Artificial conditions for the linear elasticity equations

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Abstract

In this paper, we consider the equations of linear elasticity in an exterior domain. We exhibit artificial boundary conditions on a circle, which lead to a non-coercive second order boundary value problem. In the particular case of an axisymmetric geometry, explicit computations can be performed in Fourier series proving the well-posedness except for a countable set of parameters. A perturbation argument allows to consider near-circular domains. We complete the analysis by some numerical simulations.

Keywords. Linear elasticity equations, singular perturbation, artificial boundary conditions, Ventcel condition, Dirichlet-to-Neumann map, spectral theory.

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1 Introduction

To determine the influence of geometrical perturbations, we can perform a multiscale asymptotic analysis of the equations of elasticity for a linear isotropic material with Hooke's law H. Recall that for any symmetric matrix e, H is defined by

$$He = \lambda \operatorname{tr}(e) \operatorname{Id} + 2\mu\, e,$$

where μ and λ are the Lamé constants of the material. In this work, we are more particularly interested in the numerical computation of corrective terms appearing in the evaluation of stress concentration due to the presence of geometrical defects.

Let Ω_0 be a domain of $\mathbb{R}^2$ such that the origin 0 belongs to the domain. We consider a domain Ω_ε pierced with some perturbations of size ε near well separated points x^j (see Figure 1): for N defects it can be defined as

$$\Omega_\varepsilon = \Omega_0 \setminus \cup_{j=1}^{N} \overline{\omega_\varepsilon^j}, \qquad \text{with} \qquad \omega_\varepsilon^j = x^j + \varepsilon \omega^j.$$

The case of a single perturbation was presented in [12]. The case of two relatively close inclusions is studied in [9, 8].

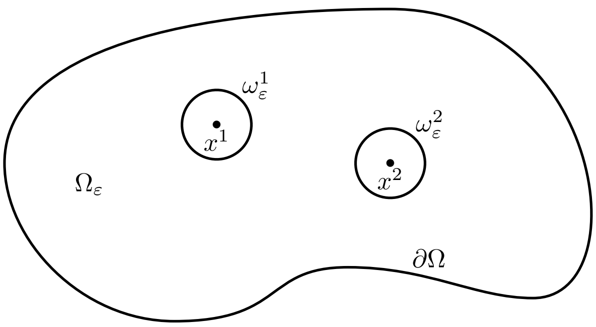

Figure 1: The perturbed domain when $N = 2$.

We assume that the domains ω^j contain the origin 0. We denote by $\mathbf{H}_\infty^j$ the unbounded domains obtained by a blow-up around each perturbation:

$$\mathbf{H}_\infty^j = \mathbb{R}^2 \setminus \overline{\omega^j}.$$

The problem we focus on is written on the perturbed domain as:

$$\left\{ \begin{array}{rcll} -\operatorname{div} \sigma(\mathbf{u}_\varepsilon) = -\mu \Delta \mathbf{u}_\varepsilon - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u}_\varepsilon & = & \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \mathbf{u}_\varepsilon & = & \mathbf{u}^d & \text{on } \Gamma_d, \\ \sigma(\mathbf{u}_\varepsilon) \cdot \mathbf{n} & = & \mathbf{g} & \text{on } \Gamma_n, \end{array} \right. \tag{1.1}$$

where $\mathbf{u}_\varepsilon$ denotes the displacement and $\sigma(\mathbf{u}_\varepsilon) = He(\mathbf{u}_\varepsilon)$ and $e(\mathbf{u}_\varepsilon)$ stand respectively for the stress tensor and the linearized strain tensor defined by

$$\sigma(\mathbf{u}) = \lambda \ \mathrm{tr}(e(\mathbf{u}))\ \mathrm{Id} + 2\mu \ e(\mathbf{u}), \qquad e(\mathbf{u}) = \frac{1}{2}\Big[D\mathbf{u} + D\mathbf{u}^{\mathsf{T}}\Big] \qquad \text{with} \qquad (D\mathbf{u})_{i,j} = \partial_i u_j.$$

Γ_d and Γ_n are the Dirichlet and Neumann boundaries of the domain respectively. Γ_n includes the boundary of the perturbation and $\mathbf{g}$ is supposed to be zero in a neighborhood of the perturbation. This problem enters the general framework of local perturbations for elliptic problems, which has been deeply studied. Among others, let us mention the following works using potential theory [33, 5, 3, 4], and the reference monographs [27, 28] for multiscale expansions. Following [11, 9], the solution of (1.1) is given at first order by

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) - \varepsilon \sum_{j=1}^{N} \left[\alpha_1^j \mathbf{v}_1^j\left(\frac{x - x^j}{\varepsilon}\right) + \alpha_2^j \mathbf{v}_2^j\left(\frac{x - x^j}{\varepsilon}\right)\right] + \mathcal{O}\left(\varepsilon^2\right), \tag{1.2}$$

with $\mathbf{u}_0$ the solution on the unperturbed domain, $\boldsymbol{\alpha}_1 = \sigma_{11}(\mathbf{u}_0)(0)$ and $\boldsymbol{\alpha}_2 = \sigma_{12}(\mathbf{u}_0)(0)$. The profiles $\mathbf{v}_1^j$ and $\mathbf{v}_2^j$ are obtained as solutions of an homogeneous Navier equation stated on the unbounded domain $\mathbf{H}_\infty^j$ with Neumann conditions on the boundary of the normalized perturbation:

$$\left\{\begin{array}{rcll} -\mu\Delta\mathbf{v}_\ell - (\lambda+\mu)\nabla\ \mathrm{div}\,\mathbf{v}_\ell &=& \mathbf{0} & \text{in } \mathbf{H}_\infty, \\ \sigma(\mathbf{v}_\ell)\cdot\mathbf{n} &=& \mathbf{G}_\ell & \text{on } \partial\omega, \\ \mathbf{v}_\ell &\to& \mathbf{0} & \text{at infinity,} \end{array}\right. \tag{1.3}$$

with $\mathbf{G}_1 = (\mathbf{n}_1, 0)$, $\mathbf{G}_2 = (0, \mathbf{n}_1)$ and $\mathbf{n}_1$ the first component of the outer normal to $\partial\mathbf{H}_\infty$ with $\omega = \omega_j$.

When the distance between the x^j can not be assumed large with respect to ε, e.g. when $\|x^i - x^j\| \approx \varepsilon^\alpha$ for $\alpha \in (0, 1)$, the order of the error term in (1.2) is reduced (see [11]).

Since Problem (1.3) is posed in an infinite domain (an exterior domain in the presented case), its numerical approximation is not straightforward. Among the techniques known to overcome this difficulty, let us mention *infinite elements*, introduced in the seventies (see [34, 7]), which directly handle the problem with a standard Galerkin formulation. The other (numerous) methods come back to a bounded domain for a classical finite element resolution. This is the case of *absorbing conditions* (mainly for wave propagation), [14, 16, 21, 32, 23] or *integral representation*, [25]. In both case, the domain is bounded with a ball, on the boundary of which a new condition – which is non-local most of the time – is imposed to get an equivalent formulation. The present work is in line with such techniques, but we restrict ourselves to differential conditions and seek therefore *approximate* boundary conditions on the artificial boundary. We choose here the boundary to be the circle $\partial\mathcal{B}_R$, where R is assumed to be large. We show in the next section that the problem is reduced to seek solutions of the following boundary value problem:

$$\left\{\begin{array}{rcll} -\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\ \mathrm{div}\,\mathbf{u} &=& \mathbf{0} & \text{in } \mathcal{B}_R \setminus \omega, \\ \sigma(\mathbf{u})\cdot\mathbf{n} &=& \mathbf{G} & \text{on } \partial\omega, \\ \frac{R(1+\nu)}{E}\sigma(\mathbf{u})\cdot\mathbf{n} + \frac{1}{2}\begin{bmatrix} -\frac{\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix}\Delta_\tau\mathbf{u} + \mathbf{u} &=& \mathbf{0} & \text{on } \partial\mathcal{B}_R, \end{array}\right. \tag{1.4}$$

with $\mathbf{G} \in \mathrm{H}^{1/2}(\partial\omega)$, ω is a $\mathcal{C}^\infty$ perturbation of the unit ball and E, ν the Young's modulus and Poisson's ratio linked to the Lamé coefficient by relation (2.9). The main result of this paper is the following theorem:

Theorem 1.1 *There is a countable set of parameters $\mathcal{S}$ such that for any $\nu \notin \mathcal{S}$ we have the following results :*

1. *Suppose $\omega = \mathcal{B}_1$. Then there is a bounded and at most countable set $\mathcal{R}_\nu$ such that for all $R \notin \mathcal{R}_\nu$ and for all $\mathbf{G} \in \mathrm{H}^{1/2}(\partial\omega)$, problem* (1.4) *admits a unique solution $\mathbf{u} \in \mathrm{H}^2(\mathcal{B}_R \setminus \omega)$.*

2. *Suppose ω is a small perturbation of $\mathcal{B}_1$ in the following sense: $\omega = (\mathrm{Id} + \boldsymbol{h})\mathcal{B}_1$ with $\boldsymbol{h} \in \mathcal{C}^\infty$ and a $\mathrm{W}^{1,\infty}$-norm strictly less than 1. Then there exist ε_ν and R_ν such that for all $R > R_\nu$, for all $\mathbf{G} \in \mathrm{H}^{1/2}(\partial\omega)$, for all $\|\boldsymbol{h}\|_{\mathrm{W}^{1,\infty}} \leq \varepsilon_\nu$, problem* (1.4) *admits a unique solution $\mathbf{u} \in \mathrm{H}^2(\mathcal{B}_R \setminus \omega)$.*

The non-standard condition on the outer ball in (1.4) is known as a Ventcel boundary condition (i.e. involving second order tangential derivatives of the displacement field). Existence and uniqueness of the solution of this boundary value problem are non trivial. We show in Section 3 that Theorem 1.1 is a consequence of a careful analysis of another reduced problem on the boundary $\partial\mathcal{B}_R$ of the following type :

$$\frac{1}{2}\begin{bmatrix} -\frac{\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \partial_\theta^2 \varphi + \varphi + \Lambda_R(\varphi) = -\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n}. \tag{1.5}$$

We point out that this Dirichlet-to-Neumann map has been studied as a pseudodifferential operator by Nakamura and Uhlmann in [30, 31] where the symbol in the pseudodifferential framework is explicited. Anyway the expression given there was not adapted to our study with parameter R, for which we need to be far more explicit.
Note that the determination of conditions of type (1.5) has connections with impedance conditions for thin layer problems, [15, 6, 20], or wall laws for rough boundaries, [1, 22, 2].

In Section 2, we derive the artificial boundary condition (1.4). In Section 3 we show existence of the solution of the boundary value problem in the sense of Theorem 1.1. The strategy of the proof consists in reducing the problem to equation (1.5), which can be analyzed after projection onto a family of finite subspaces of functions. An explicit study of the uniform solvability is then performed for the projected linear systems. We give an explicit form for the Dirichlet-to-Neumann map for the elasticity system in the particular geometric configuration where ω is a disk. Finally, we extend the result when ω is close to a disk by a perturbation method. In Section 4 we present the proof of the main Proposition 3.2 stated in Section 3. Last, we present in Section 5 numerical illustrations of our results.

2 Derivation of the artificial boundary condition

2.1 Singularities at infinity for the elasticity problem

We consider the profile problem (1.3) set in the perturbed plane $\mathbf{H}_\infty = \mathbb{R}^2 \setminus \overline{\omega}$. Let R denote a positive real number and $\mathcal{B}_R$ be the ball of radius R centered at the origin. To derive artificial boundary conditions for the linear elasticity on $\partial\mathcal{B}_R$ for large R, we need to know the precise behavior of the solution $\mathbf{v}_\ell$ of Problem (1.3) at infinity. In the following, L denotes the operator $L = \mu\Delta + (\lambda+\mu)\nabla\mathrm{div}$.
It is natural to introduce polar coordinates:

$$\mathbf{u}(r,\theta) = u_r(r,\theta)\mathbf{e}_r + u_\theta(r,\theta)\mathbf{e}_\theta,$$

with $\mathbf{e}_r = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$ and $\mathbf{e}_\theta = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$.

We recall the expression of the involved differential operators in the polar system

$$\begin{aligned}
\Delta \mathbf{u} &= \left(\partial_r^2 u_r + \frac{1}{r}\partial_r u_r - \frac{1}{r^2}u_r + \frac{1}{r^2}\partial_\theta^2 u_r - \frac{2}{r^2}\partial_\theta u_\theta\right)\mathbf{e}_r \\
&\quad + \left(\partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta - \frac{1}{r^2}u_\theta + \frac{1}{r^2}\partial_\theta^2 u_\theta + \frac{2}{r^2}\partial_\theta u_r\right)\mathbf{e}_\theta, \\
\text{div } \mathbf{u} &= \partial_r u_r + \frac{1}{r}u_r + \frac{1}{r}\partial_\theta u_\theta, \\
\nabla(\text{div } \mathbf{u}) &= \left(\partial_r^2 u_r + \frac{1}{r}\partial_r u_r - \frac{1}{r^2}u_r - \frac{1}{r^2}\partial_\theta u_\theta + \frac{1}{r}\partial_{r\theta}^2 u_\theta\right)\mathbf{e}_r \\
&\quad + \left(\frac{1}{r}\partial_{r\theta}^2 u_r + \frac{1}{r^2}\partial_\theta u_r + \frac{1}{r^2}\partial_\theta^2 u_\theta\right)\mathbf{e}_\theta.
\end{aligned}$$

Then the operator L takes the form

$$\begin{aligned}
L\mathbf{u} = &\left((\lambda+2\mu)\left[\partial_r^2 u_r + \frac{1}{r}\partial_r u_r - \frac{1}{r^2}u_r\right] + \frac{\mu}{r^2}\partial_\theta^2 u_r - \frac{\lambda+3\mu}{r^2}\partial_\theta u_\theta + \frac{\lambda+\mu}{r}\partial_{r\theta}^2 u_\theta\right)\mathbf{e}_r \\
&+ \left(\mu\left[\partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta - \frac{1}{r^2}u_\theta\right] + \frac{\lambda+2\mu}{r^2}\partial_\theta^2 u_\theta + \frac{\lambda+\mu}{r}\partial_{r\theta}^2 u_r + \frac{\lambda+3\mu}{r^2}\partial_\theta u_r\right)\mathbf{e}_\theta, \quad (2.1)
\end{aligned}$$

and the stress tensor is given by

$$\sigma(\mathbf{u}) = \begin{bmatrix} (\lambda+2\mu)\partial_r u_r + \frac{\lambda}{r}(u_r + \partial_\theta u_\theta) & \mu\left(\frac{1}{r}(\partial_\theta u_r - u_\theta) + \partial_r u_\theta\right) \\ \mu\left(\frac{1}{r}(\partial_\theta u_r - u_\theta) + \partial_r u_\theta\right) & (\lambda+2\mu)\frac{1}{r}(\partial_\theta u_\theta + u_r) + \lambda\partial_r u_r \end{bmatrix}. \quad (2.2)$$

Singularities of elliptic problems appear to be of tensorial form, see [17, 13, 29], and especially [24, 18, 19] for the elasticity system. Therefore, we seek solutions of $L\mathbf{u} = 0$ under the form

$$\mathbf{u}(r,\theta) = r^s \begin{bmatrix} \phi_r(\theta) \\ \phi_\theta(\theta) \end{bmatrix}. \quad (2.3)$$

Consequently, using (2.1) and (2.2), we have in polar coordinates

$$L\mathbf{u} = r^{s-2}\begin{bmatrix} \mu\phi_r'' + (\lambda+2\mu)(s^2-1)\phi_r + [(\lambda+\mu)s - (\lambda+3\mu)]\phi_\theta' \\ (\lambda+2\mu)\phi_\theta'' + \mu(s^2-1)\phi_\theta + [(\lambda+\mu)s + (\lambda+3\mu)]\phi_r' \end{bmatrix}, \quad (2.4)$$

$$\sigma(\mathbf{u}) = r^{s-1}\begin{bmatrix} \lambda\phi_\theta' + ((\lambda+2\mu)s + \lambda)\phi_r & \mu(\phi_r' + (s-1)\phi_\theta) \\ \mu(\phi_r' + (s-1)\phi_\theta) & (\lambda+2\mu)\phi_\theta' + (\lambda s + (\lambda+2\mu))\phi_r \end{bmatrix}. \quad (2.5)$$

