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THE COHOMOLOGY OF RESTRICTED LIE ALGEBRAS AND OF HOPF ALGEBRAS; APPLICATION TO THE STEENROD ALGEBRA

BY

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ABSTRACT

A theorem due to Drs. Milnor and Moore states that a primitively generated Hopf algebra is isomorphic to the universal enveloping algebra of its restricted Lie algebra of primitive elements. In particular, the conclusion is valid for the associated graded algebra of any Hopf algebra. In the first part of this thesis, algebraic machinery is developed which takes advantage of these results to facilitate the calculation of the cohomology of the Hopf algebra. In the second part, this machinery is applied to calculate the cohomology of the Steenrod algebra, $H^{s,t}(A)$, in the range $t-s \leq 2(p-1)(2p^2+p+2)-4$ for odd primes p and $t-s \leq 42$ for p=2. In both cases, partial information is obtained in higher dimensions. Using the Adams spectral sequence, these results are used to extend Toda's calculations of the stable homotopy groups of spheres. In particular, we find that the differentials in the Adams spectral sequence show at least a limited amount of periodicity. Part II is written in such a manner that the reader interested primarily in the topological applications need refer to Part I only for proofs.

PREFACE

I wish to express my deep gratitude to Dr. John C. Moore, who suggested the topic of this thesis, and without whose guidance and instruction I would not have been prepared to work it out. Dr. Norman E. Steenrod was also unsparing of his time, and offered helpful advice. Warmest thanks goes to Elizabeth Epstein who devoted her skill and energy to the typing of this manuscript.

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Princeton University
May, 1964

CONTENTS

I. Cohomology of restricted Lie algebras and of Hopf algebras

0. Introduction

It is well-known that the existence of the bar construction theoretically solves the problem of calculating the cohomology of an augmented algebra. It is equally well-known that the bar construction is too large to be of much practical value in calculating high dimensional homology groups. The object of the first part of this thesis is to develop an alternative, and more manageable, method for calculating the cohomology of a Hopf algebra.

A theorem due to Milnor and Moore states that any primitively generated Hopf algebra over a field of characteristic p is isomorphic to the universal enveloping algebra of its restricted Lie algebra of primitive elements. It follows that the associated graded algebra E^0A of any Hopf algebra A satisfies the conclusion. We will find a reasonably small complex with which to calculate $H^*(E^0A)$ and will devise a spectral sequence having $E_2 = H^*(E^0A)$ and $E_\infty = E^0H^*(A)$.

In the first two sections, the definitions and some of the properties of graded Hopf algebras, Lie algebras, and restricted Lie algebras are recalled. In section 3, a canonical free resolution of the ground field is obtained on the category of universal enveloping algebras of graded restricted Lie algebras. An incidental result is the obtaining of such a resolution for graded Lie algebras. In section 4, we find a method for embedding certain complexes over the ground field in the bar construction and apply this result to the resolutions of section 3. Finally, in section 5, we obtain a spectral sequence, defined for any (well-behaved) filtered algebra A, E_2 of which is $H^*(E^0A)$, and which converges (not necessarily finitely) as an algebra to $E_\infty = E^0H^*(A)$. We remark that essential use is made of the bar construction in forming the spectral sequence.

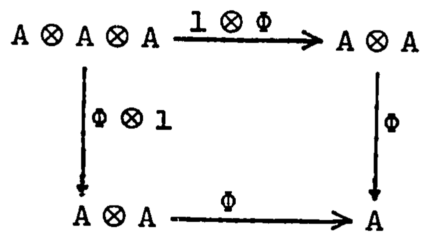
Now suppose we are given a Hopf algebra A. If we can somehow determine explicitly the structure of E^0A, we can then use our resolution of section 3, which is a good deal smaller and more easily studied than the bar construction, to calculate $H_*(E^0A)$. Using section 4, we then have representative cycles for homology classes in $\overline{B}(E^0A)$, the bar construction of E^0A. E^1 of the dual to the spectral sequence cited above is $\overline{B}(E^0A)$, and we may either study the homology spectral sequence and then dualize to obtain $E^0H^*(A)$ or dualize $\overline{B}(E^0A)$, obtain representative co-cycles for $H^*(E^0A) = E_2$ in $(\overline{B}(E^0A))^* = E_1$, and calculate $E^0H^*(A)$ using the cohomology spectral sequence.

At the conclusion of section 5, we demonstrate the applicability of our procedure to the calculation of $Ext_A(K,M)$, where M is an A-module and A is a Hopf algebra over a field K.

1. Preliminaries; Hopf algebras

Let K be a fixed commutative unitary ring. By a module over K we will understand a graded module indexed on the non-negative integers, and we denote the component of a module M concentrated in degree n by M_n. The dual of M, M^*, is defined by $M_n^* = \text{Hom}(M_n, K)$. A K-morphism $f: M \longrightarrow N$ is a sequence of morphisms $f_n: M_n \longrightarrow N_n$. Thus all morphisms are assumed to be of degree zero. When a second grading is imposed on a graded module, the new degree will be called the bidegree, and morphisms of non-zero bidegree will be allowed (in practice, we will obtain complexes by this method, the differentials having degree zero and bidegree minus one). This convention will not be in force when a module is obtained initially as bigraded. A filtration F of a module M is a sequence of submodules F_iM indexed on the integers such that $F_iM \subset F_{i+1}M$. The associated bigraded object E^0M is defined by $E^0_{r,s}M = (F_rM/F_{r-1}M)_{r+s}$ and is an example of an object given initially as bigraded. If M and N are modules, $M \otimes N$ is graded by $(M \otimes N)_n = \sum_{r+s=n} M_r \otimes N_s$. If M and N are filtered, $M \otimes N$ is given a filtration by $F_n(M \otimes N) = \sum_{r+s=n} F_rM \otimes F_sN$ (where it is assumed the filtrations on M and N are such that the sum is finite).

<u>Definitions I.1.1</u>: An algebra is a K-module A together with K-morphisms $\Phi: A \otimes A \longrightarrow A$ and $\eta: K \longrightarrow A$ such that the diagrams



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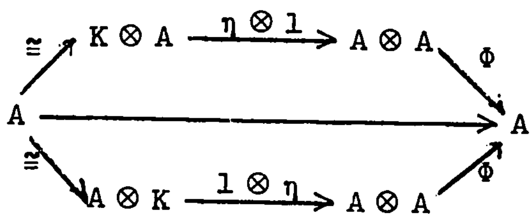
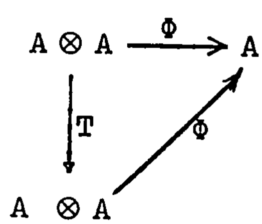


are commutative.

Reversing all the arrows gives the definition of a coalgebra. Thus all our algebras are defined to be associative and unitary, our coalgebras coassociative and unitary. A morphism of algebras $\varepsilon\colon A \longrightarrow K$ defines A as an augmented algebra; similarly, if A is a coalgebra, a morphism of coalgebras $K \longrightarrow A$ defines A as an augmented coalgebra. The algebra A is commutative if the diagram



is commutative, where $T(x \otimes y) = (-1)^{\deg x \deg y}\, y \otimes x$.

If A is a coalgebra, reversing the arrows defines cocommutativity.

We recall that a module M is of finite type if each M_n is finitely generated. If M is projective of finite type, then we may identify M with M^{**} and if M and N are projective of finite type, we may identify $M^* \otimes N^*$ with $(M \otimes N)^*$. If M is projective of finite type, so is M^*, but not necessarily conversely.

Proposition I.1.2: If A is an algebra with product Φ and the K-module A is projective of finite type then A^* is a coalgebra with coproduct Φ^* and A^* is augmented, respectively cocommutative, if and only if A is augmented, respectively commutative. Similarly, if A is a coalgebra with coproduct ψ, A^* is an algebra with product ψ^*, augmented, respectively commutative, if and only if A is augmented, respectively cocommutative.

Definitions I.1.3: If A and B are algebras with products Φ_1 and Φ_2, $A \otimes B$ is an algebra with product $(\Phi_1 \otimes \Phi_2)(1 \otimes T \otimes 1)$. If A and B are coalgebras with coproducts ψ_1 and ψ_2, $A \otimes B$ is a coalgebra with coproduct $(1 \otimes T \otimes 1)(\psi_1 \otimes \psi_2)$. Using this definition we define a Hopf algebra as a K-module A which is both an algebra and coalgebra and is such that the product is a morphism of coalgebras (equivalently, the coproduct is a morphism of algebras), the algebra unit is a coalgebra augmentation, and the coalgebra unit is an algebra augmentation. It follows that the product is a morphism of augmented algebras and the coproduct a morphism of augmented coalgebras. A is connected if the algebra unit (equivalently, the coalgebra unit) defines an isomorphism between A_o and K. By the previous proposition, if A is projective of finite type as a K-module, then A is a Hopf algebra if and only if A^* is.

Definitions I.1.4: If A is an augmented algebra $I(A) = \mathrm{Ker}\ \varepsilon$, $\varepsilon: A \longrightarrow K$, $Q(A) = I(A)/(I(A))^2$, the cokernel of $\Phi: I(A) \otimes I(A) \longrightarrow I(A)$. The elements of $Q(A)$ are called (by an abuse of language) the indecomposable elements of A. If A is an augmented coalgebra, $J(A) = \text{cokernel}\ \eta$, $\eta: K \longrightarrow A$, $P(A)$ is the kernel of $J(A) \xrightarrow{\psi} A \otimes A \longrightarrow J(A) \otimes J(A)$, that is, the elements of $J(A)$ such that $\psi(a) = a \otimes 1 + 1 \otimes a$. The elements of $P(A)$ are called the primitive elements of A. If A is a Hopf algebra and $P(A)$ contains a complete set of coset representatives for $Q(A)$, then A is said to be primitively generated.

Proposition I.1.5: If K is a field and A is a Hopf algebra which is a K-space of finite type, then $P(A^*) = Q(B)^*$ and $Q(B^*) = P(B)^*$.

Proof: This follows from the fact that over a field the dual of an exact sequence is exact.

For expansion of these definitions and proofs, see the paper of Milnor and Moore, "On the Structure of Hopf Algebras," which is also a general reference for the next section.

We complete this section by recalling the definitions of the homology and cohomology of an algebra. Given an algebra A over a field K, its cohomology $H^*(A)$ is defined as $\mathrm{Ext}_A(K,K)$ and its homology $H_*(A)$ as $\mathrm{Tor}^A(K,K)$. Let X be a free

A-resolution of K. Then $H^*(A) = H(\mathrm{Hom}_A(X,K))$ and $H_*(A) = H(K \otimes_A X)$. Since K is a field, we have the functorial equivalences:

$$H(\mathrm{Hom}_A(K,K)) = H(\mathrm{Hom}_K(K \otimes_A X,K) = \mathrm{Hom}_K(H(K \otimes_A X),K) = (H(K \otimes_A X))^*.$$

Thus $H^*(A) = (H_*(A))^*$. If A is of finite type, then $(H^*(A))^* = H_*(A)$ is also true. These results remain valid when $H_*(A)$ is given its natural coalgebra structure and $H^*(A)$ its natural algebra structure.

2. Lie algebras and restricted Lie algebras

<u>Definitions I.2.1</u>: Let K be a field of characteristic p. A Lie algebra is a vector space L together with a map $[,]: L \otimes L \longrightarrow L$ of vector spaces such that for some algebra A there exists a monomorphism $f: L \longrightarrow A$ of vector spaces such that for $x, y \in L$ $f([x,y]) = f(x)f(y) - (-1)^{\deg x \deg y} f(y)f(x)$. L^+ denotes the subset of even degree elements of L, L^- the subset of odd degree elements, unless $\operatorname{char} K = 2$, when $L^+ = L$, $L^- = \Phi$. If L is a Lie algebra together with a map $\beta: L^+ \longrightarrow L^+$ of vector spaces such that there exists an algebra A and a map $f: L \longrightarrow A$ with both $f([x,y]) = f(x)f(y) - (-1)^{\deg x \deg y} f(y)f(x)$ and $f\beta(z) = (f(z))^p$, $x, y \in L$, $z \in L^+$, then L is a restricted Lie algebra.

<u>Lemma I.2.2</u>: Let L be a vector space and $[,]: L \otimes L \longrightarrow L$ be a map of vector spaces. L is a Lie algebra if and only if

i) $[x,y] = (-1)^{mn+1} [y,x]$, $x \in L_m$, $y \in L_n$.

ii) $(-1)^{mr} [x,[y,z]] + (-1)^{mn} [y,[z,x]] + (-1)^{nr} [z,[x,y]] = 0$, $x \in L_m$, $y \in L_n$, $z \in L_r$.

iii) $[x,[x,x]] = 0$ and

iv) $[x,x] = 0$ if degree $x \equiv 0 \mod 2$ or if $\operatorname{char} K = 2$.

<u>Lemma I.2.3</u>: Let L be a Lie algebra, $\beta: L^+ \longrightarrow L^+$ be a map of vector spaces. L is a restricted Lie algebra if and only if

i) $\beta(kx) = k^p\beta(x), \quad x \in L^+, \quad k \in K$.

ii) $[\beta(x),z] = (\mathrm{ad}\ x)^p(z), \quad x \in L^+, z \in L$, where $\mathrm{ad}\ x(z) = [x,z]$.

iii) If $\deg x = \deg y$, $\beta(x+y) = \beta(x) + \beta(y) + \sum_{i=1}^{p-1} s_i(x,y)$, where $(\mathrm{ad}(\lambda x + y))^{p-1}(x) = \sum_{i=1}^{p-1} is_i(x,y)\lambda^{i-1}$, λ an indeterminate.

Necessity of the conditons follows from properties of algebras, while sufficiency is the standard proof of existence of a universal enveloping algebra for a Lie algebra, respectively, restricted Lie algebra. The proofs in chapter V of Jacobson apply, with trivial modifications, to the graded case.

Proposition I.2.4: If L is a Lie algebra, there exists an algebra $U(L)$ and a monomorphism $i: L \longrightarrow U(L)$ of Lie algebras such that if A is an algebra and $f: L \longrightarrow A$ a morphism of Lie algebras, there exists a unique morphism $\tilde{f}: U(L) \longrightarrow A$ of algebras such that $\tilde{f}i = f$; $U(L)$ is unique up to canonical isomorphism. Similarly, if L is a restricted Lie algebra, there exists an algebra $V(L)$ and a monomorphism $j: L \longrightarrow V(L)$ of restricted Lie algebras such that if $f: L \longrightarrow A$ is a morphism of restircted Lie algebras there exists a unique morphism $\tilde{f}: V(L) \longrightarrow A$ of algebras such that $\tilde{f}j = f$; $V(L)$ is unique up to canonical isomorphism.

Proof: It is only necessary to prove existence. In the first case, $U(L) = T(L)/I$ where $T(L)$ is the tensor algebra and I is the ideal generated by $\{xy - (-1)^{\deg x \deg y} yx - [x,y] \mid x,y \in L\}$. The definition of Lie algebra ensures that $i: L \longrightarrow U(L)$ is a monomorphism. In the second case, $V(L) = T(L)/J$, where J is the ideal generated by the generators of I and by $\{x^p - \beta(x) | x \in L^+\}$. Note that $V(L)$ is isomorphic to $U(L)/C$, where C is the ideal generated by $\{i(x)^p - i\beta(x) | x \in L^+\}$.

Proposition I.2.5: If L is a Lie algebra, resp. a restricted Lie algebra, then $U(L)$, resp. $V(L)$, has a natural Hopf algebra structure and $i(L) \subset P(U(L))$, resp. $j(L) \subset P(V(L))$; in particular, $U(L)$, resp. $V(L)$, is primitively generated.

Proof: Define $\psi: L \longrightarrow U(L) \otimes U(L)$ by $\psi(x) = i(x) \otimes 1 + 1 \otimes i(x)$ and apply the universal property of $U(L)$ to obtain $\tilde{\psi}: U(L) \longrightarrow U(L) \otimes U(L)$, checking first that ψ is a map of Lie algebras. The same procedure applies to the case where L is restricted.

Proposition I.2.6: If A is a Hopf algebra, $P(A)$ is a sub-Lie algebra of A, a restricted sub-Lie algebra if char $K > 0$.

Definitions I.2.7: Let L be a Lie algebra. Define a filtration of $U(L)$ by $F_{-n}U(L) = 0$, $F_0U(L) = K$, $F_1U(L) = K \cup L$ and $F_nU(L) = (F_1U(L))^n$. Define the associated bigraded object $E^0U(L)$ by $E^0_{r,s}U(L) = (F_rU(L)/F_{r-1}U(L))_{r+s}$ and let $E^0_rU(L) = \bigcup_s E^0_{r,s}U(L)$. If L is a restricted Lie algebra, the same definitions are made with $V(L)$ replacing $U(L)$.

<u>Proposition I.2.8</u>: If L is a Lie algebra, then

i) $E^0_{0,0}U(L) = E^0_0 U(L) = K$.

ii) $E^0U(L)$ is a primitively generated commutative Hopf algebra.

iii) $E^0_{r,s} \otimes E^0_{r',s'} \longrightarrow E^0_{r+r', s+s'}$ under the multiplication.

iv) $L \xrightarrow{\cong} E^0_1 U(L) \xrightarrow{\cong} Q(E^0U(L))$.

If L is a restricted Lie algebra, the proposition remains true with $V(L)$ replacing $U(L)$.

<u>Proof</u>: Clear by inspection of the definitions.

<u>Theorem I.2.9</u> (Poincaré, Birkhoff, Witt): If L is a Lie algebra and $f: E^0A(L) \longrightarrow E^0U(L)$ is the natural map induced by the injection of $L = E^0_1 U(L) \longrightarrow E^0U(L)$, then f is an isomorphism of Hopf algebras. $(A(L) = U(L^{\#})$ where $L^{\#}$ is the K-space L regarded as an Abelian Lie algebra).

For the proof, see Milnor and Moore.

<u>Corollary I.2.10:</u> If L is a restricted Lie algebra, $E^0V(L)$ is isomorphic to $E^0A(L)/I$ where I is the ideal generated by $\{x^p | x \in L^+\}$.

<u>Proof</u>: Since $V(L) = U(L)/C$ as above, $x^p = 0$ in $E^0V(L)$, $x \in L^+$.

<u>Remarks I.2.11</u>: If $E^0U(L)$ is graded by total degree, $E^0U(L)_n = \bigoplus_{r+s=n} E^0_{r,s}U(L)$, then $E^0U(L)$ is isomorphic to $U(L)$ as a vector space. If $\operatorname{char} K \neq 2$, $A(L) = E(L^-) \otimes P(L^+)$ while for $\operatorname{char} K = 2$ $A(L) = P(L)$ where E denotes the exterior algebra, P the polynomial algebra. If a basis $\{x_i\}_{i \in I}$ for L^- is indexed on a totally ordered set I and a basis $\{y_j\}_{j \in J}$ for L^+ is indexed on a totally

ordered set J , then $\{x_{i_1}\cdots x_{i_m} y_{j_1}^{r_1}\cdots y_{j_n}^{r_n} | i_1 < \cdots < i_m, j_1 < \cdots < j_n, r_k \geq 0\}$ is a basis for $U(L)$. If L is restricted, a basis for $U(L)$ is obtained by adding the requirement $r_k < p$.

Theorem I.2.12 (Milnor and Moore): If A is a primitively generated Hopf algebra over a field K , then

i) If $\operatorname{char} K = 0$, A is isomorphic to $U(P(A))$ as a Hopf algebra.

ii) If $\operatorname{char} K = p > 0$, A is isomorphic to $V(P(A))$ as a Hopf algebra.

In particular, the conclusion is valid for the associated graded Hopf algebra of any Hopf algebra A: we have $E^0_{p,q}A = (F_pA/F_{p-1}A)_{p+q}$, where $F_nA = A$, $n \geq 0$, and if $\Phi_1 = i\colon I(A) \longrightarrow I(A)$, $\Phi_2\colon I(A) \otimes I(A) \longrightarrow I(A)$ is the multiplication, and

$$\Phi_n = \Phi_2(\Phi_1 \otimes \Phi_{n-1})\colon \overbrace{I(A) \otimes \cdots \otimes I(A)}^{n \text{ factors}} \longrightarrow I(A) \text{ , then}$$

$F_{-n}A = \operatorname{Im} \Phi_n \subset I(A)$. Clearly E^0A is primitively generated.

3. Cohomology of graded restricted Lie algebras

Let L be a graded restricted Lie algebra over a field K. We will obtain here a free resolution $X(L) = V(L) \otimes \overline{X}(L)$ of the ground field K. Our procedure will be to first regard L merely as a Lie algebra and to obtain a free resolution $Y(L) = U(L) \otimes \overline{Y}(L)$ of K over U(L), and then to attach an extra piece to the resulting complex $V(L) \otimes \overline{Y}(L)$.

Thus we suppose first only that L is a Lie algebra over a field K (of any characteristic). We could generalize the classical procedure for the case of a Lie algebra concentrated in degree zero, which is to embed $U(L) \otimes E(sL)$ as a complex in the bar construction. In fact, we will carry out such an argument in the succeeding section. However, as will be seen there, such a procedure would not generalize to give us a free resolution for restricted Lie algebras. An alternative method in the classical case is described in exercise 14 of Chapter XIII of Cartan and Eilenberg. This method gives $U(L) \otimes E(sL)$ a rather peculiar K-algebra structure. Such an algebra structure will be just what is needed to effect the desired generalization.

To begin the construction, we regard L as bigraded with bidegree zero, and we let sL denote a copy of L in which all elements have bidegree one. We denote by L^+ the subspace of L consisting of the elements of even degree and by L^- the subspace of odd degree elements. If char(K) = 2, we adopt the convention that $L^+ = L$ and L^- is void. As a K-space our resolution will be $Y(L) = U(L) \otimes \overline{Y}(L)$, $\overline{Y}(L) = E(sL^+) \otimes \Gamma(sL^-)$, where $E(sL^+)$ is an exterior algebra and $\Gamma(sL^-)$ a divided polynomial algebra. The bidegree will be the homological degree. If L is Abelian, $H_*(U(L)) = E(sL^+) \otimes \Gamma(sL^-)$, and therefore no smaller free resolution could be obtained as a functor of L.

Let Λ denote the ring of dual numbers over K considered as a Hopf algebra. Thus Λ is the exterior algebra on one primitive generator d of degree zero and bidegree minus one. Then the category of bigraded K-complexes and their morphisms may be identified with that of Λ-modules and their morphisms. Let M denote the bigraded K-module $sL^+ \cup (\bigcup_{i \geq 0} \pi^i sL^-)$, where $\pi^i sL^-$ is a copy of sL^- with degree and bidegree multiplied by p^i, $p = \text{char}(K)$, and $\pi^0 sL^- = sL^-$. Let Z be the tensor algebra $T(\Lambda \otimes M)$, and give Z a structure of Λ-module by requiring that Z be an algebra over the Hopf algebra Λ, that is, by requiring $d(ab) = (da)b + (-1)^{\deg a} a(db)$, where, as usual, the exponent of the sign is the total degree. As a complex, Z has trivial homology, $H_*(Z) = K$. We will obtain $Y(L)$ as a quotient algebra of Z. We consider $M = K \otimes M$ to be contained in Z; then if $x \in L^-$, the element $\pi^i sx$ of Z will give rise to the p^ith divided power of sx, denoted by $\gamma_{p^i}(\overline{x})$, in $Y(L)$; similarly, if $y \in L^+$, the element sy will give rise to the element of $E(sL^+)$ denoted by $\langle \overline{y} \rangle$. We remark that if $\text{char}(K) = 0$, we must take $M = sL^+ \cup sL^-$; hereafter will not specify the modifications of the arguments required for this case.

Now we seek a Λ-submodule I of Z which has the property that Z/I is an algebra which is isomorphic to $Y(L)$ as a K-space. Let I be the two-sided ideal with generators of the forms:

1) $ab - (-1)^{\deg a \deg b}\, ba, \quad a \in M,\ b \in M$

$(\pi^i sx)^p, \quad x \in L^-, \quad i \geq 0$

$(sy)(sy), \quad y \in L^+$

2) $(dsx)(dsy) - (-1)^{\deg x \deg y}(dsy)(dsx) - ds[xy], \quad x \in L,\ y \in L$

3) $d\pi^i sx - (-1)^i[(dsx)(sx) - \frac{s[xx]}{2}](sx)^{p-2}(\pi sx)^{p-1}\dots(\pi^{i-1}sx)^{p-1}$,

$x \in L^-, \quad i \geq 1$

4) $(\pi^i sx)(dsy) - (dsy)(\pi^i sx) - (-1)^i\, s[xy](sx)^{p-1}\dots(\pi^{i-1}sx)^{p-1}$,

$x \in L^-, y \in L, \quad i \geq 1$

5) $(sx)(dsy) - (-1)^{\deg sx \deg y}(dsy)(sx) - s[xy], \quad x \in L, \quad y \in L$

The following lemma gives the algebra and K-space structure of Z/I. Lemma I.3.2 states that I is actually a Λ-submodule of Z, hence that Z/I is a Λ-module.

<u>Lemma I.3.1</u>: Give $Y(L)$ a K-algebra structure by requiring the product to agree with the natural one on $U(L)$ and on $\overline{Y}(L)$ and to satisfy the relations

i) $<\bar{y}_1>y_2 = y_2<\bar{y}_1> + <[\overline{y_1 y_2}]>, \quad y_1 \in L^+,\ y_2 \in L^+$

ii) $<\bar{y}>x = -x<\bar{y}> + \gamma_1([\overline{yx}]), \quad y \in L^+, \quad x \in L^-$

iii) $\gamma_i(\bar{x})y = y\gamma_i(\bar{x}) + \gamma_1([\overline{xy}])\gamma_{i-1}(\bar{x})$, $x \in L^-$, $y \in L^+$

iv) $\gamma_i(\bar{x}_1)x_2 = x_2\gamma_i(\bar{x}_1) + <[\overline{x_1x_2}]>\gamma_{i-1}(\bar{x}_1)$, $x_1 \in L^-$, $x_2 \in L^-$

Then Z/I is isomorphic to $Y(L)$ as a K-algebra.

Proof: Generators of I of the forms listed in 1) imply that we can define an isomorphism of algebras $f: \bar{Y}(L) \longrightarrow T(M)/I \cap T(M)$ by $f(<\bar{y}>) = sy$, $y \in L^+$, and $f(\gamma_{ip^j}(\bar{x})) = \frac{(\pi^j sx)^i}{i!}$, $x \in L^-$, $j \geq 0$, $1 \leq i \leq p-1$. Let $N = dsL^+ \cup dsL^-$. Generators of the form 2) imply that we can define an isomorphism of algebras $g: U(L) \longrightarrow T(N)/I \cap T(N)$ by $g(x) = dsx$, $x \in L$. Therefore we have an isomorphism of K-spaces $g \otimes f: Y(L) \longrightarrow T(N)/I \cap T(N) \otimes \otimes T(M)/I \cap T(M) \subset Z$. Let J be the subideal of I generated by those generators of I of the forms 3), 4), and 5). Then $Z/J \cong T(N) \otimes T(M)$ as a K-space: generators of the form 3) enable us to express the $d\pi^i sx$ as elements of $T(N) \otimes T(M)$ in Z/J, while generators of the forms 4) and 5) enable us to so express products ab, $a \in M$, $b \in N$. It follows that $Z/I \cong T(N)/I \cap T(N) \otimes T(M)/I \cap T(M) \cong Y(L)$ as a K-space. Identifying Z/I with $Y(L)$, the relations i) and ii) and the relations iii) and iv) with $i = 1$ follow from generators of I of the form 5). Noting that $(p-1)! \equiv -1 \mod p$, relations iii) and iv) with $i = p^j$ follow from the generators of I of the form 4). The relations iii) and iv) for $i \neq p^j$ follow from those for $i = p^j$.

Lemma I.3.2: I is a Λ-submodule of Z ; that is, $dI \subset I$.

Proof: We write $\equiv$ for congruence mod I . We must prove $da \equiv 0$, where a is a generator of I . d applied to generators of the form 2) is zero and d applied to generators of the form 5) gives generators of the form 2). Consider generators of the forms listed in 1). If $x \in L$, $y \in L$, then:

$$\begin{aligned} d(sxsy) &= (dsx)sy + (-1)^{\deg sx} sx(dsy) \\ &\equiv (-1)^{\deg x \deg sy}(sy(dsx) - s[yx]) + (-1)^{\deg sx} \\ &\qquad\qquad ((-1)^{\deg sx \deg y}(dsy)sx + s[xy]) \\ &= (-1)^{\deg sx \deg sy}((dsy)sx + (-1)^{\deg sy} sy(dsx) \\ &= d((-1)^{\deg sx \deg sy} sysx) . \end{aligned}$$

The handling of the remaining commutators in 1) is equally simple. $d(sysy) = (dsy)sy - sy(dsy) \equiv 0$ since $[y,y] = 0$, $y \in L^+$. If $x \in L^-$, an easy induction gives

i) $d(sx)^j \equiv j(dsx)(sx)^{j-1} + \frac{j(j-1)}{2} s[xx](sx)^{j-2}$, $j \geq 2$, and

ii) $d(\pi^i sx)^j \equiv j(-1)^i((dsx)sx - \frac{s[xx]}{2})(sx)^{p-2}(\pi sx)^{p-1}\ldots(\pi^{i-1}sx)^{p-1}(\pi^i sx)^{j-1}$, $i \geq 1$, $j \geq 1$.

In particular, $d(\pi^i sx)^p \equiv 0$, which completes consideration of generators of the forms listed in 1). Using ii) and generators of the forms 1) and 4), we find that if $x \in L^-$, then

iii) $d((\pi sx)^{p-1}\ldots(\pi^i sx)^{p-1}) \equiv ((dsx)(sx)^{p-1} - \frac{s[xx]}{2}(sx)^{p-2})$ $(\pi sx)^{p-2} (\pi^2 sx)^{p-1}\ldots(\pi^i sx)^{p-1}$, $i \geq 1$.

Therefore, to prove that d applied to generators of the form 3) is congruent to zero, it suffices to show that

$$d((dsx)(sx)^{p-1} - \frac{s[xx]}{2}(sx)^{p-2}) \equiv 0 \quad \text{and} \quad ((dsx)(sx)^{p-1} - \frac{s[xx]}{2}(sx)^{p-2})^2 \equiv 0 .$$

The former is easily verified using i) and noting that $[[xx],x] = 0$; the latter follows from $(s[xx])^2 \equiv 0$ and $(sx)^p \equiv 0$. It remains to consider generators of the form 4). An inductive proof gives:

$$(sx)^j(\pi sx)^{p-1}\ldots(\pi^i sx)^{p-1}(dsy) \equiv ((dsy)sx + js[xy])(sx)^{j-1}(\pi sx)^{p-1}\ldots(\pi^i sx)^{p-1} .$$

Using iii) we obtain $(sx)^{p-1}\, d((\pi sx)^{p-1}\ldots(\pi^i sx)^{p-1}) \equiv 0$. These facts imply that it suffices to prove that generators 4) with $i = 1$ are stable under d . Here we have:

$$d(\pi sx(dsy) - (dsy)\pi sx + s[xy](sx)^{p-1})$$

$$= d\pi sx(dsy) - (-1)^{\deg y}(dsy)d\pi sx + (ds[xy])(sx)^{p-1} + (-1)^{\deg s[xy]}s[xy]d(sx)^{p-1}$$

$$\equiv -(dsx)(dsy)(sx)^{p-1} + (dsx)s[xy](sx)^{p-2} + (-1)^{\deg y}(dsy)\frac{s[xx]}{2}(sx)^{p-2} + \frac{s[[xx],y]}{2}(sx)^{p-2} - s[xx]s[xy](sx)^{p-3} + (-1)^{\deg y}(dsy)(dsx)(sx)^{p-1} - (-1)^{\deg y}(dsy)\frac{s[xx]}{2}(sx)^{p-2} + (ds[xy])(sx)^{p-1} - (dsx)s[xy](sx)^{p-2} - (-1)^{\deg y}s[[xy]x](sx)^{p-2} + (-1)^{\deg y}s[xy]s[xx](sx)^{p-3} \equiv 0 ,$$

where we have used the Jacoby identity (ii) of Lemma I.2.2). This completes the proof.

We now identify $Y(L)$ with Z/I as a K-algebra. Then the lemma above implies that $Y(L)$ is an algebra over the Hopf algebra Λ . Using the algebra structure of Lemma I.3.1, $Y(L)$ may be interpreted as a complex over $U(L)$. Thus if $u \in U(L)$ and $a \in \bar{Y}(L)$, $d(u \otimes a) = (-1)^{\deg u} u \otimes da$, where $u \otimes da$ is to be written as an element of the free $U(L)$-module $Y(L)$ by making use of relations i) and ii) of Lemma I.3.1. For example, if $x \in L^+$, $y \in L^+$, then we find

$$d\langle\bar{x},\bar{y}\rangle = (d\langle\bar{x}\rangle)\langle\bar{y}\rangle - \langle\bar{x}\rangle d\langle\bar{y}\rangle = x\langle\bar{y}\rangle - \langle\bar{x}\rangle y = x\langle\bar{y}\rangle - y\langle\bar{x}\rangle - \langle\overline{[x,y]}\rangle .$$

We can now state the following theorem:

<u>Theorem I.3.3</u>: Let $Y(L) = U(L) \otimes \bar{Y}(L)$, $\bar{Y}(L) = E(sL^+) \otimes \Gamma(sL^-)$. Give $Y(L)$ an algebra structure by requiring the product to agree with the natural one on $U(L)$ and on $\bar{Y}(L)$ and to satisfy the relations:

1) $\langle\bar{y}_1\rangle y_2 = y_2\langle\bar{y}_1\rangle + \langle\overline{[y_1 y_2]}\rangle$, $y_1 \in L^+$, $y_2 \in L^+$

2) $\langle\bar{y}\rangle x = -x\langle\bar{y}\rangle + \gamma_1(\overline{[yx]})$, $y \in L^+$, $x \in L^-$

3) $\gamma_i(\bar{x})y = y\gamma_i(\bar{x}) + \gamma_1(\overline{[xy]})\gamma_{i-1}(\bar{x})$, $x \in L^-$, $y \in L^+$

4) $\gamma_i(\bar{x}_1)x_2 = x_2\gamma_i(\bar{x}_1) + \langle\overline{[x_1 x_2]}\rangle\, \gamma_{i-1}(\bar{x}_1)$, $x_1 \in L^-$, $x_2 \in L^-$.

Define a differential d on $Y(L)$ by

a) $d(ua) = (-1)^{\deg u}\, u\,da$, $u \in U(L)$, $a \in \bar{Y}(L)$

b) $d\langle\bar{y}\rangle = y$; $d\gamma_i(\bar{x}) = x\gamma_{i-1}(\bar{x}) + \frac{1}{2}\langle\overline{[xx]}\rangle\, \gamma_{i-2}(\bar{x})$ (where $\gamma_{-1}(\bar{x}) = 0$)

c) $d(ab) = (da)b + (-1)^{\deg a} a(db)$, $a \in \bar{Y}(L)$, $b \in \bar{Y}(L)$.

Note that $d(ab)$ is determined uniquely as an element of $U(L) \otimes \bar{Y}(L)$ by relations 1) through 4).

Then $Y(L)$ is a free resolution over $U(L)$ of the ground field K .

<u>Proof</u>: Regarding $Y(L)$ as Z/I, we have already proven that $Y(L)$ has the specified algebra structure and is a complex with the differential defined above (the form of $d\gamma_i(\bar{x})$, $i \neq p^j$, is easily proven to be as stated; for $i = p^j$, there is nothing to prove). The exactness proof is quite analogous to that given in Cartan and Eilenberg for the classical case and to the proof given below for our resolutions for restricted Lie algebras, and will therefore be omitted.

<u>Corollary I.3.4</u>: Let $f = \langle \bar{y}_1, \ldots, \bar{y}_n \rangle \gamma_{r_1}(\bar{x}_1) \cdots \gamma_{r_m}(\bar{x}_m) \in \bar{Y}(L)$.

Then the differential d is given explicitly by the formula:

$$
\begin{aligned}
d(f) = & \sum_{i=1}^{n} (-1)^{i+1} y_i \langle \bar{y}_1, \ldots, \hat{\bar{y}}_i, \ldots, y_n \rangle \gamma_{r_1}(\bar{x}_1) \cdots \gamma_{r_m}(\bar{x}_m) \\
& + \sum_{i<j} (-1)^{i+j} \langle \overline{[y_i y_j]}, \bar{y}_1, \ldots, \hat{\bar{y}}_i, \ldots \hat{\bar{y}}_j, \ldots \bar{y}_n \rangle \gamma_{r_1}(\bar{x}_1) \cdots \gamma_{r_m}(\bar{x}_m) \\
& + \sum_{i=1}^{m} x_i \langle \bar{y}_1, \ldots, \bar{y}_n \rangle \gamma_{r_1}(\bar{x}_1) \cdots \gamma_{r_i - 1}(\bar{x}_i) \cdots \gamma_{r_m}(\bar{x}_m) \\
& + \sum_{i=1}^{m} \tfrac{1}{2} \langle \overline{[x_i x_i]}, \bar{y}_1, \ldots, \bar{y}_n \rangle \gamma_{r_1}(\bar{x}_1) \cdots \gamma_{r_i - 2}(\bar{x}_i) \cdots \gamma_{r_m}(\bar{x}_m) \\
& + \sum_{i<j} \langle \overline{[x_i x_j]}, \bar{y}_1, \ldots, \bar{y}_n \rangle \gamma_{r_1}(\bar{x}_1) \cdots \gamma_{r_i - 1}(\bar{x}_i) \cdots \gamma_{r_j - 1}(\bar{x}_j) \cdots \gamma_{r_m}(\bar{x}_m) \\
& + \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{j+1} \langle \bar{y}_1, \ldots, \hat{\bar{y}}_j, \ldots, \bar{y}_n \rangle \gamma_1(\overline{[x_i y_j]}) \gamma_{r_1}(\bar{x}_1) \cdots \gamma_{r_i - 1}(\bar{x}_i) \cdots \gamma_{r_m}(\bar{x}_m)
\end{aligned}
$$

Note that the first two terms are precisely those for the classical case of a Lie algebra concentrated in degree zero.

Proof: The formula is easily derived by induction on m and n.

Corollary I.3.5: Let L be a restricted Lie algebra. Then $V(L) \otimes \overline{Y}(L)$ with differential K-algebra structure defined as in the theorem (replacing $U(L)$ by $V(L)$ in all statements) is a free complex over $V(L)$; its differential is given explicitly by Corollary I.3.4.

Proof: Let $J \subset Y(L)$ be the two-sided ideal with generators of the forms $y^p - \beta(y)$, $y \in L^+$, where β denotes the pth power operation. Then $V(L) \otimes \overline{Y}(L) \cong Y(L)/J$ as an algebra over the Hopf algebra Λ. The conclusions follow.

For the remainder of this section, L will denote a graded restricted Lie algebra over a field K of characteristic $p > 0$. We have a free $V(L)$-complex $W(L) = V(L) \otimes \overline{Y}(L)$. This is not a resolution, since $y^{p-1}\langle\overline{y}\rangle - \langle\overline{\beta(y)}\rangle$ is a nonbounding cycle, $y \in L^+$. We wish to enlarge this complex to obtain a free resolution over $V(L)$ of the ground field K. Let π and s be as previously defined: π multiplies degree and bidegree by p and s adds one to the bidegree. As a K-space, our resolution will be $W(L) \otimes \Gamma(s^2\pi L^+)$ and, writing $\tilde{y}$ for $s^2\pi y$, we will have $d(\gamma_1(\tilde{y})) = y^{p-1}\langle\overline{y}\rangle - \langle\overline{\beta(y)}\rangle$. If L is Abelian with all pth powers zero, then $H_*(V(L)) = \overline{Y}(L) \otimes \Gamma(s^2\pi L^+)$, and therefore no smaller free resolution could be obtained canonically.

In our construction, we will need the following concepts.

<u>Definitions I.3.6</u>: Let A be a differential K-algebra and let X be a complex. Then X is said to be a left differential A-module if X is a left module over the algebra A and $d(ax) = (da)x + (-1)^{\deg a} adx$, $a \in A$, $x \in X$. X is said to be a right differential A-module if X is a right module over the algebra A and $d(xa) = (dx)a + (-1)^{\deg x} xda$. For example, if A is a differential K-subalgebra of B, then the inclusion $A \subset B$ induces a structure of two-sided differential A-module on B.

The following elementary observation will play a crucial role in our construction.

<u>Lemma I.3.7</u>: If A is a differential K-algebra, X is a right differential A-module, and Y is a left differential A-module, then $X \otimes_A Y$ is a complex. In particular, if $Y = A \otimes Z$ is a free left differential A-module, the natural isomorphism of $X \otimes_A Y$ and $X \otimes Z$ induces a structure of complex on $X \otimes Z$.

<u>Proof</u>: $X \otimes Y$ is a complex, being a tensor product of complexes. $X \otimes_A Y = (X \otimes Y)/M$, where M is the K-submodule of $X \otimes Y$ with generators of the form $xa \otimes y - x \otimes ay$. Since X and Y are differential A-modules, M is stable under d, and therefore $X \otimes_A Y$ is a complex.

We can now obtain our free resolution of K over V(L). Let $\{y_i | i \in I\}$ be a basis for L^+ indexed on a totally ordered set I. Let $W(L) = V(L) \otimes \overline{Y}(L)$ and $W(L^+) = V(L^+) \otimes \overline{Y}(L^+)$. Our procedure will require two preliminary steps. First, we will define a structure of free left differential $W(L^+)$-module on each $F_i = W(L^+) \otimes \Gamma(\tilde{y}_i)$, $\tilde{y}_i = s^2\pi y_i$. Second, we will define a structure of right differential $W(L^+)$-module on each F_i. Then we will be able to use the lemma to induce a structure of complex on $X(L)$ via the K-space isomorphisms:

$$X(L) = W(L) \otimes \Gamma(s^2\pi L^+) \cong W(L) \otimes \Gamma(\tilde{y}_{i_1}) \otimes \Gamma(\tilde{y}_{i_2}) \otimes \cdots$$

$$\cong W(L) \otimes_{W(L^+)} F_{i_1} \otimes_{W(L^+)} F_{i_2} \otimes_{W(L^+)} \cdots$$

Step 1: Construction of F_i as a free left differential $W(L^+)$-module. Let y denote any element of $\{y_i | i \in I\}$. Let N_y be the K-space with basis $\{\pi^i\tilde{y} | i \geq 0\}$ and give the tensor algebra $T_y = T(\Lambda \otimes N_y)$ a structure of Λ-module by requiring T_y to be an algebra over the Hopf algebra Λ. Let L_y be the restricted Lie subalgebra of L^+ generated by y, and form $W(L_y) \otimes T_y$, where $W(L_y) = V(L_y) \otimes E(sL_y)$. As a tensor product of differential K-algebras, $W(L_y) \otimes T_y$ has a structure of differential K-algebra. Let $J_y \subset W(L_y) \otimes T_y$ be the two-sided ideal with generators of the forms:

1) $(\pi^i\tilde{y})^p$; $\pi^i\tilde{y}\pi^j\tilde{y} - \pi^j\tilde{y}\pi^i\tilde{y}$

2) $(d\pi^i\tilde{y}) - (-1)^i (y^{p-1}\langle\tilde{y}\rangle - \langle\overline{\beta(y)}\rangle)\, \tilde{y}^{p-1}\cdots(\pi^{i-1}\tilde{y})^{p-1}$

Noting that, by the definition of the tensor product of two algebras, $ab = (-1)^{\deg a \deg b}\, ba$, $b \in W(L_y)$, $a \in T_y$, and noting that L_y is Abelian as a Lie algebra (since $[yy] = 0 = [\beta(y)y]$), we find that

a) $\pi^{i}\tilde{y}\, d\pi^{j}\tilde{y} \equiv (d\pi^{j}\tilde{y})\pi^{i}\tilde{y} \mod J_y$

b) $(y^{p-1}\langle\bar{y}\rangle - \langle\overline{\beta(y)}\rangle)^2 = 0$, and

c) $d(\pi^{i}\tilde{y})^{j} \equiv j(d\pi^{i}\tilde{y})(\pi^{i}\tilde{y})^{j-1} \mod J_y$.

It follows that J_y is a Λ-submodule of $W(L_y) \otimes T_y$. Defining $\gamma_{ip^j}(\tilde{y}) \longrightarrow \frac{(\pi^{j}\tilde{y})^{i}}{i!}$, we obtain an isomorphism of K-algebras $W(L_y) \otimes \Gamma(\tilde{y}) \longrightarrow (W(L_y) \otimes T_y)/J_y$. Identifying $W(L_y) \otimes \Gamma(\tilde{y})$ with $(W(L_y) \otimes T_y)/J_y$, $W(L_y) \otimes \Gamma(\tilde{y})$ becomes a differential K-algebra, and is therefore also a free left differential $W(L_y)$-module. Since $W(L_y) \subset W(L^+)$, we may use Lemma I.3.7 to obtain a structure of complex on $W(L^+) \otimes \Gamma(\tilde{y}) \cong W(L^+) \otimes_{W(L_y)} W(L_y) \otimes \Gamma(\tilde{y})$. $W(L^+) \otimes \Gamma(\tilde{y})$ then becomes a free left differential $W(L^+)$-module; in fact, the differential on $W(L^+) \otimes \Gamma(\tilde{y})$ is given by the formula:

α) $d(w\gamma_r(\tilde{y})) = (dw)\gamma_r(\tilde{y}) + (-1)^{\deg w} w(y^{p-1}\langle\bar{y}\rangle - \langle\overline{\beta(y)}\rangle)\gamma_{r-1}(\tilde{y})$, $w \in W(L^+)$.

