)$ are similar for each z in Ω. It is easy to construct pointwise similar matrices which are not similar. However, it does imply that A and B are similar on some smaller surface Ω', and under certain circumstances, one can prescribe that a fixed point z is in Ω' (see [Wa],[OS],[G1]).

We wish to consider a stronger condition—local similarity. Say A and B are locally similar if for each $z \in \Omega$, there exists a neighborhood Ω' of z such that A and B are similar over the ring of holomorphic functions on Ω'. This is equivalent to asserting that A and B are similar over localization of R at $P_z = \{ f \mid f(z) = 0 \}$ for each $z \in \Omega$ (this is not obvious). We shall show (Theorem 4.1) that this is equivalent to A and B being globally similar. In Section 5, we apply this to obtain results about pointwise similarity.

In order to solve this problem, one needs to consider representations of finitely generated R-algebras. We show (Section 3) that R satisfies some very nice algebraic properties. In particular, R is Bézout, has one in the stable range, and its quotient field has trivial Brauer group. We study the problem in an algebraic setting.

The problem can be generalized to the case of a commutative ring R. One replaces Ω by a subset of $\operatorname{Spec} R$. In Section 4, we establish sufficient conditions for a local-global principle to hold (which includes rings of analytic functions). In Section 6, for a certain class of rings (including orders over Dedekind domains), we describe a method for determining by

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how much the local-global principle fails. These results have applications to various cancellation problems.

These types of problems can all be viewed as studying representations which become equivalent under certain extension of scalars. This point of view is discussed in Section 7. In particular, we give a proof of the Noether-Deuring Theorem.

2 Some Preliminary Results

In this section, we state and prove some results which will be useful later. Let R be commutative ring with 1. Then $\operatorname{Spec} R$ is the set of prime ideals of R. If Λ is an R-algebra and M and N are Λ-modules, write $M_p = M \otimes_R R_p$,where R_p is the localization of R and P for some P in $\operatorname{Spec} R$. So M_p is a Λ_p-module. The Krull dimension of R is the maximum length of a chain of prime ideals in R. We say that *one is in the stable range* of a ring S if $ax + b = 1$ implies $a + by$ is a unit for some y in S. This definition is left-right symmetric (this is not obvious). We first record some properties of zero dimensional rings (i.e., maximal ideals are minimal primes). See [GW] for proofs. In particular, the result applies to local rings.

Lemma 2.1 *Let J be the Jacobson radical of R, and assume that R/J has Krull dimension zero. Let Λ be a module finite R-algebra. Let M be a finitely generated Λ-module.*

(a) One is in the stable range of $E = \operatorname{End}_\Lambda(M)$.

(b) If N and X are finitely generated Λ-modules, then $M \oplus X \cong N \oplus X$ implies $M \cong N$.

(c) Let tM denote t copies of M. Then $tM \cong tN$ implies $M \cong N$.

(d) If M and N are finitely presented, then $M_P \cong N_P$ for all P in $\operatorname{Spec} R$ implies $M \cong N$.

Lemma 2.2 *Let Λ be a finitely generated R-algebra. Let M and N be Λ-modules which are finitely generated as R-modules.*

(a) $E = \operatorname{End}_\Lambda(M)$ is a direct limit of module finite R-algebras.

(b) If R is noetherian, then $\operatorname{Hom}_\Lambda(M, N)$ is a finitely generated R-module.

(c) If R is a Prüfer domain (i.e. finitely generated ideals are projective) and M and N are R-projective, then $\operatorname{Hom}_\Lambda(M, N)$ is finitely generated as an R-module.

Proof: (a) follows from the observation that E is the homomorphic image of a subalgebra of $M_n(R)$, the ring of $n \times n$ matrices over R. (b) is obvious. Let $\lambda_1, \cdots, \lambda_s$ be generators for A over R. Consider the exact sequence

$$0 \to \mathrm{Hom}_\Lambda(M, N) \to \mathrm{Hom}_R(M, N) \xrightarrow{\tau} \bigoplus_{i=1}^{s} \mathrm{Hom}_R(M, N),$$

where $\tau(\sigma) = (\sigma\lambda_1 - \lambda_1\sigma, \cdots, \sigma\lambda_s - \lambda_s\sigma)$. Since M and N are finitely generated projective modules, so is $\mathrm{Hom}_R(M, N)$. Since R is Prüfer, the image of τ is projective, and so $\mathrm{Hom}_\Lambda(M, N)$ is an R-summand of $\mathrm{Hom}_R(M, N)$.

In certain situations, one only wants to work with a subset Ω of $\mathrm{Spec}\, R$ (e.g., if R is a ring of functions on Ω). The next result says this is sufficient under suitable conditions.

Lemma 2.3 *Assume R is a Prüfer domain. Let Λ be a finitely generated R-algebra. Let M and N be Λ-modules which are finitely generated projective R-modules. Suppose Ω is a subset of* $\mathrm{Spec}\, R$ *such that if I is a finitely generated ideal of R, then I is contained in some element of Ω. Then $M_P \cong N_P$ for all P in Ω implies $M_P \cong N_P$ for all P in* $\mathrm{Spec}\, R$.

Proof: First assume that M and N are free. Since $M_P \cong N_P$ for P in Ω, this implies $M \cong N$ as R-modules. Thus one can define the determinant of an element in $\mathrm{Hom}_R(M, N)$. By Lemma 2.2, there exists $\sigma_1, \ldots, \sigma_s$ a set of R-generators for $\mathrm{Hom}_R(M, N)$. Define $f(x_1, \cdots, x_s) = \det(x_1\sigma_1 + \cdots + x_s\sigma_s)$. Since $M_P \cong N_P$ for P in Ω, f takes on values outside of P. Hence by hypothesis, the coefficients of f generate R as an ideal. Let P be in $\mathrm{Spec}\, R$. If R/P is infinite, then clearly f represents an element not in P, and so $M_P \cong N_P$. If R/P is finite, pass to a faithfully flat extension S in which f does represent a unit (e.g. take S to be $R[x]$, localized at the set of polynomials whose coefficients are not contained in any maximal ideal). Then $M \otimes_R S \cong N \otimes_R S$ and so by the Noether-Deuring Theorem (see Section 7), $M_P \cong N_P$ for all P.

If M and N are not free, choose projective R-modules M' and N' such that $M \oplus M'$ and $N \oplus N'$ are free R-modules of the same finite rank. We can assume that Λ is a free R-algebra. Extend the action of Λ to M' and N' by letting the generators act trivially on them. By the previous paragraph, $M \oplus M'$ is locally isomorphic to $N \oplus N'$. Clearly M' and N' are locally R-isomorphic (and hence locally Λ-isomorphic). By local cancellation (Lemma 2.1(b)), this implies M and N are locally isomorphic.

Lemma 2.4 *Let Λ be an R-algebra and M a finitely presented Λ-module.*

> *(a) If R' is a flat commutative extension of R, then* $\mathrm{Hom}_{\Lambda'}(M', N') \cong \mathrm{Hom}_\Lambda(M, N) \otimes_R R'$, *where $\Lambda' = \Lambda \otimes_R R'$.*

(b) The map $\theta: N \mapsto \mathrm{Hom}_\Lambda(M, N)$ *is an additive bijection from the category of* Λ*-modules which are summands of* tM *for some* t *to the category of finitely generated projective* $E = \mathrm{End}_\Lambda(M)$*- modules. Moreover,* θ *also induces a bijection between the genus of* M*,* $\mathrm{G}(M) = \{N \mid N_P \cong M_P \text{ for all } P\}$ *and* $\mathrm{G}(E)$.

Proof: This is well known. Note that $M \otimes_E \mathrm{Hom}_\Lambda(M, N) \cong N$ (via $m \otimes \sigma \leftrightarrow \sigma(m)$). See also [G2].