Using (2.4), we reduce the second order system $L\mathbf{u} = 0$ into a bigger system of first order. Introducing $\psi_r = \phi_r'$, $\psi_\theta = \phi_\theta'$, and $\mathbf{U} = (\phi_r, \phi_\theta, \psi_r, \psi_\theta)^{\mathsf{T}}$, we get the matricial formulation

$$\mathbf{U}' = A\mathbf{U},$$

with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{(\lambda+2\mu)(1-s^2)}{\mu} & 0 & 0 & \frac{(\lambda+3\mu)-(\lambda+\mu)s}{\mu} \\ 0 & \frac{\mu(1-s^2)}{\lambda+2\mu} & -\frac{(\lambda+3\mu)+(\lambda+\mu)s}{\lambda+2\mu} & 0 \end{bmatrix}.$$

The eigenvalues of A are $\pm i(s \pm 1)$. Hence, the polar functions ϕ_r, ϕ_θ belong to the space generated by $\cos((s\pm1)\theta)$, $\sin((s\pm1)\theta)$. The continuity for $\theta = 0$, $\theta = 2\pi$ requires s to

be an integer.
The integer s being fixed, we look for coefficients $A_r, B_r, C_r, D_r, A_\theta, B_\theta, C_\theta, D_\theta$, so that

$$\begin{aligned}\phi_r(\theta) &= A_r\cos((s-1)\theta)+B_r\sin((s-1)\theta)+C_r\cos((s+1)\theta)+D_r\sin((s+1)\theta),\\ \phi_\theta(\theta) &= A_\theta\cos((s-1)\theta)+B_\theta\sin((s-1)\theta)+C_\theta\cos((s+1)\theta)+D_\theta\sin((s+1)\theta).\end{aligned}$$

Writing $L\mathbf{u}=0$, we get :

$$\left\{\begin{aligned} aA_r+B_\theta &= 0\\ aB_r-A_\theta &= 0\\ C_r+D_\theta &= 0\\ D_r-C_\theta &= 0\end{aligned}\right. \qquad \text{with} \qquad a=\frac{(\lambda+\mu)s+(\lambda+3\mu)}{(\lambda+\mu)s-(\lambda+3\mu)}. \tag{2.6}$$

Consequently, the functions ϕ_r and ϕ_θ satisfy:

$$\begin{aligned}\phi_r(\theta) &= A_r\cos((s-1)\theta)+B_r\sin((s-1)\theta)+C_r\cos((s+1)\theta)+D_r\sin((s+1)\theta),\\ \phi_\theta(\theta) &= aB_r\cos((s-1)\theta)-aA_r\sin((s-1)\theta)+D_r\cos((s+1)\theta)-C_r\sin((s+1)\theta).\end{aligned}$$

Remark 2.1 *It can be shown that these singular functions describe the behavior at infinity of the solutions of the elasticity system in the plane. The solution $\mathbf{v}_\ell$ of* (1.3) *satisfies for any $N>0$*

$$\mathbf{v}_\ell(x)=\sum_{-N<s<0}\mathbf{v}_\ell^{[s]}(x)+\mathcal{O}_{|x|\to\infty}\big(|x|^{-N}\big), \tag{2.7}$$

where $\mathbf{v}_\ell^{[s]}$ has the structure (2.3). *The sum is extended to negative integers since $\mathbf{v}_\ell$ vanishes at infinity. Let us mention that such an expansion still holds for the derivatives of $\mathbf{v}_\ell$ at any order.*

2.2 Artificial boundary condition on $\partial\mathcal{B}_R$

To approximate Problem (1.3), we introduce the bounded domain $\mathbf{H}_\infty^R=\mathbf{H}_\infty\cap\mathcal{B}_R$, where $\mathcal{B}_R$ is the ball of radius R centered at the origin. We are looking for a boundary condition to impose on the artificial boundary $\partial\mathcal{B}_R$. Since $\mathbf{v}_\ell$ tends to 0 at infinity, a first (naive) choice consists of setting an homogeneous Dirichlet condition on $\partial\mathcal{B}_R$. However, thanks to expansion (2.7), the resulting error is of order $\mathcal{O}(R^{-1})$, which is rather poor. To improve this approximation accuracy, we seek a boundary condition which is satisfied by the leading term in (2.7) so that the error becomes of order $\mathcal{O}(R^{-2})$.
More precisely, we find a linear relation between displacement and traction on the artificial boundary $\partial\mathcal{B}_R$. If $s=-1$, the relation $L\mathbf{u}=0$ reads

$$\left\{\begin{aligned} \mu\phi_r''-2(\lambda+2\mu)\phi_\theta' &= 0,\\ (\lambda+2\mu)\phi_\theta''+2\mu\phi_r' &= 0.\end{aligned}\right. \tag{2.8}$$

To determine artificial boundary conditions on $\partial\mathcal{B}_R$, we consider $\sigma(\mathbf{u})\cdot\mathbf{n}$ and notice that $\mathbf{n}=\mathbf{e}_r$ on $\partial\mathcal{B}_R$. Using (2.5) and (2.8), we get for $s=-1$:

$$\begin{aligned}\sigma(\mathbf{u})\cdot\mathbf{e}_r &= r^{-2}\begin{bmatrix}\lambda\phi_\theta'\\ \mu\phi_r'\end{bmatrix}-2\mu r^{-2}\begin{bmatrix}\phi_r\\ \phi_\theta\end{bmatrix}\\ &= r^{-2}\begin{bmatrix}\frac{\lambda\mu}{2(\lambda+2\mu)} & 0\\ 0 & -\frac{\lambda+2\mu}{2}\end{bmatrix}\begin{bmatrix}\phi_r''\\ \phi_\theta''\end{bmatrix}-2\mu r^{-2}\begin{bmatrix}\phi_r\\ \phi_\theta\end{bmatrix}.\end{aligned}$$

Consequently

$$\sigma(\mathbf{u})\cdot\mathbf{n}+\frac{1}{2R}\begin{bmatrix}-\frac{\lambda\mu}{\lambda+2\mu} & 0\\ 0 & \lambda+2\mu\end{bmatrix}\Delta_\tau\mathbf{u}+\frac{2\mu}{R}\mathbf{u}=0.$$

Lamé's coefficients are linked to the physical parameters (Young's modulus and Poisson's ratio) through the following relations:

$$\begin{cases}\lambda &= \dfrac{\nu E}{(1+\nu)(1-2\nu)}\\ \mu &= \dfrac{E}{2(1+\nu)}\end{cases}\qquad \text{with} \qquad E>0,\ -1<\nu<0.5. \tag{2.9}$$

So that, this boundary condition of Ventcel's type rewrites on $\partial\mathcal{B}_R$:

$$\sigma(\mathbf{u})\cdot\mathbf{n}+\frac{E}{2R(1+\nu)}\begin{bmatrix}\frac{-\nu}{2(1-\nu)} & 0\\ 0 & \frac{1-\nu}{1-2\nu}\end{bmatrix}\Delta_\tau\mathbf{u}+\frac{E}{R(1+\nu)}\mathbf{u}=0.$$

We notice that

$$\frac{1-\nu}{(1+\nu)(1-2\nu)}>0, \qquad \text{and} \qquad \begin{cases}\dfrac{-\nu}{2(1-\nu^2)}<0 & \text{if} \quad \nu\in(0,0.5),\\ \dfrac{-\nu}{2(1-\nu^2)}>0 & \text{if} \quad \nu\in(-1,0).\end{cases}$$

We finally get the following boundary value problem: for $\mathbf{G}\in\mathrm{H}^{1/2}(\partial\omega)$, looking for $\mathbf{u}\in\mathrm{H}^2(\mathcal{B}_R\setminus\omega)$ such that

$$\begin{cases}-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{u} &= \mathbf{0} & \text{in } \mathcal{B}_R\setminus\omega,\\ \sigma(\mathbf{u})\cdot\mathbf{n} &= \mathbf{G} & \text{on } \partial\omega,\\ \frac{R(1+\nu)}{E}\sigma(\mathbf{u})\cdot\mathbf{n}+\frac{1}{2}\begin{bmatrix}-\frac{\nu}{2(1-\nu)} & 0\\ 0 & \frac{1-\nu}{1-2\nu}\end{bmatrix}\Delta_\tau\mathbf{u}+\mathbf{u} &= \mathbf{0} & \text{on } \partial\mathcal{B}_R.\end{cases} \tag{2.10}$$

This last equation is written on the polar basis $(\mathbf{e}_r,\mathbf{e}_\theta)$, i.e. $\mathbf{u}=(u_r,u_\theta)$. Whatever the sign of the parameter ν, the obtained approximate boundary condition leads to a non-coercive weak formulation. A similar problem has been investigated in [10] for the scalar Laplace equation.

3 Solvability of the equations

3.1 The strategy

We look for a solution $\mathbf{u}\in\mathrm{H}^2(\mathcal{B}_R\setminus\omega)$ of problem (2.10) in the form

$$\mathbf{u}=\mathbf{u}_0+\mathbf{v},$$

where $\mathbf{u}_0\in\mathrm{H}^2(\mathcal{B}_R\setminus\omega)$ lifts the Dirichlet data $\mathbf{G}\in\mathrm{H}^{1/2}(\partial\omega)$, i.e. satisfies the following problem with a Dirichlet condition on the artificial boundary :

$$\begin{cases}-\mu\Delta\mathbf{u}_0-(\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{u}_0 &= \mathbf{0} & \text{in } \mathcal{B}_R\setminus\omega,\\ \sigma(\mathbf{u}_0)\cdot\mathbf{n} &= \mathbf{G} & \text{on } \partial\omega,\\ \mathbf{u}_0 &= 0 & \text{on } \partial\mathcal{B}_R.\end{cases} \tag{3.1}$$

We define the Dirichlet-to-Neumann map Λ_R for all $s > 0$ by

$$\begin{array}{rccc} \Lambda_R: & \mathrm{H}^{s+1/2}(\partial\mathcal{B}_R) & \to & \mathrm{H}^{s-1/2}(\partial\mathcal{B}_R) \\ & \varphi & \mapsto & \frac{R(1+\nu)}{E}\,\sigma(\mathbf{v})\cdot\mathbf{n}, \end{array} \tag{3.2}$$

where $\mathbf{v}$ is the solution in $\mathrm{H}^{s+1}(\mathcal{B}_R \setminus \omega)$ of

$$\left\{ \begin{array}{rcll} -\mu\Delta\mathbf{v} - (\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{v} & = & \mathbf{0} & \text{in } \mathcal{B}_R \setminus \omega, \\ \sigma(\mathbf{v})\cdot\mathbf{n} & = & \mathbf{0} & \text{on } \partial\omega, \\ \mathbf{v} & = & \varphi & \text{on } \partial\mathcal{B}_R. \end{array} \right. \tag{3.3}$$

Let φ be the solution in $\mathrm{H}^{3/2}(\partial\omega)$ of

$$\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n} + \frac{1}{2}\begin{bmatrix} -\frac{\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix}\partial_\theta^2\varphi + \varphi + \Lambda_R(\varphi) = 0. \tag{3.4}$$

Then $\mathbf{u}$ is solution in $\mathrm{H}^2(\mathcal{B}_R \setminus \omega)$ of (2.10) if and only if $\mathbf{u}_0$ is solution in $\mathrm{H}^2(\mathcal{B}_R \setminus \omega)$ of (3.1) and $\mathbf{v}$ is solution in $\mathrm{H}^2(\mathcal{B}_R \setminus \omega)$ of (3.3) with φ solution in $\mathrm{H}^{3/2}(\partial\mathcal{B}_R)$ of (3.4).

We aim at proving that Problem (3.4) is well-posed for any $R > 0$ except in a bounded and countable set. We therefore have to reduce the problem to the following boundary problem of unknown $\varphi \in \mathrm{H}^{3/2}(\partial\mathcal{B}_R)$, and where u_0 is given:

$$\frac{1}{2}\begin{bmatrix} -\frac{\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix}\partial_\theta^2\varphi + \varphi + \Lambda_R(\varphi) = -\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n}. \tag{3.5}$$

In order to solve it, we shall give a concrete expression of Λ_R and work in polar coordinates.

Remark 3.1 *A careful analysis of the strategy before shows that we have the better following regularity properties for the unknown functions:* $\mathbf{u}_0 \in \mathrm{H}^2(\mathcal{B}_R \setminus \omega)$ *as stated, but* $\varphi \in \mathrm{H}^{5/2}(\partial\mathcal{B}_R)$ *and* $\mathbf{v} \in \mathrm{H}^3(\mathcal{B}_R \setminus \omega)$ *by elliptic regularity, since (as a direct consequence of Proposition 3.2 below)* Λ_R *is in fact an operator of order* 1*. Note also that the decoupling in two problems* (3.1) *in the one side and* (3.3)–(3.4) *on the other side was necessary in order to be able to introduce* Λ_R *as an operator (i.e. an unbounded linear application).*

3.2 A decoupled system when the inclusion is a disk

We consider the case where ω is the disk $\mathcal{B}_1$ of radius 1 centered at the origin. The boundary problem (3.3) defining the Dirichlet-to-Neumann map is then set in a ring. It is natural to write the Dirichlet datum φ as a fourier series:

$$\varphi = \begin{bmatrix} \varphi_0^r \\ \varphi_0^\theta \end{bmatrix} + \sum_{n\geq 1}\begin{bmatrix} \varphi_n^r \\ \varphi_n^\theta \end{bmatrix}\cos n\theta + \sum_{n\geq 1}\begin{bmatrix} \psi_n^r \\ \psi_n^\theta \end{bmatrix}\sin n\theta. \tag{3.6}$$

We will show in Propositions 3.2 and 3.3 below that the Dirichlet-to-Neumann map takes the form

$$\Lambda_R(\varphi) = \Lambda_R^0(\varphi) + \mathcal{R}(\varphi), \tag{3.7}$$

where the principal part Λ_R^0 is defined in polar coordinates by

$$\begin{aligned} \Lambda_R^0(\varphi) = \frac{1-\gamma}{\gamma}\begin{bmatrix} \varphi_0^r \\ 0 \end{bmatrix} + \frac{1}{1+\gamma}\sum_{n\geq 1}\begin{bmatrix} (n-\gamma)\varphi_n^r + (1-n\gamma)\psi_n^\theta \\ (n\gamma-1)\psi_n^r + (n-\gamma)\varphi_n^\theta \end{bmatrix}\cos n\theta \\ + \frac{1}{1+\gamma}\sum_{n\geq 1}\begin{bmatrix} (n-\gamma)\psi_n^r + (n\gamma-1)\varphi_n^\theta \\ (1-n\gamma)\varphi_n^r + (n-\gamma)\psi_n^\theta \end{bmatrix}\sin n\theta, \end{aligned} \tag{3.8}$$

where

$$\gamma = \frac{1-2\nu}{2(1-\nu)},$$

and the remainder $\mathcal{R}(\varphi)$ is controlled. In the following Proposition, we reformulate Problem (3.5) in terms of the unknown Fourier coefficients φ_n^r, φ_n^θ, ψ_n^r, and ψ_n^θ.

