MAXIMAL IDEALS OF STEIN ALGEBRAS

by

Andrew Markoe

1 Stein Algebras

A Frechet algebra (F-algebra) is a commutative topological algebra which is a Frechet space when considered as a linear topological space, whose pseudonorms $\|\cdot\|_j$ $j \in N$ satisfy $\|fg\|_j \leq \|f\|_j\|g\|_j$ for all $f,g \in A$.

If A is a Frechet algebra, Specm (A) will denote the set of maximal ideals of A and C-spec (A) will denote the topological space whose underlying set is the collection of non-zero continuous complex valued multiplicative linear functionals and whose topology is induced by the weak topology on $L(A,\mathbb{C})$.

There is a natural injection C-Spec (A) $\to$ Specm (A) obtained by sending a multiplicative functional to its kernel. An outstanding conjecture of E. Michael [8] asserts that the range of this map is all maximal ideals $\mathfrak{m}$ such that $A/\mathfrak{m} = \mathbb{C}$ (observe that $\mathbb{C} \approx A/\mathfrak{m}$ in a natural way; here equality is meant, not merely isomorphism). In other words, E. Michael's conjecture asserts that every multiplicative linear functional on a Frechet algebra is continuous. The conjecture is known to be true for finitely generated Frechet algebras, i.e., Frechet algebras in which polynomials in finitely many elements are dense. One purpose of this paper to investigate the validity of the conjecture for Stein algebras.

DEFINITION 1. A Frechet algebra A is said to be a Stein algebra if there is a, not necessarily reduced, Stein analytic space $(X,\mathcal{O}_X)$ such that A is isomorphic to $\Gamma(X,\mathcal{O}_X)$ as topological $\mathbb{C}$-algebras.

The Stein analytic space X (if there is no confusion, reference to the structure sheaf is omitted) appearing in the definition is unique up to analytic isomorphism. Furthermore C-Spec (A) has a natural structure as an analytic space such that $\mathcal{O}(\text{C-Spec }(A)) = A$. Thus in the sequel we always let $X = \text{C-spec }(A)$, $\mathcal{O}(X) = A$ (here $\mathcal{O}(X) = \Gamma(X,\mathcal{O}_X)$ = global sections of the structure sheaf). A is semi-simple (i.e. $\bigcap \text{Specm }(A) = (0)$) if and only if X is reduced.

The above facts about Stein algebras may be found in O. Forster [3], along with a proof of E. Michael's conjecture for Stein algebras with $\dim X < \infty$. We prove here (section 3) that E. Michael's conjecture is valid for Stein algebras with $\dim S(X) < \infty$ ($S(x)$ denotes the singular set of X). This is accomplished by characterizing the maximal ideals, Specm (A), of a Stein algebra A (section 2).

2 Maximal Ideals and $\mathcal{V}$-Filters

Let X be a set and let $\mathcal{V}$ be a collection of subsets of X, closed under finite intersections.

DEFINITION 2.

(1) A $\mathcal{V}$-filter is a subcollection $\mathfrak{F} \subset \mathcal{V}$ such that

(i) $V_1, V_2 \in \mathfrak{F} \Rightarrow V_1 \cap V_2 \in \mathfrak{F}$,

(ii) $V_1 \in \mathfrak{F}$, $V \in \mathcal{V}$ and $V_1 \subset V \Rightarrow V \in \mathfrak{F}$ and

(iii) $V \in \mathfrak{F} \Rightarrow V \neq \emptyset$.

(2) A $\mathcal{V}$-filter basis is a sub-collection $\mathfrak{B} \subset \mathcal{V}$ such that

(i) $\forall\, V_1, V_2 \in \mathfrak{B}$, $\exists\, V_3 \in \mathfrak{B}$ with $V_3 \subset V_1 \cap V_2$

(ii) $V \in \mathfrak{B} \Rightarrow V \neq \emptyset$.

(3) If $\mathfrak{B}$ is a $\mathcal{V}$-filter basis, then $\mathfrak{F}(B) = \{V \in \mathcal{V}: \exists W \in \mathfrak{B}$ with $W \subset V\}$ is a $\mathcal{V}$-filter called the $\mathcal{V}$-filter generated by $\mathfrak{B}$.