We remark that if R is a domain (or more generally reduced with only finitely many minimal primes) and M and N are finitely generated torsion free R-modules, then (a) and (b) also hold (cf. [W2, 3.5]).

If Λ is a ring, we say that *n is in the stable range of* Λ if $\alpha_1\Lambda + \cdots + \alpha_n\Lambda + \beta\Lambda = \Lambda$ implies there exist $\lambda_1, \cdots, \lambda_n \in \Lambda$ with $\Lambda = \Sigma(\alpha_i + \beta\lambda_i)\Lambda$. If this holds, write $\mathrm{sr}(\Lambda) \leq n$. The next proof is based on [G, 4.4].

Lemma 2.5 *Suppose* Λ *is a subring of* Γ *and* I *is a common two sided ideal of* Λ *and* Γ*. Then* $\mathrm{sr}(\Lambda) \leq max\{\mathrm{sr}(\Gamma), \mathrm{sr}(\Lambda/I)\} = n$.

Proof: Assume $\alpha_1\Lambda + \cdots + \alpha_n\Lambda + \beta\Lambda = \Lambda$. Since $\mathrm{sr}(\Lambda/I) \leq n$, there exist $\alpha_i' = \alpha_i + \beta a_i$ with $\alpha_i'\Lambda + \cdots + \alpha_n'\Lambda + I = \Lambda$. So we can assume $\alpha_i = \alpha_i'$. Thus

$$1 = (\Sigma\alpha_i b_i) + d$$

for some $b_i \in \Lambda$ and $d \in I$. Also, there exist c, $c_i \in \Lambda$ with

$$1 = \Sigma\alpha_i c_i + \beta c.$$

Thus $d = \Sigma\alpha_i c_i d + \beta cd$, whence

$$1 = \Sigma\alpha_i b_i + \Sigma\alpha_i c_i d + \beta cd = \Sigma\alpha_i(b_i + c_i d) + \beta cd$$

So by replacing β by βcd, we can assume $\beta \in I$. Set $e_i = b_i + c_i d$. Then $\Sigma\alpha_i e_i + \beta = 1$. Squaring this expression yields $\Sigma\alpha_i\Lambda + \beta^2\Lambda = \Lambda$. Since $\mathrm{sr}(\Gamma) \leq n$, this implies that $\Sigma(\alpha_i + \beta^2 f_i)\Gamma = \Gamma$ for some $f_i \in \Gamma$. Then $g_i = \beta f_i \in I \subset \Lambda$. Set $J = \Sigma(\alpha_i + \beta g_i)\Lambda$. Then $J\Gamma = \Gamma$, and $J \supset JI = JI\Gamma = J\Gamma I = I$. Since $J + I = \Lambda$, this implies $J = \Lambda$, as desired.

We shall need the next well known result for reference (cf. [W2]).

Lemma 2.6 *Assume* $\mathrm{sr}(\Lambda) = 1$*. If* P *is a finitely generated projective* Λ*-module, then* $\mathrm{sr}(\mathrm{End}_\Lambda(P)) = 1$*. In particular,* $\mathrm{sr}(M_n(\Lambda)) = 1$.

One can ask about the stable range of other overrings. It is apparently still open as the whether integral extensions of commutative rings R with $\mathrm{sr}(R) = 1$ also have this property. One case that is trivial to verify is the following:

Lemma 2.7 *Let R be Bézout domain (i.e. finitely generated ideals are principal) with quotient field K. If S is an overring of R contained in K, then* $\mathrm{sr}(S) \leq \mathrm{sr}(R)$. *Also S is a Bézout domain.*

We give an example to show that some hypothesis is necessary.

Example 2.8 Let $T = k[u, v]$ be the polynomial ring in two variables over a field k. Let R be the ring of polynomials $T[x]$ localized at the set of primitive polynomials (i.e. polynomials $f(x) = \sum a_i x^i$ such that $T = \sum a_i T$). It is easy to verify that $\mathrm{sr}(R) = 1$ (c.f., [VK]). Let $R' = R[w^{-1}]$, where $w = u^2 + v$. We claim that $\mathrm{sr}(R') \neq 1$. Note that $uR' + vR' = R'$, Suppose that $u + vt$ is a unit for some $t \in R'$. By multiplying by some unit of R', this yields

$$gw^n u + vs = w^m f,$$

where $m, n \geq 0$, f and g are primitive polynomials in $T[x]$ and $s \in T[x]$. By substituting in $v = 0$, we obtain

$$g_0 u^{2n+1} = u^{2m} f_0,$$

where f_0, g_0 are obtained from f, g by evaluation at $v = 0$. Thus either f_0 or g_0 is a multiple of u. However this implies that f or g is in the ideal of $T[x]$ generated by u and v. This contradicts the primitivity.

Lemma 2.9 *Let S be a ring with T a two sided ideal. If T is semiprime and artinian as a left S-module, then T is generated by a central idempotent.*

Proof: Let I be a minimal left S-ideal of T. Since T is semiprime, $TI \neq 0$, so $TI = I^2 = I$. Also if I' is a nonzero left ideal of T contained in I, then TI' is S-invariant, so as $TI' \neq 0$, $I' = I$. Thus as T is artinian as an S-module and every minimal submodule of T is a summand, it follows that T is artinian semisimple. In particular, $T = eT = Te$, where e is the identity of T. If $s \in S$, then $es = ese = se$, and the result follows.

Proposition 2.10 *Let R be a Prüfer domain such that $(R/I)/\mathrm{rad}(R/I)$ is von Neumann regular whenever $I \neq 0$. Let T be the R-torsion ideal of a module finite R-algebra Λ. Let J be the Jacobson radical of Λ. If $J \cap T = 0$, then $\Lambda = T \oplus \Lambda_0$ (as rings), where Λ_0 is the annihilator of T.*

Proof: Since R is Prüfer, T is an R-summand of Λ, whence finitely generated. So $fT = 0$ for some nonzero f in R. Let K/fR be the radical of R/fR. Thus $KT = 0$ and R/K is von Neumann regular. If P is a maximal ideal of R, then T_P is finite dimensional over R/P. Moreover, T_P is semiprime. Thus the result holds locally by Lemma 2.9, whence globally.

3 Properties of Rings of Holomorphic Functions

Throughout this section Ω will denote a noncompact Riemann surface $R = \mathrm{H}(\Omega)$, the ring of holomorphic functions on Ω, and $K = \mathrm{M}(\Omega)$, the field of meromorphic functions on Ω. If $z \in \Omega$, let $P_z = \{ f \in R \mid f(z) = 0 \}$. Let R_z denote the localization of R at P_z. This is somewhat smaller than $\tilde{R}_z$, the ring of germs of analytic functions at z, which is contained in the completion of R_z. Note that this completion is the ring of formal power series. We record some properties of R.

Lemma 3.1 ([F, Theorem 25.5]) *Given a discrete subset X of Ω and non-negative integers n_x, $x \in X$, there exists $f \in R$ such that the multiplicity of f at x is n_x.*

Lemma 3.2 *R_z is a local principal ideal domain.*

Lemma 3.3 (Strong Approximation) *Let X be a discrete subset of Ω. Then given positive integers n_x, $x \in X$ and functions f_x holomorphic about x, there exists $f \in R$ such that $f \equiv f_x \bmod (P_x)^{n_x}$. Moreover, if $f_x(x) \neq 0$ for each $x \in X$, we can choose f to be a unit of R.*

Proof: Choose $h \in R$ such that X is exactly the set of zeroes of h and that the order of the zero is n_x. Let $U_x = (\Omega - X) \cup \{ x \}$. Then $\{U_x\}$ is an open cover of Ω. Define a meromorphic function $g_x = f_x/h$ on U_x. If $x \neq y$, then $g_x - g_y$ is holomorphic on $U_x \cap U_y \subset \Omega - X$. Hence by [F, Theorem 26.3], there exists a meromorphic function g on Ω with $g - g_x$ holomorphic on U_x for all $x \in X$. Set $f = gh$. Then on U_x, $f = gh = (g - g_x)h + g_x h = (g - g_x)h + f_x$. Since $g - g_x$ is holomorphic on U_x, so is f. Thus $f \in R$. Since $h \in (P_x)^{n_x}$,$f \equiv f_x \bmod (P_x)^{n_x}$.