Proposition 3.2 *Let $\varphi \in H^{3/2}(\partial\mathcal{B}_R)$. For all $n \geq 1$, let $\Phi_n = (\varphi_n^r, \psi_n^r, \varphi_n^\theta, \psi_n^\theta)^\mathsf{T}$, corresponding to the n-th coordinates of φ in (3.6), and let $f_{n,R}$ be the 4-uplet of the decomposition of $-\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n}$ with respect to the same basis. Similarly, let $\Phi_0 = (\varphi_0^r, \varphi_0^\theta)^\mathsf{T}$ correspond to the 0-th coordinates of φ in (3.6), respectively $f_{0,R}$ be the 2-uplet of the decomposition of $-\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n}$ with respect to the last basis.*
Then equation (3.5) reads

$$P_n\Phi_n + \mathcal{R}_{n,R}\Phi_n = f_{n,R}, \tag{3.9}$$

where P_n is the matrix given by
for $n \geq 1$,

$$P_n = \frac{-n^2}{4}\begin{bmatrix} 1-2\gamma & 0 & 0 & 0 \\ 0 & 1-2\gamma & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} \end{bmatrix} + \mathrm{Id}_4 + \frac{1}{1+\gamma}\begin{bmatrix} n-\gamma & 0 & 0 & 1-n\gamma \\ 0 & n-\gamma & n\gamma-1 & 0 \\ 0 & n\gamma-1 & n-\gamma & 0 \\ 1-n\gamma & 0 & 0 & n-\gamma \end{bmatrix},$$

for $n = 0$,

$$P_0 = 0 + \mathrm{Id}_2 + \begin{bmatrix} \frac{1-\gamma}{\gamma} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\gamma} & 0 \\ 0 & 1 \end{bmatrix},$$

and the remainder $\mathcal{R}_{n,R}$ is controlled uniformly in $n \in \mathbb{N}$ for R large: there exist $R_0 > 0$ and a constant C (independent of n and R) such that for any n and $R > R_0$, we have

$$\begin{cases} \|\mathcal{R}_{n,R}\|_\infty \leq Cn^2R^{-2n+2} & \text{if} \quad n \geq 2 \\ \|\mathcal{R}_{1,R}\|_\infty \leq CR^{-4}, \\ \|\mathcal{R}_{0,R}\|_\infty \leq CR^{-2}, \end{cases} \tag{3.10}$$

PROOF. The proof of this fundamental result is postponed to Section 4. ■

Note that (3.8) is a consequence of the expression of Λ_R^0 in terms of Fourier coefficients. Using decomposition (3.6) and denoting $\Phi_n = (\varphi_n^r, \psi_n^r, \varphi_n^\theta, \psi_n^\theta)^\mathsf{T}$, this relation reads

$$\Lambda_R^0(\varphi) = P_0^0\begin{bmatrix} \varphi_0^r \\ \varphi_0^\theta \end{bmatrix} + \sum_{n\geq 1}\begin{bmatrix} \cos n\theta & \sin n\theta & 0 & 0 \\ 0 & 0 & \cos n\theta & \sin n\theta \end{bmatrix} P_n^0\Phi_n$$

with

$$P_n^0 = \frac{1}{1+\gamma}\begin{bmatrix} n-\gamma & 0 & 0 & 1-n\gamma \\ 0 & n-\gamma & n\gamma-1 & 0 \\ 0 & n\gamma-1 & n-\gamma & 0 \\ 1-n\gamma & 0 & 0 & n-\gamma \end{bmatrix} \quad \text{when } n \geq 1,$$

and

$$P_0^0 = \begin{bmatrix} \frac{1-\gamma}{\gamma} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{when } n = 0.$$

These matrices have to be considered as a definition of operator Λ_R^0 in Fourier modes, and Λ_R can be written in the equivalent usual form (3.8) by summing up the Fourier modes. Now we deal with the remainder term in (3.7):

$$\mathcal{R} = \Lambda_R - \Lambda_R^0 \tag{3.11}$$

Estimates (3.10) directly imply by summation in Fourier modes the following result:

Proposition 3.3 *For all $s \in \mathbb{R}$, there exists $C_s > 0$ such that for any $\varphi \in \mathcal{C}^\infty(\partial B_R)$ and $R > R_0$,*

$$\|\mathcal{R}(\varphi)\|_{\mathrm{H}^s(\partial\mathcal{B}_R)} \leq C_{-s} R^{-2} \|\varphi\|_{\mathrm{H}^{-s}(\partial\mathcal{B}_R)}.$$

In particular, we have by natural extension that for all $\varphi \in \mathrm{H}^{1/2}(\partial\mathcal{B}_R)$

$$\|\mathcal{R}(\varphi)\|_{\mathrm{H}^{-1/2}(\partial\mathcal{B}_R)} \leq C_{1/2} R^{-2} \|\varphi\|_{\mathrm{H}^{1/2}(\partial\mathcal{B}_R)}.$$

3.3 Wellposedness for the disk case

As a direct consequence of Propositions 3.2 and 3.3, we get, in the case where ω is the disk $\mathcal{B}_1$, the following result of solvability for equation (3.5), and therefore for the system (2.10). The expected regularity for each problem is the one stated in Subsection 3.1 when we developed our strategy : $\mathbf{u} \in \mathrm{H}^2(\mathcal{B}_R \setminus \omega)$, $\varphi \in \mathrm{H}^{3/2}(\partial\mathcal{B}_R)$ and $\mathbf{v} \in \mathrm{H}^2(\mathcal{B}_R \setminus \omega)$.

Proposition 3.4 *Let us recall that $\gamma = \frac{1-2\nu}{2(1-\nu)}$. We have*

1. *The matrix P_n is generically (with respect to γ) invertible for all n, which means that for each $\gamma \notin \mathcal{S}$, where $\mathcal{S}$ is an at most countable set of physical parameters, P_n is invertible with norm of the inverse uniformly bounded in n.*
2. *For all fixed $\gamma \notin \mathcal{S}$, there exists R_γ such that equation* (3.5) *admits a unique solution φ for all $R \geq R_\gamma$.*
3. *For all $\gamma \notin \mathcal{S}$, there exists R_γ such that the system* (2.10) *admits a unique solution $\mathbf{u}$ for all $R \geq R_\gamma$.*
4. *For all $\gamma \notin \mathcal{S}$, there exists a bounded and at most countable set $\mathcal{R}_\gamma$ such that the system* (2.10) *admits a unique solution $\mathbf{u}$ for all $R \notin \mathcal{R}_\gamma$.*

Remark 3.5 *The system* (2.10) *is expressed with the parameter ν. But since $\nu \mapsto \gamma = \frac{1-2\nu}{2(1-\nu)}$ is a strictly decreasing function of ν, it is equivalent to avoid a countable set in ν or γ variables. It is more convenient to make the computations with parameter γ, so we express the forbidden set in γ-variable.*

Proof. First we notice that the range of values of γ is $(0, 3/4)$ as $\nu \in (-1, 1/2)$ since, as already mentioned, the function $\nu \mapsto \gamma = \frac{1-2\nu}{2(1-\nu)}$ is decreasing on $(-1, 1/2)$.

Let us deal with point 1. First, we observe that P_0 is actually invertible. For each fixed $n \geq 1$, we look at the full determinant of P_n and show that only a finite number of values of γ is forbidden. We can rewrite

$$P_n = \begin{bmatrix} A_1 & 0 & 0 & B_1 \\ 0 & A_1 & -B_1 & 0 \\ 0 & -B_1 & A_2 & 0 \\ B_1 & 0 & 0 & A_2 \end{bmatrix},$$

with

$$\begin{aligned} A_1 &= -\frac{n^2}{4}(1-2\gamma) + 1 + \frac{n-\gamma}{1+\gamma} = \frac{n^2(1-\gamma+2\gamma^2)+4n+4}{4(1+\gamma)}, \\ A_2 &= \frac{-n^2}{4\gamma} + 1 + \frac{n-\gamma}{1+\gamma} = \frac{-n^2(1+\gamma)+4n\gamma+4\gamma}{4\gamma(1+\gamma)}, \\ B_1 &= \frac{1-n\gamma}{1+\gamma}. \end{aligned}$$

Then, we compute $\det P_n$ and obtain

$$\det P_n = (A_1A_2 - B_1^2)^2 = \frac{n^2}{16\gamma^2(1+\gamma)^2}\left(\sum_{j=0}^{3} n^j P_j(\gamma)\right)^2,$$

with

$$\begin{aligned} P_0(\gamma) &= -32\gamma, \\ P_1(\gamma) &= 4(1-3\gamma+2\gamma^2) = 4(1-2\gamma)(1-\gamma), \\ P_2(\gamma) &= 4(1+\gamma-2\gamma^2) = 4(1+2\gamma)(1-\gamma), \\ P_3(\gamma) &= -1+\gamma+2\gamma^2 = (-1+2\gamma)(1+\gamma). \end{aligned}$$

Let $\Delta_n(\gamma) := 16\gamma^2(1+\gamma)^2 \det P_n$. The expanded expression of Δ_n in terms of powers of n is

$$\Delta_n(\gamma) \;=\; n^2\left((-1+\gamma+2\gamma^2)n^3 + 4(1+\gamma-2\gamma^2)n^2 + 4(1-3\gamma+2\gamma^2)n - 32\gamma\right)^2.$$

We notice that $\Delta_n(\gamma)$ is a polynomial function in both n and γ of order at most 4 in γ. We want to check that Δ_n is not identically zero and for this, we look first at the value at $\gamma = 0$ for n fixed:

$$\Delta_n(0) = n^4(-n^2+4n+4)^2.$$

This is clear that Δ_n is never identically zero, and therefore is a polynomial with respect to the variable γ with at most 4 roots in the range $(0, 3/4)$. We denote this set by $\mathcal{S}_n$ and pose $\mathcal{S} = \{1/2\} \cup \bigcup_{n\in\mathbb{N}^*} \mathcal{S}_n$. Then $\mathcal{S}$ is at most countable.
Now for each γ fixed in the complementary $\mathcal{S}^c$ of $\mathcal{S}$, we notice that the matrix P_n is equivalent (in the asymptotic sense $n \to \infty$) to the matrix

$$-\frac{n^2}{4}\begin{bmatrix} 1-2\gamma & 0 & 0 & 0 \\ 0 & 1-2\gamma & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} \end{bmatrix}.$$

Recall that $\gamma \in \mathcal{S}^c$ (thus $\gamma \neq 1/2$), the previous matrix is thus invertible and it implies that there exists n_γ such that for all $n > n_\gamma$, P_n also is invertible with inverse uniformly bounded with respect to n. For each integer n in the finite set $\{1, 2, \cdots, n_\gamma\}$, P_n is again invertible since $\gamma \notin \mathcal{S}$, and the norm of the inverse can be bounded uniformly since there is only a finite number of values. At the end, we get that the norm of the inverse of P_n can be bounded uniformly in $n \in \mathbb{N}$. This is the result of uniform boundedness of point 1.

For point 2, we first fix $\gamma \notin \mathcal{S}$. For any $n \geq 0$, we notice that the term $\mathcal{R}_{n,R}$ has a norm going to 0 uniformly in n when R goes to infinity (recall that this term also depends on γ) as mentioned in (3.10). This means that, according to point 1, there exists R_γ such that for all $R > R_\gamma$, the matrix $P_n + \mathcal{R}_{n,R}$ is invertible for all $n \in \mathbb{N}$ with norm of this inverse uniformly bounded with respect to n. At the end, we have been able to solve the full problem (3.5) mode by mode with a control (uniform in n) of the norm of the inverse. This gives the result.

For point 3, we just use the reduction in the beginning of Section 3 allowing to reduce the problem on the boundary, and we get the result.

For point 4, we fix $\gamma \notin \mathcal{S}$ and look at the problem mode by mode. For fixed $n \geq 0$, we can notice that for M sufficiently large (for example $M > \max\{2n-2, 4\}$), the determinant of the matrix $R^M P_n + R^M \mathcal{R}_{n,R}$ built in the proof is a non-zero polynomial in R. This implies that for each n, only a finite number of radius give rise to a non solvable equation (3.9). Again this means that for $R \notin \mathcal{R}_\gamma$, where $\mathcal{R}_\gamma$ is an at most countable set of radii, the problem $P_n\Phi_n + \mathcal{R}_{n,R}\Phi_n = f_{n,R}$ is solvable for all $n \geq 0$, with norm of this inverse of $P_n + \mathcal{R}_{n,R}$ uniformly bounded with respect to n (recall that $f_{n,R}$ was defined in Proposition 3.2). As before we get that for $R \notin \mathcal{R}_\gamma$ the problem (3.5) is solvable and so is the problem (2.10). Notice that point 2 implies that for each $\gamma \notin \mathcal{S}$, $\mathcal{R}_\gamma$ is a bounded set. The proof is complete. ■

3.4 A perturbation result for quasi circular inclusions

We aim at extending the previous result obtained for $\omega = \mathcal{B}_1$ to close domains ω. Now, applying Point 3 of Proposition 3.4, we consider a real $R > 1$ such that the system (2.10) admits a unique solution $\mathbf{u}$ on the domain $\mathcal{B}_R \setminus \mathcal{B}_1$ as soon as $\gamma \notin \mathcal{S}$. Consider a domain ω close to $\mathcal{B}_1$ and ask if the system (2.10) admits a unique solution $\mathbf{u}$ on the domain $\mathcal{B}_R \setminus \omega$ for the same set of parameters $\mathcal{S}$. Our aim is to prove that the perturbed Dirichlet-to-Neumann map depends continuously (as operator) on smooth perturbations on the domain.
We adapt directly [10, Theorem 3.1] and consider a $\mathcal{C}^\infty$ vector field $\boldsymbol{h}$ supported in $\mathcal{B}_\rho$ for $\rho \in (1, R)$ and the application $\mathrm{T}_{\boldsymbol{h}} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\mathrm{T}_{\boldsymbol{h}} = \mathrm{Id}_{\mathbb{R}^2} + \boldsymbol{h}$. Clearly, $\mathrm{T}_{\boldsymbol{h}}$ is a diffeomorphism when the norm of $\boldsymbol{h}$ is small, then the perturbed domain $\mathcal{B}_R \setminus \omega_{\boldsymbol{h}}$, with $\omega_{\boldsymbol{h}} = \mathrm{T}_{\boldsymbol{h}}(\omega)$, is just $\mathrm{T}_{\boldsymbol{h}}(\mathcal{B}_R \setminus \omega)$.
For any $s \geq 0$, let us first define the Dirichlet-to-Neumann operator

$$\begin{aligned} \Lambda_{R,\boldsymbol{h}} : \ \mathrm{H}^{s+1/2}(\partial\mathcal{B}_R) &\rightarrow \mathrm{H}^{s-1/2}(\partial\mathcal{B}_R) \\ \boldsymbol{\phi} &\mapsto \frac{R(1+\nu)}{E}\sigma(\mathbf{v}_{\boldsymbol{h}}) \cdot \mathbf{n}, \end{aligned}$$

where $\mathbf{v}_{\boldsymbol{h}} \in \mathrm{H}^{s+1}(\mathcal{B}_R \setminus \omega_{\boldsymbol{h}})$ solves the boundary values problem (3.3) with $\omega = \omega_{\boldsymbol{h}}$.
We are now in position to state the result of this section.

Proposition 3.6 *In the previously described geometric setting, and for all $s \geq 0$, there exists C_s such that if $\|\boldsymbol{h}\|_{\mathrm{W}^{1,\infty}(\mathcal{B}_R)} < 1$ then*

$$\|\Lambda_{R,\boldsymbol{h}} - \Lambda_R\|_{\mathcal{L}(\mathrm{H}^{s+1/2}(\partial\mathcal{B}_R),\mathrm{H}^{s-1/2}(\partial\mathcal{B}_R))} \leq C_s \|\boldsymbol{h}\|_{\mathrm{W}^{1,\infty}(\mathcal{B}_R)}. \tag{3.12}$$

Proof. The difference with the situation presented in [10] is that one deals with the elasticity system instead of the Laplace equation. Hence the transported weak formation for (3.3) is more complicated: one has to transport the symmetrized gradient instead of the usual gradient. Its reads

$$e_{\boldsymbol{h}}(\mathbf{u}) = \frac{1}{2}\left(D\mathbf{u} \cdot D\mathrm{T}_{\boldsymbol{h}}^{-1} + (D\mathrm{T}_{\boldsymbol{h}}^{-1})^\mathsf{T} \cdot D\mathbf{u}^\mathsf{T}\right).$$

For example, the starting point is to write the perturbed bilinear form as

$$a(\mathbf{v}, \varphi) = \int_{\mathcal{B}_R \setminus \mathcal{B}_1} e_{\boldsymbol{h}}(\mathbf{v}) : H : e_{\boldsymbol{h}}(\varphi) \det D\mathrm{T}_{\boldsymbol{h}},$$

with $\mathbf{v} = \mathbf{u} \circ \mathrm{T}_h$. Then, one adapts straightforwardly step-by-step the proof of Theorem 3.1 in [10]. This is left to the reader. ■

3.5 Proof of the Theorem 1.1

Now we can complete the proof of Theorem 1.1. First notice that point 1 corresponding to the case when $\omega = \mathcal{B}_1$ is a rephrasing of point 3 and 4 of Proposition 3.4. We can anyway repeat the argument for point 4 in a slightly more direct formulation, which will be useful for the proof of part 2.

