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GROUND STATE ENERGY OF THE MAGNETIC LAPLACIAN ON CORNER DOMAINS

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ABSTRACT. The asymptotic behavior of the first eigenvalues of magnetic Laplacian operators with large magnetic fields and Neumann realization in smooth three-dimensional domains is characterized by model problems inside the domain or on its boundary. In two-dimensional polygonal domains, a new set of model problems on sectors has to be taken into account. In this paper, we consider the class of general corner domains. In dimension 3, they include as particular cases polyhedra and axisymmetric cones. We attach model problems not only to each point of the closure of the domain, but also to a hierarchy of "tangent substructures" associated with singular chains. We investigate spectral properties of these model problems, namely semicontinuity and existence of bounded generalized eigenfunctions. We prove estimates for the remainders of our asymptotic formula. Lower bounds are obtained with the help of an IMS type partition based on adequate two-scale coverings of the corner domain, whereas upper bounds are established by a novel construction of quasimodes, qualified as sitting or sliding according to spectral properties of local model problems. A part of our analysis extends to any dimension.

CONTENTS

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1. Introduction of the problem and main results

In this work we investigate the ground state energy of the magnetic Laplacian associated with a large magnetic field, posed on a bounded three-dimensional domain and completed by Neumann boundary conditions. This problem can be obtained by linearization from a Ginzburg-Landau equation [23]. The operator can also be viewed as a Schrödinger operator with magnetic field. The problematics of large magnetic field for the magnetic Laplacian is trivially equivalent to the semiclassical limit of the Schrödinger operator as the small parameter h tends to 0. This problem has been addressed in numerous works in various situations (smooth two- or three-dimensional domains, see *e.g.* the papers [5, 38, 29, 31, 52] and the book [24], and polygonal domains in dimension 2, see *e.g.* [33, 44, 7, 8]). Much less is known for corner three-dimensional domains, see *e.g.* [44, 50], and this is our aim to provide a unified treatment of smooth and corner domains, possibly in any space dimension n. As we will see, we have succeeded at this level of generality for $n = 2$ and 3, and have also obtained somewhat less precise results for any value of n.

The semiclassical limit of the ground state energy is provided by the infimum of *local energies* defined at each point of the closure of the domain. Local energies are ground state energies of adapted *tangent operators* at each point. The notion of tangent operator is fitting the problematics that one wants to solve. For example if one is interested in Fredholm theory for elliptic boundary value problems, tangent operators are obtained by taking the principal part of the operator frozen in each point. Another example is the semiclassical limit of the Schrödinger operator with electric field. For a rough estimate, tangent operators are then obtained by freezing the electric field at each point, and, for more information on the semiclassical limit, the Hessian at each point has to be included in the tangent operator.

In our situation, tangent operators are obtained by freezing the magnetic field at each point, that is, taking the *linear part of the magnetic potential* at each point. The domain on which the tangent operator is acting is the tangent model domain at this point. For smooth domains, this notion is obvious (the full space if the point is sitting inside the domain, and the tangent half-space if the point belongs to the boundary). For corner domains, various infinite cones have to be added to the collection of tangent domains.

Almost all known results concerning the semiclassical limit of the ground state energy rely on an *a priori* knowledge (or assumptions) on where the local energy is minimal. For instance, this is known if the domain is smooth, or if it is a polygon with openings $\leq \frac{\pi}{2}$ and constant magnetic field. In contrast, for three-dimensional polyhedra, possible configurations involving edges and corners are much more intricate, and nowadays this is impossible to know where the local energy attains its minimum. It was not even known whether the infimum is attained.

In this work, we investigate the behavior of the local energy in general 3D corner domains and we prove in particular that it attains its minimum. The properties that we show allow us to obtain an asymptotics with remainder for the ground state energy of the Schrödinger operator with magnetic field. In some situations, the remainder is optimal. We also have partial results for the natural class of n-dimensional corner domains. Let us now present our problematics and results in more detail.

1.1. **The magnetic Laplacian and its lowest eigenvalue.** The Schrödinger operator with magnetic field (also called magnetic Laplacian) in a n-dimensional space takes the form

$$(-i\nabla + \mathbf{A})^2 = \sum_{j=1}^{n}(-i\partial_{x_j} + A_j)^2,$$

where $\mathbf{A} = (A_1, \dots, A_n)$ is a given vector field and ∂_{x_j} is the partial derivatives with respect to x_j with $\mathbf{x} = (x_1, \dots, x_n)$ denoting Cartesian variables. The field $\mathbf{A}$ represents the magnetic potential. When set on a domain Ω of $\mathbb{R}^n$, this elliptic operator is completed by the magnetic Neumann boundary conditions $(-i\nabla + \mathbf{A})\psi \cdot \mathbf{n} = 0$ on $\partial\Omega$, where $\mathbf{n}$ denotes the unit normal vector to the boundary. We assume everywhere that $\mathbf{A}$ is twice differentiable on $\overline{\Omega}$

$$\mathbf{A} \in W^{2,\infty}(\Omega)^n. \tag{1.1}$$

This realization is denoted by $H(\mathbf{A}, \Omega)$. If Ω is bounded with Lipschitz boundary[1], the form domain of $H(\mathbf{A}, \Omega)$ is the standard Sobolev space $H^1(\Omega)$ and $H(\mathbf{A}, \Omega)$ is self-adjoint, non negative, and with compact resolvent. A ground state of $H(\mathbf{A}, \Omega)$ is an eigenpair (λ, ψ) associated with the lowest eigenvalue λ. If Ω is simply connected, its eigenvalues only depend on the magnetic field defined as follows, cf. [24, §1.1]. If $\omega_{\mathbf{A}}$ denotes the 1-form associated with the vector field $\mathbf{A}$

$$\omega_{\mathbf{A}} = \sum_{j=1}^{n} A_j \,\mathrm{d}x_j, \tag{1.2}$$

the corresponding 2-form $\sigma_{\mathbf{B}}$

$$\sigma_{\mathbf{B}} = \mathrm{d}\omega_{\mathbf{A}} = \sum_{j<k} B_{jk} \,\mathrm{d}x_j \wedge \mathrm{d}x_k \tag{1.3}$$

is called the magnetic field. In dimension $n = 2$ or $n = 3$, $\sigma_{\mathbf{B}}$ can be identified with

$$\mathbf{B} = \operatorname{curl} \mathbf{A}. \tag{1.4}$$

When the domain Ω is simply connected (which will be assumed everywhere unless otherwise stated), the eigenvectors corresponding to two different instances of $\mathbf{A}$ for the same $\mathbf{B}$ are deduced from each other by a *gauge transform* and the eigenvalues depend on $\mathbf{B}$ only.

Introducing a (small) parameter $h > 0$ and setting

$$H_h(\mathbf{A}, \Omega) = (-ih\nabla + \mathbf{A})^2 \quad \text{with magnetic Neumann b.c. on } \partial\Omega,$$

we get the relation

$$H_h(\mathbf{A}, \Omega) = h^2 H\Big(\frac{\mathbf{A}}{h}, \Omega\Big) \tag{1.5}$$

linking the problem with large magnetic field to the semiclassical limit $h \to 0$ for the Schrödinger operator with magnetic potential. Reminding that eigenvalues depend only on the magnetic field, we denote by $\lambda_h = \lambda_h(\mathbf{B}, \Omega)$ the smallest eigenvalue of $H_h(\mathbf{A}, \Omega)$ and by ψ_h an associated eigenvector, so that

$$\begin{cases} (-ih\nabla + \mathbf{A})^2\psi_h = \lambda_h\psi_h & \text{in } \Omega\,, \\ (-ih\nabla + \mathbf{A})\psi_h \cdot \mathbf{n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.6}$$

[1] Or more generally if Ω is a finite union of bounded Lipschitz domains, cf. [39, Chapter 1] for instance.

The behavior of $\lambda_h(\mathbf{B},\Omega)$ as $h \to 0$ clearly provide equivalent information about the lowest eigenvalue of $H(\breve{\mathbf{A}},\Omega)$ when $\breve{\mathbf{B}}$ is large, especially in the parametric case when $\breve{\mathbf{B}} = B\mathbf{B}$ where the real number B tends to $+\infty$ and $\mathbf{B}$ is a chosen reference magnetic field.

From now on, we consider that $\mathbf{B}$ is fixed. We assume that it is smooth enough and, unless otherwise mentioned, does not vanish on $\overline{\Omega}$. The question of the semiclassical behavior of $\lambda_h(\mathbf{B},\Omega)$ has been considered in many papers for a variety of domains, with constant or variable magnetic fields: Smooth domains [5, 37, 29, 22, 2, 51] and polygons [33, 44, 6, 7, 8] in dimension $n = 2$, and mainly smooth domains [38, 30, 31, 52, 24] in dimension $n = 3$. Until now, three-dimensional non-smooth domains are only addressed in two particular configurations—rectangular cuboids [44] and lenses [47, Chap. 8] and [50], with special orientation of the (constant) magnetic field. We give more detail and references about the state of the art in Section 2.

1.2. **Local ground state energies.** Let us make precise what we call local energy in the three-dimensional setting. The domains that we are considering are members of a very general class of corner domains defined by recursion over the dimension n (these definitions are set in Section 3). In the three-dimensional case, each point $\mathbf{x}$ in the closure of a corner domain Ω is associated with a dilation invariant, tangent open set $\Pi_{\mathbf{x}}$, according to the following cases:

(1) If $\mathbf{x}$ is an interior point, $\Pi_{\mathbf{x}} = \mathbb{R}^3$,
(2) If $\mathbf{x}$ belongs to a *face* $\mathbf{f}$ (*i.e.*, a connected component of the smooth part of $\partial\Omega$), $\Pi_{\mathbf{x}}$ is a half-space,
(3) If $\mathbf{x}$ belongs to an *edge* $\mathbf{e}$, $\Pi_{\mathbf{x}}$ is an infinite wedge,
(4) If $\mathbf{x}$ is a *vertex* $\mathbf{v}$, $\Pi_{\mathbf{x}}$ is an infinite cone.

Let $\mathbf{B}_{\mathbf{x}}$ be the magnetic field frozen at $\mathbf{x}$. The tangent operator at $\mathbf{x}$ is the magnetic Laplacian $H(\mathbf{A}_{\mathbf{x}},\Pi_{\mathbf{x}})$ where $\mathbf{A}_{\mathbf{x}}$ is the linear approximation of $\mathbf{A}$ at $\mathbf{x}$, so that

$$\operatorname{curl}\mathbf{A}_{\mathbf{x}} = \mathbf{B}_{\mathbf{x}}\,.$$

We define the *local energy* $E(\mathbf{B}_{\mathbf{x}},\Pi_{\mathbf{x}})$ at $\mathbf{x}$ as the ground state energy of the tangent operator $H(\mathbf{A}_{\mathbf{x}},\Pi_{\mathbf{x}})$ and we introduce the global quantity *(lowest local energy)*

$$\mathscr{E}(\mathbf{B},\Omega) := \inf_{\mathbf{x}\in\overline{\Omega}} E(\mathbf{B}_{\mathbf{x}},\Pi_{\mathbf{x}}). \tag{1.7}$$

One of our objectives is to show the existence of a minimizer for these ground state energies, reached for some tangent geometry and associated with suitable generalized eigenfunctions.

The tangent operators $H(\mathbf{A}_{\mathbf{x}},\Pi_{\mathbf{x}})$ are magnetic Laplacians set on unbounded domains and with constant magnetic field. So they have mainly an essential spectrum and, only in some cases when $\mathbf{x}$ is a vertex, discrete spectrum. This fact makes it difficult to study continuity properties of the ground energy and to construct quasimodes for the initial operator.

In the regular case, the tangent operators are magnetic Laplacians associated respectively with interior points and boundary points, acting respectively on the full space and on half-spaces. The spectrum of the operator on the full space is well-known and corresponds to Landau modes. The case of the half-spaces has also been investigated for a long time ([38, 31]): The ground state energy depends now on the angle between the (constant) magnetic field and the boundary of the half-space. It is continuous and increasing with this angle, so that the ground state is minimal for a magnetic field tangent to the boundary, and maximal for a magnetic field normal to the boundary. In all cases, it is possible to find a bounded generalized eigenfunction satisfying locally the boundary conditions.

For two dimensional domains with corners, new tangent model operators have to be considered, now acting on infinite sectors ([44, 6]). For openings $\leq \frac{\pi}{2}$, the ground state energy is an eigenvalue strictly less than in the regular case for the same value of $\mathbf{B}$. But for larger openings in 2D and conical or polyhedral singularities in 3D, it becomes harder to compare ground states energies, and for a given tangent operator, it is not clear whether there exist associated generalized eigenfunctions. Moreover, it is not clear anymore whether the infimum of the ground state energies over all tangent operators is reached.

In this work, for two or three dimensions of space, we provide positive answers to the questions of existence for a minimum in (1.7) and for related generalized eigenvectors attached to the minimum energy. First we have proved very general continuity and semicontinuity properties for the function $\mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ as described now. Let $\mathfrak{F}$ be the set of faces $\mathbf{f}$, $\mathfrak{E}$ the set of edges $\mathbf{e}$ and $\mathfrak{V}$ the set of vertices of Ω. They form a partition of the closure of Ω, called stratification

$$\overline{\Omega} = \Omega \cup \Big(\bigcup_{\mathbf{f}\in\mathfrak{F}} \mathbf{f} \Big) \cup \Big(\bigcup_{\mathbf{e}\in\mathfrak{E}} \mathbf{e} \Big) \cup \Big(\bigcup_{\mathbf{v}\in\mathfrak{V}} \mathbf{v} \Big). \tag{1.8}$$

The sets Ω, $\mathbf{f}$, $\mathbf{e}$ and $\mathbf{v}$ are open sets called the strata of $\overline{\Omega}$, compare with [40] and [42, Ch. 9]. We denote them by $\mathbf{t}$ and their set by $\mathfrak{T}$. We will show the following facts

(a) For each stratum $\mathbf{t} \in \mathfrak{T}$, the function $\mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is continuous on $\mathbf{t}$.
(b) The function $\mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is lower semicontinuous on $\overline{\Omega}$.

As a consequence, the infimum determining the limit $\mathscr{E}(\mathbf{B}, \Omega)$ in (1.7) is a minimum

$$\mathscr{E}(\mathbf{B}, \Omega) = \min_{\mathbf{x}\in\overline{\Omega}} E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})\,. \tag{1.9}$$

From this we can deduce in particular that $\mathscr{E}(\mathbf{B}, \Omega) > 0$ as soon as $\mathbf{B}$ is positive and continuous on $\overline{\Omega}$.

But we need more than properties a) and b) to show an upper bound for $\lambda_h(\mathbf{B}, \Omega)$ as $h \to 0$. We need to construct quasimodes in any case. For this we define a second level of energy attached to each point $\mathbf{x} \in \overline{\Omega}$ which we denote by $\mathscr{E}^*(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ and call *energy on tangent substructures*. This quantity has been introduced on the emblematic example of edges in [49]: If $\mathbf{x}$ belongs to an edge, then $\Pi_\mathbf{x}$ is a wedge. This wedge has two faces defining two half-spaces $\Pi^\pm_\mathbf{x}$ in a natural way: This provides, in addition with the full space $\mathbb{R}^3$, what we call the *tangent substructures* of $\Pi_\mathbf{x}$. In this situation $\mathscr{E}^*(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is defined as

$$\mathscr{E}^*(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x}) = \min\big\{E(\mathbf{B}_\mathbf{x}, \Pi^+_\mathbf{x}), E(\mathbf{B}_\mathbf{x}, \Pi^-_\mathbf{x}), E(\mathbf{B}_\mathbf{x}, \mathbb{R}^3)\big\}.$$

For a general point $\mathbf{x} \in \overline{\Omega}$, $\mathscr{E}^*(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is the infimum of local energies associated with the tangent substructures of $\Pi_\mathbf{x}$, that is all cones $\Pi_\mathbf{y}$ associated with points $\mathbf{y} \in \overline{\Pi}_\mathbf{x} \setminus \mathbf{t}_0$ where $\mathbf{t}_0$ is the stratum of $\Pi_\mathbf{x}$ containing the origin (for the example of a wedge, $\mathbf{t}_0$ is its edge). Equivalently, $\mathscr{E}^*(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ yields the infimum of $\liminf_{\mathbf{y}\to\mathbf{x}} E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{y})$ for points $\mathbf{y} \in \overline{\Omega}$ which are not in the same stratum as $\mathbf{x}$. We show that $E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x}) \leq \mathscr{E}^*(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$. This may be understood as a monotonicity property of the ground state energy for a tangent cone and its tangent substructures.