Moreover, if $f_x(x) \neq 0$ for each $x \in X$ then $f_x \equiv e^{d_x} \bmod (P_x)^{n_x}$ for some analytic d_x. Hence by the previous paragraph, there exists $d \in R$ with $d \equiv d_x \bmod (P_x)^{n_x}$ for each x, and so $f = e^d \equiv f_x \bmod (P_x)^{n_x}$, with f a unit.

Recall that a ring is *Bézout* if every finitely generated ideal is principal.

Lemma 3.4 *(a) If f,$g \in R$ with no common zero, then $f + gh$ is a unit of R for some $h \in R$. (b) R is Bézout.*

Proof: For part (a), let X be the set of zeroes of g. This is discrete (if $g \neq 0$). Since f does not vanish on X, by the previous result, there exists a unit $u \in R$ such that $u \equiv f \bmod (P_x)^{n_x}$, where n_x is the multiplicity of the zero of g at x. Hence $h = (u - f)/g \in R$, and $u = f + gh$, as desired.

For part (b), let f, $g \in R$. Let X be the set of common zeroes of f and g. Choose h such that h vanishes only on X, and the order of the zero is the minimum of the orders for f and g. Then f/h and $g/h \in R$ and have no common zeroes. So by (a), $1 = a(f/h) + b(g/h)$ in R. Hence

$fR + gR = hR$. We wish to apply these results to certain extensions of R by means of the following:

Proposition 3.5 *Let K' be a finite dimensional field extension of K. Then there exists a finite branched covering Ω' of Ω such that K' is the field of meromorphic functions of Ω'. If R' is the ring of holomorphic functions on Ω', then R' is the integral closure of R in K'.*

Proof: The first statement is [F, Theorem 8.12]. The fact that R' is the integral closure of R follows from [F, Theorems 8.2 and 8.3].

Corollary 3.6 *Let $\bar{R}$ be the integral closure of R in the algebraic closure of K. Then*

(a) $\bar{R}$ is Bézout,

(b) $\mathrm{sr}(\bar{R}) = 1$, and

(c) $\bar{R}$ satisfies the primitive criterion (i.e. given $f(x) = \sum a_i x^i \in \bar{R}[x]$ with $\bar{R} = \sum a_i \bar{R}$, then f represents a unit in $\bar{R}$).

Proof: (a) and (b) follow from the two previous results. Now (c) follows from (a) and (b) by [G3, Lemma 5.2] .

Note that R itself does not in general satisfy the primitive criterion (see [EG, Example 5.5].)

The next result shows that no division rings arise over K. The following proof is based on a letter of M. Artin. By an *R-order* in a K-algebra A, we mean an integral subalgebra Λ such that $K\Lambda = A$.

Proposition 3.7 *Let A be a simple finite dimensional K-algebra. If Γ is maximal R-order of A, then $\Gamma \cong M_n(R')$ (and $A \cong M_n(K')$), where K' is the center of A and R' is the integral closure of R in K'. In particular, K has trivial Brauer group.*

Proof: By Lemma 3.5, we can assume $K = K'$. Since $\tilde{R}_x$ is a discrete valuation ring with algebraically closed residue field and the group of units is divisible, it follows that the Brauer group of its quotient field $\tilde{K}_x$ is trivial (this also completes the proof if Ω is simply connected—use Lemma 3.1 instead of the fact that $\tilde{R}_x$ is a local pid.)

Suppose $\dim A = n^2$. Then $\tilde{\Gamma}_x = \Gamma \otimes_R \tilde{R}_x$ is a maximal order in $\tilde{A}_x \cong M_n(\tilde{K}_x)$. Since $\tilde{R}_x$ is a pid, $\tilde{\Gamma}_x \cong M_n(\tilde{R}_x)$. Thus there exists an open cover $\mathcal{O}$ of Ω such that for $U \in \mathcal{O}$, $\phi_U : \Gamma_U \to M_n(R_U)$ is an isomorphism, where R_U is the ring of holomorphic functions on U. If $U, V \in \mathcal{O}$ with $U \cap V$ nonempty, then ϕ_U and ϕ_V are two representations of $\Gamma_{U\cap V} = \Gamma \otimes_R R_{U\cap V}$ onto $M_n(R_{U\cap V})$. Since $R_{U\cap V}$ is Bézout, any two representations are equivalent. Hence $\alpha(U,V)\phi_V = \phi_U\alpha(U,V)$ for some $\alpha(U,V) \in \mathrm{GL}_n(R_{U\cap V})$. Moreover α is uniquely determined up to a scalar. It is straightforward to verify that $\alpha \in \mathrm{H}^1(M, \mathrm{PGL}_n)$, where PGL_n is the (nonabelian) sheaf associated to $\mathrm{PGL}_n(R)$.

Consider the sequences of sheaves

$$1 \to Z \to \mathfrak{A} \stackrel{exp}{\to} \mathfrak{A}^* \to 1, \text{ and}$$

$$1 \to \mathfrak{A}^* \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1,$$

where $\mathfrak{A}$ is the sheaf of germs of analytic functions and $\mathfrak{A}^*$ is the sheaf on nonvanishing germs of analytic functions. Since $\mathrm{H}^2(\Omega, \mathfrak{A}) = 0$ (c.f., [H, p. 178]) and $\mathrm{H}^3(\Omega, Z) = 0$ (by dimension), it follows that $\mathrm{H}^2(\Omega, \mathfrak{A}^*) = 0$. Since $\mathrm{H}^1(X, \mathrm{GL}_n) = 0$ (c.f., [F, Corollary 30.5]), we have $\mathrm{H}^1(X, \mathrm{PGL}_n) = 0$. Hence $\alpha(U, V) = \beta(U)\beta(V)^{-1}$ for some $\beta \in \mathrm{H}^0(X, \mathrm{PGL}_n)$. Now replace ϕ_U by $\beta(U)^{-1}\phi_U\beta(U)$ (this is independent of the lift of $\beta(U)$ to $\mathrm{GL}_n(R_U)$). Then $\phi_U = \phi_V$ on $U \cap V$. Thus ϕ defines a global map from Γ into $M_n(R)$. Since ϕ is locally an isomorphism, it is globally, and the result follows.

In the case Ω is a compact Riemann surface, the triviality of the Brauer group is a classical result of Tsen. In fact Tsen proves that the field satisfies certain stronger properties. We do not know if this is still true in the noncompact case. One can derive results about quadratic forms and the Witt ring of K from Proposition 3.7. For example, it follows that any quadratic form in two variables is universal (i.e. $ax^2 + by^2 = c$ always has a solution for $ab \neq 0$ in K), and so any quadratic form is a sum of hyperbolic planes and either a one or two dimensional space.

We need to record some other properties of R. Recall that the Krull dimension of a commutative ring is the maximum length of a chain of prime ideals.

Proposition 3.8 *If I is a nonzero ideal of R, then $S_I = (R/I)/\operatorname{rad}(R/I)$ is von Neumann regular.*

Proof: Choose $0 \neq f \in I$. By Lemma 3.3,

$$R/fR \cong \prod_{x \in Z(f)} R/(P_x)^{n_x},$$

where $Z(f)$ are the zeroes of f and n_x is the multiplicity of the zero of f at x. Hence S_I is a direct product of fields, and the result follows.

Proposition 3.9 *Let K' be a finite dimensional field extension of K, and let R' denote the integral closure of R in K. Suppose $R \subset S \subset R'$ and S has quotient field K'.*

(a) There exists $0 \neq \delta \in R'$ with $\delta R' \subset S$.