(4) The collection of $\mathcal{V}$-filters is ordered by $\subset$. A maximal element is called a $\mathcal{V}$-ultra filter.

PROPOSITION 1. Given a $\mathcal{V}$-filter $\mathfrak{F}$, there exists a $\mathcal{V}$-ultrafilter $\mathfrak{F}_o$ with $\mathfrak{F} \subset \mathfrak{F}_o$.

The proof is an easy application of Zorn's lemma and is omitted.

EXAMPLES

(1) If $\mathcal{V} = \mathcal{P}(X) =$ all subsets of X, then $\mathcal{V}$-filters are the classical filters.

(2) If X is an analytic space and if $\mathcal{V} = \mathcal{V}(X) = \{V(f_1,\ldots,f_k): \{f_1,\ldots,f_k\} \subset \mathcal{O}(X)\}$ then $\mathcal{V} = \mathcal{V}(X)$ is closed under finite intersections and $\mathcal{V}$-filters are well defined. This is the class of $\mathcal{V}$-filters used in the sequel.

We note that if X is not reduced, an element $f \in \mathcal{O}(X)$ is not a complex valued function. In this case $V(f) =$ Support $(\mathcal{O}_X/f \cdot \mathcal{O}_X)$ $= \{x \in X: f(x) \in \mathfrak{m}(x) =$ maximal ideal at $x\}$. Also $V(f_1,\ldots,f_k)$ $= V(f_1) \cap \ldots \cap V(f_k)$.

In this connection it is useful to let $r = r(x): \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m(x) = \mathbb{C}$ be the canonical homomorphism. Then $x \mapsto r(x)f(x)$ is an analytic function on the reduction of X, denoted by rf and $V(f) = V(rf)$.

In the sequel A is a Stein algebra, $X =$ C-Spec (A) is the associated Stein space and $\mathcal{V}(X)$ is as defined in example (2), above.

THEOREM 1. (Characterization of Maximal Ideals of a Stein Algebra)

Let A be a Stein algebra, $X =$ C-Spec (A).

(1) If $\mathfrak{F}$ is a $\mathcal{V}(X)$-ultrafilter, then $M(\mathfrak{F}) = \{f \in \mathcal{O}(X) : rf|_V = 0$ for some $V \in \mathcal{V}\}$ is a maximal ideal of $\mathcal{O}(X) = A$.

(2) If $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}(X) = A$ (i.e., $m \in \operatorname{Specm}(A)$), then $\mathfrak{B} = \{V(f_1,\dots,f_k) : \{f_1,\dots,f_k\} \subset m\}$ is a $\mathcal{V}(X)$-filterbasis and $\mathfrak{m} = M(\mathfrak{F})$ where $\mathfrak{F}$ is a $\mathcal{V}(X)$-ultrafilter containing $\mathfrak{F}(\mathfrak{B})$.

(3) If $\mathfrak{F}$ is a $\mathcal{V}(X)$ ultrafilter, then $\bigcap \mathfrak{F}$ is either empty or a singleton.

(4) $M(\mathfrak{F})$ is a dense maximal ideal ($\mathfrak{F}$ a $\mathcal{V}(X)$-ultrafilter) if and only if $\bigcap \mathfrak{F} = \emptyset$.

(5) $M(\mathfrak{F})$ is a closed maximal ideal if and only if $\bigcap \mathfrak{F} = \{x\}$, and in this case $M(\mathfrak{F}) = m(x)$.

Before proceeding to the proof, we give two preliminary results.

LEMMA 1 (Corona Property) If X is a Stein space and if $\{f_1,\dots,f_k\} \subset \mathcal{O}(X)$ such that $V(f_1,\dots,f_k) = \emptyset$, then there are $\{g_1,\dots,g_k\} \subset \mathcal{O}(X)$ with $\sum_{i=1}^{k} f_i g_i = 1$.