The quantity $\mathscr{E}^*(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ has a spectral interpretation: For a vertex $\mathbf{x}$ of Ω, $\mathscr{E}^*(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ is the bottom of the essential spectrum of $H(\mathbf{A}_\mathbf{x},\Pi_\mathbf{x})$ so that if $E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x}) < \mathscr{E}^*(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$, there exists an eigenfunction associated with $E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$. For $\mathbf{x}$ other than a vertex, the interpretation of $\mathscr{E}^*(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ is less standard: We show that if $E(\mathbf{B},\Pi_\mathbf{x}) < \mathscr{E}^*(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$, then there exists a bounded *generalized eigenfunction* associated with $E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$.

However, it remains possible that $E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ equals $\mathscr{E}^*(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$. This case seems at first glance to be problematic, but we provide a solution issued from the recursive properties of corner domains: We show that there always exists a tangent substructure of $\Pi_\mathbf{x}$ providing generalized eigenfunctions for the same level of energy.

1.3. **Asymptotic formulas with remainders.**

• *Case of 3D domains.* A thorough investigation of local energies $E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ and $\mathscr{E}^*(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ allows us to find asymptotic formulas with remainders for the ground state energy $\lambda_h(\mathbf{B},\Omega)$ of the magnetic Laplacian on any 3D corner domain Ω as $h\to 0$. Our remainders depend on the singularities of Ω: The convergence rate is improved in the case of *polyhedral domains* in which, in contrast with conical domains, the main curvatures at any smooth point of the boundary remain uniformly bounded. Our main results can be stated as follows (Theorems 5.1 and 9.1) as $h\to 0$

$$\big|\lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega)\big| \le \begin{cases} C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\, h^{11/10}, & \Omega \text{ corner domain,}\\ C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\, h^{5/4}, & \Omega \text{ polyhedral domain.}\end{cases} \tag{1.10}$$

Here the constant C_Ω only depends on the domain Ω (and not on $\mathbf{A}$, nor on h). Note that the lower bound in (1.10) for the polyhedral case coincides with the one obtained in the smooth case in dimensions 2 and 3 when no further assumptions are done, see the state of the art below.

Besides, if $\mathbf{B}$ cancels somewhere in $\overline{\Omega}$, the lowest local energy $\mathscr{E}(\mathbf{B},\Omega)$ is zero, and we obtain the upper bound in any 3D corner domain Ω (Theorem 9.1)

$$\lambda_h(\mathbf{B},\Omega) \le C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\, h^{4/3}, \tag{1.11}$$

which, in view of [28, 20], is optimal. Indeed, we also improve the upper bound in (1.10) recovering the power $h^{4/3}$ for general potentials that are 3 times differentiable in polyhedral domains, namely

$$\lambda_h(\mathbf{B},\Omega) \le h\mathscr{E}(\mathbf{B},\Omega) + \begin{cases} C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)}\big)\, h^{9/8}, & \Omega \text{ corner domain,}\\ C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)}\big)\, h^{4/3}, & \Omega \text{ polyhedral domain.}\end{cases} \tag{1.12}$$

Note that the $h^{4/3}$ rate was known for smooth three-dimensional domains, [31, Proposition 6.1 & Remark 6.2] and that (1.12) extends this result to polyhedral domains without loss.

Two-dimensional corner domains are curvilinear polygons. The curvature of their boundary satisfies the same property of uniform boundedness than polyhedral domains. That is why the asymptotic formulas with remainder in $h^{5/4}$ (and even $h^{4/3}$ for the upper bound) are valid.

With the point of view of large magnetic fields in the parametric case $\breve{\mathbf{B}} = B\mathbf{B}$, the identity (1.5) used with $h = B^{-1}$ provides

$$\lambda(\breve{\mathbf{B}},\Omega) = B^2\lambda_{B^{-1}}(\mathbf{B},\Omega), \tag{1.13}$$

therefore (1.10) yields obviously as $B \to \infty$

$$\left|\lambda(\breve{\mathbf{B}},\Omega) - B\mathscr{E}(\mathbf{B},\Omega)\right| \le \begin{cases} C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\, B^{9/10}, & \Omega \text{ corner domain,}\\ C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\, B^{3/4}, & \Omega \text{ polyhedral domain,}\end{cases} \tag{1.14}$$

where $\mathbf{A}$ is a potential associated with $\mathbf{B}$. Note that $B\mathscr{E}(\mathbf{B},\Omega) = \mathscr{E}(\breve{\mathbf{B}},\Omega)$ by homogeneity. In the same spirit, improved upper bounds (1.12) can be written as

$$\lambda(\breve{\mathbf{B}},\Omega) \le B\mathscr{E}(\mathbf{B},\Omega) + \begin{cases} C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)}\big)\, B^{7/8}, & \Omega \text{ corner domain,}\\ C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)}\big)\, B^{2/3}, & \Omega \text{ polyhedral domain.}\end{cases} \tag{1.15}$$

• *Estimates involving* $\mathbf{B}$ *only.* In formulas (1.10) the remainder estimates depend on the magnetic potential $\mathbf{A}$. It is possible to obtain estimates depending on the magnetic field $\mathbf{B}$ and not on the potential as soon as Ω is simply connected. For this, we consider $\mathbf{B}$ as a datum and associate a potential $\mathbf{A}$ with it. Operators $\mathscr{A} : \mathbf{B} \mapsto \mathbf{A}$ lifting the curl (*i.e.*, such that $\operatorname{curl}\circ\mathscr{A} = \mathbb{I}$) and satisfying suitable estimates do exist in the literature. We quote [16] in which it is proved that such lifting can be constructed as a pseudo-differential operator of order -1. As a consequence $\mathscr{A}$ is continuous between Hölder classes of non integer order:

$$\forall \ell \in \mathbb{N},\ \forall \alpha \in (0,1), \quad \exists K_{\ell,\alpha} > 0, \quad \|\mathscr{A}\mathbf{B}\|_{W^{\ell+1+\alpha,\infty}(\Omega)} \le K_{\ell,\alpha}\|\mathbf{B}\|_{W^{\ell+\alpha,\infty}(\Omega)}\,.$$

Choosing $\mathbf{A} = \mathscr{A}\mathbf{B}$ with $\ell = 2$ and $\alpha > 0$ in (1.10), or with $\ell = 3$ and $\alpha > 0$ in (1.12), we obtain remainder estimates depending on $\mathbf{B}$ only.

• *Generalization to n-dimensional corner domains.* We have also obtained a weaker result valid in any space dimension n, $n \ge 4$. Combining Sections 4.4 and 5.3 we can see that the quotient $\lambda_h(\mathbf{B},\Omega)/h$ converges to $\mathscr{E}(\mathbf{B},\Omega)$ as $h \to 0$ and that a general lower bound with remainder is valid, giving back

$$-\,C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\, h^{5/4} \le \lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega) \tag{1.16}$$

for a n-dimensional polyhedral domain.

• *Generalization to non simply connected domains.* If Ω is not simply connected, the first eigenvalue of the operator $H(\mathbf{A},\Omega)$ will depend on $\mathbf{A}$, and not only on $\mathbf{B}$. A manifestation of this is the Aharonov Bohm effect, see [26] for instance. Our results (1.10)–(1.11) still hold for the first eigenvalue $\lambda_h = \lambda_h(\mathbf{A},\Omega)$ of $H_h(\mathbf{A},\Omega)$. Note that, in contrast, the ground state energies of tangent operators $H(\mathbf{A}_\mathbf{x},\Pi_\mathbf{x})$ only depend on the (constant) magnetic field $\mathbf{B}_\mathbf{x}$ because the potential $\mathbf{A}_\mathbf{x}$ is linear by definition. Therefore the lowest local energy only depends on the magnetic field and can still be denoted by $\mathscr{E}(\mathbf{B},\Omega)$ even in the non simply connected case.

1.4. **Contents of the paper.** In the first part of the paper (sections 2 to 5) we introduce classes of corner domains with attached atlantes, prove some fundamental properties, and deduce a lower bound and a rough upper bound for the quotient $\lambda_h(\mathbf{B},\Omega)/h$. The second part of the paper (sections 6 to 9) relies on more specific features of the (two- and) three-dimensional model magnetic Laplacians, and is devoted to the proof of several different upper bounds. The last part of the paper (sections 10 to 12) deals with improvements and generalizations in various directions.

• *Part I.* In Section 2 we place our results in the framework of existing literature. In Section 3 we introduce the class of corner domains defined recursively on the space dimension $n \geq 1$, alongside with their tangent cones and singular chains $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \ldots)$. We particularize these notions in the case of three-dimensional domains and prove weighted estimates for local maps and their derivatives. The weights are powers of the distance to conical vertices around which one main curvature blows up. We investigate a special class of functions acting on singular chains in which will enter the local energy. In Section 4, we introduce the tangent operators for magnetic Laplacians and establish weighted estimates of the linearization error. We deduce a rough general upper bound for the quotient $\lambda_h(\mathbf{B}, \Omega)/h$ for corner domains in any dimension $n \geq 2$.

In Section 5 we prove for 3D corner domains the lower bound $h\mathscr{E}(\mathbf{B}, \Omega) - Ch^{11/10} \leq \lambda_h(\mathbf{B}, \Omega)$ by an IMS formula based on a two-scale partition of unity. In polyhedra, a one-scale standard partition can be used, which yields the improved lower bound $h\mathscr{E}(\mathbf{B}, \Omega) - Ch^{5/4} \leq \lambda_h(\mathbf{B}, \Omega)$. We can generalize these results to corner domains in any dimension n, letting appear the power $1 + 1/(3 \cdot 2^{\nu+1} - 2)$ of h with an integer $\nu \in [0, n]$ depending on Ω.

• *Part II.* In Section 6 we introduce the lowest energy on tangent substructures $\mathscr{E}^*(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ and we classify magnetic model problems on three-dimensional tangent cones (taxonomy): We characterize as much as possible their ground state energy, their lowest energy on tangent substructures, and their essential spectrum. We show in Section 7 that to each point $\mathbf{x}_0$ in $\overline{\Omega}$ is associated a tangent structure $\Pi_\mathbb{X}$ (characterized by a singular chain $\mathbb{X}$ originating at $\mathbf{x}_0$) for which the tangent operator $H(\mathbf{A}_\mathbb{X}, \Pi_\mathbb{X})$ possesses suitable bounded generalized eigenvectors (said *admissible*) with energy $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$. Section 8 is devoted to the investigation of various continuity properties of the local ground energy $E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$.

In Section 9 we prove the upper bounds

$$\lambda_h(\mathbf{B}, \Omega) \leq h\mathscr{E}(\mathbf{B}, \Omega) + Ch^\kappa, \tag{1.17}$$

with $\kappa = 11/10$ or $\kappa = 5/4$ depending on whether Ω is a corner domain or a polyhedral domain, by a construction of quasimodes based on admissible generalized eigenvectors for tangent problems. Our construction critically depends on the length ν of the singular chain $\mathbb{X}$ that provides the generalized eigenvector. When $\nu = 1$, we are in the classical situation: It suffices to concentrate the support of the quasimode around $\mathbf{x}_0$, and we qualify it as *sitting*. When $\nu = 2$, the chain has the form $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$: Our quasimode is decentered in the direction provided by $\mathbf{x}_1$, has a two-scale structure in general, and we qualify it as *sliding*. When $\nu = 3$, the chain has the form $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ and our quasimode is *doubly sliding*. In dimension $n = 3$, considering chains of length $\nu \leq 3$ is sufficient to conclude.

• *Part III.* To show the improved upper bounds (1.12), we revisit, in Section 10, admissible generalized eigenvectors by analyzing the stability of their structure under perturbation. In Section 11, we prove refined upper bounds of type (1.17) with improved rates $\kappa = 9/8$ and $\kappa = 4/3$ when Ω is a general corner domain and a polyhedral domain, respectively, but with a constant C involving now the norm $W^{3,\infty}$ of the magnetic potential instead of the norm $W^{2,\infty}$. This proof is based on the same stratification as the previous one, combined with a new classification depending on the number of directions along which the admissible generalized eigenvector is exponentially decaying. We conclude our paper in Section 12.

1.5. **Notations.** We denote by $\langle\cdot,\cdot\rangle_{\mathcal{O}}$ the L^2 Hilbert product on the open set $\mathcal{O}$ of $\mathbb{R}^n$

$$\big\langle f,g\big\rangle_{\mathcal{O}}=\int_{\mathcal{O}}f(\mathbf{x})\,\overline{g}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

When there is no confusion, we simply write $\langle f,g\rangle$ and $\|f\|=\langle f,f\rangle^{1/2}$.

For a generic (unbounded) self-adjoint operator L we denote by $\mathrm{Dom}(L)$ its domain and $\mathfrak{S}(L)$ its spectrum. Likewise the domain of a quadratic form q is denoted by $\mathrm{Dom}(q)$.

Domains as open simply connected subsets of $\mathbb{R}^n$ are in general denoted by $\mathcal{O}$ if they are generic, Π if they are invariant by dilatation (cones) and Ω if they are bounded.

The quadratic forms of interest are those associated with magnetic Laplacians, namely, for a positive constant h, a smooth magnetic potential $\mathbf{A}$, and a generic domain $\mathcal{O}$

$$q_h[\mathbf{A},\mathcal{O}](f):=\big\langle(-ih\nabla+\mathbf{A})f,(-ih\nabla+\mathbf{A})f\big\rangle_{\mathcal{O}}=\int_{\mathcal{O}}(-ih\nabla+\mathbf{A})f\cdot\overline{(-ih\nabla+\mathbf{A})f}\,\mathrm{d}\mathbf{x},\tag{1.18}$$

with its domain $\mathrm{Dom}(q_h[\mathbf{A},\mathcal{O}])=\{f\in L^2(\mathcal{O}),\ (-ih\nabla+\mathbf{A})f\in L^2(\mathcal{O})\}$. For a bounded domain Ω, $\mathrm{Dom}(q_h[\mathbf{A},\Omega])$ coincides with $H^1(\Omega)$. For $h=1$, we omit the index h, denoting the quadratic form by $q[\mathbf{A},\mathcal{O}]$. In the same way we introduce the following notation for Rayleigh quotients

$$\mathscr{Q}_h[\mathbf{A},\mathcal{O}](f)=\frac{q_h[\mathbf{A},\mathcal{O}](f)}{\langle f,f\rangle_{\mathcal{O}}},\quad f\in\mathrm{Dom}(q_h[\mathbf{A},\mathcal{O}]),\ f\neq0,\tag{1.19}$$

and recall that, by the min-max principle

$$\lambda_h(\mathbf{B},\Omega)=\min_{f\in\mathrm{Dom}(q_h[\mathbf{A},\Omega])\setminus\{0\}}\mathscr{Q}_h[\mathbf{A},\Omega](f)\,.\tag{1.20}$$

In relation with changes of variables, we will also use the more general form with metric:

$$q_h[\mathbf{A},\mathcal{O},\mathrm{G}](f)=\int_{\mathcal{O}}(-ih\nabla+\mathbf{A})f\cdot\mathrm{G}\big(\overline{(-ih\nabla+\mathbf{A})f}\big)\ |\mathrm{G}|^{-1/2}\,\mathrm{d}\mathbf{x},\tag{1.21}$$

where G is a smooth function with values in 3×3 positive symmetric matrices and $|\mathrm{G}|=\det\mathrm{G}$. Its domain is $\mathrm{Dom}(q_h[\mathbf{A},\mathcal{O},\mathrm{G}])=\{f\in L^2_{\mathrm{G}}(\mathcal{O}),\ \mathrm{G}^{1/2}(-ih\nabla+\mathbf{A})f\in L^2_{\mathrm{G}}(\mathcal{O})\}$, where $L^2_{\mathrm{G}}(\mathcal{O})$ is the space of the square-integrable functions for the weight $|\mathrm{G}|^{-1/2}$ and $\mathrm{G}^{1/2}$ is the square root of the matrix G. The corresponding Rayleigh quotient is denoted by $\mathscr{Q}_h[\mathbf{A},\mathcal{O},\mathrm{G}]$.