(b) R' is a finitely generated R-module.

Proof: (a) follows just from the fact that K' is separable. Just choose α in S with $K' = K[\alpha]$, and take $\delta \in R$ to be the discriminant of α.

For (b), observe that for each x, R_x is pid so there exist $\lambda_{x,i} \in R'$, $1 < i < [K' : K]$, with $R' = \sum R\lambda_{x,i}$. Choose $\lambda_i \in R'$ so that λ_i approximates $\lambda_{x,i}$ as closely as possible at each x in the zeros of δ. Let $T = \sum R\lambda_i + R[\alpha]$. If x is not a zero of δ, then $T_x = R_x[\alpha] = R'_x$; while if x is a zero of δ then $T_x = \sum R_x\lambda_i = R'_x$ (by Nakayama's lemma). From this, it is easy to deduce that $T = R'$ is finitely generated.

4 One-Dimensional Rings Satisfying a Local-Global Principle

In this section R will denote an integrally closed integral domain with quotient field K satisfying the following conditions for any finite dimensional extension K' of K:

(4.1a) The integral closure R' of R in K' is Bézout.

(4.1b) $\mathrm{sr}(R') = 1$.

(4.1c) $Br(K') = 0$.

(4.1d) If I is a nonzero ideal of R, then $(R/I)/\operatorname{rad}(R/I)$ is von Neumann regular.

(4.1e) If S is an R-subalgebra of R' with quotient field K', then $\delta R' \subset S$ for some nonzero $\delta \in R$.

Examples of such rings include the ring of all algebraic integers, the ring of holomorphic functions on a noncompact Riemann surface (see the previous section), and semilocal domains whose quotient field is algebraically closed, and the ring of all algebraic integers. The crucial conditions for our purposes are (a) and (b). It may be possible to eliminate (c), and this is possible when considering the problem of matrix similarity. One can avoid (e) by working in characteristic zero or in orders in separable K-algebras.

Fix a subset Ω of $\operatorname{Spec} R$ such that if $r \in R$ is not a unit, then $r \in P$ for some P in Ω (e.g., if R is the ring of holomorphic functions on Ω then Ω suffices). The main result of this section is a local-global principle for modules over R-algebras.

Theorem 4.1 *Let Λ be a finitely generated R-algebra (where R satisfies (4.1a-e)). Let M and N be Λ-modules which are finitely generated free R-modules. The following are equivalent:*

(i) $M_p \cong N_P$ for all $P \in \Omega$.

(ii) $M_p \cong N_P$ for all $P \in \operatorname{Spec} R$

(iii) $M \cong N$.

Proof: Clearly (iii) implies (i). Since R is Bézout, (i) implies (ii) by Lemma 2.3.

So assume (ii) holds. By Lemmas 2.2 and 2.4, we can assume $M = \Lambda$ and N is a projective Λ-module. Let $A = \Lambda \otimes_R K$. Since Λ is a free R-module, Λ embeds in A. Let J be the Jacobson radical of A, and set $I = \Lambda \cap J$. Since I is nilpotent, $\Lambda/I \cong N/IN$ if and only if $\Lambda \cong N$. Moreover, since Λ/I is R-torsionfree, it is in fact R-free. So we can assume A is a semisimple K-algebra. By (4.1c), $A = \bigoplus M(n_i, K_i)$, where K_i is a finite dimensional field extension of K. Let R_i be the integral closure of R in K_i, and set $T = \bigoplus R_i$. Then $T\Lambda$ is a module finite T-algebra. So by (4.1a), $T\Lambda = \Gamma \cong \bigoplus M(n_i, R_i)$. Since Γ is finitely generated over T and $A = K\Lambda$, there exists $0 \neq d \in R$ with $d\Gamma \subset R\Lambda$. Let $Z = \Lambda \cap T$. Let Z_i be the projection of Z onto R_i. Since $KZ = T$, Z_i and R_i have the same quotient field. Thus $0 \neq fR_i \subset Z_i$ for some $0 \neq f \in R$. Let e_i be the central idempotent in R_i. Then $ge_i \in \Lambda$ for some $0 \neq g \in R$. Hence $gZ_i \subset gZe_i \subset \Lambda$. Thus $gfd\Gamma \subset gfT\Lambda \subset \Lambda$. Set $c = gdf$.

Let R_c be the ring obtained from R by inverting all elements of R relatively prime to c. Thus every maximal ideal of R_c contains c, and so R_c modulo its Jacobson radical is zero dimensional. Thus by Lemma 2.1, $\Lambda \otimes_R R_c = \Lambda_c \cong N_c$. Since each R_i is Bézout, $\Gamma N \cong \Gamma\Lambda = \Gamma$, and so we can assume that $N \subset \Gamma N = \Gamma$. Since $\Lambda_c \cong N_c$, it follows that $N_c = \Lambda_c \alpha$ for some $\alpha \in A$. Since $\Gamma_c = \Gamma_c N_c = \Gamma_c \alpha$, this implies α is a unit in Γ_c. Without loss of generality, we can also assume that $\alpha \in \Gamma$. Hence $\Gamma = \alpha\Gamma + c\Gamma$. By (4.1b) and Lemma 2.6, $\mathrm{sr}(\Gamma) = 1$, and so $\alpha + c\gamma$ is a unit in Γ. Now set $L = \Lambda(\alpha + c\gamma)$. Note that if $c \notin P$, then $L_P = \Gamma_P = \Lambda_P = N_P$ (as $c\Gamma \subset \Lambda$). Also $\beta = (\alpha + c\gamma)\alpha^{-1} \equiv 1 \bmod c\Gamma_c$. Hence β is a unit in Λ_c. Thus $L_c = \Lambda_c\alpha = N_c$. Thus $N = L = \Lambda(\alpha + c\gamma) \cong \Lambda$, as desired.

Corollary 4.2 *Assume the hypotheses of the theorem,*

(i) If $tM \cong tN$, *then* $M \cong N$.

(ii) If $M \oplus X \cong N \oplus X$ *for* X *a finitely generated* Λ*-module, then* $M \cong N$.

Proof: The results hold locally (c.f., [GW]), whence globally by the theorem.

One other observation will be useful later.

Proposition 4.3 *If* Λ *is a module finite* R*-algebra, then* $\mathrm{sr}(\Lambda) = 1$.

Proof: Since Λ is module finite, it is a homomorphic image of a subalgebra of $M_n(R)$. So we can assume $\Lambda \subset M_n(R)$. Moreover, we can assume that the nilradical of $\Lambda = 0$. Hence Λ is an order in a semisimple K-algebra

A. Let Γ be a maximal R-order in A. Then $\Gamma = \bigoplus M(n_i, R_i)$ where R_i is the integral closure of R in a finite dimensional field extension. As in the proof of the theorem, $0 \neq c\Gamma \subset \Lambda$, for some $0 \neq c \in R$. Since $\Lambda/c\Gamma$ is a module finite R/c algebra, and R satisfies (4.1d), it follows from [GW] that $\mathrm{sr}(\Lambda/c\Gamma) = 1$. By Lemma 2.6, $\mathrm{sr}(\Gamma) = 1$. Hence by Lemma 2.5, $\mathrm{sr}(\Lambda) = 1$.

One can give another proof using the theorem and results in [G3].

There is a cohomological interpretation of Theorem 4.1 which we state without proof.

Corollary 4.4 *Let R be the ring of analytic functions on a noncompact Riemann surface. Let Λ be a module finite free R-algebra. Let $\mathcal{G}$ be the sheaf associated to the group of units of Λ. Then* $\mathrm{H}^1(\Omega, \mathcal{G}) = 0$.

It is also worthwhile to note that when R is the ring of analytic functions on a noncompact Riemann surface, there are several notions of local isomorphism. One can consider the localization, the ring of germs at a point, or the ring of formal power series. Since the latter two are faithfully flat extensions of the first, it follows by Section 7 that all of these notions are the same.