PROOF. We first work locally. If $x \in X$, there is a Stein neighborhood U of x , a polydisc neighborhood Δ of 0 in some $\mathbb{C}^n$ and a coherent ideal sheaf $I \subset \mathcal{O}_\Delta$ such that we can identify $(U,\mathcal{O}_U)$ and $(\operatorname{Supp}(\mathcal{O}_\Delta/I),\mathcal{O}_\Delta/I)$. Here x is identified with $0 \in \Delta$. Under this identification, the homomorphism $r(x) : \mathcal{O}_{X,x} \to \mathbb{C}$ can be factored as follows:

$$\begin{tikzcd} \mathcal{O}_{X,x} = \mathcal{O}_{\Delta,0}/I_0 \arrow[r, "S"] \arrow[dr, "r"'] & \mathcal{O}_{\Delta,0}/\operatorname{Rad}(I_0) \arrow[d] \\ & \mathcal{O}_{\Delta,0}/m(0) = \mathbb{C} \end{tikzcd}$$

Since $r(x)f_i(x) \not\equiv 0$ for some i (depending on x), there is a germ $h_{i,x} \in \mathcal{O}_{\Delta,0}/\mathrm{Rad}(I_0)$ with $(\rho(x)f_i(x))h_{i,x} = 1$. As the diagram shows, there is a germ $g_{i,x} \in \mathcal{O}_{X,x}$ with $\rho(g_{i,x}) = h_{i,x}$. Hence $\rho(f_i(x)g_{i,x}-1 = 0$. This means that $f_i(x)g_{i,x}-1 \in \mathrm{Rad}(I_0)/I_0$. By taking representatives of these germs in $\mathcal{O}_{\Delta,0}$, say $f_i^*(x)$, $g_{i,x}^*$, we obtain that $(f_i^*(x)g_{i,x}^*-1^j \in I_0$ for some integer $j > 0$.

Hence $(f_i^*(x)g_{i,x}^*)^j + \ldots + \binom{j}{i} f_i^*(x)g_{i,x}^* + (-1)^j \in I_0$. In other words, $f_i^*(x)(f_i^*(x)^{j-1}g_{i,x}^{*\,j} + \ldots + \binom{j}{1}g_{i,x}^*) + (-1)^j \in I_0$.

If we set $G_{i,x} = (-1)^{j+1}(f_i^*(x)^{j-1}g_{i,x}^{*\,j} + \ldots + \binom{j}{1}g_{i,x}^*) + I_0 \in \mathcal{O}_{X,x}$, then $f_i(x)G_{i,x} = 1 \in \mathcal{O}_{X,x}$.

Thus the sheaf homomorphism $\mathcal{O}_X^k \to \mathcal{O}_X$ defined by $(g_1,\ldots,g_k) \mapsto \sum_{i=1}^{k} f_i g_i$ is surjective. Since X is Stein, Cartan's theorem B holds, so that the homomorphism remains surjective on the section level. This completes the proof. We remark that lemma 1 is proved in [4] (VIII A Corollary 16). The additional complications here are caused by the possible existence of nilpotents.

LEMMA 2 If X is a Stein space, then an ideal $I \subset \mathcal{O}(X)$ is dense if and only if $V(I) = \emptyset$ (here $V(I) = \cap\{V(f): f \in I\}$).

PROOF. If I is dense and $x \in X$, there is an $f \in \mathcal{O}(X)$ with $(rf)(x) \neq 0$ (approximate 1 uniformly on the compact $\{x\}$). Hence $V(I) = \emptyset$.

Conversely, suppose $V(I) = \emptyset$. Since X is Stein we can write $X = \cup X_n$ where X_n is Stein and open, $\mathcal{O}(X)$ is dense in $\mathcal{O}(X_n)$ and $X_n \subset\subset X_{n+1}$. Since $V(I) = \emptyset$ and since $\overline{X}_{n+1}$ is compact, there exist finitely many $f_1,\ldots,f_k \in I$ with $V(f_1,\ldots,f_k) \cap \overline{X}_{n+1} = \emptyset$. By the corona property for Stein spaces there exist $g_1,\dots,g_k \in \mathcal{O}(X_{n+1})$ with $\sum_{i=1}^{k} f_i|_{X_{n+1}} g_i = 1|_{X_{n+1}}$. But $\mathcal{O}(X)$ is dense in $\mathcal{O}(X_{n+1})$ and $\bar{X}_n$ is compact in X_{n+1}. Here there are sequences $(g_i^{(\nu)}) \subset \mathcal{O}(X)$ with $\lim_{\nu\to\infty} g_i^{(\nu)} = g_i$, uniformly on $\bar{X}_n$. Since I is an ideal, $(\Sigma f_i g_i^{(\nu)}) \subset I$ and converges uniformly to 1 on $\bar{X}_n$, as $\nu \to \infty$. Since I is an ideal, any $f \in \mathcal{O}(X)$ can be approximated uniformly well on $\bar{X}_n$: Since the $\bar{X}_n$ form a compact exhaustion of X, the conclusion follows.