The domain of the magnetic Laplacian with Neumann boundary conditions on the set $\mathcal{O}$ is

$$\begin{aligned}\mathrm{Dom}\,(H_h(\mathbf{A},\mathcal{O}))=\big\{&f\in\mathrm{Dom}(q_h[\mathbf{A},\mathcal{O}]),\\&(-ih\nabla+\mathbf{A})^2f\in L^2(\mathcal{O})\ \text{ and }\ (-ih\nabla+\mathbf{A})f\cdot\mathbf{n}=0\text{ on }\partial\mathcal{O}\big\}\,.\end{aligned}\tag{1.22}$$

We will also use the space of the functions which are *locally*[2] in the domain of $H_h(\mathbf{A},\mathcal{O})$:

$$\begin{aligned}\mathrm{Dom}_{\,\mathrm{loc}}\,(H_h(\mathbf{A},\mathcal{O})):=\{&f\in H^1_{\mathrm{loc}}(\overline{\mathcal{O}}),\\&(-ih\nabla+\mathbf{A})^2f\in H^0_{\mathrm{loc}}(\overline{\mathcal{O}})\ \text{ and }\ (-ih\nabla+\mathbf{A})f\cdot\mathbf{n}=0\text{ on }\partial\mathcal{O}\}.\end{aligned}\tag{1.23}$$

When $h=1$, we omit the index h in (1.22) and (1.23).

[2]Here $H^m_{\mathrm{loc}}(\overline{\mathcal{O}})$ denotes for $m=0,1$ the space of functions which are in $H^m(\mathcal{O}\cap\mathcal{B})$ for any ball $\mathcal{B}$.

2. State of the art

Here we collect some results of the literature about the semiclassical limit for the first eigenvalue of the magnetic Laplacian depending on the geometry of the domain and the variation of the magnetic field. We briefly mention the case when the domain has no boundary, before reviewing in more detail what is known on bounded domains $\Omega \subset \mathbb{R}^n$ with Neumann boundary conditions depending on the dimension $n \in \{2,3\}$. To keep this section relatively short, we focus more on results related with our problematics, *i.e.*, the general asymptotic behavior of the ground state energy without any further assumption on the minimum local energy.

2.1. Without boundary or with Dirichlet conditions.

Here M is either a compact Riemannian manifold without boundary or $\mathbb{R}^n$, and $H_h(\mathbf{A}, M)$ is the magnetic Laplacian associated with the 1-form $\omega_{\mathbf{A}}$ defined in (1.2). In this general framework, the magnetic field $\mathbf{B}$ is the antisymmetric matrix corresponding to the 2-form $\sigma_{\mathbf{B}}$ introduced in (1.3). Then for each $\mathbf{x} \in M$ the local energy at $\mathbf{x}$ is the intensity

$$b(\mathbf{x}) := \tfrac{1}{2}\,\mathrm{Tr}([\mathbf{B}^*(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x})]^{1/2}) \tag{2.1}$$

and $\mathscr{E}(\mathbf{B}, M) = b_0 := \inf_{\mathbf{x}\in M} b(\mathbf{x})$. It is proved by Helffer and Mohamed in [28] that if b_0 is positive and under a condition at infinity if $M = \mathbb{R}^n$, then

$$-Ch^{5/4} \le \lambda_h(\mathbf{B}, M) - h\mathscr{E}(\mathbf{B}, M) \le Ch^{4/3}\,.$$

Note that more precise results can be proved in dimension 2 when b admits a unique positive non-degenerate minimum [27, 54]. Finally, the case of Dirichlet boundary conditions is very close to the case without boundary, see [28, 29] and Section 12.4.

2.2. Neumann conditions in dimension 2.

In contrast, when Neumann boundary conditions are applied on the boundary, the local energy drops significantly as was established in [55] by Saint-James and de Gennes as early as 1963. In this review of the dimension $n = 2$, we classify the domains in two categories: those with a regular boundary and those with a polygonal boundary.

2.2.1. *Regular domains.* Let $\Omega \subset \mathbb{R}^2$ be a regular domain and B be a regular non-vanishing scalar magnetic field on $\overline{\Omega}$. To each $\mathbf{x} \in \overline{\Omega}$ is associated a tangent problem. According to whether $\mathbf{x}$ is an interior point or a boundary point, the tangent problem is the magnetic Laplacian on the plane $\mathbb{R}^2$ or the half-plane $\Pi_{\mathbf{x}}$ tangent to Ω at $\mathbf{x}$, with the constant magnetic field $B_{\mathbf{x}} \equiv B(\mathbf{x})$. The associated spectral quantities $E(B_{\mathbf{x}}, \mathbb{R}^2)$ and $E(B_{\mathbf{x}}, \Pi_{\mathbf{x}})$ are respectively equal to $|B_{\mathbf{x}}|$ and $|B_{\mathbf{x}}|\Theta_0$ where $\Theta_0 := E(1, \mathbb{R}^2_+)$ is a universal constant whose value is close to 0.59 (see [55]). With the quantities

$$b = \inf_{\mathbf{x}\in\overline{\Omega}} |B(\mathbf{x})|, \quad b' = \inf_{\mathbf{x}\in\partial\Omega} |B(\mathbf{x})|, \quad \text{and} \quad \mathscr{E}(B,\Omega) = \min(b, b'\Theta_0) \tag{2.2}$$

the asymptotic limit

$$\lim_{h\to 0} \frac{\lambda_h(B,\Omega)}{h} = \mathscr{E}(B,\Omega) \tag{2.3}$$

is proved by Lu and Pan in [37]. Improvements of this result depend on the geometry and the variation of the magnetic field as we describe now.

• *Constant magnetic field.* If the magnetic field is constant and normalized to 1, then $\mathscr{E}(B,\Omega) = \Theta_0$. The following estimate is proved by Helffer and Morame:

$$-Ch^{3/2} \leq \lambda_h(1,\Omega) - h\Theta_0 \leq Ch^{3/2}\,,$$

for h small enough [29, §10], while the upper bound was already given by Bernoff and Sternberg [5]. This result is improved in [29, §11] in which a two-term asymptotics is proved, showing that a remainder in $O(h^{3/2})$ is optimal. Under the additional assumption that the curvature of the boundary admits a unique and non-degenerate maximum, a complete expansion of $\lambda_h(1,\Omega)$ is provided by Fournais and Helffer [22].

• *Variable magnetic field.* In [29, §9], several different estimates for remainders are proved, function of the place where the local energy attains its minimum: In any case

$$-Ch^{\kappa^-} \leq \lambda_h(B,\Omega) - h\mathscr{E}(B,\Omega) \leq Ch^{\kappa^+}.$$

with (a) $\kappa^- = \kappa^+ = 2$ if the minimum is attained inside the domain and (b) $\kappa^- = 5/4$, $\kappa^+ = 3/2$ if the minimum is attained on the boundary. Under non-degeneracy hypotheses, the optimality in the first case (a) is a consequence of [27], whereas the eigenvalue asymptotics provided in [51, 53] yields that the upper bound in the latter case (b) is sharp.

2.2.2. *Polygonal domains.* Let Ω be a curvilinear polygon and let $\mathfrak{V}$ be the (finite) set of its vertices. In this case, new model operators appear on infinite sectors Π_{x} tangent to Ω at vertices $\mathsf{x} \in \mathfrak{V}$. By homogeneity $E(B_{\mathsf{x}},\Pi_{\mathsf{x}}) = |B(\mathsf{x})|E(1,\Pi_{\mathsf{x}})$ and by rotation invariance, $E(1,\Pi_{\mathsf{x}})$ only depends on the opening $\alpha(\mathsf{x})$ of the sector Π_{x}. Let $\mathcal{S}_\alpha$ be a model sector of opening $\alpha \in (0,2\pi)$. Then

$$\mathscr{E}(B,\Omega) = \min\big(b, b'\Theta_0, \min_{\mathsf{x}\in\mathfrak{V}} |B(\mathsf{x})|\, E(1,\mathcal{S}_{\alpha(\mathsf{x})})\big)\,.$$

In [6, §11], it is proved that $-Ch^{5/4} \leq \lambda_h(B,\Omega) - h\mathscr{E}(B,\Omega) \leq Ch^{9/8}$. Moreover, under the assumption that a corner attracts the minimum energy

$$\mathscr{E}(B,\Omega) < \min(b, b'\Theta_0), \tag{2.4}$$

the asymptotics provided in [7] yield the sharp estimates from above and below with power $h^{3/2}$.

From [33, 6] follows that for all $\alpha \in (0, \frac{\pi}{2}]$ there holds

$$E(1,\mathcal{S}_\alpha) < \Theta_0. \tag{2.5}$$

Therefore condition (2.4) holds for constant magnetic fields as soon as there is an angle opening $\alpha_{\mathsf{x}} \leq \frac{\pi}{2}$. Finite element computations by Galerkin projection as presented in [8] suggest that (2.5) still holds for all $\alpha \in (0,\pi)$. Let us finally mention that if Ω has straight sides and B is constant, the convergence of $\lambda_h(B,\Omega)$ to $h\mathscr{E}(B,\Omega)$ is exponential.

2.3. **Neumann conditions in dimension 3.**

2.3.1. *Regular domains.* For a continuous magnetic field $\mathbf{B}$ it is known ([38] and [30]) that (2.3) holds. In that case

$$\mathscr{E}(\mathbf{B},\Omega) = \min\Big(\inf_{\mathbf{x}\in\Omega}|\mathbf{B}(\mathbf{x})|, \inf_{\mathbf{x}\in\partial\Omega}|\mathbf{B}(\mathbf{x})|\sigma(\theta(\mathbf{x}))\Big),$$

where $\theta(\mathbf{x}) \in [0, \frac{\pi}{2}]$ denotes the unoriented angle between the magnetic field and the boundary at point $\mathbf{x}$, and the quantity $\sigma(\theta)$ is the bottom of the spectrum of a model problem, cf. § 6.2.

• *Constant magnetic field.* Here the magnetic field $\mathbf{B}$ is assumed without restriction to be unitary. Then there exists a non-empty set Σ of $\partial\Omega$ on which $\mathbf{B}(\mathbf{x})$ is tangent to the boundary, which implies that $\mathscr{E}(\mathbf{B},\Omega) = \Theta_0$. Then Theorem 1.1 of [31] states that

$$|\lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega)| \le Ch^{4/3}.$$

Under some extra assumptions on Σ, Theorem 1.2 of [31] yields a two-term asymptotics for $\lambda_h(\mathbf{B},\Omega)$ showing the optimality of the previous estimate.

• *Variable magnetic field.* For a smooth non-vanishing magnetic field there holds [24, Theorem 9.1.1] (see also [38]) $|\lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega)| \le Ch^{5/4}$. In [31, Remark 6.2], the upper bound is improved to $Ch^{4/3}$. Finally, under extra assumptions, a three-term quasimode is constructed in [52], providing the sharp upper bound $Ch^{3/2}$.

2.3.2. *Singular domains.* Until now, two examples of non-smooth domains have been addressed in the literature. In both cases, the magnetic field $\mathbf{B}$ is assumed to be constant.

• *Rectangular cuboids.* This case is considered by Pan [44]: The asymptotic limit (2.3) holds for such a domain and there exists a vertex $\mathbf{v} \in \mathfrak{V}$ such that $\mathscr{E}(\mathbf{B},\Omega) = E(\mathbf{B},\Pi_{\mathbf{v}})$. Moreover, in the case where the magnetic field is tangent to a face but is not tangent to any edge, there holds

$$E(\mathbf{B},\Pi_{\mathbf{v}}) < \inf_{\mathbf{x}\in\overline{\Omega}\setminus\mathfrak{V}} E(\mathbf{B},\Pi_{\mathbf{x}}).$$

• *Lenses.* The domain Ω is supposed to have two faces separated by an edge $\mathbf{e}$ that is a regular loop contained in the plane $x_3 = 0$. The magnetic field considered is $\mathbf{B} = (0,0,1)$. It is proved in [47] that, if the opening angle α of the lens is constant and $\le 0.38\pi$,

$$\inf_{\mathbf{x}\in\mathbf{e}} E(\mathbf{B},\Pi_{\mathbf{x}}) < \inf_{\mathbf{x}\in\overline{\Omega}\setminus\mathbf{e}} E(\mathbf{B},\Pi_{\mathbf{x}})$$

and that the asymptotic limit (2.3) holds with an estimate in $Ch^{5/4}$ from above and below. When the opening angle of the lens is variable and under some non-degeneracy hypotheses, a complete eigenvalue asymptotics is obtained in [50] resulting into the optimal error estimate in $Ch^{3/2}$.

3. Domains with corners and their singular chains

For the sake of completeness and for ease of further discussion, in the same spirit as in [18, Section 2], we introduce here a recursive definition of two intertwining classes of domains

a) $\mathfrak{P}_n$, a class of infinite open cones in $\mathbb{R}^n$.
b) $\mathfrak{D}(M)$, a class of bounded connected open subsets of a smooth manifold without boundary — actually, $M = \mathbb{R}^n$ or $M = \mathbb{S}^n$, with $\mathbb{S}^n$ the unit sphere of $\mathbb{R}^{n+1}$,

3.1. **Tangent cones and corner domains.** We call a *cone* any open subset Π of $\mathbb{R}^n$ satisfying

$$\forall\rho > 0 \text{ and } \mathbf{x}\in\Pi, \quad \rho\mathbf{x}\in\Pi,$$

and the *section* of the cone Π is its subset $\Pi\cap\mathbb{S}^{n-1}$. Note that $\mathbb{S}^0 = \{-1, 1\}$.

Definition 3.1 (TANGENT CONE)**.** Let Ω be an open subset of $M = \mathbb{R}^n$ or $M = \mathbb{S}^n$. Let $\mathbf{x}_0 \in \overline{\Omega}$. The cone $\Pi_{\mathbf{x}_0}$ is said to be *tangent to* Ω at $\mathbf{x}_0$ if there exists a local $\mathscr{C}^\infty$ diffeomorphism $\mathrm{U}^{\mathbf{x}_0}$ which maps a neighborhood $\mathcal{U}_{\mathbf{x}_0}$ of $\mathbf{x}_0$ in M onto a neighborhood $\mathcal{V}_{\mathbf{x}_0}$ of $\mathbf{0}$ in $\mathbb{R}^n$ and such that

$$\mathrm{U}^{\mathbf{x}_0}(\mathbf{x}_0) = \mathbf{0}, \quad \mathrm{U}^{\mathbf{x}_0}(\mathcal{U}_{\mathbf{x}_0}\cap\Omega) = \mathcal{V}_{\mathbf{x}_0}\cap\Pi_{\mathbf{x}_0} \quad \text{and} \quad \mathrm{U}^{\mathbf{x}_0}(\mathcal{U}_{\mathbf{x}_0}\cap\partial\Omega) = \mathcal{V}_{\mathbf{x}_0}\cap\partial\Pi_{\mathbf{x}_0}. \tag{3.1}$$

We denote by $\mathrm{J}^{\mathbf{x}_0}$ the Jacobian of the inverse of $\mathrm{U}^{\mathbf{x}_0}$, that is

$$\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}) := \mathrm{d}_{\mathbf{v}}(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}), \qquad \forall\mathbf{v}\in\mathcal{V}_{\mathbf{x}_0}\,. \tag{3.2}$$

We assume without restriction that the Jacobian at $\mathbf{0}$ is the identity matrix: $\mathrm{J}^{\mathbf{x}_0}(\mathbf{0}) = \mathbb{I}_n$. The open set $\mathcal{U}_{\mathbf{x}_0}$ is called a *map-neighborhood* and $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ a *local map*.

The metric associated with the local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ is denoted by $\mathrm{G}^{\mathbf{x}_0}$ and defined as

$$\mathrm{G}^{\mathbf{x}_0} = (\mathrm{J}^{\mathbf{x}_0})^{-1}((\mathrm{J}^{\mathbf{x}_0})^{-1})^\top. \tag{3.3}$$

The metric $\mathrm{G}^{\mathbf{x}_0}$ at $\mathbf{0}$ is the identity matrix.