We close this section by observing that the result hold for modules as well as lattices.

Corollary 4.5 *Let R satisfy the hypotheses of (4.1). If Λ is a module finite R-algebra and M and N are finitely presented Λ-modules such that $M_p \cong N_p$ for all maximal ideals P of R (or a sufficiently large subset), then $M \cong N$.*

Proof: By Lemma 2.4, we can assume M and N are projective. Let T be the R-torsion ideal of R and J the Jacobson radical of Λ. So $M \cong N$ if and only if $M/JM \cong N/JN$. Hence we can assume $J = 0$. By Proposition 2.10, $\Lambda = \Lambda_0 \oplus T$. Thus we can consider the two cases separately. If $\Lambda = \Lambda_0$, Theorem 4.1 applies. If $\Lambda = T$, the result follows by [GW] or Lemma 2.1.

5 Pointwise Equivalence of Representations

In this section, we consider a weaker condition than local equivalence of modules (or representations). Let R be a commutative ring with 1, and fix a subset Ω of $\operatorname{Spec} R$. If $a \in R$, let $a(P)$ denote its image in R/P (and similarly for polynomials, matrices, etc.) Let $K(P)$ denote the quotient field of R/P. If Λ is an R-algebra and M is a Λ-module, set $M(P) = M \otimes_R K(P)$. So $M(P)$ is a $\Lambda(P) = \Lambda \otimes_R K(P)$ module. If M and N are Λ-modules such that $M(P) \cong N(P)$ as $\Lambda(P)$-modules for all $P \in \Omega$, we say M and N are pointwise isomorphic on Ω. This is equivalent to saying that $M \otimes_R S \cong N \otimes_R S$ as $\Lambda \otimes_R S$-modules, where S is the direct product of the $K(P)$, P in Ω. Since $K(P) = R_P/PR_P$, $M_P \cong N_P$ as Λ_P-modules obviously implies $M(P) \cong N(P)$. It is easy to see that $M(P) \cong N(P)$ does not imply $M_P \cong N_P$ (e.g., take $M = R$ and $N = R/P$). In fact, even assuming $M(P) \cong N(P)$ for all $P \in \Omega$ does not imply $M_Q \cong N_Q$ for all $Q \in \Omega$. Choose a maximal ideal P of R with $z \in P - P^2$. Consider the representations of $R[x]$ given be the two matrices,

$$\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & z^2 \\ 0 & 0 \end{pmatrix}.$$

The corresponding modules M and N satisfy $M_Q \cong N_Q$ for all Q with z not in Q, $M(P) \cong N(P)$, but M_P is not isomorphic to N_P. Another example is obtained by considering

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A^{\mathsf{T}}.$$

Theorem 5.1 *Let R be an integral domain with quotient field K. Assume Λ is a module finite R-algebra and M and N are finitely presented Λ-modules. Then the following are equivalent:*

(i) $M(P) \cong N(P)$ for a dense subset Ω of Spec R *(i.e., $\bigcap P = 0, P \in \Omega$).*

(ii) $M \otimes_R K \cong N \otimes_R K$ as $\Lambda \otimes_R K$-modules.

(iii) $M(P) \cong N(P)$ for a dense open subset of Spec R.

Proof: Assume (i) holds and set $S = \prod K(P)$, $P \in \Omega$. Since Ω is dense, K embeds in $K \otimes_R S = T$. Thus $M \otimes_K T \cong N \otimes_K T$ as $A \otimes_K T$-modules where $A = \Lambda \otimes_R K$ is finite dimensional K-algebra. By the Noether-Deuring theorem (see Section 7), this implies $M \otimes_R K \cong N \otimes_R K$ as A-modules. So (ii) holds.

If $M \otimes_R K \cong N \otimes_R K$ as $\Lambda \otimes_R K$-modules, then we can assume the isomorphism is given by $\sigma \otimes 1$ for some $\sigma \in \mathrm{Hom}_\Lambda(M, N)$. $L = \ker \sigma$ and $N/\sigma(M)$ are both torsion modules. Since N is finitely presented, there exists $0 \neq d \in R$ with $dN \subset \sigma(M)$. Set $R' = R[1/d]$. Then σ induces a surjection from $M \otimes_R R'$ onto $N \otimes_R R'$. Since $N \otimes_R R'$ is finitely presented, $L \otimes_R R'$ must be finitely generated (as an R'-module). Thus there exists some nonzero multiple f of d with $fL = 0$. Thus $\sigma(P)$ induces an isomorphism form $M(P)$ to $N(P)$ for any $P \in \Omega = \{ Q \in \mathrm{Spec}\, R \mid f \notin Q \}$. This is the desired dense (and open) subset of R.

See [G1] or [OS] for some what different proof in the matrix case. One can generalize this to rings other than domains.

In [Wa], [OS], and [G1], various conditions in the matrix case were discussed which forced pointwise equivalence at P to imply local equivalence on some neighborhood of P (which is the same as equivalence over R_P). These can be extended.

Let Λ be a finitely generated R-algebra. If M and N are Λ-modules which are finitely generated as R-modules, define $\nu_P(M,N)$ to be the smallest nonnegative integer ν such that

$$\phi\colon \mathrm{Hom}_{\Lambda_P}(M_P, N_P) \to \mathrm{Hom}_{\Lambda_P}(M(P), N(P))$$

and

$$\phi_\nu\colon \mathrm{Hom}_{\Lambda_P}(M_P/P^{\nu+1}M_P, N_P/P^{\nu+1}N_P) \to \mathrm{Hom}_{\Lambda_P}(M(P), N(P))$$

have the same image. If no such integer exists, set $\nu_P(M,N) = \infty$. It follows from the Artin-Rees Lemma (c.f., [G1]) that if R_P is noetherian, then $\nu_P(M,N)$ is finite. The following generalizes results in [Wa], [OS], and [G1].

Lemma 5.2 *If $\nu_P(M,N) = 0$ and $M_P \cong N_P$ as R_P-modules, then $M(P) \cong N(P)$ implies $M_P \cong N_P$ as Λ_P-modules.*

Proof: Let α be an isomorphism form $M(P)$ to $N(P)$. Since $v_P(M,N) = 0$, there exists a Λ_P-homomorphism β from M_P to N_P such that the following diagram commutes:

$$\begin{array}{ccc} M_P & \xrightarrow{\beta} & N_P \\ \downarrow & & \downarrow \\ M(P) & \xrightarrow{\alpha} & N(P). \end{array}$$

Since α is surjective, it follows from Nakayama's lemma that β is surjective. Since $M_P \cong N_P$ as R_P-modules, this implies β is injective. Hence $M_P \cong N_P$.

More generally, the proof of Lemma 5.2 shows that if $\nu_P(M,N) = \nu$ and $M_P \cong N_P$ as R-modules, then $M_P/P^{\nu+1}M_P \cong N_P/P^{\nu+1}N_P$ implies $M_P \cong N_P$ (c.f., [G1, Theorem 3.2].) Examples where $\nu_P = 0$ include the case where M is projective or $\Lambda = RG$, G a finite group, and M and N are permutation modules.

In the case R_P is a principal ideal domain, one can explicitly compute $\nu_P(M,N)$. We do this only in the torsion free case. So assume R is a local principal ideal domain with quotient field K, Λ is a finitely generated R-algebra M and N are Λ-lattices (i.e., Λ-modules which are finitely generated R torsion free modules). Let $H = \mathrm{Hom}_R(M,N)$. Let $x_1, \ldots, x_t$ be generators for Λ over R. Then there is an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(M,N) \xrightarrow{T} tH,$$

where $T(\sigma) = (\sigma x_1 - x_1\sigma, \ldots, \sigma x_t - x_t\sigma)$. So T is a linear transformation between two free R-modules. Hence T has a matrix representation as $\mathrm{Diag}\,(\mathrm{p}^{\mathrm{e}_1}, \ldots, \mathrm{p}^{\mathrm{e}_s}, 0, \ldots, 0)$ with $e_1 \le e_2 \le \cdots \le e_s$, where $P = pR$ is the maximal ideal of R. By tensoring this sequence with R/P^f, it is easy to see that $\nu_P(M,N) = e_s$ and that $s = \operatorname{rank} T$.