<u>Step 2</u>: Definition of F_i as a right differential $W(L^+)$-module. We continue with the notation of step 1 and consider the complex $W(L^+) \otimes \Gamma(\tilde{y})$ with differential given by formula α). Since $W(L^+)$ is itself a right differential $W(L^+)$-module, to define a structure of right differential $W(L^+)$-module on $W(L^+) \otimes \Gamma(\tilde{y})$ it suffices to define $\gamma_r(\tilde{y})\langle\bar{z}\rangle$ and $\gamma_r(\tilde{y})z$, $z \in L^+$. We define $\gamma_r(\tilde{y})\langle\bar{z}\rangle = \langle\bar{z}\rangle\gamma_r(\tilde{y})$. This already determines $\gamma_r(\tilde{y})z$: we must have $d(\gamma_r(\tilde{y})\langle\bar{z}\rangle) = (d\gamma_r(\tilde{y}))\langle\bar{z}\rangle + \gamma_r(\tilde{y})z$. Using formula α) and our definition of $\gamma_r(\tilde{y})\langle\bar{z}\rangle$, this implies that

$$\gamma_r(\tilde{y})z = z\gamma_r(\tilde{y}) - \langle\bar{z}\rangle(y^{p-1}\langle\bar{y}\rangle - \langle\overline{\beta(y)}\rangle)\gamma_{r-1}(\tilde{y}) - (y^{p-1}\langle\bar{y}\rangle - \langle\overline{\beta(y)}\rangle)\langle\bar{z}\rangle\gamma_{r-1}(\tilde{y})$$
$$= z\gamma_r(\tilde{y}) - (\langle\bar{z}\rangle y^{p-1}\langle\bar{y}\rangle + y^{p-1}\langle\bar{y},\bar{z}\rangle)\gamma_{r-1}(\tilde{y}) .$$

We must evaluate $\langle\bar{z}\rangle y^{p-1}$ using the algebra structure of $W(L^+)$. By an easy induction using 1) of Theorem I.3.3, we find

$$\langle\bar{z}\rangle y^j = \sum_{i=o}^{j} (-1)^i \, (i,j-i) y^{j-i} \, \langle\overline{(\mathrm{ad}\, y)^i(z)}\rangle , \qquad 1 \leq j \leq p-1 .$$

Here $(\mathrm{ad}\, y)^i(z) = \overbrace{[y[y[\ldots[yz]\ldots]]]}^{i \text{ factors}}$, $(\mathrm{ad}\, y)^o(z) = z$. If $j = p-1$, $(-1)^i \, (i,p-1-i) \equiv 1 \bmod p$, $0 \leq i \leq p-1$, and we find, therefore, that:

$$\beta) \qquad \gamma_r(\tilde{y})z = z\gamma_r(\tilde{y}) - \sum_{i=1}^{p-1} y^{p-1-i} \, \langle\overline{(\mathrm{ad}\, y)^i(z)},\bar{y}\rangle\gamma_{r-1}(\tilde{y}) .$$

Note that if $z \in L_y$, then $\gamma_i(\tilde{y})z = z\gamma_i(\tilde{y})$, which is in agreement with the algebra structure of $W(L_y) \otimes \Gamma(\tilde{y})$ utilized in Step 1.

We can now define $X(L) = W(L) \otimes \Gamma(s^2\pi L^+)$ as a complex via the K-space isomorphism $X(L) \cong W(L) \otimes_{W(L^+)} (\bigotimes_{i \in I}{}_{W(L^+)} F_i)$, where F_i precedes F_j if $i < j$. That $X(L)$ is thereby given a structure of complex follows from Lemma I.3.7, since $F_i = W(L^+) \otimes \Gamma(y_i)$ is a two-sided differential $W(L^+)$-module. We give a formal description of the complex $X(L)$ in the following theorem:

<u>Theorem I.3.8.</u> Let $X(L)$ be the free $V(L)$-module $V(L) \otimes \bar{X}(L)$, $\bar{X}(L) = E(sL^+) \otimes \Gamma(sL^-) \otimes \Gamma(s^2\pi L^+)$. Let $\{y_i | i \in I\}$ be a basis for L^+ indexed on a totally ordered set I and let $\tilde{y}_i = s^2\pi y_i$. Identify $\Gamma(s^2\pi L^+)$ as a K-space with $\bigotimes_{i \in I} \Gamma(\tilde{y}_i)$, where $\Gamma(\tilde{y}_i)$ precedes $\Gamma(\tilde{y}_j)$ if $i < j$. Let $W(L) = V(L) \otimes \bar{Y}(L)$,

so that $X(L) = W(L) \otimes \Gamma(s^2\pi L^+)$. Give $W(L)$ its structure of differential K-algebra derived in Corollary I.3.5. Give $X(L)$ a structure of right $W(L^+)$-module by defining

1) $\gamma_r(\tilde{y}_i)\langle\bar{z}\rangle = \langle\bar{z}\rangle\gamma_r(\tilde{y}_i)$, $z \in L^+$, and

2) $\gamma_r(\tilde{y}_i)z = z\gamma_r(\tilde{y}_i) - \sum_{k=1}^{p-1} y_i^{p-1-k}\langle\overline{(\mathrm{ad}\, y_i)^k(z)}, \bar{y}_i\rangle\,\gamma_{r-1}(\tilde{y}_i)$, $z \in L^+$.

Then we can define a differential on $X(L)$ by

3) $d\gamma_r(\tilde{y}_i) = (y_i^{p-1}\langle\bar{y}_i\rangle - \langle\overline{\beta(y_i)}\rangle)\gamma_{r-1}(\tilde{y}_i)$

4) $d(\gamma_{r_1}(\tilde{y}_{i_1})\cdots\gamma_{r_n}(\tilde{y}_{i_n})) = \sum_{j=1}^{n} \gamma_{r_1}(\tilde{y}_{i_1})\cdots(d\gamma_{r_j}(\tilde{y}_{i_j}))\cdots\gamma_{r_n}(\tilde{y}_{i_n})$,

which is to be determined as an element of the $V(L)$-module $X(L)$ by means of 1) and 2), and

5) $d(w\Phi) = (dw)\Phi + (-1)^{\deg w}\, w d\Phi$, $w \in W(L)$, $\Phi \in \Gamma(s^2\pi L^+)$,

where $wd\Phi$ is to be determined as an element of the $V(L)$-module $X(L)$ by means of 1) and 2) and the algebra structure of $W(L)$.

$X(L)$ with this differential is a free $V(L)$-complex.

<u>Proof</u>: The proof consists only in verifying that the theorem accurately describes the structure of complex induced on $X(L)$ by the isomorphism $X(L) \cong W(L) \otimes_{W(L^+)} (\bigotimes_{i \in I} {}_{W(L^+)} (W(L^+) \otimes \Gamma(\tilde{y}_i))$, and this follows from formula α) of Step 1 and β) of Step 2.

We make no attempt to derive an explicit formula for the differential on $X(L)$, as its form is quite complicated in the general case. We remark that the differential depends on the choice of the ordering of the set I: if we interchange the order of two basis elements, the formula for the differential is changed.

We must prove that $X(L)$ is actually a free resolution of K over $V(L)$. We will do this by first proving the result for the case of an Abelian restricted Lie algebra with zero pth powers and then filtering $X(L)$ in such a manner that $E^0X(L) = X(L^{\#})$, where $L^{\#}$ is the underlying K-space of L regarded as an Abelian restricted Lie algebra with zero pth powers.

We let $X = X(L)$ and note that $X_0 = V(L)$ and $X_1 \xrightarrow{d} V(L) \xrightarrow{\varepsilon} K \longrightarrow 0$ is exact, where ε is the augmentation. We first prove the

Lemma I.3.9: Let L be Abelian with zero pth powers. Then X is a free resolution of K over $V(L)$.

Proof: We must prove that $H_*(X) = K$. Let $g = \langle\bar{z}_1,\ldots,\bar{z}_n\rangle\gamma_{r_1}(\bar{x}_1)\ldots\gamma_{r_m}(\bar{x}_m)\gamma_{s_1}(\tilde{y}_1)\ldots\gamma_{s_\ell}(\tilde{y}_\ell) \in \bar{X}(L)$. In this case, inspection of Theorem I.3.8 shows that the differential is independent of the order in which factors of $\Gamma(s^2\pi L^+)$ are written, and is in fact given by the explicit formula:

$$\begin{aligned} d(g) = &\sum_{i=1}^{n} (-1)^{i+1} z_i\langle\bar{z}_1,\ldots,\hat{\bar{z}}_i,\ldots,\bar{z}_n\rangle\gamma_{r_1}(\bar{x}_1)\ldots\gamma_{r_m}(\bar{x}_m)\gamma_{s_1}(\tilde{y}_1)\ldots\gamma_{s_\ell}(\tilde{y}_\ell) \\ &+ \sum_{i=1}^{m} x_i\langle\bar{z}_1,\ldots,\bar{z}_n\rangle\,\gamma_{r_1}(\bar{x}_1)\ldots\gamma_{r_i-1}(\bar{x}_i)\ldots\gamma_{r_m}(\bar{x}_m)\gamma_{s_1}(\tilde{y}_1)\ldots\gamma_{s_\ell}(\tilde{y}_\ell) \\ &+ \sum_{i=1}^{\ell} y_i^{p-1}\langle\bar{z}_1,\ldots,\bar{z}_n,\bar{y}_i\rangle\gamma_{r_1}(\bar{x}_1)\ldots\gamma_{r_m}(\bar{x}_m)\gamma_{s_1}(\tilde{y}_1)\ldots\gamma_{s_i-1}(\tilde{y}_i)\ldots\gamma_{s_\ell}(\tilde{y}_\ell)\,. \end{aligned}$$

We will prove the result by obtaining a contracting homotopy s, that is, a morphism of K-modules $X \longrightarrow X$ which satisfies $sd+ds = i-\varepsilon$, where I denotes the identity map. Suppose first that L has one generator $y \in L^{+} = L^{\cdot}$. Then:

$$d(1) = 0 \qquad\qquad s(1) = 0$$

$$d(y^{j}\gamma_{i}(\tilde{y})) = 0 \qquad\qquad s(y^{j}\gamma_{i}(\tilde{y})) = y^{j-1}\langle\bar{y}\rangle\gamma_{i}(\tilde{y}), \quad 1 \leq j \leq p-1,\ 0 \leq i$$

$$d(y^{j-1}\langle\bar{y}\rangle\gamma_{i}(\tilde{y})) = y^{j}\gamma_{i}(\tilde{y}) \qquad\qquad s(y^{j-1}\langle\bar{y}\rangle\gamma_{i}(\tilde{y})) = 0, \quad 1 \leq j \leq p-1,\ 0 \leq i$$

$$d(y^{p-1}\langle\bar{y}\rangle\gamma_{i}(\tilde{y})) = 0 \qquad\qquad s(y^{p-1}\langle\bar{y}\rangle\gamma_{i}(\tilde{y})) = \gamma_{i+1}(\tilde{y}), \quad 0 \leq i$$

$$d(\gamma_{i+1}(\tilde{y})) = y^{p-1}\langle\bar{y}\rangle\gamma_{i}(\tilde{y}) \qquad\qquad s(\gamma_{i+1}(\tilde{y})) = 0, \quad 0 \leq i$$

s so defined clearly satisfies $sd+ds = i-\varepsilon$. Next, suppose that L has one generator $x \in L^{-} = L$. Then $s(1) = 0$, $s(x\gamma_{i}(\bar{x})) = \gamma_{i+1}(\bar{x})$, $s(\gamma_{i+1}(\bar{x})) = 0$, $0 \leq i$, defines the desired contracting homotopy. Now suppose $L = M \oplus N$, where $\dim M = 1$. We may identify X with $X_1 \otimes X_2$, $X_1 = X(M)$, $X_2 = X(N)$, and then $d = d_1 \otimes i_2 + i_1 \otimes d_2$. Let s_1 be the contracting homotopy just constructed on X_1 and assume as an induction hypothesis that we have a contracting homotopy s_2 on X_2. Define s on X by $s = s_1 \otimes i_2 + \varepsilon_1 \otimes s_2$. Then we find:

$$ds = d_1 s_1 \otimes i_2 + d_1\varepsilon_1 \otimes s_2 - s_1 \otimes d_2 + \varepsilon_1 \otimes d_2 s_2 \quad \text{and}$$

$$sd = s_1 d_1 \otimes i_2 + s_1 \otimes d_2 - \varepsilon_1 d_1 \otimes s_2 + \varepsilon_1 \otimes s_2 d_2 \,.$$

Since $d_1\varepsilon_1 = \varepsilon_1 d_1 = 0$ and s_1 and s_2 are contracting homotopies,

$$ds+sd = (i_1-\varepsilon_1) \otimes i_2 + \varepsilon_1 \otimes (i_2-\varepsilon_2) = i_1 \otimes i_2 - \varepsilon_1 \otimes \varepsilon_2 = i-\varepsilon \,.$$

By finite and transfinite induction, this completes the proof.

<u>Theorem I.3.10</u>: X is a free resolution of K over $V(L)$.

<u>Proof</u>: We must prove that the complex

$$X': \dots \longrightarrow X_n \longrightarrow \dots \longrightarrow X_1 \longrightarrow \overline{X}_o \longrightarrow 0 \text{ is exact,}$$

where $\overline{X}_o = \operatorname{Ker} \varepsilon$, the augmentation ideal of $V(L)$.

We define a filtration on $X(L)$ as follows:

1) $V(L)$ is given the filtration defined in Definitions I.2.7.

2) $\overline{X}$ is filtered by $F_q\overline{X} = 0$ for $q < 0$, $F_o\overline{X} = K$, and, if $q > 0$, $<\overline{z}_1,\dots,\overline{z}_n>\gamma_{r_1}(\overline{x}_1)\dots\gamma_{r_m}(\overline{x}_m)\gamma_{s_1}(\tilde{y}_1)\dots\gamma_{s_\ell}(\tilde{y}_\ell) \in F_q\overline{X}$ if and only if $n + \sum_{v=1}^{m} r_v + p\sum_{v=1}^{\ell} s_v \leq q$ and $\Sigma k_v q_v \in F_q\overline{X}$ if and only if some $q_v \in F_q\overline{X}$, where q_v is a basis element of $\overline{X}$ and $k_v \in K$.

3) $F_qX = \sum_{i+j=q} F_iV(L) \otimes F_j\overline{X}$.

Then X' is filtered by $F_qX' = F_qX$ if $q \neq 0$, $F_oX' = 0$. Using Theorem I.3.8, it is easily seen that F_qX' is a subcomplex of X' and of F_tX' for $t > q$. Thus E^oX' is a complex.
$E^o_{q,r}X' = (F_qX'/F_{q+1}X')_{q+r}$, where $q+r$ is the homological dimension; grade E^oX' by total degree: $E^o_nX' = \bigoplus_{q+r=n} E^o_{q,r}X'$. Then, using Corollary I.2.10 and the definition of X' as a complex, we find that E^oX' is precisely the complex $X'(L^\#)$, where $L^\#$ is the Abelian Lie algebra with zero <u>pth</u> powers on the underlying space of L. Therefore $E^1X' = H_*(E^oX') = 0$ by the lemma. It follows that $H_*(X') = 0$, as was to be proven.

We complete this section by defining a diagonal map D on our resolution X. Let ψ denote the natural coproduct in $V(L)$ defined by requiring that the elements of L be primitive. Define a structure of $V(L)$-module on $X \otimes X$ by $u(a\otimes b) = \psi(u)\cdot a\otimes b$, where the product on the right is defined by the composition

$$(V(L) \otimes V(L)) \otimes (X \otimes X) \xrightarrow{1\otimes T\otimes 1} (V(L) \otimes X) \otimes (V(L) \otimes X) \longrightarrow X \otimes X .$$

If $\bar{d}$ denotes the differential $d\otimes 1 + 1\otimes d$ on $X \otimes X$, then it is easily verified that $\bar{d}(\psi(u)\cdot a\otimes b) = (-1)^{\deg u}\,\psi(u)\bar{d}(a\otimes b)$, that is, $X \otimes X$ is a differential $V(L)$-module. Noting that, by Theorem I.3.8, the differential on X is defined formally as if X were a differential K-algebra, provided that we write elements of $\bar{X}$ with "factors" in the correct order, we may define a diagonal map D on X as follows:

1) D is a morphism of $V(L)$-modules: $D(ua) = (-1)^{\deg u}\,\psi(u)D(a)$.

2) $D\langle\bar{y}\rangle = \langle\bar{y}\rangle \otimes 1 + 1 \otimes \langle\bar{y}\rangle$

$$D(\gamma_r(\bar{x})) = \sum_{i+j=r} \gamma_i(\bar{x}) \otimes \gamma_j(\bar{x})$$

$$D(\gamma_s(\tilde{y})) = \sum_{j+k=s} \gamma_j(\tilde{y}) \otimes \gamma_k(\tilde{y}) + \sum_{i=1}^{p-1} \sum_{j+k=s-1} (-1)^i y^{i-1} \langle\bar{y}\rangle\gamma_j(\tilde{y}) \otimes y^{p-1-i}\langle\bar{y}\rangle\gamma_k(\tilde{y}) .$$

3) $D(ab) = D(a)D(b)$, where ab is an element of a basis for $\bar{X}$ with factors written in an order consistent with $\bar{X}(L) = E(sL^+) \otimes \Gamma(sL^-) \otimes (\bigotimes_{i \in I} \Gamma(\tilde{y}_i))$, $\{\tilde{y}_i\ i \in I\}$ being a basis for L^+ indexed on a totally ordered set I; the product on the right is formally the same as that defined on a tensor product

of algebras: $(X \otimes X) \otimes (X \otimes X) \xrightarrow{1\otimes T\otimes 1} (X \otimes X) \otimes (X \otimes X) \xrightarrow{\Phi\otimes\Phi} X \otimes X$.
In our case, $\Phi: X \otimes X \longrightarrow X$ is to be determined by the algebra structure on $W(L) = V(L) \otimes \overline{Y}(L)$ and the right $W(L^+)$-module structure on X .

<u>Theorem I.3.11</u>: $Dd = \overline{d}D$, $\overline{d} = d\otimes 1 + 1\otimes d$.

<u>Proof</u>: Since $Dd(ua) = \psi(u)Dd(a)$ and $\overline{d}D(ua) = \psi(u)\overline{d}D(a)$, $u \in V(L)$, $a \in \overline{X}$, it suffices to prove the result on elements of $\overline{X}$. Since if ab is a basis element of $\overline{X}$

$$Dd(ab) = (Dda)Db + (-1)^{\deg a} Da(Ddb) \quad \text{and}$$
$$\overline{d}D(ab) = (\overline{d}Da)Db + (-1)^{\deg a} Da(\overline{d}Db) ,$$

it suffices to prove the result on elements of the forms $\langle\tilde{y}\rangle$, $\gamma_r(\overline{x})$, and $\gamma_s(\tilde{y})$. Here we find:

1) $Dd\langle\tilde{y}\rangle = D(y) = \psi(y) = y\otimes 1 + 1\otimes y = \overline{d}D\langle\tilde{y}\rangle$.

2)
$$\begin{aligned} Dd\gamma_r(\overline{x}) &= D(x\gamma_{r-1}(\overline{x}) + \tfrac{1}{2}\langle[\overline{xx}]\rangle\, \gamma_{r-2}(\overline{x})) \\ &= (x\otimes 1 + 1\otimes x) \sum_{i+j=r-1} \gamma_i(\overline{x}) \otimes \gamma_j(\overline{x}) \\ &\quad + \tfrac{1}{2}(\langle[\overline{xx}]\rangle \otimes 1 + 1\otimes\langle[\overline{xx}]\rangle) \sum_{i+j=r-2} \gamma_i(\overline{x}) \otimes \gamma_j(\overline{x}) \\ &= \sum_{i+j=r} d\gamma_i(\overline{x}) \otimes \gamma_j(\overline{x}) + \sum_{i+j=r} \gamma_i(\overline{x}) \otimes d\gamma_j(\overline{x}) \\ &= \overline{d}D\gamma_r(\overline{x}) . \end{aligned}$$

3) Noting that $\psi(y^{p-1}) = \sum_{i=o}^{p-1} (i,p-1-i)y^i \otimes y^{p-1-i}$ and that $(i,p-1-i) \equiv -(i+1,p-i-2) \bmod p$, and therefore $(i,p-1-i) \equiv (-1)^i \bmod p$, a simple but tedious calculation gives $Dd\gamma_s(\tilde{y}) = \overline{d}D\gamma_s(\tilde{y})$.

Remarks I.3.12: D is cocommutative, $D = TD$. D is coassociative on the subcomplex $V(L) \otimes E(sL^+) \otimes \Gamma(sL^-)$. D is coassociative on X if and only if $p = 2$, since it is easily verified that $(D\otimes 1)D(\gamma_r(\tilde{y})) = (1\otimes D)D(\gamma_r(\tilde{y}))$, $r \geq 2$, if and only if $p = 2$. The dual complex $X^* = V(L)^* \otimes \bar{X}^*$ is therefore a commutative differential algebra, associative if and only if $p = 2$, and the homology of $\bar{X}^*$ is $H^*(V(L))$. Note that the induced product on $H^*(V(L))$ must be associative, even though the product on $\bar{X}^*$ is not.

4. Embedding of resolutions in the bar construction.

Let A be an augmented graded algebra over a commutative unitary ring K. We will find sufficient conditions for a free complex X over K to be embeddable in $B(A)$. The result will be used to embed $X(L)$ in $B(V(L))$, where $X(L)$ is the resolution obtained in the previous section.

We recall the definition and properties of the bar construction. Let $B(A) = A \otimes T(I(A))$ and $\overline{B}(A) = K \otimes_A B(A) \cong T(I(A))$. $B(A)$ is bigraded, with bidegree $(a \otimes \overline{a}_1 \otimes \ldots \otimes \overline{a}_n) = n$, $\overline{a}_i = a - \varepsilon(a_i)$. We will write elements of $B(A)$ in the form $a[a_1|\ldots|a_n]$ and we let $[\ ] = 1$. Define an augmentation $\varepsilon: B(A) \longrightarrow K$ by $\varepsilon(1) = 1$, $\varepsilon(a[a_1|\ldots|a_n]) = 0$ if $n > 0$. Define a contracting homotopy $S: B(A) \longrightarrow B(A)$ by $S(a[a_1|\ldots|a_n]) = [\overline{a}|a_1|\ldots|a_n]$. A boundary d may then be defined inductively by $d(1) = 0$, $d(a[a_1|\ldots|a_n]) = (-1)^{\deg a}$ $ad[a_1|\ldots|a_n]$, and $dS + Sd = 1 - \varepsilon$. It follows that:

1) $$d(a[a_1|\ldots|a_n]) = (-1)^{\deg a}\, a(a_1[a_2|\ldots|a_n] + \sum_{1 \leq r < n} (-1)^{\lambda(r)}[a_1|\ldots|a_r\, a_{r+1}|\ldots a_n])$$

where $\lambda(r) = \sum_{1 \leq i \leq r} \deg[a_i]$. If $\overline{d} = 1 \otimes_A d$ on $\overline{B}(A)$,

2) $$\overline{d}([a_1|\ldots|a_n]) = \sum_{1 \leq r < n} (-1)^{\lambda(r)}[a_1|\ldots|a_r\, a_{r+1}|\ldots|a_n].$$

$B(A)$ is a free resolution of K over A, $H(\overline{B}(A)) = \mathrm{Tor}^A(K,K) = H_*(A)$.
The following property will be used:

<u>Lemma I.4.1</u>: If $x \in Z_qB(A) \cap \mathrm{Ker}\,\varepsilon$, then there exists one and only one $y \in \overline{B}(A)$ such that $d(y) = x$, and $y = S(x)$.

<u>Proof</u>: Clearly $x = dS(x)$. If $d(y') = x$, $d(y'') = 0$, $y'' = y' - S(x)$. But $y'' \in \overline{B}_{q+1}(A)$, $q \geq 0$, hence $S(y'') = \varepsilon(y'') = 0$. Therefore $y'' = (Sd + dS - \varepsilon)(y'') = 0$.

<u>Proposition I.4.2:</u> Let $X = A \otimes \overline{X}$ be a free complex over K such that $X_o = A$, $X_o \longrightarrow K$ is the augmentation of A, and $Z_qX \cap \overline{X} = \Phi$ for all $q > 0$. Then there exists a unique monomorphism of complexes $f\colon X \longrightarrow B(A)$ lying over the identity map of K, and satisfying $f(\overline{X}) \subset \overline{B}(A)$.

<u>Proof</u>: Let $f_o = i\colon X_o \longrightarrow B_o(A)$, so that the diagram

$$\begin{array}{ccc} X_o & \xrightarrow{f_o} & B_o(A) \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\ K & \xrightarrow{i} & K \end{array}$$

commutes. Let δ denote the differential in X. If $x \in \overline{X}_1$, $0 \neq \delta(x) \in I(A)$, $\varepsilon f_o \delta(x) = \varepsilon\delta(x) = 0$. Let $f_1(x) = Sf_o\delta(x)$, and extend f_1 to X_1 by the requirement that f_1 be a morphism of A-modules. $df_1 = f_o\delta$ and f_1 is a monomorphism. Suppose f_q has been constructed, $q \geq 1$. Let $x \in \overline{X}_{q+1}$; $\delta(x) \neq 0$, hence $f_q\delta(x) \neq 0$, and $df_q\delta(x) = f_{q-1}\delta^2(x) = 0$. Define $f_{q+1}(x) = Sf_q\delta(x)$ and extend to X_{q+1} as before. Clearly $df_{q+1} = f_q\delta$ and f_{q+1} is a monomorphism. The uniqueness of f follows from the lemma.

Before applying this result to restricted Lie algebras, we obtain some further properties of the bar construction. We define an (m,n)-shuffle as a permutation π of the $m+n$ integers $1,2,\ldots,m+n$ which satisfies $\pi(i) < \pi(j)$ if $1 \leq i < j \leq m$ and if $m+1 \leq i < j \leq m+n$. Using this concept we define a commutative multiplication in $\overline{B}(A)$ by $1*x = x$ and

$$[a_1|\ldots|a_n]*[a_{m+1}|\ldots|a_{m+n}] = \sum_{\pi} (-1)^{\varepsilon(\pi)}[a_{\pi(1)}|\ldots|a_{\pi(m+n)}]$$

where the sum is taken over all (m,n)-shuffles and $\varepsilon(\pi) = \Sigma \deg[a_i] \deg[a_{m+j}]$ summed over all pairs $(i,m+j)$ such that $\pi(i) > \pi(m+j)$, that is, such that π moves a_i past a_{m+j}. If A is commutative, $\overline{d}(x*y) = \overline{d}(x)*y + (-1)^{\deg x} x*\overline{d}(y)$. For $a \in I(A)$, $\deg a \equiv 0 \bmod 2$, define maps $\sigma(a)$ and $\tau(a)$ of $B(A)$ to itself as follows:

$$\sigma(a)(a_o\{a_1|\ldots|a_n\}) = (-1)^{\deg a_o} a_o\{a\}*\{a_1|\ldots|a_n\} ;$$

$$\tau(a)(a_o\{a_1|\ldots|a_n\}) = a_o a\{a_1|\ldots|a_n\} -$$

$$\sum_{1 \leq i \leq n} a_o\{a_1|\ldots|a_{i-1}|[a,a_i]|a_{i+1}|\ldots|a_n\} ,$$

where we have written $[a_1|\ldots|a_n] = \{a_1|\ldots|a_n\}$ to avoid confusion with the bracket product. Let $\zeta(a) = d\sigma(a) + \sigma(a)d - \tau(a)$.

<u>Lemma I.4.3</u>: $\zeta(a) = 0$ for all $a \in I(A)$ such that $\deg a \equiv 0$ mod 2 .

Proof: $\zeta(a)(a_o x) = a_o\zeta(a)(x)$; therefore it suffices to prove the result on elements $x = \{a_1|\dots|a_n\} \in \overline{B}(A)$. If $n = 0$, $\zeta(a)(\{\ \}) = a\{\ \} - a\{\ \} = 0$. Assume the result for $n-1$, $n > 0$. Then $\varepsilon(\{a_1|\dots|a_n\}) = 0$, and $Sd + dS = 1$ on $\zeta(a)\{a_1|\dots|a_n\}$. We must prove that $S\zeta(a)(\{a_1|\dots|a_n\}) = 0$ and $Sd\zeta(a)(\{a_1|\dots|a_n\}) = 0$. Calculating mod ker S :

1) $d\sigma(a)(\{a_1|\dots|a_n\}) \equiv a\{a_1|\dots|a_n\} + (-1)^{\deg\{a_1\}} a_1\{a\}*\{a_2|\dots|a_n\}$;

$\sigma(a)d(\{a_1|\dots|a_n\}) \equiv (-1)^{\deg a_1} a_1\{a\}*\{a_2|\dots|a_n\}$;

$-\tau(a)(\{a_1|\dots|a_n\}) \equiv -a\{a_1|\dots|a_n\}$.

Thus $S\zeta(a)(\{a_1|\dots|a_n\}) = 0$.

2) $d\zeta(a) = d\sigma(a)d - d\tau(a) = \tau(a)d - d\tau(a)$ on $\{a_1|\dots|a_n\}$

by application of the induction hypothesis.

$$d\tau(a)(\{a_1|\dots|a_n\}) = d(a\{a_1|\dots|a_n\} - \sum_{1 \le i \le n} \{a_1|\dots|[a,a_i]|\dots|a_n\})$$

$$\equiv aa_1\{a_2|\dots|a_n\} + \sum_{1 \le r < n} (-1)^{\lambda(r)} a\{a_1|\dots|a_r a_{r+1}|\dots|a_n\}$$

$$- [a,a_1]\{a_2|\dots|a_n\} - \sum_{2 \le i \le n} a_1\{a_2|\dots|[a,a_i]|\dots|a_n\} ;$$

$$\tau(a)d(\{a_1|\dots|a_n\}) = \tau(a)(a_1\{a_2|\dots|a_n\} +$$

$$+ \sum_{1 \le r < n} (-1)^{\lambda(r)}\{a_1|\dots|a_r a_{r+1}|\dots|a_n\})$$

$$\equiv a_1 a\{a_2|\dots|a_n\} - \sum_{2 \le i \le n} a_1\{a_2|\dots|[a,a_i]|\dots|a_n\}$$

$$+ \sum_{1 \le r < n} (-1)^{\lambda(r)} a\{a_1|\dots|a_r a_{r+1}|\dots|a_n\} .$$

Since $aa_1 - [a,a_1] = a_1 a$, $S(\tau(a)d - d\tau(a))(\{a_1|\dots|a_n\}) = 0$ as desired.

Next, let $\{a\}^k = \{a|\dots|a\}$ k factors, for $a \in I(A)$ and $\deg a \equiv 1 \bmod 2$, $\{a\}^{-1} = 0$. Define

$$\sigma_k(a)(a_o\{a_1|\dots|a_n\}) = a_o\{a\}^k * \{a_1|\dots|a_n\} \quad \text{and}$$

$$\tau_k(a)(a_o\{a_1|\dots|a_n\}) = (-1)^{\deg a_o} a_o(a\{a\}^{k-1}*\{a_1|\dots|a_n\} + \tfrac{1}{2}\{[a,a]\}*$$

$$\{a\}^{k-2}*\{a_1|\dots|a_n\} + \sum_{1 \le i \le n} (-1)^{\lambda(i-1)}\{a\}^{k-1} *$$

$$\{a_1|\dots|[a,a_i]|\dots|a_n\})$$

Let $\zeta_k(a) = d\sigma_k(a) - \sigma_k(a)d - \tau_k(a)$. To simplify notation, note that $\frac{1}{2}\{[a,a]\} = \{a^2\}$. In characteristic 2, for $a \in I(A)$ define $\sigma_k(a)$, $\tau_k(a)$, and $\zeta_k(a)$ as above but with $\{a^2\}$ replacing $\frac{1}{2}\{[a,a]\}$. Then the proof of the following lemma gives $\zeta_k(a) = 0$ for all a .

Lemma I.4.4: $\zeta_k(a) = 0$ for all $a \in I(A)$ such that $\deg a \equiv 1 \mod 2$.

Proof: $\zeta_k(a)(a_o x) = (-1)^{\deg a_o} a_o \zeta_k(a)(x)$. $\zeta_k(a)([\ ]) = 0$, since $\tau_k(a)([\ ]) = a[a]^{k-1} + [a^2] * [a]^{k-2} = d\sigma_k(a)([\ ]) = d[a]^k$. Let $x = [a_1|\dots|a_n]$ and assume the result for elements in $\overline{B}_{n-1}(A)$.

1) $S\zeta_k(a)(x) = 0$: Calculating mod ker S , we have

$$d\sigma_k(a)[a_1|\dots|a_n] = d[a]^k * [a_1|\dots|a_n]$$

$$\equiv a[a]^{k-1} * [a_1|\dots|a_n] + a_1[a]^k * [a_2|\dots|a_n] ,$$

$$-\sigma_k(a)d[a_1|\dots|a_n] \equiv - a_1[a]^k * [a_2|\dots|a_n] , \text{ and}$$

$$-\tau_k(a)[a_1|\dots|a_n] \equiv - a[a]^{k-1} * [a_1|\dots|a_n] .$$

2) $Sd\zeta_k(a)(x) = - S(d\sigma_k(a)d + d\tau_k(a))(x) = - S(\tau_k(a)d + d\tau_k(a))(x) = 0$:

Calculating mod ker S , we have

$$\tau_k(a)d[a_1|\dots|a_n] \equiv (-1)^{\deg a_1} a_1(a[a]^{k-1} * [a_2|\dots|a_n] + [a^2] * [a]^{k-2} * [a_2|\dots|a_n]) + (-1)^{\deg a_1} a_1 \sum_{2 \le i < n} (-1)^{\lambda(i-2)} [a]^{k-1} * [a_2|\dots|[a,a_i]|\dots|a_n] + a \sum_{1 \le r < n} (-1)^{\lambda(r)} [a]^{k-1} * [a_1|\dots|a_r a_{r+1}|\dots|a_n] ,$$ while

$$d(a\{a\}^{k-1} * \{a_1|\dots|a_n\}) \equiv - a^2\{a\}^{k-2} * \{a_1|\dots|a_n\} - aa_1\{a\}^{k-1} *$$
$$* \{a_2|\dots|a_n\} - a\{a^2\} * \{a\}^{k-3} * \{a_1|\dots|a_n\} - a \sum_{1 \le r \le n} (-1)^{\lambda(r-1)}$$
$$\{a\}^{k-2} * \{a_1|\dots|[a,a_r]|\dots|a_n\} - a \sum_{1 \le r < n} (-1)^{\lambda(r)}\{a\}^{k-1} *$$
$$*\{a_1|\dots|a_r a_{r+1}|\dots|a_n\} ,$$

$$d(\{a^2\} * \{a\}^{k-2} * \{a_1|\dots|a_n\}) \equiv a^2\{a\}^{k-2} * \{a_1|\dots|a_n\} + a\{a^2\} * \{a\}^{k-3} *$$
$$* \{a_1|\dots|a_n\} + (-1)^{\deg \{a_1\}} a_1\{a^2\} * \{a\}^{k-2} * \{a_2|\dots|a_n\} , \text{ and}$$

$$d(\sum_{1 \le i \le n} (-1)^{\nu(i)}\{a\}^{k-1} * \{a_1|\dots|[a,a_i]|\dots|a_n\}) \equiv (-1)^{\lambda(0)}[a,a_1]$$
$$\{a\}^{k-1} * \{a_2|\dots|a_n\} + a_1 \sum_{2 \le i \le n} (-1)^{\lambda(i-1)}\{a\}^{k-1} *$$
$$* \{a_2|\dots|[a,a_i]|\dots|a_n\} + a \sum_{1 \le i \le n} (-1)^{\lambda(i-1)}\{a\}^{k-2} *$$
$$* \{a_1|\dots|[a,a_i]|\dots|a_n\} .$$

Noting that $\lambda(0) = 0$ and that $\lambda(i) = \lambda(i-1) + \deg \{a_1\}$, adding the last terms gives $d\tau(a)\{a_1|\dots|a_n\} \equiv - \tau(a)d\{a_1|\dots|a_n\}$, mod ker S , as was to be shown.

Now define $\gamma_k(a) = \{a\}^k$ for $a \in I(A)$ such that deg a is odd. Clearly $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x) * \gamma_j(y)$, and, since there exist (k,h) (k,h)-shuffles, $\gamma_k(a)\gamma_h(a) = (k,h)\gamma_{k+h}(a)$. $d(\gamma_k(a)) = a\gamma_{k-1}(a) + \{a^2\} * \gamma_{k-2}(a)$. Let $I^+(A)$ be the subset of $I(A)$ consisting of the elements of even degree, $I^-(A)$ the subset of elements of odd degree. Then (ignoring the algebra structure of A)

$E(sI^+(A)) \otimes \Gamma(sI^-(A))$ is embedded as an algebra in $\overline{B}(A)$ via the map $\overline{f}$ given by

$$\overline{f}(<\overline{y}_1,\ldots,\overline{y}_n>\gamma_{r_1}(\overline{x}_1)\ldots\gamma_{r_m}(\overline{x}_m)) = \{y_1\}*\ldots*\{y_n\} * \{x_1\}^{r_1}*\ldots*\{x_m\}^{r_m},$$

where s is the suspension as defined in the previous section, $\overline{y}_i = s(y_i)$, $\overline{x}_i = s(x_i)$. Extend $\overline{f}$ to a map f of $A \otimes E(sI^+(A)) \otimes \Gamma(sI^-(A)) = M(A)$ into $B(A)$ by requiring f to be a morphism of A-modules. We identify $M(A)$ with its image in $B(A)$. Let $y \in I^+(A)$, $x \in I^-(A)$. We have:

1) $\sigma(y)(a<\overline{y}_1,\ldots,\overline{y}_n>\gamma_{r_1}(\overline{x}_1)\ldots\gamma_{r_m}(\overline{x}_m)) = (-1)^{\deg a}\, a<\overline{y},\overline{y}_1,\ldots,\overline{y}_n>$
$\gamma_{r_1}(\overline{x}_1)\ldots\gamma_{r_m}(\overline{x}_m)$

2) $\tau(y)(a<\overline{y}_1,\ldots,\overline{y}_n>\gamma_{r_1}(\overline{x}_1)\ldots\gamma_{r_m}(\overline{x}_m)) =$

$$= ay<\overline{y}_1,\ldots,\overline{y}_n>\gamma_{r_1}(\overline{x}_1)\ldots\gamma_{r_m}(\overline{x}_m)$$
$$+ \sum_{1\le j\le n}(-1)^{j+1}\, a<\overline{[y,y_j]},\overline{y}_1,\ldots,\hat{\overline{y}}_j,\ldots\overline{y}_n>\gamma_{r_1}(\overline{x}_1)\ldots\gamma_{r_m}(\overline{x}_m)$$
$$+ \sum_{1\le i\le m} a<\overline{y}_1,\ldots,\overline{y}_n>\gamma_1(\overline{[x_i,y]})\gamma_{r_1}(\overline{x}_1)\ldots\gamma_{r_i-1}(\overline{x}_i)\ldots\gamma_{r_m}(\overline{x}_m)$$

3) $d\sigma(y) = \tau(y) - \sigma(y)d$

4) $\sigma_k(x)(a<\overline{y}_1,\ldots,\overline{y}_n>\gamma_{r_1}(\overline{x}_1)\ldots\gamma_{r_m}(\overline{x}_m)) = a<\overline{y}_1,\ldots,\overline{y}_n>$
$\gamma_k(\overline{x})\gamma_{r_1}(\overline{x}_1)\ldots\gamma_{r_m}(\overline{x}_m)$

5) $\tau_k(x)(a < \bar{y}_1,\ldots,\bar{y}_n > \gamma_{r_1}(\bar{x}_1)\ldots\gamma_{r_m}(\bar{x}_m) = (-1)^{\deg a}\, ax < \bar{y}_1,\ldots,\bar{y}_n >$

$\gamma_{k-1}(\bar{x})\gamma_{r_1}(\bar{x}_1)\ \ldots\ \gamma_{r_m}(\bar{x}_m) + (-1)^{\deg a}\, a\cdot\frac{1}{2}<\overline{[x,x]},\bar{y}_1,\ldots\bar{y}_n>\gamma_{k-2}(\bar{x})$

$\gamma_{r_1}(\bar{x}_1)\ldots\gamma_{r_m}(\bar{x}_m) + (-1)^{\deg a}\, a \sum_{1 \le j \le n} (-1)^{j+1}$

$< \bar{y}_1,\ldots,\hat{\bar{y}}_j,\ldots\bar{y}_n > \gamma_1(\overline{[x,y_j]})\gamma_{k-1}(\bar{x})\gamma_{r_1}(\bar{x}_1)\ldots\gamma_{r_m}(\bar{x}_m)$

$+ (-1)^{\deg a}\, a \sum_{1 \le i \le m} <\overline{[x,x_i]},\bar{y}_1,\ldots,\bar{y}_n> \gamma_{k-1}(\bar{x})$

$\gamma_{r_1}(\bar{x})\ldots\gamma_{r_i-1}(\bar{x}_i)\ldots\gamma_{r_m}(\bar{x}_m)$

6) $d\sigma_k(x) = \tau_k(x) + \sigma_k(x)d$

These formulae prove, by induction on the bidegree, that $M(A)$ is a subcomplex of $B(A)$ on which d is defined by the formula given in Corollary I.3.4. It follows that for any Lie algebra $L \subset A$, $A \otimes \bar{Y}(L)$, $\bar{Y}(L) = E(sL^+) \otimes \Gamma(sL^-)$, is a subcomplex of $M(A) \subset B(A)$. If $A = U(L)$, this gives the embedding of the free resolution $A \otimes \bar{Y}$ in $B(A)$. If $A = V(L)$, this gives the embedding of the subcomplex $A \otimes \bar{Y}$ of $X(L)$ in $B(A)$. We remark that the method of proof here is a generalization of that in Cartan and Eilenberg for the case of Lie algebras concentrated in degree zero.

Define a diagonal map $\bar{D}$ on $\bar{B}(A)$ by $\bar{D}\{a_1|\ldots|a_n\} = \sum_{i=0}^{n} (-1)^{\varepsilon}\{a_1|\ldots|a_i\} \otimes \{a_{i+1}|\ldots|a_n\}$, $\varepsilon = \deg\{a_1|\ldots|a_i\}\deg\{a_{i+1}|\ldots|a_n\}$.

$\bar{D}\,\bar{d} = (\bar{d} \otimes 1 + 1 \otimes \bar{d})\bar{D}$ (where $(1 \otimes \bar{d})(a \otimes b) = (-1)^{\deg a}\, a \otimes \bar{d}(b)$).

Giving $\overline{B}(A) \otimes \overline{B}(A)$ its algebra structure induced by the $*$ product in $\overline{B}(A)$, $\overline{D}$ is seen to be a morphism of algebras. Let $L \subset A$ be a Lie algebra and give $\overline{Y}(L)$ a structure as a Hopf algebra by requiring that $s(x)$ be primitive, $x \in L$. Then the embedding of $\overline{Y}$ in $\overline{B}(A)$ constructed above is a monomorphism of Hopf algebras. If A is itself a Hopf algebra with diagonal map ψ, $\overline{D}$ may be extended to $D\colon B(A) \longrightarrow B(A)$ by $D(ax) = (-1)^{\deg a}\psi(a)D(x)$, where the latter product is defined in the obvious way. Now if $A = U(L)$ or $A = V(L)$, the embedding of the complex $A \otimes \overline{Y}$ in $B(A)$ clearly carries the diagonal map D defined in the previous section to that just constructed on $B(A)$: $(f \otimes f)D = Df$, $f\colon A \otimes F \longrightarrow B(A)$.

Finally, let $A = V(L)$ for a restricted Lie algebra L. Let X be the free resolution of K constructed in section 3. By Proposition I.4.2, there exists a unique embedding f of X in $B(A)$, and $f|A \otimes F$ has been determined. Now $fd(\gamma_1(\tilde{z})) = f(z^{p-1}\langle\overline{z}\rangle - \langle\overline{\beta(z)}\rangle)$, $dfd(\gamma_1(\tilde{z})) = 0$, hence $f(\gamma_1(\tilde{z})) = Sfd(\gamma_1(\tilde{z})) = \{z^{p-1}|z\}$. Using Theorem I.38 to determine $d(\langle\overline{y}\rangle\gamma_1(\tilde{z}))$, we find

$$f(\langle\overline{y}\rangle\gamma_1(\tilde{z})) = \{y|z^{p-1}|z\} + \sum_{j=0}^{p-1}(\{z^{p-1-j}|z|(\operatorname{ad} z)^j(y)\} - \{z^{p-1-j}|(\operatorname{ad} z)^j(y)|z\}) .$$

In particular, $f(\langle\overline{z}\rangle\gamma_1(\tilde{z})) = \{z|z^{p-1}|z\} = \{z\} * \{z^{p-1}|z\}$. Let $\{z^{p-1}|z\}^k = \{z^{p-1}|z|\ldots|z^{p-1}|z\}$, k factors $\{z^{p-1}|z\}$. Then $f(\langle\overline{z}\rangle\gamma_i(\tilde{z})) = \{z\} * \{z^{p-1}|z\}^i$ implies $f(\gamma_{i+1}(\tilde{z})) = \{z^{p-1}|z\}^{i+1}$, and $f(\gamma_{i+1}(\tilde{z})) = \{z^{p-1}|z\}^{i+1}$ implies $f(<\bar{z}>\gamma_{i+1}(\tilde{z})) = \{z\} * \{z^{p-1}|z\}^{i+1}$. Inductively, these formulae hold. We go no further in the general case, since Theorem I.3.8 defines a method for determining the differential in X and, knowing the differential in X, Proposition I.4.2 tells how to obtain the embedding of X in B(A). The diagonal map on X is carried over by the formula $(f \otimes f)D = D'f$. D' so defined does not coincide with the diagonal D defined above on B(A) if char $K > 2$.

Suppose char $K = 2$. In this case the resolution X and its embedding in B(A) take quite simple forms. Here $\overline{X} = E(sL) \otimes \Gamma(s^2\pi L)$ is naturally isomorphic as an algebra to $\Gamma(sL)$, and the diagonal map D of Theorem I.3.11 clearly gives $\Gamma(sL)$ its natural structure as a hopf algebra. Identifying $\overline{X}$ with $\Gamma(sL)$, Theorem I.3.8 implies that d is given by

$$d(\gamma_{r_1}(\bar{y}_1)\cdots\gamma_{r_n}(\bar{y}_n)) = \sum_{i=1}^{n} y_i\gamma_{r_1}(\bar{y}_1)\cdots\gamma_{r_i-1}(\bar{y}_i)\cdots\gamma_{r_n}(\bar{y}_n)$$

$$+ \sum_{i=1}^{n} \gamma_1(\overline{\xi(y_i)})\gamma_{r_1}(\bar{y}_1)\cdots\gamma_{r_i-2}(\bar{y}_i)\cdots\gamma_{r_n}(\bar{y}_n)$$

$$+ \sum_{i<j} \gamma_1(\overline{[y_i,y_j]})\gamma_{r_1}(\bar{y}_1)\cdots\gamma_{r_i-1}(\bar{y}_i)\cdots\gamma_{r_j-1}(\bar{y}_j)\cdots\gamma_{r_n}(\bar{y}_n) .$$

Now Lemma I.4.4 holds for all $a \in L$, provided that $\{a^2\} = \{\xi(a)\}$ replaces $\frac{1}{2}\{[a,a]\}$ in the definition of $\tau(a)$. It follows easily that the embedding $\bar{f}\colon \overline{X} \longrightarrow \overline{B}(A)$ is given by $\bar{f}(\gamma_{r_1}(\bar{y}_1)\cdots\gamma_{r_n}(\bar{y}_n)) = \{y_1\}^{r_1} * \cdots * \{y_n\}^{r_n}$ and that $(f \otimes f)D = Df$, that is, the diagonal map of Theorem I.3.11 is taken into the diagonal map constructed above on B(A).