PROOF OF POINT 1 OF THEOREM 1.1 WHEN $R \to \infty$ (ALTERNATIVE).
As seen when the general strategy was explained in Section 3.1, solvability of the problem (1.4) is equivalent to solve equation (3.5) on $\partial\mathcal{B}_R$. We introduce the following operator

$$\mathcal{P}_R^0\varphi = \frac{1}{2}\begin{bmatrix} -\frac{\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \partial_\theta^2\varphi + \varphi + \Lambda_R^0(\varphi),$$

and we work now with parameter γ instead of parameter ν. Then from point 1 of Proposition 3.4 and Remark 3.1, we know that for any $\gamma \notin \mathcal{S}$, fixed from now on, the problem

$$\mathcal{P}_R^0\varphi = -\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n}$$

is well posed with a unique solution $\varphi \in \mathrm{H}^{5/2}(\partial\mathcal{B}_R) \subset \mathrm{H}^2(\partial\mathcal{B}_R)$. This is due to the fact that $\mathcal{P}_R^0$ is elliptic of order 2 and invertible from $\mathrm{H}^{s+3/2}(\partial\mathcal{B}_R)$ to $\mathrm{H}^{s-1/2}(\partial\mathcal{B}_R)$ for all $s \in \mathbb{R}$, according to the Fourier mode decomposition given in Proposition 3.2. Recall that ellipticity of $\mathcal{P}_R^0$ is a consequence of the fact that Λ_0 is of order 1. Another consequence is that $\mathcal{P}_R^0$ has compact resolvent, and therefore a discrete spectrum as an unbounded operator in $L^2(\partial\mathcal{B}_R)$ (this explains partly the countability argument and the introduction of the countable set $\mathcal{R}_\gamma$). The invertibility of $\mathcal{P}_R^0$ is then equivalent to say that 0 is not in the spectrum of $\mathcal{P}_R^0$. If we introduce the so-called resolvent at 0 defined by abuse by $(\mathcal{P}_R^0)^{-1} : \mathrm{L}^2(\partial\mathcal{B}_R) \longrightarrow \mathrm{L}^2(\partial\mathcal{B}_R)$, we can write

$$\varphi = (\mathcal{P}_R^0)^{-1}\left(-\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n}\right),$$

recalling that $-\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n} \in \mathrm{H}^{1/2}(\partial\mathcal{B}_R)$. Since the spectrum is discrete, and 0 is not in the spectrum, there exists ε_γ such that for all bounded operator $\mathcal{Q}_R : \mathrm{H}^2(\partial\mathcal{B}_R) \longrightarrow \mathrm{L}^2(\partial\mathcal{B}_R)$ with norm strictly less than ε_γ, there exists also a unique solution to the problem

$$(\mathcal{P}_R^0 + \mathcal{Q}_R)\varphi = -\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n}.$$

For this it is sufficient to take $\varepsilon_\gamma = \left\|\mathcal{P}_R^0\right\|^{-1}_{\mathcal{L}(\mathrm{H}^2(\partial\mathcal{B}_R),\mathrm{L}^2(\partial\mathcal{B}_R))}$ and the corresponding inverse is then given by

$$(\mathcal{P}_R^0 + \mathcal{Q}_R)^{-1} = (\mathcal{P}_R^0)^{-1}(\mathrm{Id} + \mathcal{Q}_R(\mathcal{P}_R^0)^{-1})^{-1},$$

where the first inverse is well defined using Neumann series and $\left\|\mathcal{Q}_R(\mathcal{P}_R^0)^{-1}\right\| < 1$.

We apply this strategy to $\mathcal{Q}_R = \mathcal{R} = \Lambda_R - \Lambda_R^0$ for which we showed in Proposition 3.3 that

$$\|\mathcal{R}\|_{\mathcal{L}(\mathrm{L}^2(\partial\mathcal{B}_R),\mathrm{L}^2(\partial\mathcal{B}_R))} \le C_0(\gamma)R^{-2}$$

(recall that the norm of operator $\mathcal{P}_R^0$ is *independant* of R from its expression in Fourier modes). We therefore get that for R such that $C_0(\gamma)R^{-2} < \varepsilon_\gamma$, Problem (3.5) is well posed, with a solution $\varphi \in \mathrm{H}^{5/2}(\partial\mathcal{B}_R) \subset \mathrm{H}^{3/2}(\partial\mathcal{B}_R)$. This concludes the alternative proof of the part 1 of Theorem 1.1 concerning the existence of R_γ, called R_ν in the statement. ■

Proof of part 2 of Theorem 1.1.
We apply exactly the same argument in the case when $\omega = \omega_{\boldsymbol{h}}$ is close to the unit ball in the sense of Subsection 3.4. In that case, we want to solve on $\partial\mathcal{B}_R$:

$$\frac{1}{2}\begin{bmatrix} -\frac{\nu}{2(1-\nu)} & 0 \\ 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \partial_\theta^2 \varphi + \varphi + \Lambda_{R,\boldsymbol{h}}(\varphi) = -\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n}.$$

This can be rewritten as

$$(\mathcal{P}_R^0 + \mathcal{Q}_R)\varphi = -\frac{R(1+\nu)}{E}\sigma(\mathbf{u}_0)\cdot\mathbf{n}, \qquad \text{with } \mathcal{Q}_R = (\Lambda_R - \Lambda_R^0) + (\Lambda_{R,h} - \Lambda_R).$$

We use Propositions 3.3 and 3.6 to get

$$\|\mathcal{Q}_R\|_{\mathcal{L}(\mathrm{H}^1(\partial\mathcal{B}_R),L^2(\partial\mathcal{B}_R))} \le C_1(\gamma)R^{-2} + C_{1/2}\|\boldsymbol{h}\|_{\mathrm{W}^{1,\infty}(\mathcal{B}_R)},$$

where $C_1(\gamma)$ and $C_{1/2}$ are defined in Propositions 3.3 and 3.6 respectively. Choosing R_γ such that $C_1(\gamma)R_\gamma^{-2} < \varepsilon_\gamma/2$ and $C_{1/2}\|\boldsymbol{h}\|_{\mathrm{W}^{1,\infty}(\mathcal{B}_R)} < \varepsilon_\gamma/2$ give the result of part 2 of Theorem 1.1, with $R_\nu = R_\gamma$ and $\varepsilon_\nu = C_{1/2}^{-1}\varepsilon_\gamma/2$ there. ■

4 Proof of Proposition 3.2

In this section, we give the proof of Proposition 3.2. This will be done in several steps. The first one is to analyze equation $L\mathbf{v} = 0$ in Fourier modes and seek solutions which will appear to have a special form. The second one is to explicit mode by mode the expression of

$$\frac{r(1+\nu)}{E}\sigma(\mathbf{v})\cdot\mathbf{n} \qquad \text{on } \partial\mathcal{B}_r.$$

The third one is to use boundary conditions in (3.3) for $r = 1$ and $r = R$ to get the expression of the Dirichlet-to-Neumann map mode by mode (this will be splitted into cases $n \ge 2$, $n = 1$ and $n = 0$).

4.1 First step : solutions of $L\mathbf{v} = 0$ in Fourier modes

We shall look for solutions $\mathbf{v}$ in polar coordinates in the form of a Fourier series:

$$\mathbf{v} = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} e_0^r(r) \\ e_0^\theta(r) \end{bmatrix} + \sum_{n\ge1} \begin{bmatrix} c_n^r(r) \\ c_n^\theta(r) \end{bmatrix} \cos n\theta + \sum_{n\ge1} \begin{bmatrix} d_n^r(r) \\ d_n^\theta(r) \end{bmatrix} \sin n\theta. \tag{4.1}$$

We first stick to the case when $n \ge 1$. It will appear later that we have to separate the study between the cases $n = 1$ and $n \ge 2$. Using the expression (2.1) of the elasticity operator in polar coordinates, the coefficients $(c_n^r, c_n^\theta, d_n^r, d_n^\theta)$ satisfy

$$\begin{aligned}
(\lambda+2\mu)\left(c_n^{r\,\prime\prime} + \frac{1}{r}c_n^{r\,\prime} - \frac{1}{r^2}c_n^r\right) - \frac{\mu n^2}{r^2}c_n^r - \frac{\lambda+3\mu}{r^2}nd_n^\theta + \frac{\lambda+\mu}{r}nd_n^{\theta\,\prime} &= 0, &(4.2)\\
(\lambda+2\mu)\left(d_n^{r\,\prime\prime} + \frac{1}{r}d_n^{r\,\prime} - \frac{1}{r^2}d_n^r\right) - \frac{\mu n^2}{r^2}d_n^r + \frac{\lambda+3\mu}{r^2}nc_n^\theta - \frac{\lambda+\mu}{r}nc_n^{\theta\,\prime} &= 0, &(4.3)\\
\mu\left(c_n^{\theta\,\prime\prime} + \frac{1}{r}c_n^{\theta\,\prime} - \frac{1}{r^2}c_n^\theta\right) - \frac{\lambda+2\mu}{r^2}n^2c_n^\theta + \frac{\lambda+\mu}{r}nd_n^{r\,\prime} + \frac{\lambda+3\mu}{r^2}nd_n^r &= 0, &(4.4)\\
\mu\left(d_n^{\theta\,\prime\prime} + \frac{1}{r}d_n^{\theta\,\prime} - \frac{1}{r^2}d_n^\theta\right) - \frac{\lambda+2\mu}{r^2}n^2d_n^\theta - \frac{\lambda+\mu}{r}nc_n^{r\,\prime} - \frac{\lambda+3\mu}{r^2}nc_n^r &= 0. &(4.5)
\end{aligned}$$

This is a system of four differential equations which are coupled two by two. We introduce

$$\gamma = \frac{\mu}{\lambda + 2\mu} = \frac{1-2\nu}{2(1-\nu)} \quad \in \Big(0, \frac{3}{4}\Big). \tag{4.6}$$

Then relations (4.2)–(4.5) rewrite

$$\left\{ \begin{array}{rcl} \left({c_n^r}'' + \frac{1}{r}{c_n^r}' - \frac{1}{r^2}c_n^r - \frac{n}{r^2}d_n^\theta + \frac{n}{r}{d_n^\theta}'\right) - \frac{\gamma n}{r}\left(\frac{n}{r}c_n^r + \frac{1}{r}d_n^\theta + {d_n^\theta}'\right) &=& 0, \\ \gamma\left({d_n^\theta}'' + \frac{1}{r}{d_n^\theta}' - \frac{1}{r^2}d_n^\theta - \frac{n}{r^2}c_n^r + \frac{n}{r}{c_n^r}'\right) - \frac{n}{r}\left(\frac{n}{r}d_n^\theta + \frac{1}{r}c_n^r + {c_n^r}'\right) &=& 0, \end{array} \right. \tag{4.7}$$

$$\left\{ \begin{array}{rcl} \left({d_n^r}'' + \frac{1}{r}{d_n^r}' - \frac{1}{r^2}d_n^r + \frac{n}{r^2}c_n^\theta - \frac{n}{r}{c_n^\theta}'\right) - \frac{\gamma n}{r}\left(\frac{n}{r}d_n^r - \frac{1}{r}c_n^\theta - {c_n^\theta}'\right) &=& 0, \\ \gamma\left({c_n^\theta}'' + \frac{1}{r}{c_n^\theta}' - \frac{1}{r^2}c_n^\theta + \frac{n}{r^2}d_n^r - \frac{n}{r}{d_n^r}'\right) - \frac{n}{r}\left(\frac{n}{r}c_n^\theta - \frac{1}{r}d_n^r - {d_n^r}'\right) &=& 0. \end{array} \right. \tag{4.8}$$

Mimicking the form of solutions in Fourier series for the Laplace problem, we look for solutions of the form

$$c_n^r(r) = \beta_r r^\alpha \qquad \text{and} \qquad d_n^\theta(r) = \beta_\theta r^\alpha, \tag{4.9}$$

$$d_n^r(r) = \tilde\beta_r r^{\tilde\alpha} \qquad \text{and} \qquad c_n^\theta(r) = \tilde\beta_\theta r^{\tilde\alpha}, \tag{4.10}$$

where the parameters α, β and $\tilde\alpha, \tilde\beta$ have to be determined. Plugging these expressions into (4.7)–(4.8), we obtain

$$\left\{ \begin{array}{rcl} \big((\alpha^2-1)\beta_r + n(\alpha-1)\beta_\theta\big) - \gamma n\left(n\beta_r + (\alpha+1)\beta_\theta\right) &=& 0, \\ \gamma\big((\alpha^2-1)\beta_\theta + n(\alpha-1)\beta_r\big) - n\left(n\beta_\theta + (\alpha+1)\beta_r\right) &=& 0, \end{array} \right. \tag{4.11}$$

$$\left\{ \begin{array}{rcl} \Big((\tilde\alpha^2-1)\tilde\beta_r - n(\tilde\alpha-1)\tilde\beta_\theta\Big) - \gamma n\Big(n\tilde\beta_r - (\tilde\alpha+1)\tilde\beta_\theta\Big) &=& 0, \\ \gamma\Big((\tilde\alpha^2-1)\tilde\beta_\theta - n(\tilde\alpha-1)\tilde\beta_r\Big) - n\Big(n\tilde\beta_\theta - (\tilde\alpha+1)\tilde\beta_r\Big) &=& 0. \end{array} \right. \tag{4.12}$$

In a matricial form, these equations read

$$M(\alpha)\begin{bmatrix}\beta_r\\ \beta_\theta\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad \text{and} \qquad \widetilde{M}(\tilde\alpha)\begin{bmatrix}\tilde\beta_r\\ \tilde\beta_\theta\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \tag{4.13}$$

with

$$M(\alpha) = \begin{bmatrix} \alpha^2 - 1 - \gamma n^2 & n(\alpha - 1 - \gamma(\alpha+1)) \\ n(\gamma(\alpha-1) - (\alpha+1)) & \gamma(\alpha^2-1) - n^2 \end{bmatrix},$$

and

$$\widetilde{M}(\alpha) = \begin{bmatrix} \alpha^2 - 1 - \gamma n^2 & -n(\alpha - 1 - \gamma(\alpha+1)) \\ -n(\gamma(\alpha-1) - (\alpha+1)) & \gamma(\alpha^2-1) - n^2 \end{bmatrix}.$$

Thus, the determinants of the later two matrices $M(\alpha)$ and $\widetilde{M}(\alpha)$ involve the same bi-quadratic expression in α and are given by

$$\det M(\alpha) = \det \widetilde{M}(\alpha) = \gamma\left(\alpha^4 - 2\alpha^2(1+n^2) + (n^2-1)^2\right). \tag{4.14}$$

These determinants cancel for

$$\alpha_n^{\pm\pm} = \pm n \pm 1.$$

At this point we notice that we have 4 roots when $n \geq 2$ and only 3 in the case when $n = 1$ for which $\alpha_1^{-+} = \alpha_1^{+-} = 0$. We study the two cases separately.