If N' is another Λ-lattice we get a corresponding map T'. If $N' \otimes_R K \cong N \otimes_R K$ and $N'/P^{\nu+1}N' \cong N/P^{\nu+1}N$, where $\nu = \nu_P(M,N)$, then T and T' are equivalent over K and also over $R/P^{\nu+1}$. Hence they are equivalent over R. Then $\nu_P(M,N) = \nu_P(M,N')$. Combining this observation with Lemma 5.2 yields:

Proposition 5.3 *Let R be an integral domain with quotient field K. Suppose Λ is a finitely generated R-algebra, M and N are Λ-modules, and $P \in \operatorname{Spec} R$ such that R_P is a principal ideal domain and M_P and N_P are R_P-free modules of finite rank. Set $\nu = \nu_P(M,N)$. Then $M_P \cong N_P$ if and only if $M_P/P^{\nu+1}M_P \cong N_P/P^{\nu+1}N_P$ and $M(0) \cong N(0)$.*

Proposition 5.3 shows that ν depends only on M not on N. This is not true if R_P is not a principal ideal domain (see [G1]). However, one special case does apply.

Proposition 5.4 *Let R be an integral domain with quotient field K. Let Λ be a finitely generated R-algebra. Assume M and N are Λ-modules such that M_P and N_P are free R_P-modules. If $\nu_P(M,N) = 0$, then $M(0) \cong N(0)$ and $M(P) \cong N(P)$ implies $M_P \cong N_P$.*

Proof: This is proved in the same manner as the previous result. Instead of using the invariant factors, quote [G1, Theorem 3.1].

If A is an $n \times n$ matrix over R, then A determines an $R[x]$-module M isomorphic as an R-module to nR, where x acts on M via multiplication by A. Two matrices determine isomorphic modules if and only if they are similar. Thus, one can define $\nu_P(A,B)$ for a pair of square matrices. There is a canonical form for matrices with $\nu_P(A,A) = 0$.

Proposition 5.5 (G1, Theorem 5.2) *Let A be an $n \times n$ matrix over R. Then $\nu_P(A,A) = 0$ if and only if A is similar over R_P to*

$$\begin{pmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_t \end{pmatrix}$$

where C_i is the companion matrix of $f_i(x)$ in $R_P[x]$ and $f_i | f_{i+1}$.

We can obtain global versions of the preceding results by using Section 4. Let us say a commutative ring R is a *weak LG-ring* if whenever M and N are Λ-lattices, then $M_P \cong N_P$ for all $P \in \operatorname{Spec} R$ implies $M \cong N$. In particular, this includes the rings in Section 4, but also includes other classes of rings. In particular, semilocal rings, or more generally rings R with $R/\operatorname{rad} R$ von Neumann regular satisfy this. See [EG] for other examples.

Theorem 5.6 *Let R be a weak LG Prüfer domain with quotient field K. Suppose Λ is a finitely generated R-algebra and M and N are Λ-lattices with $M \otimes_R K \cong N \otimes_R K$ as $\Lambda \otimes_R K$-modules. Set $\Omega' = \{ P \in \operatorname{Spec} R \mid \nu_P(M, M) = 0$ and $M(P) \cong N(P) \}$. Then there exists a in R such that a is not in P for any P in Ω' with $M \otimes_R R[a^{-1}] \cong N \otimes_R R[a^{-1}]$. In particular, M and N are isomorphic an a dense open subset Ω'' of* $\operatorname{Spec} R$ *with $\Omega'' \supset \Omega'$. If $\Omega' = \operatorname{Spec} R$, then $M \cong N$.*

Proof: Let R' be the ring obtained from R by inverting all elements t in R such that t is not in any element of Ω'. It is an easy exercise to prove that R' is also a weak LG-ring. By Proposition 5.4, $M_P \cong N_P$ for all P in Ω'. Observe that if $t \in R'$ is not a unit, then $t \in PR'$ for some $P \in \Omega'$. Since R (and so R') is Bézout (by the weak LG property), Lemma 2.3 implies that $M \otimes R'_P \cong N \otimes_R R'_P$ as $\Lambda \otimes_R R'_P$-modules for all P in $\operatorname{Spec} R'$. Hence $M \otimes_R R' \cong N \otimes_R R'$. Suppose ϕ is an isomorphism. Without loss of generality, $\phi \in \operatorname{Hom}_\Lambda(M, N)$. Since M and N are R-free of the same rank, $a = \det \phi$ is defined. Since ϕ is an isomorphism on R', a is a unit in R', i.e., a is not in P for any P in Ω'. Let $\Omega'' = \{ P$ in $\operatorname{Spec} R \mid a$ is not in $P \}$. The last statement follows for if $\Omega' = \operatorname{Spec} R$, then $R = R'$.

In particular, we can obtain global versions of the matrix results of Wasow, Ostrowski, Friedland, Ohm and Schneider and the author. We state these only for rings of analytic functions. There are obvious versions for a larger class of rings as well as for sets of matrices.

Theorem 5.7 *Let Ω be a noncompact Riemann surface with R its ring of analytic functions. Let A and B be two $n \times n$ matrices over R. Let $\Omega' = \{ z \in \Omega \mid \nu_z(A, B) = 0$ and $A(z)$ and $B(z)$ are similar$\}$, and assume Ω' nonempty.*

(a) (Generalization of Wasow) There exists an open codiscrete submanifold $\Omega_0 \supset \Omega'$ of Ω such that A and B are similar over R_0, the ring of analytic functions on Ω_0.

(b) If $\Omega' = \Omega$, then A and B are similar over R.

(c) (Generalization of Ostrowski) Let $\Omega_1 = \{ z \mid \nu_z(A, A) = 0 \}$. Then A is similar to C, the rational canonical form on Ω_1 (i.e., over the ring R_1 of analytic functions on Ω_1). Moreover, Ω_1 is an open codiscrete submanifold of Ω.

(d) A is similar to C over R if and only if $\nu_z(A, A) = 0$ for all z.

Proof: (a) is just a restatement of Theorem 5.6 in a special case. Now (b) follows from (a).

By [G1, Theorem 5.2], $\nu_z(A, A) = 0$ if and only if A is similar over R_z to C. Then (c) follows from (a). In particular, if $\Omega_1 = \Omega$, the this implies A is similar to C. Conversely, by [G1, Theorem 5.2], $\nu_z(C, C) = 0$. So if A is similar to C, then $\nu_z(A, A) = 0$ also.

The earlier results mentioned above merely asserted the existence of a neighborhood of a point $z \in \Omega'$ (or Ω_1) satisfying the conclusion.

6 The Genus Class Group and Cancellation

Let Λ be a module finite R-algebra. If M is a finitely presented Λ-module (or R is reduced with only finitely many minimal primes and M is a Λ-module which is a finitely generated torsion free R-module), define the genus of M, $\mathrm{G}(M)$ to be the collection of finitely presented Λ-modules N with $M_P \cong N_P$ for all P in $\operatorname{Spec} R$. By Lemma 2.4, this is in one to one correspondence with $\mathrm{G}(E)$, where $E = \operatorname{End}_\Lambda(M)$. One can put a group structure on finitely generated projective E-modules (via $K_0(E)$) which via the bijection of Lemma 2.4 imposes one on

$$\operatorname{Div} M = \{ N \mid N \text{ is a } \Lambda\text{-summand of } sM \text{ for some } s > 0 \} \supset \mathrm{G}(M).$$

We wish to give a more explicit description of this group structure in a special case. The next result is essentially [W1, Theorem 3.2], (see also [G2]). Write $M|N$ to indicate M is isomorphic to a summand of N.