We summarize the results obtained. Let A be a Hopf algebra over a field K. Let $L \subset A$ be a Lie algebra and define $\overline{Y}(L) = E(sL^+) \otimes \Gamma(sL^-)$. Then $A \otimes \overline{Y}(L)$ is a complex with differential d determined by the formula in Corollary I.3.4 and diagonal map D as defined above Theorem I.3.11. Proposition I.4.2 gives an embedding of complexes $f: A \otimes F(L) \longrightarrow B(A)$. $\overline{B}(A)$ is given a structure of commutative algebra by use of the shuffle product defined on page I-4.3. $B(A)$ is given the diagonal map D defined on pages I-4.9, I-4.10. Then we have the

<u>Theorem I.4.5</u>: $f: A \otimes \overline{Y}(L) \longrightarrow B(A)$ is obtained as follows:

i) $f(ax) = (-1)^{\deg a} af(x), \quad a \in A, \quad x \in \overline{Y}(L)$.

ii) $f(< \overline{y} >) = [y], \quad \overline{y} = s(y), \quad y \in L^+$

$$f(\gamma_r(\overline{x})) = [x]^r = [\underbrace{x| \ldots |x}_{r \text{ factors}}], \quad \overline{x} = sx, \quad x \in L^-$$

iii) $f(xy) = f(x)*f(y), \quad x,y \in \overline{Y}(L)$

f satisfies $(f \otimes f)\, D(x) = Df(x), \quad x \in A \otimes \overline{Y}(L)$.

If $A = V(L)$ and $X(L)$ is the free resolution of K $A \otimes \overline{Y}(L) \otimes \Gamma(s^2\pi\, L^+)$, $f: X(L) \longrightarrow B(A)$ is determined by Theorem I.4.5 on $A \otimes \overline{Y}(L)$ and satisfies:

<u>Proposition I.4.6</u>: If $\tilde{z} = s^2\pi(z)$, $z \in L^+$, then $f(\gamma_r(\tilde{z})) = [z^{p-1}|z]^r$. If $x \in \overline{Y}(L) \otimes \Gamma(s^2\pi\, L^+)$, $f(x) = Sfd(x)$, where $d(x)$ is to be determined by use of Theorem I.3.8, and S

is the contracting homotopy of $B(V(L))$.

Finally, we have

<u>Theorem I.4.7</u>: Let $A = V(L)$, where char $K = 2$. Identify $\overline{X}(L)$ with $\Gamma(sL)$. Then $f: X(L) \longrightarrow B(A)$ is given by

i) $f(ax) = af(x), \quad x \in \overline{X}(L), \quad a \in A$

ii) $f(\gamma_r(\overline{y})) = [y]^r, \quad \overline{y} = s(y), \quad y \in L$

iii) $f(xy) = f(x)*f(y), \quad x,y \in \Gamma(sL)$

f satisfies $(f \otimes f)D(x) = Df(x), \quad x \in X(L)$.

5. A spectral sequence

Let A be a filtered augmented graded algebra over a unitary commutative ring K. Assume that A is projective of finite type as a K-module and that the filtration satisfies $F_i A = A$ for $i \geq 0$, $F_{-1}A = I(A)$, and, of course, $F_p A \cdot F_q A \subset F_{p+q} A$. Suppose also that either $(A/F_p A)_{p+q}$ is flat for all p and q or that each $0 \longrightarrow F_p A \longrightarrow A \longrightarrow A/F_p A \longrightarrow 0$ is split exact. Define $E^o_{r,s}(A) = (F_r A/F_{r-1}A)_{r+s}$. We will construct a spectral sequence $E_r(A)$ such that $E_2(A) = H^*(E^o A)$ and $E_\infty(A) = E^o H^*(A)$. Under additional hypotheses, we will obtain an interpretation of each E_r.

We obtain first the dual to the desired spectral sequence. Define a filtration on $\overline{B}(A)$ by $F_p T^n(I(A)) = \sum_{i_1+\dots+i_n+n=p,\, i_j \leq -1} F_{i_1} I(A) \otimes \dots \otimes F_{i_n} I(A)$ for $p \leq 0$ and $F_p\overline{B}(A) = \overline{B}(A)$ for $p \geq 0$. Since $F_p I(A) \cdot F_q I(A) \subset F_{p+q} I(A)$, $\overline{d} F_p \overline{B}(A) \subset F_{p-1}\overline{B}(A)$. Thus $E^o = E^1$ in the resulting spectral sequence, $E^1_{r,s} = (F_r\overline{B}(A)/F_{r-1}\overline{B}(A))_{r+s}$ where $r+s$ is the bidegree (except in certain signs, the grading induced by that of A will be of no further concern, all maps conserving this degree). We consider the spectral sequence to commence with E^1 and continue to let E^o denote the associated graded algebra of A with respect to the given filtration. Defining $E^o_p = \cup_q E^o_{p,q}$, we bi-grade $\overline{B}(E^o)$ by $\overline{B}_{r,n-r}(E^o) = \sum_{i_1+\dots+i_n+n=r,\, i_j \leq -1} E^o_{i_1} \otimes \dots \otimes E^o_{i_n}$.

Due to the assumption that $A/F_p A$ is flat or that $0 \longrightarrow F_p A \longrightarrow A \longrightarrow A/F_p A \longrightarrow 0$ is split exact, we may identify

$\overline{B}(E^o)$ with E^1 as a bigraded K-module, and then $d_1: E^1_{r,s} \longrightarrow E^1_{r-1,s}$ agrees with the differential on $\overline{B}(E^o)$. Therefore $E^2 = H_*(E^o)$ as a bigraded K-module.

To dualize, let A^* be the coalgebra dual to A, with coproduct Φ^* and augmentation ε^*. Let $J(A^*) = \text{coker}\ \varepsilon^* = I(A)^*$, $\overline{C}(A^*) = T(J(A^*))$. Write elements of $\overline{C}(A^*)$ in the form $[\alpha_1|\ldots|\alpha_n]$. Then if $\Phi^*(\alpha_r) = \sum_s \alpha'_{r,s} \otimes \alpha''_{r,s}$, the coboundary $\overline{\delta}$ in $\overline{C}(A^*)$ dual to $\overline{d}$ in $\overline{B}(A)$ is given by the formula

$$1)\quad \overline{\delta}[\alpha_1|\ldots|\alpha_{n-1}] = \sum_{r,s} (-1)^{\lambda(r,s)}[\alpha_1|\ldots|\alpha'_{r,s}|\alpha''_{r,s}|\ldots|\alpha_{n-1}],$$

$$\lambda(r,s) = \deg \alpha''_{r,s} \deg[\alpha'_{r,s}] + \sum_{r+1 \leq i \leq n-1} \deg[\alpha_i].$$

(Here we have defined $<\alpha\otimes\beta, a\otimes b> = (-1)^{\deg \beta \deg a}<\alpha,a><\beta,b>$ for the pairing of $A^* \otimes A^*$ and $A \otimes A$, hence

$$<[\alpha_1|\ldots|\alpha_n],[a_1|\ldots|a_n]> = (-1)^{\sum_i \deg[a_i] \sum_{j>i} \deg[\alpha_j]} \prod_k <[\alpha_k],[a_k]>.)$$

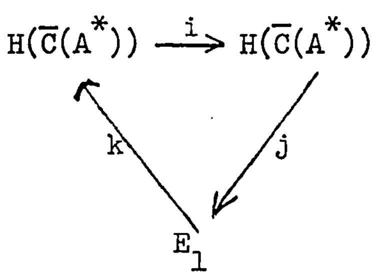
$\overline{C}(A^*)$ is the cobar construction, $H(\overline{C}(A^*)) = H^*(A)$.

Define a filtration on A^* by $F^pA^* = 0$, $p \geq 0$, $F^pA^* = [I(A)/F_{p-1}I(A)]^*$, $p \leq 0$. Then define a filtration of $\overline{C}(A^*)$ by $F^pT^nJ(A^*) = \sum_{i_1+\ldots+i_n+n=p,\, i_j \leq -1} F^{i_1}J(A^*) \otimes\ldots\otimes F^{i_n} J(A^*)$,

or, equivalently, $F^p\overline{C}(A^*) = [\overline{B}(A)/F_{p-1}\overline{B}(A)]^*$. Then $E_o = E_1 = \overline{C}((E^oA)^*)$, $E_1^{p,q} = (F^p\overline{C}(A^*)/F^{p+1}\overline{C}(A^*))_{p+q} = (E^1_{p,q})^*$, and $E_2 = H^*(E^o)$. $H^*(A)$ is filtered by $F^pH^*(A) = \ell(H(F^p\overline{C}(A^*)))$ where ℓ is induced by

$F^p\overline{C}(A^*) \subset \overline{C}(A^*)$. $E^o_{p,q}H^*(A) = (F^pH^*(A)/F^{p-1}H^*(A))_{p+q}$ and is isomorphic to $E^{p,q}_\infty = Z^{p,q}_\infty/B^{p,q}_\infty$, where $Z^{p,q}_\infty = \text{Ker } k$, $k\colon E^{p,q}_1 \longrightarrow H(F^{p+1}\overline{C}(A^*))_{p+q+1}$, and $B^{p,q}_\infty = j\ (\text{Ker } \ell)$, $j\colon H(F^p\overline{C}(A^*))_{p+q} \longrightarrow E^{p,q}_1$. Clearly some condition is necessary on our filtrations in order for the spectral sequence to determine $H^*(A)$. Suppose A satisfies the condition $\varprojlim A/F_pA = A$, or $\varinjlim F^pA^* = J(A^*)$ or (since direct limits commute with tensor products) $\varinjlim F^p\overline{C}(A^*) = \overline{C}(A^*)$. The last condition is the statement that the filtration of $\overline{C}(A^*)$ is complete, in the terminology of Eilenberg and Moore, "Limits and Spectral Sequences," and in this case our spectral sequence does in a sense determine $H^*(A)$.

We consider now the behavior of products in our spectral sequence. The product $[\alpha_1|\ldots|\alpha_n][\alpha_{n+1}|\ldots|\alpha_{n+m}] = (-1)^{\varepsilon}[\alpha_1|\ldots|\alpha_{n+m}]$, $\varepsilon = \deg[\alpha_1|\ldots|\alpha_n]\ \deg[\alpha_{n+1}|\ldots|\alpha_{n+m}]$, is dual to the diagonal map $\overline{D}$ of $\overline{B}(A)$ defined in the previous section. Therefore $\overline{\delta}(xy) = \overline{\delta}(x)\cdot y + (-1)^{\deg x}\, x\overline{\delta}(y)$, where $\deg\ x$ is the total degree $n + \sum_{1 \le i \le n} \deg \alpha_i$, $x = [\alpha_1|\ldots|\alpha_n]$. Thus $\overline{C}(A^*)$ is a differential graded algebra, and an algebra structure is induced on $H^*(A)$. Consider the exact couple



i is easily verified to be a transduction, $i(xy) = xi(y) = i(x)y$. If $x = i(a)$, $y = i(b)$, $xy = i(ab)$ defines $i(H(\overline{C}(A^*)))$ as a graded algebra, and $i|i(H(\overline{C}(A^*)))$ is a transduction. The process may be iterated to obtain the product in $D_r = i^{r-1}(H(\overline{C}(A^*)))$. If $\overline{x}$ denotes the image of $x \in \overline{C}(A^*)$ in E_1, $\overline{x}\,\overline{y} = \overline{xy}$. Suppose $k(\overline{x}) = i^{r-1}(a)$, $k(\overline{y}) = i^{r-1}(b)$. Then $k(\overline{x}\,\overline{y})$ is the cohomology class of $\overline{\delta}(x)\cdot y + (-1)^{\deg x} x\overline{\delta}(y)$, which is in D_r, say $i^{r-1}(c) = k(\overline{x}\,\overline{y})$. $j(c) = j(a)y + (-1)^{\deg x} xj(b)$, and $j^{(r)}k^{(r)}(\overline{x}\,\overline{y}) = j^{(r)}k^{(r)}(\overline{x})\cdot\overline{y} + (-1)^{\deg x}\,\overline{x}j^{(r)}k^{(r)}(\overline{y})$. Therefore each E_r is a differential graded algebra. It is also clear that $j^{(r)}(xy) = j^{(r)}(x)j^{(r)}(y)$ and that $k^{(r)}[j^{(r)}(x)\overline{y}] = (-1)^{\deg x} xk^{(r)}(\overline{y})$ and $k^{(r)}[\,\overline{x}j^{(r)}(y)] = k^{(r)}(\overline{x})y$. (We have followed Massey, "Products in Exact Couples" here.) Note that Z_{r+1} is a subalgebra of E_r and that B_{r+1} is an ideal in Z_{r+1}. Thus each E_{r+1} is the quotient of a subalgebra of E_r by an ideal. Under the hypothesis $\varprojlim A/F_pA = A$, the spectral sequence converges (not necessarily finitely) as an algebra to E_∞.

We remark that the same product structure in the spectral sequence could be obtained by dualization from any diagonal map giving $\overline{B}(A)$ a structure as a differential coalgebra, and that if A is a Hopf algebra, the products are commutative (see Cartan, Séminaire Cartan 1958/59, Exposé 12).

Now assume $A = \varprojlim A/F_pA$. Then $\bigcap_p F_pA = 0$, and if we define a weight function w on A by $w(a) = -p$ if $a \in F_pA$ and $a \notin F_{p-1}A$, we have $0 < w(a) < \infty$ for $a \in I(A)$. Suppose that A

is a free K-module and that $I(A)$ possesses a basis $\{a_i\}$ satisfying $w(\Sigma k_i a_i) = \min w(a_i)$, $k_i \in K$. Under these hypotheses we can obtain a simple algebraic interpretation of each E_r. Let A^t, $t \geq 0$, be the filtered algebra which has the same underlying filtered K-module as A and has product Φ_t induced by that of A by the formula $\Phi_t(a_i \otimes a_j) = a_i a_j \mod F_{p-t-1}A$ if $w(a_i) + w(a_j) = p$; that is, if $a_i a_j = \Sigma k_\ell a_\ell$ in A, then $a_i a_j$ in A^t is $\Sigma k_m a_m$ taken over those m such that $0 \leq w(a_m) - (w(a_i) + w(a_j)) \leq t$. We may identify $E^0_{p,q}$ with the free K-module having as basis those a_i of weight $-p$ and degree $p+q$, and if E^0 is graded by total degree, $E^0_n = \bigoplus_{p+q=n} E^0_{p,q}$, then $E^0 = A^0$. Further, $E^0(A) = E^0(A^t)$ as a bigraded algebra for all t. If we consider our spectral sequence for each A^t, it is easily seen that $E_r(A^t) = E_r(A)$, $1 \leq r \leq t+2$, and $E^{t+2}_{p,q}(A) = E^{t+2}_{p,q}(A^t) = E^\infty_{p,q}(A^t) = E^0_{p,q}H^*(A^t)$, $t \geq 0$.

We now revert to the hypotheses on A which are stated at the beginning of the section and assume in addition that K is a field. Let M be a left A-module which is of finite type. Then M^* is a right A-module, the operations of A being defined by $(fa)(m) = f(am)$, $f \in M^*$, $a \in A$, and $m \in M$. Thus the A-module structure on M^* is given by a map $M^* \otimes A \longrightarrow M^*$ satisfying $M^*_i \otimes A_q \longrightarrow M^*_{i-q}$. Let Y be any A-projective resolution of K regarded as a right A-module with trivial operations. Then we have functorial equivalences:

$$\mathrm{Tor}^A(K,M)^* = \mathrm{Hom}_K(H(Y \otimes_A M),K) = H(\mathrm{Hom}_K(Y \otimes_A M,K))$$
$$= H(\mathrm{Hom}_A(Y,\mathrm{Hom}_K(M,K))) = \mathrm{Ext}_A(K,M^*) .$$

Since K is a field, $\mathrm{tor}^A(K,M)^{**} = \mathrm{tor}^A(K,M)$, and therefore $\mathrm{Ext}_A(K,M^*)^* = \mathrm{tor}^A(K,M)$ is also true. Note further that we have shown that $\mathrm{Ext}_A(K,M^*)$ may be computed as $H((Y \otimes_A M)^*)$.

We will prove that M may be filtered in such a manner that E^0M is a left E^0A module and that we may define a spectral sequence E_2 of which is $\mathrm{Ext}_{E^0A}(K,(E^0M)^*)$ and E_∞ of which is $E^0\,\mathrm{Ext}_A(K,M^*)$. Thus define a filtration on M by $F_rM = F_rA\cdot M$ and let $E^0_{r,s}M = (F_rM/F_{r-1}M)_{r+s}$. That E^0M has a naturally induced E^0A-module structure is obvious. To define the desired spectral sequence, we need a slight modification of the bar construction. Let $B(A)^0 = T(I(A)) \otimes A$ considered as a right A-module and with the obvious differential:

$$d[a_n|\cdots|a_1]a = (-1)^{\deg a}([a_n|\cdots|a_2]a_1 + \sum_{1 \le r < n} (-1)^{\lambda(r)}[a_n|\cdots|a_{r+1}a_r|\cdots|a_1])a$$

where $\lambda(r) = \sum_{1 \le i \le r} \deg[a_i]$. $B(A)^0$ is a free resolution of K regarded as a right A-module. We filter $\overline{B}(A)^0$ exactly as we filtered $\overline{B}(A)$. Then we give the tensor product filtration (page I-1.1) to $B(A)^0 \otimes_A M$, which is isomorphic to $\overline{B}(A)^0 \otimes M$ as a K-space (but not as a complex). $d(F_r(\overline{B}(A)^0 \otimes M)) \subset F_{r-1}(\overline{B}(A)^0 \otimes M)$ and therefore $E^0 = E^1$ in the resulting spectral sequence. Clearly we may identify $E^1_{r,s} = (F_r(\overline{B}(A)^0 \otimes M)/F_{r-1}\overline{B}(A)^0 \otimes M)_{r+s}$ with $\bigoplus_{i+j=r} \overline{B}_{i,s}(E^0A) \otimes E^0_jM$, where $E^0_jM = \bigcup_k E^0_{j,k}M$. Therefore $E^2_{r,s} = \mathrm{tor}^{E^0A}_{r,s}(K,E^0M)$. The dual of this spectral sequence may be obtained by filtering the complex $\overline{C}(A^*)^o \otimes M^*$ dual to $\overline{B}(A)^o \otimes M$ by $F^p(\overline{C}(A^*)^o \otimes M^*) = [\overline{B}(A) \otimes M/F_{p-1}(\overline{B}(A) \otimes M)]^*$. The resulting spectral sequence satisfies $E_2 = \mathrm{Ext}_{E^oA}(K,(E^oM)^*)$ and $E_\infty = E^o\ \mathrm{Ext}_A(K,M^*)$.

Next we show that each term of the spectral sequence E_rM^* just constructed may be given a structure of left differential E_r-module, where E_r is a term of the spectral sequence converging to $E^oH^*(A)$. We define a left $\overline{C}(A^*)^o$-module structure on the complex $\overline{C}(A^*)^o \otimes M^*$ by $[\alpha_n|\ldots|\alpha_1][\beta_q|\ldots|\beta_1]m^* = [\alpha_n|\ldots|\beta_1]m^*$. Noting that if $[\alpha_n|\ldots|\alpha_1]m^* \in \overline{C}(A^*)^o \otimes M^*$, then

$$\delta[\alpha_n|\ldots|\alpha_1]m^* = (\delta[\alpha_n|\ldots|\alpha_1])m^* + \sum_t (-1)^{\varepsilon(t)}[\alpha_n|\ldots|\alpha_1|\alpha_t]m_t^*,$$

where the A^*-comodule structure of M^* dual to the A-module structure of M is given by $m^* \longrightarrow \sum_t \alpha_t \otimes m_t^*$ and where $\varepsilon(t) = \deg[\alpha_n|\ldots|\alpha_1] + \deg[\alpha_t]\deg m_t^*$, it is easily seen that $\delta(fg) = (\delta f)g + (-1)^{\deg f} f\delta g$, $f \in \overline{C}(A^*)^o$, $g \in \overline{C}(A^*)^o \otimes M^*$. (Note that no signs are to be introduced in defining the product in $\overline{C}(A^*)^o$; this is to be expected, since $\overline{C}(A^*)^o$ may be thought of as the opposite differential algebra of $\overline{C}(A^*)$.) Now the proof that each $E_r(M^*)$ is a left differential E_r-module goes through just as in the special case $M^* = K$. Summarizing, we have the

<u>Theorem I.5.1</u>: Let A be a filtered algebra of finite type over a field K, and suppose that $A = \varprojlim A/F_pA$. Let M be a left A-module of finite type. Then there exists a spectral sequence $\{E_rM^*\}$, E_2 of which is $\mathrm{Ext}_{E^oA}(K,(E^oM)^*)$ and which converges to

$E_\infty = E^0 \, Ext_A(K,M^*)$. Each $E_r M^*$ is a left differential $E_r K$ module. Dually, there exists a spectral sequence $\{E^r M\}$, E^2 of which is $tor^{E^0 A}(K,E^0 M)$ and which converges to $E^\infty = E^0 \, tor^A(K,M)$. Each $E^r M$ is a left differential comodule over $E^r K$.

If A is a Hopf algebra with the product filtration (page I-2.5), then $E^0 A$ is the universal enveloping algebra of its restricted Lie algebra of primitive elements (unrestricted if char K = 0) . If M is a left A-module, then $tor^{E^0 A}(K,E^0 M)$ may be computed by means of the complex $X^0 \otimes_{E^0 A} E^0 M$. Here X^0 is the opposite complex to the complex obtained in section I.3, and is defined by simply reversing the order in which factors are written. The embedding $f^0: X^0 \longrightarrow B(E^0 A)^0$ opposite to that obtained in section I.4 allows explicit computation of the differentials in the homology spectral sequence. Actually we need compute $d_r(X)$ only on those elements $x \in E^r M$ which are so situated dimensionally that it is possible for x^* to be a summand of $\delta_r(y^*)$, where y^* is an $E_r K$-module generator of $E_r M$. Dualizing, this gives the differentials in $E_r M$ and allows computation of E_∞ . We remark that our results remain true if we start with a right A-module M ; in this case $B(A)$, $C(A^*)$, and $X(E^0 A)$ are to be used instead of $B(A)^0$, $C(A^*)^0$, and $X(E^0 A)^0$. We have stated our results for left A-modules M , since this is the case in the main application we have in mind, namely the case where A is the Steenrod algebra, $M = H^*(X;Z_p)$ for some space X , and $M^* = H_*(X;Z_p)$.

II. Application to the Steenrod algebra

0. Introduction

Knowledge of the cohomology of the Steenrod algebra is needed for the study of nth order cohomology operations and of the Adams spectral sequence. We will determine $H^{s,t}(A)$ in the range $t-s \leq 2(p-1)(2p^2+p+2)$ for odd primes p and $t-s \leq 42$ for p=2 by applying the machinery developed in Part I. We restate fundamental theorems from Part I for the special case under consideration and give specific references to auxiliary results used. Thus the reader interested primarily in topological applications need only refer to Part I for proofs.

Section 1 is devoted to a review of known results on the Steenrod algebra A. Of particular importance in the sequel will be Milnor's elegant results on the structure of the Steenrod algebra. Using Milnor's results, we determine the structure of the associated graded algebra E^0A in section 2. In section 3, we begin the study of $H^*(E^0A)$ by describing the form of our free E^0A - resolution X of Z_p, obtained in section I -3, and by obtaining part of the (non-associative) algebra structure of $\overline{X}^*=(Z_p \otimes_A X)^*$ for the case $p > 2$. In section 4 we determine $H^{s,t}(E^0A)$ in the range $t-s \leq 2(p-1)(2p^2+p+2)-4$ for the case of odd primes. In section 5 we determine $H^{s,t}(E^0A)$ in the range $t-s \leq 164$ for the case p=2. In both cases, these calculations make use of a sequence of spectral sequences quite analogous to that constructed by Adams to facilitate calculation of $H^*(A)$ using the cobar construction. These sections also define various indecomposable elements of $H^*(E^cA)$ lying outside the cited range; in the case p=2, it is likely that these include all indecomposable elements. In section 6 we come to the main theorems of the thesis. These completely describe $H^{s,t}(A)$ in the range $t-s \leq 2(p-1)(2p^2+p+2)-4$ for the case of odd primes and $t-s \leq 42$ for the case p=2, and are obtained by explicit computation of the differentials in the spectral sequence passing from $H^*(E^0A)$ to $H^*(A)$. In both cases, partial information is obtained in higher dimensions. It will be noted that we have used the complete range of our calculation of $H^*(E^0A)$ in the case of odd primes but only part of the range in the case p=2. The reason is that while calculation of the differentials presents no particular difficulty in either case, $H^*(A)$ differs relatively little from $H^*(E^0A)$ in the case of odd primes but differs radically in the case p=2. In fact, for p=2, δ_2 is non-zero on every indecomposable element of $H^*(E^0A)$ in dimension $s \geq 2$. Extension of these calculations would be tedious, but not prohibitively so, and the calculations are considerably simpler than would be the case using the cobar construction.

In section 7, we garner the obvious corollaries for the stable homotopy groups of spheres. These are obtained by combining the algebraic properties of the Adams spectral sequence with Toda's calculations of these groups. We show that the differentials in the Adams spectral sequence satisfy a limited amount of periodicity and obtain nearly complete results on $\pi_m^S(S; Z_p)$ in the range $m < 2(p-1)(p^2+2p)-3$ in the case of odd primes and $m < 29$ in the case p=2.

In appendices, we depict our results graphically for the cases p=2 and p=3.

1. The Steenrod algebra

We recall first the axiomatic definition of the Steenrod powers. Let p be an odd prime. Then $P^i: H^q(X;Z_p) \longrightarrow H^{q+2i(p-1)}(X;Z_p)$ is a Z_p-morphism defined for all $i \geq 0$, $q \geq 0$ and spaces X satisfying:

i) P^i is a natural transformation of functors

ii) $P^0 = 1$

iii) If $\deg x = 2i$, $P^i x = x^p$

iv) If $2i > \deg x$, $P^i x = 0$

v) $P^i(xy) = \sum_{j+k=i} P^i x \cup P^k y$

These characterize the P^i uniquely. Existence is proven in Steenrod, chapter 7. Let δ denote the Bockstein coboundary operator associated with the exact coefficient sequence

$$0 \longrightarrow Z_p \longrightarrow Z_{p^2} \longrightarrow Z_p \longrightarrow 0 .$$

For $p = 2$, $Sq^i: H^q(X;Z_2) \longrightarrow H^{q+i}(X;Z_2)$ is the Z_2-morphism satisfying

i)' Sq^i is a natural transformation of functors

ii)' $Sq^0 = 1$

iii)' If $\deg x = i$, $Sq^i x = x^2$

iv)' If $i > \deg x$, $Sq^i x = 0$

v)' $Sq^i(xy) = \sum_{j+k=i} Sq^i x \cup Sq^k x$

The axioms imply that $Sq^1 = \delta$, the Bockstein coboundary operator associated with $0 \longrightarrow Z_2 \longrightarrow Z_4 \longrightarrow Z_2 \longrightarrow 0$. To simplify statements we write $P^i = Sq^i$ for $p = 2$.

Recall that for all p :

vi) $\delta^2 = 0$

vii) $\delta(xy) = (\delta x)y + (-1)^{\deg x} x\delta y$

viii) δ is a natural transformation of functors

Axioms i) through viii) and i)' through v)' imply

ix) If $i < pj$, $P^iP^j = \sum_{t=0}^{[i/p]} (-1)^{i+t}(i-pt, pj-(i+j)+t-1)P^{i+j-t}P^t$

x) If $i \leq pj$, $P^i\delta P^j = \sum_{t=0}^{[i-1/p]} (-1)^{i-1+t}(i-1-pt, pj-(i+j)+t)P^{i+j-t}\delta P^t$

ix)' If $i < 2j$, $Sq^iSq^j = \sum_{t=0}^{[i/2]} (i-2t, j-i+t-1)Sq^{i+j-t}Sq^t$

The Steenrod algebra $A(p)$ is defined as follows: the free associative algebra $F(p)$ generated by the P^i and δ acts on the cohomology of any space. Let $I(p)$ denote the ideal of all $f \in F$ such that $f(x) = 0$ for all cohomology classes x of any space. Then $A(p) \equiv F(p)/I(p)$. $A(p)$ is connected and associative, but not commutative. It is known that vi), ix) and x) give all relations, i.e., all generators of $I(p)$ (proofs are in Adem's paper).

A monomial of $A(p)$, $p > 2$, has the form $\delta^{\varepsilon_0}P^{s_1}\delta^{\varepsilon_1} \dots P^{s_k}\delta^{\varepsilon_k}$, $\varepsilon_i = 0$ or 1, $s_i = 1,2,3,\dots$. Such a monomial is called admissible if $s_i \geq ps_{i+1} + \varepsilon_i$ for $i \geq 1$. For $p = 2$, a monomial has the form $P^{s_1}P^{s_2} \dots P^{s_k}$ and is called admissible if $s_i \geq 2s_{i+1}$, $i \geq 1$. The admissible monomials form a vector space basis for $A(p)$, all p. The elements P^i, $i \neq p^k$, are decomposable and therefore the P^{p^i} (and δ if $p > 2$) generate $A(p)$.

For $\theta \in A(p)$, there is a unique element $\psi(\theta) \in A(p) \otimes A(p)$ such that if $\psi(\theta) = \Sigma\, \theta_i' \otimes \theta_i''$, $\theta(xy) = \Sigma(-1)^{\deg \theta_i'' \deg x}\, \theta_i' x \cup \theta_i'' y$. The map ψ is given on generators by $\psi(P^i) = \sum_{j+k=i} P^j \otimes P^k$ and $\psi(\delta) = \delta \otimes 1 + 1 \otimes \delta$ and is a morphism of algebras. It follows that $A(p)$ is a (coassociative) cocommutative Hopf algebra.

Since $A(p)$ is of finite type, $A(p)_*$ is a commutative (associative and coassociative) Hopf algebra. Let

$M_k = P^{p^{k-1}} P^{p^{k-2}} \cdots P^p P^1$, $M_0 = P^0 = 1$ and for $p \neq 2$, $M_k' = M_k \delta$

and let $\xi_k \in A(p)_*$ be the dual of M_k, $\tau_k \in A(p)_*$ be the dual of M_k'. Note that $\deg \xi_k = 2(p^k - 1)$, $p \neq 2$, $\deg \tau_k = 2p^k - 1$, $\xi_0 = 1$, and $\deg \xi_k = 2^k - 1$, $p = 2$. Then $A(p)_* \cong E(\tau_0, \tau_1, \ldots) \otimes P(\xi_1, \xi_2, \ldots)$.

This is proven by writing monomials as $\tau_0^{\varepsilon_0} \xi_1^{r_1} \tau_1^{\varepsilon_1} \cdots \xi_k^{r_k} \tau_k^{\varepsilon_k}$, $\varepsilon_i = 0$ or 1, $r_i \geq 0$ and finding a one-to-one correspondence between the sequences $(\varepsilon_0, r_1, \varepsilon_1, \ldots, \varepsilon_k)$ and the sequences corresponding to admissible monomials in $A(p)$ after first proving that the natural morphism of algebras $E(\tau_0, \tau_1, \ldots) \otimes P(\xi_1, \xi_2, \ldots) \longrightarrow A(p)_*$ is an epimorphism.

If $\Phi: A(p) \otimes A(p) \longrightarrow A(p)$ is the multiplication, $\Phi_*: A(p)_* \longrightarrow A(p)_* \otimes A(p)_*$ is given on generators by

$$\Phi_*(\xi_k) = \sum_{i=o}^{k} \xi_{k-i}^{p^i} \otimes \xi_i \quad \text{and} \quad \Phi_*(\tau_k) = \tau_k \otimes 1 + \sum_{i=o}^{k} \xi_{k-i}^{p^i} \otimes \tau_i .$$

Let $R = (r_1,\ldots,r_k)$ be a finite sequence of non-negative integers and $\xi(R) = \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_k^{r_k}$. Let $E = (\varepsilon_o,\ldots,\varepsilon_k)$, $\varepsilon_i = 0$ or 1 and $\tau(E) = \tau_o^{\varepsilon_o} \cdots \tau_k^{\varepsilon_k}$. Then $\{\tau(E)\xi(R)\}$ is a Z_p-basis for $A(p)_*$. Denote the dual basis by $\{\rho(E,R)\}, < \rho(E,R), \tau(E')\xi(R')> = \delta_{E,E'}\delta_{R,R'}$. $\rho(0,(i)) = P^i$, and we therefore define $\rho(0,R) = P^R$. To avoid lengthy superscripts later, we write $P^R = P(R)$. Let Q_k denote the dual of τ_k, and note that $Q_o = \delta$. $\{Q_o^{\varepsilon_o} Q_1^{\varepsilon_1} \cdots Q_k^{\varepsilon_k} P^R | \varepsilon_i = 0 \text{ or } 1, \ k < \infty\}$ is a Z_p-basis for $A(p)$ which is, up to sign, the same as the basis $\{\rho(E,R)\}$ dual to $\{\tau(E)\xi(R)\}$.

To describe the multiplication in terms of this basis, we define $R - S = (r_1 - s_1, r_2 - s_2,\ldots)$ or $(0,0,\ldots)$ if $r_i - s_i < 0$ for any i. We also consider infinite matrices of non-negative integers, almost all of which are zero, and with leading term omitted.

For such a matrix $X = \begin{Vmatrix} * & x_{o1} & x_{o2} & \cdots \\ x_{1o} & x_{11} & \vdots & \cdots \\ x_{2o} & \vdots & \cdots\vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{Vmatrix}$

define $R(X) = (r_1, r_2, \ldots)$, $S(X) = (s_1, s_2, \ldots)$ and $T(X) = (t_1, t_2, \ldots)$ by $r_i = \sum_j p^j x_{ij}$, $s_j = \sum_i x_{ij}$, $t_n = \sum_{i+j=n} x_{ij}$. Define also

$$B(X) = \prod_n (x_{n,o}, x_{n-1,1}, \ldots, x_{o,n}) = \prod_n t_n! / \prod_{i,j} x_{ij}!.$$

<u>Theorem II.1.1</u>: In terms of the basis $\{Q_o^{\varepsilon_o} \ldots Q_k^{\varepsilon_k} P(R)\}$ of $A(p)$, the product is given by

i) $[Q_i Q_j] = 0$

ii) $[P(R)Q_i] = Q_{i+1} P(R - (p^i, o, o, \ldots)) + Q_{i+2} P(R - (o, p^i, o, \ldots) + \ldots$

iii) $P(R)P(S) = \sum_{R(X)=R,\ S(X)=S} B(X) P(T(X))$

It follows that

iv) $Q_{i+1} = [P^{p^i}, Q_i]$

v) $P^i P^j = \sum_{x=o}^{\min(j,[i/p])} (i-px, j-x)\, P(i+j-(p+1)x, x)$

vi) If $r_1 < p$, $r_2 < p, \ldots$ then $P(R)P(S) = (r_1, s_1)(r_2, s_2) \ldots P(R+S)$

vii) $P(o, \ldots, o, r_i = 1) = [P^{p^{i-1}}, P(o, \ldots, o, r_{i-1} = 1)]$

$\{Q_o, P^{p^i}\}$ forms a basis for the indecomposable elements. The coproduct is given by

viii) $\psi(Q_k) = Q_k \otimes 1 + 1 \otimes Q_k$

ix) $\psi(P(R)) = \sum_{R_1+R_2=R} P(R_1) \otimes P(R_2)$

$\{Q_o, Q_1, \ldots, P(o,\ldots,o,r_k = 1)\}$ forms a basis for the primitive elements. ($P(o,\ldots,o,r_k = 1)$ denotes $P(R)$ where $r_i = 0$, $i \neq k$, and $r_k = 1$.)

The above theorem will be used repeatedly in what follows. Note that $\{P(R)\}$ forms a basis for a subalgebra $A^+(p)$ of $A(p)$ and $A^+(2) = A(2)$.

Theorem II.1.2: Let $A = A(p)$. Let $A(n) \subset A$ be the subalgebra generated by $\{Q_o, P^1, \ldots, P^{p^{n-1}}\}$ (by $\{P^1, \ldots, P^{2^{n-1}}\}$ if $p = 2$). Then $A(n)$ has the finite dimension $2^{n+1}p^{n(n+1)/2}$ ($2^{n(n+1)/2}$ if $p = 2$). $\{Q_o^{\varepsilon_o} \ldots Q_n^{\varepsilon_n} P(r_1,\ldots,r_n) | \varepsilon_i = 0$ or 1, $0 \leq r_i < p^{n-i+1}\}$ ($\{P(r_1,\ldots,r_n) | 0 \leq r_i < 2^{n-i+1}\}$ if $p = 2$) forms a basis for $A(n)$. A is the union of the $A(n)$ and therefore every element of A of deg > 0 is nilpotent. Further, each $A(n)$ is a sub-Hopf algebra of A.

Expansion of definitions and proofs for this section are to be found in Adem's paper, "The relations on Steenrod powers of cohomology classes," in Milnor's paper, "The Steenrod algebra and its dual," and in Steenrod's book, Cohomology Operations.

2. The associated graded algebra of the Steenrod algebra

In this section we determine explicitly the structure of the associated graded algebra E^o of the Steenrod algebra A for any prime p.

Let $I \subset A$ be the augmentation ideal.

Let $\Phi_1 : I \longrightarrow I$ be the identity,

$\Phi_2 : I \otimes I \longrightarrow I$ the multiplication, and

$\Phi_n = \Phi_{n-1}(1 \otimes \dots \otimes 1 \otimes \Phi_2) : \overbrace{I \otimes \dots \otimes I}^{n \text{ factors}} \longrightarrow I$. Define

$F_i A = A$, $i \geq 0$; $F_{-i}A = \operatorname{Im} \Phi_i$, $i > 0$.

Then $E^o_{i,j} = (F_i/F_{i-1})_{i+j}$. We also use the notation $E^o_q = \sum_{i+j=q} E^o_{i,j}$. In the latter notation, E^o is isomorphic to A as a Z_p-space, and is a primitively generated connected Hopf algebra under the induced product and coproduct. It follows from theorem I.2.12 that $E^o \cong V(P(E^o))$. We will find $P(E^o)$ as a restricted Lie algebra. For $x \in A$, we define the weight of x, $w(x)$, as that integer n such that $x \in F_{-n}A$, $x \notin F_{-n-1}A$. The crucial point is the determination of $w(P(R))$. The following lemma will be needed.

<u>Lemma II.2.1</u>: Let $n, m_1, \dots, m_k$ be non-negative integers such that $\sum_{i=1}^{k} m_i = n$ and let $n = \sum_{j=o}^{\infty} a_j p^j$, $m_i = \sum_{j=o}^{\infty} b_{ij} p^j$ be their p-adic expansions. Then the multinomial coefficient $(m_1, \dots, m_k)$ is zero mod p if and only if $\sum_i b_{ij} \neq a_j$ for some j . If $\sum_i b_{ij} = a_j$ for all j , then $(m_1, \dots, m_k) \equiv \prod_j (b_{1j}, \dots, b_{kj})$ mod p .

Proof: Consider the polynomial algebra $Z_p[x_1,\ldots,x_k]$.

$(x_1+\ldots+x_k)^n = \sum_{i_1+\ldots+i_k=n} (i_1,\ldots,i_k)\, x_1^{i_1}\ldots x_k^{i_k}$. Thus

$(m_1,\ldots,m_k)$ is the coefficient of $x_1^{m_1}\ldots x_k^{m_k}$ in $(x_1+\ldots+x_k)^n$.

Clearly $(x_1+\ldots+x_k)^p \equiv x_1^p+\ldots+x_k^p \bmod p$, hence $(x_1+\ldots+x_k)^{p^j} \equiv$ $x_1^{p^j}+\ldots+x_k^{p^j} \bmod p$. It follows that $(x_1+\ldots+x_k)^n \equiv$

$$\prod_j (x_1^{p^j}+\ldots+x_k^{p^j})^{a_j} = \prod_j \sum_{\ell_{1j}+\ldots+\ell_{kj}=a_j} (\ell_{1j},\ldots,\ell_{kj})\, x_1^{\ell_{1j}p^j}\ldots x_k^{\ell_{kj}p^j} .$$

Since also $x_1^{m_1}\ldots x_k^{m_k} = x_1^{\sum_j b_{1j}p^j}\ldots x_k^{\sum_j b_{kj}p^j}$, we obtain $(m_1,\ldots,m_k) \equiv$ $\prod_j (b_{1j},\ldots,b_{kj}) \bmod p$ if the latter product occurs in $\{\prod_j (\ell_{1j},\ldots,\ell_{kj}) \mid \ell_{1j}+\ldots+\ell_{kj} = a_j\}$, that is, if $\sum b_{ij} = a_j$ for all j , and is zero otherwise.

Theorem II.2.2: Let $v(R) = \sum_{i=1}^{\infty} \sum_{j=o}^{\infty} i\, a_{ij}$ where $r_i = \sum_{j=o}^{\infty} a_{ij}p^j$ is the p-adic expansion. Then $v(R) = w(P(R))$.

Proof: We will prove (Lemma II.2.4) that if $kP(R)$ is a summand of $P(S)\cdot P(T)$, $k \neq 0$, then $v(R) \geq v(S) + V(T)$. It follows inductively that if $kP(R)$ is a summand of $P(S_1)\ldots P(S_q)$, then $v(R) \geq \sum_{i=1}^{q} v(S_i)$. If each $P(S_i) = P^{p^j}$ for some j , then $v(S_i) = w(P(S_i)) = 1$, and therefore $v(R) \geq w(P(R))$. To prove the opposite inequality, we will show (Lemma II.2.8) that there exists an element $\alpha(R)$ of $I(A) \otimes \ldots \otimes I(A)$, $v(R)$ copies, such that $\Phi_{v(R)}(\alpha(R)) = P(R) + \varepsilon$, where ε is a sum of terms $kP(S)$, each S satisfying $v(S) > v(R)$. Then, letting $A(n)$ be the <u>nth</u> subalgebra of A as defined in Theorem II.1.2, and noting that it suffices to prove the result for $P(R) \in A(n)$ for arbitrary fixed n, we define $m = \max_{P(S) \in A(n)} v(S)$, and proceed by induction on $m - v(R)$. If $m - v(R) = 0$, Lemma II.2.8 gives $w(P(R)) \geq v(R)$, while if $m - v(R) > 0$, we may apply the induction hypothesis to each summand $kP(S)$ of $\Phi_{v(R)}(\alpha(R))$, obtaining $w(\varepsilon) > v(R)$, hence $w(P(R)) \geq v(R)$.

<u>Corollary II.2.3</u>: $w(\sum_i k_i P(R_i)) = \min_i w(P(R_i))$.

<u>Proof</u>: By Lemma II.2.4, no $P(R_i)$ is a summand of a product of $v(R_i)+1$ elements of $I(A)$. Since $w(P(R_i)) = v(R_i)$, this implies the result.

<u>Lemma II.2.4</u>: If $kP(T)$, $k \neq 0$, is a summand of $P(R)\cdot P(S)$, then $v(T) \geq v(R) + v(S)$.

<u>Proof</u>: $P(R)\cdot P(S) = \sum_{R(X)=R, S(X)=S} B(X)T(X)$ as stated in Theorem II.1.1. We must show that $B(X) \not\equiv 0 \bmod p$, $R(X) = R$, and $S(X) = S$ imply $v(T) \geq v(R) + v(S)$, $T = T(X)$. We denote entries of X by x_{ij}. Let $r_i = \sum_\ell a_{i\ell} p^\ell$, $s_j = \sum_\ell b_{j\ell} p^\ell$, $t_k = \sum_\ell c_{k\ell} p^\ell$, and $x_{ij} = \sum_\ell e_{ij\ell} p^\ell$ be the p-adic expansions of these integers. $t_k = \sum_{i+j=k} x_{ij}$, and $B(X) = \prod_k (x_{o,n}, x_{1,n-1}, \ldots, x_{n,o})$. Lemma II.2.1 gives immediately

that $B(X) \not\equiv 0 \bmod p$ if and only if $\sum_{i+j=k} e_{ij\ell} = c_{k\ell}$ for all k and ℓ. Now $r_i = \sum_j x_{ij} p^j$ and $s_j = \sum_i x_{ij}$, hence

$\sum_m (\sum_{j+\ell=m} e_{ij\ell}) p^m = \sum_m a_{im} p^m$ and $\sum_\ell (\sum_i e_{ij\ell}) p^\ell = \sum_\ell b_{j\ell} p^\ell$, where, of course, the left-hand sides of the last two equations need not be p-adic expansions. By the properties of the p-adic expansion of an integer, $\sum_m (\sum_{j+\ell=m} e_{ij\ell}) \geq \sum_m a_{im}$ and $\sum_\ell (\sum_i e_{ij\ell}) \geq \sum_\ell b_{j\ell}$, with equality holding if and only if $\sum_{j+\ell=m} e_{ij\ell} = a_{im}$ for each m, respectively, $\sum_i e_{ij\ell} = b_{j\ell}$ for each ℓ. Therefore

$$v(R) + v(S) = \sum_i \sum_m i(a_{im} + b_{im}) \leq \sum_i \sum_m \sum_{j+\ell=m} i e_{ij\ell} + \sum_j \sum_\ell \sum_i j e_{ij\ell}$$

$$= \sum_k \sum_{i+j=k} \sum_\ell k e_{ij\ell} = \sum_k \sum_\ell k c_{k\ell} = v(T),$$

as was to be shown. Note that we have also proven that $v(T) = v(R) + v(S)$ if and only if $\sum_{j+\ell=m} e_{ij\ell} = a_{im}$ for all i and m and $\sum_i e_{ij\ell} = b_{j\ell}$ for all j and ℓ. This last criterion will be applied repeatedly in the proofs of the next three lemmas.

<u>Notations II.2.5</u>: P_j^i will denote $P(R)$, where $r_j = p^i$ and $r_k = 0,\ k \neq j$. Thus $P_1^i = P^{p^i}$, and P_j^0 is primitive. Given any two sequences R and S, $T(P(R),P(S))$ or $T(R,S)$ will denote $T(X)$, X being the matrix with entries $r_{i,0} = r_i$, $x_{0,j} = s_j$, and $x_{i,j} = 0$ if both $i > 0$ and $j > 0$.