Remarks The proof of theorem 1, given below, can be done for any analytic space for which lemmas 1 and 2 are valid.

It is thus interesting to observe that lemmas 1 and 2, individually, characterize Stein open subspaces of Stein spaces; i.e., if either lemma 1 or lemma 2 hold for X and if X is an open subset of a Stein space, then X is also Stein [7]. Hence the hypothesis that A is a Stein algebra in theorem 1 is very general.

PROOF OF THEOREM 1.

(1) Let F be a $\mathcal{V}(X)$-ultrafilter and let $M(\mathfrak{F}) = \{f \in \mathcal{O}(X) : rf|_V = 0$ for some $V \in \mathfrak{F}\}$. Firstly, $M(\mathfrak{F})$ is an $\mathcal{O}(X)$-ideal. This follows because $\mathfrak{F}$ is closed under finite intersections. Also we see that $M(\mathfrak{F})$ is a proper ideal $\subset \mathcal{O}(X)$ since every $f \in M(\mathfrak{F})$ has a zero $(\emptyset \notin \mathfrak{F})$.

To show that $M(\mathfrak{F})$ is a maximal ideal, we first show that

(*) for all $h \notin M(\mathfrak{F})$, there exists $V \in \mathfrak{F}$ with $V(h) \cap V = \emptyset$.

Suppose for contradiction that (*) did not hold. Then there exists $h \notin M(\mathfrak{F})$ such that $V(h) \cap V \neq \emptyset$ for all $V \in \mathfrak{F}$. Clearly then, $\mathfrak{F} \cup \{V(h) \cap V : V \in \mathfrak{F}\}$ is a $\mathcal{V}(X)$ filter basis. Since $\mathfrak{F}$ is a $\mathcal{V}(X)$ ultrafilter, $V(h) \cap V \in \mathfrak{F}$ for all $V \in \mathfrak{F}$. This implies that $h \in M(\mathfrak{F})$, a contradiction. Hence (*) holds.

Now if $h \notin M(\mathfrak{F})$, by (*) there exist $f_1,\ldots,f_k \in M(\mathfrak{F})$ such that $V(h,f_1,\ldots,f_k) = \emptyset$. The corona property (lemma 1) gives the existence of $g,g_1,\ldots,g_k \in \mathcal{O}(X)$ such that $gh + \sum_{i=1}^{k} f_i g_i = 1$. This shows that $\mathcal{O}(X)h + M(\mathfrak{F}) = \mathcal{O}(X)$. Since this is true for all $h \notin M(\mathfrak{F})$, $M(\mathfrak{F})$ is maximal.

(2) Let m be a maximal $\mathcal{O}(X)$-ideal. It is easy to show that $\mathcal{B} = \{V(f_1,\ldots,f_k)\colon \{f_1,\ldots,f_k\} \subset m\}$ is a $\mathcal{V}(X)$-filterbasis, once it is observed that $V(f_1,\ldots,f_k) \neq \emptyset$ for $\{f_1,\ldots,f_k\} \subset m$. But m is maximal, so the corona property implies that $V(f_1,\ldots,f_k) \neq \emptyset$ for $\{f_1,\ldots,f_k\} \subset m$.

Now if $\mathfrak{F}$ is an ultrafilter containing $\mathcal{B}$ it is clear that $m \subset M(\mathfrak{F})$. By the maximality of m and $M(\mathfrak{F})$, $m = M(\mathfrak{F})$.

(3) To prove (3) assume that $\mathfrak{F}$ is a $\mathcal{V}(X)$-ultrafilter and that $x,y \in \bigcap\mathfrak{F}$. Were $x \neq y$ there would exist $f \in \mathcal{O}(X)$ with $rf(x) = 0$, $rf(y) = 1$, since X is Stein. Hence $\mathfrak{F} \cup \{V(f) \cap V\colon V \in \mathfrak{F}\}$ would be a $\mathcal{V}(X)$-filterbasis. Since $\mathfrak{F}$ is an ultrafilter, $V(f) \cap V \in \mathfrak{F}$ for all $V \in \mathfrak{F}$. Since $x,y \in \bigcap\mathfrak{F}$ this yields $x,y \in V(f)$, contradicting $rf(y) = 1$. Hence (3) is proved.