Lemma 6.1 *Let R be a commutative ring of Krull dimension one with only a finite number of minimal primes. Let Λ be a module finite R-algebra. Assume that A, B, and C are finitely generated Λ-modules such that either they are finitely presented or A is reduced and A, B, and C are R torsion-free. If $C_P|A_P$ and $C_P|B_P$ for each minimal prime P and $C_P|A_P$ or $C_P|B_P$ for each maximal prime P, then $C|A \oplus B$.*

Corollary 6.2 *Let R, Λ and A be as in 6.1. If B_1, $B_2 \in \mathrm{G}(A)$, there exist C_1, $C_2 \in \mathrm{G}(A)$ with $B_1 \oplus B_2 \cong A \oplus C_1$ and $B_1 \oplus C_2 \cong A \oplus A$.*

Now assume R and Λ are as above and M satisfies the conditions of Lemma 6.1. If $N \in \mathrm{G}(M)$, let

$$[N] = \{ N' \in \mathrm{G}(M) \mid N' \oplus kM \cong N \oplus kM \text{ for some } k \}$$

(in fact $k = 1$ suffices). Now define $[N_1] + [N_2] = [N_3]$, where $N_3 \oplus M \cong N_1 \oplus N_2$. This makes $\tilde{\mathrm{G}}(M) = \{ [N] \mid N \in \mathrm{G}(M) \}$ into an abelian group.

We wish to describe $\tilde{\mathrm{G}}(M)$ and obtain some consequences. If $\Lambda \subset \Gamma$ are two R-algebras with a common ideal I, we can compare $\tilde{\mathrm{G}}(\Lambda)$ and $\tilde{\mathrm{G}}(\Gamma)$ via a result of Milnor (see [B, p. 482]). In fact, in the case of interest for us, we can derive this fairly easily. The following will unify certain classical results for orders over Dedekind domains (c.f., [CR] and [G2]), ring orders (see [L], [WW]), and the results of Section 4. Let $\mathrm{U}(\Lambda)$ denote the group of units of Λ.

So for the rest of this section, assume that $\Lambda \subset \Gamma$ are rings such that:

(1) $\Gamma = \bigoplus \mathrm{End}_{R_i}(P_i)$, where R_i is a one-dimensional Prüfer domain and P_i is a finitely generated projective R-module,

(2) Γ is integral over Z, the center of Λ.

(3) There exists an ideal I of Z such that I contains a regular element and $IR \subset \Lambda$.

We wish to study certain Λ-modules. Let $\mathrm{Lat}\,\Lambda$ denote the category of finitely generated Λ-modules which are Z torsionfree (an alternative description is as follows: let K_i denote the quotient field of R_i, and set $K = \bigoplus K_i$; M is in $\mathrm{Lat}\,\Lambda$ if M embeds in $KM = K \otimes_Z M$). Let $\Gamma M = \{\sum \gamma_m m \mid \gamma_m \in \Gamma,\ m \in M\} \subset KM$. So ΓM is a Γ-lattice. Since the genus of ΓM is well understood (in terms of the Picard group of the R_i), we focus our attention on the kernel of $\mathrm{G}(M) \to \mathrm{G}(\Gamma M)$. We show that $\mathrm{G}(M)$ and $\tilde{\mathrm{G}}(M)$ coincide in the case under discussion.

Let $\mathrm{D}(M) = \{N \in \mathrm{G}(M) \mid \Gamma N \cong \Gamma M\}$. We wish to describe $\mathrm{D}(M)$. Set $E = \mathrm{End}_\Lambda(M) \subset F = \mathrm{End}_\Gamma(\Gamma M) \subset \mathrm{End}_A(KM) = B$, where $A = K \otimes_Z \Lambda$. Note $IF \subset E$. If $N \in \mathrm{D}(M)$, then we may assume $N \subset \Gamma N = \Gamma M$. Let Z_I denote the localization of Z at the set of regular elements which are relatively prime to I. Then Z_I is zero dimensional modulo its radical. Hence by [GW], $N_I \cong M_I = (Z_I \otimes_Z M)$. Thus there exists $\alpha \in B$ with $N_I = M_I\alpha$. Since $\Gamma N = \Gamma M$, this implies $\alpha \in \mathrm{U}(F_I)$. Conversely given $\alpha \in \mathrm{U}(F_I)$, define $N_\alpha = M_I\alpha \cap \Gamma M$. Note if $P \in \mathrm{Spec}\, Z$, then $(N_\alpha)_P = M_P$ if P does not contain I, while if $P \supset I$, then $(N_\alpha)_P = (M_P)\alpha$. Hence $N_\alpha \in \mathrm{G}(M)$. Also $\Gamma N_\alpha = \Gamma M$. Thus $N_\alpha \in \mathrm{D}(M)$. It is straightforward to compute that $N_\alpha \cong N_\beta \Leftrightarrow \mathrm{U}(E_I)\alpha\mathrm{U}(F) = \mathrm{U}(E_I)\beta\mathrm{U}(F)$. Note that if $\alpha \equiv \beta \bmod IF_I$, then $N_\alpha = N_\beta$. Thus we obtain:

Proposition 6.3 *There is bijection between* $\mathrm{D}(M)$ *and the set of double cosets* $\mathrm{U}(E/I)\backslash\mathrm{U}(F/I)/\mathrm{U}^*(F/I)$, *where* $\mathrm{U}^*(F/I)$ *is the image of* $\mathrm{U}(F)$ *in* $\mathrm{U}(F/I)$.

Since $F \cong \mathrm{End}_\Gamma(\Gamma M) \cong \bigoplus \mathrm{End}_{R_i}(\tilde{P}_i)$ where $\tilde{P}_i$ is a finitely generated projective R-module, we can define the determinant $\nu\colon F \to T = \bigoplus R_i \to T/IT$. Since F/IF is a direct sum of matrix rings over zero dimensional rings, every element of determinant 1 is a product of elementary matrices and hence is in $\mathrm{U}^*(F/I)$. Combining this with Proposition 6.3 and applying ν yields:

Corollary 6.4 $\mathrm{D}(M)$ *is in one to one correspondence with* $\mathrm{U}(T/I)/\mathrm{U}^*(T/I)\lambda(M)$, *where* $\lambda(M)$ *is the subgroup of* $\mathrm{U}(I)$ *equal to* $\nu(\mathrm{U}(E/I))$.

Note that if T/I is finite, this implies $\mathrm{D}(M)$ is finite. Corollary 6.4 induces a group structure on $\mathrm{D}(M)$. To see that it is the same as the earlier one, we note that $N_\alpha \oplus N_\beta \cong N_{\alpha\beta} \oplus M$ (set $L = M \oplus M$, note both $N_\alpha \oplus N_\beta$ and $N_{\alpha\beta} \oplus M$ are in $\mathrm{D}(L)$, and compute $\nu(N_\alpha \oplus N_\beta) = \nu(N_{\alpha\beta} \oplus M)$.)

Theorem 6.5 *Suppose M_1 and M_2 are faithful Λ-lattices. Then*

$$\lambda(M_1 \oplus M_2) = \lambda(M_1)\lambda(M_2).$$

Proof: Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(E/IE)$, where $E = \mathrm{End}_\Lambda(M_1 \oplus M_2)$. If $\alpha^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, then $dd' + eb' = 1$ in E_2/IE_2, where $E_2 = \mathrm{End}_\Lambda(M_2)$. Since E_2 is integral over Z/I which is zero dimensional, it follows that $\mathrm{sr}(E_2/I) = 1$. Hence $u = d + cb'e$ is a unit of E_2/IE_2 for some e. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & b'e \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} = \begin{pmatrix} a^* & b^* \\ 0 & 1 \end{pmatrix}.$$

Hence $\nu\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \nu(u)\nu(a^*) \in \lambda(M_1)\lambda(M_2)$.