Lemma II.2.6: $[P^i_j P^k_\ell] = \delta_{i,k+\ell} P^k_{\ell+j} + \varepsilon$, $i \geq k$, where ε denotes a sum of terms $kP(S)$, each satisfying $v(S) > j + \ell$. In particular, $[P^{i+j-1}_1 [P^{i+j-2}_1 [\ldots [P^{i+2}_1 [P^{i+1}_1 P^i_1]] \ldots]]] = P^i_j + \varepsilon$ for all i and j .

Proof: Theorem I.1.1 and the criterion developed in the proof of Lemma II.2.4 show that if $i \geq k$ and $P^i_j \neq P^k_\ell$, then $P^i_j P^k_\ell = P(T(P^i_j, P^k_\ell)) + \delta_{i,k+\ell} P^k_{j+\ell} + \varepsilon$, while $P^k_\ell P^i_j = P(T(P^i_j, P^k_\ell)) + \varepsilon$.

Lemma II.2.7: $(P^i_j)^a = a!P(T_a) + \varepsilon$, $1 \leq a \leq p$, where T_a is the sequence $t_j = ap^i$, $t_k = 0$ if $k \neq j$, and ε is a sum of terms $kP(S)$, each satisfying $v(S) > aj$. In particular, $(P^i_j)^p = \varepsilon$.

Proof: $T_a = T(P(T_{a-1}), P^i_j)$. By Theorem I.1.1 and the criterion of Lemma II.2.4, $P(T_{a-1}) \cdot P^i_j = ((a-1)p^i, p^i) T_a + \varepsilon$ and by Lemma II.2.1, $((a-1)p^i, p^i) \equiv a \bmod p$. The result follows by induction on a .

Lemma II.2.8: Let R be any sequence and let $r_i = \Sigma\, a_{ij} p^j$ be the p-adic expansions of the r_i . Then

$$(P^o_1)^{a_{1,o}} (P^o_2)^{a_{2,o}} \ldots (P^1_1)^{a_{1,1}} (P^1_2)^{a_{2,1}} \ldots (P^j_1)^{a_{1,j}} (P^j_2)^{a_{2,j}} \ldots = kP(R) + \varepsilon,\ k \neq 0 ,$$

where ε is a sum of terms $k'P(S)$, each satisfying $v(S) > v(R)$. Together with Lemma II.2.6, this implies that there exists $\alpha(R) \in I(A) \otimes \ldots \otimes I(A)$, $v(R)$ copies, such that $\Phi_{v(R)}(\alpha(R)) = P(R) + \varepsilon$.

Proof: Let R_j be the sequence $r_i = a_{ij} p^j$, and let S_j be the sequence $s_i = \sum_{k > j} a_{ik} p^k$. By Theorem I.1.1 and the criterion of Lemma II.2.4, $P(R_o)P(S_o) = P(R)$ and $P(R_i)P(S_i) = P(S_{i-1}) + \varepsilon_i$, where $v(R_i) + v(S_i) < v(T)$ if $kP(T)$ is a term of ε_i . For some n, $S_n = 0$, and therefore $P(R_o)\ldots P(R_n) = P(R) + \varepsilon$. Now $P^j_k P^j_\ell = T(P^j_k, P^j_\ell) + \varepsilon$, where if $kP(T)$ is a term of ε, $v(T) > k+\ell$.

Applying Lemma II.2.7, we find $(P^j_1)^{a_{1,j}}(P^j_2)^{a_{2,j}}\ldots = k_j P(R_j) + \varepsilon$, $k_j \neq 0$, where $v(T) > \sum_i ia_{ij}$ if $kP(T)$ is a term of ε. This completes the proof of Lemma II.2.8 and of Theorem II.2.2.

Theorem II.2.9: A basis for the primitive elements of E^o is $\{Q_i \mid i \geq 0\} \cup \{P^j_k \mid j \geq 0, k \geq 1\}$. The bracket product and pth powers are given by:

i) $[Q_i, Q_j] = 0$

ii) $[P^j_k, Q_i] = \delta_{j,i}\, Q_{i+k}$

iii) $[P^i_j, P^k_\ell] = \delta_{i,k+\ell}\, P^k_{j+\ell}$, $i \geq k$.

iv) $\beta(P^j_k) = 0$.

Proof: The Q_i are primitive in A. Since $\psi(P(R)) = \sum_{R_1+R_2=R} P(R_1) \otimes P(R_2)$ in A, Theorem II.2.2 gives immediately that each P^i_j is primitive in E^o and that no other $P(R)$ is. Relations i) hold in A. $[P^j_k, Q_i] = Q_{i+k}P(o,\ldots,r_k = p^j - p^i)$ in A, and relations ii) follow. iii) and iv) are consequences of Lemmas II.2.6 and II.2.7.

Corollary II.2.10: $(\mathrm{Ad}\, P^j_k)^i = 0$ in E^o if $i \geq 2$.

Corollary II.2.11: $E^o \cong V(P(E^o)) = V$.

Proof: As remarked above, this is true a priori. A simple direct proof is possible, however. Since E^o is primitively generated, the inclusion $P(E^o) \subset E^o$ induces an epimorphism of Hopf algebras $f: V \longrightarrow E^o$. Defining $P^i_j < P^k_\ell$ if $i < k$ or if $i = k$ and $j < \ell$, Remarks I.2.11 state that

$$\{Q_o^{\varepsilon_o} \cdots Q_n^{\varepsilon_n} \cdots (P^o_1)^{a_{1,o}} (P^o_2)^{a_{2,o}} \cdots (P^j_1)^{a_{1,j}} (P^j_2)^{a_{2,j}} \cdots \mid \varepsilon_i = 0 \text{ or } 1,$$
$$0 \leq a_{i,j} < p\}$$

is a basis for V. Lemma II.2.8 implies that the same set is a basis for E^o, and therefore that f is an isomorphism.

3. The cohomology of the associated graded algebra of the Steenrod algebra; introduction.

In section I.3, we obtained a canonical free resolution of the ground field on the category of restricted Lie algebras (over some field). In this section we describe this complex and its dual for the case of the restricted Lie algebra L of primitive elements of the associated graded algebra E^0 of the Steenrod algebra A. Recall that by corollary II.2.11, $E^0 \cong V(L)$. The structure of L is given in theorem II.2.9.

The complex $X = X(L)$ is defined as the free $V(L)$-module $X = V(L) \otimes \overline{X}$, $\overline{X} = E(sL^+) \otimes \Gamma(sL^-) \otimes \Gamma(s^2\pi L^+)$. Here we are regarding E^0 as graded by total degree, $E^0_n = \underset{i+j=n}{\oplus} E^0_{i,j}$. L^+ denotes the sub-Lie algebra of L generated by $\{P^i_j\}$, L^- the sub-Lie algebra generated by $\{Q_i\}$. Elements of L are given bidegree o, s is the map which raises the bidegree of elements by one and π is the map which multiplies the degree of elements by p: $s(P^i_j) = \overline{P}^i_j$ has bigrading $(1, 2p^i(p^j-1))$ or $(1, 2^i(2^j-1))$ if $p = 2$, where 1 is the bidegree (or homological dimension), $s(Q_k) = \overline{Q}_k$ has bigrading $(1, 2p^k-1)$, and $s^2\pi(P^i_j) = \tilde{P}^i_j$ has bigrading $(2, 2p^{i+1}(p^j-1))$ or $(2, 2^{i+1}(2^j-1))$ if $p = 2$. Γ denotes a divided polynomial algebra, E an exterior algebra. $V(L) \otimes E(sL^+) \otimes \Gamma(sL^-)$ is given a Z_p-algebra structure by giving $V(L)$ and $E(sL^+) \otimes \Gamma(sL^-)$ their natural structures and by relations 1) through 4) below. Then X is given a structure of right $V(L^+) \otimes E(sL^+)$-module by 5) and 6):

1) $< \bar{P}^i_k > P^j_\ell = P^j_\ell < \bar{P}^i_k > + <\overline{[P^i_k P^j_\ell]}>$

2) $< \bar{P}^i_k > Q_j = -Q_j < \bar{P}^i_k > + \gamma_1(\overline{[P^i_k Q_j]})$

3) $\gamma_r(\bar{Q}_j)P^i_k = P^i_k \gamma_r(\bar{Q}_j) - \gamma_1(\overline{[P^i_k Q_j]})\gamma_{r-1}(\bar{Q}_j)$

4) $\gamma_r(\bar{Q}_i)Q_j = Q_j \gamma_r(\bar{Q}_i)$

5) $\gamma_r(\tilde{P}^i_k)P^j_\ell = P^j_\ell \gamma_r(\tilde{P}^i_k) + (P^i_k)^{p-2} <\bar{P}^i_k, \overline{[P^i_k P^j_\ell]}> \gamma_{r-1}(\tilde{P}^i_k)$

6) $\gamma_r(\tilde{P}^i_k) < \bar{P}^j_\ell > = < \bar{P}^j_\ell > \gamma_r(\tilde{P}^i_k)$

Then, writing $\Gamma(s^2\pi\ L^+) = \Gamma(\tilde{P}^o_1) \otimes \Gamma(\tilde{P}^o_2) \otimes \ldots \otimes \Gamma(\tilde{P}^1_1) \otimes \Gamma(\tilde{P}^1_2) \otimes \ldots$, X is given a differential d by defining:

a) $d(ux) = (-1)^{\deg u}\, ud(x), \quad u \in V(L), \quad x \in \bar{X}$

b) $d(< \bar{P}^i_j >) = P^i_j$

$d(\gamma_r(\bar{Q}_i)) = Q_i\ \gamma_{r-1}(\bar{Q}_i)$

$d(\gamma_r(\tilde{P}^i_j)) = (P^i_j)^{p-1} <\bar{P}^i_j> \gamma_{r-1}(\tilde{P}^i_j)$

c) $d(xy) = d(x)\cdot y + (-1)^{\deg x}\, xd(y)$, where xy is a basis element of $\bar{X}$ with factors written in the prescribed order. $d(xy)$ is determined as an element of $V(L) \otimes \bar{X}$ by making use of relations 1) through 6). Theorem I.3.8 states that X is a complex, $d^2 = 0$, while theorem I.3.10 states that X is a resolution of Z_p over $V(L)$.

We consider $\overline{d} = 1 \otimes d$ on $\overline{X} = Z_p \otimes_{V(L)} X$. Here we must treat the cases $p > 2$ and $p = 2$ separately. Suppose first that $p > 2$. Using 1), 3), and 5), we find that if $1 \leq s \leq p-1$:

7) $< \overline{P}^i_k > (P^j_\ell)^s = (P^j_\ell)^s < \overline{P}^i_k > + s(P^j_\ell)^{s-1} <\overline{[P^i_k P^j_\ell]}>$

8) $\gamma_r(\overline{Q}_k)(P^j_\ell)^s = \sum_{i=o}^{s} (-1)^i \frac{s!}{(s-i)!} (P^j_\ell)^{s-i} \gamma_i(\overline{[P^j_\ell Q_k]}) \gamma_{r-i}(\overline{Q}_k)$, $r \geq s$

9) $\gamma_r(\tilde{P}^i_k)(P^j_\ell)^s = (P^j_\ell)^s \gamma_r(\tilde{P}^i_k) + \sum_{t=o}^{s-1} (P^j_\ell)^t (P^i_k)^{p-2} (P^j_\ell)^{s-1-t}$

$$<\overline{P}^i_k, \overline{[P^i_k P^j_\ell]}> \gamma_{r-1}(\tilde{P}^i_k)$$

If $f \in \Gamma(s^2\pi L^+)$, then, using 9), we find $\overline{d}(f) = 0$, and therefore $\Gamma(s^2\pi L^+) \subset H_*(V(L))$. If $g = <\overline{P}^{i_1}_{k_1},\ldots,\overline{P}^{i_n}_{k_n}> \gamma_{r_1}(\overline{Q}_{j_1})\cdots\gamma_{r_m}(\overline{Q}_{j_m})$, then

α) $\overline{d}(g) = \sum_{s<t} (-1)^{s+t} <\overline{[P^{i_s}_{k_s} P^{i_t}_{k_t}]}, \overline{P}^{i_1}_{k_1},\ldots,\hat{\overline{P}}^{i_s}_{k_s},\ldots,\hat{\overline{P}}^{i_t}_{k_t},\ldots,\overline{P}^{i_n}_{k_n}>$

$$\gamma_{r_1}(\overline{Q}_{j_1})\cdots\gamma_{r_m}(\overline{Q}_{j_m})$$

$$+ \sum_{s=1}^{n} \sum_{t=1}^{m} (-1)^s <\overline{P}^{i_1}_{k_1},\ldots,\hat{\overline{P}}^{i_s}_{k_s},\ldots,\overline{P}^{i_n}_{k_n}> \gamma_1(\overline{[P^{i_s}_{k_s} Q_{j_t}]})$$

$$\gamma_{r_1}(\overline{Q}_{j_1})\cdots\gamma_{r_t-1}(\overline{Q}_{j_t})\cdots\gamma_{r_m}(\overline{Q}_{j_m})$$

by Corollary I.3.4. $d(gf) = d(g)\cdot f + (-1)^{\deg g} gd(f)$, f and g as above, where the image of $gd(f)$ in $\overline{X}$ may be non-zero and is to be determined by use of 1) through 9). For example:

$\overline{d}(\gamma_r(\overline{Q}_i)\gamma_s(\tilde{P}^i_j)) = - < \overline{P}^i_j > \gamma_{p-1}(\overline{Q}_{i+j})\gamma_{r-p+1}(\overline{Q}_i)\gamma_{s-1}(\tilde{P}^i_j)$, $r \geq p-1$

We make no attempt to give an explicit formula for $\overline{d}(gf)$.

Next, consider $\bar{d}$ in the case $p = 2$. As remarked at the end of section I.4, $\bar{X}$ is naturally isomorphic as an algebra (but not as a coalgebra) to $\Gamma(sL)$ under $<\bar{P}^i_j> \longrightarrow \gamma_1(\bar{P}^i_j)$, $\gamma_1(\tilde{P}^i_j) \longrightarrow \gamma_2(\bar{P}^i_j)$. If we identify $\bar{X}$ with $\Gamma(sL)$, then 1) and 5) give that if $g = \gamma_{r_1}(\bar{P}^{i_1}_{j_1}) \dots \gamma_{r_n}(\bar{P}^{i_n}_{j_n})$, then

$$\beta) \quad \bar{d}(g) = \sum_{s<t} \gamma_1(\overline{[P^{i_s}_{j_s} P^{i_t}_{j_t}]})\gamma_{r_1}(\bar{P}^{i_1}_{j_1}) \dots \gamma_{r_s - 1}(\bar{P}^{i_s}_{j_s}) \dots \gamma_{r_t - 1}(\bar{P}^{i_t}_{j_t}) \dots \gamma_{r_n}(\bar{P}^{i_n}_{j_n}) .$$

Now we describe the diagonal map D for our complexes and obtain the structure of the dual of $\bar{X}$. D is defined by

i) $D(ux) = \psi(u)D(x)$, $u \in V(L)$, $x \in \bar{X}$, where ψ is the coproduct in $V(L)$.

ii) $D(<\bar{P}^i_j>) = <\bar{P}^i_j> \otimes 1 + 1 \otimes <\bar{P}^i_j>$

$$D(\gamma_r(\bar{Q}_i)) = \sum_{j+k=r} \gamma_j(\bar{Q}_i) \otimes \gamma_k(\bar{Q}_i)$$

$$D(\gamma_r(\tilde{P}^i_j)) = \sum_{k+\ell=r} \gamma_k(\tilde{P}^i_j) \otimes \gamma_\ell(\tilde{P}^i_j) + \sum_{k=1}^{p-1} \sum_{\ell+m=r-1} (-1)^k (P^i_j)^{k-1} <\bar{P}^i_j> \gamma_\ell(\tilde{P}^i_j) \otimes (P^i_j)^{p-1-k} <\bar{P}^i_j> \gamma_m(\tilde{P}^i_j)$$

iii) $D(xy) = D(x)D(y)$, where xy is a basis element of $\bar{X}$ with factors written in the prescribed order and the latter product is to be determined as an element of $X \otimes X$ by making use of relations 1) through 9) above.

By Theorem I.3.11, $Dd = (d\otimes 1 + 1\otimes d)D$. By Remarks I.3.12, D is cocommutative and is coassociative if and only if $p = 2$, but is always coassociative on the subcomplex $V(L) \otimes \overline{Y}$, $\overline{Y} = E(sL^+) \otimes \Gamma(sL^-)$.

Before proceeding, we recall the method by which the algebra structure is obtained on the dual of a coalgebra C with coproduct ψ. Let $\{a_i\}$ be a basis for C and suppose $\psi(a_i) = \sum_{j,k} b_i^{j,k} a_j \otimes a_k$, the $b_i^{j,k}$ being elements of the ground ring. Let $\{a_i^*\}$ denote the dual basis of C^*. Then $a_j^* a_k^* = \sum_i (-1)^{\deg a_j \deg a_k} b_i^{j,k} a_i^*$.

Now let $\overline{X}^*$ denote the complex dual to $\overline{X}$ and denote its differential by δ. Consider first the case $p = 2$. Here

$$D(\gamma_r(\tilde{P}_j^i)) = \sum_{k+\ell=r} \gamma_k(\tilde{P}_j^i) \otimes \gamma_\ell(\tilde{P}_j^i) + \sum_{k+\ell=r-1} \langle \overline{P}_j^i \rangle \gamma_k(\tilde{P}_j^i) \otimes \langle \overline{P}_j^i \rangle \gamma_\ell(\tilde{P}_j^i) .$$

Identifying $\overline{X}$ with $\Gamma(sL^-)$ in the manner described above,

$D(\gamma_r(\overline{P}_j^i)) = \sum_{k+\ell=r} \gamma_k(\overline{P}_j^i) \otimes \gamma_\ell(\overline{P}_j^i)$; that is, D gives the natural coalgebra structure on $\Gamma(sL)$. Thus $\overline{X}^* = P((sL)^*)$ as an algebra. $\overline{X}^*$ is determined as a differential algebra by noting that

$$\delta(R_j^i) = \sum_{k=1}^{j-1} R_{j-k}^{i+k} R_k^i , \quad \text{where} \quad R_j^i = \gamma_1(\overline{P}_j^i)^* .$$

Now consider the case $p > 2$. As a K-space $\overline{X}^*$ is isomorphic to $E((sL^+)^*) \otimes P((sL^-)^*) \otimes P((s^2\pi L^+)^*)$. Let $R_j^i = \langle \overline{P}_j^i \rangle^*$, $S_k = \gamma_1(\overline{Q}_k)^*$, and $\tilde{R}_j^i = \gamma_1(\tilde{P}_j^i)^*$. Elements of the dual basis of $\overline{X}^*$ will be written by juxtaposition of these symbols; e.g., $R^i_j(S_k)^n(\tilde{R}^\ell_h)^m = (\langle\bar{P}^i_j\rangle\,\gamma_n(\bar{Q}_k)\,\gamma_m(\tilde{P}^\ell_h))^*$. Clearly $\delta(\tilde{R}^i_j) = 0$. $\bar{d}(\langle P^{i+k}_{j-k}, P^i_k\rangle) = -\langle\bar{P}^i_j\rangle$ and $\bar{d}(\langle\bar{P}^k_{i-k}\rangle\,\gamma_1(\bar{Q}_k)) = -\gamma_1(\bar{Q}_i)$. Therefore we may take $\delta(R^i_j) = \sum_{k=1}^{j-1} R^{i+k}_{j-k} R^i_k$ and $\delta(S_i) = \sum_{k=0}^{i=1} R^k_{i-k} S_k$ (where, for convenience, we have replaced the differential obtained by dualization by its negative). Of course, to calculate $H^*(V(L))$ using $\bar{X}^*$ we must know the algebra structure of $\bar{X}^*$. Using 9), it is easily verified that $P\{\tilde{R}^i_j\}$ has its natural algebra structure and therefore $P\{\tilde{R}^i_j\} \subset H^*(V(L))$ as an algebra. $\bar{Y}^*$ with its natural algebra structure is a quotient differential algebra of $\bar{X}^*$. The coproduct on $\bar{X}$ differs from the natural coalgebra structure only in that some extra summands of $D(gf)$, $g \in \bar{Y}$, $f \in \Gamma(s^2\pi L^+)$, derived from the summands

$$\sum_{k=1}^{p-1} \sum_{\ell+m=r-1} (-1)^k (P^i_j)^{k-1} \langle\bar{P}^i_j\rangle\,\gamma_\ell(\tilde{P}^i_j) \otimes (P^i_j)^{p-1-k} \langle\bar{P}^i_j\rangle\,\gamma_m(\tilde{P}^i_j) \quad \text{of}$$

$D(\gamma_r(\tilde{P}^i_j))$, may be non-zero in $\bar{X}$. If $f = \gamma_r(\tilde{P}^i_j)$, this occurs only if g has at least $p-2$ factors each having non-zero bracket product with P^i_j, that is, if g has as a factor $\langle\bar{P}^{k_1}_{\ell_1},\ldots,\bar{P}^{k_n}_{\ell_n}\rangle\,\gamma_m(\bar{Q}_{\cdot})$, where either $k_t+\ell_t = i$ or $k_t = i+j$, $1 \le t \le n$, and $n+m \ge p-2$. Then the precise form of $\bar{D}(gf)$ is to be determined by use of relations 7) and 8). Dualizing, we note first that if μ and ν are basis elements of $\bar{Y}^*$ and $\mu\nu = \Phi \neq 0$ in $\bar{Y}^*$ with its natural algebra structure, then $\mu\nu = \Phi$ in $\bar{X}^*$. Therefore, writing basis elements of $\bar{X}^*$ by juxtaposition is, up to sign, consistent with the algebra structure on $\bar{X}^*$. We will consider the dual basis to consist of the appropriate iterated products. The signs then become consistent if, in obtaining the product by dualization, we take the sign to be plus on those summands of products which would occur had $\bar{X}^*$ its natural algebra structure, but take the sign to be minus on the other summands. The algebra structure on $\bar{X}^*$ differs from the natural one only in that if R^i_j is a factor of each of two basis elements μ and ν, then it is possible that $\mu\nu \neq 0$. This can occur only if μ has n factors of the form R^i_{j+k}, $R^{i-\ell}_{j+\ell}$, or S_{i+j} and ν has m such factors, where $m+n \geq p-2$. If $\Phi = R^{i_1}_{j_1} \cdots R^{i_n}_{j_n} S^{\ell_1}_{k_1} \cdots S^{\ell_m}_{k_m}$ and $\psi \in P\{\tilde{R}^i_j\}$, then the differential $\delta(\Phi\psi)$ is given by the formula:

$$\gamma) \qquad \delta(\Phi\psi) = \sum_{t=1}^{n} (-1)^{t+1}\, \delta(R^{i_t}_{j_t}) \cdot (R^{i_1}_{j_1} \cdots \hat{R}^{i_t}_{j_t} \cdots R^{i_n}_{j_n} S^{\ell_1}_{k_1} \cdots S^{\ell_m}_{k_m} \psi)$$
$$+ \sum_{t=1}^{m} \ell_t \delta(S_{k_t}) \cdot (R^{i_1}_{j_1} \cdots R^{i_n}_{j_n} S^{\ell_1}_{k_1} \cdots S^{\ell_t - 1}_{k_t} \cdots S^{\ell_m}_{k_m} \psi),$$

where $\delta(R^{i_t}_{j_t})$ and $\delta(S_{k_t})$ are as given above and where we must watch out for products which differ from the natural ones. We catalog in the next proposition such of these products as we shall need later.

<u>Proposition II.3.1</u>: The following relations hold in $\bar{X}^*$:

$$1) \qquad (R^i_j S^{\ell}_{i+j})(R^i_j S^m_{i+j}) = -\frac{1}{\ell+1} S^{p-2}_i \tilde{R}^i_j, \quad \ell+m = p-2$$
$$= -S^{p-2}_i S_{i+j} \tilde{R}^i_j, \quad \ell = p-1,\ m = 0$$
$$= 0, \quad \ell+m > p-2,\ \ell \leq p-2,\ m \leq p-2$$

2) $$(R^i_{j+k}R^i_j S^\ell_{i+j})(R^i_j S^m_{i+j}) = \frac{1}{\ell+2} R^{i+k}_j S^{p-3}_i \tilde{R}^i_j \,, \quad \ell+m = p-3$$

$$= -\frac{1}{\ell+1}(R^i_{j+k}S^{p-2}_i + R^{i+j}_k S^{p-3}_i S_{i+j})\tilde{R}^i_j \,, \quad \ell+m=p-2,\ m > 0$$

$$= (R^i_{j+k}S^{p-2}_i + 2R^{i+j}_k S^{p-3}_i S_{i+j})\tilde{R}^i_j \,, \quad \ell = p-2,\ m = 0$$

$$= -(R^i_{j+k}S^{p-2}_i S_{i+j} + R^{i+j}_k S^{p-3}_i S^2_{i+j})\tilde{R}^i_j \,, \quad \ell = p-1,\ m = 0$$

$$= 0 \,, \quad \ell+m > p-2,\ \ell \leq p-3,\ m \leq p-2 .$$

3) $$(R^i_j R^{i-k}_{j+k} S^\ell_{i+j})(R^i_j S^m_{i+j}) = \frac{1}{\ell+2} R^{i-k}_k S^{p-3}_i \tilde{R}^i_j \,, \quad \ell+m = p-3$$

$$= \frac{1}{\ell+1}(R^{i-k}_{j+k}S^{p-2}_i - R^{i-k}_k S^{p-3}_i S_{i+j})\tilde{R}^i_j \,, \quad \ell+m=p-2,\ m > 0$$

$$= -(R^{i-k}_{j+k}S^{p-2}_i - 2R^{i-k}_k S^{p-3}_i S_{i+j})\tilde{R}^i_j \,, \quad \ell=p-2,\ m = 0$$

$$= (R^{i-k}_{j+k}S^{p-2}_i S_{i+j} - R^{i-k}_k S^{p-3}_i S^2_{i+j})\tilde{R}^i_j \,, \quad \ell=p-1,\ m = 0$$

$$= 0 \,, \quad \ell+m > p-2,\ \ell \leq p-3,\ m \leq p-2 .$$

4) $$(R^i_{j+k}R^i_j S^\ell_{i+j})(R^i_{j+k}R^i_j S^m_{i+j}) = \frac{1}{(\ell+1)(\ell+2)} R^{i+j}_k R^i_{j+k} S^{p-3}_i \tilde{R}^i_j \,, \quad \ell+m=p-3$$

$$= 0 \,, \quad \ell+m > p-3,\ \ell \leq p-3,\ m \leq p-3$$

5) $$(R^i_j R^{i-k}_{j+k} S^\ell_{i+j})(R^i_j R^{i-k}_{j+k} S^m_{i+j}) = \frac{-1}{(\ell+1)(\ell+2)} R^{i-k}_{j+k} R^{i-k}_k S^{p-3}_i \tilde{R}^i_j \,, \quad \ell+m=p-3$$

$$= 0 \,, \quad \ell+m > p-3,\ \ell \leq p-3,\ m \leq p-3$$

6) If $p = 3$: $(R^i_{j+k}R^i_j)R^i_j = - R^{i+j}_k\tilde{R}^i_j$

$$(R^i_jR^{i-k}_{j+k})R^i_j = - R^{i-k}_k\tilde{R}^i_j$$

$$(R^i_{j+k}R^i_j)(R^i_jR^{i-\ell}_{j+\ell}) = - (R^{i+j}_kR^{i-\ell}_{j+\ell} + R^i_{j+k}R^{i-k}_k)\tilde{R}^i_j$$

$$(R^i_{j+k}R^i_j)(R^i_{j+k}R^i_j) = - R^{i+j}_kR^i_{j+k}\tilde{R}^i_j$$

$$(R^i_jR^{i-k}_{j+k})(R^i_jR^{i-k}_{j+k}) = - R^{i-k}_{j+k}R^{i-k}_k\tilde{R}^i_j$$

Proof: Consider the relations in 1). Suppose first that $\ell+m \geq p-2$, $\ell \leq p-2$, and $m \leq p-2$. Then

$$\sum_{t=p-2-m}^{\ell} (-1)^{t+1} \gamma_{\ell-t}(\overline{Q}_{i+j})\gamma_t(\overline{Q}_i)(P^i_j)^t \langle\bar{P}^i_j\rangle \otimes \gamma_{m-p+2+t}(\overline{Q}_{i+j})\gamma_{p-2-t}(\overline{Q}_i)(P^i_j)^{p-2-t}\langle\bar{P}^i_j\rangle$$

is a summand of $D(\gamma_{\ell+m-p+2}(\overline{Q}_{i+j})\ \gamma_{p-2}(\overline{Q}_i)\ \gamma_1(\tilde{P}^i_j))$, and, using 8) and $t!(p-2-t)! \equiv (-1)^t \frac{1}{t+1} \bmod p$, this sum is found to be

$$\left(\sum_{t=p-2-m}^{\ell} \frac{1}{t+1}\,(\ell-t,t)(m-p+2+t,p-2-t)\right) \langle\bar{P}^i_j\rangle\ \gamma_\ell(\overline{Q}_{i+j}) \otimes \langle\bar{P}^i_j\rangle\ \gamma_m(\overline{Q}_{i+j})\ .$$

The coefficient is easily verified to be $\frac{1}{\ell+1}$ if $\ell = p-2-m$ and zero if $\ell > p-2-m$. Similarly we find that $\langle\bar{P}^i_j\rangle\ \gamma_{p-1}(\overline{Q}_{i+j}) \otimes \langle\bar{P}^i_j\rangle$ is a summand of $D(\gamma_{p-2}(\overline{Q}_i)\ \gamma_1(\overline{Q}_{i+j})\ \gamma_1(\tilde{P}^i_j))$. These facts give the relations in 1). The proofs of the remaining relations are similar and will be omitted.

4. The cohomology of E^oA, $p > 2$

In this section we define certain generators of $H^*(E^oA)$, and, using a sequence of spectral sequences, compute $H^{s,t}(E^oA)$ completely for small s and t.

$E^oA \cong V(L)$, where L is the restricted Lie algebra of primitive elements of E^oA. $H^*(E^oA)$ is the homology of the complex $\bar{X}^*$, the form of which was studied in the previous section: as a Z_p-space, $\bar{X}^* = E\{R_j^i\} \otimes P\{\tilde{R}_j^i\}$; the differential is given by

$$\delta(R_j^i) = \sum_{k=1}^{j-1} R_{j-k}^{i+k} R_k^i, \quad \delta(S_i) = \sum_{k=o}^{i-1} R_{i-k}^k S_k, \quad \delta(\tilde{R}_j^i) = 0,$$ and by

formula γ) of the preceding section. For convenience, we define $e_{i,j} = \delta(R_j^i)$, $i \geq 0$, $j \geq 2$, and $f_i = \delta(S_i)$, $i \geq 2$.

We denote by b_j^i the cohomology class of $\tilde{R}_j^i$, by a_o the class of S_o, and by a_i the class of S_i^p, $i > 0$. Clearly there are no relations among the a_i or among the b_j^i and therefore $P\{a_i\} \subset H^*(E^oA)$ and $P\{b_j^i\} \subset H^*(E^oA)$. If $i > 1$, $\delta(f_i S_i^{p-1}) = - f_i(f_i S_i^{p-2})$, and, using relations 1) of Proposition II.3.1, the latter product is $\sum_{k=o}^{i-1} S_k^p \tilde{R}_{i-k}^k$. Therefore $a_o^p b_i^o + \sum_{k=1}^{i-1} a_k b_{i-k}^k = 0$. Similarly, we find $\delta(R_1^o S_1^{p-1}) = -(R_1^o S_o)(R_1^o S_1^{p-2}) = S_o^{p-1} \tilde{R}_1^o$, and therefore $a_o^{p-1} b_1^o = 0$. These relations generate all others between elements of $\{a_i\}$ and of $\{b_j^i\}$.

Next we consider certain cohomology classes defined by cochains in $E\{R_j^i\}$. Write basis elements of $E\{R_j^i\}$ in the form $\prod_{k=1}^{n} R_{j_k}^{i_k}$, where $i_k > i_\ell$ or $i_k = i_\ell$ and $j_k > j_\ell$ if $k < \ell$. Let h be such a basis element and suppose h satisfies the following conditions:

1) If $R_j^i \in \{R_{j_k}^{i_k}\}$, then R_t^i or $R_{j-t}^{i+t} \in \{R_{j_k}^{i_k}\}$, $1 \leq t \leq j-1$

2) If $R_j^i \in \{R_{j_k}^{i_k}\}$, then $\{R_{j_k}^{i_k}\}$ contains less than $p-1$ elements of the form R_{j+t}^{i-t} or R_{j+t}^{i}, $t > 0$.

3) For no k and ℓ is $i_k = i_\ell + j_\ell$.

4) h cannot be factored as $\pm h'h''$, where both h' and h'' satisfy 1), 2), and 3).

Conditions 1) and 2) imply $e_{i_k,j_k}(R_{j_1}^{i_1}\cdots\hat{R}_{j_k}^{i_k}\cdots R_{j_n}^{i_n}) = 0$, and therefore h is a cocycle. Condition 3) states that h is not a coboundary and is not even a summand of a coboundary. Condition 4) states that h is not decomposable as a product of other such cocycles. We will let h denote both the cocycle and its cohomology class. I conjecture that each such h is in fact indecomposable, and that no other indecomposable cohomology classes are represented by cocycles lying in $E\{R_j^i\}$. An inductive proof of these conjectures using the spectral sequences set up below should be possible. The first conjecture could perhaps be proven by dualizing: the dual basis element h_* of $E(sL^+)$ is a cycle which is not a summand of a boundary by 1), 2), and 3); by 4), if $\pm h'_* \otimes h''_*$ is a summand of $D(h_*)$, then either h' or h'' does not satify 1) and therefore h'_* or h''_* is at least a summand of a boundary; to prove that h_* is primitive it would suffice to prove that h'_* or h''_* is a boundary.

Next we consider products of certain of the elements above with the S_ℓ . Suppose that h is a cocycle of the type found above and that h satisfies

a) $\prod_{t=o}^{\ell-1} R^t_{\ell-t}$ is a factor of h .

b) $\{R^{i_k}_{j_k}\}$ contains less than p-1-m elements of the form

$R^{t-u}_{\ell-t+u}$ or $R^t_{\ell-t+u}$, $0 \leq t \leq \ell-1$.

a) and b) imply that $(f_\ell S^{m-1}_\ell)h = 0$ and that hS^m_ℓ is a cocycle which is not a summand of a coboundary. Note that a) and b) also imply that $\ell < p-m$: by b), $\{R^{i_k}_{j_k}\}$ contains less than p-1-m elements $R^{\ell-1-u}_{u+1}$ and each $R^t_{\ell-t}$, $0 \leq t \leq \ell-2$, is of this form. I conjecture that all such cocycles represent indecomposable classes. The methods cited above could be used to attempt a proof of this.

We summarize the information obtained so far on $H^*(V(L))$ in the

<u>Proposition II.4.1</u>: The following cocycles define distinct non-zero cohomology classes of $H^*(V(L))$:

i) $\tilde{R}^i_j$, $i \geq 0$, $j \geq 1$; S_o and S^p_i if $i > 0$

ii) $h = \prod_{k=1}^{n} R_{j_k}^{i_k}$, where h satisfies 1) through 4) above

iii) $h S^m_\ell$, where h is a cocycle of ii) which also satisfies a) and b) above

None of the classes defined by these cocycles is decomposable in terms of the others. If b^i_j, a_o, and a_i denote the classes of $\tilde{R}^i_j$, S_o, and S^p_i, then the relations among them are generated by:

iv) $$a_o^p b_i^o + \sum_{k=1}^{i-1} a_k b_{i-k}^k = 0 \quad \text{and} \quad a_o^{p-1} b_1^o = 0$$

The classes in ii) and iii) satisfy at least those relations implied by the algebra structure of $\overline{X}^*$.

<u>Conjecture II.4.2</u>: The cohomology classes listed in Proposition II.4.1 are all indecomposable.

There are generators of $H^*(V(L))$ not listed in the proposition. To further study $H^*(V(L))$ we introduce a sequence of spectral sequences. These are essentially of the same type as those introduced by Adams in order to facilitate calculation of the cohomology of the Steenrod algebra using the cobar construction. The setting up of our spectral sequences is quite simple. Let $\overline{X}^*_n$ be the subcomplex and subalgebra of $\overline{X}^*$ generated by $\{R^i_j, S_k, \tilde{R}^\ell_h \mid j \leq n,\ k \leq n-1,\ h \leq n\}$. It is easily verified that $\overline{X}^*_n = E\{R^i_j \mid j \leq n\} \otimes P\{S_k \mid k \leq n-1\} \otimes P\{\tilde{R}^\ell_h \mid h \leq n\}$ as a Z_p-space.

Let $Z_n = E\{R_n^i\} \otimes P(S_{n-1}) \otimes P\{\tilde{R}_n^i\}$ and identify $\bar{X}_n^*$ with $\bar{X}_{n-1}^* \otimes Z_n$ as a Z_p-space. Then filter $\bar{X}_n^*$ by $x \otimes z \in F^s\bar{X}_n^*$ if and only if x has homological degree greater than or equal to s, where $x \in \bar{X}_{n-1}^*$ and $z \in Z_n$, and $\Sigma\, x_i \otimes z_i \in F^s\bar{X}_n^*$ if and only if some $x_i \otimes z_i \in F^s\bar{X}_n^*$, $x_i \in \bar{X}_{n-1}^*$, $z_i \in Z_n$. It is easily seen that $F^s\bar{X}_n^*$ is a subcomplex and subalgebra of $F^{s-1}\bar{X}_n^*$, and therefore that ${}_nE_o$ is a complex and an algebra, where ${}_nE_o^{s,t} = (F^s\bar{X}_n^*/F^{s+1}\bar{X}_n^*)_{s+t}$, $s+t$ being the total homological degree. Examination of the algebra structure of $\bar{X}_n^*$ shows that if ${}_nE_o$ is graded by total degree, ${}_nE_o^r = \bigoplus_{s+t=r} {}_nE_o^{s,t}$, then ${}_nE_o \cong \bar{X}_{n-1}^* \otimes Z_n$ as an algebra, where X_{n-1}^* has its algebra structure as a subalgebra of $\bar{X}^*$ and Z_n has its natural algebra structure. Consider the resulting spectral sequence. $\delta_o = 0$, hence ${}_nE_1^{s,t} = {}_nE_o^{s,t}$. δ_1 is given by $\delta_1(R_j^i) = e_{i,j}$ if $j < n$, $\delta_1(S_k) = f_k$ if $k < n-1$, and by $\delta_1(R_n^i) = \delta_1(S_{n-1}) = 0$. Therefore ${}_nE_2^{s,t} = H^s(\bar{X}_{n-1}^*) \otimes Z_n^t$, and ${}_nE_2$ is an associative differential algebra. δ_2 is given by $\delta_2(R_n^i) = e_{i,n}$ and $\delta_2(S_{n-1}) = f_{n-1}$. The original non-associative algebra structure on $\bar{X}_n^*$ can give rise to non-zero higher differentials. For example, $f_n S_n^{p-1} \in {}_{n+1}E_2^{2,p-1}$ and $\delta_2(f_n S_n^{p-1}) = 0 \in {}_{n+1}E_2^{4,p-2}$, but $\delta_p(f_n S_n^{p-1}) = \sum_{k=o}^{n-1} S_k^p \tilde{R}_{n-k}^k \in {}_{n+1}E_2^{p+2,o}$. As in this example, we will use the same notation for elements of ${}_nE_r$ as for the corresponding cochains of $\bar{X}_n^*$ when this is convenient. All $\delta_r = 0$, $r > p$, as is easily verified by considering the differential and algebra structure of $\bar{X}_n^*$. Therefore ${}_nE_{p+1}^{s,t} = {}_nE_\infty^{s,t} = E_{s,t}^o H^*(\bar{X}_n^*)$. Since $H^{s,t}(\bar{X}_n^*) = H^{s,t}(\bar{X}^*)$ for $t < 2p^{n-1} - 1$, in order to compute $H^*(\bar{X}^*) = H^*(V(L))$ it suffices to calculate the $H^*(\bar{X}_n^*)$ successively.

Before proceeding with the calculations, we give n interpretation of these spectral sequences. Let A_n^* be the Hopf subalgebra of the dual of the Steenrod algebra generated by $\{1, \xi_1, \ldots, \xi_n, \tau_o, \ldots, \tau_{n-1}\}$ (see page II-1.3 for the notation), let $C_n = (A_n^*)^*$, and let $B_n = (A_n^* /\!/ A_{n-1}^*)^* \subset C_n$. Let $E^oC_n = E_n^o$ and note that $E^oB_n = B_n$ (with grading by total degree). $E^oC_n = V(L_n)$, $E^oB_n = V(L_n')$, where L_n' is, of course, Abelian. Then we have ${}_nE_2^{s,t} = H^s(V(L_{n-1})) \otimes H^t(V(L_n'))$, and ${}_nE_\infty^{s,t} = E_{s,t}^o\, H^*(V(L_n))$.

We now begin the calculation of the spectral sequences. $\overline{X}_1^* = H^*(\overline{X}_1^*) = Z_1$ as an algebra, hence ${}_2E_2 = Z_1 \otimes Z_2$ as an algebra. The differential on ${}_2E_2$ is given by $\delta_2(R_2^i) = R_1^{i+1}\, R_1^i$ and $\delta_2(S_1) = R_1^oS_o$. The notation $x{:}y$ will mean the cohomology class x with representative cocycle y. Then we find:

<u>Proposition II.4.3</u>: A basis for the indecomposable elements of ${}_2E_3$ consists of

i) f_2: $R_2^oS_o + R_1^1S_1$; $e_{i,3}$: $R_1^{i+2}R_2^i + R_2^{i+1}R_1^i$, $i \geq 0$

ii) b_j^i: $\tilde{R}_j^i$, $j = 1$ or 2, $i \geq 0$

iii) a_o: S_o, a_1: S_1^p

iv) $h_{i+1}(j,k)$: $(R_2^{i+1})^{j-1}\, R_1^{i+1}\, (R_2^i)^{k-1}$, $j = 1$ or 2, $k = 1$ or 2, $i \geq 0$

v) $h_o^\ell(j,1)$: $(R_2^o)^{j-1}\, R_1^oS_1^\ell$, $j = 1$ or 2, $0 \leq \ell \leq p-1$

The abbreviated notations $h_i = h_i(1,1)$, $g_1^\ell = h_o^\ell(1,1)$, and $g_2^\ell = h_o^\ell(2,1)$ will also be used, as will be $h_{i+1}^o(j,k) = h_{i+1}(j,k)$.

Commutativity, associativity, and the following relations determine ${}_2E_3$ as an algebra (unless explicitly restricted, i,j,k,ℓ, etc. take all values consistent with the list of generators):

1. a) $f_2^2 = 0$

   b) $h_2 f_2 = e_{o,3} a_o$

   c) $e_{i+1,3} h_i = h_{i+3}\, e_{i,3}$

2. a) $e_{i,3} h_i^{\ell}(2,k) = 0$

   b) $e_{i,3} h_{i+1}(2,2) = 0$

   c) $e_{i,3} h_{i+2}(j,2) = 0$

3. a) $f_2 h_1(j,2) = 0$

   b) $f_2 g_1^{p-2} = 0$

4. $h_i^{\ell}(j,k) h_i^{\ell'}(j',k') = 0$

5. a) $h_{i+1} h_i = 0$

   b) $h_{i+1}(j,2) h_i^{\ell}(2,k) = 0$

   c) $h_{i+1}(j,1) h_i^{\ell}(2,k) = h_{i+1}(j,2) h_i^{\ell}(1,k)$

   d) $h_{i+2} h_{i+1}(1,2) = h_{i+1}(2,1) h_i$

6. a) $e_{i,3} h_{i+1} = -h_{i+2} h_{i+1}(1,2) - h_{i+1}(2,1) h_i$

   b) $e_{i,3} h_{i+1}(2,1) = h_{i+2}(1,2) h_{i+1}(1,2)$

   c) $e_{i,3} h_{i+1}(1,2) = h_{i+1}(2,1) h_i(2,1)$

   d) $e_{i,3} h_i^{\ell}(1,k) = h_{i+2} h_i^{\ell}(2,k)$

   e) $e_{i,3} h_{i+2}(j,1) = h_{i+2}(j,2) h_i$

   f) $e_{i,3} e_{i+1,3} = -2h_{i+2}(2,1) h_{i+1}(1,2)$

   g) $(e_{i,3})^2 = -2h_{i+2}(1,2) h_i(2,1)$

   h) $e_{o,3} f_2 = -2h_1(2,1) g_1^1$

7. a) $g_1^{\ell} a_o = 0, \quad 0 \leq \ell \leq p-2$

b) $(\ell+1) g_2^{\ell} a_o = h_1 g_1^{\ell+1}, \quad 0 \leq \ell \leq p-2$

8. a) $f_2 g_1^{\ell} = g_2^{\ell} a_o + h_1 g_1^{\ell+1}, \quad 0 \leq \ell \leq p-3$

b) $f_2 g_1^{p-1} = -g_2^{p-1} a_o$

c) $f_2 g_2^{\ell} = h_1 g_2^{\ell+1}, \quad 0 \leq \ell \leq p-2$

d) $f_2 g_2^{p-1} = h_1 h_o(2,1) a_1$

e) $h_1(j,1) f_2 = h_1(j,2) a_o$

Proof: The proof of the proposition consists only of routine inspection. That the cocycles of i) through v) give indecomposable classes is clear from the definition of ${}_2E_2$ as a complex. That no other indecomposable classes occur is easily seen by considering all possible cochains. Many of the relations listed are implied by the algebra structure of ${}_2E_2$. The others are easy to derive. For example, $\delta_2(R_2^{i+2} R_2^{i}) = h_{i+3}\, e_{i,3} - e_{i+1,3} h_i$ and $\delta_2(r_2^1 S_1) = h_2 f_2 - e_{o,3} a_o$, which proves 1b) and 1c). That the relations listed generate all others is seen by examining all possible products.

To determine the differentials in ${}_2E_r$, $r > 2$, we need consider only those generators of ${}_2E_3$ represented by cocycles of ${}_2E_2$ which are not cocycles in $\bar{\bar{X}}_2^*$. We easily find that $\delta_r = 0, \; 2 < r < p$. The next proposition gives the form of δ_p and describes the structure of ${}_2E_\infty = {}_2E_{p+1}$ by stating how it differs from that of ${}_2E_3$.