4.1.1 Case $n \geq 2$

Consider the 4 roots $\alpha_n^{\pm\pm}$. For each of them and modulo a multiplicative constant, the coordinates $\beta_{r,n}^{\pm\pm}, \beta_{\theta,n}^{\pm\pm}, \tilde\beta_{r,n}^{\pm\pm}, \tilde\beta_{\theta,n}^{\pm\pm}$ of vectors satisfying (4.13) are defined by

$$\begin{aligned} \beta_{\theta,n}^{\pm\pm} &= \frac{1}{\gamma-1}\left(\gamma - \frac{(\alpha_n^{\pm\pm})^2-1}{n^2}\right), & \beta_{r,n}^{\pm\pm} &= \frac{\alpha_n^{\pm\pm}-1-\gamma(\alpha_n^{\pm\pm}+1)}{n(\gamma-1)}, \\ \tilde\beta_{\theta,n}^{\pm\pm} &= \beta_{\theta,n}^{\pm\pm}, & \tilde\beta_{r,n}^{\pm\pm} &= -\beta_{r,n}^{\pm\pm}. \end{aligned}$$

After simplification, we have

$$\beta_{\theta,n}^{\pm\pm} = 1 - \frac{\pm\pm 2}{n(\gamma-1)}, \qquad \beta_{r,n}^{\pm\pm} = -\frac{\alpha_n^{\pm\pm}+1}{n} - \frac{2}{n(\gamma-1)}.$$

For each choice $\alpha_n^{\pm\pm} = \pm n \pm 1$, we choose explicitly coefficients $\beta_{\theta,n}^{\pm\pm}, \beta_{r,n}^{\pm\pm}$ by

$$\left\{ \begin{array}{llllll} \alpha_n^{++} = n+1, & \beta_{\theta,n}^{++} = 1 - \frac{2}{n(\gamma-1)}, & \beta_{r,n}^{++} = -1 - \frac{2\gamma}{n(\gamma-1)}, \\ \alpha_n^{+-} = n-1, & \beta_{\theta,n}^{+-} = 1, & \beta_{r,n}^{+-} = -1, \\ \alpha_n^{-+} = -n+1, & \beta_{\theta,n}^{-+} = 1 + \frac{2}{n(\gamma-1)}, & \beta_{r,n}^{-+} = 1 - \frac{2\gamma}{n(\gamma-1)}, \\ \alpha_n^{--} = -n-1, & \beta_{\theta,n}^{--} = 1, & \beta_{r,n}^{--} = 1. \end{array} \right. \tag{4.15}$$

Then the functions $c_n^r, d_n^\theta, d_n^r, c_n^\theta$ take the form

$$\begin{aligned} c_n^r(r) &= \beta_{r,n}^{--}A_n^{--}r^{-n-1} + \beta_{r,n}^{-+}A_n^{-+}r^{-n+1} + \beta_{r,n}^{+-}A_n^{+-}r^{n-1} + \beta_{r,n}^{++}A_n^{++}r^{n+1}, \\ d_n^\theta(r) &= \beta_{\theta,n}^{--}A_n^{--}r^{-n-1} + \beta_{\theta,n}^{-+}A_n^{-+}r^{-n+1} + \beta_{\theta,n}^{+-}A_n^{+-}r^{n-1} + \beta_{\theta,n}^{++}A_n^{++}r^{n+1}, \\ d_n^r(r) &= -\beta_{r,n}^{--}B_n^{--}r^{-n-1} - \beta_{r,n}^{-+}B_n^{-+}r^{-n+1} - \beta_{r,n}^{+-}B_n^{+-}r^{n-1} - \beta_{r,n}^{++}B_n^{++}r^{n+1}, \\ c_n^\theta(r) &= \beta_{\theta,n}^{--}B_n^{--}r^{-n-1} + \beta_{\theta,n}^{-+}B_n^{-+}r^{-n+1} + \beta_{\theta,n}^{+-}B_n^{+-}r^{n-1} + \beta_{\theta,n}^{++}B_n^{++}r^{n+1}. \end{aligned}$$

After simplification using (4.15), we obtain

$$c_n^r(r) = A_n^{--}r^{-n-1} + \beta_{r,n}^{-+}A_n^{-+}r^{-n+1} - A_n^{+-}r^{n-1} + \beta_{r,n}^{++}A_n^{++}r^{n+1}, \tag{4.16}$$
$$d_n^\theta(r) = A_n^{--}r^{-n-1} + \beta_{\theta,n}^{-+}A_n^{-+}r^{-n+1} + A_n^{+-}r^{n-1} + \beta_{\theta,n}^{++}A_n^{++}r^{n+1}, \tag{4.17}$$
$$d_n^r(r) = -B_n^{--}r^{-n-1} - \beta_{r,n}^{-+}B_n^{-+}r^{-n+1} + B_n^{+-}r^{n-1} - \beta_{r,n}^{++}B_n^{++}r^{n+1}, \tag{4.18}$$
$$c_n^\theta(r) = B_n^{--}r^{-n-1} + \beta_{\theta,n}^{-+}B_n^{-+}r^{-n+1} + B_n^{+-}r^{n-1} + \beta_{\theta,n}^{++}B_n^{++}r^{n+1}. \tag{4.19}$$

For each $n \geq 2$, the system (4.7) and respectively (4.8) are systems of two linear equations of second order, therefore with a vector space of solutions of dimension four. On the other hand the solutions described in (4.16)–(4.17), respectively (4.18)–(4.19) span each a vector space of dimension four as the four real numbers $A_n^{\pm\pm}$ move. We have therefore found all the solutions of respectively (4.7) and (4.8) for $n \geq 2$.

4.1.2 Case $n = 1$

In this subsection, we keep the subscript n although $n = 1$. In this case, we have 3 roots $\alpha_n^{++} = 2$, $\alpha_n^{00} = 0$ and $\alpha_n^{--} = -2$. Associated to each of them we can choose coordinates of eigenvectors in the following way :

$$\left\{ \begin{array}{lll} \alpha_n^{++} = 2, & \beta_{\theta,n}^{++} = 1 - \frac{2}{\gamma-1}, & \beta_{r,n}^{++} = -1 - \frac{2\gamma}{\gamma-1}, \\ \alpha_n^{00} = 0, & \beta_{\theta,n}^{00} = 1, & \beta_{r,n}^{00} = -1, \\ \alpha_n^{--} = -2, & \beta_{\theta,n}^{--} = 1, & \beta_{r,n}^{--} = 1. \end{array} \right. \tag{4.20}$$

Then (for $n = 1$) the functions $c_n^r, d_n^\theta, d_n^r, c_n^\theta$ take the form

$$\begin{array}{rcrcrcr} c_n^r(r) &=& \beta_{r,n}^{--}A_n^{--}r^{-n-1} &+& \beta_{r,n}^{00}A_n^{00}r^{n-1} &+& \beta_{r,n}^{++}A_n^{++}r^{n+1}, \\ d_n^\theta(r) &=& \beta_{\theta,n}^{--}A_n^{--}r^{-n-1} &+& \beta_{\theta,n}^{00}A_n^{00}r^{n-1} &+& \beta_{\theta,n}^{++}A_n^{++}r^{n+1}, \\ d_n^r(r) &=& -\beta_{r,n}^{--}B_n^{--}r^{-n-1} &-& \beta_{r,n}^{00}B_n^{00}r^{n-1} &-& \beta_{r,n}^{++}B_n^{++}r^{n+1}, \\ c_n^\theta(r) &=& \beta_{\theta,n}^{--}B_n^{--}r^{-n-1} &+& \beta_{\theta,n}^{00}B_n^{00}r^{n-1} &+& \beta_{\theta,n}^{++}B_n^{++}r^{n+1}. \end{array}$$

After simplification using (4.20), we obtain

$$c_n^r(r) = A_n^{--}r^{-2} - A_n^{00} + \beta_{r,n}^{++}A_n^{++}r^2, \tag{4.21}$$
$$d_n^\theta(r) = A_n^{--}r^{-2} + A_n^{00} + \beta_{\theta,n}^{++}A_n^{++}r^2, \tag{4.22}$$
$$d_n^r(r) = -B_n^{--}r^{-2} + B_n^{00} - \beta_{r,n}^{++}B_n^{++}r^2, \tag{4.23}$$
$$c_n^\theta(r) = B_n^{--}r^{-2} + B_n^{00} + \beta_{\theta,n}^{++}B_n^{++}r^2. \tag{4.24}$$

Let us first prove that we have found the full space of solutions of (4.7) and (4.8) for $n = 1$.

Lemma 4.1 *The space of solutions of systems* (4.7) *and* (4.8) *are a 3 dimension space and any solution reads* (4.21)–(4.24).

PROOF. We prove this result for system (4.7). The proof is essentially similar for the system (4.8).
We use the change of variables $r = e^x$ and define

$$c_n^r(r) = f(\ln r), \qquad d_n^\theta(r) = g(\ln r).$$

System (4.7) can be rewritten after simplification:

$$\begin{cases} (f'' - f - g + g') - \gamma(f + g + g') = 0, \\ \gamma(g'' - g - f + f') - (g + f + f') = 0. \end{cases} \tag{4.25}$$

Denoting by $\mathbf{U} = (f, g, f', g')^\intercal$, we write (4.25) on the form $\mathbf{U}' = M\mathbf{U}$ with

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1+\gamma & 1+\gamma & 0 & \gamma - 1 \\ 1 + \frac{1}{\gamma} & 1 + \frac{1}{\gamma} & \frac{1}{\gamma} - 1 & 0 \end{bmatrix}.$$

We have $\det(M - X\mathrm{Id}_4) = X^2(X-2)(X+2)$. We notice that we find again the solutions $e^{2x} = r^2$, $e^{-2x} = r^{-2}$, and the constant functions. Computing a Jordan decomposition for M, we have

$$M = PTP^{-1} \quad \text{with} \quad T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 3\gamma - 1 & 1 & 1 & 0 \\ 3 - \gamma & 1 & -1 & 0 \\ 2(3\gamma - 1) & -2 & 0 & 1 \\ 2(3 - \gamma) & -2 & 0 & -1 \end{bmatrix}.$$

The vector $\mathbf{V} = P^{-1}\mathbf{U}$ satisfies the equation $\mathbf{V}' = T\mathbf{V}$ whose general solution is given by

$$\mathbf{V}(x) = \begin{bmatrix} ae^{2x} \\ be^{-2x} \\ c \\ cx + d \end{bmatrix}, \qquad \text{with} \quad a, b, c, d \in \mathbb{R}^4.$$

We infer

$$\mathbf{U}(x) = P\mathbf{V}(x) = \begin{bmatrix} a(3\gamma-1)e^{2x} + be^{-2x} + c \\ a(3-\gamma)e^{2x} + be^{-2x} - c \\ 2a(3\gamma-1)e^{2x} - 2be^{-2x} + cx + d \\ 2a(3-\gamma)e^{2x} - 2be^{-2x} - cx - d \end{bmatrix}.$$

Coming back to the initial r-variable, the first two relations give the general expression for c_n^r and d_n^θ:

$$\begin{bmatrix} c_n^r(r) \\ d_n^\theta(r) \end{bmatrix} = \begin{bmatrix} a(3\gamma-1)r^2 + br^{-2} + c \\ a(3-\gamma)r^2 + br^{-2} - c \end{bmatrix} = \begin{bmatrix} a(1-\gamma)\beta_{r,n}^{++} r^2 + br^{-2} + c \\ a(1-\gamma)\beta_{\theta,n}^{++} r^2 + br^{-2} - c \end{bmatrix},$$

since $\beta_{r,n}^{++} = \frac{1-3\gamma}{\gamma-1}$ and $\beta_{\theta,n}^{++} = \frac{\gamma-3}{\gamma-1}$. This achieves the proof of the lemma. ∎

4.1.3 Case $n = 0$

The system (2.1) then simply reads

$$(\lambda+2\mu)\left(e_0^{r\prime\prime} + \frac{1}{r}e_0^{r\prime} - \frac{1}{r^2}e_0^r\right) = 0, \tag{4.26}$$

$$\mu\left(e_0^{\theta\prime\prime} + \frac{1}{r}e_0^{\theta\prime} - \frac{1}{r^2}e_0^\theta\right) = 0. \tag{4.27}$$

This is a decoupled system of two second order equations, for which the solutions are easily shown to be

$$e_0^r(r) = A_0^{--}r^{-1} + A_0^{++}r, \tag{4.28}$$

$$e_0^\theta(r) = B_0^{--}r^{-1} + B_0^{++}r. \tag{4.29}$$

Thus we have found the full vector space of solutions for $n = 0$.

4.2 Expression of the Neumann operator

In this section, we give the expression of the Neumann operator

$$\frac{r(1+\nu)}{E}\sigma(\mathbf{v})\cdot\mathbf{n}(r),$$

when $\mathbf{v}$ has the form (4.1) first, and then in the special case when $\mathbf{v}$ is a solution of $L\mathbf{v}$ and satisfies (4.16)–(4.19), (4.21)–(4.24) and (4.28)–(4.29). Using first (2.2) and (4.1), we can write

$$\sigma(\mathbf{v})\cdot\mathbf{n}(r) = \begin{bmatrix} (\lambda+2\mu)e_0^{r\prime}(r) + \frac{\lambda}{r}e_0^r(r) \\ \mu\left(e_0^{\theta\prime}(r) - \frac{1}{r}e_0^\theta(r)\right) \end{bmatrix} + \sum_{n\geq 1}\begin{bmatrix} (\lambda+2\mu)c_n^{r\prime}(r) + \frac{\lambda}{r}c_n^r(r) + \frac{\lambda}{r}nd_n^\theta(r) \\ \mu\left(\frac{n}{r}d_n^r(r) - \frac{1}{r}c_n^\theta(r) + c_n^{\theta\prime}(r)\right) \end{bmatrix}\cos n\theta$$
$$+ \sum_{n\geq 1}\begin{bmatrix} (\lambda+2\mu)d_n^{r\prime}(r) + \frac{\lambda}{r}d_n^r(r) - \frac{\lambda}{r}nc_n^\theta(r) \\ \mu\left(-\frac{n}{r}c_n^r(r) - \frac{1}{r}d_n^\theta(r) + d_n^{\theta\prime}(r)\right) \end{bmatrix}\sin n\theta.$$

Using definition (4.6) of γ, we have $\frac{\lambda}{\lambda+2\mu} = 1 - 2\gamma$ and we deduce

$$r\sigma(\mathbf{v}) \cdot \mathbf{n}(r) = \begin{bmatrix} \lambda + 2\mu & 0 \\ 0 & \mu \end{bmatrix} \Bigg(\begin{bmatrix} {e_0^r}'(r)r + (1-2\gamma)e_0^r(r) \\ {e_0^\theta}'(r)r - e_0^\theta(r) \end{bmatrix} + \sum_{n\geq 1} \begin{bmatrix} {c_n^r}'(r)r + (1-2\gamma)c_n^r(r) + n(1-2\gamma)d_n^\theta(r) \\ nd_n^r(r) - c_n^\theta(r) + {c_n^\theta}'(r)r \end{bmatrix} \cos n\theta + \sum_{n\geq 1} \begin{bmatrix} {d_n^r}'(r)r + (1-2\gamma)d_n^r(r) - n(1-2\gamma)c_n^\theta(r) \\ -nc_n^r(r) - d_n^\theta(r) + {d_n^\theta}'(r)r \end{bmatrix} \sin n\theta \Bigg). \quad (4.30)$$

Using (2.9) we also get that

$$\frac{1+\nu}{E}\mu = \frac{1}{2}, \qquad \text{and} \qquad \frac{1+\nu}{E}(\lambda + 2\mu) = \frac{1}{2\gamma}, \quad (4.31)$$

and therefore

$$\frac{1+\nu}{E} r\sigma(\mathbf{v}) \cdot \mathbf{n}(r) = \begin{bmatrix} \frac{1}{2\gamma} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \Bigg(\begin{bmatrix} {e_0^r}'(r)r + (1-2\gamma)e_0^r(r) \\ {e_0^\theta}'(r)r - e_0^\theta(r) \end{bmatrix} + \sum_{n\geq 1} \begin{bmatrix} {c_n^r}'(r)r + (1-2\gamma)c_n^r(r) + n(1-2\gamma)d_n^\theta(r) \\ nd_n^r(r) - c_n^\theta(r) + {c_n^\theta}'(r)r \end{bmatrix} \cos n\theta + \sum_{n\geq 1} \begin{bmatrix} {d_n^r}'(r)r + (1-2\gamma)d_n^r(r) - n(1-2\gamma)c_n^\theta(r) \\ -nc_n^r(r) - d_n^\theta(r) + {d_n^\theta}'(r)r \end{bmatrix} \sin n\theta \Bigg).$$

For convenience we shall now write

$$\forall n \geq 1, \quad \begin{cases} c_1(n,r) &=& {c_n^r}'(r)r + (1-2\gamma)c_n^r(r) + n(1-2\gamma)d_n^\theta(r), \\ s_2(n,r) &=& -nc_n^r(r) - d_n^\theta(r) + {d_n^\theta}'(r)r, \end{cases} \quad (4.32)$$

$$\forall n \geq 1, \quad \begin{cases} s_1(n,r) &=& {d_n^r}'(r)r + (1-2\gamma)d_n^r(r) - n(1-2\gamma)c_n^\theta(r), \\ c_2(n,r) &=& nd_n^r(r) - c_n^\theta(r) + {c_n^\theta}'(r)r, \end{cases} \quad (4.33)$$

$$c_1(r) = {e_0^r}'(r)r + (1-2\gamma)e_0^r(r), \quad (4.34)$$

$$c_2(r) = {e_0^\theta}'(r)r - e_0^\theta(r), \quad (4.35)$$

so that

$$\frac{1+\nu}{E} r\sigma(\mathbf{v}) \cdot \mathbf{n}(r) = \begin{bmatrix} \frac{1}{2\gamma} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \Bigg(\begin{bmatrix} c_1(r) \\ c_2(r) \end{bmatrix} + \sum_{n\geq 1} \begin{bmatrix} c_1(n,r) \\ c_2(n,r) \end{bmatrix} \cos n\theta + \sum_{n\geq 1} \begin{bmatrix} s_1(n,r) \\ s_2(n,r) \end{bmatrix} \sin n\theta \Bigg). \quad (4.36)$$

In the following, we compute the coefficients $c_1(n,r)$, $s_2(n,r)$, $s_1(n,r)$, $c_2(n,r)$, $c_1(r)$, and $c_2(r)$. Again we split the study depending on the cases $n \geq 2$, $n = 1$ and $n = 0$.