(4) By lemma 2, $M(\mathfrak{F})$ is a dense maximal ideal if and only if $V(M(\mathfrak{F})) = \emptyset$. But an easy calculation shows $V(M(\mathfrak{F})) = \bigcap\mathfrak{F}$.

(5) If $M(\mathfrak{F})$ is maximal and closed, $\bigcap\mathfrak{F}$ cannot be empty. By (3) there exists $x \in X$ with $\bigcap\mathfrak{F} = \{x\}$. The converse is also easily proved.

To show that $M(\mathfrak{F}) = m(x)$, just observe that $M(\mathfrak{F}) \subset m(x)$, if $\bigcap\mathfrak{F} = \{x\}$. Since $M(\mathfrak{F})$ is maximal and $m(x) \neq \mathcal{O}(X)$ it follows that $m(x) = M(\mathfrak{F})$.

<u>Remarks</u> When X is an open Riemann surface the maximal ideals of $\mathcal{O}(X)$ have already been extensively investigated using $\mathcal{V}(X)$-filters. M. Henriksen

[5] used the concept of Δ-filter to this end. S. Kakutani [6] also used a concept equivalent to that of $\mathcal{U}(X)$-filter in this study.

Of course open Riemann surfaces are precisely the class of one dimensional Stein manifolds. Here discrete sequences are the proper subvarieties, so Δ-filters are the natural ones to use in this case ($\Delta = \{X\} \cup \{D \subset X: D$ is discrete$\}$). In higher dimensions one must resort to the use of higher dimensional subvarieties, as shown above.

3 E. Michael's Conjecture for Stein Algebras

In his thesis [8], E. Michael posed the following conjecture. If A is a commutative Frechet algebra, then every multiplicative linear functional is continuous. Already in [8], Michael affirmatively answered this question for $A = \mathcal{O}(X)$ where X is a domain in $\mathbb{C}^n$ whose envelope of holomorphy is schlicht. One purpose of this section is to indicate that, in the light of more recent developments in complex analysis, Michael essentially answered his question for semi-simple Stein algebras where $\dim(\text{C-Spec}(A)) < \infty$. Another purpose of this section is to indicate the relationship to Stein algebras of R. Arens result [1] that Michael's conjecture is true on every finitely generated Frechet algebra. Finally, and most important, we prove a generalization of Forster's theorem [3] that every (possibly non semi-simple) Stein algebra A with $\dim(\text{C-Spec}(A)) < \infty$ satisfies Michael's conjecture. The generalization is to the case where it is only required that $\dim(S(\text{C-Spec}(A))) < \infty$ (here $S(X)$ denotes the singular set of the analytic space X).

Our first result is a characterization of reduced Stein spaces of bounded dimension.

PROPOSITION 2. Let X be a reduced Stein space. Then $\dim X < \infty$ if and only if the following condition is satisfied:

II there exists $\{f_1,\dots,f_m\} \subset \mathcal{O}(X)$ with $V(f_1 - \lambda_1,\dots,f_m - \lambda_m)$ compact for every $(\lambda_1,\dots,\lambda_m) \in \mathbb{C}^m$.

PROOF. Suppose II holds. Then the map $F = (f_1,\dots,f_m): X \to \mathbb{C}^m$ is holomorphic and has compact fibers.

Let $x_0 \in X$. By proposition 2, p. 161 of Gunning and Rossi [4], there is a neighborhood U of $F^{-1}(F(x_0))$ and an $\varepsilon > 0$, such that $F|_U: U \to \Delta(F(x_0), (\varepsilon,\dots,\varepsilon))$ is proper.

The dimension estimate in the proper mapping theorem (p. 162 [4]) shows that $\dim_{F(x_0)} F(U) = \max\{\dim_x F: F(x) \text{ near } F(x_0)\}$. Since X is Stein and the level sets of F are compact it follows that $\dim_X F = \dim_x X$. Hence $m \geq \dim_{F(x_0)} F(U) = \dim_{x_0} X$.