Note that the tangent cone $\Pi_{\mathbf{x}_0}$ does not depend on the choice of the map-neighborhood $\mathcal{U}_{\mathbf{x}_0}$ or the local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ because of the constraint $\mathrm{J}^{\mathbf{x}_0}(\mathbf{0}) = \mathbb{I}_n$. Therefore when there exists a tangent cone to Ω at $\mathbf{x}_0$, it is unique.

Definition 3.2 (CLASS OF CORNER DOMAINS)**.** The classes of corner domains $\mathfrak{D}(M)$ ($M = \mathbb{R}^n$ or $M = \mathbb{S}^n$) and tangent cones $\mathfrak{P}_n$ are defined as follow:

INITIALIZATION: $\mathfrak{P}_0$ has one element, $\{0\}$. $\mathfrak{D}(\mathbb{S}^0)$ is formed by all subsets of $\mathbb{S}^0$.

RECURRENCE: For $n \geq 1$,

(1) $\Pi\in\mathfrak{P}_n$ if and only if the section of Π belongs to $\mathfrak{D}(\mathbb{S}^{n-1})$,
(2) $\Omega\in\mathfrak{D}(M)$ if and only if for any $\mathbf{x}_0\in\overline{\Omega}$, there exists a tangent cone $\Pi_{\mathbf{x}_0}\in\mathfrak{P}_n$ to Ω at $\mathbf{x}_0$.

Polyhedral domains and polyhedral cones form important subclasses of $\mathfrak{D}(M)$ and $\mathfrak{P}_n$.

Definition 3.3 (CLASS OF POLYHEDRAL CONES AND DOMAINS)**.** The classes of polyhedral domains $\overline{\mathfrak{D}}(M)$ ($M = \mathbb{R}^n$ or $M = \mathbb{S}^n$) and polyhedral cones $\overline{\mathfrak{P}}_n$ are defined as follow:

(1) The cone $\Pi\in\mathfrak{P}_n$ is a polyhedral cone if its boundary is contained in a finite union of subspaces of codimension 1. We write $\Pi\in\overline{\mathfrak{P}}_n$.
(2) The domain $\Omega\in\mathfrak{D}(M)$ is a polyhedral domain if all its tangent cones $\Pi_{\mathbf{x}}$ are polyhedral. We write $\Omega\in\overline{\mathfrak{D}}(M)$.

Here is a rapid description of corner domains in lower dimensions.

Example 3.4. In dimensions $n = 1, 2, 3$ we have:

- The elements of $\mathfrak{P}_1$ are $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$.
- The elements of $\mathfrak{D}(\mathbb{S}^1)$ are $\mathbb{S}^1$ and all open intervals $\mathcal{I}\subset\mathbb{S}^1$ such that $\overline{\mathcal{I}}\neq\mathbb{S}^1$.

- The elements of $\mathfrak{P}_2$ are $\mathbb{R}^2$ and all sectors with opening $\alpha \in (0, 2\pi)$, including half-spaces.
- The elements of $\mathfrak{D}(\mathbb{R}^2)$ are curvilinear polygons with piecewise non-tangent smooth sides (corner angles $\alpha \neq 0, \pi, 2\pi$). Note that $\mathfrak{D}(\mathbb{R}^2)$ includes smooth domains.
- The elements of $\mathfrak{D}(\mathbb{S}^2)$ are $\mathbb{S}^2$ and all curvilinear polygons with piecewise non-tangent smooth sides in the sphere $\mathbb{S}^2$.
- The elements of $\mathfrak{P}_3$ are all cones with section in $\mathfrak{D}(\mathbb{S}^2)$. This includes $\mathbb{R}^3$, half-spaces, dihedra and many different cones like octants or axisymmetric cones.
- The elements of $\mathfrak{D}(\mathbb{R}^3)$ are tangent in each point $\mathbf{x}_0$ to a cone $\Pi_{\mathbf{x}_0} \in \mathfrak{P}_3$. Note that the nature of the section of the tangent cone determines whether the 3D domain has a vertex, an edge, or is regular near $\mathbf{x}_0$.

We will give later on § 3.5 a more exhaustive description of the class $\mathfrak{D}(\mathbb{R}^3)$ of 3D corner domains.

Remark 3.5. In dimension 2, the cones are sectors. So their sides are contained in one-dimensional subspaces, and they are "polyhedral". We deduce that

$$\mathfrak{P}_2 = \overline{\mathfrak{P}}_2 \quad \text{and} \quad \mathfrak{D}(M) = \overline{\mathfrak{D}}(M) \ \text{ for } \ M = \mathbb{R}^2 \text{ or } \mathbb{S}^2. \tag{3.4}$$

In dimension 3, a non-degenerate axisymmetric cone (*i.e.*, different from $\mathbb{R}^3$ or a half-space) is not polyhedral, whereas an octant is.

3.2. **Admissible atlantes.** We are going to introduce the notion of admissible atlas for a corner domain, so that the associated diffeomorphisms satisfy some uniformity properties. We need some definition and preliminary result first.

Notation 3.6. For $\mathbf{v} \in \mathbb{R}^n$, we denote by $\langle \mathbf{v} \rangle$ the vector space generated by $\mathbf{v}$. For $r > 0$, we denote by $\mathrm{N}_r(\mathbf{v}) := r^{-1}\mathbf{v}$ the scaling of ratio r^{-1}. Note that $\mathrm{N}_{r^{-1}} = \mathrm{N}_r^{-1}$.

The following lemma illustrates the coherence of Definition 3.1.

Lemma 3.7. *Let Ω be an open subset of M and $\mathbf{x}_0 \in \overline{\Omega}$ such that there exists a tangent cone $\Pi_{\mathbf{x}_0} \in \mathfrak{P}_n$ to Ω at $\mathbf{x}_0$ with map-neighborhood $\mathcal{U}_{\mathbf{x}_0}$. Then for all $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0} \cap \overline{\Omega}$ there exists a tangent cone $\Pi_{\mathbf{u}_0} \in \mathfrak{P}_n$ to Ω at $\mathbf{u}_0$.*

Proof. Let $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0} \cap \overline{\Omega}$. We have to prove that there exists a tangent cone $\Pi_{\mathbf{u}_0}$ at $\mathbf{u}_0$ in the sense of Definition 3.1 and that $\Pi_{\mathbf{u}_0} \in \mathfrak{P}_n$. Let $\widehat{\Omega}_{\mathbf{x}_0} = \Pi_{\mathbf{x}_0} \cap \mathbb{S}^{n-1}$ be the section of $\Pi_{\mathbf{x}_0}$. Let $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ be a local map and $\mathbf{v}_0 = \mathrm{U}^{\mathbf{x}_0}(\mathbf{u}_0) \in \overline{\Pi}_{\mathbf{x}_0}$. We denote by $(r(\mathbf{v}_0), \theta(\mathbf{v}_0)) \in (0, +\infty) \times \overline{\widehat{\Omega}}_{\mathbf{x}_0}$ its polar coordinates:

$$r(\mathbf{v}_0) := \|\mathbf{v}_0\| \qquad \text{and} \qquad \theta(\mathbf{v}_0) := \frac{\mathbf{v}_0}{\|\mathbf{v}_0\|}\,. \tag{3.5}$$

By the recursive definition there exists a tangent cone $\Pi_{\theta(\mathbf{v}_0)} \in \mathfrak{P}_{n-1}$ to $\widehat{\Omega}_{\mathbf{x}_0}$ at $\theta(\mathbf{v}_0)$. Let $\mathrm{U}^{\theta(\mathbf{v}_0)}$ be an associated diffeomorphism which sends a map-neighborhood $\mathcal{U}_{\theta(\mathbf{v}_0)}$ of $\theta(\mathbf{v}_0)$ onto a neighborhood $\mathcal{V}_{\theta(\mathbf{v}_0)}$ of $\mathbf{0} \in \mathbb{R}^{n-1}$. We may assume without restriction that there exists a n-dimensional ball with center $\theta(\mathbf{v}_0)$ and radius $\rho_1 \in (0,1)$ such that

$$\mathcal{U}_{\theta(\mathbf{v}_0)} = \mathcal{B}(\theta(\mathbf{v}_0), \rho_1) \cap \mathbb{S}^{n-1}. \tag{3.6}$$

Then we set[3] $\mathcal{U}_{(1,\theta(\mathbf{v}_0))} = \mathcal{B}(\theta(\mathbf{v}_0),\rho_1)$ and define on $\mathcal{U}_{(1,\theta(\mathbf{v}_0))}$ the diffeomorphism—using polar coordinates $(r(\mathbf{v}),\theta(\mathbf{v}))$:

$$\mathrm{U}^{(1,\theta(\mathbf{v}_0))}: \mathbf{v} \mapsto \left(r(\mathbf{v})-1, \mathrm{U}^{\theta(\mathbf{v}_0)}(\theta(\mathbf{v}))\right). \tag{3.7}$$

There holds $\mathrm{d}_{(1,\theta(\mathbf{v}_0))}\mathrm{U}^{(1,\theta(\mathbf{v}_0))} = \mathbb{I}_n$. Define

$$\Pi_{\mathbf{v}_0} := \langle \mathbf{v}_0 \rangle \times \Pi_{\theta(\mathbf{v}_0)}. \tag{3.8}$$

Notice that $\Pi_{\mathbf{v}_0} \in \mathfrak{P}_n$. It is the tangent cone to $\Pi_{\mathbf{x}_0}$ at the point $(1,\theta(\mathbf{v}_0))$ and $\mathrm{U}^{(1,\theta(\mathbf{v}_0))}$ maps $\mathcal{U}_{(1,\theta(\mathbf{v}_0))}$ on a neighborhood of $\mathbf{0} \in \mathbb{R}^n$. Let

$$\mathrm{U}^{\mathbf{v}_0} := \mathrm{N}^{-1}_{r(\mathbf{v}_0)} \circ \mathrm{U}^{(1,\theta(\mathbf{v}_0))} \circ \mathrm{N}_{r(\mathbf{v}_0)}. \tag{3.9}$$

Then $\mathrm{U}^{\mathbf{v}_0}$ is a diffeomorphism defined on

$$\mathcal{U}_{\mathbf{v}_0} := \|\mathbf{v}_0\|\, \mathcal{U}_{(1,\theta(\mathbf{v}_0))} = \mathcal{B}(\mathbf{v}_0, \rho_1\|\mathbf{v}_0\|). \tag{3.10}$$

Let us define

$$\mathcal{U}_{\mathbf{u}_0} := (\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathcal{U}_{\mathbf{v}_0}). \tag{3.11}$$

It is a neighborhood of $\mathbf{u}_0$. Let

$$\mathrm{U}^{\mathbf{u}_0}(\mathbf{u}) := \mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0)\left(\mathrm{U}^{\mathbf{v}_0} \circ \mathrm{U}^{\mathbf{x}_0}(\mathbf{u})\right) \tag{3.12}$$

be defined for $\mathbf{u} \in \mathcal{U}_{\mathbf{u}_0}$. Note that the differential of $\mathrm{U}^{\mathbf{u}_0}$ at the point $\mathbf{u}_0$ is the identity matrix $\mathbb{I}_n$. Let us set finally

$$\Pi_{\mathbf{u}_0} := \mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0)(\Pi_{\mathbf{v}_0}). \tag{3.13}$$

Then the map-neighborhood $\mathcal{U}_{\mathbf{u}_0}$, the diffeomorphism $\mathrm{U}^{\mathbf{u}_0}$ and the cone $\Pi_{\mathbf{u}_0}$ satisfy the requirements of Definition 3.1 and $\Pi_{\mathbf{u}_0}$ is the tangent cone to Ω at $\mathbf{u}_0$. Since $\Pi_{\mathbf{v}_0} \in \mathfrak{P}_n$, there holds $\Pi_{\mathbf{u}_0} \in \mathfrak{P}_n$. □

Remark 3.8. If the tangent cone $\Pi_{\mathbf{x}_0}$ is *polyhedral*, the procedure for constructing $\mathrm{U}^{\mathbf{u}_0}$ can be simplified as follows: We define $\mathbf{v}_0$ and its polar coordinates $(r(\mathbf{v}_0),\theta(\mathbf{v}_0))$ as before. Since $\Pi_{\mathbf{x}_0}$ is polyhedral, the ball $\mathcal{B}(\theta(\mathbf{v}_0),\rho_1)$ (3.6) is such that the set $\widetilde{\mathcal{U}} := \overline{\mathcal{B}(\theta(\mathbf{v}_0),\rho_1)} \cap \Pi_{\mathbf{x}_0}$ is homogeneous with respect to $\theta(\mathbf{v}_0)$, that is

$$\mathbf{v} \in \widetilde{\mathcal{U}} \;\text{ and }\; \rho \in \left[0, \frac{\rho_1}{\|\mathbf{v}-\theta(\mathbf{v}_0)\|}\right] \quad \Longrightarrow \quad \rho\mathbf{v} + (1-\rho)\theta(\mathbf{v}_0) \in \widetilde{\mathcal{U}}.$$

The set $\widetilde{\mathcal{V}} := \{\mathbf{v} \in \mathbb{R}^n |\; \mathbf{v} + \theta(\mathbf{v}_0) \in \widetilde{\mathcal{U}}\}$ defines a polyhedral cone $\widetilde{\Pi}$ in a natural way by $\{\mathbf{v} \in \mathbb{R}^n |\; \exists \rho > 0\; \rho\mathbf{v} \in \widetilde{\mathcal{V}}\}$. Defining $\mathrm{U}^{\mathbf{v}_0}$ as the translation $\mathrm{T}_{\mathbf{v}_0} : \mathbf{v} \mapsto \mathbf{v} - \mathbf{v}_0$, we find that $\widetilde{\Pi} = \Pi_{\mathbf{v}_0}$. Then, with this simple definition of $\mathrm{U}^{\mathbf{v}_0}$ we still define $\mathrm{U}^{\mathbf{u}_0}$ by (3.12). On the other hand, by uniqueness of tangent cones, the new definition of $\Pi_{\mathbf{v}_0}$ coincides with the old one (3.8). Finally, $\Pi_{\mathbf{u}_0}$ is still defined by (3.13).

Lemma 3.9. *Let $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ be a local map with image a neighborhood $\mathcal{V}_{\mathbf{x}_0}$ of $\mathbf{0}$, and such that $\mathrm{J}^{\mathbf{x}_0}(\mathbf{0}) = \mathbb{I}_n$. There exists $r_0 > 0$ such that $\mathcal{B}(\mathbf{0}, r_0) \subset \mathcal{V}_{\mathbf{x}_0}$ and for any $\mathbf{v}, \mathbf{v}' \in \mathcal{B}(\mathbf{0}, r_0)$*

$$\|\mathbf{u}' - \mathbf{u} - (\mathbf{v}' - \mathbf{v})\| \le \tfrac{1}{2}\|\mathbf{v}' - \mathbf{v}\|, \qquad \textit{with} \quad \mathbf{u} = (\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}), \quad \mathbf{u}' = (\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}'). \tag{3.14}$$

[3] We distinguish between the point $\theta(\mathbf{v}_0) \in \widehat{\Omega}_{\mathbf{x}_0}$ and its polar coordinates $(1,\theta(\mathbf{v}_0))$.