4.2.1 Case $n \geq 2$

In that case, replacing c_n^r and d_n^θ by their expression (4.16)–(4.17) in (4.32)–(4.33), we get

$$\begin{aligned} c_1(n,r) &= -2\gamma(n+1)A_n^{--}r^{-n-1} + \Big((-n+2-2\gamma)\beta_{r,n}^{-+} + n(1-2\gamma)\beta_{\theta,n}^{-+}\Big) A_n^{-+}r^{-n+1} \\ &\quad +2\gamma(1-n)A_n^{+-}r^{n-1} + \Big((n+2-2\gamma)\beta_{r,n}^{++} + n(1-2\gamma)\beta_{\theta,n}^{++}\Big) A_n^{++}r^{n+1} \\ &= -2\gamma(n+1)A_n^{--}r^{-n-1} + 2\gamma\left(\frac{2}{n} - 1 - n\right) A_n^{-+}r^{-n+1} \\ &\quad +2\gamma(1-n)A_n^{+-}r^{n-1} + 2\gamma\left(\frac{2}{n} + 1 - n\right) A_n^{++}r^{n+1}, \end{aligned} \quad (4.37)$$

$$\begin{aligned} s_2(n,r) &= -2(n+1)A_n^{--}r^{-n-1} - n(\beta_{r,n}^{-+} + \beta_{\theta,n}^{-+})A_n^{-+}r^{-n+1} \\ &\quad +2(n-1)A_n^{+-}r^{n-1} - n(\beta_{r,n}^{++} - \beta_{\theta,n}^{++})A_n^{++}r^{n+1} \\ &= -2(n+1)A_n^{--}r^{-n-1} + 2(1-n)A_n^{-+}r^{-n+1} \\ &\quad +2(n-1)A_n^{+-}r^{n-1} + 2(n+1)A_n^{++}r^{n+1}, \end{aligned} \tag{4.38}$$

and similarly with c_n^θ and d_n^r, we get

$$\begin{aligned} s_1(n,r) &= -\Bigg(-2\gamma(n+1)B_n^{--}r^{-n-1} + 2\gamma\left(\frac{2}{n} - 1 - n\right)B_n^{-+}r^{-n+1} \\ &\quad +2\gamma(1-n)B_n^{+-}r^{n-1} + 2\gamma\left(\frac{2}{n} + 1 - n\right)B_n^{++}r^{n+1}\Bigg), \end{aligned} \tag{4.39}$$

$$\begin{aligned} c_2(n,r) &= -2(n+1)B_n^{--}r^{-n-1} - 2(n-1)B_n^{-+}r^{-n+1} \\ &\quad -2(1-n)B_n^{+-}r^{n-1} + 2(n+1)B_n^{++}r^{n+1}. \end{aligned} \tag{4.40}$$

4.2.2 Case $n = 1$

A similar computation replacing c_n^r and d_n^θ in (4.32)–(4.33) by their expression (4.21)–(4.24) gives in the case $n = 1$,

$$\begin{aligned} c_1(n,r) &= -4\gamma A_n^{--}r^{-2} + 4\gamma A_n^{++}r^2, \\ s_2(n,r) &= -4A_n^{--}r^{-2} + 4A_n^{++}r^2, \\ s_1(n,r) &= 4\gamma B_n^{--}r^{-2} - 4\gamma B_n^{++}r^2, \\ c_2(n,r) &= -4B_n^{--}r^{-2} + 4B_n^{++}r^2. \end{aligned} \tag{4.41}$$

Note that the constant terms A_n^{00} and B_n^{00} do not appear.

4.2.3 Case $n = 0$

Using the expressions (4.28)–(4.29) in (4.34)–(4.35), we deduce

$$\begin{aligned} c_1(r) &= -2\gamma A_n^{--}r^{-1} + 2(1-\gamma)A_n^{++}r, \\ c_2(r) &= -2B_n^{--}r^{-1}. \end{aligned} \tag{4.42}$$

4.3 DtN map and Ventcel conditions for modes $n \geq 2$

Now we are in position to compute the expression of the Dirichlet-to-Neumann map for each mode $n \geq 2$ and prove the corresponding part of Proposition 3.2. The strategy is the following : We first get relations between the $A^{\pm\pm}(n)$ thanks to the property in the inner disk of radius $r = 1$. Then we get an expression of the component of φ with respect to the predominant terms $A^{+\pm}(n)$ thanks to the Dirichlet condition in (3.3), and we control the remainder terms (Lemma 4.2). Eventually we give the expression of $\Lambda_R(\varphi)$ with respect to φ for mode n.

4.3.1 Neumann conditions on the inner circle

In this subsection we fix $n \geq 2$. In (3.3), we use the Neumann condition on the inner circle $\partial\omega$ of radius $r = 1$, in order to get relations between the terms $A^{\pm\pm}(n)$ (resp. $B^{\pm\pm}(n)$). Neumann conditions also read

$$\frac{1+\nu}{E} r\sigma(\mathbf{v}) \cdot \mathbf{n}|_{r=1} = 0.$$

This implies that for $r = 1$:

$$c_1(n,r) = s_2(n,r) = 0 \qquad \text{and} \qquad s_1(n,r) = c_2(n,r) = 0.$$

Using (4.37)–(4.40), these equalities can be written in matrix form

$$M_n^- \begin{bmatrix} A_n^{--} \\ A_n^{-+} \end{bmatrix} = M_n^+ \begin{bmatrix} A_n^{+-} \\ A_n^{++} \end{bmatrix}, \qquad \text{and} \qquad M_n^- \begin{bmatrix} B_n^{--} \\ B_n^{-+} \end{bmatrix} = M_n^+ \begin{bmatrix} B_n^{+-} \\ B_n^{++} \end{bmatrix}, \tag{4.43}$$

with the same matrices $M_n^\pm$ for the two linear systems defined by

$$M_n^- = 2 \begin{bmatrix} -\gamma(n+1) & \gamma(\frac{2}{n}-1-n) \\ n+1 & n-1 \end{bmatrix}, \qquad M_n^+ = -2 \begin{bmatrix} \gamma(1-n) & \gamma(\frac{2}{n}+1-n) \\ 1-n & -(n+1) \end{bmatrix}.$$

We have

$$\det M_n^- = \det M_n^+ = 8\gamma \left(n - \frac{1}{n} \right).$$

We deduce

$$(M_n^-)^{-1} = \frac{1}{4\gamma(n-\frac{1}{n})} \begin{bmatrix} n-1 & -\gamma(\frac{2}{n}-1-n) \\ -(n+1) & -\gamma(n+1) \end{bmatrix} = \mathcal{O}(1).$$

Then, denoting $M_n = (M_n^-)^{-1} M_n^+$, we find

$$M_n \;=\; \begin{bmatrix} n-1 & n \\ -n & -(n+1) \end{bmatrix}. \tag{4.44}$$

With matrix M_n given by (4.44), relations (4.43) read

$$\begin{bmatrix} A_n^{--} \\ A_n^{-+} \end{bmatrix} = M_n \begin{bmatrix} A_n^{+-} \\ A_n^{++} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} B_n^{--} \\ B_n^{-+} \end{bmatrix} = M_n \begin{bmatrix} B_n^{+-} \\ B_n^{++} \end{bmatrix}.$$

Let us define

$$M_n(R) = R^{-2n} \begin{bmatrix} n-1 & nR^{-2} \\ -nR^2 & -(n+1) \end{bmatrix}, \tag{4.45}$$

then

$$\begin{bmatrix} A_n^{--} R^{-n-1} \\ A_n^{-+} R^{-n+1} \end{bmatrix} \;=\; M_n(R) \begin{bmatrix} A_n^{+-} R^{n-1} \\ A_n^{++} R^{n+1} \end{bmatrix}, \quad \begin{bmatrix} B_n^{--} R^{-n-1} \\ B_n^{-+} R^{-n+1} \end{bmatrix} \;=\; M_n(R) \begin{bmatrix} B_n^{+-} R^{n-1} \\ B_n^{++} R^{n+1} \end{bmatrix}. \tag{4.46}$$

4.3.2 Dirichlet condition on $r = R$

The aim of this subsection is to give the relation between the n-th component of φ and the terms $A^{+\pm}(n)$, $B^{+\pm}(n)$, and to control the other terms. For this we use the Dirichlet boundary condition $\mathbf{v} = \varphi$ on $\partial\mathcal{B}_R$ in (3.3). Decomposing the datum φ and the solution $\mathbf{v}$ of (3.3) according to (3.6) and (4.1), we have

$$c_n^r(R) = \varphi_n^r, \qquad d_n^r(R) = \psi_n^r, \qquad c_n^\theta(R) = \varphi_n^\theta, \qquad d_n^\theta(R) = \psi_n^\theta. \tag{4.47}$$

Using expressions (4.16)–(4.17), we have

$$\begin{aligned} \varphi_n^r &= A_n^{--} R^{-n-1} + \beta_{r,n}^{-+} A_n^{-+} R^{-n+1} - A_n^{+-} R^{n-1} + \beta_{r,n}^{++} A_n^{++} R^{n+1}, \\ \psi_n^\theta &= A_n^{--} R^{-n-1} + \beta_{\theta,n}^{-+} A_n^{-+} R^{-n+1} + A_n^{+-} R^{n-1} + \beta_{\theta,n}^{++} A_n^{++} R^{n+1}. \end{aligned}$$

Using (4.46), we deduce

$$\begin{aligned}
\begin{bmatrix}\varphi_n^r\\ \psi_n^\theta\end{bmatrix} &= \begin{bmatrix}1 & \beta_{r,n}^{-+}\\ 1 & \beta_{\theta,n}^{-+}\end{bmatrix}\begin{bmatrix}A_n^{--}R^{-n-1}\\ A_n^{-+}R^{-n+1}\end{bmatrix} + \begin{bmatrix}-1 & \beta_{r,n}^{++}\\ 1 & \beta_{\theta,n}^{++}\end{bmatrix}\begin{bmatrix}A_n^{+-}R^{n-1}\\ A_n^{++}R^{n+1}\end{bmatrix}\\
&= M_A\left(\mathrm{Id}_2 + M_A^{-1}N_A\ M_n(R)\right)\begin{bmatrix}A_n^{+-}R^{n-1}\\ A_n^{++}R^{n+1}\end{bmatrix},
\end{aligned}$$

with

$$M_A = \begin{bmatrix}-1 & \beta_{r,n}^{++}\\ 1 & \beta_{\theta,n}^{++}\end{bmatrix} = \begin{bmatrix}-1 & -1-\frac{2\gamma}{n(\gamma-1)}\\ 1 & 1-\frac{2}{n(\gamma-1)}\end{bmatrix}, \qquad N_A = \begin{bmatrix}1 & \beta_{r,n}^{-+}\\ 1 & \beta_{\theta,n}^{-+}\end{bmatrix} = \begin{bmatrix}1 & 1-\frac{2\gamma}{n(\gamma-1)}\\ 1 & 1+\frac{2}{n(\gamma-1)}\end{bmatrix}.$$

We notice that

$$\det M_A = \det N_A = \frac{2(\gamma+1)}{n(\gamma-1)}, \qquad M_A^{-1} = \frac{n(\gamma-1)}{2(\gamma+1)}\begin{bmatrix}1-\frac{2}{n(\gamma-1)} & 1+\frac{2\gamma}{n(\gamma-1)}\\ -1 & -1\end{bmatrix}.$$

Defining $N := M_A^{-1}N_A$, we compute

$$N = \frac{\gamma-1}{\gamma+1}\begin{bmatrix}n+1 & n+\frac{4\gamma}{n(\gamma-1)^2}\\ -n & 1-n\end{bmatrix}. \tag{4.48}$$

Thus we have

$$\begin{bmatrix}A_n^{+-}R^{n-1}\\ A_n^{++}R^{n+1}\end{bmatrix} = \left(\mathrm{Id}_2 + Q_n(R)\right)^{-1} M_A^{-1}\begin{bmatrix}\varphi_n^r\\ \psi_n^\theta\end{bmatrix}, \tag{4.49}$$

with

$$Q_n(R) = N\ M_n(R), \qquad \|Q_n(R)\|_\infty = n^2R^{2-2n}. \tag{4.50}$$

Similarly, using expressions (4.18)–(4.19), we have

$$\begin{aligned}
\psi_n^r &= -B_n^{--}R^{-n-1} - \beta_{r,n}^{-+}B_n^{-+}R^{-n+1} + B_n^{+-}R^{n-1} - \beta_{r,n}^{++}B_n^{++}R^{n+1},\\
\varphi_n^\theta &= B_n^{--}R^{-n-1} + \beta_{\theta,n}^{-+}B_n^{-+}R^{-n+1} + B_n^{+-}R^{n-1} + \beta_{\theta,n}^{++}B_n^{++}R^{n+1}.
\end{aligned}$$

Using (4.46), we deduce

$$\begin{aligned}
\begin{bmatrix}\psi_n^r\\ \varphi_n^\theta\end{bmatrix} &= \begin{bmatrix}-1 & -\beta_{r,n}^{-+}\\ 1 & \beta_{\theta,n}^{-+}\end{bmatrix}\begin{bmatrix}B_n^{--}R^{-n-1}\\ B_n^{-+}R^{-n+1}\end{bmatrix} + \begin{bmatrix}1 & -\beta_{r,n}^{++}\\ 1 & \beta_{\theta,n}^{++}\end{bmatrix}\begin{bmatrix}B_n^{+-}R^{n-1}\\ B_n^{++}R^{n+1}\end{bmatrix}\\
&= M_B\left(\mathrm{Id}_2 + M_B^{-1}N_B\ M_n(R)\right)\begin{bmatrix}B_n^{+-}R^{n-1}\\ B_n^{++}R^{n+1}\end{bmatrix},
\end{aligned}$$

with

$$M_B = \begin{bmatrix}1 & -\beta_{r,n}^{++}\\ 1 & \beta_{\theta,n}^{++}\end{bmatrix} = \begin{bmatrix}1 & 1+\frac{2\gamma}{n(\gamma-1)}\\ 1 & 1-\frac{2}{n(\gamma-1)}\end{bmatrix}, \qquad N_B = \begin{bmatrix}-1 & -\beta_{r,n}^{-+}\\ 1 & \beta_{\theta,n}^{-+}\end{bmatrix} = \begin{bmatrix}-1 & -1+\frac{2\gamma}{n(\gamma-1)}\\ 1 & 1+\frac{2}{n(\gamma-1)}\end{bmatrix}.$$

Thus

$$\det M_B = \det N_B = -\frac{2(\gamma+1)}{n(\gamma-1)}, \qquad M_B^{-1} = -\frac{n(\gamma-1)}{2(\gamma+1)}\begin{bmatrix}1-\frac{2}{n(\gamma-1)} & -1-\frac{2\gamma}{n(\gamma-1)}\\ -1 & 1\end{bmatrix}.$$

We therefore have

$$\begin{bmatrix} B_n^{+-}R^{n-1} \\ B_n^{++}R^{n+1} \end{bmatrix} = (\mathrm{Id}_2 + Q_n(R))^{-1}\, M_B^{-1} \begin{bmatrix} \psi_n^r \\ \varphi_n^\theta \end{bmatrix}, \tag{4.51}$$

with the same matrix $Q_n(R)$ defined in (4.50) for coefficients $A_n^{+,\pm}R^{n\pm1}$.
Note that from (4.46), (4.49) and (4.51), we also have

$$\begin{bmatrix} A_n^{--}R^{-n-1} \\ A_n^{-+}R^{-n+1} \end{bmatrix} = M_n(R)\,(\mathrm{Id}_2 + Q_n(R))^{-1}\, M_A^{-1} \begin{bmatrix} \varphi_n^r \\ \psi_n^\theta \end{bmatrix}, \tag{4.52}$$

and

$$\begin{bmatrix} B_n^{--}R^{-n-1} \\ B_n^{-+}R^{-n+1} \end{bmatrix} = M_n(R)\,(\mathrm{Id}_2 + Q_n(R))^{-1}\, M_B^{-1} \begin{bmatrix} \psi_n^r \\ \varphi_n^\theta \end{bmatrix}. \tag{4.53}$$

From (4.49)–(4.53) and the expression (4.45) of $M_n(R)$, we get the following first rough estimate:

Lemma 4.2 *There exist $R_0 > 0$ and a constant C such that for any $R > R_0$ and any $n \geq 2$, we have with $\Phi_n = (\varphi_n^r, \psi_n^r, \varphi_n^\theta, \psi_n^\theta)^\mathsf{T}$:*

$$|A_n^{+\pm}R^{n\pm1}| \leq Cn\|\Phi_n\|_\infty, \qquad |B_n^{+\pm}R^{n\pm1}| \leq Cn\|\Phi_n\|_\infty,$$
$$|A_n^{-\pm}R^{-n\pm1}| \leq C(n^3R^{2-2n})\|\Phi_n\|_\infty, \qquad |B_n^{-\pm}R^{-n\pm1}| \leq C(n^3R^{2-2n})\|\Phi_n\|_\infty.$$

4.3.3 DtN operator and Ventcel boundary condition on $r = R$

Now we are in position to give the expression of the Dirichlet-to-Neumann operator as an operator acting on the mode $n \geq 2$ of φ, and exhibit the Ventcel boundary condition (3.5) in mode n as in Proposition 3.2.