Conversely suppose that X is of bounded dimension. Let X_k be the union of all irreducible components of X of dimension k. Then if $\dim X = n$, $X = X_1 \cup\dots\cup X_n$ and each X_k is a pure dimensional Stein space. Hence there is a holomorphic homeomorphism $(f_{k,1},\dots,f_{k,2k+2}): X_k \to \mathbb{C}^{2k+2}$. Each $f_{k,j}$ extends holomorphically to X, so $\{f_{k,j}: 1 \leq k \leq h, 1 \leq j \leq 2k+2\}$ is a finite set of holomorphic functions on X which satisfy II.

THEOREM 2 (Forster [3]) If X is a Stein analytic space of bounded dimension, then E. Michael's conjecture is true on $\mathcal{O}(X)$.

PROOF. Let $\tilde{X}$ be the reduction of X. By local considerations (c.f., proof of lemma 1, section 2) $\mathcal{O}_{\tilde{X}}$ is a coherent $\mathcal{O}_X$-module and the sheaf homomorphism $\mathcal{O}_X \xrightarrow{r} \mathcal{O}_{\tilde{X}}$ is surjective. Since X is Stein, Cartan's theorem B implies that $\mathcal{O}(X) \xrightarrow{r} \mathcal{O}(\tilde{X})$ is surjective.

Since $\dim X < \infty$, there exist $f_1,\ldots,f_m \in \mathcal{O}(\tilde{X})$ satisfying II of the above proposition. Let $g_1,\ldots,g_m \in \mathcal{O}(X)$ with $r(g_i) = f_i$.

Suppose for contradiction that $\phi\colon \mathcal{O}(X) \to \mathbb{C}$ were a discontinuous multiplicative linear functional. Thus $\ker\phi$ would be dense in $\mathcal{O}(X)$ and one would be able to approximate 1 uniformly on

$$V(g_1 - \phi(g_1),\ldots,g_m - \phi(g_m)) = V(f_1 - \phi(g_1),\ldots,f_m - \phi(g_m))$$

(which is compact by II). Hence there is some $h \in \ker\phi$ with $V(h,g_1 - \phi(g_1),\ldots,g_m - \phi(g_m)) = \emptyset$. By the corona property (lemma 1) and the fact that h and $g_i - \phi(g_i)$ are in $\ker\phi$ we get that $1 \in \ker\phi$. This contradicts the maximality of the ideal $\ker\phi$, thus completing the proof.

The assertion, before, that E. Michael had the essential ideas to prove Forster's theorem in the semi-simple case (i.e., X reduced) is borne out by proposition 2 and the following result from [8].

PROPOSITION (c.f., Proposition 12.5 [8]).

Let X be a completely regular topological space and let A be a sub-algebra of $C(X)$ containing 1. Then every non-zero multiplicative linear functional is a point evaluation on X provided that A satisfies I and II below.

I for all $\{f_1,\ldots,f_n\} \subset A$ with $V(f_1,\ldots,f_n) = \emptyset$, there exists $\{g_1,\dots,g_n\} \subset A$ such that $\sum f_i g_i = 1$.

<u>II</u> There is a finite set $\{h_1,\dots,h_m\} \subset A$ with $V(h_1 - \lambda_1,\dots h_n - \lambda_m)$ compact for any $(\lambda_1,\dots,\lambda_m) \in \mathbb{C}^m$.

Here we have let $V(f_1,\dots,f_n) = \{x \in X: f_i(x) = 0,\ i = 1,\dots,n\}$ as in the case $A = \mathcal{O}(X)$.

<u>Remark</u> In Michael's paper [8] the hypothesis that every nonvanishing $f \in A$ has $\frac{1}{f} \in A$ also is included. This is unnecessary, being a consequence of <u>I</u>.

The proof of this proposition is very much like the proof of the corollary to proposition 2, and is omitted.

Hence Michael was very close to having Forster's theorem; he only needed the results on Stein spaces that were to come later (in particular, Cartan's theorems to get the corona property (<u>I</u>) and the embedding theorem to get the characterization of bounded dimension (II).