Corollary 6.6 *If $M \oplus X \cong N \oplus X$ where X is a summand of kM for some k, then $M \cong N$.*

Proof: Without loss of generality $X = kM$ and M is faithful. Then by local cancellation, $N \in \mathrm{G}(M)$. Since Γ is Morita equivalent to T, we also have $\Gamma N \cong \Gamma M$. So $N \in \mathrm{D}(M)$. Thus $N \cong N_\alpha$ for some α. Set $L = (k+1)M$. Observe that $N_\alpha \oplus kM = L_\beta$ where $\beta = \mathrm{diag}(\alpha, 1, \ldots, 1)$. Hence $L_\beta \cong L$ implies $\nu(\beta) \in \lambda(L)\mathrm{U}^*(T/I) = \lambda(M)\mathrm{U}^*(T/I)$, and so $N_\alpha \cong M$.

In particular, this implies $\mathrm{G}(M) = \tilde{\mathrm{G}}(M)$.

Corollary 6.7 *If $M \oplus X \cong N \oplus X$, for some lattice X then $M \oplus \Gamma \cong N \oplus \Gamma$.*

Proof: First assume M is faithful. As in the previous proof, $N \cong N_\alpha$ for some α. Thus $\nu(\alpha) \in \mathrm{U}(T/I) = \lambda(M \oplus \Gamma) = \lambda(M)\lambda(\Gamma)$. In the general case, we can replace M by $M \oplus \Gamma$, and then apply Corollary 6.6.

There are many similar results that can be derived by these techniques. We state some without proof. Most of these can be found in [G2] for orders over Dedekind domains. The proofs are essentially unchanged except that we use the fact that T/I is zero dimensional instead of the fact that in [G2], T/I is artinian.

Theorem 6.8 *(a) If M_P is isomorphic to a summand of N_P for all $P \in \mathrm{Spec}\, Z$, then $N \cong M' \oplus N'$ for some $M' \in \mathrm{G}(M)$.*

(b) If $L \in \mathrm{G}(M \oplus N)$, then $L \cong M' \oplus N'$ for some $M' \in \mathrm{G}(M)$ and $N' \in \mathrm{G}(N)$.

(c) If $L \in \mathrm{G}(tM)$, then $L \cong (t-1)M \oplus M'$.

Theorem 6.9 *If M_P is isomorphic to a summand of N_P for all P in* $\operatorname{Spec} Z$ *and the multiplicity of each A-composition factor in KN is strictly larger than KM, then M is isomorphic to a summand of N. In particular, if M_P is isomorphic to a summand of N_P for all P and F is a faithful Λ-lattice, then M is a summand of $N \oplus F$.*

Note that the results of Section 4 follow from Corollary 6.4. For if $\mathrm{sr}(T) = 1$, then $\mathrm{U}^*(T/I) = \mathrm{U}(T/I)$ and so $|\mathrm{D}(M)| = 1$. So if $N \in \mathrm{G}(M)$, then T Bézout implies $\Gamma N \cong \Gamma M$, whence $N \in \mathrm{D}(M)$, and so $N \cong M$.

Corollary 6.10 *If M and X are faithful lattices, then the following sequence is exact:*

$$0 \to \mathrm{D}(M, X) \to \mathrm{G}(M) \xrightarrow{\phi} \mathrm{G}(M \oplus X) \to 0$$

where $\phi(N) = N \oplus X$. Moreover,

$$\mathrm{D}(M, X) \cong \lambda(X)/\lambda(X) \cap \mathrm{U}^*(T/I)\lambda(M).$$

In particular, if $\Lambda = R$, we obtain the results of [WW] on stable isomorphism classes.

Corollary 6.11 *If M is a faithful R-lattice, then $M \oplus R \cong N \oplus R$ if and only if $N \cong N_\alpha$, where $\nu(\alpha) \in \mathrm{U}^*(T/I)\lambda(M)\lambda(R)$. If M has constant rank t, then $M \oplus R \cong N \oplus R$ implies $tM \cong tN$.*

Note that if M has rank t, then $\lambda(M) \supset \lambda(R)^t$.

7 The Noether-Deuring Theorem

As we observed earlier, most of the problems discussed here can be phrased in terms of ring extensions.

We fix some notation for this section. Let R be a commutative ring and Λ a module finite R-algebra. If R' is a commutative extension of R, let $\Lambda' = R' \otimes_R \Lambda$. If M is a Λ-module, then $M' = R' \otimes_R \Lambda$ is a Λ'-module. The question addressed here is: does $M' \cong N'$ imply $M \cong N$? The answer in general is no. However, there is a positive answer when R is a field. This was proved be Noether and Deuring. There have been many extensions by Reiner and Zassenhaus, Roggenkamp, Grothendieck, and others.

Theorem 7.1 (Grothendieck) *If R is a local ring with maximal ideal P, R' is faithfully flat, and M and N are finitely presented Λ-modules, then $M' \cong N'$ implies $M \cong N$.*

Proof: Let $\bar{R} = R/P$, $\bar{\Lambda} = \Lambda/P\Lambda$, $\bar{M} = M/PM$, $\bar{N} = N/PN$. Now $M' \cong N'$ clearly implies $\dim \bar{M} = \dim \bar{N}$. Since M and N are finitely presented, the isomorphism between M' and N' is given by $\sum s_i \otimes \sigma_i$, where $\sigma_i \in \mathrm{Hom}_\Lambda(M,N)$ and $s_i \in R'$. Define $f(x_1,\ldots,x_n) = \det(\sum x_i\sigma_i) \in \bar{R}[x_i,\ldots,x_t]$, where $\bar{\sigma}_i$ maps $\bar{M}$ into $\bar{N}$. By hypothesis $f(\bar{s}_i,\ldots,\bar{s}_n) \neq 0$. Hence $f \neq 0$. If $\bar{R}$ is infinite, this implies $f(\bar{r}_1,\ldots,\bar{r}_n) \neq 0$ for some $r_i \in \bar{R}$. Thus $\sigma = \sum r_i\sigma_i$ is surjection from M to N. Similarly, there exists a surjection τ from N to M. Hence $\tau\sigma$ is a surjection from M to itself. Thus $\tau\sigma$ is an automorphism and so $M \cong N$. If $\bar{R}$ is finite, pass to a free rank t finitely generated extension R'' so that the residue field of R'' is sufficiently large that f represents a nonzero element in the residue field of R'' (take $R'' = R[x]/g(x)$), where $g(x)$ is irreducible of large degree). Then the argument above shows $M'' \cong N''$, and so $tM \cong tN$ as Λ-modules. Then $M \cong N$ by [GW].

If the assumption that R is local is dropped, the result is no longer true. The obstruction to this is exactly $\mathrm{G}(M)$. Also, note the same proof shows that $M'|N'$ implies $M|N$.

Corollary 7.2 *If R' is a faithfully flat commutative extension and M and N are finitely presented Λ-modules, then $M' \cong N'$ implies $N \in \mathrm{G}(M)$. Conversely, if $N \in \mathrm{G}(M)$, there exists a faithfully flat extension R' with $M' \cong N'$.*

Proof: The first statement is an immediate consequence of the theorem. For the second one, take R' to be the direct product of the localization of R (other extensions will suffice).

If faithfully flat is replaced by faithful finitely generated projective, then by a result of Bass it follows that $M' \cong N'$ implies $tM \cong tN$ for some t. It is apparently still open as to whether the converse is true. It is when R is Dedekind [G2] and when $\Lambda = M = R$ [BG]. If only finite generation is assumed then by an example of S. Wiegand, the corollary is false even for R semilocal.

One can also derive similar results for modules as in Section 4. See [G2].

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