Recall that $\Lambda_R(\varphi) = \frac{R(1+\nu)}{E}\sigma(\mathbf{v})\cdot\mathbf{n}_{|r=R}$. We use (4.36), with the explicit components computed in (4.37)–(4.40). The first component for mode n of $\Lambda_R(\varphi)$ is then

$$\begin{aligned}
&\begin{bmatrix} 1-n \\ \frac{2}{n}+1-n \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} A_n^{+-}R^{n-1} \\ A_n^{++}R^{n+1} \end{bmatrix} \cos n\theta - \begin{bmatrix} B_n^{+-}R^{n-1} \\ B_n^{++}R^{n+1} \end{bmatrix} \sin n\theta \right) \\
&\qquad + \begin{bmatrix} -n-1 \\ \frac{2}{n}-1-n \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} A_n^{--}R^{-n-1} \\ A_n^{-+}R^{-n+1} \end{bmatrix} \cos n\theta - \begin{bmatrix} B_n^{--}R^{-n-1} \\ B_n^{-+}R^{-n+1} \end{bmatrix} \sin n\theta \right) \\
&= \left(\begin{bmatrix} 1-n \\ \frac{2}{n}+1-n \end{bmatrix}^\mathsf{T} + \begin{bmatrix} -n-1 \\ \frac{2}{n}-1-n \end{bmatrix}^\mathsf{T} M_n(R) \right) \\
&\qquad \times \left(\begin{bmatrix} A_n^{+-}R^{n-1} \\ A_n^{++}R^{n+1} \end{bmatrix} \cos n\theta - \begin{bmatrix} B_n^{+-}R^{n-1} \\ B_n^{++}R^{n+1} \end{bmatrix} \sin n\theta \right) \\
&= \left(\begin{bmatrix} 1-n \\ \frac{2}{n}+1-n \end{bmatrix}^\mathsf{T} + \begin{bmatrix} -n-1 \\ \frac{2}{n}-1-n \end{bmatrix}^\mathsf{T} M_n(R) \right) \\
&\qquad \times (\mathrm{Id}_2 + Q_n(R))^{-1} \left(M_A^{-1} \begin{bmatrix} \varphi_n^r \\ \psi_n^\theta \end{bmatrix} \cos n\theta - M_B^{-1} \begin{bmatrix} \psi_n^r \\ \varphi_n^\theta \end{bmatrix} \sin n\theta \right).
\end{aligned} \tag{4.54}$$

Let us now deal with the second component in mode $n \geq 2$ of $\Lambda_R(\varphi)$. This is

$$\begin{aligned}
&\begin{bmatrix} n-1 \\ n+1 \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} B_n^{+-}R^{n-1} \\ B_n^{++}R^{n+1} \end{bmatrix} \cos n\theta + \begin{bmatrix} A_n^{+-}R^{n-1} \\ A_n^{++}R^{n+1} \end{bmatrix} \sin n\theta \right) \\
&\qquad + \begin{bmatrix} -n-1 \\ 1-n \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} B_n^{--}R^{-n-1} \\ B_n^{-+}R^{-n+1} \end{bmatrix} \cos n\theta + \begin{bmatrix} A_n^{--}R^{-n-1} \\ A_n^{-+}R^{-n+1} \end{bmatrix} \sin n\theta \right)
\end{aligned}$$

$$= \left(\begin{bmatrix} n-1 \\ n+1 \end{bmatrix}^{\mathsf{T}} + \begin{bmatrix} -n-1 \\ 1-n \end{bmatrix}^{\mathsf{T}} M_n(R) \right) \left(\begin{bmatrix} B_n^{+-} R^{n-1} \\ B_n^{++} R^{n+1} \end{bmatrix} \cos n\theta + \begin{bmatrix} A_n^{+-} R^{n-1} \\ A_n^{++} R^{n+1} \end{bmatrix} \sin n\theta \right)$$

$$= \left(\begin{bmatrix} n-1 \\ n+1 \end{bmatrix}^{\mathsf{T}} + \begin{bmatrix} -n-1 \\ 1-n \end{bmatrix}^{\mathsf{T}} M_n(R) \right)$$
$$\times (\mathrm{Id}_2 + Q_n(R))^{-1} \left(M_B^{-1} \begin{bmatrix} \psi_n^r \\ \varphi_n^\theta \end{bmatrix} \cos n\theta + M_A^{-1} \begin{bmatrix} \varphi_n^r \\ \psi_n^\theta \end{bmatrix} \sin n\theta \right). \quad (4.55)$$

Let us denote

$$V_1 = \begin{bmatrix} 1-n \\ \frac{2}{n} + 1 - n \end{bmatrix} \qquad \text{and} \qquad V_2 = \begin{bmatrix} n-1 \\ n+1 \end{bmatrix}. \quad (4.56)$$

Thus a simple computation gives

$$\begin{array}{rclrcl} V_1^{\mathsf{T}} M_A^{-1} & = & \frac{1}{1+\gamma} \left[n-\gamma, 1-n\gamma \right], & V_1^{\mathsf{T}} M_B^{-1} & = & \frac{1}{1+\gamma} \left[-(n-\gamma), 1-n\gamma \right], \\ V_2^{\mathsf{T}} M_A^{-1} & = & \frac{1}{1+\gamma} \left[1-n\gamma, n-\gamma \right], & V_2^{\mathsf{T}} M_B^{-1} & = & \frac{1}{1+\gamma} \left[-(1-n\gamma), n-\gamma \right]. \end{array}$$

The main term for mode $n \geq 2$ for the Dirichlet-to-Neumann operator is then

$$\frac{1}{1+\gamma} \begin{bmatrix} \cos n\theta \left((n-\gamma)\varphi_n^r + (1-n\gamma)\psi_n^\theta \right) + \sin n\theta \left((n-\gamma)\psi_n^r + (n\gamma-1)\varphi_n^\theta \right) \\ \cos n\theta \left((n\gamma-1)\psi_n^r + (n-\gamma)\varphi_n^\theta \right) + \sin n\theta \left((1-n\gamma)\varphi_n^r + (n-\gamma)\psi_n^\theta \right) \end{bmatrix}.$$

Defining for each $n \geq 2$ the vector $\Phi_n = (\varphi_n^r, \psi_n^r, \varphi_n^\theta, \psi_n^\theta)^{\mathsf{T}}$, we get that in the corresponding basis, the matrix for mode n of $\Lambda_R(\varphi)$ is given of the form $\Lambda_n + \mathcal{R}_{n,R}$, with

$$\Lambda_n = \frac{1}{1+\gamma} \begin{bmatrix} n-\gamma & 0 & 0 & 1-n\gamma \\ 0 & n-\gamma & n\gamma-1 & 0 \\ 0 & n\gamma-1 & n-\gamma & 0 \\ 1-n\gamma & 0 & 0 & n-\gamma \end{bmatrix}. \quad (4.57)$$

From (4.55) and the expressions (4.45) and (4.50) of matrices $M_n(R)$ and $Q_n(R)$, we get

$$\|\mathcal{R}_{n,R}\|_\infty \leq C_\gamma n^2 R^{-2n+2},$$

for a fixed constant C_γ.

We can now give the expression of the Ventcel boundary condition in mode n as a matrix in the same basis. Using the decomposition (3.6) and relation (4.6), we can rewrite the left-hand-side of (3.5) in the form $(P_n + \mathcal{R}_{n,R})\Phi_n$, with P_n given by

$$P_n = \frac{-n^2}{4} \begin{bmatrix} \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ 0 & \frac{\nu}{1-\nu} & 0 & 0 \\ 0 & 0 & \frac{2(1-\nu)}{1-2\nu} & 0 \\ 0 & 0 & 0 & \frac{2(1-\nu)}{1-2\nu} \end{bmatrix} + \mathrm{Id}_4 + \frac{1}{1+\gamma} \begin{bmatrix} n-\gamma & 0 & 0 & 1-n\gamma \\ 0 & n-\gamma & n\gamma-1 & 0 \\ 0 & n\gamma-1 & n-\gamma & 0 \\ 1-n\gamma & 0 & 0 & n-\gamma \end{bmatrix}.$$

Using (4.6), we have,

$$\nu = \frac{1-2\gamma}{2(1-\gamma)}, \qquad \frac{\nu}{1-\nu} = 1-2\gamma, \qquad \frac{1-\nu}{1-2\nu} = \frac{1}{2\gamma},$$

so that P_n can be written in the following form

$$P_n = \frac{-n^2}{4} \begin{bmatrix} 1-2\gamma & 0 & 0 & 0 \\ 0 & 1-2\gamma & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} \end{bmatrix} + \mathrm{Id}_4 + \frac{1}{1+\gamma} \begin{bmatrix} n-\gamma & 0 & 0 & 1-n\gamma \\ 0 & n-\gamma & n\gamma-1 & 0 \\ 0 & n\gamma-1 & n-\gamma & 0 \\ 1-n\gamma & 0 & 0 & n-\gamma \end{bmatrix}.$$

This completes the proof of Proposition 3.2 in case $n \geq 2$.

4.4 DtN map and Ventcel conditions for modes $n = 1, 0$

We follow now the same strategy for the mode $n = 1$ and the constant term $n = 0$ in (4.36).

4.4.1 Case $n = 1$

The first step is to use Neumann conditions on the inner circle of radius $r = 1$.

$$\frac{1+\nu}{E} r\sigma(\mathbf{v}) \cdot \mathbf{n}|_{r=1} = 0.$$

This implies that for $n = 1$, $r = 1$,

$$c_1(n,r) = s_2(n,r) = 0 \qquad \text{and} \qquad s_1(n,r) = c_2(n,r) = 0.$$

Using (4.41) we directly get

$$A_n^{--} = A_n^{++} \qquad \text{and} \qquad B_n^{--} = B_n^{++} \qquad \text{when } n = 1,$$

so that

$$A_n^{--} R^{-2} = R^{-4} A_n^{++} R^2, \qquad B_n^{--} R^{-2} = R^{-4} B_n^{++2} \qquad \text{when } n = 1. \tag{4.58}$$

Now we apply the Dirichlet condition in (3.3) on $r = R$, which reads

$$c_n^r(R) = \varphi_n^r, \qquad d_n^r(R) = \psi_n^r, \qquad c_n^\theta(R) = \varphi_n^\theta, \qquad d_n^\theta(R) = \psi_n^\theta. \tag{4.59}$$

Using (4.21)–(4.24), we get

$$\begin{array}{rcrcrcl} \varphi_n^r &=& A_n^{--} R^{-2} &-& A_n^{00} &+& \beta_{r,n}^{++} A_n^{++} R^2, \\ \psi_n^\theta &=& A_n^{--} R^{-2} &+& A_n^{00} &+& \beta_{\theta,n}^{++} A_n^{++} R^2, \\ \psi_n^r &=& -B_n^{--} R^{-2} &+& B_n^{00} &-& \beta_{r,n}^{++} B_n^{++} R^2, \\ \varphi_n^\theta &=& B_n^{--} R^{-2} &+& B_n^{00} &+& \beta_{\theta,n}^{++} B_n^{++} R^2. \end{array}$$

Adding the first two equalites and substracting the last ones give:

$$\begin{array}{rcl} \varphi_n^r + \psi_n^\theta &=& 2A_n^{--} R^{-2} + S^{++} A_n^{++} R^2, \\ \varphi_n^\theta - \psi_n^r &=& 2B_n^{--} R^{-2} + S^{++} B_n^{++} R^2, \end{array}$$

with

$$S^{++} = \beta_{r,n}^{++} + \beta_{\theta,n}^{++} = \frac{-2(\gamma+1)}{n(\gamma-1)}. \tag{4.60}$$

Using (4.58), we deduce

$$\begin{array}{rcl} \varphi_n^r + \psi_n^\theta &=& S^{++}(1 + 2(S^{++})^{-1} R^{-4}) A_n^{++} R^2, \\ \varphi_n^\theta - \psi_n^r &=& S^{++}(1 + 2(S^{++})^{-1} R^{-4}) B_n^{++} R^2, \end{array}$$

so that denoting the scalar $Q_1(R) = 2(S^{++})^{-1} R^{-4}$, we can write

$$\begin{array}{rcl} A_n^{++} R^2 &=& \left(S^{++}(1 + Q_1(R))\right)^{-1} (\varphi_n^r + \psi_n^\theta), \\ B_n^{++} R^2 &=& \left(S^{++}(1 + Q_1(R))\right)^{-1} (\varphi_n^\theta - \psi_n^r). \end{array} \tag{4.61}$$

Note that, using (4.58), we get in that case

$$\begin{array}{rcl} A_n^{--} R^{-2} &=& R^{-4}(1 + Q_n(R))^{-1} (S^{++})^{-1} (\varphi_n^r + \psi_n^\theta), \\ B_n^{--} R^{-2} &=& R^{-4}(1 + Q_n(R))^{-1} (S^{++})^{-1} (\varphi_n^\theta - \psi_n^r). \end{array} \tag{4.62}$$

We therefore have proven the following lemma :

Lemma 4.3 *There exist $R_0 > 0$ and a constant C such that for any $R > R_0$ and $n = 1$, we have with $\Phi_n = (\varphi_n^r, \psi_n^r, \varphi_n^\theta, \psi_n^\theta)^\mathsf{T}$:*

$$\begin{array}{rclcrcl} |A_n^{++}R^2| & \leq & C\|\Phi_n\|_\infty, & & |B_n^{++}R^2| & \leq & C\|\Phi_n\|_\infty, \\ |A_n^{--}R^{-2}| & \leq & CR^{-4}\|\Phi_n\|_\infty, & & |B_n^{-\pm}R^{-2}| & \leq & R^{-4}\|\Phi_n\|_\infty. \end{array}$$

Note that we don't have any control on A_n^{00}. It will appear later that it is not necessary.