Proof. Let r_1 be such that $\mathbf{v}, \mathbf{v}' \in \mathcal{B}(\mathbf{0}, r_1) \subset \mathcal{V}_{\mathbf{x}_0}$. A Taylor expansion of $(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}')$ around $\mathbf{v}$ gives

$$\|(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}') - (\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}) - \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})(\mathbf{v}' - \mathbf{v})\| \leq \tfrac{1}{2}\,\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}' - \mathbf{v}\|^2.$$

Another Taylor expansion of $\mathrm{J}^{\mathbf{x}_0}(\mathbf{v})$ around $\mathbf{0}$ gives

$$\|\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}) - \mathrm{J}^{\mathbf{x}_0}(\mathbf{0})\| \leq \|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}\|\ .$$

Since $(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}) = \mathbf{u}$, $(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathbf{v}') = \mathbf{u}'$ and $\mathrm{J}^{\mathbf{x}_0}(\mathbf{0}) = \mathbb{I}_n$, we deduce

$$\|(\mathbf{u}' - \mathbf{u}) - (\mathbf{v}' - \mathbf{v})\| \leq \Big(\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}\| \ + \tfrac{1}{2}\,\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}' - \mathbf{v}\|\Big)\ \|\mathbf{v}' - \mathbf{v}\|.$$

If we choose $r_0 \leq \min\big\{r_1,\ 1/(4\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))})\big\}$, we have

$$\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}\| \ + \tfrac{1}{2}\,\|\mathrm{dJ}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},r_1))}\|\mathbf{v}' - \mathbf{v}\| \leq \tfrac{1}{2}, \qquad \forall \mathbf{v}, \mathbf{v}' \in \mathcal{B}(\mathbf{0}, r_0),$$

which ends the proof. □

Proposition 3.10. *(i) The domain Ω belongs to $\mathfrak{D}(\mathbb{R}^n)$ if and only if there exist a finite set $\mathfrak{X} \subset \overline{\Omega}$ and, for each $\mathbf{x}_0 \in \mathfrak{X}$, a cone $\Pi_{\mathbf{x}_0} \in \mathfrak{P}_n$ and a local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ such that (3.1) holds, with the condition that, moreover, $\cup_{\mathbf{x}_0 \in \mathfrak{X}} \mathcal{U}_{\mathbf{x}_0} \supset \overline{\Omega}$.*

(ii) The equivalence (i) still holds if one requires moreover that for all $\mathbf{x}_0 \in \mathfrak{X}$ and all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}_{\mathbf{x}_0}$, (3.14) holds.

Proof. (i) The “if” direction is a consequence of the definition of $\mathfrak{D}(\mathbb{R}^n)$ and, in particular, the fact that $\overline{\Omega}$ is compact and can be covered by a finite number of map-neighborhoods. The “only if” direction is a consequence of Lemma 3.7.

(ii) is then a consequence of Lemma 3.9 (and of the compactness of $\overline{\Omega}$, of course). □

Definition 3.11 (ADMISSIBLE ATLAS)**.** Let $\Omega \in \mathfrak{D}(M)$. An atlas $(\mathcal{U}_{\mathbf{x}}, \mathrm{U}^{\mathbf{x}})_{\mathbf{x} \in \overline{\Omega}}$ is called *admissible* if it comes from the following recursive procedure:

1. Take a finite set $\mathfrak{X} \subset \overline{\Omega}$ as in Proposition 3.10 together with the associated map-neighborhoods and diffeomorphisms $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ for $\mathbf{x}_0 \in \mathfrak{X}$, satisfying moreover (3.14).
2. We assume moreover that for each $\mathbf{x}_0 \in \mathfrak{X}$ the map-neighborhood $\mathcal{U}_{\mathbf{x}_0}$ contains a ball $\mathcal{B}(\mathbf{x}_0, 2R_{\mathbf{x}_0})$ for some $R_{\mathbf{x}_0} > 0$ and that the balls with half-radius $\mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$ cover $\overline{\Omega}$.
3. All the other map-neighborhoods and diffeomorphisms $(\mathcal{U}_{\mathbf{x}}, \mathrm{U}^{\mathbf{x}})$ with $\mathbf{x} \in \overline{\Omega} \setminus \mathfrak{X}$ are constructed by the recursive procedure (3.5)–(3.12), based on admissible atlantes for the sections $\widehat{\Omega}_{\mathbf{x}_0}$ associated with the set of reference points $\mathbf{x}_0 \in \mathfrak{X}$. In the polyhedral case, the straightforward construction described in Remark 3.8 is preferred.

As a direct consequence of Lemmas 3.7, 3.9, and Proposition 3.10, we obtain the existence of admissible atlantes.

Theorem 3.12. *Let Ω be a corner domain in $\mathfrak{D}(M)$. Then Ω admits an admissible atlas.*

For an admissible atlas, we can express the derivative of the diffeomorphism as follows: Let $\mathbf{x}_0 \in \mathfrak{X}$, $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0}$ and $\mathbf{v}_0 := \mathrm{U}^{\mathbf{x}_0}(\mathbf{u}_0)$. Differentiating (3.12), we get

$$\forall \mathbf{v} \in \mathcal{V}_{\mathbf{u}_0}, \quad \mathrm{J}^{\mathbf{u}_0}(\mathbf{v}) = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})\ \mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v}))\ (\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))^{-1}\,, \tag{3.15}$$

and (3.9) provides:

$$\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v})) = \mathrm{J}^{(1,\theta(\mathbf{v}_0))}\left(\mathrm{U}^{(1,\theta(\mathbf{v}_0))}\big(\tfrac{\mathbf{v}}{\|\mathbf{v}_0\|}\big)\right). \tag{3.16}$$

3.3. **Estimates for local Jacobian matrices.** We give in Proposition 3.13 several estimates for the Jacobians $\mathrm{J}^{\mathbf{x}_0}$ (3.2) and the metric $\mathrm{G}^{\mathbf{x}_0}$ (3.3) of all the diffeomorphisms contained in an admissible atlas of a corner domain Ω. All estimates are consequence of local bounds in L^∞ norm on the derivative of Jacobian functions. We denote for any $\mathbf{x}_0 \in \overline{\Omega}$

$$\mathrm{K}^{\mathbf{x}_0}(\mathbf{v}) = \mathrm{d}_{\mathbf{v}}\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}), \quad \mathbf{v} \in \mathcal{V}_{\mathbf{x}_0}. \tag{3.17}$$

After considering the case of reference points $\mathbf{x}_0 \in \mathfrak{X}$, we deal with points $\mathbf{u}_0 \in \overline{\Omega}$ close to a reference point $\mathbf{x}_0$ such that $\Pi_{\mathbf{x}_0} \in \overline{\mathfrak{P}}_n$: in that case the quantities $\mathrm{K}^{\mathbf{u}_0}$ for $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0}$ remain bounded uniformly in $\mathcal{U}_{\mathbf{x}_0}$. The next estimate is a global version of the first one when assuming that $\Omega \in \overline{\mathfrak{D}}(M)$. The last estimate deals with points $\mathbf{u}_0$ close to a reference point $\mathbf{x}_0$ such that the section $\widehat{\Omega}_{\mathbf{x}_0}$ of $\Pi_{\mathbf{x}_0}$ is polyhedral[4]: in that case we show that for $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0}$, the quantity $\mathrm{K}^{\mathbf{u}_0}$ is controlled by $\|\mathbf{u}_0 - \mathbf{x}_0\|^{-1}$. These estimates will be useful when using change of variables on quadratic form defined on corner domains in dimension 3. An important feature of these estimates is a recursive control of their domain of validity: In each case we exhibit such domains as balls with explicit centers and implicit radii. The principle is to start from the finite number of reference points $\mathbf{x}_0 \in \mathfrak{X}$ provided by an admissible atlas and proceed with points $\mathbf{u}_0$ which are not in this set using Lemma 3.7 and Remark 3.8. The outcome is that estimates are valid in a ball around $\mathbf{u}_0$ with radius $\rho(\mathbf{u}_0)$ proportional to the distance $\operatorname{dist}(\mathbf{u}_0, \mathfrak{X})$ of $\mathbf{u}_0$ to the set of reference points, the proportion ratio $\rho(\widehat{\mathbf{u}}_1)$ being a similar radius associated with the section $\widehat{\Omega}_{\mathbf{x}_0} \in \mathfrak{D}(\mathbb{S}^{n-1})$.

Proposition 3.13. *Let $\Omega \in \mathfrak{D}(M)$ and $(\mathcal{U}_{\mathbf{x}}, \mathrm{U}^{\mathbf{x}})_{\mathbf{x}\in\overline{\Omega}}$ be an admissible atlas with set of reference points $\mathfrak{X} \subset \overline{\Omega}$. Then we have the following assertions:*

(a) Let $\mathbf{x}_0 \in \mathfrak{X}$. With $R_{\mathbf{x}_0}$ introduced in Definition 3.11, there exists $c(\mathbf{x}_0)$ such that

$$\begin{aligned}&\|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},R_{\mathbf{x}_0}))} \le c(\mathbf{x}_0),\\ &\|\mathrm{J}^{\mathbf{x}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{x}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le r\,c(\mathbf{x}_0) \quad \textit{for all } r \le R_{\mathbf{x}_0}.\end{aligned} \tag{3.18}$$

(b) Let $\mathbf{x}_0 \in \mathfrak{X}$ such that $\Pi_{\mathbf{x}_0} \in \overline{\mathfrak{P}}_n$. Then there exists a constant $c(\mathbf{x}_0)$ such that for all $\mathbf{u}_0 \in \overline{\Omega} \cap \mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$, $\mathbf{u}_0 \neq \mathbf{x}_0$, there holds, denoting $\widehat{\mathbf{u}}_1 := \mathrm{U}^{\mathbf{x}_0}\mathbf{u}_0/\|\mathrm{U}^{\mathbf{x}_0}\mathbf{u}_0\| \in \widehat{\Omega}_{\mathbf{x}_0}$

$$\begin{aligned}&\|\mathrm{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\rho(\mathbf{u}_0)))} \le c(\mathbf{x}_0) \quad \textit{with} \quad \rho(\mathbf{u}_0) = \tfrac{1}{3}\rho(\widehat{\mathbf{u}}_1)\,\|\mathbf{u}_0 - \mathbf{x}_0\|,\\ &\|\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le r\,c(\mathbf{x}_0) \quad \textit{for all } r \le \rho(\mathbf{u}_0).\end{aligned} \tag{3.19}$$

(c) Let $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^n)$, then there exists $c(\Omega)$ such that for all $\mathbf{u}_0 \in \overline{\Omega}$, there holds, with $\widehat{\mathbf{u}}_1$ as above,

$$\begin{aligned}&\|\mathrm{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\rho(\mathbf{u}_0)))} \le c(\Omega) \quad \textit{with} \quad \rho(\mathbf{u}_0) = \tfrac{1}{3}\rho(\widehat{\mathbf{u}}_1)\operatorname{dist}(\mathbf{u}_0, \mathfrak{X}),\\ &\|\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le r\,c(\Omega) \quad \textit{for all } r \le \rho(\mathbf{u}_0).\end{aligned} \tag{3.20}$$

[4] But this does not imply that the tangent cone $\Pi_{\mathbf{x}_0}$ is polyhedral.

(d) Let $\mathbf{x}_0 \in \mathfrak{X}$ *be such that the section* $\widehat{\Omega}_{\mathbf{x}_0} = \Pi_{\mathbf{x}_0} \cap \mathbb{S}^{n-1}$ *belongs to* $\overline{\mathfrak{D}}(\mathbb{S}^{n-1})$*. Then there exists* $c(\mathbf{x}_0)$ *such that for all* $\mathbf{u}_0 \in \overline{\Omega} \cap \mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$*,* $\mathbf{u}_0 \neq \mathbf{x}_0$*, there holds,*

$$
\begin{aligned}
&\|\mathrm{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\rho(\mathbf{u}_0)))} \le \frac{1}{\|\mathbf{u}_0 - \mathbf{x}_0\|}\, c(\mathbf{x}_0) \quad \textit{with} \quad \rho(\mathbf{u}_0) = \tfrac{1}{3}\rho(\widehat{\mathbf{u}}_1)\,\|\mathbf{u}_0 - \mathbf{x}_0\|,\\
&\|\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n\,\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n\,\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le \frac{r}{\|\mathbf{u}_0 - \mathbf{x}_0\|}\, c(\mathbf{x}_0) \quad \textit{for all } r \le \rho(\mathbf{u}_0)\,.
\end{aligned}
\tag{3.21}
$$

Proof. *(a)* The estimate for $\mathrm{K}^{\mathbf{x}_0}$ in (3.18) comes from the definition of a map-neighborhood. The bound in (3.18) on $\mathrm{J}^{\mathbf{x}_0} - \mathbb{I}_n$ follows immediately because of the Taylor estimate

$$
\|\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}) - \mathbb{I}_n\,\| \le \|\mathbf{v}\|\,\|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\|\mathbf{v}\|))}, \quad \mathbf{v} \in \mathcal{V}_{\mathbf{x}_0}\,. \tag{3.22}
$$

Concerning the bound (3.18) on $\mathrm{G}^{\mathbf{x}_0} - \mathbb{I}_n$, we rely on the Taylor estimate

$$
\|\mathrm{G}^{\mathbf{x}_0}(\mathbf{v}) - \mathbb{I}_n\,\| \le \|\mathbf{v}\|\,\|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\|\mathbf{v}\|))}\,\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\|^3_{L^\infty(\mathcal{B}(\mathbf{0},\|\mathbf{v}\|))}\,. \tag{3.23}
$$

(b) Since $\Pi_{\mathbf{x}_0}$ is polyhedral, we can take advantage of Remark 3.8: For $\mathbf{u}_0$ in the ball $\mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$, the local map $(\mathcal{U}_{\mathbf{u}_0}, \mathrm{U}^{\mathbf{u}_0})$ is defined by (3.10)–(3.12) where, for some $\rho_1 < 1$,

$$
\mathbf{v}_0 = \mathrm{U}^{\mathbf{x}_0}(\mathbf{u}_0), \quad \mathcal{U}_{\mathbf{v}_0} = \mathcal{B}(\mathbf{v}_0, \rho_1\|\mathbf{v}_0\|), \quad \text{and} \quad \mathrm{U}^{\mathbf{v}_0}(\mathbf{v}) = \mathbf{v} - \mathbf{v}_0\,.
$$

Note that the radius ρ_1 is the radius $\rho(\widehat{\mathbf{u}}_1)$ of a map neighborhood of $\widehat{\mathbf{u}}_1 := \mathbf{v}_0/\|\mathbf{v}_0\|$, which plays the same role as $\rho(\mathbf{u}_0)$ in one dimension less.

We recall that our admissible atlas satisfies Condition (1) of Definition 3.11. Applying (3.14) with the couples $\{(\mathbf{u}, \mathbf{u}_0), (\mathbf{v}, \mathbf{v}_0)\}$ and $\{(\mathbf{u}_0, \mathbf{x}_0), (\mathbf{v}_0, \mathbf{0})\}$, we deduce that $\mathcal{U}_{\mathbf{u}_0}$ contains the ball $\mathcal{B}(\mathbf{u}_0, \frac{1}{3}\rho_1\|\mathbf{u}_0 - \mathbf{x}_0\|)$. On the other hand, in this case (3.15) reduces to

$$
\forall \mathbf{v} \in \mathcal{V}_{\mathbf{u}_0}, \quad \mathrm{J}^{\mathbf{u}_0}(\mathbf{v}) = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})\,(\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))^{-1}. \tag{3.24}
$$

Thus, we deduce from the above formula that

$$
\|\mathrm{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{V}_{\mathbf{u}_0})} \le \|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{V}_{\mathbf{x}_0})}\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\|_{L^\infty(\mathcal{U}_{\mathbf{x}_0})}\,. \tag{3.25}
$$

All of this proves estimate for $\mathrm{K}^{\mathbf{u}_0}$ in (3.19).

The bound in (3.19) on $\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n$ follows immediately because of the Taylor estimate (3.22) where $\mathbf{x}_0$ is replaced by $\mathbf{u}_0$. Concerning the bound on $\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n$, we start from the Taylor estimate (3.23) where we replace $\mathbf{x}_0$ by $\mathbf{u}_0$. It remains to bound $\|(\mathrm{J}^{\mathbf{u}_0})^{-1}\|$. We note that we have, thanks to (3.24)

$$
\mathrm{J}^{\mathbf{u}_0}(\mathbf{v})^{-1} = (\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))\,(\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}))^{-1}\,.
$$

Whence the bound (3.19) on $\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n$.