Proposition II.4.4: δ_p is zero on all generators of ${}_2E_p = {}_2E_3$ except for the following:

i) $\delta_p(g_1^{p-1}) = a_o^{p-1}b_1^o$

ii) $\delta_p(g_2^{p-2}) = h_1 a_o^{p-2}b_1^o$

iii) $\delta_p(g_2^{p-1}) = -f_2 a_o^{p-2}b_1^o$

iv) If $p = 3$, $\delta_p(h_{i+1}(2,2)) = h_{i+2}h_i b_1^{i+1}$

A basis for the indecomposable elements of ${}_2E_{p+1} = {}_2E_\infty$ is obtained by deleting g_1^{p-1}, g_2^{p-2}, g_2^{p-1}, and, if $p = 3$, $h_{i+1}(2,2)$ from the basis for the indecomposable elements of ${}_2E_3$. The algebra structure on ${}_2E_\infty$ differs from that of ${}_2E_3$ only in that all relations involving the deleted basis elements must be omitted from those listed in Proposition II.4.3 and the following relations must be added to the list:

9. a) $a_o^{p-1}\, b_1^o = 0$

b) $h_1\, a_o^{p-2}\, b_1^o = 0$

c) $f_2\, a_o^{p-2}\, b_1^o = 0$

d) If $p = 3$, $h_{i+2}h_i\, b_1^{i+1} = 0$

Proof: The cochains of $\bar{X}_2^*$ which represent indecomposable elements of ${}_2E_3$ are all cocycles in $\bar{X}_2^*$ except for those representing g_1^{p-1}, g_2^{p-1}, g_2^{p-2}, and, if $p = 3$, $h_{i+1}(2,2)$. Using Proposition II.3.1, we find that in $\bar{X}^*$:

i) $\delta(R_1^o S_1^{p-1}) = -(R_1^o S_o)(R_1^o S_1^{p-2}) = S_o^{p-1}\, \tilde{R}_1^o$

ii) $\delta(R_2^o R_1^o S_1^{p-2}) = (R_1^1 R_1^o)(R_1^o S_1^{p-2}) - 2(R_1^o S_o)(R_2^o R_1^o S_1^{p-3}) = R_1^1 S_o^{p-2}\tilde{R}_1^o$

iii) $\delta(R_2^oR_1^oS_1^{p-1}) = (R_1^1R_1^o)(R_1^oS_1^{p-1}) - (R_1^oS_o)(R_2^oR_1^oS_1^{p-2}) = -(R_1^1S_1 + R_2^oS_o)S_o^{p-2}\tilde{R}_1^o$

iv) If $p = 3$, $\delta(R_2^{i+1}R_1^{i+1}R_2^i) = (R_1^{i+2}R_1^{i+1})(R_1^{i+1}R_2^i) + (R_1^{i+1}R_1^i)(R_2^{i+1}R_1^{i+1}) =$

$$= R_1^{i+2}R_1^i\tilde{R}_1^{i+1} \ .$$

This proves the statements as to the form of δ_p (checking that the change in filtration degree is p in each of i) through iv)). The statements about indecomposable elements and relations follow from the easily verified facts that if $\delta_p(a) = b \neq 0$, then the annihilator of a is included in the annihilator of b, and that precisely those elements of the ideal generated by g_1^{p-1}, g_2^{p-2}, g_2^{p-1}, and, if $p = 3$, $h_{i+1}(2,2)$ which are also in the ideal generated by all other indecomposable elements of ${}_2E_3$ are nonbounding cocycles.

We have now determined ${}_2E_\infty = E^oH^*(\overline{X}_2^*)$. In order to proceed to the next spectral sequence $\{{}_3E_r\}$ we must first determine the algebra structure of $H^*(\overline{X}_2^*)$. In general, one must first find representative cocycles in $H^*(\overline{X}_n^*)$ and then study the algebra structure, since the "elements of $\overline{X}_n^*$" representing non-zero elements of ${}_nE_\infty$ need not be cocycles in $\overline{X}_n^*$ (see Remarks II.4.7 below for examples of such behavior). In the case $n = 2$, all indecomposable classes of ${}_2E_\infty$ are represented by cocycles of $\overline{X}_2^*$ and we need study only such relations as resulted in ${}_2E_3$ from the algebra structure of ${}_2E_2$. We use the same notation for generators of $H^*(\overline{X}_2^*)$ as for generators of ${}_2E_\infty$. We state how the algebra structure of $H^*(\overline{X}_2^*)$ differs from that of ${}_2E_\infty$ in the following proposition, the proof of which depends only on the definitions of the generating cohomology classes and on the products in Proposition II.3.1.

<u>Proposition II.4.5</u>: The algebra structure of $H^*(\overline{X}_2^*)$ differs from that of ${}_2E_\infty$ only in that relations 2), 3), and 4) (listed in Proposition II.4.3) are replaced by:

2'. a) $e_{i,3}h_i^\ell(2,k) = 0$ if $p > 3$ and $\ell < p-3$

$e_{o,3}g_2^{p-3} = -h_1(2,1)a_o^{p-3}b_1^o$

$e_{i,3}h_i(2,1) = -h_{i+1}(2,1)b_1^i$ if $p = 3$

b) $e_{i,3}h_{i+1}(2,2) = 0 \quad (p > 3)$

c) $e_{i,3}h_{i+2}(j,2) = 0$ if $p > 3$

$e_{i,3}h_{i+2}(1,2) = -h_{i+1}(1,2)b_1^{i+2}$ if $p = 3$

3'. a) $f_2h_1(j,2) = 0$ if $p > 3$

$f_2h_1(1,2) = -g_1^1b_1^1$ if $p = 3$

b) $f_2g_1^{p-2} = 0$

4'. a) If $p > 3$, $h_{i+1}(j,k)h_{i+1}(j,k) = 0$

If $p = 3$: $h_ih_i = 0$

$h_i(j,1)h_i(2,1) = -h_{i+1}(1,j)b_1^i$

$h_i(1,k)h_i(1,2) = -h_{i-1}(k,1)b_1^i$

$h_i(2,1)h_i(1,2) = -e_{i-1,3}b_1^i$

b) $g_1^\ell g_1^m = 0$ if $\ell+m \neq p-2$

$g_1^\ell g_1^{p-2-\ell} = -\frac{1}{\ell+1}\, a_o^{p-2}b_1^o$

c) $g_2^{\ell} g_1^{m} = 0$ if $\ell+m < p-3$ or $\ell+m > p-2$

$$g_2^{\ell} g_1^{p-3-\ell} = \frac{1}{\ell+2} h_1 a_o^{p-3} b_1^o$$

$$g_2^{\ell} g_1^{p-2-\ell} = -\frac{1}{\ell+1} f_2 a_o^{p-3} b_1^o$$

d) $g_2^{\ell} g_2^{m} = 0$ if $\ell+m \neq p-3$

$$g_2^{\ell} g_2^{p-3-\ell} = \frac{1}{(\ell+2)(\ell+1)} h_1(1,2) a_o^{p-3} b_1^o$$

To recapitulate, $H^*(\overline{X}_2^*)$ is generated by all cohomology classes listed in i) - iv) of Proposition II.4.3 except those deleted due to Proposition II.4.4. The algebra structure of $H^*(\overline{X}_2^*)$ is determined by relations 1 and 5-8 of Proposition II.4.3 (with relations involving deleted generators omitted), by relations 9 of Proposition II.4.4, and by relations 2'-4' of Proposition II.4.5.

We can now begin the calculation of the spectral sequence $\{{}_3E_r\}$. ${}_3E_2$ is the differential algebra $H^*(\overline{X}_2^*) \otimes Z_3$ with differential determined by $\delta_2(R_3^i) = e_{i,3}$, $\delta_2(S_2) = f_2$, and $\delta_2(\tilde{R}_3^i) = 0$. The image of $H^*(\overline{X}_2^*)$ in ${}_3E_3$ is therefore $H^*(\overline{X}_2^*)/I$, where I is the ideal in $H^*(\overline{X}_2^*)$ generated by $\{e_{i,3}\}$ and f_2. The following proposition lists the indecomposable elements of ${}_3E_3$. Its proof consists of a rather tedious inspection of the structure of $H^*(\overline{X}_2^*)$, in particular, of the annihilators of generators of the ideal I.

<u>Proposition II.4.6</u>: A basis for the indecomposable elements of ${}_3E_3$ which are not in $H^*(\overline{X}_2^*)$ consists of:

i) Φ_3: $R_3^o a_o + h_2 S_2$; $\varepsilon_{i,4}$: $R_3^{i+1} h_i + h_{i+3} R_3^i$

ii) a_2: S_2^p ; b_3^i: $\tilde{R}_3^i$

iii) γ: $f_2 S_2^{p-1}$

μ^ℓ: $g_1^{p-2} S_2^\ell$, $1 \leq \ell \leq p-1$

$\nu^{\ell,m}$: $h_2 g_1^\ell S_2^m$, $0 \leq \ell \leq p-3$, $2 \leq m \leq p-1$

g_3^ℓ: $R_3^o g_2^\ell$, $0 \leq \ell \leq p-4$, $p > 3$

k_i^ℓ: $h_1(i,2) S_2^\ell$, $1 \leq \ell \leq p-1$, $p > 3$

j^ℓ: $h_1(1,2) h_o R_3^o S_2^\ell$, $1 \leq \ell \leq p-1$, $p > 3$

λ^ℓ: $h_2 a_o^{p-3} b_1^o S_2^{\ell+1} + (\ell+1) a_o^{p-2} b_1^o R_3^o S_2^\ell$, $1 \leq \ell \leq p-2$

iv) $R_3^o h_2 g_1^\ell S_2^m$, $0 \leq \ell \leq p-3$, $0 \leq m \leq p-1$

$R_3^i h_{i+2} h_i$, $i > 0$

v) If $p > 3$:

$R_3^{i+1} h_{i+1}(2,1)$

$R_3^i h_{i+2}(1,2)$

$(R_3^{i+1})^j (R_3^i)^k (R_3^{i-1})^\ell \; h_{i+1}(2,2) S_2^m$, $i \geq \ell$ and $j,k,\ell = 0$ or 1,

$m = 0$ if $i > 0$, $0 \leq m \leq p-1$ if $i = 0$, and $j+k+\ell > 0$.

$(R_3^{i+2})^{\ell_1} R_3^i (R_3^{i-2})^{\ell_2} h_{i+2}(j,1) h_i(1,k) S_2^m$, $i \geq 2\ell_2$, $\ell_1 < j$, $\ell_2 < k$,

$j+k > 2$, $m = 0$ if $i \geq k$, $0 \leq m \leq p-1$ if $i < k$.

Remarks II.4.7: In i) and iii) above, we have used Greek letters to name certain cocycles of ${}_3E_2$ whose corresponding cochains in $\bar{X}_3^*$ are not cocycles. Thus the cochains corresponding to Φ_3 and $\varepsilon_{i,4}$ in $\bar{X}_3^*$ are $R_3^o S_o + R_1^2 S_2$ and $R_3^{i+1} R_1^i + R_1^{i+3} R_3^i$, and these must be extended to f_3 and $e_{i,4}$ to obtain the representative cocycles in $\bar{X}_3^*$ for the elements of $H^*(\bar{X}_3^*)$ corresponding to the elements Φ_3

and $\varepsilon_{i,4}$ of ${}_3E_\infty$. We note further that the relations $\Phi_3^2 = 0$, $h_3\Phi_3 = \varepsilon_{0,4}a_0$, and $\varepsilon_{i+1,4}h_i = h_{i+4}\varepsilon_{i,4}$ hold in ${}_3E_3$ and ${}_3E_\infty$, and the corresponding relations with f_3 replacing Φ_3, $e_{i,4}$ replacing $\varepsilon_{i,4}$, hold in $H^*(\bar{X}_3^*)$. Therefore Φ_4: $h_3S_3 + R_4^o a_o$ and $\varepsilon_{i,5}$: $R_4^{i+1}h_i + h_{i+4}R_4^i$ are cocycles of ${}_4E_2$. The analogous phenomena occur in each higher spectral sequence.

Conventions II.4.8: The letter t will always denote the grading derived from that of the Steenrod algebra and the letter s will always denote the homological degree. The notation $x \in (s,t)$ will mean that x is an element (of any group under consideration) with homological degree s and grading t. For example, $h_i \in (1,2p^i(p-1))$. Since $t-s$ is the total degree in the Adams spectral sequence, this dimension will be of particular interest to us.

Since $R_3^o h_2 h_o \in (3,2(p-1)(2p^2+p+2))$, the smallest value of $t-s$ taken by the indecomposable elements of iv) and v) is $2(2p^2+p+2)(p-1)-3$. All indecomposable elements listed in iii) except k_2^{p-2}, k_2^{p-1}, j^{p-2} and j^{p-1} have lower values of $t-s$. In the next proposition, we shall determine all non-zero higher differentials in the range $t-s \leq 2(2p^2+p+2)(p-1)-3$.

Proposition II.4.9: The following list gives all non-zero higher differentials on those indecomposable elements of ${}_3E_3$ satisfying $t-s \leq 2(2p^2+p+2)(p-1)-3$.

(In i) - iv) below, $k_i^o = h_i(i,2)$, $j^o = h_1 g_3^o$):

i) $\delta_3(\mu^\ell) = -\ell(\ell-1)k_1^{\ell-2}a_1$, $\ell \geq 2$

ii) $\delta_3(\nu^{\ell,m}) = -\dfrac{m(m-1)(\ell+3)}{\ell+1}\, g_1^{\ell+2}k_1^{m-2}$, $0 \leq \ell \leq p-4$

iii) $\delta_3(\lambda^{\ell}) = (\ell+1)\ell(\ell-1)j^{\ell-2}a_1$, $\ell \geq 2$

iv) $\delta_4(\nu^{p-3,m}) = -m(m-3,2)k_2^{m-3}a_1$, $m \geq 3$

v) $\delta_{p-2}(k_2^{p-3}) = -2h_2g_1^{p-3}b_1^1$

vi) $\delta_{p-1}(k_1^{p-2}) = g_1^{p-2}b_1^1$

$\delta_{p-1}(k_1^{p-1}) = \mu^1b_1^1$

vii) $\delta_{p-1}(j^{p-3}) = -\frac{1}{4}h_2a_o^{p-3}b_1^ob_1^1$

viii) $\delta_p(\gamma) = a_o^pb_2^o + a_1b_1^1$

Proof: i) - iv) are proven by making explicit use of the definition of the differentials in a spectral sequence. We give the proof of i), the proofs of ii) - iv) being similar. μ^{ℓ} is represented by the image in ${}_3E_o^{p-1,\ell}$ of the cochain $R_1^oS_1^{p-2}S_2^{\ell} \in F^{p-1}\bar{X}_3^*$. In $F^{p-1}\bar{X}_3^*$, $\delta(R_1^oS_1^{p-2}S_2^{\ell}) = \ell f_2R_1^oS_1^{p-2}S_2^{\ell-1}$, which, since $\delta_2(\mu^{\ell}) = 0$, must be congruent in $F^{p-1}\bar{X}_3^*$ to an element of $F^{p+2}\bar{X}_3^* \subset F^{p-1}\bar{X}_3^*$.
$\delta(\ell R_2^oS_1^{p-1}S_2^{\ell-1}) = \ell f_2R_1^oS_1^{p-2}S_2^{\ell-1} + \ell(\ell-1)R_1^1R_2^oS_1^pS_2^{\ell-2}$. Therefore $\delta(R_1^oS_1^{p-2}S_2^{\ell}) \equiv -\ell(\ell-1)R_1^1R_2^oS_1^pS_2^{\ell-2}$ in $F^{p-1}\bar{X}_3^*$. By definition, the image of $-\ell(\ell-1)R_1^1R_2^oS_1^pS_2^{\ell-2}$ in ${}_3E_o^{p+2,\ell-2}$ represents $\delta_3(\mu^{\ell})$ in ${}_3E_3$, and therefore $\delta_3(\mu^{\ell}) = -\ell(\ell-1)h_1^{\ell-2}(1,2)a_1$ as claimed. The proofs of v) - viii) are similar to the proof of Proposition II.4.4, and depend only on the product structure of $\bar{X}_3^*$.

Remarks II.4.10: We may easily calculate that $\delta_{p-2}(k_2^{p-2}) = 2g_1^{p-3}b_1^1\Phi_3$ and $\delta_{p-2}(k_2^{p-1}) = 2\nu^{p-3,2}b_1^1$. It is also true that $\delta_{p-1}(j^{p-2}) \neq 0$ and $\delta_{p-1}(j^{p-1}) \neq 0$, but the precise calculation is somewhat more difficult, requiring use of formula 9) on page II-3.3, and I have not carried out these computations.

A long inspection shows that, at least in the range $t-s < 2(p-1)(2p^2 + p+2)-3$, no new indecomposable elements arise in the calculation of the spectral sequence. In the same range, the only generators of the annihilator of Φ_3 are g_2^{p-3} and $h_1^{p-3}(1,2)$. The relations $g_2^{p-3}\Phi_3 = 0$ and $h_1^{p-3}(1,2)\Phi_3 = 0$ pass to the relations $g_2^{p-3}f_3 = 0$ and $h_1^{p-3}(1,2)f_3 = 0$ in $H^*(\overline{X}_3^*)$, and therefore ω: $g_2^{p-3}S_3$ and χ: $h_1^{p-3}(1,2)S_3$ are indecomposable in ${}_4E_3$. Clearly both survive to ${}_4E_\infty$. At this point, we know all possible indecomposable elements in the cited range. We will find all relations (in this range) among these elements. We first discuss the form of the image of $H^*(\overline{X}_2^*)$ in ${}_3E_\infty$. Note that, by the definition of the filtration on $\overline{X}_3^*$, the image of $H^*(\overline{X}_2^*)$ in ${}_3E_\infty$ passes unchanged to $H^*(\overline{X}_3^*)$.

The letter q will denote the number $2(p-1)$ for the remainder of this section.

Theorem II.4.11: The image of $H^*(\overline{X}_2^*)$ in $H^*(E^oA)$ is generated by the following elements:

i) a_o: $S_o \in (1,1)$; a_1: $S_1^p \in (p,pq+p)$

ii) b_1^i: $\tilde{R}_1^i \in (2,p^{i+1}q)$; $b_2^i \in (2,(p^{i+2} + p^{i+1})q)$

iii) h_i: $R_1^i \in (1,p^iq)$

iv) $h_i(2,1)$: $R_2^iR_1^i \in (2,p^i(p+2)q)$

v) $h_{i+1}(1,2)$: $R_1^{i+1}R_2^i \in (2,p^i(2p+1)q)$

vi) $h_{i+1}(2,2)$: $R_2^{i+1}R_1^{i+1}R_2^i \in (3,p^i(p^2+3p+1)q)$

vii) g_1^ℓ: $R_1^oS_1^\ell \in (\ell+1,(\ell+1)q+\ell)$, $0 \leq \ell \leq p-2$

viii) g_2^ℓ: $R_2^oR_1^oS_1^\ell \in (\ell+2, (p+\ell+2)q+\ell)$, $0 \leq \ell \leq p-3$

($g_1^o = h_o$, $g_2^o = h_o(2,1)$; h_i may be written $h_i(1,1)$ to simplify the statements of relations.)

These elements satisfy at least the relations:

1) $g_i^\ell g_j^m = 0$ except for the cases

$$g_1^\ell g_1^{p-2-\ell} = -\frac{1}{\ell+1} a_o^{p-2} b_1^o$$

$$g_2^\ell g_1^{p-3-\ell} = \frac{1}{\ell+1} h_1 a_o^{p-3} b_1^o$$

$$g_2^\ell g_2^{p-3-\ell} = \frac{1}{(\ell+2)(\ell+1)} h_1(1,2) a_o^{p-3} b_1^o = 0 \text{ unless } p = 3$$

2) $h_i(j,k)h_i(j',k') = 0$ except for the cases

$h_i(2,1)h_i(j,1) = -h_{i+1}(1,j)b_1^i$ if $p = 3$

$h_i(1,2)h_i(1,k) = -h_{i-1}(k,1)b_1^i$ if $p = 3$

3) $h_1(i,j)g_1^\ell = 0$, $h_1(i,j)g_2^\ell = 0$, $\ell > 0$

$h_{i+1}(j,k)h_i(j',k') = 0$ except for the cases

$h_{i+1}(j,2)h_i(1,j) = h_{i+1}(j,1)h_i(2,j)$

4) $h_2 g_2^\ell = 0$

$h_{i+2}h_i(2,k) = 0$, $h_{i+2}(j,2)h_i = 0$, $h_{i+2}(1,2)h_i(2,1) = 0$

5) $g_i^\ell a_o = 0$, $h_1(j,2)a_o = 0$, $h_1(j,2)a_1 = 0$

6) $a_o^{p-1}b_1^o = 0$, $h_1 a_o^{p-2} b_1^o = 0$, $g_1^{p-2} b_1^1 = 0$, $a_o^p b_2^o + a_1 b_1^1 = 0$

$h_1(2,1)a_o^{p-3}b_1^o = 0$, $h_2 g_1^{p-3} b_1^1 = 0$

If $p = 3$: $h_{i+1}(2,1)b_1^i = 0$, $h_{i+1}(1,2)b_1^{i+2} = 0$, $h_{i+2}h_ib_1^{i+1} = 0$. No other relations hold in the range $s \leq 3$ or in the range $t-s \leq (2p^2 + p+2)q-3$.

Proof: The listed relations follow from the structure of $H^*(\overline{X}_2^*)$ and from Proposition II.4.9, except for $h_1(2,1)g_1^\ell = 0$, which holds since $f_3g_2^\ell = (\ell+3)g_3^\ell a_o$, $(\ell+1)g_3^\ell h_1 a_o = h_1(2,1)g_2^{\ell+1}$, and $\frac{(\ell+2)(\ell+1)}{2} a_o^2 g_3^\ell = h_1(2,1)g_1^{\ell+2}$ in $H^*(\overline{X}_3^*)$, $0 \leq \ell \leq p-4$. The last statement follows from the facts that no other elements belong both to the image of $H^*(\overline{X}_2^*)$ and to the ideal generated by f_3 in $H^*(\overline{X}_3^*)$, and that any further relations must result either from higher differentials being non-zero due to the non-associative algebra structure of $\overline{X}^*$, which can occur only for $s \geq p$ and $p > 3$, or from intersections of the image of $H^*(\overline{X}_2^*)$ with the ideal generated by $\{e_{1,4}\}$ in $H^*(\overline{X}_3^*)$.

Remarks II.4.12: We note that $\varepsilon_{i,4}h_{i+2}h_{i+1}(2,1) = 0$ in ${}_3E_3$ and that the corresponding relation in $H^*(\overline{X}_3^*)$ is $e_{i,4}h_{i+2}h_{i+1}(2,1) =$ $= h_{i+1}(2,1)h_i(2,2)$. Therefore $h_{i+1}(2,1)h_i(2,2) = 0$ in $H^*(V(L))$. Other relations can arise in a like manner.

For $s \leq 2$, the only indecomposable elements in $H^*(\overline{X}_3^*)$ which are not in the image of $H^*(\overline{X}_2^*)$ are b_3^i, $e_{i,4}$, and f_3. Clearly the analogous statements are true for $H^*(\overline{X}_n^*)$, $n > 3$, and we therefore have determined $H^{s,t}(E^oA)$ for $s \leq 2$. The next theorem will completely describe $H^{s,t}(V(L))$ for $t-s \leq q(2p^2+p+2)-4$. For notational simplicity, we let $k_1^o = h_1(1,2)$, $k_2^o = h_1(2,2)$, and $c = h_1(2,1)$.

<u>Theorem II.4.13</u>: In dimensions $t-s \leq (2p^2 + p+2)q-4$, $q = 2(p-1)$, a basis for those indecomposable elements of $H^{s,t}(E^oA)$ not in the image of $H^*(\overline{X}_2^*)$ is given by:

i) a_2: $S_2^p \in (p,(p^2+p)q + p)$

ii) g_3^ℓ: $R_3^oR_2^oR_1^oS_1^\ell \in (\ell+3,(p^2+2p+\ell+3)q + \ell)$, $0 \leq \ell \leq p-4$

iii) k_1^ℓ: $R_1^1R_2^oS_2^\ell \in (\ell+2,((\ell+2)p+\ell+1)q + \ell)$, $1 \leq \ell \leq p-3$

iv) k_2^ℓ: $R_2^1R_1^1R_2^oS_2^\ell \in (\ell+3,(p^2 + (\ell+3)p+\ell+1)q + \ell)$, $1 \leq \ell \leq p-4$

v) j^ℓ: $R_1^1R_3^oR_2^oR_1^oS_2^\ell \in (\ell+4,(p^2 + (\ell+3)p+\ell+3)q + \ell)$, $1 \leq \ell \leq p-4$

vi) u: $R_1^oS_1^{p-2}S_2 - R_2^oS_1^{p-1} \in (p,2pq + p-1)$

vii) w: $R_2^oR_1^oS_1^{p-3}S_3 + R_3^oR_2^oS_1^{p-2} - R_3^oR_1^oS_1^{p-3}S_2 \in (p,(p^2+3p)q+p-2)$

viii) x: $R_1^1R_2^oS_2^{p-3}S_3 - \frac{1}{2}R_1^1R_3^oS_2^{p-2} - \frac{1}{2}R_2^1R_2^oS_2^{p-2} \in (p,(2p^2 +p-1)q + p-2)$

In the cited range, a minimal set of relations is given by those relations holding in the image of $H^*(\overline{X}_2^*)$ and by the following relations (where $k_1^o = h_1(1,2)$, $k_2^o = h_1(2,2)$, and $c = h_1(2,1)$):

1. a) $g_3^\ell g_1^m = 0$, $\ell+m \neq p-4$

$$g_3^\ell g_1^{p-4-\ell} = -\frac{1}{\ell+3}\, c a_o^{p-4} b_1^o$$

b) $g_3^\ell g_2^m = 0$

2. a) $g_3^\ell k_1^m = 0$, $m < p-3$

2. b) $g_2^{\ell}k_1^m = 0$

c) $g_1^{\ell}k_1^m = 0, \quad \ell > 0$

d) $g_2^{\ell}k_2^m = 0$

e) $g_1^{\ell}k_2^m = 0$

3. a) $k_1^{\ell}k_1^m = 0, \quad \ell+m \neq p-3$

$$k_1^{\ell}k_1^{p-3-\ell} = -\frac{1}{(\ell+2)(\ell+1)}\, g_2^{p-3}b_1^1$$

b) $k_1^{\ell}k_2^m = 0, \quad \ell+m < p-4$

4. a) $g_1^m j^{\ell} = 0$

b) $g_2^m j^{\ell} = 0$

c) $k_1^m j^{\ell} = 0$

5. a) $h_1 g_3^{\ell} = 0, \quad \ell > 0$

b) $h_1 j^{\ell} = 0$

c) $h_1 k_1^{\ell} = 0, \quad \ell < p-3$

$$h_1 k_1^{p-3} = -\, g_1^{p-3}b_1^1$$

d) $h_1 k_2^{\ell} = 0, \quad \ell < p-4$

$$h_1 k_2^{p-4} = -\, h_2 g_1^{p-4}b_1^1$$

6. a) $h_2 k_1^{\ell} = 0$

b) $ck_1^{\ell} = 0, \quad \ell < p-4$

$$ck_1^{p-4} = h_2 g_1^{p-4}b_1^1$$

7. a) $g_3^\ell a_o = 0$

b) $k_i^\ell a_o = 0$

c) $k_i^\ell a_1 = 0$

d) $j^\ell a_o = 0$

e) $j^\ell a_1 = 0$

f) $h_2 a_o^{p-3} b_1^o b_1^1 = 0$

8. a) $ua_o = h_1 a_1$

b) $ub_1^1 = - a_o^{p-1} h_1 b_2^o$

c) $ug_1^{\ell+1} = - a_1 g_2^\ell \quad (ug_1^o \neq 0)$

d) $ug_2^\ell = 0$

e) $ug_3^\ell = 0$

f) $uk_i^\ell = 0$

g) $uj^\ell = 0, \quad \ell < p-4$

h) $uh_1 = 0$

i) $uh_2 = 0$

j) $uc = 0$

k) $uw = 0$ if $p > 3$; if $p = 3$, $uw = h_2(a_1 b_2^o - a_2 b_1^o)$

9. a) $wa_o^2 = ca_1$

b) $h_1 w = 0$ if $p > 3$; if $p = 3$, $h_1 w = h_o x - h_2 a_o b_2^o$

c) $wg_1^{\ell+2} = a_1 g_3^\ell, \quad \ell \geq 0 \quad (g_1^o w \neq 0,\ g_1^1 w \neq 0)$

9. d) $wg_2^\ell = 0$

e) $wk_1^\ell = 0, \quad \ell < p-3$

10. a) $xa_o = 0$

b) $xg_1^\ell = 0, \quad \ell > 0$

Proof: We first comment on the identification of the indecomposable elements. u corresponds to $\mu^1 \in {}_3E_\infty$. w and x correspond to ω and χ found in ${}_4E_\infty$. The elements in $H^*(\bar{X}^*)$ corresponding to λ^1 and $\nu^{p-3,2}$ of ${}_3E_\infty$ are not indecomposable: upon choosing representative cocycles ℓ and n in $\bar{X}_3^*$ for λ^1 and $\nu^{p-3,2}$, we find that $g_1^1 w \equiv \ell$ and $a_o w \equiv n$ in $\bar{X}_4^*$. There are no other possible indecomposable elements in the range of $t-s$ under consideration. The listed relations are found by studying the products of the representative cocycles in $\bar{X}^*$. For example, $g_3^\ell a_o = \frac{1}{\ell+3} f_3 g_2^\ell$, $h_2 k_1^{\ell+1} = \frac{\ell+1}{\ell+3} f_3 k_1^{\ell+1}$, $h_1 u = g_1^{p-2} f_3$, and $g_3^\ell g_1^{p-3-\ell} = \frac{1}{\ell+2} f_3 h_1 a_o^{p-4} b_1^o$ in $\bar{X}_3^*$. Relations in ${}_3E_3$ and ${}_4E_3$ are used as guides in seeking the relations in $H^*(\bar{X}_4^*)$.

5. The cohomology of E^0A, $p = 2$.

In this section, we define certain generators of $H^*(E^0A)$, and, using a sequence of spectral sequences, compute $H^{s,t}(E^0A)$ completely for small s at t.

$E^0A \cong V(L)$, where L is the restricted Lie algebra of primitive elements of E^0A. $H^*(E^0A)$ is the homology of the complex $\bar{X}^*$, the form of which was studied in section II.3: $\bar{X}^*$ is the differential algebra $P\{R^i_j\}$ with differential determined by $\delta(R^i_j) = e_{i,j}$, $\quad e_{i,j} = \sum_{k=1}^{j-1} R^{i+k}_{j-k} R^i_k$.

We denote by b^i_j the cohomology class of $(R^i_j)^2$.

$\delta(e_{i,j}R^i_j) = e^2_{i,j} = \sum_{k=1}^{j-1} (R^{i+k}_{j-k})^2 (R^i_k)^2$, and therefore $\sum_{k=1}^{j-1} b^{i+k}_{j-k} b^i_k = 0$.

No other relations hold among the elements b^i_j. Let h_i denote the cohomology class of R^i_1, so that $(h_i)^2 = b^i_1$. $\delta(R^i_2) = R^{i+1}_1 R^i_1$ and therefore $h_{i+1}h_i = 0$. No other relations hold among the elements h_i.

To find other generators of $H^*(V(L))$, it will be convenient to work in the dual complex $\Gamma(sL)$ with differential d given by formula β) of page II-3.4. We write $\bar{P}^i_j$ for $\gamma_1(\bar{P}^i_j)$, and note that $\bar{P}^i_j\, \gamma_r(\bar{P}^i_j) = (r+1)\gamma_{r+1}(\bar{P}^i_j)$. Let $\alpha = \gamma_{r_1}(\bar{P}^{i_1}_{j_1})\ldots\gamma_{r_n}(\bar{P}^{i_n}_{j_n})$. Since $d(xy) = d(yx)$, we may write factors in any convenient order. We assume $s < t$ implies that $i_s + j_s < i_t + j_t$ or $i_s + j_s = i_t + j_t$ and $i_s > i_t$, and then α is said to be written with factors in lexicographic order. We will consider d as being the sum of elementary operations, one for each relation σ: $[P^a_b P^{a+b}_c] = P^a_{b+c}$. Thus $d(\alpha) = \sum_\sigma \sigma(\alpha)$, where $\sigma(\alpha) = 0$ unless the corresponding relation gives rise to the non-zero summand $\sigma(\alpha)$ of $d(\alpha)$. $r(P^i_j)$ will denote the divided power to which $\bar{P}^i_j$ occurs in α. When defining chains in the proofs below, we write explicitly only those factors which differ from the corresponding factors of α and let ε represent the remaining factors. We prove first the

<u>Lemma II.5.1</u>: Suppose α is a cycle and there exists a triple (a,b,c) such that $[P^{i_a}_{j_a} P^{i_b}_{j_b}] = P^{i_c}_{j_c}$ and both $r(P^{i_a}_{j_a}) > 0$ and $r(P^{i_b}_{j_b}) > 0$. Then α is a boundary.

<u>Proof</u>: Let c be the smallest integer such that (a,b,c) is such a triple. Since α is a cycle, $r(P^{i_c}_{j_c}) \equiv 1 \bmod 2$. Let

$\beta = \gamma_{r_a+1}(\bar{P}^{i_a}_{j_a})\gamma_{r_b+1}(\bar{P}^{i_b}_{j_b})\gamma_{r_c-1}(\bar{P}^{i_c}_{j_c})\varepsilon$, $r_x = r(P^{i_x}_{j_x})$, $x = a, b,$ or c.

$d(\beta) = \alpha + \sum_e \sigma_e(\beta)$, σ_e: $[P^{i_c}_e \, P^{i_c+e}_{j_c-e}] = P^{i_c}_{j_c}$, $e \neq j_a$, summed over e such that $r_e = r(P^{i_c}_e) > 0$ and $s_e = r(P^{i_c+e}_{j_c-e}) > 0$. That no other terms occur follows from the choice of c.

$d\rho_e(\beta) = 0$; for each e, let

$\beta_e = \gamma_{r_a+2}(\bar{P}^{i_a}_{j_a})\gamma_{r_b+2}(\bar{P}^{i_b}_{j_b})\gamma_{r_e-1}(\bar{P}^{i_c}_e)\gamma_{s_e-1}(\bar{P}^{i_c+e}_{j_c-e})\gamma_{r_c-1}(\bar{P}^{i_c}_{j_c})\,\varepsilon$.

$d(\beta_e) = \sigma_e(\beta) + \sigma_e(\beta_e) + \sum_{e' \neq e} \sigma_{e'}(\beta_e)$. $\sigma_{e'}(\beta_e) = \sigma_e(\beta_{e'})$, and therefore $d(\beta + \sum_e \beta_e) = \sum_e \sigma_e(\beta_e)$. We repeat the argument on each $\sigma_e(\beta_e)$, noting that $\sigma_e(\beta_e)$ differs from α only in that r_a, r_b, r_e, and s_e are replaced by r_a+2, r_b+2, r_e-2, and s_e-2. Iterating, we find that α is congruent to a sum of terms satisfying the same hypotheses as α but which are such that $r_e = 0$ or $s_e = 0$ for all e . It follows that α is a boundary.

Now we suppose $\alpha = \gamma_{r_1}(\bar{P}^{i_1}_{j_1})\cdots\gamma_{r_n}(\bar{P}^{i_n}_{j_n})$ satisfies the condition $r(P^i_j) \leq 1$ if $j > 1$. We shall find necessary and sufficient conditions for such an α to be a nonbounding cycle. If α is a nonbounding cycle, then by the previous lemma, no i_a is equal to any $i_b + j_b$. Define $S(\alpha) = \{i_1,\ldots,i_n\}$ and $T(\alpha) = \{i_1,\ldots,i_n, i_1+j_1,\ldots,i_n+j_n\}$. Let $y(\alpha)$ be the number of distinct integers in $T(\alpha)$ and let $z(\alpha) = 2n(\alpha) - y(\alpha)$, where $n(\alpha)$ is the number of P^i_j such that $r(P^i_j) > 0$. Thus $z(\alpha)$ measures the number of duplications $i_a = i_b$ or $i_a + j_a = i_b + j_b$ in $T(\alpha)$. The next two lemmas give necessary conditions for α to be a nonbounding cycle.

Lemma II.5.2: Let α be a nonbounding cycle as above. Suppose $z(\alpha) > 0$ and let a be minimal such that $i_a = i_b$ or $i_a + j_a = i_b + j_b$ for some b . Then $j_a = 1$, there exists one and only one b such that $i_a + j_a = i_b + j_b$, there exists one and only one c such that $i_a = i_c$, and

$\alpha \equiv \gamma_{r_a+1}(\bar{P}^{i_a}_1)\hat{\bar{P}}^{i_b}_{j_b}\hat{\bar{P}}^{i_c}_{j_c}\bar{P}^{i_b}_{j_b+j_c-1}\varepsilon$. It follows that every non-zero homology class containing a cycle of the form α contains such a cycle with $z = 0$.

<u>Proof</u>: Let b be minimal such that $i_a = i_b$ or $i_a + j_a = i_b + j_b$. If $i_a = i_b$, then $d(\gamma_{r_a+1}(\bar{P}^{i_a}_{j_a})\bar{P}^{i_a+j_a}_{j_b-j_a}\hat{\bar{P}}^{i_b}_{j_b}\,\varepsilon) = \alpha$, since for no c is $i_c + j_c = i_a + j_a$. Therefore $i_a + j_a = i_b + j_b$. Here we consider first the case $j_a = 1$ and proceed by induction on $n(\alpha)$.

Let $\beta = \bar{P}^{i_b}_{j_b-1}\,\gamma_{r_a+1}(\bar{P}^{i_a}_{1})\hat{\bar{P}}^{i_b}_{j_b}\,\varepsilon$. $d(\beta) = \alpha + \sum_c \sigma_c(\beta)$,

σ_c: $[P^{i_b}_{j_b-1}P^{i_c}_{j_c}] = P^{i_b}_{j_b+j_c-1}$, summed over c such that $i_c = i_a$.

$n(\sigma_c(\beta)) = n(\alpha)-1$, and no σ_c can occur if $n(\alpha) = 2$. The result holds for each $\sigma_c(\alpha)$ by the induction hypothesis on $n(\alpha)$, hence if there exist three or more such c, each $\sigma_c(\beta)$ is a boundary. If there exists only one such c, then there can exist no $f > b$ such that $i_f + j_f = i_a + 1$, since otherwise $d(\beta_c) = \sigma_c(\alpha)$,

$\beta_c = \bar{P}^{i_f}_{j_f-1}\,\gamma_{r_a+2}(\bar{P}^{i_a}_{1})\hat{\bar{P}}^{i_b}_{j_b}\,\hat{\bar{P}}^{i_f}_{j_f}\,\hat{\bar{P}}^{i_c}_{j_c}\,\bar{P}^{i_b}_{j_b+j_c-1}\,\varepsilon$. Therefore α satisfies the conclusion of the lemma in this case. Suppose there exist c and e such that $i_c = i_e = i_a$. Then, by the induction hypothesis on $\sigma_c(\beta)$ and $\sigma_e(\beta)$, there exists exactly one $f > b$ such that $i_f + j_f = i_a + 1$ and there exists no $g > b$ such that $i_g = i_b$. Let β_c be as above, form β_e similarly, and let

$$\Phi = \bar{P}^{i_f}_{j_f-j_b}\,\gamma_{r_a+2}(\bar{P}^{i_a}_{1})\,\hat{\bar{P}}^{i_b}_{j_b}\,\hat{\bar{P}}^{i_f}_{j_f}\,\hat{\bar{P}}^{i_c}_{j_c}\,\bar{P}^{i_b}_{j_b+j_c-1}\,\hat{\bar{P}}^{i_e}_{j_e}\,\bar{P}^{i_b}_{j_b+j_e-1}\,\varepsilon .$$

If $\tau_c: [P^{i_f}_{j_f-1}\ P^{i_c}_{j_c}] = P^{i_f}_{j_c+j_f-1}$ and $\tau_e: [P^{i_f}_{j_f-1}\ P^{i_e}_{j_e}] = P^{i_f}_{j_e+j_f-1}$, then $d(\beta_c) = \sigma_c(\beta) + \tau_e(\beta_c)$, $d(\beta_e) = \sigma_e(\beta) + \tau_c(\beta_e)$ and $d(\Phi) = \tau_e(\beta_c) + \tau_c(\beta_e)$. Therefore $\sigma_c(\beta) + \sigma_e(\beta)$ and α are boundaries. It remains to consider the case $i_a + j_a = i_b + j_b$, $j_a > 1$. Let $\psi = \overline{P}^{i_a}_{j_a-1}\ \overline{P}^{i_a+j_a-1}_{1}\ \hat{\overline{P}}^{i_a}_{j_a}\ \varepsilon$. $d(\psi) = \alpha$ unless there exists c such that $i_c = i_a + j_a-1$ or $i_c + j_c = i_a + j_a-1$. Suppose the first case obtains, $d(\psi) = \alpha + \sum_c \sigma_c(\psi)$, $\sigma_c: [P^{i_a}_{j_a-1}\ P^{i_c}_{j_c}] = P^{i_a}_{j_a+j_c-1}$. By the case $j_a = 1$, the result holds for each $\sigma_c(\psi)$. Therefore there are exactly two such c , say c and e , there is just one $b > a$ such that $i_b + j_b = i_a + j_a$ and there is no $g > a$ with $i_g = i_a$. Form β_c, β_e , and Φ as above, with i_a replaced by $i_a + j_a-1$ (r_a by zero) and with i_b, j_b, i_f , and j_f replaced by i_a, j_a, i_b , and j_b of our present notation. Then $d(\beta_c + \beta_e + \Phi) = \sigma_c(\psi) + \sigma_e(\psi)$ and α is a boundary. Thus there exists c such that $i_c + j_c = i_a + j_a-1$. There can be only one such c by the choice of a , and there is no $e > c$ with $i_e = i_c$. Now $d(\psi) = \alpha + \rho(\psi)$,

$\rho: [P^{i_c}_{j_c}\ P^{i_a + j_a-1}_{1}] = P^{i_c}_{j_c + 1}$. If $j_c = 1$ and $r_c > 1$, the result holds for $\rho(\psi)$ by the case $j_a = 1$, and therefore there exists no $e > a$ such that $i_e = i_a$. If $r_c = 1$, there exists no $e > a$ such that $i_e = i_a$ since otherwise $\rho(\psi)$ would be a boundary by the case $i_a = i_b$. Therefore

$d(\overline{P}^{i_b}_{j_b-j_a}\ \gamma_2(\overline{P}^{i_a}_{j_a})\hat{\overline{P}}^{i_b}_{j_b}\ \varepsilon) = \alpha$ if there exists c such that

$i_c + j_c = i_a + j_a - 1$. This completes the proof. Note that the conclusion implies that if for any e and f $i_e = i_f$ or $i_e + j_e = i_f + j_f$ and both $j_e > 1$ and $j_f > 1$, then α is a boundary. We have used this fact at several points in the proof.

Lemma II.5.3: Let α be a nonbounding cycle of the form described above Lemma II.5.2. Then either $T(\alpha)$ is a sequence of integers or $\alpha = \prod_p \alpha_p$ where each $T(\alpha_p)$ is a sequence and $T(\alpha_p) \cap T(\alpha_q) = \Phi$.

Proof: By the previous lemma, we may assume $z(\alpha) = 0$. Let $i = \min\{i_a\}$ and suppose there exists t such that $i < t < i_n + j_n$ but $t \notin T(\alpha)$. Let u be the largest integer, $i \leq u < t$, such that $u \in T(\alpha)$. If $u = i_a$ for some a , then $i_a + j_a > t$,

$j_a > t - u$, and $d(\overline{P}^{i_a}_{t-i_a} \; \overline{\overline{P}}^{t}_{i_a+j_a-t} \; \hat{\overline{P}}^{i_a}_{j_a} \; \varepsilon) = \alpha$. Thus $u = i_a + j_a$ for some a . Suppose there exists $b > a$ such that $i_b + j_b > t$ but $i_b < t$. Then $d(\overline{P}^{i_b}_{t-i_b} \; \overline{\overline{P}}^{t}_{i_b+j_b-t} \; \hat{\overline{P}}^{i_b}_{j_b} \; \varepsilon) = \alpha$. Therefore $\alpha = \alpha_1\alpha_2$,

$\alpha_1 = \gamma_{r_1}(\overline{P}^{i_1}_{j_1})\ldots\gamma_{r_a}(\overline{P}^{i_a}_{j_a})$, $\alpha_2 = \gamma_{r_{a+1}}(\overline{P}^{i_{a+1}}_{j_{a+1}})\ldots\gamma_{r_n}(\overline{P}^{i_n}_{j_n})$, where $k \in T(\alpha_1)$ implies $i \leq k \leq u$ and $k \in T(\alpha_2)$ implies $i' \leq k \leq i_n + j_n$, $i' = \min\{i_{a+1},\ldots,i_n\}$. Since the argument is valid for every such t , this completes the proof.

Theorem II.5.4: Let $\alpha = \gamma_{r_1}(\overline{P}^{i_1}_{j_1})\ldots\gamma_{r_n}(\overline{P}^{i_n}_{j_n})$ where $r_a = 1$ if $j_a > 1$. Let $S(\alpha) = \{i_1,\ldots,i_n\}$ and $T(\alpha) = \{i_1,\ldots,i_n,i_1+j_1,\ldots,i_n+j_n\}$.

Then α is a nonbounding cycle if and only if:

i) For no a and b is $i_b = i_a + j_a$.

ii) If $a < b$, $i_a + j_a = i_b + j_b$ if and only if $j_a = 1$ and there exists exactly one c such that $i_c = i_a$; if $a < c$, $i_a = i_c$ if and only if $j_a = 1$ and there exists exactly one b such that $i_b + j_b = i_a + j_a$.

iii) Either $T(\alpha)$ is a sequence or $\alpha = \prod_p \alpha_p$ where each $T(\alpha_p)$ is a sequence and $T(\alpha_p) \cap T(\alpha_q) = \Phi$.

If α satisfies i) - iii), the homology class $\bar{\alpha}$ of α is determined by $S(\alpha)$, $T(\alpha)$, and $R(\alpha) = \gamma_{r_1-t_1}(\bar{P}_{j_1}^{i_1}) \cdots \gamma_{r_n-t_n}(\bar{P}_{j_n}^{i_n})$, where $t_a = 1$ unless $j_a = 1$ and there exists b such that $i_b + j_b = i_a + 1$, when $t_a = 0$.