Now we focus on what happens on the circle $r = R$. We give the expression for mode $n = 1$ of the components of $\Lambda_R(\varphi) = \frac{R(1+\nu)}{E}\sigma(\mathbf{v}) \cdot \mathbf{n}_{|r=R}$. We use (4.36), with the explicit components computed in (4.41). The first component for mode $n = 1$ of $\Lambda_R(\varphi)$ is therefore

$$\begin{aligned} &\frac{1}{2\gamma}\left(c_1(n,r)\cos\theta + s_1(n,r)\sin\theta\right) \\ &= 2((-A_n^{--}R^{-2} + A_n^{++}R^2)\cos\theta + (B_n^{--}R^{-2} - B_n^{++}R^2)\sin\theta) \\ &= 2(S^{++})^{-1}(1 - R^{-4})(1 + Q_1(R))^{-1}\left((\varphi_n^r + \psi_n^\theta)\cos\theta + (\psi_n^r - \varphi_n^\theta)\sin\theta\right), \end{aligned}$$

where we used (4.58) and (4.61) for the last equality. An easy computation gives

$$2(S^{++})^{-1} = \frac{1-\gamma}{1+\gamma},$$

so that the first component for mode 1 of $\Lambda_R(\varphi)$ is given by

$$\frac{1-\gamma}{1+\gamma}(1 - R^{-4})(1 + Q_1(R))^{-1}\left((\varphi_n^r + \psi_n^\theta)\cos\theta + (\psi_n^r - \varphi_n^\theta)\sin\theta\right). \tag{4.63}$$

Working similarly, we get that the second component for mode 1 of $\Lambda_R(\varphi)$ which is given from is given from (4.36) and (4.41) by

$$2\left((-B_n^{--}R^{-2} + B_n^{++}R^2)\cos\theta + (-A_n^{--}R^{-2} + A_n^{++}R^2)\sin\theta\right).$$

After similar computations this is equal to

$$\frac{1-\gamma}{1+\gamma}(1 + Q_1(R))^{-1}\left((\varphi_n^\theta - \psi_n^r)\cos\theta + (\varphi_n^r + \psi_n^\theta)\sin\theta\right). \tag{4.64}$$

Using the vector $\Phi_n = (\varphi_n^r, \psi_n^r, \varphi_n^\theta, \psi_n^\theta)^\mathsf{T}$ for $n = 1$, we get that the matrix of $\Lambda_R(\varphi)$ for mode $n = 1$ in this basis is of the form $\Lambda_n + \mathcal{R}_{n,R}$ with

$$\Lambda_n = \frac{1}{1+\gamma}\begin{bmatrix} 1-\gamma & 0 & 0 & 1-\gamma \\ 0 & 1-\gamma & \gamma-1 & 0 \\ 0 & \gamma-1 & 1-\gamma & 0 \\ 1-\gamma & 0 & 0 & 1-\gamma \end{bmatrix}.$$

From (4.63) and (4.64), and the expression of matrix $Q_n(R)$, we get that for $n = 1$,

$$|\mathcal{R}_{n,R}| \leq C_\gamma R^{-4},$$

for a fixed constant C_γ. Note that this coincides with the general expression for $n \geq 2$ given in (4.57). As a direct consequence, we get the same result as in case $n \geq 2$: Using the decomposition (3.6), we can rewrite the left-hand-side of (3.5) for mode $n = 1$ in the form $(P_n + \mathcal{R}_{n,R})\Phi_n$, with P_n given by

$$P_n = \frac{-n^2}{4}\begin{bmatrix} 1-2\gamma & 0 & 0 & 0 \\ 0 & 1-2\gamma & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma} \end{bmatrix} + \mathrm{Id}_4 + \frac{1}{1+\gamma}\begin{bmatrix} n-\gamma & 0 & 0 & 1-n\gamma \\ 0 & n-\gamma & n\gamma-1 & 0 \\ 0 & n\gamma-1 & n-\gamma & 0 \\ 1-n\gamma & 0 & 0 & n-\gamma \end{bmatrix}.$$

This completes the proof of Proposition 3.2 in case $n = 1$.

4.4.2 Case $n = 0$

We follow the same strategy in this much simpler case. We first use Neumann conditions on the inner circle of radius $r = 1$:

$$\frac{1+\nu}{E} r\sigma(\mathbf{v}) \cdot \mathbf{n}|_{r=1} = 0.$$

Using (4.36), this implies that

$$c_1(r) = 0 \qquad \text{and} \qquad c_2(r) = 0.$$

Using (4.42), we directly get

$$\gamma A_0^{--} = (1-\gamma) A_0^{++}, \qquad B_0^{--} = 0. \tag{4.65}$$

This can be rewritten

$$A_0^{--} R^{-1} = \frac{1-\gamma}{\gamma} R^{-2} A_0^{++} R, \qquad B_0^{--} R^{-1} = 0. \tag{4.66}$$

Now we apply the Dirichlet condition in (3.3) on $r = R$, which tells that

$$e_0^r(R) = \varphi_0^r, \qquad e_0^\theta(R) = \varphi_0^\theta.$$

Using (4.28)–(4.29) we get

$$\varphi_0^r = A_0^{--} R^{-1} + A_0^{++} R, \qquad \varphi_0^\theta = B_0^{--} R^{-1} + B_0^{++} R.$$

Using (4.66) we therefore get

$$\varphi_0^r = \left(1 + \frac{1-\gamma}{\gamma} R^{-2}\right) A_0^{++} R, \qquad \varphi_0^\theta = B_0^{++} R,$$

and therefore

$$A_0^{++} R = \left(1 + \frac{1-\gamma}{\gamma} R^{-2}\right)^{-1} \varphi_0^r, \qquad B_0^{++} R = \varphi_0^\theta. \tag{4.67}$$

Thus we have proven the following Lemma

Lemma 4.4 *There exist $R_0 > 0$ and a constant C such that for any $R > R_0$, we have with $\Phi_0 = (\varphi_0^r, \varphi_0^\theta)^T$:*

$$\begin{array}{rclcrcl} |A_0^{++} R| & \leq & C\|\Phi_0\|_\infty, & & |B_0^{++} R| & \leq & C\|\Phi_0\|_\infty, \\ |A_0^{--} R^{-1}| & \leq & CR^{-2}\|\Phi_0\|_\infty, & & B_0^{--} R^{-2} & = & 0. \end{array}$$

Now we look at what happens on the circle $r = R$. We give the expression for mode $n = 0$ of the components of $\Lambda_R(\varphi) = \frac{R(1+\nu)}{E}\sigma(\mathbf{v}) \cdot \mathbf{n}_{|r=R}$. We use (4.36), with the explicit expressions computed in (4.42). The components for mode $n = 0$ of $\Lambda_R(\varphi)$ are (for $r = R$)

$$\begin{bmatrix} \frac{1}{2\gamma} c_1(r) \\ \frac{1}{2} c_2(r) \end{bmatrix} = \begin{bmatrix} \frac{1}{2\gamma}\left(-2\gamma A_0^{--} R^{-1} + 2(1-\gamma) A_0^{++} R\right) \\ \frac{1}{2}(-2B_0^{--} R^{-1}) \end{bmatrix}$$

Using (4.66) and (4.67), we get that for $r = R$,

$$\begin{aligned} \begin{bmatrix} \frac{1}{2\gamma} c_1(r) \\ \frac{1}{2} c_2(r) \end{bmatrix} &= \begin{bmatrix} \frac{1-\gamma}{\gamma}\left(1 - R^{-2}\right) A_0^{++} R \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1-\gamma}{\gamma}\left(1 - R^{-2}\right)(1 + \frac{1-\gamma}{\gamma} R^{-2})^{-1} \varphi_0^r \\ 0 \end{bmatrix}. \end{aligned}$$

We now introduce the vector $\Phi_0 = (\varphi_0^r, \varphi_0^\theta,)^\mathsf{T}$. With respect to this decomposition, we get that the matrix of $\Lambda_R(\varphi)$ for mode $n = 0$ is of the form $\Lambda_n + \mathcal{R}_0$ with

$$\Lambda_0 = \begin{bmatrix} \frac{1-\gamma}{\gamma} & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \|\mathcal{R}_0\|_\infty \leq CR^{-2}.$$

Using the decomposition (3.6), we can rewrite the left-hand-side of (3.5) for mode $n = 0$ in the form $(P_n + \mathcal{R}_{n,R})\Phi_n$, with P_n given by

$$P_0 = \mathrm{Id}_2 + \begin{bmatrix} \frac{1-\gamma}{\gamma} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\gamma} & 0 \\ 0 & 1 \end{bmatrix}$$

This completes the proof of Proposition 3.2 in case $n = 0$.

5 Numerical results

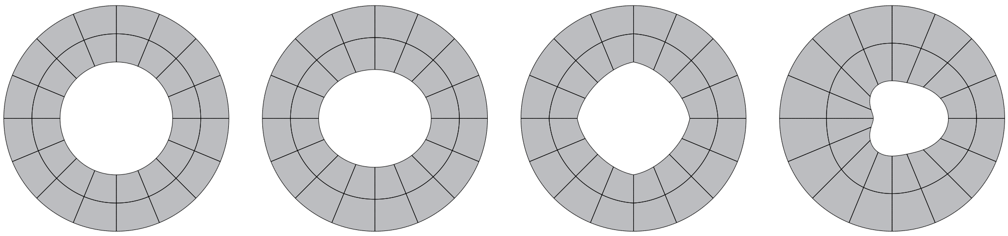

Figure 2: Disk, ellipse with eccentricity 0.5, "generic" domains ($R = 2$ and $\mathbb{Q}_6$ Meshes).

5.1 Fixed parameters – various geometries

In this section, we have set the mechanical paramaters as follows

$$\lambda = 0.5769230769, \qquad \mu = 0.3846153396.$$

We consider Problem (1.4) for various interior domains ω, see Figure 2. The exterior radius R is varying. In order to detect the forbidden values of R, we investigate the norm of the inverse of the operator L_R associated with Problem (1.4).

The finite element resolution is performed using the library Mélina [26], with isoparametric $\mathbb{Q}_6$ lagrangian elements (the mesh for the presented results is made up of 16 such elements). In Figure 3, we have plotted the norm of the inverse of the operator L_R associated with Problem (1.4) with respect to the external radius R (in logarithmic coordinates) in the case where ω is the unit disk or en ellipse close to this disk. It turns out that, as expected, no forbidden ratio is encountered for "large" values of R. A zoom near the small values of R is shown in the left plot, where several singular radii are present. The forbidden radii are close to each other for the first two cases, which is in accordance with the perturbation arguments developed in Subsection 3.4. In Figure 4, we show the dependence of the forbidden radii with respect to the eccentricity of the ellipse.

In Figure 5, we present the same results for more generic geometry: the third and fourth domains plotted on Figure 2.

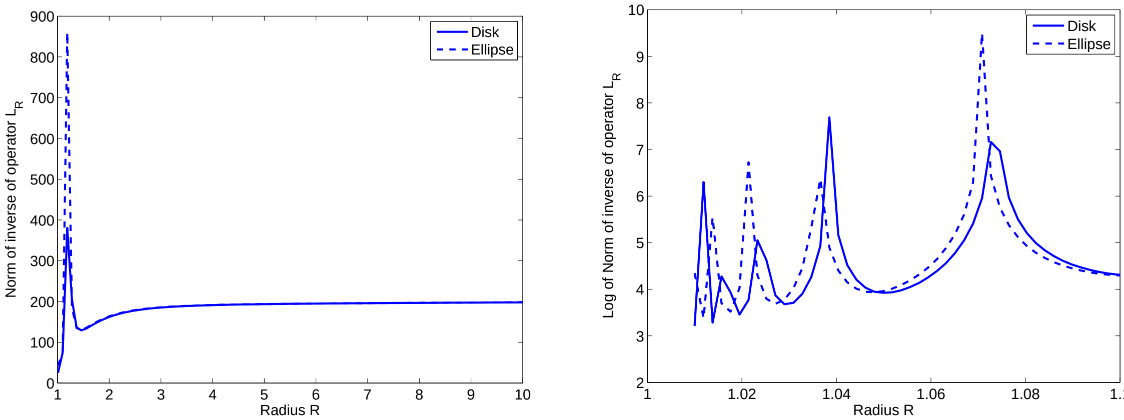

Figure 3: Norm of resolvent for the disk and for an "almost disk".

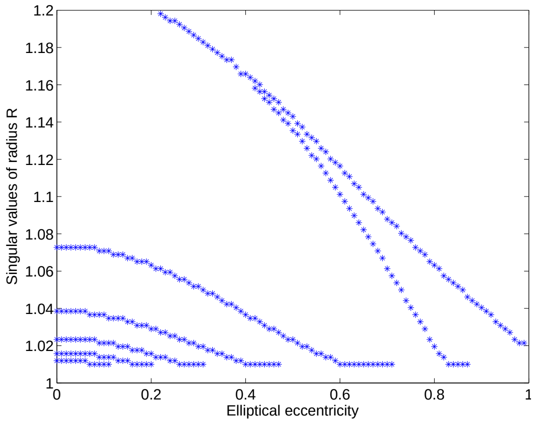

Figure 4: Dependence of singular values of R for an ellipse ω with respect to its eccentricity.

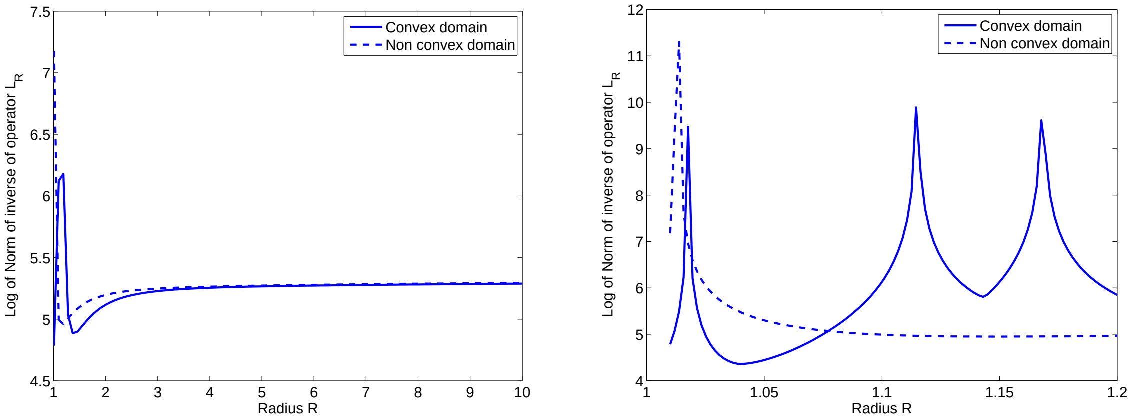

Figure 5: Norm of resolvent for the "generic" domains of Figure 2 ($\nu = 0.5$).

5.2 Fixed geometry – moving parameters

In this part, we fix the Young modulus: $E = 1$, and the Poisson ratio ν is varying. We present the results obtained for two values of ν in Figure 6. Let us mention that more forbidden radii are observed for $\nu < 0$ than for $\nu > 0$.

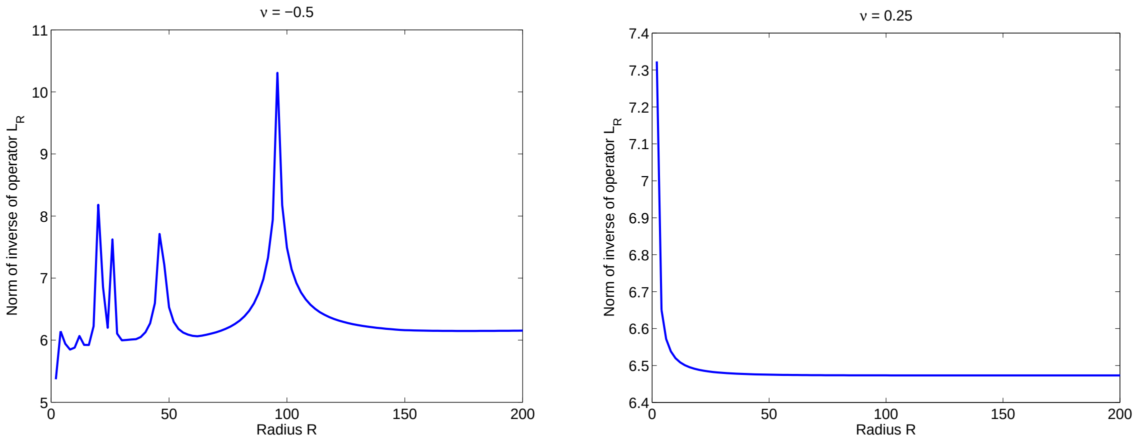

Figure 6: Norm of resolvent for a disk and different values ν.

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