(c) Applying Proposition 3.10 to $\Omega \in \overline{\mathfrak{D}}(M)$, we deduce from (3.25):

$$
\sup_{x\in\overline{\Omega}} \|\mathrm{K}^{\mathbf{x}}\|_{L^\infty(\mathcal{V}_{\mathbf{x}})} \le \max_{\mathbf{x}_0\in\mathfrak{X}} \big(\|\mathrm{K}^{\mathbf{x}_0}\|_{L^\infty(\mathcal{V}_{\mathbf{x}})}\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\|_{L^\infty(\mathcal{U}_{\mathbf{x}})}\big) < +\infty. \tag{3.26}
$$

(d) Differentiating (3.15) with respect to $\mathbf{v}$ yields

$$
\mathrm{K}^{\mathbf{u}_0}(\mathbf{v}) = \mathrm{K}^{\mathbf{x}_0}(\mathbf{v})\;\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v}))\;(\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))^{-1} + \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})\;\mathrm{d}_{\mathbf{v}}\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v}))\;\;(\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))^{-1}. \tag{3.27}
$$

Using in turn (3.16) we calculate

$$\begin{aligned}
\mathrm{d}_{\mathbf{v}}\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v})) &= \mathrm{d}_{\mathbf{v}}\Big\{\mathrm{J}^{(1,\theta(\mathbf{v}_0))}\Big(\mathrm{U}^{(1,\theta(\mathbf{v}_0))}(\tfrac{\mathbf{v}}{\|\mathbf{v}_0\|})\Big)\Big\} \\
&= \tfrac{1}{\|\mathbf{v}_0\|}\,\mathrm{K}^{(1,\theta(\mathbf{v}_0))}\Big(\mathrm{U}^{(1,\theta(\mathbf{v}_0))}(\tfrac{\mathbf{v}}{\|\mathbf{v}_0\|})\Big)\,\Big(\mathrm{J}^{(1,\theta(\mathbf{v}_0))}\Big(\mathrm{U}^{(1,\theta(\mathbf{v}_0)}(\tfrac{\mathbf{v}}{\|\mathbf{v}_0\|})\Big)\Big)^{-1}. \qquad (3.28)
\end{aligned}$$

Recall that $\mathrm{U}^{(1,\theta)}$ is deduced from U^{θ} by formula (3.7) on the domain $\mathcal{U}_{(1,\theta(\mathbf{v}_0))} = \mathcal{B}(\theta(\mathbf{v}_0),\rho_0)$, cf. (3.6). Therefore there exists a constant $c(\rho_0)\geq 1$ such that

$$\|\mathrm{J}^{(1,\theta)}\|_{L^\infty(\mathcal{V}_{(1,\theta)})} \leq c(\rho_0)\|\mathrm{J}^{\theta}\|_{L^\infty(\mathcal{V}_\theta)} \quad\text{and}\quad \|\mathrm{K}^{(1,\theta)}\|_{L^\infty(\mathcal{V}_{(1,\theta)})} \leq c(\rho_0)\|\mathrm{K}^{\theta}\|_{L^\infty(\mathcal{V}_\theta)}\,.$$

We deduce

$$\|\mathrm{K}^{\mathbf{u}_0}\| \leq c'(\rho_0)\left(\|\mathrm{K}^{\mathbf{x}_0}\|\,\|\mathrm{J}^{\theta(\mathbf{v}_0)}\|\,\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\| + \frac{\|(\mathrm{J}^{\theta(\mathbf{v}_0)})^{-1}\|}{\|\mathbf{v}_0\|}\,\|\mathrm{J}^{\mathbf{x}_0}\|\,\|\mathrm{K}^{\theta(\mathbf{v}_0)}\|\,\|(\mathrm{J}^{\mathbf{x}_0})^{-1}\|\right) \qquad (3.29)$$

where we have omitted the mention of the L^∞ norms. Since the section $\widehat{\Omega}_{\mathbf{x}_0}$ belongs to $\overline{\mathfrak{D}}(\mathbb{S}^{n-1})$, we deduce from *(c)* and (3.26) applied to the section $\widehat{\Omega}_{\mathbf{x}_0}$ that

$$\sup_{\theta\in\overline{\widehat{\Omega}}_{\mathbf{x}_0}} \|\mathrm{J}^{\theta}\|_{L^\infty(\mathcal{V}_\theta)} < +\infty \quad\text{and}\quad \sup_{\theta\in\overline{\widehat{\Omega}}_{\mathbf{x}_0}} \|\mathrm{K}^{\theta}\|_{L^\infty(\mathcal{V}_\theta)} < +\infty\,.$$

Therefore the r.h.s. of (3.29) is controlled by $c(\mathbf{x}_0)/\|\mathbf{v}_0\|$. Using (3.14) we obtain that $\|\mathbf{v}_0\|\simeq\|\mathbf{u}_0-\mathbf{x}_0\|$, whence the bound (3.21) on $\mathrm{K}^{\mathbf{u}_0}$. The bound (3.21) for $\mathrm{J}^{\mathbf{u}_0}-\mathbb{I}_n$ follows immediately as in point *(a)*. Finally, to prove the bound on $\mathrm{G}^{\mathbf{u}_0}-\mathbb{I}_n$, we combine the Taylor estimate (3.23) (at $\mathbf{u}_0$) with the estimate of $\mathrm{K}^{\mathbf{u}_0}$ in (3.21) and the formula for $(\mathrm{J}^{\mathbf{u}_0})^{-1}$

$$(\mathrm{J}^{\mathbf{u}_0}(\mathbf{v}))^{-1} = (\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0))\,(\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v})))^{-1}\,(\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}))^{-1}\,,$$

deduced from (3.15). It remains to use (3.16) to bound $(\mathrm{J}^{\mathbf{v}_0}(\mathrm{U}^{\mathbf{v}_0}(\mathbf{v})))^{-1}$, which ends the proof. □

Remark 3.14. In dimension $n=2$, domains $\Omega\in\mathfrak{D}(\mathbb{R}^2)$ are always in case *(b)* or *(c)* of Proposition 3.13 since $\mathfrak{D}(\mathbb{R}^2)=\overline{\mathfrak{D}}(\mathbb{R}^2)$, cf. (3.4). In dimension $n=3$, Proposition 3.13 still covers all possibilities: Indeed, since $\mathfrak{D}(\mathbb{S}^2)=\overline{\mathfrak{D}}(\mathbb{S}^2)$, one is at least in case *(d)*. In higher dimensions $n\geq 4$, Proposition 3.13 does not provide estimates for all possible singular points. General estimates would involve distance to non-discrete sets of points, see (3.36) later on. However Proposition 3.13 is sufficient for the core of our investigation, which, for independent reasons, is limited to dimension $n\leq 3$.

Remark 3.15. We can use the computation of $\mathrm{K}^{\mathbf{u}_0}$ in the proof of Proposition 3.13 to obtain estimates for its differentials $\mathrm{d}^\ell\mathrm{K}^{\mathbf{u}_0}$, $\ell=1,2,\dots$ Note that in (3.29), the worst term is $1/\|\mathbf{v}_0\|$. By differentiating ℓ times (3.27), we obtain an upper bound in $1/\|\mathbf{v}_0\|^{\ell+1}$. Thus we have the following improvements in Proposition 3.13:

(1) In cases *(a)*, *(b)* and *(c)*, the estimates for $\mathrm{K}^{\mathbf{x}_0}$ and $\mathrm{K}^{\mathbf{u}_0}$ are still valid for their differentials $\mathrm{d}^\ell\mathrm{K}^{\mathbf{x}_0}$ and $\mathrm{d}^\ell\mathrm{K}^{\mathbf{u}_0}$, respectively.

(2) Let $\mathbf{x}_0\in\mathfrak{X}$ such that $\widehat{\Omega}_{\mathbf{x}_0}=\Pi_{\mathbf{x}_0}\cap\mathbb{S}^{n-1}$ belongs to $\overline{\mathfrak{D}}(\mathbb{S}^{n-1})$. Then there exists $c(\mathbf{x}_0)$ such that for all $\mathbf{u}_0\in\overline{\Omega}\cap\mathcal{B}(\mathbf{x}_0,R_{\mathbf{x}_0})$, $\mathbf{u}_0\neq\mathbf{x}_0$, there holds, with $\widehat{\mathbf{u}}_1:=\mathrm{U}^{\mathbf{x}_0}\mathbf{u}_0/\|\mathrm{U}^{\mathbf{x}_0}\mathbf{u}_0\|$

$$\|\mathrm{d}^\ell\mathrm{K}^{\mathbf{u}_0}\|_{L^\infty(\mathcal{B}(\mathbf{0},\rho(\mathbf{u}_0)))} \leq \frac{1}{\|\mathbf{u}_0-\mathbf{x}_0\|^{\ell+1}}\,c(\mathbf{x}_0) \quad\text{with}\quad \rho(\mathbf{u}_0)=\tfrac13\rho(\widehat{\mathbf{u}}_1)\,\|\mathbf{u}_0-\mathbf{x}_0\|. \qquad (3.30)$$

3.4. **Strata and singular chains.** In this section, we exhibit a canonical structure of tangent cones and corner domains.

Definition 3.16. Let $\mathfrak{O}_n$ denote the group of orthogonal linear transformations of $\mathbb{R}^n$.

a) We say that a cone Π is *equivalent* to another cone Π' and denote $\Pi \equiv \Pi'$ if there exists $\underline{\mathrm{U}} \in \mathfrak{O}_n$ such that $\underline{\mathrm{U}}\Pi = \Pi'$.
b) Let $\Pi \in \mathfrak{P}_n$. If Π is equivalent to $\mathbb{R}^{n-d} \times \Gamma$ with $\Gamma \in \mathfrak{P}_d$ and d is minimal for such an equivalence, Γ is said to be *a minimal reduced cone* associated with Π and we denote by $d(\Pi) := d$ the *reduced dimension* of the cone Π.
c) Let $\mathbf{x} \in \overline{\Omega}$ and let $\Pi_{\mathbf{x}}$ be its tangent cone. We denote by $d_0(\mathbf{x})$ the dimension of the minimal reduced cone associated with $\Pi_{\mathbf{x}}$.

Remark 3.17. If there exists a linear isomorphism between Π and Π' then $d(\Pi) = d(\Pi')$.

3.4.1. *Recursive definition of the singular chains.* A singular chain $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_p) \in \mathfrak{C}(\Omega)$ (with p a non negative integer) is a finite collection of points according to the following recursive definition.

Initialization: $\mathbf{x}_0 \in \overline{\Omega}$,

- Let $C_{\mathbf{x}_0}$ be the tangent cone to Ω at $\mathbf{x}_0$ (here $C_{\mathbf{x}_0} = \Pi_{\mathbf{x}_0}$).
- Let $\Gamma_{\mathbf{x}_0} \in \mathfrak{P}_{d_0}$ be its minimal reduced cone: $C_{\mathbf{x}_0} = \underline{\mathrm{U}}^0(\mathbb{R}^{n-d_0} \times \Gamma_{\mathbf{x}_0})$.
- Alternative:
  - If $p = 0$, stop here.
  - If $p > 0$, then[5] $d_0 > 0$ and let $\Omega_{\mathbf{x}_0} \in \mathfrak{D}(\mathbb{S}^{d_0-1})$ be the section of $\Gamma_{\mathbf{x}_0}$

Recurrence: $\mathbf{x}_j \in \overline{\Omega}_{\mathbf{x}_0,\ldots,\mathbf{x}_{j-1}} \in \mathfrak{D}(\mathbb{S}^{d_{j-1}-1})$. If $d_{j-1} = 1$, stop here ($p = j$). If not:

- Let $C_{\mathbf{x}_0,\ldots,\mathbf{x}_j}$ be the tangent cone to $\Omega_{\mathbf{x}_0,\ldots,\mathbf{x}_{j-1}}$ at $\mathbf{x}_j$,
- Let $\Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_j} \in \mathfrak{P}_{d_j}$ be its minimal reduced cone: $C_{\mathbf{x}_0,\ldots,\mathbf{x}_j} = \underline{\mathrm{U}}^j(\mathbb{R}^{d_{j-1}-1-d_j} \times \Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_j})$.
- Alternative:
  - If $p = j$, stop here.
  - If $p > j$, then $d_j > 0$ and let $\Omega_{\mathbf{x}_0,\ldots,\mathbf{x}_j} \in \mathfrak{D}(\mathbb{S}^{d_j-1})$ be the section of $\Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_j}$.

Note that $n \geq d_0 > d_1 > \ldots > d_p$. Hence $p \leq n$. Note also that for $p = 0$, we obtain the trivial one element chain $(\mathbf{x}_0)$ for any $\mathbf{x}_0 \in \overline{\Omega}$.

Definition 3.18. For any $\mathbf{x} \in \overline{\Omega}$, we denote by $\mathfrak{C}_{\mathbf{x}}(\Omega)$ the subset of chains $\mathbb{X} \in \mathfrak{C}(\Omega)$ originating at $\mathbf{x}$, *i.e.*, the set of chains $\mathbb{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_p)$ with $\mathbf{x}_0 = \mathbf{x}$. Note that the one element chain $(\mathbf{x})$ belongs to $\mathfrak{C}_{\mathbf{x}}(\Omega)$. We also set

$$\mathfrak{C}^*_{\mathbf{x}}(\Omega) = \{\mathbb{X} \in \mathfrak{C}_{\mathbf{x}}(\Omega),\ p > 0\} = \mathfrak{C}_{\mathbf{x}}(\Omega) \setminus \{(\mathbf{x})\}. \tag{3.31}$$

[5] If $d_0 = 0$, we have necessarily $p = 0$.

We set finally, with the notation $\langle \mathbf{y} \rangle$ for the vector space generated by $\mathbf{y}$,
(3.32)
$$\Pi_{\mathbb{X}} = \begin{cases} C_{\mathbf{x}_0} = \Pi_{\mathbf{x}_0} & \text{if } p = 0, \\ \underline{\mathrm{U}}^0\big(\mathbb{R}^{n-d_0} \times \langle \mathbf{x}_1 \rangle \times C_{\mathbf{x}_0,\mathbf{x}_1}\big) & \text{if } p = 1, \\ \underline{\mathrm{U}}^0\Big(\mathbb{R}^{n-d_0} \times \langle \mathbf{x}_1 \rangle \times \ldots \times \underline{\mathrm{U}}^{p-1}\big(\mathbb{R}^{d_{p-2}-1-d_{p-1}} \times \langle \mathbf{x}_p \rangle \times C_{\mathbf{x}_0,\ldots,\mathbf{x}_p}\big) \ldots\Big) & \text{if } p \geq 2. \end{cases}$$

Note that if $d_p = 0$, the cone $C_{\mathbf{x}_0,\ldots,\mathbf{x}_p}$ coincides with $\mathbb{R}^{d_{p-1}-1}$, leading to $\Pi_{\mathbb{X}} = \mathbb{R}^n$.

Definition 3.19. Let $\mathbb{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_p)$ be a chain in $\mathfrak{C}(\Omega)$.

(i) The cone $\Pi_{\mathbb{X}}$ defined in (3.32) is called a *tangent structure* [of Ω] at $\mathbf{x}_0$, and if $\mathbb{X} \neq (\mathbf{x}_0)$, $\Pi_{\mathbb{X}}$ is called a *tangent substructure* of $\Pi_{\mathbf{x}_0}$.

(ii) Let $\mathbb{X}' = (\mathbf{x}'_0, \ldots, \mathbf{x}'_{p'})$ be another chain in $\mathfrak{C}(\Omega)$. We say that $\mathbb{X}'$ is equivalent to $\mathbb{X}$ if $\mathbf{x}'_0 = \mathbf{x}_0$ and $\Pi_{\mathbb{X}'} = \Pi_{\mathbb{X}}$.

This notion of equivalence is well suited to the class of operators that we consider in this paper.

3.4.2. *Strata of a corner domain.* For $d \in \{0, \ldots, n\}$, let

$$\mathfrak{A}_d(\Omega) = \{\mathbf{x} \in \overline{\Omega}, \quad d_0(\mathbf{x}) = d\}. \tag{3.33}$$

The strata of $\overline{\Omega}$ are the connected components of $\mathfrak{A}_d(\Omega)$, for $d \in \{0, \ldots, n\}$. They are denoted by $\mathbf{t}$ and their set by $\mathfrak{T}$.

Examples:

- $\mathfrak{A}_0(\Omega)$ coincides with Ω.
- $\mathfrak{A}_1(\Omega)$ is the subset of $\partial\Omega$ of the regular points of the boundary (the corresponding strata being the faces in dimension $n = 3$ and the sides in dimension $n = 2$).
- If $n = 2$, $\mathfrak{A}_2(\Omega)$ is the set of corners.
- If $n = 3$, $\mathfrak{A}_2(\Omega)$ is the set of edge points.
- If $n = 3$, $\mathfrak{A}_3(\Omega)$ is the set of corners.