<u>Proof</u>: The previous three lemmas imply the necessity of conditions i) - iii). Conversely, suppose these conditions are satisfied. If no $j_a > 1$, there is nothing to prove. Choose any a such that $j_a > 1$ and let $\omega = \bar{P}_h^{i_a} \bar{P}_{j_a-h}^{i_a+h} \wedge \bar{P}_{j_a}^{i_a} \varepsilon$ for any h such that $1 \leq h \leq j_a - 1$. (If $h = 1$ and $r(P_1^{i_a}) > 0$ or if $j_a - h = 1$ and $r(P_1^{i_a+h}) > 0$, then ω is to be interpreted as the chain with the corresponding r raised by one.) Then $d(\omega) = \alpha + \alpha' + \beta$, where α' satisfies the hypotheses on α and β (if present) is a boundary: by iii), $i_a + h \in T(\alpha)$ and by ii) there exists one and only one b such that $i_b + j_b$ or i_b equals $i_a + h$ and $t_b = 1$; let $\alpha' = \sigma(\omega)$,

$\sigma: [P_{j_b}^{i_b} P_{j_a-h}^{i_a+h}] = P_{j_a+j_b-h}^{i_b}$ or $[P_h^{i_a} P_{j_b}^{i_b}] = P_{h+j_b}^{i_a}$. If there exists c such that $i_c + j_c$ or i_c equals $i_a + h$ and $t_c = 0$ (hence $j_c = 1$) , then the resulting term β of $d(\omega)$ is a boundary by Lemma II.5.2. That $R(\alpha') = R(\alpha)$, $S(\alpha') = S(\alpha)$, and $T(\alpha') = T(\alpha)$ is easily checked. Clearly this implies that α is not a boundary. It remains to prove that every nonbounding cycle α' such that $R(\alpha') = R(\alpha)$, $S(\alpha') = S(\alpha)$, and $T(\alpha') = T(\alpha)$ is congruent to α . By Lemma II.4.2, we may assume $z(\alpha) = 0$ and $z(\alpha') = 0$. Then we have $T(\alpha') = T(\alpha)$, $n(\alpha') = n(\alpha)$ and we may write $\alpha' = \gamma_{s_1}(\overline{P}_{v_1}^{u_1})\ldots\gamma_n(\overline{P}_{v_n}^{u_n})$. Let a be the smallest integer such that $\gamma_{r_a}(\overline{P}_{j_a}^{i_a}) \neq \gamma_{s_a}(\overline{P}_{v_a}^{u_a})$ and proceed by induction on $n-a$. Clearly $i_a + j_a = u_a + v_a$. Since $R(\alpha') = R(\alpha)$ and $z(\alpha') = z(\alpha) = 0$, we may assume $i_a > u_a$, $r_a = 1$, and $s_a = 1$. Consider

$$\omega = \gamma_{r_1}(\overline{P}_{j_1}^{i_1})\ldots\gamma_{r_{a-1}}(\overline{P}_{j_{a-1}}^{i_{a-1}})\overline{P}_{v_a-j_a}^{u_a}\ \overline{P}_{j_a}^{i_a}\gamma_{s_{a+1}}(\overline{P}_{v_{a+1}}^{u_{a+1}})\ldots\gamma_{s_n}(\overline{P}_{v_n}^{u_n}) .$$

$d(\omega) = \alpha' + \alpha''$, where $\alpha'' = \sigma(\omega)$, $\sigma: [P_{v_a-j_a}^{u_a}\ P_{v_b}^{u_b}] = P_{v_a+v_b-j_a}^{u_a}$, which occurs since for some $b > a$, $u_b = i_a$ (because $i_a \in S(\alpha')$) . The first a factors of γ'' agree with those of γ and, by induction, the result is proven.

<u>Corollary II.5.5</u>: Let $\alpha = \gamma_{r_1}(\overline{P}_{j_1}^{i_1})\ldots\gamma_{r_n}(\overline{P}_{j_n}^{i_n})$, $r_a = 1$ if $j_a > 1$, be a nonbounding cycle. Then the homology class $\overline{\alpha}$ of α is primitive if and only if $R(\alpha) = 1$, $T(\alpha)$ is a sequence, and α cannot be expressed as $\alpha_1\alpha_2$ where both $T(\alpha_1)$ and $T(\alpha_2)$ are sequences.

Proof: If no $j_a > 1$, $\overline{\alpha}$ is primitive if and only if $\alpha = \overline{P}_1^i$. Let $\alpha' = (\overline{P}_{j_1}^{i_1})^{t_1} \cdots (\overline{P}_{j_n}^{i_n})^{t_n}$, where t_a is as defined in the theorem. Then α' and $R(\alpha)$ are nonbounding cycles such that $\alpha' \otimes R(\alpha)$ is a summand of $D(\gamma)$. If $\overline{\alpha}$ is primitive, $\alpha = \alpha'$. The necessity of the conditions is now clear, and sufficiency is obvious.

Remarks II.5.6: We obtain here a canonical representative cycle for each of the primitive homology classes given by the corollary. Let $\alpha = \overline{P}_{j_1}^{i_1} \cdots \overline{P}_{j_n}^{i_n}$ represent such a class. If $n = 1$, $\alpha = \overline{P}_1^i$. Suppose we have determined a canonical representative cycle for each class of homological dimension less than n, $n > 1$. We will prove that $\alpha \equiv \beta \overline{P}^i_{i_n+j_n-i}$, $i = \min\{i_a\}$. Then β is a nonbounding cycle and $\beta \equiv \prod_p \beta_p$, where each β_p is the canonical representative of a primitive class. $(\prod_p \beta_p)\overline{P}^i_{i_n+j_n-i}$ is the desired cycle. Thus suppose $i = i_a$. If $a = n$, we are finished. Proceed by induction on $n - a$. There exists $b > a$ such that $i_b < j_a + i$ (since otherwise $T(\overline{P}_{j_1}^{i_1} \cdots \overline{P}_{j_a}^{i_a})$ would be a sequence). Then $d(\overline{P}_{j_1}^{i_1} \cdots \overline{P}^i_{i_b-i}\overline{P}^{i_b}_{j_a+i-i_b}\hat{\overline{P}}^i_{j_a} \cdots \overline{P}_{j_n}^{i_n}) = \alpha + \alpha'$, α' resulting from $[P^i_{i_b-i}P^{i_b}_{j_b}] = P^i_{i_b+j_b-i}$. Since $n-b < n-a$, this proves the result.

Let a_n denote the number of primitive homology classes $\overline{\alpha}$ satisfying $T(\alpha) = \{i, i+1, \ldots, i+2n-1\}$ for fixed i. Using the result just obtained, each such class has a canonical representative cycle of the form $\prod_{p=1}^{m} \beta_p \bar{P}^{i}_{2n-1}$. a_n is the number of possible choices for $\prod_{p=1}^{m} \beta_p$. If $p = 1$, β_1 is primitive and there are a_{n-1} such choices. If β_1 has homological degree j , there are a_j choices for β_1 , and, since $\prod_{p=2}^{m} \beta_p$ has homological degree $n-1-j$, there are a_{n-j} choices for $\prod_{p=2}^{m} \beta_p$. Thus if $n \geq 2$,

$$a_n = a_{n-1} + \sum_{j=1}^{n-2} a_j a_{n-j} = \sum_{j=o}^{n} a_j a_{n-j} \text{ , where } a_o = 0 \text{ and } a_1 = 1 .$$

It follows that $a_n = \frac{1}{n}(n-1,n-1)$ for $n \geq 2$ (as is seen by forming a power series $y = \sum_{i=o}^{\infty} a_i x^i$; y satisfies $a_o = 0$, $a_1 = 1$, and $y^2-y+x = 0$, hence $y = \frac{1}{2} \pm \frac{\sqrt{1-4x}}{2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - x}$; expanding $\sqrt{\frac{1}{4} -x}$ in a binomial series, the result is obtained).

Now if $\alpha = \bar{P}^{i_1}_{j_1} \cdots \bar{P}^{i_n}_{j_n}$ represents a primitive class $\bar{\alpha}$, $\bar{\alpha}$ is determined by $i = \min\{i_a\}$ and by $S' = \{s_2-i,\ldots,s_n-i\}$, where $S = \{i,s_2,\ldots,s_n\}$ with elements of S written in increasing order, $s_a < s_b$ if $a < b$. We denote the cohomology class dual to $\bar{\alpha}$ by $h_i(S')$ if $n \geq 2$ and by h_i if $n = 1$. There is just one representative cocycle for $h_i(S')$, namely the sum of the cochains dual to the chains representing $\bar{\alpha}$. If $n = 2$, the cohomology classes of this form are just $h_i(1)$ represented by $R_1^{i+1} R_3^{i} + R_2^{i} R_2^{i+1}$. If $n = 3$, the cohomology classes of this form are just $h_i(1,3)$ represented by $R_1^{i+1} R_1^{i+3} R_5^{i} + R_1^{i+1} R_4^{i} R_2^{i+3} + R_2^{i} R_1^{i+3} R_4^{i+1} + R_2^{i} R_3^{i+1} R_2^{i+3}$ and $h_i(1,2)$ represented by

$R_1^{i+2}R_3^{i+1}R_5^{i} + R_2^{i+1}R_2^{i+2}R_5^{i} + R_1^{i+2}R_4^{i}R_4^{i+1} + R_3^{i}R_2^{i+2}R_4^{i+1} + R_2^{i+1}R_4^{i}R_3^{i+2} + R_3^{i}R_3^{i+1}R_3^{i+2}$.

Extensive calculations in homology have led me to the

Conjecture II.5.7: The elements b_j^i $(j > 1)$, h_i, and $h_i(S')$ defined above form a basis for the indecomposable elements of $H^*(V(L))$.

To further study $H^*(V(L))$, we introduce a sequence of spectral sequences. These are, as in the case $p > 2$, essentially of the same type as those introduced by Adams to facilitate the calculation of the cohomology of the Steenrod algebra using the cobar construction. The setting up of these spectral sequences is a triviality. Let $\overline{X}_n^*$ be the differential sub-algebra of $\overline{X}^*$ generated by $\{R_j^i | j \leq n\}$. As an algebra, we may take $\overline{X}_n^* = \overline{X}_{n-1}^* \otimes Z_n$, $Z_n = P\{R_n^i\}$. Filter $\overline{X}_n^*$ by $x \otimes z \in F^s\overline{X}_n^*$ if and only if x has homological degree greater than or equal to s, $x \in \overline{X}_{n-1}^*$, $z \in Z_n$, and $\Sigma\, x_i \otimes z_i \in F^s\overline{X}_n^*$ if and only if some $x_i \otimes z_i \in F^s\overline{X}_n^*$, $x_i \in \overline{X}_{n-1}^*$, $z_i \in Z_n$. Then $F^s\overline{X}_n^*$ is a differential sub-algebra of $F^{s-1}\overline{X}_n^*$, and therefore ${}_nE_o$ is a differential algebra, ${}_nE_o^{s,t} = (F^s\overline{X}_n^*/F^{s+1}\overline{X}_n^*)_{s+t}$, $s+t$ being the total homological degree. Consider the resulting spectral sequence. $\delta_o = 0$, hence ${}_nE_o^{s,t} = {}_nE_1^{s,t}$. δ_1 is given by $\delta_1(R_j^i) = e_{i,j}$ if $j < n$ and $\delta_1(R_n^i) = 0$. Therefore ${}_nE_2^{s,t} = H^s(\overline{X}_{n-1}^*) \otimes Z_n^t$. δ_2 is given by $\delta_2(R_n^i) = e_{i,n}$ and convergence is immediate, ${}_nE_3^{s,t} = {}_nE_\infty = E_{s,t}^o H^*(\overline{X}_n^*)$. Since $H^{s,t}(\overline{X}_n^*) = H^{s,t}(\overline{X}^*)$ for $t < 2^n-1$, in order to compute $H^*(\overline{X}^*)$, it suffices to calculate the $H^*(\overline{X}_n^*)$ successively.

As in the case $p > 2$, the spectral sequences $\{{}_nE_r\}$ have an obvious interpretation: Let A_n^* be the Hopf subalgebra of the dual of the Steenrod algebra generated by $\{1,\xi_1,\dots,\xi_n\}$ (see page II-1.3 for the notation), let $C_n = (A_n^*)^*$ and let $B_n = (A_n^* /\!/ A_{n-1}^*)^* \subset C_n$. Let $E^0C_n = E_n^0$ and note that $E^0B_n = B_n$ (with grading by total degree). Then $E_n^0 = V(L_n)$, $E^0B_n = V(L_n')$ where L_n' is Abelian, and we have ${}_nE_2^{s,t} = H^s(V(L_{n-1})) \otimes H^t(V(L_n'))$ and ${}_nE_\infty^{s,t} = E^0_{s,t}H^*(V(L_n))$.

We now proceed with the calculation of the spectral sequences. Recall that b_j^i is the cohomology class of $(R_j^i)^2$, h_i of R_1^i , and $h_i(S')$ of the dual of that primitive class $\overline{\alpha}$ of $H_*(\overline{X}^*)$ found in Corollary II.5.5 which satisfies $S(\alpha) = \{i,s_1' + i,\dots,s_n' + i\}$, where $S' = \{s_1',\dots,s_n'\}$. Recall also that $e_{i,j}$ denotes $\sum_{k=1}^{j-1} R_k^i R_{j-k}^{i+k}$. The notation $\varepsilon_{i,j} = h_i R_{j-1}^{i+1} + R_{j-1}^i h_{i+j-1}$ will be needed for $j > 3$. The symbol $\Phi_i(S')$ will denote the cochain defined as follows: by Remarks II.5.6, the class $\overline{\alpha}$ dual to $h_i(S')$, $S' = \{s_1',\dots,s_n'\}$, has a canonical representative cycle $\prod_p \beta_p \overline{P}^i_{2n+1}$; $\Phi_i(S')$ will denote the product of the unique cocycle representing the class dual to $\prod_p \beta_p$ and of R^i_{2n+1} .

We will calculate $H^*(\overline{X}_2^*)$ and $H^*(\overline{X}_3^*)$ completely and will partially calculate $H^*(\overline{X}_4^*)$ and $H^*(\overline{X}_5^*)$. Proofs will be brief, but we will discuss certain phenomena arising in the computation of the spectral sequences in Remarks II.5.10 and II.5.14. We will conclude by stating the structure of $H^{s,t}(E^0A)$ for small s and t .

<u>Lemma II.5.8</u>: $H^*(\bar{X}_2^*)$ is generated by $\{h_i, b_2^i, e_{i,3}\}$. All relations among these elements are generated by the following:

1. $h_i e_{i+1,3} = h_{i+3} e_{i,3}$
2. $e_{i,3}^2 = h_i^2 b_2^{i+1} + b_2^i h_{i+2}^2$
3. $e_{i,3} e_{i+1,3} = h_i b_2^{i+1} h_{i+3}$
4. $h_{i+1} e_{i,3} = 0$
5. $h_{i+1} h_i = 0$

<u>Proof</u>: Calculating in $\bar{X}_2^*$, $\delta(R_2^i R_2^{i+2}) = h_i e_{i+1,3} + h_{i+3} e_{i,3}$, $\delta(R_2^i R_2^{i+1}) = h_{i+1} e_{i,3}$, and $\delta(R_2^i R_2^{i+1} R_2^{i+2}) = e_{i,3} e_{i+1,3} + h_i b_2^{i+1} h_{i+3}$ (where we have used the same notation for a cohomology class and its representative cocycle).

<u>Lemma II.5.9</u>: A basis for those indecomposable elements of ${}_3E_3 = {}_3E_\infty$ not in $H^*(\bar{X}_2^*)$ is given by $\{b_3^i, \varepsilon_{i,4}, \Phi_i(1)\}$ $(\Phi_i(1) = h_{i+1} R_3^i)$. The algebra structure of ${}_3E_\infty$ is determined by the relations:

1. $h_{i+1} h_i = 0$, $h_i^2 b_2^{i+1} + b_2^i h_{i+2}^2 = 0$, $h_i b_2^{i+1} h_{i+3} = 0$
2. $h_i \varepsilon_{i+1,4} = h_{i+4} \varepsilon_{i,4}$
3. $\varepsilon_{i,4}^2 = h_i^2 b_3^{i+1} + b_3^i h_{i+3}^2$
4. $\varepsilon_{i,4} \varepsilon_{i+1,4} = h_i b_3^{i+1} h_{i+4} + \Phi_i(1) \Phi_{i+2}(1)$
5. $h_{i+1} \varepsilon_{i,4} = h_{i+3} \Phi_i(1)$

   $h_{i+2} \varepsilon_{i,4} = h_i \Phi_{i+1}(1)$

6. $\Phi_i(1)\varepsilon_{i,4} = h_{i+1}b_3^i h_{i+3}$

$\Phi_{i+1}(1)\varepsilon_{i,4} = h_i b_3^{i+1} h_{i+2}$

7. $h_i\Phi_i(1) = 0$

$h_{i+2}\Phi_i(1) = 0$

8. $\Phi_i(1)^2 = h_{i+1}^2 b_3^i$

$\Phi_i(1)\Phi_{i+1}(1) = 0$

Proof: That no other indecomposable elements occur is clear. The relations follow easily from $\delta_2(R_3^i) = e_{i,3}$ and the algebra structure of $H^*(\bar{X}_2^*)$.

Remarks II.5.10: We are using Greek letters to denote elements of ${}_nE_\infty$, the corresponding cochains of which are not cocycles in $\bar{X}_n^*$. Thus $\Phi_i(1)$ corresponds to $R_1^{i+1}R_3^i$ which must be lifted to $R_1^{i+1}R_3^i + R_2^i R_2^{i+1}$ to obtain a cocycle in $\bar{X}_3^*$. Those relations in ${}_nE_\infty$ involving at least one element $\varepsilon_{i,n+1}$ or $\Phi_i(S')$ must be studied in $\bar{X}_n^*$ with $e_{i,n+1}$ and $h_i(S')$ replacing $\varepsilon_{i,n+1}$ and $\Phi_i(S')$ (and similarly with any other such elements which may arise). Relations in ${}_nE_\infty$ not involving such elements pass unchanged to $H^*(\bar{X}_n^*)$. Note that, for simplicity, we are using the same notation for elements represented by a given cochain of $\bar{X}^*$ no matter in which spectral sequence or $H^*(\bar{X}_n^*)$ they are considered. Relations of the form $h_i\varepsilon_{i+1,n+1} = h_{i+n+1}\varepsilon_{i,n+1}$ occur in each ${}_nE_\infty$, give rise to the relations $h_i e_{i+1,n+1} = h_{i+n+1}e_{i,n+1}$ in $H^*(\bar{X}_n^*)$, and therefore $\varepsilon_{i,n+2}$ is a cocycle of ${}_{n+1}E_2$.

Lemma II.5.11: The algebra structure of $H^*(\overline{X}_3^*)$ is given by relations 1 , 2 , 4 , and 5 of Lemma II.5.9 with $\varepsilon_{i,4}$ replaced by $e_{i,4}$ and $\Phi_i(1)$ replaced by $h_i(1)$ and by the following relations:

3'. $$e_{i,4}^2 = h_i^2 b_3^{i+1} + b_2^i b_2^{i+2} + b_3^i h_{i+3}^2$$

6'. $$h_i(1)e_{i,4} = h_{i+1}b_3^i h_{i+3} + b_2^i h_{i+1}(1)$$
$$h_{i+1}(1)e_{i,4} = h_i b_3^{i+1} h_{i+2} + b_2^{i+2} h_i(1)$$

7'. $$h_i h_i(1) = b_2^i h_{i+2}$$
$$h_{i+2} h_i(1) = h_i b_2^{i+1}$$

8'. $$h_i(1)^2 = h_{i+1}^2 b_3^i + b_2^i b_2^{i+1}$$
$$h_i(1)h_{i+1}(1) = b_2^{i+1} e_{i,4}$$

Proof: In $\overline{X}_3^*$, $\delta(R_2^i R_3^i) = h_i h_i(1) + b_2^i h_{i+2}$ and $\delta(R_2^i R_3^{i+1}) = h_{i+2} h_i(1) + h_i b_2^{i+1}$, which prove 7' . The proofs of the remaining relations are equally simple.

We furnish no proofs for the remainder of this section, since the methods are obvious. We shall not list the relations in ${}_nE_3$ and in $H^*(\overline{X}_n^*)$ holding among those elements in the image of $H^*(\overline{X}_{n-1}^*)$, as these are given by the statement that the image in ${}_nE_\infty$ and in $H^*(\overline{X}_n^*)$ of $H^*(\overline{X}_{n-1}^*)$ is $H^*(\overline{X}_{n-1}^*)/I$, where I is the ideal in $H^*(\overline{X}_{n-1}^*)$ generated by $\{e_{i,n}\}$.

Lemma II.5.12: A basis for those indecomposable elements of ${}_4E_3$ not in $H^*(\overline{X}_3^*)$ is given by $\{b_4^i, \varepsilon_{i,5}, \psi_i(4), \chi_i(4)\}$ where $\psi_i(4)$ is represented by $h_i(1)h_{i+5}R_4^{i+3} + h_{i+1}h_{i+4}(1)R_4^i$ and $\chi_i(4)$ is represented by $h_{i+1}h_{i+4}h_{i+7}R_4^iR_4^{i+5}$. Those relations in ${}_4E_3$ not holding in the image of $H^*(\overline{X}_3^*)$ and not involving $\chi_i(4)$ are generated by:

1. $h_i\varepsilon_{i+1,5} = h_{i+5}\varepsilon_{i,5}$

2. $h_{i+1}h_{i+3}\varepsilon_{i,5} = 0$

   $h_{i+1}(1)\varepsilon_{i,5} = 0$

3. $h_{i+1}h_{i+5}(1)\varepsilon_{i,5} = h_{i+6}h_i(1)\varepsilon_{i+3,5}$

4. $h_{i+2}\varepsilon_{i,5}\varepsilon_{i+1,5} = h_ih_{i+2}b_4^{i+1}h_{i+5}$

   $h_{i+3}\varepsilon_{i,5}\varepsilon_{i+1,5} = h_ih_{i+3}b_4^{i+1}h_{i+5}$

5. $\varepsilon_{i,5}^2 = h_i^2b_4^{i+1} + b_4^ih_{i+4}^2$

6. $\varepsilon_{i,5}\psi_i(4) + b_2^i\varepsilon_{i+1,5}\varepsilon_{i+2,5} = h_{i+1}h_{i+6}(b_2^ib_4^{i+2} + b_4^ib_2^{i+4})$

   $\varepsilon_{i+2,5}\psi_i(4) + b_2^{i+5}\varepsilon_{i,5}\varepsilon_{i+1,5} = h_ih_{i+5}(b_2^{i+1}b_4^{i+3} + b_4^{i+1}b_2^{i+5})$

7. $\varepsilon_{i+3,5}\psi_i(4) = h_i(1)h_{i+5}h_{i+7}b_4^{i+3}$

   $\varepsilon_{i,5}\psi_{i+1}(4) = h_ih_{i+2}h_{i+5}(1)b_4^{i+1}$

8. $h_{i+3}\psi_i(4) = 0$

9. $h_i\psi_i(4) = h_{i+5}b_2^i\varepsilon_{i+2,5}$

$h_{i+2}\psi_i(4) = h_i(1)h_{i+5}\varepsilon_{i+2,5}$

$h_{i+4}\psi_i(4) = h_{i+1}h_{i+4}(1)\varepsilon_{i,5}$

$h_{i+6}\psi_i(4) = h_{i+1}b_2^{i+5}\varepsilon_{i,5}$

10. $h_{i+1}(1)\psi_i(4) = 0$

$h_{i+2}(1)\psi_i(4) = 0$

$h_{i+3}(1)\psi_i(4) = 0$

11. $b_2^{i+2}\psi_i(4) = h_i b_3^{i+1}h_{i+5}\varepsilon_{i+2,5}$

$b_2^{i+3}\psi_i(4) = h_{i+1}b_3^{i+3}h_{i+6}\varepsilon_{i,5}$

<u>Lemma II.5.13</u>: With $\varepsilon_{i,4}$ and $\psi_i(4)$ replaced by $e_{i,4}$ and $f_i(4)$, relations 1 - 3 and 8 - 11 of the previous lemma hold in $H^*(\overline{X}_4^*)$. Relations 4 - 7 correspond to the following relations in $H^*(\overline{X}_4^*)$ ($f_i(4)$ is represented by $h_i(1)(R_1^{i+5}R_4^{i+3} + R_2^{i+5}R_3^{i+3}) + h_{i+4}(1)$ $(R_2^iR_3^{i+1} + R_1^{i+1}R_4^i)$):

4'. $h_{i+2}e_{i,5}e_{i+1,5} = h_ih_{i+2}b_4^{i+1}h_{i+5} + b_3^{i+2}h_i(1)h_{i+5}$

$h_{i+3}e_{i,5}e_{i+1,5} = h_ih_{i+3}b_4^{i+1}h_{i+5} + h_ib_3^{i+1}h_{i+3}(1)$

5'. $e_{i,5}^2 = h_i^2b_4^{i+1} + b_2^ib_3^{i+2} + b_3^ib_2^{i+3} + b_4^ih_{i+4}^2$

6'. $e_{i,5}f_i(4) + b_2^ie_{i+1,5}e_{i+2,5} = h_{i+1}h_{i+6}(b_2^ib_4^{i+2} + b_3^ib_3^{i+3} + b_4^ib_2^{i+4})$

$e_{i+2,5}f_i(4) + b_2^{i+5}e_{i,5}e_{i+1,5} = h_ih_{i+5}(b_2^{i+1}b_4^{i+3} + b_3^{i+1}b_3^{i+4} + b_4^{i+1}b_2^{i+5})$

7'. $e_{i+3,5}f_i(4) = h_i(1)h_{i+5}h_{i+7}b_4^{i+3} + h_i(1)b_3^{i+3}h_{i+5}(1)$

$e_{i,5}f_{i+1}(4) = h_ih_{i+2}h_{i+5}(1)b_4^{i+1} + h_i(1)b_3^{i+2}h_{i+5}(1)$

Remarks II.5.14: We comment first on the significance of the elements $\psi_i(4) \in {}_4E_\infty$ and $f_i(4) \in H^*(\overline{X}_4^*)$ and of the relations involving these elements. Relations 6' imply that $h_{i+1}h_{i+6}e_{i,6}^2 = 0$ and $h_ih_{i+5}e_{i+1,6}^2 = 0$ in $H^*(\overline{X}_5^*)$. But these relations must hold because in ${}_6E_2$ we have $h_{i+1}h_{i+6}e_{i,6}R_6^i = h_{i+1}e_{i,6}\varepsilon_{i,7}$ and $h_ih_{i+5}e_{i+1,6}R_6^{i+1} = h_{i+5}e_{i+1,6}\varepsilon_{i,7}$, both of which must be cocycles in ${}_6E_2$. Relations 7' of course give relations in the image of $H^*(\overline{X}_4^*)$ in ${}_5E_3$. Calculating in $\overline{X}_6^*$, we find that the relation $h_{i+1}h_{i+5}\varepsilon_{i,7} = 0$ in ${}_6E_3$ passes to the relation $h_{i+1}h_{i+5}e_{i,7} = f_i(4)$. Then the interpretation of relations 8 - 11 becomes obvious: 8 implies that $\Phi_i(1,3,5)$ is indecomposable in ${}_7E_3$; 9 - 11 are necessary in order that the products of certain cocycles in $H^*(\overline{X}_3^*)$ with the cocycles $\Phi_i(1,2,5)$ and $\Phi_i(1,3,4)$ of ${}_7E_2$ be cocycles. Next, we note that because the relation $h_{i+1}h_{i+5}(1)\varepsilon_{i,5} = h_{i+6}h_i(1)\varepsilon_{i+3,5}$ occurs in ${}_4E_3$ and passes to the corresponding relation with $e_{i,5}$ and $e_{i+3,5}$ replacing $\varepsilon_{i,5}$ and $\varepsilon_{i+3,5}$ in $H^*(\overline{X}_4^*)$, the element $h_{i+1}h_{i+5}(1)R_5^i + h_{i+6}h_i(1)R_5^{i+3}$ of ${}_5E_2$ is a cocycle which represents an indecomposable element $\psi_i(5)$ of ${}_5E_\infty$. $\psi_i(5)$ gives rise to an element $f_i(5)$ of $H^*(\overline{X}_5^*)$. Then the relation $h_{i+1}h_{i+6}\varepsilon_{i,8} = 0$ in ${}_7E_3$ yields the relation $h_{i+1}h_{i+6}e_{i,8} = f_i(5)$ in $H^*(\overline{X}_7^*)$. Finally, we note the significance of the elements $X_i(4) \in {}_4E_\infty$ and of the elements $g_i(4) \in H^*(\overline{X}_4^*)$ to which they give rise. In ${}_5E_3$, we have the relations $h_{i+1}h_{i+6}(1)\varepsilon_{i,6} = 0$ and $h_i(1)h_{i+7}\varepsilon_{i+3,6} = 0$. These give rise to $h_{i+1}h_{i+6}(1)e_{i,6} = g_i(4) = h_i(1)h_{i+7}e_{i+3,6}$ in $H^*(\bar{X}_5^*)$. Further, the resulting indecomposable elements $\omega_i(6) = h_{i+1}h_{i+6}(1)R_6^i + h_i(1)h_{i+7}R_6^{i+3}$ of ${}_6E_3$ give rise to elements $w_i(6) \in H^*(\bar{X}_6^*)$. Then the relations $h_{i+1}h_{i+7}\ \varepsilon_{i,9} = 0$ in ${}_8E_3$ pass to the relations $h_{i+1}h_{i+7}e_{i,9} = w_i(6)$ in $H^*(\bar{X}_8^*)$. I conjecture that this behavior generalizes as follows: In ${}_{n-1}E_3$, $n \geq 4$, we have $h_{i+1}h_{i+n}h_{i+2n-1}e_{i,n}e_{i+n+1,n} = 0$. There result indecomposable elements $X_i(n) = h_{i+1}h_{i+n}h_{i+2n-1}R_n^iR_n^{i+n+1}$ of ${}_nE_3$ and these give rise to elements $g_i(n)$ of $H^*(\bar{X}_n^*)$. The relations $h_{i+1}h_{i+n+2}(1)\varepsilon_{i,n+2} = 0$ and $h_i(1)h_{i+n+3}\varepsilon_{i+3,n+2} = 0$ in ${}_{n+1}E_3$ give rise to $h_{i+1}h_{i+n+2}(1)e_{i,n+2} = g_i(n) = h_i(1)h_{i+n+3}e_{i+3,n+2}$ in $H^*(\bar{X}_{n+1}^*)$. The resulting indecomposable elements $\omega_i(n+2) = h_{i+1}h_{i+n+2}(1)R_{n+2}^i + h_i(1)h_{i+n+3}R_{n+2}^{i+3}$ of ${}_{n+2}E_\infty$ pass to $w_i(n+2) \in H^*(\bar{X}_{n+2}^*)$. Then the relations $h_{i+1}h_{i+n+3}\varepsilon_{i,n+5} = 0$ in ${}_{n+4}E_\infty$ pass to the relations $h_{i+1}h_{i+n+3}e_{i,n+5} = w_i(n+2)$ in $H^*(\bar{X}_{n+4}^*)$.

<u>Conventions II.5.15</u>: The letter t will denote the grading derived from that of the Steenrod algebra and the letter s will denote the homological degree. The notation $x \in (s,t)$ will mean that x is an element (of any group under consideration) with homological degree s and grading t. For example, $h_i \in (1,2^i)$. Since $t-s$ is the total degree in the Adams spectral sequence, this dimension will be of particular interest to us.

Lemma II.5.16: A basis for those indecomposable elements of ${}_5E_3$ not in $H^*(\bar{X}_4^*)$ is given by $\{b_5^i, \varepsilon_{i,6}, \Phi_i(1,3), \Phi_i(1,2), \psi_i(5), \chi_i(5)\}$. Aside from those relations holding in the image of $H^*(\bar{X}_4^*)$, all elements of a defining set of relations for ${}_5E_3$ in the range $t-s < 165$ are included among the following:

1. $h_i \varepsilon_{i+1,6} = h_{i+6} \varepsilon_{i,6}$

2. $h_{i+1} h_{i+4} \varepsilon_{i,6} = 0$

3. $\varepsilon_{i,6}^2 = h_i^2 b_5^{i+1} + h_{i+5}^2 b_5^i$

4. $h_{i+2}(1) \varepsilon_{i,6} = h_i \Phi_{i+1}(1,2)$

   $h_{i+1}(1) \varepsilon_{i,6} = \Phi_i(1,2) h_{i+5}$

   $h_{i+2} h_{i+4} \varepsilon_{i,6} = h_i \Phi_{i+1}(1,3)$

   $h_{i+1} h_{i+3} \varepsilon_{i,6} = \Phi_i(1,3) h_{i+5}$

5. $h_i \Phi_i(1,2) = 0$

   $h_{i+4} \Phi_i(1,2) = 0$

6. $h_i \Phi_i(1,3) = 0$

   $h_{i+2} \Phi_i(1,3) = 0$

   $h_{i+4} \Phi_i(1,3) = 0$

7. $h_{i+1} h_{i+3} \Phi_i(1,2) = h_{i+1}(1) \Phi_i(1,3)$

8. $h_{i+1}^2 \Phi_i(1,2) = b_2^{i+1} \Phi_i(1,3)$

   $h_{i+3}^2 \Phi_i(1,2) = b_2^{i+2} \Phi_i(1,3)$

9. $b_2^{i+3}\Phi_i(1,2) = b_3^{i+2}\Phi_i(1,3)$

$b_2^i\Phi_i(1,2) = b_3^i\Phi_i(1,3)$

10. $b_2^i\Phi_{i+1}(1,3) = h_i(1)h_{i+4}\varepsilon_{i,6}$

$b_2^{i+4}\Phi_i(1,3) = h_{i+1}h_{i+3}(1)\varepsilon_{i,6}$

11. $h_i(1)\Phi_i(1,2) = 0$

$h_{i+2}(1)\Phi_i(1,2) = 0$

$h_i(1)\Phi_i(1,3) = 0$

$h_{i+2}(1)\Phi_i(1,3) = 0$

12. $h_i(1)\Phi_{i+1}(1,2) = b_3^{i+1}h_{i+4}\varepsilon_{i,6}$

$h_{i+3}(1)\Phi_i(1,2) = h_{i+1}b_3^{i+2}\varepsilon_{i,6}$

$h_i(1)\Phi_{i+1}(1,3) = b_2^{i+1}h_{i+4}\varepsilon_{i,6}$

$h_{i+3}(1)\Phi_i(1,3) = h_{i+1}b_2^{i+3}\varepsilon_{i,6}$

13. $h_i(1)\Phi_i(1,2) = 0$

$h_{i+2}(1)\Phi_i(1,2) = 0$

$h_i(1)\Phi_i(1,3) = 0$

$h_{i+2}(1)\Phi_i(1,3) = 0$

14. $\Phi_i^2(1,2) = h_{i+1}^2(1)b_5^i$

$\Phi_i^2(1,3) = h_{i+1}^2h_{i+3}^2b_5^i$

$\Phi_i(1,2)\Phi_i(1,3) = b_2^{i+1}h_{i+3}^2b_5^i$

15. $\Phi_i(1,2)\Phi_{i+1}(1,2) = 0$

$\Phi_i(1,3)\Phi_{i+1}(1,3) = 0$

$\Phi_i(1,2)\Phi_{i+1}(1,3) = 0$

$\Phi_i(1,3)\Phi_{i+1}(1,2) = 0$

<u>Lemma II.5.17</u>: Relations 1, 4, 7, and 12 of the previous lemma hold in $H^*(\bar{X}_5^*)$ (with $\varepsilon_{i,6}$, $\Phi_i(1,2)$, and $\Phi_i(1,3)$ replaced by $e_{i,6}, h_i(1,2)$, and $h_i(1,3)$). The relations in $H^*(\bar{X}_5^*)$ corresponding to 2, 3, and 10 are:

2'. $h_{i+1}h_{i+4}e_{i,6} = h_i(1)h_{i+3}(1)$

3'. $e_{i,6}^2 = h_i^2 b_5^{i+1} + \sum_{j=2}^{4} b_j^i b_{6-j}^{i+j} + b_5^i h_{i+5}^2$

10'. $b_2^i h_{i+1}(1,3) = h_i(1)h_{i+4}e_{i,6} + h_{i+1}b_3^i h_{i+3}(1)$

$b_2^{i+4}h_i(1,3) = h_{i+1}h_{i+3}(1)e_{i,6} + h_{i+4}b_3^{i+3}h_i(1)$

The products on the left in relations 15 are elements of $I\{e_{i,6}\} \subset H^*(\bar{X}_5^*)$. The relations in $H^*(\bar{X}_5^*)$ corresponding to 5, 6, 8 - 11, 13, and 14 are given in the following theorem.

We have verified the truth of Conjecture II.5.7 in at least the range $t-s < 165$. The smallest value of $t-s$ taken by an indecomposable element in ${}_7E_3$ which is not in the image of $H^*(\bar{X}_6^*)$ is 165, since $\Phi_0(1,3,5) \in (4,169)$. Thus $H^*(E^0A)$ is completely determined in the range $t-s < 165$. The following theorem summarizes our results.

<u>Theorem II.5.18</u>: At least in the range $t-s < 165$, the only indecomposable elements of $H^*(E^0A)$ which are in the image of $H^*(\overline{X}_6^*)$ are the following:

i) $b_j^i \in (2, 2^{i+1}(2^j-1))$, $2 \leq j \leq 6$, $0 \leq i$

ii) $h_i \in (1, 2^i)$, $0 \leq i$

iii) $h_i(1) \in (2, 9\cdot 2^i)$, $0 \leq i$

iv) $h_i(1,3) \in (3, 41\cdot 2^i)$, $0 \leq i$

v) $h_i(1,2) \in (3, 49\cdot 2^i)$, $0 \leq i$

b_j^i and h_i are represented by $(R_j^i)^2$ and R_1^i; representative cocycles for $h_i(1)$, $h_i(1,3)$, and $h_i(1,2)$ are named on page II-5.11. The only other indecomposable elements of $H^*(E^0A)$ satisfying $s \leq 3$ are b_j^i, $j > 6$. There are no other indecomposable elements of $H^*(E^0A)$ which satisfy $t-s < 165$. The listed elements satisfy at least the following relations, and all other relations in the range $s < 4$ and in the range $t-s < 165$ are implied by these (we let $b_1^i = (h_i)^2$ to simplify the statements of certain relations):

1. $\sum_{j=1}^{k-1} b_j^i b_{k-j}^{i+j} = 0$

2. $h_i h_{i+1} = 0$

3. $h_i h_{i+1}(1) = 0$

$h_i h_i(1) = h_{i+2} b_2^i$

$h_{i+2} h_i(1) = h_i b_2^{i+1}$

$h_{i+3} h_i(1) = 0$

4. $h_i(1)^2 = b_1^{i+1} b_3^i + b_2^i b_2^{i+1}$

$h_i(1) h_{i+1}(1) = 0$

$h_i(1) h_{i+2}(1) = h_i b_3^{i+1} h_{i+4}$

$h_i(1) h_{i+3}(1) = 0$

5. $b_2^i h_{i+1}(1) = h_{i+1} b_3^i h_{i+3}$

$b_2^{i+2} h_i(1) = h_i b_3^{i+1} h_{i+2}$

6. $h_i h_{i+1}(1,3) = 0$

$h_i h_i(1,3) = b_2^i h_{i+2}(1)$

$h_{i+2} h_i(1,3) = h_i b_3^{i+1} h_{i+4}$

$h_{i+4} h_i(1,3) = b_2^{i+3} h_i(1)$

$h_{i+5} h_i(1,3) = 0$

7. $h_i h_{i+1}(1,2) = 0$

$h_i h_i(1,2) = b_3^i h_{i+2}(1) + h_{i+2} b_4^i h_{i+4}$

$h_{i+4} h_i(1,2) = b_3^{i+2} h_i(1) + h_i b_4^{i+1} h_{i+2}$

$h_{i+5} h_i(1,2) = 0$

8. $h_{i+1}h_{i+3}h_i(1,2) = h_{i+1}(1)h_i(1,3)$

$b_1^{i+1}h_i(1,2) = b_2^{i+1}h_i(1,3) + h_{i+4}b_3^{i+1}h_i(1)$

$b_1^{i+3}h_i(1,2) = b_2^{i+2}h_i(1,3) + h_i b_3^{i+1}h_{i+2}(1)$

$b_2^{i}h_i(1,2) = b_3^{i}h_i(1,3) + h_{i+4}b_4^{i}h_i(1)$

$b_2^{i+3}h_i(1,2) = b_3^{i+2}h_i(1,3) + h_i b_4^{i+1}h_{i+2}(1)$

9. $b_2^{i}h_{i+1}(1,3) = h_{i+1}b_3^{i}h_{i+3}(1)$

$b_2^{i+4}h_i(1,3) = h_{i+4}b_3^{i+3}h_i(1)$

10. $h_i(1)h_{i+1}(1,3) = 0$

$h_i(1)h_i(1,3) = h_{i+4}(b_1^{i+1}b_4^{i} + b_2^{i}b_3^{i+1})$

$h_{i+2}(1)h_i(1,3) = h_i(b_1^{i+3}b_4^{i+1} + b_3^{i+1}b_2^{i+3})$

$h_{i+3}(1)h_i(1,3) = 0$

11. $h_i(1)h_{i+1}(1,2) = 0$

$h_i(1)h_i(1,2) = h_{i+4}(b_2^{i+1}b_4^{i} + b_3^{i}b_3^{i+1})$

$h_{i+2}(1)h_i(1,2) = h_i(b_2^{i+2}b_4^{i+1} + b_3^{i+1}b_3^{i+2})$

$h_{i+3}(1)h_i(1,2) = 0$

12. $h_i(1,3)^2 = b_1^{i+1}(b_1^{i+3}b_5^{i} + b_4^{i}b_2^{i+3}) + b_2^{i}(b_1^{i+3}b_4^{i+1} + b_3^{i+1}b_2^{i+3})$

$h_i(1,3)h_{i+1}(1,3) = 0$

13. $h_i(1,2)^2 = b_1^{i+2}(b_3^{i+1}b_5^{i} + b_4^{i}b_4^{i+1}) + b_2^{i+1}(b_2^{i+2}b_5^{i} + b_4^{i}b_3^{i+2}) + b_3^{i}(b_2^{i+2}b_4^{i+1} + b_3^{i+1}b_3^{i+2})$

$h_i(1,2)h_{i+1}(1,2) = 0$

14. $h_i(1,2)h_{i+1}(1,3) = 0$

$h_i(1,3)h_{i+1}(1,2) = 0$

We remark that none of the relations 3 through 14 were derived in $H^*(\overline{X}_4^*)$; that is, the relations holding in the image of $H^*(\overline{X}_4^*)$ in ${}_5E_3$ and not in the image of $H^*(\overline{X}_3^*)$ were all implied by certain relations found in $H^*(\overline{X}_n^*)$, $n > 4$. I conjecture that such behavior occurs in each $H^*(\overline{X}_{2n}^*)$, $n \geq 2$. This is in line with Conjecture II.5.7, which, if true, would imply that, aside from the b_{2n}^i , no indecomposable elements arising in ${}_{2n}E_\infty$ could survive to non-zero elements of $H^*(E^oA)$.

6. The cohomology of the Steenrod algebra

In the previous sections, we have obtained a good deal of information about the structure of $H^*(E^0A)$. In this section, we study the spectral sequence passing from $H^*(E^0A)$ to $H^*(A)$.

We note first that E^0A is actually bigraded, $E^0_{p,q}A = (F_pA/F_{p-1}A)_{p+q}$ (the filtration of A is defined on page II-2.1). Defining a weight function w on A by $w(x) = n$ if $x \in F_{-n}A$, $x \notin F_{-n-1}A$, we have $w(xy) \geq w(x) + w(y)$, and, by Corollary II.2.3, $w(\Sigma k_i x_i) = \min_i w(x_i)$, $k_i \in Z_p$, $x_i \in A$. Using Lemma II.2.8, we may consider E^0A and A to have essentially the same bases, namely the set of elements $\prod_{i,j} (P^i_j)^{a_{i,j}} \prod_k (Q_k)^{b_k}$ written with P^i_j preceding P^k_ℓ if $i < k$ or if $i = k$ and $j < \ell$ and satisfying $0 \leq a_{i,j} < p$ and $0 \leq b_k \leq 1$. If such an element is considered in A, then by Theorem II.2.2 it has weight $\sum_{i,j} j a_{i,j} + \sum_k (k+1)b_k$. An element of weight w and degree t belongs to $E^0_{-w,t+w}A$ when considered as an element of E^0A.

Now we recall the definition of the spectral sequence. The bar construction of A, $\bar{B}(A)$, is the tensor algebra $T(I(A))$ as a Z_p-space and may be given a weight function by $w[x_1|\ldots|x_n] = \sum_i w(x_i)$, $w(\Sigma k_i\alpha_i) = \min_i w(\alpha_i)$, $k_i \in Z_p$, $\alpha_i \in \bar{B}(A)$. $\bar{B}(A)$ is then filtered by $[x_1|\ldots|x_n] \in F_u\bar{B}(A)$ if $w[x_1|\ldots|x_n] \geq -u+n$, $\Sigma k_i\alpha_i \in F_u\bar{B}(A)$ if $\alpha_i \in F_u\bar{B}(A)$ for some i. By definition of the boundary in $\bar{B}(A)$ (see page I-4.1), $d(F_uA) \subset F_{u-1}A$, and therefore $E^0 = E^1$ in the resulting spectral sequence. We consider the spectral sequence to start with E^1, $E^1_{u,v} = (F_u\overline{B}(A)/F_{u-1}\overline{B}(A))_{u+v}$. Give $\overline{B}(A)$ the basis derived from that of A described above. Then $E^1_{u,v}$ may be considered to have as a basis those basis elements α of $\overline{B}(A)$ which satisfy $\alpha \in T^{u+v}(I(A))$ and $w(\alpha) = v$. Then it is easily seen that E^1 may be identified with $\overline{B}(E^0A)$ as a complex, $E^2 = H_*(E^0A)$. The dual $\{E_r\}$ of the spectral sequence $\{E^r\}$ just defined is a spectral sequence of differential algebras satisfying $E_2 = H^*(E^0A)$ and $E_\infty = E^0H^*(A)$. Further details of the construction of the spectral sequence are given in section I-5. Note that d_r in the homology spectral sequence lowers the filtration degree by r and therefore raises the weight by $r-1$. If we define a weight function in $\overline{C}(A^*) = \overline{B}(A)^*$ by $w(\alpha^*) = w(\alpha)$ if α is a basis element of $\overline{B}(A)$ and $w(\sum_i \alpha_i^*) = \min_i w(\alpha_i^*)$, then since δ_r in the cohomology spectral sequence raises the filtration degree by r , it lowers weights by $r-1$. We remark that E^r and E_r are trigraded objects, having a filtration degree u , a complementary degree v and a grading t derived from that of the Steenrod algebra. The letter s will denote the homological degree $u+v$.