R. Arens [1] has shown that every finitely generated Frechet algebra has the property in Michael's conjecture. However, a Stein algebra A with $\dim(\text{C-Spec}(A)) < \infty$ is finitely generated if and only if the local embedding dimensions of C-Spec(A) are bounded (see [2] S. 137, Satz 3, for details). Hence, this case is subsumed under that of Forster's theorem.

We can now give a nice generalization of Forster's theorem. But first the following result is needed.

PROPOSITION 3 Let V be a closed analytic subspace of the Stein space X , let $\rho: \mathcal{O}_X \to \mathcal{O}_V$ denote the restriction homomorphism and let m be a dense maximal ideal of $\mathcal{O}(X)$ such that $\rho(m) \neq \mathcal{O}(V)$.

Then $\rho(m)$ is a dense maximal ideal of $\mathcal{O}(V)$ and the quotient fields $\mathcal{O}(X)/m$ and $\mathcal{O}(V)/\rho(m)$ are naturally isomorphic as $\mathbb{C}$-algebras.

PROOF. By Cartan's theorem B, $\rho: \mathcal{O}(X) \to \mathcal{O}(V)$ is surjective. This easily implies that $\rho(m)$ is an $\mathcal{O}(V)$-ideal.

If J is an $\mathcal{O}(V)$-ideal, with $\rho(M) \subset J$, then $\rho^{-1}(J)$ is an $\mathcal{O}(X)$ ideal. Also $m \subset \rho^{-1}(J)$, so since m is maximal, either $m = \rho^{-1}(J)$ or $\mathcal{O}(X) = \rho^{-1}(J)$. Hence either $J = \rho(m)$ or $J = \mathcal{O}(V)$. Since $\rho(m) \neq \mathcal{O}(V)$, this proves that $\rho(m)$ is maximal.

Now, $V(\rho(m)) = V(m) \cap V = \emptyset \cap V = \emptyset$ (by lemma 2, since m is dense). Again by lemma 2, $\rho(m)$ is dense.

To show that the quotient fields are naturally isomorphic, observe the exact sequence

$$0 \to \rho^{-1}(\rho(m))/m \to \mathcal{O}(X)/m \xrightarrow{\rho} \mathcal{O}(V)/\rho(m) \to 0 .$$

Now, $m \subset \rho^{-1}\rho(m)$, so by maximality, either $m = \rho^{-1}\rho(m)$ or $\mathcal{O}(X) = \rho^{-1}\rho(m)$. But the latter possibility contradicts the hypothesis that $\rho(m) \neq \mathcal{O}(V)$.

Hence $0 \to 0 = m/m \to \mathcal{O}(X)/m \to \mathcal{O}(V)/\rho(m) \to 0$ is an exact sequence of $\mathbb{C}$-algebras.

COROLLARY If V is a closed analytic subspace of the Stein space X and if $\mathcal{O}(V)$ has no discontinuous multiplicative functionals, then a sufficient condition for a multiplicative $\mathcal{O}(X)$-functional ϕ to be continuous is that $\rho(\ker \phi) \neq \mathcal{O}(V)$.

PROOF. Let $m = \ker \phi$. Suppose for contradiction that m were dense. Then m satisfies the hypotheses of the proposition, so that $\rho(m)$ is a dense maximal ideal in $\mathcal{O}(V)$ with $\mathcal{O}(V)/\rho(m) = \mathcal{O}(X)/m = \mathbb{C}$. Thus there is a discontinuous multiplicative functional on $\mathcal{O}(V)$; a contradiction.

THEOREM 3 If X is a Stein space and if $\dim S(X) < \infty$, then every multiplicative linear functional on $\mathcal{O}(X)$ is continuous.

PROOF. Assume for contradiction that ϕ is a discontinuous multiplicative functional on $\mathcal{O}(X)$.

Since $\dim S(X)$ is bounded, proposition 3 applies to $S(X) = V$ (by theorem 2).

Thus we must have $\rho(\ker \phi) = \mathcal{O}(S(X))$ where $\rho: \mathcal{O}(X) \to \mathcal{O}(S(X))$ is the restriction homomorphism.

In particular, there is an $f_0 \in \ker \phi$ with $\rho(f_0) = 1$. Hence $V(f_0) \cap S(X) = \emptyset$.