Proposition 3.20. *Let $\mathbf{t} \in \mathfrak{A}_d(\Omega)$ be a stratum. Then $\mathbf{t}$ is a smooth submanifold*[6] *of codimension d. In particular $\mathfrak{A}_n(\Omega)$ is a finite subset of $\partial\Omega$.*

Proof. Let $\mathbf{x}_0 \in \mathbf{t}$ and $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ be an associated local map. The tangent cone at $\mathbf{x}_0$ writes $\Pi_{\mathbf{x}_0} = \underline{\mathrm{U}}\left(\mathbb{R}^{n-d} \times \Gamma_{\mathbf{x}_0}\right)$, with $\Gamma_{\mathbf{x}_0} \in \mathfrak{P}_d$. For simplicity, we may assume that $\underline{\mathrm{U}} = \mathbb{I}_n$. Denote by π the orthogonal projection on $\mathbb{R}^{n-d}$ and set $\pi^\perp := \mathbb{I}_n - \pi$. Let $\mathbf{u} \in \mathcal{U}_{\mathbf{x}_0}$ and $\mathbf{v} = \mathrm{U}^{\mathbf{x}_0}(\mathbf{u})$. According as $\pi^\perp(\mathbf{v})$ is 0 or not, the tangent cone $\Pi_{\mathbf{v}}$ at $\mathbf{v}$ to $\Pi_{\mathbf{x}_0}$ has distinct expressions.

(1) If $\pi^\perp(\mathbf{v}) = 0$, then $\mathrm{U}^{\mathbf{v}}$ can be taken as the translation by $\mathbf{v}$ and $\Pi_{\mathbf{v}} = \Pi_{\mathbf{x}_0}$.

[6] This means that for each $\mathbf{x}_0 \in \mathbf{t}$ there exists a neighborhood $\mathcal{U} \subset \mathbf{t}$ of $\mathbf{x}_0$ and an associate local diffeomorphism from $\mathcal{U}$ onto an open set in $\mathbb{R}^{n-d}$.

(2) If $\pi^\perp(\mathbf{v}) \neq 0$, we introduce the cylindrical coordinates $(r(\mathbf{v}), \theta(\mathbf{v}), \pi(\mathbf{v}))$ of $\mathbf{v}$ with:

$$r(\mathbf{v}) = \|\pi^\perp(\mathbf{v})\|, \quad \theta(\mathbf{v}) = \frac{\pi^\perp(\mathbf{v})}{\|\pi^\perp(\mathbf{v})\|} \in \overline{\Omega}_{\mathbf{x}_0} \quad \text{with} \quad \Omega_{\mathbf{x}_0} = \Gamma_{\mathbf{x}_0} \cap \mathbb{S}^{d-1}\,. \tag{3.34}$$

Let $\Pi_{\theta(\mathbf{v})} \in \mathfrak{P}_{d-1}$ be the tangent cone to $\Omega_{\mathbf{x}_0}$ at $\theta(\mathbf{v})$. We have, cf. proof of Lemma 3.7,

$$\Pi_{\mathbf{v}} := \mathbb{R}^{n-d} \times \langle \pi^\perp(\mathbf{v}) \rangle \times \Pi_{\theta(\mathbf{v})}\,. \tag{3.35}$$

In any case, the tangent cone $\Pi_{\mathbf{u}}$ is linked to $\Pi_{\mathbf{v}}$ by the formula $\Pi_{\mathbf{u}} = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})(\Pi_{\mathbf{v}})$. We deduce:

(1) If $\pi^\perp(\mathbf{v}) = 0$, then $d(\Pi_{\mathbf{u}}) = d(\Pi_{\mathbf{x}_0})$ (cf. Remark 3.17), therefore $d_0(\mathbf{u}) = d_0(\mathbf{x}_0) = d$ and $\mathbf{u} \in \mathfrak{A}_d(\Omega)$.

(2) If $\pi^\perp(\mathbf{v}) \neq 0$, then $d(\Pi_{\mathbf{u}}) = d(\Pi_{\mathbf{v}})$ and we have $d_0(\mathbf{u}) \leq d-1 < d_0(\mathbf{x}_0) = d$.

Therefore $\mathbf{u} \in \mathfrak{A}_d(\Omega)$ if and only if $\pi^\perp(\mathbf{v}) = 0$. We conclude that

$$\mathfrak{A}_d(\Omega) \cap \mathcal{U}_{\mathbf{x}_0} = (\mathrm{U}^{\mathbf{x}_0})^{-1}(\pi(\mathcal{V}_{\mathbf{x}_0})).$$

Hence the stratum $\mathbf{t}$ is a smooth submanifold of codimension d. □

Remark 3.21. Let Ω be a corner domain and $\mathfrak{X}$ be the set of reference points of an admissible atlas, cf. Definition 3.11. Let $\mathbf{x}_0 \in \mathfrak{X}$. As a consequence of the above proof we find that for any $\mathbf{u}_0 \in \mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$, $d_0(\mathbf{u}_0) \leq d_0(\mathbf{x}_0)$. Thus, in particular, the set of corners $\mathfrak{A}_n(\Omega)$ is contained in $\mathfrak{X}$.

3.4.3. *Topology on singular chains.* Here we introduce a distance on equivalence classes of the set of chains $\mathfrak{C}(\Omega)$, for the equivalence already introduced in Definition 3.19. This will allow to introduce natural notions of continuity and lower semicontinuity on chains.

Let us denote by $\mathsf{BGL}(n)$ the ring of linear isomorphisms L with norm $\|L\| \leq 1$, where

$$\|L\| = \max_{\mathbf{x} \in \mathbb{R}^n \setminus \{0\}} \frac{\|L\mathbf{x}\|}{\|\mathbf{x}\|}\,.$$

Definition 3.22. Let $\mathbb{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_p)$ and $\mathbb{X}' = (\mathbf{x}'_0, \ldots, \mathbf{x}'_{p'})$ be two singular chains in $\mathfrak{C}(\Omega)$. We define the distance $\mathbb{D}(\mathbb{X}, \mathbb{X}') \in \mathbb{R}_+ \cup \{+\infty\}$ as

$$\mathbb{D}(\mathbb{X}, \mathbb{X}') = \|\mathbf{x}_0 - \mathbf{x}'_0\| + \frac{1}{2}\Big\{ \min_{\substack{L \in \mathsf{BGL}(n) \\ L\Pi_{\mathbb{X}} = \Pi_{\mathbb{X}'}}} \|L - \mathbb{I}_n\| + \min_{\substack{L \in \mathsf{BGL}(n) \\ L\Pi_{\mathbb{X}'} = \Pi_{\mathbb{X}}}} \|L - \mathbb{I}_n\| \Big\}\,,$$

where the second term is set to $+\infty$ if $\Pi_{\mathbb{X}}$ and $\Pi_{\mathbb{X}'}$ do not belong to the same orbit for the action of $\mathsf{BGL}(n)$ on $\mathfrak{P}_n$.

Remark 3.23. (a) The distance $\mathbb{D}(\mathbb{X}, \mathbb{X}')$ is zero if and only if the chains $\mathbb{X}$ and $\mathbb{X}'$ are equivalent.

(b) As a consequence of the proof of Proposition 3.20, the strata of $\overline{\Omega}$ are contained in orbits of the natural action of $\mathsf{BGL}(n)$ on chains.

(c) For strata of polyhedral domains, the distance $\mathbb{D}$ between chains of length 1 is equivalent to the standard distance in $\mathbb{R}^n$. This is no longer true for strata containing conical points in their closure.

We define a partial order on chains.

Definition 3.24. Let $\mathbb{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_p)$ and $\mathbb{X}' = (\mathbf{x}'_0, \ldots, \mathbf{x}'_{p'})$ be two singular chains in $\mathfrak{C}(\Omega)$. We say that $\mathbb{X} \leq \mathbb{X}'$ if $p \leq p'$ and $\mathbf{x}_j = \mathbf{x}'_j$ for all $0 \leq j \leq p$.

Theorem 3.25. *Let Ω be a corner domain in $\mathfrak{D}(M)$ with $M = \mathbb{R}^n$ or $\mathbb{S}^n$, and $F : \mathfrak{C}(\Omega) \to \mathbb{R}$ be a function such that*

(i) F is continuous on $\mathfrak{C}(\Omega)$ for the distance $\mathbb{D}$

(ii) F is order-preserving on $\mathfrak{C}(\Omega)$ (i.e., $\mathbb{X} \leq \mathbb{X}'$ implies $F(\mathbb{X}) \leq F(\mathbb{X}')$).

Then for all chain $\mathbb{X} = (\mathbf{x}_0, \ldots, \mathbf{x}_p) \cup \{\emptyset\}$, the function (with the convention that $\Omega_\emptyset = \Omega$)

$$\overline{\Omega}_{\mathbf{x}_0,\ldots,\mathbf{x}_p} \ni \mathbf{x} \longmapsto F((\mathbf{x}_0, \ldots, \mathbf{x}_p, \mathbf{x}))$$

is lower semicontinuous. In particular $\overline{\Omega} \ni \mathbf{x} \mapsto F((\mathbf{x}))$ is lower semicontinuous.

Proof. The proof is recursive over the dimension n.

Initialization. $n = 1$. Let Ω belong to $\mathfrak{D}(M)$ with $M = \mathbb{R}$ or $\mathbb{S}^1$. Then Ω is an open interval $(\mathbf{c}, \mathbf{c}')$. The chains in $\mathfrak{C}(\Omega)$ are

- $\mathbb{X} = (\mathbf{x}_0)$ for $\mathbf{x}_0 \in (\mathbf{c}, \mathbf{c}')$ with $\Pi_\mathbb{X} = \mathbb{R}$,
- $\mathbb{X} = (\mathbf{x}_0)$ for $\mathbf{x}_0 = \mathbf{c}$ and $\mathbf{x}_0 = \mathbf{c}'$, with $\Pi_\mathbb{X} = \mathbb{R}_+$ and $\mathbb{R}_-$, respectively,
- $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$ for $\mathbf{x}_0 = \mathbf{c}$ or $\mathbf{x}_0 = \mathbf{c}'$, and $\mathbf{x}_1 = 1$, with $\Pi_\mathbb{X} = \mathbb{R}$.

The function F is continuous on $\mathfrak{C}(\Omega)$. By definition of the distance $\mathbb{D}$ there holds

$$\mathbb{D}\big((\mathbf{x}), (\mathbf{c}, 1)\big) = \|\mathbf{x} - \mathbf{c}\| \quad \text{and} \quad \mathbb{D}\big((\mathbf{x}), (\mathbf{c}', 1)\big) = \|\mathbf{x} - \mathbf{c}'\|, \quad \forall \mathbf{x} \in (\mathbf{c}, \mathbf{c}')\,.$$

Therefore, as $\mathbf{x} \to \mathbf{c}$, with $\mathbf{x} \neq \mathbf{c}$, $F((\mathbf{x}))$ tends to $F((\mathbf{c}, 1))$. By assumption $F((\mathbf{c}, 1)) \geq F((\mathbf{c}))$, and the same at the other end $\mathbf{c}'$. This proves that F is lower semicontinuous on $\overline{\Omega} = [\mathbf{c}, \mathbf{c}']$.

Recurrence. We assume that Theorem 3.25 holds for any dimension $n^\star < n$. Let us prove it for the dimension n.

a) Let $\mathbb{X}_0$ be a non-empty chain in $\mathfrak{C}(\Omega)$. Then $\Omega_{\mathbb{X}_0}$ belongs to $\mathfrak{D}(\mathbb{S}^{n^\star})$ for a $n^\star < n$. The chains $\mathbb{Y} \in \mathfrak{C}(\Omega_{\mathbb{X}_0})$ correspond to the chains $(\mathbb{X}_0, \mathbb{Y})$ in $\mathfrak{C}(\Omega)$ and the corresponding tangent substructures $\Pi_\mathbb{Y} \in \mathfrak{P}_{n^\star}$ and $\Pi_{\mathbb{X}_0,\mathbb{Y}} \in \mathfrak{P}_n$ are linked by a relation of the type, cf. (3.32)

$$\Pi_{\mathbb{X}_0,\mathbb{Y}} = \underline{\mathrm{U}}^0\big(\mathbb{R}^{n-d_0} \times \langle \mathbf{x}_1 \rangle \times \ldots \times \Pi_\mathbb{Y}\big).$$

Hence the distances $\mathbb{D}\big((\mathbb{X}_0, \mathbb{Y}), (\mathbb{X}_0, \mathbb{Y}')\big)$ and $\mathbb{D}\big(\mathbb{Y}, \mathbb{Y}'\big)$ can be compared:

$$\begin{aligned}
\mathbb{D}\big((\mathbb{X}_0, \mathbb{Y}), (\mathbb{X}_0, \mathbb{Y}')\big) &= \frac{1}{2}\Bigg\{ \min_{\substack{L \in \mathsf{BGL}(n) \\ L\Pi_{\mathbb{X}_0,\mathbb{Y}} = \Pi_{\mathbb{X}_0,\mathbb{Y}'}}} \|L - \mathbb{I}_n\| + \min_{\substack{L \in \mathsf{BGL}(n) \\ L\Pi_{\mathbb{X}_0,\mathbb{Y}'} = \Pi_{\mathbb{X}_0,\mathbb{Y}}}} \|L - \mathbb{I}_n\| \Bigg\} \\
&\leq \frac{1}{2}\Bigg\{ \min_{\substack{L^\star \in \mathsf{BGL}(n^\star) \\ L^\star\Pi_\mathbb{Y} = \Pi_{\mathbb{Y}'}}} \|L^\star - \mathbb{I}_{n^\star}\| + \min_{\substack{L^\star \in \mathsf{BGL}(n^\star) \\ L^\star\Pi_{\mathbb{Y}'} = \Pi_\mathbb{Y}}} \|L^\star - \mathbb{I}_{n^\star}\| \Bigg\} \\
&\leq \mathbb{D}\big(\mathbb{Y}, \mathbb{Y}'\big).
\end{aligned}$$

Let us define the function $F^\star$ on $\mathfrak{C}(\Omega_{\mathbb{X}_0})$ by the partial application

$$F^\star(\mathbb{Y}) = F((\mathbb{X}_0, \mathbb{Y})), \quad \mathbb{Y} \in \mathfrak{C}(\Omega_{\mathbb{X}_0}).$$

Since F is continuous on $\mathfrak{C}(\Omega)$, the above inequality between distances proves that $F^\star$ is continuous on $\mathfrak{C}(\Omega_{\mathbb{X}_0})$. Likewise the monotonicity property is obviously transported from F to $F^\star$. Therefore the recurrence assumption provides the lower semicontinuity of $F^\star$ on $\overline{\Omega}_{\mathbb{X}_0}$, hence of $\mathbf{x} \mapsto F((\mathbb{X}_0, \mathbf{x}))$ on the same set.

b) It remains to prove that $\mathbf{x} \mapsto F((\mathbf{x}))$ is lower semicontinuous on $\overline{\Omega}$. Let $\mathbf{x}_0 \in \overline{\Omega}$. At this point we follow the proof of Proposition 3.20. For any $\mathbf{u} \in \mathcal{U}_{\mathbf{x}_0}$, we define π, $\pi^\perp$ and $\mathbf{v}$ like there and encounter the same two cases:

(1) If $\pi^\perp(\mathbf{v}) = 0$, then $\Pi_{\mathbf{v}} = \Pi_{\mathbf{x}_0}$. Hence $\Pi_{\mathbf{u}} = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})(\Pi_{\mathbf{x}_0})$. Since $\mathrm{J}^{\mathbf{x}_0}(\mathbf{v})$ tends to $\mathbb{I}_n$ as $\mathbf{v} \to \mathbf{0}$, the distance $\mathbb{D}((\mathbf{x}_0), (\mathbf{u}))$ tends to 0 as $\mathbf{u}$ tends to $\mathbf{x}_0$. By the continuity assumption, $F((\mathbf{u}))$ tends to $F((\mathbf{x}_0))$.