The definition of the spectral sequence relies heavily on the bar construction, but we have calculated $H^*(E^0A)$ in the complex $\overline{X}^*$. To compute the differentials δ_r of the cohomology spectral sequence directly, we would need a canonical embedding of $\overline{X}^*$ in $\overline{C}((E^0A)^*)$. I have not been able to obtain such an embedding, so we will follow the alternative procedure of calculating the differentials d_r in the homology spectral sequence and dualizing to obtain the differentials δ_r. In low dimensions, the calculations are not difficult; sample calculations are given in the following lemmas.

<u>Lemma II.6.1</u>: For all primes p, $\delta_p(b_j^i) = h_{i+1}b_{j-1}^{i+1} - h_{i+j}b_{j-1}^i$, $j \geq 2$ (here if $p = 2$, $b_1^i = (h_i)^2$).

<u>Proof</u>: We note first that the stated differentials are consistent, that is, $\delta_p(b_j^i)$ is a cocycle of E_p. $b_j^i \in H^{u,v,t}(E^oA)$, where $u+v = 2$, $v = w(b_j^i) = pj$ and $t = 2p^{i+1}(p^j-1)$ or $2^{i+1}(2^j-1)$ if $p = 2$. Ignoring the grading t, we have $b_j^i \in E_2^{2-pj,pj}$ and $(b_j^i)^* \in E^2_{2-pj,pj}$. $h_{i+1}b_{j-1}^{i+1}$ has weight $pj - p+1$, hence $h_{i+1}b_{j-1}^{i+1} \in E_2^{2-pj+p,pj-p+1}$ and $(h_{i+1}b_{j-1}^{i+1})^* \in E^2_{2-pj+p,pj-p+1}$, and similarly for $h_{i+j}b_{j-1}^i$. We easily see that there are no other elements of $E_2^{2-pj+p,pj-p+1}$, and therefore to determine $\delta_p(b_j^i)$ it suffices to determine $d_p(h_{i+1}b_{j-1}^{i+1})^*$ and $d_p(h_{i+j}b_{j-1}^i)^*$. $(h_{i+1}b_{j-1}^{i+1})^*$ is represented by the cycle $< \overline{P}_1^{i+1} > \gamma_1(\tilde{P}_{j-1}^{i+1}) \in \overline{X}^*$ and $(h_{i+j}b_{j-1}^i)^*$ is represented by $< \overline{P}_1^{i+j} > \gamma_1(\tilde{P}_{j-1}^i) \in \overline{X}^*$. To compute d_p, we need representative cycles in $\overline{B}(E^oA)$. These are obtained by making use of the canonical embedding $f: \overline{X} \longrightarrow \overline{B}(A)$ described in Theorems I.4.5 and I.4.7 and Proposition I.4.6 on pages I-4.12 and I-4.13. In fact, $f < \overline{P}_j^i > \gamma_1(\tilde{P}_\ell^k) = \{P_j^i\}*\{(P_\ell^k)^{p-1}|P_\ell^k\}$, where we are writing $\{x_1|\ldots|x_n\} = [x_1|\ldots|x_n]$ to avoid confusion with the bracket product and where $*$ denotes the shuffle product defined on page I-4.3. Using our identification of the bases of $\overline{B}(A)$ and $\overline{B}(E^oA)$, we consider the representative cycles of $(h_{i+1}b_{j-1}^{i+1})^*$ and $(h_{i+j}b_{j-1}^i)^*$ in $\overline{B}(E^oA)$ as chains in $F^{2-pj+p}A$. Then our claim is that the boundaries of these chains in this complex are congruent to

plus or minus the chain $\{(P_j^i)^{p-1}|P_j^i\}$ modulo $F^{1-pj}A$. Using Lemma I.4.3, which holds in the bar construction of any supplemented algebra, we find that

$$d\{P_1^{i+j}\}*\{(P_{j-1}^i)^{p-1}|P_{j-1}^i\} = \{P_1^{i+j}\}*\{(P_{j-1}^i)^p\} - \{[P_1^{i+j}(P_{j-1}^i)^{p-1}]|P_{j-1}^i\} - \{(P_{j-1}^i)^{p-1}|[P_1^{i+j}P_{j-1}^i]\} \text{ in } \overline{B}(A).$$

Using Theorem II.1.1, which describes the product in A, and Theorem II.2.2, which gives the weight function in A, we find that

$$[P_1^{i+j}(P_{j-1}^i)^{p-1}] = \sum_{k=1}^{p-1} (k-1)!(P_{j-1}^i)^{p-1-k}(P_j^i)^k(P_1^{i+j-1})^{p-k}$$ plus summands of weight greater than $1+(p-1)j$, that $[P_1^{i+j}P_{j-1}^i] = -P_j^i(P_1^{i+j-1})^{p-1}$ plus summands of weight greater than $j+p-1$, and that $(P_{j-1}^i)^p$ has no summand of weight less than pj unless $j = 2$ when the only such summand is $P_1^{i-1}(P_2^{i-1})^{p-1}$.

$d\{P_1^{i+2}\}*\{P_1^{i-1}|(P_2^{i-1})^{p-1}\} = \{P_1^{i+2}\}*\{P_1^{i-1}(P_2^{i-1})^{p-1}\}$ plus summands in $F^{1-2p}A$, and therefore we have in all cases

$$d\{P_1^{i+j}\}*\{(P_{j-1}^i)^{p-1}|P_{j-1}^i\} \equiv -\sum_{k=1}^{p-1} (k-1)!\{(P_{j-1}^i)^{p-1-k}(P_j^i)^k(P_1^{i+j-1})^{p-k}$$
$$+ \{(P_{j-1}^i)^{p-1}|P_j^i(P_1^{i+j-1})^{p-1}\} \mod F^{1-pj}A$$

All summands on the right have weight pj, hence are in $F^{2-pj}A$. The homology class of $E^p_{2-pj,pj}$ represented by the chain on the right is by definition $d_p(h_{i+2}b_{j-1}^i)^*$. Calculating in $\overline{B}(E^0A)$, it is easily found that the sum of the boundaries of the chains

$\{P_{j-1}^i|(P_{j-1}^i)^{p-2}|P_j^i(P_1^{i+j-1})^{p-1}\}$,

$-\{P_{j-1}^i\}*\{(P_{j-1}^i)^{p-2}(P_1^{i+j-1})^{p-1}|P_j^i\}$,

$-\sum_{k=2}^{p-1} (k-1)!\{(P_{j-1}^i)^{p-1-k}(P_j^i)^{k-1}(P_1^{i+j-1})^{p-j}|P_j^i|P_{j-1}^i\}$, and

$\sum_{k=2}^{p-1} (k-1)!\{(P_{j-1}^i)^{p-1-k}(P_j^i)^{k-1}(P_1^{i+j-1})^{p-k}|P_{j-1}^i|P_j^i\}$

is the negative of the chain above minus $\{(P_j^i)^{p-1}|P_j^i\}$. Therefore $d_p(h_{i+2}b_{j-1}^i)^* = -(b_j^i)^*$. Similarly we find, for $j \geq 3$,

$$\begin{aligned} d\{P_1^{i+1}\}*\{(P_{j-1}^{i+1})^{p-1}|P_{j-1}^{i+1}\} &= \{P_1^{i+1}\}*\{(P_{j-1}^{i+1})^p\} \\ &- \{[P_1^{i+1}(P_{j-1}^{i+1})^{p-1}]|P_{j-1}^{i+1}\} \\ &- \{(P_{j-1}^{i+1})^{p-1}|[P_1^{i+1}P_{j-1}^{i+1}]\} \\ &\equiv \{(P_1^i)^{p-1}P_j^i(P_{j-1}^{i+1})^{p-2}|P_{j-1}^{i+1}\} \\ &- \{(P_{j-1}^{i+1})^{p-1}|(P_1^i)^{p-1}P_j^i\} \bmod F^{1-2p}A \\ &\equiv \{(P_j^i)^{p-1}|P_j^i\} \quad \text{in } \overline{B}(E^oA) \end{aligned}$$

If $j=2$,

$$\begin{aligned} d\{P_1^{i+1}\}*\{(P_1^{i+1})^{p-1}|P_1^{i+1}\} &= \{P_1^{i+1}\}*\{(P_1^{i+1})^p\} \\ &\equiv \{P_1^{i+1}\}*\{P_1^i(P_2^i)^{p-1}\} \bmod F^{1-2p}A \\ &\equiv \{(P_2^i)^{p-1}|P_2^i\} \quad \text{in } \overline{B}(E^oA) . \end{aligned}$$

Thus in all cases $d_p(h_{i+1}b_{j-1}^{i+1})^* = (b_j^i)^*$ and the conclusion follows.

<u>Lemma II.6.2</u>: If $p > 2$, $\delta_p(a_i) = -a_{i-1}h_i$ if $i > 1$ and $\delta_p(a_1) = -a_o^3h_1$.

<u>Proof</u>: It suffices to prove $d_p(a_{i-1}h_i)^* = -(a_i)^*$, since $a_{i-1}h_i$ is the only element of $E_2^{3-2i,2i+1}$. $(a_{i-1}h_i)^*$ is represented by $< \overline{P}_1^i > \gamma_p(\overline{Q}_{i-1})$ in $\overline{X}^*$, and

$f < \overline{P}_1^i > \gamma_p(\overline{Q}_{i-1}) = \{P_1^i\}*\{Q_{i-1}\}^p$, $\{Q_{i-1}\}^p = \{Q_{i-1}|\cdots|Q_{i-1}\}$, p factors.

$$\begin{aligned} d\{Q_{i-1}\}^p*\{P_1^i\} &= -\{Q_{i-1}\}^{p-1}*\{[P_1^iQ_{i-1}]\} \\ &= -\{Q_{i-1}\}^{p-1}*\{Q_i\ P((p-1)p^{i-1})\} \end{aligned}$$

by Lemma I.4.4 and Theorem II.1.1.

$$d\{Q_{i-1}\}^{p-1}*\{Q_i|P((p-1)p^{i-1})\} = \{Q_{i-1}\}^{p-1}*\{Q_iP((p-1)p^{i-1})\}$$

$$- \{Q_{i-1}\}^{p-2}*\{Q_i|Q_iP(p-2)p^i)\}$$

Proceeding inductively, using $\{Q_{i-1}\}^k*\{\overbrace{Q_i|\cdots|Q_i}^{p-k\text{ factors}}|P(kp^i)\}$,

$d\{Q_{i-1}\}^p*\{P_1^i\} \equiv - \{Q_i\}^p = - f(\gamma_p(Q_i)) = - (a_i)^*$. The proof that $\delta_p(a_1) = - a_0^3h_1$ is identical.

<u>Lemma II.6.3</u>: If $p = 2$, $\delta_2(h_i(1)) = h_i(h_{i+2})^2$, $\delta_2(h_i(1,3)) = h_i(1)(h_{i+4})^2 + h_ih_{i+2}h_{i+2}(1)$, and $\delta_2(h_i(1,2)) = h_{i+3}h_i(1,3)$.

<u>Proof</u>: In each case, there are no other possible non-zero summands of δ_2 . We prove only $\delta_2(h_i(1)) = h_i(h_{i+2})^2$, the proofs of the remaining statements being similar.

$(h_i(h_{i+2})^2)^*$ is represented by $\gamma_1(\bar{P}_1^i)\gamma_2(\bar{P}_1^{i+2})$, and

$f\gamma_1(\bar{P}_1^i)\gamma_2(\bar{P}_1^{i+2}) = \{P_1^i\}*\{P_1^{i+2}\}^2$. In $\bar{B}(A)$,

$$d\{P_1^i\}*\{P_1^{i+2}\}^2 = \{P_1^i\}*\{(P_1^{i+2})^2\} + \{P_1^{i+2}\}*\{[P_1^iP_1^{i+2}]\}$$

$$\equiv \{P_1^i\}*\{P_1^{i+1}P_2^{i+1}\} + \{P_1^{i+2}\}*\{P_2^iP_1^{i+1}\} \bmod F^{-3}A .$$

Computing in $\bar{B}(E^oA)$,

$d\{P_1^i\}*\{P_1^{i+1}|P_2^{i+1}\} = \{P_1^i\}*\{P_1^{i+1}P_2^{i+1}\} + \{P_2^i|P_2^{i+1}\} + \{P_1^{i+1}|P_3^i\}$ and

$d\{P_1^{i+2}\}*\{P_2^i|P_1^{i+1}\} = \{P_1^{i+2}\}*\{P_2^iP_1^{i+1}\} + \{P_2^i|P_2^{i+1}\} + \{P_3^i|P_1^{i+1}\}$.

Adding, $d\{P_1^i\}*\{P_1^{i+2}\}^2 \equiv \{P_1^{i+1}\}*\{P_3^i\} \bmod F^{-3}A$, and therefore $d_2(h_i(h_{i+2})^2)^* = h_i(1)^*$ as was to be proven.

The differentials computed so far include all that are non-zero on elements satisfying $s \leq 2$ and completely determine δ_p in the range in which $H^*(E^oA)$ has been computed. The following results, due initially to Adams [1] in the case $p = 2$ and to Liulevicius [13] in the case $p > 2$, are now clear.

Theorem II.6.4: If $p = 2$, then the elements h_i generate $H^{s,t}(A)$ in the range $s \leq 2$. These elements satisfy only the relations $h_i h_{i+1} = 0$, $h_i h_{i+1} h_j = 0$, $h_i^2 h_{i+2} = (h_{i+1})^3$, and $h_i h_{i+2}^2 = 0$ in the range $s \leq 3$.

Theorem II.6.5: If $p > 3$, then the elements a_o, h_i, $h_i(2,1)$, $h_{i+1}(1,2)$, b_1^i, and g_1^1 generate $H^{s,t}(A)$ in the range $s \leq 2$. In the range $s \leq 3$, all relations among these elements are generated by the following:

$a_o h_o = 0$, $a_o g_1^1 = 0$, $a_o h_o(2,1) = 0$, $a_o h_1(1,2) = 0$, $h_1 g_1^1 = 0$;

$h_i h_i = 0$, $h_i h_{i+1} = 0$, $h_i h_{i+1}(2,1) = 0$, $h_i h_{i+1}(1,2) = 0$;

$h_{i+1} h_i(2,1) = h_{i+1}(1,2) h_i$, $h_{i+1}(1,2) h_{i+2} = 0$, $h_i(1,2) h_{i+2} = 0$;

if $p > 3$, $h_i h_i(2,1) = 0$, $h_{i+1} h_{i+1}(1,2) = 0$, $h_o g_1^1 = 0$;

if $p = 3$, $h_i h_i(2,1) = -h_{i+1} b_1^i$, $h_{i+1} h_{i+1}(1,2) = -h_i b_1^{i+1}$, $h_o g_1^1 = -a_o b_1^o$;

$h_{i+1} b_1^{i+1} = h_{i+2} b_1^i$. Note that only the last of these relations was derived by use of the spectral sequence.

Remarks II.6.6: To avoid proliferation of notation, we are denoting elements of $H^*(A)$ by the same symbols as the corresponding surviving elements of $H^*(E^oA)$. Our notation in Theorem II.6.4 agrees with that of Adams. $a_o \longrightarrow a_o$, $h_i \longrightarrow h_i$, $\mu_i \longrightarrow 2h_i(2,1)$, $\nu_i \longrightarrow 2h_{i+1}(1,2)$, $\lambda_i \longrightarrow b_1^i$, $\rho \longrightarrow 2g_1^1$ gives the correspondence of our notation with that of Liulevicius, in the sense that, identifying $\bar{B}(E^oA)$ with $\bar{B}(A)$ as vector spaces, the epimorphism $f^*: \bar{\mathfrak{A}}((E^oA)^*) \longrightarrow \bar{X}^*$ dual to the embedding $f: \bar{X} \longrightarrow \bar{B}(E^oA)$ sends representative cocycles to representative cocycles.

Next we describe a method by which many differentials in our spectral sequences can be computed up to non-zero constant without recourse to explicit calculation. There are many elements x in $H^*(E^oA)$ satisfying $a_o^i x \neq 0$ for all i if $p > 2$ or $h_o^i x \neq 0$ for all i if $p = 2$. Results due to Adams [4] and Liulevicius [14] state that $H^{s,t}(A) = 0$ if s is sufficiently large relative to $t - s$. It follows that for all such elements x, either $\delta_r(x) \neq 0$ for some r or $a_o^i x = \delta_r(y)$ for some r, i, and y. Explicitly, the cited results are

<u>Theorem II.6.7</u>: If $p = 2$, $H^{s,t}(A) = 0$ if $0 < t - s < f(s) - s$, where $f(s)$ is defined by $f(4n) = 12n - 1$, $f(4n+1) = 12n + 2$, $f(4n+2) = 12n + 4$, and $f(4n+3) = 12n + 6$.

<u>Theorem II.6.8</u>: If $p > 3$, $H^{s,t}(A) = 0$ if $0 < t - s < 2(p-1)s-2$.

Either by use of these results or by explicit computation, we may extend Lemmas II.6.1 and II.6.2 to obtain the following results in the case $p > 2$:

<u>Theorem II.6.9</u>: Suppose $p > 2$. Then we have:

1. $\delta_{p^{n+1}}(b_j^i)^{p^n} = h_{i+n+1}(b_{j-1}^{i+1})^{p^n} - h_{i+j+n}(b_{j-1}^i)^{p^n}$, $n \geq 0,\ j \geq 2,\ i \geq 0$

2. $\delta_{p^{n+1}}(a_1)^{p^n} = -(a_o)^{p^{n+1}} h_{n+1}$, $n \geq 0$

3. $\delta_{p^{n+1}}(a_i)^{p^n} = -(a_{i-1})^{p^n} h_{n+i}$, $n \geq 0,\ i \geq 2$

All other non-zero differentials in the range $t - s \leq 2(p-1)(2p^2+p+2)-4$ are determined by the statement that E_r is a differential algebra and by:

4. $\delta_{2p-1}(h_1 a_2) = a_o^p c$, $c = h_1(2,1)$

5. $\delta_{2p-1}(h_2 a_2) = a_o^p d$, $d = h_2(1,2)$

6. $\delta_{2p-1}(a_1^\ell a_2 u) = a_o^{p+1} a_1^\ell w$, $0 \leq \ell \leq p-4$ and $\ell = p-2$ if $p > 3$;

   $\delta_5(a_1 a_2 u) = a_o^4 m$, $m = a_1 w + a_o a_2 b_1$ if $p = 3$

7. $\delta_{p^2-3p+3}(a_1^{p-3} w) = a_o^{p^2-2p-2} h_2 b_1$

8. $\delta_{p^2-2p+2}(a_1^{p-2} u) = a_o^{p^2-p-1} b_1^1$

9. $\delta_{p^2-2p+2}(a_1^{p-3} u b_2^o) = a_o^{p^2-2p-1}(b_1^1)^2$

At this point we can state the main theorem of this thesis for the case $p > 2$. The proof consists only of a tedious comparison of Theorems II.4.11 and II.4.13 with Theorem II.6.9. The notation $y \in (s,t)$ will mean $y \in \bigcup_{u+v=s} E_\infty^{u,v,t} = E_\infty^{s,t}$.

Theorem II.6.10: Suppose $p > 2$. Then the following elements of $H^*(E^0A)$ survive to E_∞. These elements are linearly independent over Z_p and include all elements of a basis for $E_\infty^{s,t}$ in the range $0 \leq t - s \leq (2p^2 + p+2)q-4$, $q = 2(p-1)$, ($j \geq 0$ and $k \geq 0$ unless otherwise specified).

1. $a_0^i \in (i,i)$, $0 \leq i$

2. a) $a_1^j h_0 (b_1^0)^k \in (jp+2k+1, (jp+kp+1)q+jp)$

   b) $a_1^j g_1^\ell (b_1^0)^k \in (jp+2k+\ell+1, (jp+kp+\ell+1)q+jp+\ell)$, $1 \leq \ell \leq p-2$

   c) $a_0^i a_1^j (b_1^0)^k \in (jp+2k+i, (jp+kp)q+jp+i)$, $0 \leq i \leq p-2$, $1 \leq k$

   d) $a_0^i h_1 \in (i+1, pq+i)$, $0 \leq i \leq p-1$

   $a_0^i h_1 (b_1^0)^k \in (2k+i+1, (p+kp)q+i)$, $0 \leq i \leq p-3$, $k \geq 1$

   e) $a_1^j g_2^\ell (b_1^0)^k \in (jp+2k+\ell+2, (jp+kp+p+\ell+2)q+jp+\ell)$, $0 \leq \ell \leq p-3$

   f) $a_0^i a_1^j u \in (jp+p+i, (jp+2p)q+jp+p+i-1)$, $0 \leq i \leq p$, $j \not\equiv p-2 \bmod p$

   $a_0^i a_1^j u (b_1^0)^k \in (jp+2k+p+i, (jp+kp+2p)q+jp+p+i-1)$, $0 \leq i \leq p-2$, $1 \leq k$

   g) $a_1^j h_0 u (b_1^0)^k \in (jp+2k+p+1, (jp+kp+2p+1)q+jp+p-1)$

3. a) $k_1^\ell (b_1^0)^k \in (2k+\ell+2, (kp+\ell p+2p)q+\ell)$, $0 \leq \ell \leq p-3$

   b) $h_0 k_1^\ell (b_1^0)^k \in (2k+\ell+3, (kp+\ell p+2p+1)q+\ell)$, $0 \leq \ell \leq p-3$

4. a) $a_0^i b_1^1 \in (i+2, p^2 q+i)$, $0 \leq i \leq p^2-p-2$

   b) $a_0^i h_2 \in (i+1, p^2 q+i)$, $0 \leq i \leq p^2-1$

5. a) $a_o^i b_1^1 (b_1^o)^k \in (2k+i+2, (p^2+kp)q+i)$, $0 \leq i \leq p-3$, $1 \leq k$

b) $h_o b_1^1 (b_1^o)^k \in (2k+3, (p^2+kp+1)q)$, $k = 0$ if $p = 3$

c) $g_1^\ell b_1^1 (b_1^o)^k \in (2k+\ell+3, (p^2+kp+\ell+1)q+\ell)$, $1 \leq \ell \leq p-3$

d) $g_2^\ell b_1^1 (b_1^o)^k \in (2k+\ell+4, (p^2+kp+p+\ell+2)q+\ell)$, $0 \leq \ell \leq p-4$

e) $a_o^i h_2 b_1^o \in (i+3, (p^2+p)q+i)$, $0 \leq i \leq p-2$

$a_o^i h_2 (b_1^o)^k \in (2k+i+1, (p^2+kp)q+i)$, $0 \leq i \leq p-4$, $2 \leq k$

f) $h_o h_2 \in (2, (p^2+1)q)$

g) $g_1^\ell h_2 \in (\ell+2, (p^2+\ell+1)q+\ell)$, $1 \leq \ell \leq p-2$

6. $(k_1^\ell)^2 (b_1^o)^k \in (2k+p+1, (p^2+kp+2p-1)q+p-3)$, $\ell = \frac{p-3}{2}$, $k = 0$ if $p = 3$

7. a) $k_1^\ell b_1^1 (b_1^o)^k \in (2k+\ell+4, (p^2+kp+\ell p+2p+\ell+1)q+\ell)$, $0 \leq \ell \leq p-3$, $p > 3$

b) $h_o k_1^\ell b_1^1 (b_1^o)^k \in (2k+\ell+5, (p^2+kp+\ell p+2p+\ell+2)q+\ell)$, $0 \leq \ell \leq p-3$, $p > 3$

8. a) $a_o^i h_1 b_2^o \in (i+3, (p^2+2p)q+i)$, $0 \leq i \leq p-1$

$a_o^i h_1 b_2^o (b_1^o)^k \in (2k+i+3, (p^2+kp+2p)q+i)$, $0 \leq i \leq p-3$, $1 \leq k$

b) $a_o^i a_2 h_1 (b_1^o)^k \in (2k+p+i+1, (p^2+kp+2p)q+p+i)$, $0 \leq i \leq p-3$, $1 \leq k$

c) $a_o^i c \in (i+2, (p^2+2p)q+i)$, $0 \leq i \leq p-1$, $c = h_1(2,1)$

$a_o^i c (b_1^o)^k \in (2k+i+2, (p^2+kp+2p)q+i)$, $0 \leq i \leq p-4$, $1 \leq k$

9. a) $a_1^j e_\ell (b_1^o)^k \in (jp+2k+p+\ell+2, (p^2+jp+kp+2p+\ell)q+jp+p+\ell-1)$,

where $e_1 = h_o(a_1 b_2 - a_2 b)$ and $e_\ell = g_1^{\ell-1}(a_1 b_2 - a_2 b)$, $2 \leq \ell \leq p-1$

b) $a_o^i a_1^j f (b_1^o)^k \in (jp+2k+p+4, (p^2+jp+kp+3p)q+jp+p)$, $0 \leq i \leq p-2$,

where $f = b(a_1 b_2 - a_2 b)$

10. a) $a_1^j g_2^\ell b_2^o (b_1^o)^k \in (jp+2k+\ell+4, (p^2+jp+kp+2p+\ell+2)q+jp+\ell)$,

$0 \le \ell \le p-3$, $j \ge 1$ if $\ell = 0$

b) $a_1^j a_2 g_2^\ell (b_1^o)^k \in (jp+2k+p+\ell+2, (p^2+jp+kp+2p+\ell+2)q+jp+\ell+p)$, $0 \le \ell \le p-3$

c) $a_o^i a_1^j u b_2^o \in (jp+p+i+2, (p^2+jp+3p)q+jp+p+i-1)$, $0 \le i \le p$, $j \not\equiv p-3 \bmod p$

$a_o^i a_1^j u b_2^o (b_1^o)^k \in (jp+2k+p+i+2, (p^2+jp+kp+3p)q+jp+p+i-1)$, $0 \le i \le p-2$, $1 \le k$

d) $a_1^j h_o u b_2^o (b_1^o)^k \in (jp+2k+p+3, (p^2+jp+kp+3p+1)q+jp+p-1)$

e) $a_1^j a_2 h_o u (b_1^o)^k \in (jp+2k+2p+1, (p^2+jp+kp+3p+1)q+jp+2p-1)$

11. a) $a_1^j g_3^\ell (b_1^o)^k \in (jp+2k+\ell+3, (p^2+jp+kp+2p+\ell+3)q+jp+\ell)$, $0 \le \ell \le p-4$

b) $h_1 g_3^o (b_1^o)^k \in (2k+4, (p^2+kp+3p+3)q)$, $p > 3$

c) $j^\ell (b_1^o)^k \in (2k+\ell+4, (p^2+kp+\ell p+3p+\ell+3)q+\ell)$, $1 \le \ell \le p-4$

12. a) $a_o^i a_1^j w \in (jp+p+i, (p^2+jp+3p)q+jp+p+i-2)$, $0 < i < p$, $j \not\equiv p-3 \bmod p$ if $p > 3$

$a_o^i a_1 m \in (6+i, 21q+6+i-2)$, $0 \le i \le 3$, $m = a_1 w + a_o a_2 b_1$ if $p = 3$

$a_o^i a_1^j w (b_1^o)^k \in (jp+2k+p+i, (p^2+jp+kp+3p)q+jp+p+i-2)$, $0 \le i \le p-2$, $1 \le k$

b) $a_1^j h_o w (b_1^o)^k \in (jp+2k+p+1, (p^2+jp+kp+3p+1)q+jp+p-2)$

c) $a_1^j g_1^1 w (b_1^o)^k \in (jp+2k+p+2, (p^2+jp+kp+3p+2)q+jp+p-1)$

13. a) $k_1^\ell b_2^o (b_1^o)^k \in (2k+\ell+4, (p^2+kp+\ell p+3p)q+\ell)$, $0 \le \ell \le p-3$

b) $a_2 k_1^\ell (b_1^o)^k \in (2k+p+\ell+2, (p^2+kp+\ell p+3p)q+p+\ell)$, $0 \le \ell \le p-3$

c) $h_o k_1^\ell b_2^o (b_1^o)^k \in (2k+\ell+5, (p^2+kp+\ell p+3p+1)q+\ell)$, $0 \le \ell \le p-3$

d) $a_2 h_o k_1^\ell (b_1^o)^k \in (2k+p+\ell+3, (p^2+kp+\ell p+3p+1)q+p+\ell)$, $0 \le \ell \le p-3$

e) $k_2^\ell (b_1^o)^k \in (2k+\ell+3, (p^2+kp+\ell p+3p+\ell+1)q+\ell)$, $0 \le \ell \le p-4$

14. a) $a_o^i(b_1^1)^2 \in (i+4, 2p^2q+i)$, $0 \le i \le p^2-2p-2$

$a_o^i(b_1^1)^2(b_1^o)^k \in (2k+i+4, (2p^2+kp)q+i)$, $0 \le i \le p-3$, $1 \le k$

b) $h_o(b_1^1)^2(b_1^o)^k \in (2k+5, (2p^2+kp+1)q)$

c) $g_1^\ell(b_1^1)^2(b_1^o)^k \in (2k+\ell+5, (2p^2+kp+\ell+1)q+\ell)$

15. a) $a_o^i h_2 b_1^1 \in (i+3, 2p^2q+i)$, $0 \le i \le p^2-2p-3$

$a_o^i h_2 b_1^1(b_1^o)^k \in (2k+i+3, (2p^2+kp)q+i)$, $0 \le i \le p-4$, $1 \le k$

b) $h_o h_2 b_1^1(b_1^o)^k \in (2k+4, (2p^2+kp+1)q)$, $p > 3$

c) $g_1^\ell h_2 b_1^1(b_1^o)^k \in (2k+\ell+4, (2p^2+kp+\ell+1)q+\ell)$, $1 \le \ell \le p-4$

16. a) $a_o^i \ell \in (p^2-p+i+2, 2p^2q+p^2-p+i)$, $0 \le i \le p^2-2$,

where $\ell = a_1^{p-2}(a_1b_2-a_2b)$

b) $a_o^i a_1^{p-3} a_2 u \in (p^2-p+i, 2p^2q+p^2-p+i-1)$, $0 \le i \le p^2+p$

17. a) $x(b_1^o)^k \in (2k+p, (2p^2+kp+p-1)q+p-2)$

b) $h_o x(b_1^o)^k \in (2k+p+1, (2p^2+kp+p)q+p-2)$

18. a) $a_o^i h_2 b_2^o \in (i+3, (2p^2+p)q+i)$, $0 \le i \le p-1$

b) $h_o h_2 b_2^o \in (4, (2p^2+p)q)$

c) $g_1^\ell h_2 b_2^o \in (\ell+4, (2p^2+p)q+\ell)$

d) $a_o^i d \in (2+i, (2p^2+p)q+i)$, $0 \le i \le p-1$, $d = h_2(1,2)$

19. a) $a_2 h_o b_1^1 \in (p+3, (2p^2+p+1)q)$

b) $a_2 g_1^\ell b_1^1 \in (p+\ell+3, (2p^2+p+\ell+1)q)$, $1 \le \ell \le p-3$

20. a) $a_2h_oh_2 \in (p+2,(2p^2+p+1)q)$

b) $a_2g_1^{\ell}h_2 \in (p+\ell+2,(2p^2+p+\ell+1)q), \quad 1 \leq \ell \leq p-2$

Remarks II.6.12: The relations in E_∞ among the listed elements are those implied by Theorems II.4.11, II.4.13, and II.6.10. $E_\infty = E^oH^*(A)$ and the relations in $H^*(A)$ among corresponding elements (corresponding in the sense of being represented in $\overline{C}(A)$ by a lifting of a representative cocyle of $E_1 = \overline{C}(E^oA))$ may differ. However, this can occur only if the product of two elements x and y lands in a group $E_\infty^{s,t}$ which contains an element of lower weight than $w(x)+w(y)$, that is, of higher filtration degree. As this occurs quite rarely, the products in $H^*(A)$ can at most differ but slightly from those in E_∞.

We now proceed to the case $p = 2$. Here we may prove:

Theorem II.6.13: Suppose $p = 2$. Then we have:

1. $\delta_{2^{n+1}}(b_j^i)^{2^n} = h_{i+n+1}(b_{j-1}^{i+1})^{2^n} + h_{i+j+n}(b_{j-1}^i)^{2^n}, \quad n \geq 0,\ j \geq 3,\ i \geq 0$

2. $\delta_2(b_2^i) = (h_{i+1})^3 + (h_i)^2h_{i+2}, \quad i \geq 0$

3. $\delta_{2^{n+1}}(b_2^o)^{2^n} = (h_o)^{2^{n+1}}h_{n+2}, \quad n \geq 0$

4. $\delta_2(h_i(1)) = h_i(h_{i+2})^2$

5. $\delta_2(h_i(1,3)) = h_i(1)(h_{i+4})^2 + h_ih_{i+2}h_{i+2}(1)$

6. $\delta_2(h_i(1,2)) = h_{i+3}h_i(1,3)$

7. $\delta_4(h_{i+2}b_3^i) = (h_i)^2(h_{i+3})^2$

8. $\delta_{2^{n+2}}(h_{n+2}(b_3^o)^{2^n}) = (h_o)^{2^{n+1}}(h_{n+3})^2$, $n \geq 0$

9. $\delta_4(h_o b_4^o h_3) = (h_o)^3 h_2(1) + h_o b_2^o (h_{i+4})^2$

There are no other non-zero differentials in the range $t - s \leq 43$.

Before stating the main theorem, we discuss certain general phenomena which occur in the calculation of $H^*(A)$.

Remarks II.6.14: In [3], Adams has proven that in a neighborhood N_o of the line $t = 3s$, $H^{s,t}(A) \cong H^{s+4,t+12}(A)$ and in successively larger neighborhoods N_k, $H^{s,t}(A) \cong H^{s+4\cdot 2^k, t+12\cdot 2^k}(A)$. These periodicity isomorphisms, where defined, are given by $x \longrightarrow < x, h_o^{2^{k+2}}, h_{k+3} >$. The operation on the right is the Massey triple product defined as follows (see Massey, [17]): Let x,y,z be classes such that $xy = yz = 0$; let x',y',z' be representative cocycles (in $\overline{C}(A^*)$) and suppose $\delta(u) = x'y'$, $\delta(v) = y'z'$. Then $< x,y,z >$ is the class of $uz' - (-1)^{\deg x} x'v$, which is well defined as an element of $H^*(A)/[xH^*(A) + H^*(A)z]$. Now in our procedure, for x near the line $t = 3s$, we have $xh_o^{2^{k+2}} = 0$ in $E_{2^{n+1}}$, and therefore, constructing Massey products using the complex $E_{2^{n+2}}$, we have $< x, h_o^{2^{k+2}}, h_{k+3} > = (b_2^o)^{2^{k+1}} x$, since $xh_o^{2^{k+2}}$ is zero in $E_{2^{n+2}}$ not by being a cocycle but by virtue of of the algebra structure. Explicit study of the region of periodicity is carried out by use of 3 and 8 of the previous theorem. We have:

$$(b_2^0)^{2^n}(b_3^0)^{2^n} \xrightarrow{\delta_{2^{n+1}}} (h_0)^{2^{n+1}} h_{n+2}(b_3^0)^{2^n} + (b_2^0)^{2^{n+1}} h_{n+3} + (b_2^0)^{2^n} h_{n+1}(b_2^1)^{2^n}$$

$$\begin{array}{ccc} \downarrow \delta_{2^{n+2}} & \downarrow \delta_{2^{n+2}} & \downarrow \delta_{2^{n+2}} \\ (h_0)^{2^{n+2}} & (h_{n+3})^2 & 0 \end{array}$$

$$(b_2^0)^{2^n+2^{n+1}}(b_3^0)^{2^n} \xrightarrow{\delta_{2^{n+1}}} (h_0)^{2^{n+1}} h_{n+2}(b_2^0)^{2^{n+1}}(b_3^0)^{2^n} + (b_2^0)^{2^{n+2}} h_{n+3}$$

$$+ (b_2^0)^{2^n+2^{n+1}} h_{n+1}(b_2^1)^{2^n} ,$$

and in this case all three summands on the right are cocycles in $E_{2^{n+2}}$. Let $g = (b_2^1)^2$ and $P^i = (b_2^0)^{2i}$; these diagrams have the interpretation that $P^{2^n} h_{n+3}$ is a summand of a cocycle of $E_{2^{n+2}}$ which is congruent to $P^{2^{n-1}} h_{n+1}(g)^{2^{n-1}}$, but that $P^{2^{n+1}} h_{n+3}$ and $P^{2^{n-1}+2^n} h_{n+1}(g)^{2^{n-1}}$ represent distinct non-zero elements of $E_{2^{n+2}}$. Thus $P^{2^{n+1}}$, regarded as a periodicity map, may be considered as the first such which acts on h_{n+3}. Further, if $xh_0^{2^{n+1}} = 0$, then $P^{2^{n+1}} xh_{n+3} = P^{2^{n-1}+2^n} xh_{n+1}(g)^{2^n}$. Finally we note that P^i is a transduction on elements of positive total degree $t - s$, in the sense that if $P^i x \neq 0$ and $P^i y \neq 0$, then $P^i xy = xP^i y = yP^i x$. This fact can be of service in the study of the differentials in the Adams spectral sequence.

As a co rollary of the above discussion, we note the following consequence for $H^{s,t}(A)$, $t - s \equiv 7 \bmod 8$:

<u>Corollary II.6.15</u>: The elements $P^{j \cdot 2^{n+1}} h_0^i h_{n+3}$, $n \geq 0$, $j \geq 0$, and $0 \leq i < 2^{n+2}$ survive to E_∞, where $P^{2^{n+1}} = (b_2^0)^{2^{n+2}}$ and is

is to be interpreted as a periodicity map. $P^k h_o^i h_{n+3}$, $k \not\equiv 0 \bmod 2^n$, does not survive and $P^{j\cdot 2^n} h_o^i h_{n+3}$, $j \equiv 1 \bmod 2$, may be taken to be equal to $P^{(j-1)2^n+2^{n-1}} h_{n+1} h_o^i (g)^{2^{n-1}}$, where $g = (b_2^1)^2$, and is therefore zero if $i \geq 3$.

We now state our main theorem for the case p=2. The proof is by a comparison of Theorems II.5.18 and II.6.13.

<u>Theorem II.6.16</u>: Suppose p=2. Then the following elements of $H^*(E^oA)$ survive to E_∞ . These elements are linearly independent over Z_2 and include all elements of a basis for $E_\infty^{s,t}$ in the range $0 \leq t-s \leq 42$. (The notation $P^i x$ means $(b_2^o)^{2i}x$; $i \geq 0$ and $n \geq 0$ in the expressions below.)

1. h_o^j , $j \geq 0$
2. a) $P^i h_1$; $P^i h_1^2$
   b) $P^i h_o^j h_2$, $0 \leq j \leq 2$; h_2^2 , h_2^3
   c) $P^{2i} h_o^j h_3$, $0 \leq j \leq 3$; h_3^2 , $h_o h_3^2$, h_3^3
   d) $P^{4i} h_o^j h_4$, $0 \leq j \leq 7$; $h_o^j h_4^2$, $0 \leq j \leq 3$; h_4^3 , $h_o h_4^3$
   e) $P^{8i} h_o^j h_5$, $0 \leq j \leq 15$; $h_o^i h_5^2$, $0 \leq j \leq 7$; $h_o^i h_5^3$, $0 \leq j \leq 3$
3. a) $h_1 h_4$; $h_1^2 h_4$; $h_o^j h_2 h_4$, $0 \leq j \leq 2$; $h_1 h_4^2$
   b) $h_1 h_5^k$, $0 \leq k \leq 3$
   c) $h_o^j h_2 h_5$, $h_o^j h_2 h_5^2$, $0 \leq j \leq 2$; $h_2^2 h_5$, $h_2^3 h_5$
   d) $h_o^j h_3 h_5$, $0 \leq j \leq 3$, $h_1 h_3 h_5$
4. a) $P^i c_o$, $c_o = h_1 h_o(1)$
   b) $P^i h_1 c_o$
5. a) $P^i h_o^j d_o$, $0 \leq j \leq 2$, $d_o = h_o(1)^2$
   b) $P^i h_1 d_o$, $P^i h_1^2 d_o$

c) $P^i h_0^j h_2 d_0$, $0 \leq j \leq 2$

6\. a) $P^i g^n$, $P^i h_0 g$, $P^i h_0^2 g$; $g=(b_2^1)^2$; $P^1 g = d_0^2$

b) $h_1 g$, $h_2 g$, $h_0 h_2 g$, h_2^2 ; $h_1 g = h_2 f_0$

7\. a) $P^i e_0 g^n$, $e_0 = b_2^1 h_0(1)$; $h_0 e_0 = h_2 d_0$

b) f_0 , $h_0 f_0$, $f_0 g$, $h_0 f_0 g$, $f_0 = h_2^2 b_3^0$; $h_0 f_0 = h_1 e_0$

c) $P^i d_0 e_0 g^n$, $P^i h_0 d_0 e_0$, $P^i h_0^2 d_0 e_0$; $h_0 d_0 e_0 = P^1 h_2 g$

d) $P^i e_0^2 g^n$, $P^i h_0 e_0^2$, $P^i h_0^2 e_0^2$; $e_0^2 = d_0 g$; $h_0^2 e_0^2 = P^1 h_2^2 g$

8\. a) $P^i j g^n$, $P^i h_0 j$, $P^i h_0^2 j$, $P^i h_0 j g$, $P^i h_0^2 j g$, $j = h_0 h_0^2 (1) b_3^0$; $h_0 j = P^1 f_0$

b) $P^i k g^n$, $P^i h_0 k$, $P^i h_0^2 k$, $k = h_2 h_0^2 (1) b_3^0$; $h_0 k = h_2 j$, $h_0^2 k = P^1 h_1 g$

c) $P^i \ell g^n$, $P^i h_0 \ell$, $P^i h_0^2 \ell$, $\ell = h_0 (b_2^1)^2 b_3^0$; $h_0 \ell = h_2 k = f_0 d_0$, $P^1 \ell = j d_0$

d) $P^i m g^n$, $P^i h_0 m$, $P^i h_0^2 m$, $m = h_2 (b_2^1)^2 b_3^0$; $h_0 m = h_2 \ell = e_0 f_0$, $P^1 m = k d_0$, $h_2 m = f_0 g$

e) $P^j i$, $P^j h_0 i$, $i = h_2 b_3^0 (b_2^0)^2$; $h_2 i = h_0 j$

9\. a) c_1 , $c_1 = h_2 h_1 (1)$

b) $h_2 c_1$

c) $c_1 g$

10\. a) $h_4 c_0$, $h_1 h_4 c_0$

b) $h_5 c_0$, $h_1 h_5 c_0$

11\. a) d_1 , $d_1 = h_1(1)^2$

b) p , $h_0 p$, $p = h_0 h_3 b_3^1$; $h_0 p = h_1 d_1$

c) $h_2 d_1$; $h_2 d_1 = h_4 g$

d) $h_3 d_1$

e) e_1 , $e_1 = b_2^2 h_1 (1)$; $h_1 e_1 = h_3 d_1$

f) $h_0^j f_1$, $0 \leq j \leq 2$, $f_1 = h_3^2 b_3^1$; $h_0 f_1 = h_3 p$

g) $h_2 e_1$; $h_2 e_1 = h_1 f_1$

12\. a) n , $n = h_2 h_1 (1) b_3^0$

b) $h_2 n$

c) $h_2^2 n$; $h_2^2 n = f_0 c_1$

13. a) q, $q=h_1h_3(b_3^o)^2$

b) $P^i r$, $r=h_2^2\,(b_3^o)^2$

c) h_1q ; $h_1q=h_2r$

d) P^iu , P^ih_1u ; $u=h_1h_o^2(1)(b_3^o)^2$; $h_1u=P^1q$

e) P^iv , $v=h_1h_o(1)b_2^1(b_3^o)^2$

14. $h_o^jc_2$, $0 \leq j \leq 2$, $c_2=h_3h_2(1)$

Remarks II.6.17 : The calculation of $H^{s,t}(A)$ in the range $t-s \leq 17$ is due initially to Adams [4]. The algebra structure of E_∞ is easily determined by means of Theorems II.5.18 and II.6.13 and from those relations explicitly stated ; this structure can differ but little from that of $H^*(A)$. The dimensions of the listed elements may be read off the chart in Appendix A , $t-s \leq 42$; the dimensions of the remaining elements are determined by $P^1=(b_2^o)^2 \in H^{4,8}(E^oA)$ and $g=(b_2^1)^2 \in H^{4,24}(E^oA)$.

7. Stable homotopy groups of spheres

Due to the existence of the Adams spectral sequence, the results of the previous section are applicable to the computation of stable homotopy groups of spheres. We will combine Toda's results on these groups with the information obtained on $H^*(A)$ to compute a part of the Adams spectral sequence. This will determine completely some of the stable groups beyond the range of Toda's calculations.

We recall Adams' results. Let X be a space and S^nX its iterated suspension. The stable homotopy groups $\pi_m^s(X)$ are defined as $\varinjlim \pi_{m+n}(S^nX)$. We let $\pi_m^s(X;Z_p)$ denote $\pi_m^s(S)/K_m^p$, where K_m^p is the subgroup consisting of elements whose order is finite and prime to p. $\pi^s(X;Z_p)$ denotes the graded group with components $\pi_m^s(X;Z_p)$. By means of the join product (Adams [1] or Douady in [9]), we may give $\pi^s(X;Z_p)$ a structure of left $\pi^s(S;Z_p)$-module structure, where the join product gives $\pi^s(S;Z_p)$ a ring structure which differs only in sign from that given by composition. Suppose $H_*(X)$ is of finite type. Then we have:

Theorem II.7.1: There exists a spectral sequence $\{E_rX\}$ with differentials $\delta_r\colon E_r^{s,t}X \longrightarrow E_r^{s+r,t+r-1}X$ satisfying the following properties:

1. E_2X is canonically isomorphic to $\mathrm{Ext}_A(Z_p, H_*(X;Z_p))$ as a left $H^*(A)$-module, $H^*(A) = E_2S$.

2. E_rX is a differential left E_rS-module in the sense that $\delta_r(uv) = \delta_r(u)v + (-1)^{t-s}u\delta_r(v)$, $u \in E_r^{s,t}S$, $v \in E_rX$.