Let $X = \cup X_i$ be the decomposition of X into irreducible components. Since $V(f_0) \cap S(X) = \emptyset$, if we let $V_i = V(f_0) \cap X_i$, then the V_i are disjoint closed analytic subspaces of X .

Now we use theorem 1 to obtain a $\mathcal{V}(X)$-ultrafilter $\mathfrak{F}$ with $\ker \phi = M(\mathfrak{F})$.

For any $V \in \mathfrak{F}$, $V \cap V_i \neq \emptyset$ for infinitely many i . For otherwise there exists $V_0 \in \mathfrak{F}$ and an $n > 0$ with $V_0 \cap V_i = \emptyset$ for $i > n$. Then let $W = V_1 \cup \ldots \cup V_n$. Thus W is a closed analytic subspace of X and $\dim W < \infty$. Furthermore given any $V \in \mathfrak{F}$, $\emptyset \neq V \cap V_o \cap V(f_o)$. This is because $f_o \in \ker \phi$, so $V(f_o) \in \mathfrak{F}$ ($\mathfrak{F}$ being an ultrafilter). But $V \cap V_o \cap V(f_o) = (V \cap V_o \cap W) \cup ((V \cap V_o) \cap \bigcup_{i>n} V_i) = V \cap V_o \cap W$. Thus

$V \cap V_o \cap W \neq \emptyset$ for any $V \in \mathfrak{F}$. Hence for any $f \in \ker \phi$, rf has a zero on W. This means that $\rho(\ker \phi) \neq \mathcal{O}(W)$ where $\rho: \mathcal{O}(X) \to \mathcal{O}(W)$ is the restriction homomorphism. Thus the conditions of the corollary to proposition 3 are satisfied, so ϕ is continuous. This contradicts the hypothesis of this paragraph. Hence for all $V \in \mathfrak{F}$, $V \cap V_i \neq \emptyset$ for infinitely many i.

In particular $V(f_o) \in \mathfrak{F}$, so $V_i \neq \emptyset$ for infinitely many i. Since the V_i are disjoint we can construct $F \in \mathcal{O}(V(f_o))$ with $rF|_{V_i} = i$ if $V_i \neq \emptyset$. Since $V(f_0)$ is a closed subspace of X, and since X is Stein there exists an $F \in \mathcal{O}(X)$ which restricts to F on $V(f_0)$.

We now obtain the following contradiction: $F - \phi(F) \in \ker \phi$ so there exists $V \in \mathfrak{F}$ with $rF = \phi(F)$ on V. But $V \cap V_i \neq \emptyset$ for infinitely many i. Hence, by the construction of F we cannot have $rF = \phi(F)$ on all the $V \cap V_i$. This contradiction proves the theorem.

<u>Remark</u> By induction, the above result remains valid for any Stein space X such that the kth iterate of the singular set is of bounded dimension. However, the techniques used here do not seem to extend to the case where every iterate of the singular set remains unbounded in dimension.

REFERENCES

1. R. Arens, <u>Dense Inverse Limit Rings</u>, Mich. Math. J. 5 (1958), pp. 169-182.

2. H. Behnke and P. Thullen, "Theorie der Funktionen mehrerer komplexer Veränderlichen", zweite, erweiterte auflage, Ergebnisse der Mathematik und ihrer Grezgebiete, band 51, Springer Verlag, Berlin, 1970.

3. O. Forster, <u>Uniqueness of Topology in Stein Algebras</u>, "Function Algebras", F. Birtel, ed., pp. 157-163, Scott Foresman, Chicago, 1966.

4. R. Gunning and H. Rossi, "Analytic Functions of Several Complex Variables", Prentice-Hall, Englewood Cliffs, 1965.

5. M. Henriksen, <u>On the Ideal Structure of the Ring of Entire Functions</u>, Pac. J. Math. 2 (1952), 179-184.

6. S. Kakutani, <u>Rings of Analytic Functions</u> in "Lectures on Functions of a Complex Variable", W. Kaplan, ed., Univ. of Mich. 1955.

7. A. Markoe, The Corona Property and Stein Sets, manuscript, copies available from the author.

8. E. Michael, <u>Locally Multiplicatively Convex Topological Algebras</u>, Mem. AMS #11, 1952.

University of Washington