3. $\{E_r X\}$ converges to $E_\infty X = E^0\pi^S(X;Z_p)$, where $E^0\pi^S(S;Z_p)$ is the associated graded ring of $\pi^S(S;Z_p)$ with respect to a suitable filtration and $E^0\pi^S(X;Z_p)$ is a left $E^0\pi^S(S;Z_p)$-module.

Remarks II.7.2: Our statement of the theorem differs from that of Adams in that $\mathrm{Ext}_A(Z_p,H_*(X;Z_p))$ replaces $\mathrm{Ext}_A(H^*(X;Z_p),Z_p)$. The details of this modification are given by Douady in [9]. It is easily seen that the $H^*(A)$-module structure defined by Douady agrees with that given at the end of section I-5. The procedure to be followed in calculating E_2X starting from $E^0H^*(X;Z_p)$ is described in that section.

Now we restrict ourselves to the case of spheres and let $E_r = E_rS$. Then each E_r is a differential ring. $\pi_0^S(S;Z_p) \cong Z$, and $\pi_m^S(S;Z_p)$ is of finite order, $m > 0$. Multiplication in E_∞ by the element $a_0 \in E_\infty^{1,1}$ corresponds to multiplication by p in $\pi^S(S;Z_p)$, $p > 2$, and multiplication by h_0 in E_∞ corresponds to multiplication by 2 in $\pi^S(S;Z_2)$.

Consider first the case $p > 2$. Let $q = 2(p-1)$ and write $\pi(m)$ for $\pi_m^S(S;Z_p)$. In [24], Toda proves:

Theorem II.7.3: Suppose $p > 2$. Then the indecomposable elements of $\pi^S(S;Z_p)$ in the range $0 < m < p^2q-3$ are:

i) $\alpha_r \in \pi(rq-1)$, $p\alpha_r = 0$, $1 \leq r < p^2$, $r \not\equiv 0 \bmod p$

ii) $\alpha'_{rp} \in \pi(rpq-1)$, $p^2\alpha'_{rp} = 0$, $1 \leq r < p-1$

iii) $\beta_r \in \pi((rp+r-1)q-2)$, $p\beta_r = 0$, $1 \leq r < p$.

In the cited range, these elements satisfy the relations $\alpha_r\alpha_s = 0$, $\alpha_r\alpha'_s = 0$, and, if $s > 1$, $\alpha_r\beta_s = 0$ and $\alpha'_r\beta_s = 0$. Further, the group $\pi(p^2q-3)$ is Z_p or zero depending on whether or not $\alpha_1\beta_1^p = 0$, and the group $\pi(p^2q-2)$ is Z_p or zero.

Remarks II.7.4: The statement about $\pi(p^2q-2)$ was proven modulo a conjecture, the truth of which is implied by results of Liulevicius [1] and Shimada and Yamanoshita [20].

A basis for $E_2^{s,t} = H^{s,t}(A)$ in the range $0 < t - s < p^2q$ is given by the elements listed in 2, 3, and 4 of Theorem II.6.10. There is only one pattern of differentials in the Adams spectral sequence which is consistent with Toda's results. This is given by

Theorem II.7.5: Up to non-zero constant (the same constant in each of a) - c) below) we have:

a) $\delta_2(h_1) = a_o b_1^o$

b) $\delta_2(a_1^j g_2^\ell) = - a_1^j g_1^{\ell+1} b_1^o$, $0 \le \ell \le p-3$, $0 \le j$

c) $\delta_2(a_1^j u) = a_1^{j+1} b_1^o$, $0 \le j$, $j \not\equiv p-2 \bmod p$

d) $\delta_2(h_2) = a_o b_1^1$

e) $\delta_3(a_o^{p^2-p-2} h_2) = a_1^{p-1} b_1^o$

f) $\delta_{2p-1}(b_1^1) = h_o (b_1^o)^p$ or zero

Proof: That the constants are the same in a) - c) follows from the relations $a_o u b_1^o = h_1(a_1 b_1^o)$ and $g_1^\ell u b_1^o = g_2^{\ell-1}(a_1 b_1^o)$. In the range $0 < m \le p^2q-1$, these differentials are implied by Theorem II.7.4.

Our claim is that b) and c) hold without restriction on j. This follows easily since $\delta_2(h_o a_1^j u) = - h_o a_1^j \delta_2(u) = - h_o a_1^{j+1} b_1^o$ implies $\delta_2(a_1^j u) = a_1^{j+1} b_1^o$, and $a_1^j g_2^\ell = a_1^{j-1} g_1^{\ell+1} u$ then implies $\delta_2(a_1^j g_2^\ell) = - a_1^j g_1^{\ell+1} b_1^o$.

Remarks II.7.6: The result $\delta_2(h_i) = a_o b_1^{i-1}$ for all $i > 0$ has been proven by Liulevicius; a proof is given in Gershenson [11]. A problem equivalent to determining $\delta_{2p-1}(b_1^1)$ is stated in Toda [24].

Corollary II.7.7: Referring to Theorem II.6.10, the elements of 2) - 7) which can survive to E_∞ are as follows (elements in parentheses may survive, elements not in parentheses must survive):

2'. a) $h_o(b_1^o)^k$ (must survive, $0 \leq k < p$)

$a_1^j h_o$ (must survive, $0 \leq j < 2p+2$, $0 \leq j < 2p$ if $p = 3$)

b) $a_1^j g_1^\ell$ (must survive, $0 \leq j < 2p+1$, $0 \leq j < 2p-1$ if $p = 3$)

c) $(b_1^o)^k$ (must survive, $0 \leq k < 2p$)

$(a_o^i a_1^{jp+p-1} b_1^o$, $j \geq 1$, $0 \leq i \leq p-2)$

d) $a_o^{p-2} h_1$, $a_o^{p-1} h_1$

f) $a_o^{p-1} a_i^j u$, $a_o^p a_1^j u$ (must survive, $0 \leq j < 2p$), $j \not\equiv p-2 \bmod p$

3'. a) $k_1^\ell (b_1^o)^k$ (must survive, $0 \leq k < p$)

b) $h_o h_1^\ell (b_1^o)^k$ (must survive, $0 \leq k < p$)

4'. a) (b_1^1)

b) $a_o^{p^2-3} h_2$, $a_o^{p^2-2} h_2$, $a_o^{p^2-1} h_2$

5'. a) $(b_1^1(b_1^o)^k)$

b) $h_o b_1^1(b_1^o)^k$ (must survive, $0 \leq k < p$)

c) $g_1^\ell b_1^1$

e) $a_o^{p-3}h_2 b_1^o$, $a_o^{p-2}h_2 b_1^o$

f) $h_o h_2$

g) $g_1^\ell h_2$

6'. $(k_1^\ell)^2(b_1^o)^k$, $\ell = \frac{p-3}{2}$ (must survive, $0 \leq k < p$), $k=0$ if $p=3$

7'. a) $(k_1^\ell b_1^1(b_1^o)^k)$, $p > 3$

b) $h_o k_1^\ell b_1^1(b_1^o)^k$ (must survive, $0 \leq k < p$), $p > 3$

This describes $\pi(m)$ up to determination of $\delta_{2p-1}(b_1^1)$, $0 < m < (p^2+2)q-3$.

Remarks II.7.8: A correspondence between surviving elements of E_∞ and the elements of Theorem II.7.3 is given by $a_1^j h_o \longleftrightarrow \alpha_{1+pj}$, $a_1^j g_1^\ell \longleftrightarrow \alpha_{\ell+1+pj}$, $b_1^o \longleftrightarrow \beta_1$, $k_1^\ell \longleftrightarrow B_{\ell+2}$, and $a_o^{p-2}a_1^j h_1 \longleftrightarrow \alpha'_{(j+1)p}$ (where we have used the relation $a_o u = a_1 h_1$). The only other generators of $\pi^s(S;Z_p)$ in the range $0 < m < (p^2+2)q-3$ are $a_1^j h_o \longleftrightarrow \alpha_{1+pj}$, $a_1^j g_1^\ell \longleftrightarrow \alpha_{\ell+1+pj}$, $j = p$ and $j = p+1$; $(b_1^1 \longleftrightarrow \gamma)$; $a_o^{p^2-3}h_2 \longleftrightarrow \alpha''_{p^2}$, $h_o h_2 \longleftrightarrow \varepsilon_1$, $g_1^\ell h_2 \longleftrightarrow \varepsilon_{\ell+1}$; $a_o^{p-3}h_2 b_1^1 \longleftrightarrow \Phi$. The relations in $E^o\pi^s(S;Z_p)$ involving the elements of $\pi^s(S;Z_p)$ thus defined, are easily determined; for example, $\varepsilon_r\alpha_s = 0$, $\varepsilon_r\alpha'_s = 0$, and $\varepsilon_r\alpha''_{p^2} = 0$. Further,

$\alpha_r \gamma \neq 0$, $1 \leq r \leq p-2$, even if b_1^1 does not actually survive, but $\alpha_r \gamma = 0$, $r > p-2$. We note that the elements α_r were defined by Toda as elements of the toric construction $\{\alpha_{r-1}, p, \alpha_1\}$. The corresponding algebraic operation is the Massey triple product defined in Remarks II.6.14. In $\overline{X}(E^o A)^*$, $\delta(S_1) = R_1^o S_o$, $\delta(S_1^r) = rR_1^o S_1^{r-1} S_o$, and therefore $< a_1^j g_1^{\ell-1}, a_o, h_o > = (\ell+1) a_1^j g_1^{\ell}$, $1 \leq \ell \leq p-3$ $(g_1^o = h_o)$, which is, up to constant, in agreement with Toda's result. A detailed study of the relationship between Massey products in $H^*(A)$ and toric constructions in $\pi^s(S;Z_p)$ would be of interest.

There are many possible non-zero differentials among the elements 8) - 20) of Theorem I.6.10: $\delta_2(a_1^j g_2^{\ell}) = a_1^j g_2^{\ell+1} b_2^o$ and $\delta_2(a_1^j w) = a_1^j u b_2^o$ $(j \not\equiv p-3 \bmod p)$, or alternatively $\delta_2(a_1^j u b_2^o) = a_1^j f$ describe possible patterns. I conjecture that $\delta_2(a_1^{p-3} a_2 u) = \ell$ and $\delta_3(a_o^{p^2-1} a_1^{p-3} a_2 u) = a_1^{2p-1} b_1^1$ (up to constant); this seems plausible since in E_p of the previous spectral sequence $a_o^{p+1}\ell = - a_o a_1^p b_1^1$ and $a_o^{p+1} a_1^{p-3} a_2 u = a_1^p h_2$. In any case $a_o^{p^2+p-2} a_1^{p-3} a_2 u$ survives to E_∞ and therefore $\pi(p^2 q-1)$ is at least Z_{p^3}. $a_o^{p-1} c$ and e_1 if $p > 3$ are other elements of $H^*(A)$ which must survive to E_∞.

We now consider the case $p = 2$, and we let $\pi(m) = \pi_m^s(Z;Z_2)$. Toda [25] has proven:

<u>Theorem II.7.9</u>: Suppose $p = 2$; then the groups $\pi(m)$, $0 < m \leq 19$, are:

1. $\pi(e) = \pi(5) = \pi(12) = \pi(13) = 0$;

2. $\pi(1) = Z_2 = \{\eta\}$, $\pi(2) = Z_2 = \{\eta^2\}$ $\qquad (\eta \longleftrightarrow h_1)$

3. $\pi(3) = Z_8 = \{\nu\}$, $\pi(6) = Z_2 = \{\nu^2\}$ $(\nu \longleftrightarrow h_2)$

4. $\pi(7) = Z_{16} = \{\sigma\}$ $(\sigma \longleftrightarrow h_3)$

5. $\pi(8) = Z_2+Z_2 = \{\bar{\nu}\} + \{\varepsilon\}$ $(\bar{\nu} \longleftrightarrow h_1h_3,\ \varepsilon \longleftrightarrow c_o)$

6. $\pi(9) = (Z_2)^3 = \{\nu^3\} + \{\mu\} + \{\eta\varepsilon\}$ $(\mu \longleftrightarrow P^1h_1)$

7. $\pi(10) = Z_2 = \{\eta\mu\}$

8. $\pi(11) = Z_8 = \{\zeta\}$ $(\zeta \longleftrightarrow P^1h_2)$

9. $\pi(14) = Z_2+Z_2 = \{\sigma^2\} + \{X\}$ $(X \longleftrightarrow d_o)$

10. $\pi(15) = Z_{32}+Z_2 = \{\rho\} + \{\eta X\}$ $(\rho \longleftrightarrow h_o^3h_4)$

11. $\pi(16) = Z_2+Z_2 = \{\eta^*\} + \{\eta\rho\}$ $(\eta^* \longleftrightarrow h_1h_4,\ \eta\rho \longleftrightarrow P^1c_o)$

12. $\pi(17) = (Z_2)^4 = \{\eta\eta^*\} + \{\nu X\} + \{\eta^2\rho\} + \{\bar{\mu}\}$ $(\bar{\mu} \longleftrightarrow P^2h_1)$

13. $\pi(18) = Z_8+Z_2 = \{\nu^*\} + \{\eta\bar{\mu}\}$ $(\nu^* \longleftrightarrow h_2h_4)$

14. $\pi(19) = Z_8+Z_2 = \{\zeta\} + \{\bar{\sigma}\}$ $(\bar{\zeta} \longleftrightarrow P^2h_2,\ \bar{\sigma} \longleftrightarrow c_1)$

where $(Z_2)^3 = Z_2+Z_2+Z_2$, etc., and the notation $\{X\}$ means the cyclic group generated by X. These elements satisfy the relations:

$\eta^3 = 4\nu$, $\eta\sigma = \bar{\nu}+\varepsilon$, $\eta\bar{\nu} = \nu^3$, $\eta^2\mu = 4\zeta$, $\eta^2\eta^* = 4\nu^*$, $\eta^2\bar{\mu} = 4\bar{\zeta}$, $\eta\rho = \sigma\mu$, $\eta^2\rho = \varepsilon\mu$, $\eta\bar{\mu} = \mu^2$; all products not mentioned here or in the definition of the groups are zero in the range $m \leq 19$.

The correspondence on the right relates the group generators on the left to survivors to E_∞ in the Adams spectral sequence. That the elements listed are survivors follows from the fact that there is only one pattern of differentials consistent with the stated group structures, namely that given in

<u>Theorem II.7.10</u> : For all $i \geq 0$ and $n \geq 0$:

a) $\delta_2(h_4)=h_o h_3^2$; $\delta_3(h_o h_4)=h_o d_o$

b) $\delta_r(P^i d_o)=0$ for all r ; $\delta_r(P^i g^n)=0$ for all r

c) $\delta_2(P^i e_o)=P^i h_1^2 d_o$

d) $\delta_2(f_o) = h_o h_2 d_o$, $\delta_2(P^i j) = P^{i+1} h_2 d_o$

e) $\delta_2(P^i k) = P^{i+1} h_o g$

f) $\delta_2(P^i \ell) = P^i h_o d_o e_o$

g) $\delta_2(P^i m) = P^i h_o e_o^2$

<u>Proof</u>: a) and $\delta_r(g)=0$, $\delta_2(e_o)=h_1^2 d_o$, and $\delta_2(f_o)=h_o h_2 d$ are implied by the requirement of consistency with Theorem II.7.9. $\delta_r(P^i d_o)=0$ is clear from dimensional considerations and $P^i g^n = P^i (d_o)^{2n} = (d_o)^{2n-1} P^i d_o$ implies $\delta_r(P^i g^n)=0$. $h_o^2 j = P^1 h_o f = P^1 h_1 e_o = e_o P^1 h_1$, hence $P^i h_o^2 j = e_o P^{i+1} h_1$. This implies $\delta_2(P^i h_o^2 j) = h_1^2 d_o P^{i+1} h_1 = P^{i+1} h_o^2 h_2 d_o$, hence $\delta_2(P^i j) = P^{i+1} h_2 d_o$. The proofs of the remaining statements are equally simple.

<u>Remarks II.7.11</u>: The differentials a) and b) were obtained by Adams, who has proven $\delta_2(h_i)=h_o h_{i-1}^2$ for $i \geq 4$. Note that the group extensions from $E^o \pi^s(S;Z_2)$ to $\pi^s(S;Z_2)$ are non-trivial. The product $h_1 \cdot h_3 = h_1 h_3$ lifts to $\eta\sigma = \bar{\nu} + \varepsilon$; the relations $h_1 \cdot h_o^3 h_4 = 0$, $h_3 \cdot P^1 h_1 = 0$, and the fact that $P^1 c_o \neq 0$ lifts to $\eta\rho = \sigma\mu \neq 0$; the relations $h_1^2 \cdot h_o^3 h_4 = 0$ then lift to $\eta^2 \rho = \eta\sigma\mu = \varepsilon\mu$ (corresponding to $c_o P^1 n_1 = P^1 h_1 c_o$). Each other relation in $E^o \pi^s(S;Z_2)$ lifts to the same relation between corresponding elements of $\pi^s(S;Z_2)$. Toda defines many of the generators in terms of toric constructions, and the fact that $P^1 x = \langle x, h_o^4, h_3 \rangle$ shows agreement with the corresponding Massey products in most cases. However $\{\nu,\eta,\nu\} = \bar{\nu}$ but $\langle h_2, h_1, h_2 \rangle = 0$. The preceding theorem states that the differentials actually have at least some periodicity.

Our results imply the following

Corollary II.7.13: Let $\pi(m) = \pi_m^s(S;Z_2)$. Then, naming generators in E_∞ of the Adams' spectral sequence, we have:

1. $\pi(20) = Z_8 = \{g\}$
2. $\pi(21) = Z_2+Z_2 = \{h_3^3\} + \{h_1 g\}$
3. $\pi(22) = Z_2+Z_2 = \{h_2 c_1\} + \{P^1 d_o\}$ ~~, unless~~

   ~~$\delta_5(h_4 c_o) = P^1 h_o d_o$ or $\delta_6(h_4 c_o) = P^1 h_o^2 d_o$~~
4. $\pi(23) = Z_2+Z_2+Z_4+Z_{16} = \{h_4 c_o\} + \{P^1 h_1 d_o\} + \{h_2 g\} + \{P^2 h_3\}$

   ~~unless $\delta_r(h_4 c_o) \neq 0$, $r = 5$ or 6~~
5. $\pi(24) = Z_2+Z_2 = \{h_1 h_4 c_o\} + \{P^2 c_o\}$
6. $\pi(25) = Z_2+Z_2 = \{P^2 h_1 c_o\} + \{P^3 h_1\}$
7. $\pi(26) = Z_2+Z_2 = \{h_2^2 g\} + \{P^3 h_1^2\}$
8. $\pi(27) = Z_8 = \{P^3 h_2\}$
9. $\pi(28) = Z_2 = \{P^1 g\}$

Of course, our results limit the order of $\pi(m)$, $29 \leq m \leq 42$. More important, the techniques developed allow calculation of $H^{s,t}(A)$ in higher dimensions without an unreasonably large amount of tedious computation.

What we require now are general procedures for calculating the differentials in the Adams spectral sequence. It is possible that the differentials within the region of periodicity are periodic and that certain subgroups of $\pi^s(S;Z_2)$ show periodicity. In the case of odd primes, the latter possibility is also open. In any case, the machinery developed in this thesis gives weight to the Adams spectral sequence as a practical device for the calculation of stable homotopy groups.

Appendix A: The cohomology of the Steenrod algebra, p = 2

The following graphs describe $H^{s,t}(A)$ in the range $t - s \leq 42$ for the case $p = 2$. The notation is that of Theorem II.6.16. There are no non-zero elements of $H^{s,t}(A)$ in those dimensions of s and t which are omitted. We state the known differentials in the Adams spectral sequence and for $t - s \leq 29$ name the largest possibility for $\pi_{t-s}(S) = \pi^{s}_{t-s}(S;Z_2)$, the notation $\leq G$ meaning that the relevant group is at most G.

A.2

$H^*(A)$

$p = 2$

s \ $t-s$	17	18	19	20	21	22	23	24	25	26	27	28	29
15											$P^3h_0^2h_2$		
14										$P^3h_1^2$	$P^3h_0h_2$		
13									P^3h_1		P^3h_2		
12							$P^2h_0^3h_3$		$P^2h_1c_0$				
11			$P^2h_0^2h_2$				$P^2h_0^2h_3$	P^2c_0	$P^1h_0^2h_2d_0$				
10		$P^2h_1^2$	$P^2h_0h_2$			$P^1h_0^2d_0$	$P^2h_0h_3$	$P^1h_1^2d_0$	$P^1h_0h_2d_0$			$P^1h_0^2g$	
9	P^2h_1		P^2h_2			$P^1h_0d_0$	P^2h_3 / $P^1h_1d_0$		$P^1h_2d_0$	h_0^2j		P^1h_0g	h_0^2k
8	$P^1h_1c_0$					P^1d_0	h_0i		P^1e_0	h_0j		P^1g	h_0k
7	$h_0^2h_2d_0$						i			j			k
6	$h_0h_2d_0$			h_0^2g			h_0h_2g			h_2^2g			
5	h_2d_0	h_0f_0		h_0g	h_1g		h_2g	$h_1h_4c_0$					
4	e_0	f_0 / $h_0^2h_2h_4$		g		h_2c_1	h_4c_0						
3	$h_1^2h_4$	$h_0h_2h_4$	c_1		h_3^3								
2		h_2h_4											
δ_r	$\delta_2(e_0)=h_1^2d_0$; $\delta_2(f_0)=h_0h_2d_0$					$\delta_2(k)=P^1h_0d_0$		$\delta_2(P^1e_0)=P^1h_1^2d_0$	$\delta_2(j)=P^1h_2d_0$			$\delta_2(k)=P^1h_0g$	
$\pi_{t-s}(S)$	$(Z_2)^4$	Z_8+Z_2	Z_8+Z_2	Z_8	$(Z_2)^2$	Z_2 + ~~Z_2~~	Z_4+Z_2 $+Z_{16}$ ~~Z_2~~	$(Z_2)^2$	$(Z_2)^2$	$(Z_2)^2$	Z_8	Z_2	$\leq Z_2$

s \ $t-s$	0	1	2	3	6	7	8	9	10	11	14	15	16
8	h_0^8											$h_0^7h_4$	
7	h_0^7									$P^1h_0^2h_2$		$h_0^6h_4$	P^1c_0
6	h_0^6								$P^1h_1^2$	$P^1h_0h_2$	$h_0^2d_0$	$h_0^5h_4$	$h_1^2d_0$
5	h_0^5							P^1h_1		P^1h_2	h_0d_0	h_1d_0 / $h_0^4h_4$	
4	h_0^4					$h_0^3h_3$		h_1c_0			d_0	$h_0^3h_4$	
3	h_0^3			$h_0^2h_2$		$h_0^2h_3$	c_0	h_2^3			$h_0h_3^2$	$h_0^2h_4$	
2	h_0^2		h_1^2	h_0h_2	h_2^2	h_0h_3	h_1h_3				h_3^2	h_0h_4	h_1h_4
1	h_0	h_1		h_2		h_3						h_4	
0	1												
δ_r											$\delta_2(h_4)=h_0h_3^2$; $\delta_3(h_0h_4)=h_0d_0$		
$\pi_{t-s}(S)$	Z	Z_2	Z_2	Z_8	Z_2	Z_{16}	$(Z_2)^2$	$(Z_2)^3$	Z_2	Z_8	$(Z_2)^2$	$Z_{32}+Z_2$	$(Z_2)^2$

A : 3

$H^{*}(A)$

$p = 2$

s \ $t-s$	30	31	32	33	34	35	36	37	38	39	40	41	42
22													$P^5h_1^2$
21												P^5h_1	
20										$P^4h_0^3h_3$		$P^4h_1c_0$	
19						$P^4h_0^3h_2$				$P^4h_0^2h_3$	P^4c_0	$P^3h_0^2h_2d_0$	
18					$P^4h_1^2$	$P^4h_0h_2$			$P^3h_0^2d_0$	$P^4h_0h_3$	$P^3h_1^2d_0$	$P^3h_0h_2d_0$	
17				P^4h_1		P^4h_2			$P^3h_0d_0$	P^4h_3 / $P^3h_1d_0$		$P^3h_2d_0$	$P^2h_0^2j$
16		$h_0^{15}h_5$		$P^3h_1c_0$					P^3d_0			P^3e_0	P^2h_0j
15		$h_0^{14}h_5$	P^3c_0	$P^2h_0^2h_2d_0$									P^2j
14	$P^2h_0^2d_0$	$h_0^{13}h_5$	$P^2h_1^2d_0$	$P^2h_0h_2d_0$			$P^2h_0^2g$			$P^1h_0^2d_0e_0$			$P^1h_0^2e_0^2$
13	$P^2h_0d_0$	$h_0^{12}h_5$ / $P^2h_1d_0$		$P^2h_2d_0$	$P^1h_0^2j$		P^2h_0g	$P^2h_0^2k$		$P^1h_0d_0e_0$	$P^1h_0^2l$		$P^1h_0e_0^2$
12	P^2d_0	$h_0^{11}h_5$		P^2e_0	P^1h_0j		P^2g	P^1h_0k		$P^1d_0e_0$	P^1h_0l		$P^1e_0^2$
11		$h_0^{10}h_5$			P^1j			P^1k			P^1l		
10		$h_0^9h_5$, $h_0^2d_0e_0$			$h_0^2e_0^2$				P^1r		h_1u		
9		$h_0^8h_5$, $h_0d_0e_0$	h_0^2l		$h_0e_0^2$	h_0^2m			h_0f_0g	u			v
8		$h_0^7h_5$, d_0e_0	h_0l		e_0^2	h_0m		e_0g	f_0g		g^2		
7		$h_0^6h_5$	l	h_1q		m		h_2^2n		c_1g			
6	r	$h_0^5h_5$	q		h_2n						$h_0^2f_1$		
5	$h_0^3h_4^2$	$h_0^4h_5$, n		h_0p		h_2d_1			$h_0^3h_3h_5$	h_3d_1	$h_1h_5c_0$, h_0f_1	h_2e_1	
4	$h_0^2h_4^2$	$h_0^3h_5$	d_1	p	$h_0^2h_2h_5$				$h_0^2h_3h_5$, e_1	h_5c_0	$h_2^3h_5$, f_1	$h_0^2c_2$	
3	$h_0h_4^2$	$h_0^2h_5$, $h_1h_4^2$		$h_1^2h_5$	$h_0h_2h_5$			$h_2^2h_5$	$h_0h_3h_5$	$h_1h_3h_5$		h_0c_2	
2	h_4^2	h_0h_5	h_1h_5		h_2h_5				h_3h_5			c_2	
1		h_5											

$\delta_2(l) = h_0d_0e_0$ $\quad \delta_2(P^1j) = P^2h_2d$ $\quad \delta_2(P^1k) = P^2h_0g$ $\quad \delta_2(P^1l) = P^1h_0d_0e_0$ $\quad \delta_2(P^3e_0) = P^3h_1^2d_0$

$\delta_2(m) = h_0e_0^2$ $\quad \delta_2(P^2j) = P^3h_2d_0$

Appendix B: The cohomology of the Steenrod algebra, p = 3

The following graphs describe $H^{s,t}(A)$ in the range $t-s \leq 88$ for the case $p = 3$. The notation is essentially that of Theorem II.6.10, but to simplify the writing of elements, we have replaced b_1^0 by b, b_1^1 by b_1, b_2^0 by b_2, k_1^0 by k, g_1^1 by g_1, and g_2^0 by g_2. We emphasize that the elements as written do not necessarily represent products, since a_1, a_2, b_2, etc., are not survivors to E_∞ in the spectral sequence passing from $H^*(E^0A)$ to $H^*(A)$. There are no non-zero elements of $H^{s,t}(A)$ in those dimensions of s and t which are omitted. For $0 < t-s \leq 62$, we state the known and possible non-zero differentials in the Adams spectral sequence, and state the structure of $\pi_{t-s}(S) = \pi_{t-s}(S;Z_3)$. The generators of most of these groups are defined in Theorem II.7.3 and Remarks II.7.8. In dimensions $63 \leq t-s \leq 88$, we name possible survivors to E_∞, those elements written without parentheses being known to survive.

$H^*(A)$

$p = 3$

s \ t−s	23	25	26	27	28	29	30	31	32	33	34	35	36
9											$a_0 a_1^2 b$	$a_0^8 h_2$	
8								$a_1^2 g_1$	$a_0 a_1 b^2$		$a_1^2 b$	$a_0^7 h_2$, $a_1 h_0 b^2$	
7				$a_1^2 h_0$		$a_1 g_1 b$	$a_0 b^3$		$a_1 b^2$	$h_0 b^3$		$a_0^6 h_2$	
6	$a_0^3 u$	$a_1 h_0 b$		$g_1 b^2$			b^3			$a_0 u b$	$a_0^4 b_1$	$a_0^5 h_2$	$h_0 u b$
5	$a_0^2 u$, $h_0 b^2$						$a_1 g_2$	$h_1 b^2$		$u b$	$a_0^3 b_1$	$a_0^4 h_2$	
4	$a_0 u$		$h_0 u$		$g_2 b$						$a_0^2 b_1$	$a_0^3 h_2$	k ⊢
3	u					$h_0 k$					$a_0 b_1$	$a_0^2 h_2$	
2			k								b_1	$a_0 h_2$	
1												h_2	
δ_r	$\delta_2(u) = a_1 b$		$\delta_2(h_0 u) = a_1 h_0 b$	$\delta_2(g_2 b) = -g_1 b^2$		$\delta_2(a_1 g_2) = -a_1 g_1 b$	$\delta_2(h_1 b^2) = a_0 b^3$		$\delta_2(ub) = a_1 b^2$; $\delta_5(b_1) = h_0 b^3$ or 0		$\delta_2(h_2) = a_0 b_1$; $\delta_3(a_0^4 h_2) = a_1^2 b$	$\delta_2(h_0 u b) = a_1 h_0 b^2$	
$\pi_{t-s}(S)$	$Z_3 + Z_9$ $\alpha_1 \beta_1^2$ α'_6	0	Z_3 β_2	Z_3 α_7	0	Z_3 $\alpha_1 \beta_2$	Z_3 β_1^3	Z_3 α_8	0	Z_3 or 0 $(\alpha_1 \beta_1^3)$	Z_3 or 0 (γ)	Z_{27} α'_9	Z_3 $\beta_1 \beta_2$

s \ t−s	0	3	7	10	11	13	15	17	18	19	20	21	22
6	a_0^6												$a_0 a_1 b$
5	a_0^5									$a_1 g_1$	$a_0 b^2$		$a_1 b$
4	a_0^4						$a_1 h_0$	$g_1 b$			b^2		
3	a_0^3			$a_0 b$	$a_0^2 h_1$	$h_0 b$						$h_1 b$	
2	a_0^2		g_1	b	$a_0 h_1$				g_2				
1	a_0	h_0			h_1								
0	1												
δ_r				$\delta_2(h_1) = a_0 b$				$\delta_2(g_2) = -g_1 b$			$\delta_2(h_1 b) = a_0 b^2$		
$\pi_{t-s}(S)$	Z ι	Z_3 α_1	Z_3 α_2	Z_3 β_1	Z_9 α'_3	Z_3 $\alpha_1 \beta_1$	Z_3 α_4	0	0	Z_3 α_5	Z_3 β_1^2	0	

B.3

$H^*(A)$

$p = 3$

s \ $t-s$	37	38	39	40	41	42	43	44	45	46	47	48	49
12										$a_0a_1^3b$	$a_0^3a_1^2u$		$a_1^3h_0b$
11							$a_1^3g_1$	$a_0a_1^2b^2$		a_1^3b	$a_0^2a_1^2u$ $a_1^2h_0b^2$		$a_1g_1b^3$
10			$a_1^3h_0$		$a_1^2g_1b$	$a_0a_1b^3$		$a_1^2b^2$	$a_1h_0b^3$		$a_0a_1^2u$ g_1b^4		
9	$a_1^2h_0b$		$a_1g_1b^2$	a_0b^4		a_1b^3	h_0b^4		a_0a_1ub		a_1^2u	a_1h_0ub	
8	g_1b^3			b^4		$a_1^2g_2$	a_0ub^2		a_1ub	h_0ub^2		g_2b^3	
7		a_1h_0u		a_1g_2b	h_1b^3		ub^2						h_0kb^2
6		g_2b^2								kb^2			
5			h_0kb										
4								bb_1	a_0h_2b				
3	h_0b_1					g_1h_2			h_2b				
2		h_0h_2											
1													
δ_r	$\delta_2(g_2b^2)=-g_1b^3$ $\delta_2(a_1h_0u)=a_1^2h_0b$		$\delta_2(a_1g_2b)=-a_1g_1b^2$	$\delta_2(h_1b^3)=a_0b^4$	$\delta_2(a_1^2g_2)=-a_1^2g_1b$	$\delta_2(ub^2)=a_1b^3$	$\delta_5(bb_1)=h_0b^4$ or 0	$\delta_2(a_1ub)=a_1^2b^2$	$\delta_2(h_0ub^2)=a_1h_0b^3$	$\delta_2(a_1^2u)=a_1^3b$	$\delta_2(g_2b^3)=-g_1b^4$ $\delta_2(a_1h_0ub)=a_1^2h_0b^2$		
$\pi_{t-s}(S)$	Z_3 $\alpha_1\gamma$	Z_3 ϵ_1	Z_3+Z_3 $\alpha_1\beta_1\beta_2$ α_{10}	Z_3 β_1^4	0	Z_3 ϵ_2	Z_3+Z_3 or Z_3 $(\alpha_1\beta_1^4)$ α_{11}	Z_3 or 0 $(\beta_1\gamma)$	Z_9 ϕ	Z_3 $\beta_1^2\beta_2$	Z_9 α'_{12}	0	Z_3 $\alpha_1\beta_1^2\beta_2$

$H^*(A)$

$p = 3$

s \ t−s	50	51	52	53	54	55	56	57	58	59	60	61	62
16													
15									$a_0a_1^4b$	$a_0^3a_1^3u$		$a_1^4h_0b$	
14						$a_1^4g_1$	$a_0a_1^3b^2$		a_1^4b	$a_0^2a_1^3u$ $a_1^3h_0b^2$		$a_1^2g_1b^3$	$a_0a_1b^5$
13		$a_1^4h_0$		$a_1^3g_1b$	$a_0a_1^2b^3$		$a_1^3b^2$	$a_1^2h_0b^3$		$a_0a_1^3u$ $a_1g_1b^4$	a_0b^6		a_1b^5 $a_1^3h_0u$
12		$a_1^2g_1b^2$	$a_0a_1b^4$		$a_1^2b^3$	$a_1h_0b^4$		g_1b^5 $a_0a_1^2ub$		a_1^3u	b^6 $a_1^2h_0ub$		$a_1^2g_2b^2$
11	a_0b^5		a_1b^4	h_0b^5	$a_1^3g_2$	$a_0a_1ub^2$		a_1^2ub	$a_1h_0ub^2$		$a_1g_2b^3$	h_1b^5	
10	b^5 $a_1^2h_0u$		$a_1^2g_2b$	a_0ub^3		a_1ub^2	h_0ub^3		g_2b^4				
9	$a_1g_2b^2$	h_1b^4		ub^3						h_0kb^3			
8							kb^3						
7													
6					b^2b_1							e_1	
5								$a_0^2h_1b_2$					
4			k^2					$a_0h_1b_2$	a_0^2c		kb_1		
3								h_1b_2	a_0c				
2									c				
1													

δ_r:

$\delta_2(a_1g_2b^2) = -a_1g_1b^3$

$\delta_2(a_1ub^2) = a_1^2b^3$

$\delta_2(a_1^3u) = a_1^4b$

$\delta_2(a_1^2g_2b^2) = -a_1^2g_1b^3$

$\delta_2(ub^3) = a_1b^4$

$\delta_2(a_1^2ub) = a_1^3b^2$

$\delta_2(a_1g_2b^3) = -a_1g_1b^4$

$\delta_2(a_1^2h_0u) = a_1^3h_0b$

$\delta_2(h_0ub^3) = a_1h_0b^4$

$\delta_2(g_2b^4) = -g_1b^5$

$\delta_2(a_1^3h_0u) = a_1^4h_0b$

$\delta_2(a_1^3g_2) = -a_1^3g_1b$

$\delta_2(a_1^2h_0ub) = a_1^3h_0b^2$

$\delta_2(a_1h_0ub^2) = a_1^2h_0b^3$

$\delta_2(h_1b^4) = a_0b^5$

$\delta_2(h_1b^5) = a_0b^6$

$\delta_2(a_1^2g_2b) = -a_1^2g_1b^2$

$\delta_5(h_1b_2) = kb^3$ or 0

$\delta_5(bb_1^2) = h_0b^5$ or 0

$\delta_2(c) = a_0h_1b_2$ or 0

$\delta_5(kb_1) = h_0kb^3$ or 0

$(\delta_3(c) = a_0^2h_1b_2$ or $0)$

$\delta_6(e_1) = b^6$ or 0

	50	51	52	53	54	55	56	57	58	59	60	61	62
$\pi_{t-s}(S)$	Z_3 β_1^5	Z_3 α_{13}	Z_3 β_2^2	Z_3 or 0 $(\alpha_1\beta_1^5)$	Z_3 or 0 $(\beta_1^2\gamma)$	Z_3 α_{14}	Z_3 or 0 $(\beta_1^3\beta_2)$	Z_{27}, Z_9, Z_3, or 0	Z_{27}, Z_9, Z_3, or 0	Z_3+Z_9 or Z_9 $(\alpha_1\beta_1^3\beta_2)$ α'_{15}	Z_3+Z_3, Z_3, or 0 $(\beta_2\gamma)$ (β_1^6)	Z_3 or 0	0

B.5

$H^*(A)$

$p = 3$

s \ t-s	63	64	65	66	67	68	69	70	71	72	73	74	75
19													$a_1^6 h_0$
18								$a_0 a_1^5 b$	$a_0^{12} a_2 u$		$a_1^5 h_0 b$		$a_1^4 g_1 b^2$
17					$a_1^5 g_1$	$a_0 a_1^4 b^2$		$a_1^5 b$	$a_0^{11} a_2 u$ $a_1^4 h_0 b^2$		$a_1^3 g_1 b^3$	$a_0 a_1^2 b^5$	$a_1 h_0 b^6$
16	$a_1^5 h_0$		$a_1^4 g_1 b$	$a_0 a_1^3 b^3$		$a_1^4 b^2$	$a_1^3 h_0 b^3$		$a_0^{10} a_2 u$ $a_1^2 g_1 b^4$	$a_0 a_1 b^6$		$a_1^2 b^5$ $a_1^4 h_0 u$	$a_0 a_1 u b^4$
15	$a_1^3 g_1 b^2$	$a_0 a_1^2 b^4$		$a_1^3 b^3$	$a_1^2 h_0 b^4$		$a_1 g_1 b^5$ $a_0 a_1^3 u b$	$a_0^7 l$ $a_0 b^7$	$a_0^9 a_2 u$	$a_1 b^6$ $a_1^3 h_0 u b$	$h_0 b^7$	$a_1^3 g_2 b^2$	$a_1 u b^4$
14		$a_1^2 b^4$	$a_1 h_0 b^5$	$a_1^4 g_2$	$g_1 b^6$ $a_0 a_1^2 u b^2$		$a_1^3 u b$	b^7 / $a_0^6 l$ $a_1^2 h_0 u b^2$	$a_0^8 a_2 u$	$a_1^2 g_2 b^3$	$a_0 u b^5$		
13	$h_0 b^6$	$a_1^3 g_2 b$	$a_0 a_1 u b^3$		$a_1^2 u b^2$	$a_1 h_0 u b^3$		$a_0^5 l$ $a_1 g_2 b^4$	$a_0^7 a_2 u$ $h_1 b^6$		$u b^5$		
12	$a_0 u b^4$		$a_1 u b^3$	$h_0 u b^4$		$g_2 b^5$		$a_0^4 l$	$a_0^6 a_2 u$				
11	$u b^4$						$h_0 k b^4$	$a_0^3 l$	$a_0^5 a_2 u$				
10				$k b^4$				$a_0^2 l$	$a_0^4 a_2 u$			$b^4 b_1$	
9								$a_0 l$	$a_0^3 a_2 u$		$a_1 e_1$		$e_2 b$
8		$b^3 b_1$				$a_0 f$		l	$e_1 b$ / $a_0^2 a_2 u$				
7			e_2			f			$a_0 a_2 u$			$a_2 h_0 u$	
6							$a_2 h_1 b$		$a_2 u$	$h_0 u b_2$		$g_2 b_2 b$	
5				$a_2 g_2$	$h_1 b_2 b$	$a_0 b_1^2$						$a_2 k$	$k h_0 b_2$
4						b_1^2					$h_0 w$		
3							$h_2 b_1$						
s / t-s	63	64	65	66	67	68	69	70	71	72	73	74	75
elements (possibly) surviving to E_∞	$a_1^5 h_0$ $(h_0 b^6)$	$(b^3 b_1)$	(e_2)	$(a_2 g_2)$ $(k b^4)$	$(h_1 b_2 b)$ $(a_1^5 g_1)$	$(a_0 f)$ (f) (b_1^2) $a_0 b_1^2$	$(a_2 h_1 b)$ $(h_0 k b^4)$	(b^7) (l) $(a_0^i l)$ $(a_1^5 b)$ $(a_0 a_1^5 b)$	$(e_1 b)$ $(a_2 u)$ $(a_0^i a_2 u)$ $a_0^{10} a_2 u$ $a_0^{11} a_2 u$ $a_0^{12} a_2 u$	$(h_0 u b_2)$	$(h_0 b^7)$ $(h_0 l)$ $(h_0 w)$	$(b^4 b_1)$ $(a_2 h_0 u)$ $(g_2 b_2 b)$ $(a_2 k)$	$(e_2 b)$ $(k h_0 b_2)$
$\pi_{t-s}(S)$, $t-s \le 65$	$Z_3 + Z_3$ or Z_3 α_{16} $(\alpha_1 \beta_1^6)$	Z_3 or 0 $(\beta_1^3 \gamma)$	Z_3 or 0										

B.6

$H^*(A)$

$p = 3$

s \ t-s	76	77	78	79	80	81	82	83	84	85	86	87	88
22												$a_1^7 h_0$	
21							$a_0 a_1^6 b$	$a_0^3 a_1^5 u$		$a_1^6 h_0 b$		$a_1^5 g_1 b^2$	$a_0 a_1^4 b^4$
20				$a_1^6 q_1$	$a_0 a_1^5 b^2$		$a_1^6 b$	$a_1^5 h_0 b^2$ $a_0^2 a_1^5 u$		$a_1^4 g_1 b^3$	$a_0 a_1^3 b^5$		$a_1^4 b^4$
19		$a_1^5 g_1 b$	$a_0 a_1^4 b^3$		$a_1^5 b^2$	$a_1^4 h_0 b^3$		$a_1^3 g_1 b^4$ $a_0 a_1^5 u$	$a_0 a_1^2 b^6$		$a_1^3 b^5$ $a_1^5 h_0 u$	$a_1^2 h_0 b^6$	$a_1^5 g_2 b$
18	$a_0 a_1^3 b^4$		$a_1^4 b^3$	$a_1^3 h_0 b^4$		$a_1^2 g_1 b^5$ $a_0 a_1^4 u b$	$a_0 a_1 b^7$	$a_1^5 u$	$a_1^2 b^6$ $a_1^4 h_0 u b$	$a_1 h_0 b^7$	$a_1^4 g_2 b^2$	$g_1 b^8$ $a_0 a_1^2 u b^4$	
17	$a_1^3 b^4$	$a_1^2 h_0 b^5$	$a_1^5 g_2$	$a_1 g_1 b^6$ $a_0 a_1^3 u b^2$	$a_0 b^8$	$a_1^4 u b$	$a_1 b^7$ $a_1^3 h_0 u b^2$	$h_0 b^8$	$a_1^3 g_2 b^3$	$a_0 a_1 u b^5$		$a_1^2 u b^4$	$a_1 h_0 u b^5$
16	$a_1^4 g_2 b$	$g_1 b^7$ $a_0 a_1^2 u b^3$		$a_1^3 u b^2$	b^8 $a_1^2 h_0 u b^3$		$a_1^2 g_2 b^4$	$a_0 u b^6$		$a_1 u b^5$	$h_0 u b^6$		$g_2 b^7$
15		$a_1^2 u b^3$	$a_1 h_0 u b^4$		$a_1 g_2 b^5$	$h_1 b^7$		$u b^6$					
14	$h_0 u b^5$		$g_2 b^6$								$k b^6$		
13				$h_0 k b^5$									
12	$k b^5$								$b^5 b_1$	$a_1^2 e_1$		$a_1 e_2 b$	$a_0 f b^2$
11					$a_0 a_1 f$	$a_0^3 a_1 u b_2$		$a_1 e_1 b$		$e_2 b^2$			$f b^2$
10		$a_1 e_2$	$a_0 f b$		$a_1 f$	$a_0^2 a_1 u b_2$ $e_1 b^2$					$a_1 a_2 h_0 u$		$a_1 a_2 g_2 b$ $a_1^2 g_2 b_2$
9			$f b$			$a_0 a_2 u b$ $a_0 a_1 u b_2$	$a_0^3 m$		$a_2 h_0 u b$ $a_1 h_0 u b_2$		$a_2 g_2 b^2$ $a_1 g_2 b_2 b$	$h_1 b_2 b^3$	
8			$a_1 a_2 g_2$	$a_2 h_1 b^2$ $a_0 u b_2 b$		$a_2 u b$ $a_1 u b_2$	$a_0^2 m$ $h_0 u b_2 b$		$g_2 b_2 b^2$			$a_2 k h_0 b$	$b^2 b_1^2$
7	$a_2 g_2 b$ $a_1 g_2 b_2$	$h_1 b_2 b^2$		$u b_2 b$			$a_0 m$		$a_2 k b$	$h_0 m$ $k h_0 b_2 b$		$g_1 w b$	
6		$a_2 k h_0$	$b b_1^2$		$a_0 w b$		m $k b_2 b$	$h_0 w b$		$a_2 h_0 b_1$	$k^2 b_1$		
5		$g_1 w$			$w b$	$a_0^2 h_2 b_2$		$h_0 b_1 b_2$	–		$a_2 h_0 h_2$		$X b$ $g_1 h_2 b_2$
4						$h_0 X$ $a_0 h_2 b_2$	$a_0^2 d$		$h_0 h_2 b_2$				
3			X			$h_2 b_2$	$a_0 d$						
2							d						
elements (possibly) surviving to E_∞, $s > 12$				$(a_1^6 q_1)$ $(h_0 k b^5)$	(b^8)			$(h_0 b^8)$ $(a_0^2 a_1^5 u)$ $(a_0^3 a_1^5 u)$			$(k b^6)$	$(a_1^7 h_0)$	

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