(2) If $\pi^\perp(\mathbf{v}) \neq 0$, let $\mathbf{x}_1$ be the element of $\overline{\Omega}_{\mathbf{x}_0}$ defined by $\mathbf{x}_1 = \pi^\perp(\mathbf{v})\,\|\pi^\perp(\mathbf{v})\|^{-1}$. Let $\Pi_{\mathbf{x}_1} \in \mathfrak{P}_{d-1}$ be the tangent cone to $\Omega_{\mathbf{x}_0}$ at $\mathbf{x}_1$. We find

$$\Pi_{\mathbf{v}} = \mathbb{R}^{n-d} \times \langle \pi^\perp(\mathbf{v}) \rangle \times \Pi_{\mathbf{x}_1} = \Pi_{\mathbf{x}_0,\mathbf{x}_1}\,.$$

Hence $\Pi_{\mathbf{u}} = \mathrm{J}^{\mathbf{x}_0}(\mathbf{v})(\Pi_{\mathbf{x}_0,\mathbf{x}_1})$. Like before, we deduce that the distance $\mathbb{D}((\mathbf{x}_0,\mathbf{x}_1),(\mathbf{u}))$ tends to 0 as $\mathbf{u}$ tends to $\mathbf{x}_0$. By the continuity assumption, $F((\mathbf{u}))$ tends to $F((\mathbf{x}_0,\mathbf{x}_1))$, which by the monotonicity assumption, is larger than $F((\mathbf{x}_0))$.

This ends the proof of the theorem. $\square$

3.4.4. *Singular chains and admissible atlantes.* The aim of this section is to provide an overview of map-neighborhoods and Jacobian estimates in the framework of singular chains. In their generality, these facts are not needed for our study of magnetic Laplacians, which is restricted to dimension $n \leq 3$ for distinct reasons that we will explain later on. Nevertheless, full generality sheds some light on the recursive process present in the very definition of admissible atlantes and in the domain of validity of estimates in Proposition 3.13.

• *Chains of atlantes.* Denote by $\mathfrak{X}(\Omega)$ the set of reference points of an admissible atlas for a corner domain Ω. The chain of atlantes of a corner domain Ω is defined as follows:

(0) Start from the set $\mathfrak{X}(\Omega)$ of reference points $\mathbf{x}_0 \in \overline{\Omega}$, as in Definition 3.11.

(1) For each $\mathbf{x}_0 \in \mathfrak{X}(\Omega)$, choose an admissible atlas of the section $\Omega_{\mathbf{x}_0} \in \mathfrak{D}(\mathbb{S}^{d_0-1})$, with set $\mathfrak{X}(\Omega_{\mathbf{x}_0})$ of reference points $\mathbf{x}_1 \in \overline{\Omega}_{\mathbf{x}_0}$.

(2) For each $\mathbf{x}_1 \in \mathfrak{X}(\Omega_{\mathbf{x}_0})$, choose an admissible atlas of the section $\Omega_{\mathbf{x}_0,\mathbf{x}_1} \in \mathfrak{D}(\mathbb{S}^{d_1-1})$, with set $\mathfrak{X}(\Omega_{\mathbf{x}_0,\mathbf{x}_1})$ of reference points $\mathbf{x}_2 \in \overline{\Omega}_{\mathbf{x}_0,\mathbf{x}_1}$. And so on...

• *Cylindrical coordinates.* The natural coordinates associated with chains of atlantes are recursively defined cylindrical coordinates. Let $\mathbf{u}_0 \in \overline{\Omega}$.

(1) If $\mathbf{u}_0 \notin \mathfrak{X}(\Omega)$, pick $\mathbf{x}_0 \in \mathfrak{X}(\Omega)$ such that $\mathbf{u}_0 \in \mathcal{B}^n(\mathbf{x}_0, R_{\mathbf{x}_0})$ (n-dimensional ball). Then define $\mathbf{v}_0 = \mathrm{U}^{\mathbf{x}_0}\mathbf{u}_0$ and, if $d_0 > 0$, its cylindrical coordinates

$$\pi_0(\mathbf{v}_0) \in \mathbb{R}^{n-d_0}, \quad r(\mathbf{v}_0) = \|\mathbf{v}_0 - \pi_0(\mathbf{v}_0)\|, \quad \text{and} \quad \mathbf{u}_1 = \frac{\mathbf{v}_0 - \pi_0(\mathbf{v}_0)}{r(\mathbf{v}_0)} \in \overline{\Omega}_{\mathbf{x}_0}\,.$$

If $d_0 = 0$, $\pi_0 = \mathbb{I}_n$, then stop.

(2) If $\mathbf{u}_1 \notin \mathfrak{X}(\Omega_{\mathbf{x}_0})$, pick $\mathbf{x}_1 \in \mathfrak{X}(\Omega_{\mathbf{x}_0})$ such that $\mathbf{u}_1 \in \mathcal{B}^{d_0}(\mathbf{x}_1, R_{\mathbf{x}_1}) \cap \mathbb{S}^{d_0-1}$. Then define $\mathbf{v}_1 = \mathrm{U}^{\mathbf{x}_0,\mathbf{x}_1}\mathbf{u}_1$ and, if $d_1 > 0$, its cylindrical coordinates

$$\pi_1(\mathbf{v}_1) \in \mathbb{R}^{d_0-1-d_1}, \quad r(\mathbf{v}_1) = \|\mathbf{v}_1 - \pi_1(\mathbf{v}_1)\|, \quad \text{and} \quad \mathbf{u}_2 = \frac{\mathbf{v}_1 - \pi_1(\mathbf{v}_1)}{r(\mathbf{v}_1)} \in \overline{\Omega}_{\mathbf{x}_0,\mathbf{x}_1}.$$

If $d_1 = 0$, $\pi_1 = \mathbb{I}_n$, then stop. And so on...

Let $\mathbf{v}_{p_*}$ be the last element of the sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots$. In any case $p_* \le n$.

• *Local maps.* The local maps are recursively constructed using the natural coordinates associated with chains.

(0) If $\mathbf{u}_0 = \mathbf{x}_0 \in \mathfrak{X}(\Omega)$, use the local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ and stop.

(1) If $\mathbf{u}_0 \notin \mathfrak{X}(\Omega)$, a local map $(\mathcal{U}_{\mathbf{u}_0}, \mathrm{U}^{\mathbf{u}_0})$ is defined by the formulas hereafter. The map neighborhood $\mathcal{U}_{\mathbf{u}_0}$ can be chosen as $(\mathrm{U}^{\mathbf{x}_0})^{-1}(\mathcal{U}_{\mathbf{v}_0})$ with

$$\mathcal{U}_{\mathbf{v}_0} = \mathcal{B}^{n-d_0}(\pi_0(\mathbf{v}_0), R_{\mathbf{x}_0}) \times r(\mathbf{v}_0)\,\mathcal{U}_{(1,\mathbf{u}_1)}, \quad \mathcal{U}_{(1,\mathbf{u}_1)} = \mathcal{B}^{d_0}(\mathbf{u}_1, \rho_1), \quad \mathcal{U}_{\mathbf{u}_1} = \mathcal{U}_{(1,\mathbf{u}_1)} \cap \mathbb{S}^{d_0-1}.$$

The diffeomorphism $\mathrm{U}^{\mathbf{u}_0}$ is defined by $\mathrm{J}^{\mathbf{x}_0}(\mathbf{v}_0)\,(\mathrm{U}^{\mathbf{v}_0} \circ \mathrm{U}^{\mathbf{x}_0})$ with

$$\mathrm{U}^{\mathbf{v}_0} = \big(\mathrm{T}_{\pi_0(\mathbf{v}_0)}\,,\, \mathrm{N}^{-1}_{r(\mathbf{v}_0)} \circ \mathrm{U}^{(1,\mathbf{u}_1)} \circ \mathrm{N}_{r(\mathbf{v}_0)}\big) \quad \text{and} \quad \mathrm{U}^{(1,\mathbf{u}_1)} = (\mathrm{T}_1\,,\, \mathrm{U}^{\mathbf{u}_1})\,,$$

where $\mathrm{T}_{\pi_0(\mathbf{v}_0)}$ is the translation $\mathbf{v} \mapsto \mathbf{v} - \pi_0(\mathbf{v}_0)$ in $\mathbb{R}^{n-d_0}$, and T_1 is the translation by 1 for the radius in polar coordinates. If $\mathbf{u}_1 = \mathbf{x}_1 \in \mathfrak{X}(\Omega_{\mathbf{x}_0})$, stop.

(2) If $\mathbf{u}_1 \notin \mathfrak{X}(\Omega_{\mathbf{x}_0})$, a local map $(\mathcal{U}_{\mathbf{u}_1}, \mathrm{U}^{\mathbf{u}_1})$ is defined like in step (1), replacing $\mathbf{x}_0$ by $\mathbf{x}_1$, $\mathbf{v}_0$ by $\mathbf{v}_1$, $\mathcal{B}^{n-d_0}$ by $\mathcal{B}^{d_0-1-d_1}$, $\pi_0(\mathbf{v}_0)$ by $\pi_1(\mathbf{v}_1)$, $\mathcal{B}^{d_0}$ by $\mathcal{B}^{d_1}$, and finally $\mathbf{u}_1$ by $\mathbf{u}_2$...

• *Estimates on Jacobian matrices.* Let $\mathbf{u}_0 \in \overline{\Omega}$. As explained in Remark 3.8, as soon as a polyhedral cone $\Gamma_{\mathbf{x}_0,\ldots,\mathbf{x}_p}$ is reached in the construction, the corresponding diffeomorphism $\mathrm{U}^{(1,\mathbf{u}_{p+1})}$ is chosen as a translation, so it is the same for $\mathrm{U}^{\mathbf{u}_{p+1}}$, and the norm of its differential is bounded. By recursion, this implies the estimate for the differential $\mathrm{K}^{\mathbf{u}_0}$ of $\mathrm{J}^{\mathbf{u}_0}$

$$\|\mathrm{K}^{\mathbf{u}_0}\| \le \frac{c(\Omega)}{r(\mathbf{v}_0)\ \cdots\ r(\mathbf{v}_{p-1})} \tag{3.36}$$

with the convention that if $p - 1 < 0$, the denominator is 1.The same estimate is valid if $\mathbf{u}_p \in \mathfrak{X}(\Omega_{\mathbf{x}_0,\ldots,\mathbf{x}_{p-1}})$ with the convention that $\Omega_{\mathbf{x}_0,\ldots,\mathbf{x}_{p-1}} = \Omega$ if $p-1<0$. Note that $p = 0$ for any $\mathbf{u}_0$ if the domain Ω is polyhedral. In turn, the domain of validity of estimates (3.36) is (at least) a ball centered at $\mathbf{u}_0$ of radius

$$\rho(\mathbf{u}_0) = r(\Omega)\, r(\mathbf{v}_0)\ \cdots\ r(\mathbf{v}_{p_*})\,. \tag{3.37}$$

3.5. **3D domains.** In this section we refine our analysis for the particular case of 3D domains. In each case we provide an exhaustive description of the possible singular chains. We also give the consequences of Proposition 3.13.

3.5.1. *Faces, edges and corners.*

Definition 3.26. Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$. We denote by $\mathfrak{F}$ the set of the connected components of $\mathfrak{A}_1(\Omega)$ (faces), $\mathfrak{E}$ those of $\mathfrak{A}_2(\Omega)$ (edges) and $\mathfrak{V}$ the finite set $\mathfrak{A}_3(\Omega)$ (corners).
Let $\mathbf{x}_0 \in \mathfrak{A}_d(\Omega)$ with $d < 3$, then $\Pi_{\mathbf{x}_0} \in \overline{\mathfrak{P}}_3$. Let $\mathbf{x}_0 \in \mathfrak{V}$, we distinguish between two cases:

(1) If $\Pi_{\mathbf{x}_0} \in \overline{\mathfrak{P}}_3$, then $\mathbf{x}_0$ is a polyhedral corner.
(2) If $\Pi_{\mathbf{x}_0} \notin \overline{\mathfrak{P}}_3$, then $\mathbf{x}_0$ is a conical corner. We denote by $\mathfrak{V}^\circ$ the set of conical corners.

Combining Proposition 3.13 and Remark 3.5, we obtain local estimates for the Jacobian matrix and the metric issued from changes of variables pertaining to an admissible atlas:

Corollary 3.27. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $(\mathcal{U}_{\mathbf{x}}, \mathrm{U}^{\mathbf{x}})_{\mathbf{x}\in\overline{\Omega}}$ be an admissible atlas. Note that the set of its reference points $\mathfrak{X}$ contains $\mathfrak{V}$ (cf. Remark 3.21), thus in particular the set of conical corners $\mathfrak{V}^\circ$. There exists $c(\Omega) > 0$ such that*

(a) for all $\mathbf{x}_0 \in \mathfrak{X}$, there holds

$$\|\mathrm{J}^{\mathbf{x}_0} - \mathbb{I}_3\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{x}_0} - \mathbb{I}_3\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le r\, c(\Omega), \quad \text{for all } r \le R_{\mathbf{x}_0},$$

(b) for all $\mathbf{u}_0 \in \overline{\Omega} \setminus \mathfrak{X}$, there holds

$$\|\mathrm{J}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} + \|\mathrm{G}^{\mathbf{u}_0} - \mathbb{I}_n\|_{L^\infty(\mathcal{B}(\mathbf{0},r))} \le \frac{r}{d_{\mathfrak{V}^\circ}(\mathbf{u}_0)}\, c(\Omega), \quad \text{for all } r \le \rho(\mathbf{u}_0),$$

with $\rho(\mathbf{u}_0)$ as in Proposition 3.13 and

$$d_{\mathfrak{V}^\circ}(\mathbf{u}_0) = \begin{cases} 1 & \text{if} \quad \mathfrak{V}^\circ = \emptyset, \\ \operatorname{dist}(\mathbf{u}_0, \mathfrak{V}^\circ) & \text{else}. \end{cases} \tag{3.38}$$

Remark 3.28. Note that estimate *(b)* blows up when we get closer to a conical point without reaching it, while at any conical point $\mathbf{x}_0 \in \mathfrak{V}^\circ$, we have the good estimate *(a)*. This will lead to distinct analyses depending on how far $\mathbf{x}_0$ is from $\mathfrak{V}^\circ$.

3.5.2. *Singular chains of 3D corner domains.*

Proposition 3.29. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$. Then chains of length ≤ 3 are sufficient to describe all equivalence classes of the set of chains $\mathfrak{C}(\Omega)$. If moreover $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$, chains of length 2 are sufficient.*

Proof. Let $\mathbf{x}_0 \in \overline{\Omega}$. In Description 3.30 we enumerate all chains starting from $\mathbf{x}_0$ with their tangent substructures according as $\mathbf{x}_0$ is an interior point, a face point, an edge point, or a vertex.

Description 3.30.

(1) Interior point $\mathbf{x}_0 \in \Omega$. Only one chain in $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$: $\mathbb{X} = (\mathbf{x}_0)$. $\Pi_{\mathbb{X}} \equiv \mathbb{R}^3$.
(2) Let $\mathbf{x}_0$ belong to a face. There are two chains in $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$:
  (a) $\mathbb{X} = (\mathbf{x}_0)$ with $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}_0}$, the tangent half-space. $\Pi_{\mathbb{X}} \equiv \mathbb{R}^2 \times \mathbb{R}_+$.
  (b) $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$ where $\mathbf{x}_1 = 1$ is the only element in $\mathbb{R}_+ \cap \mathbb{S}^0$. Thus $\Pi_{\mathbb{X}} = \mathbb{R}^3$.
(3) Let $\mathbf{x}_0$ belong to an edge. There are three possible lengths for chains in $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$:
  (a) $\mathbb{X} = (\mathbf{x}_0)$ with $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}_0}$, the tangent wedge (which is not a half-plane). The reduced cone of $\Pi_{\mathbf{x}_0}$ is a sector $\Gamma_{\mathbf{x}_0}$ the section of which is an interval $\mathcal{I}_{\mathbf{x}_0} \subset \mathbb{S}^1$.
  (b) $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$ where $\mathbf{x}_1 \in \overline{\mathcal{I}}_{\mathbf{x}_0}$.
    (i) If $\mathbf{x}_1$ is interior to $\mathcal{I}_{\mathbf{x}_0}$, $\Pi_{\mathbb{X}} = \mathbb{R}^3$. No further chain.
    (ii) If $\mathbf{x}_1$ is a boundary point of $\mathcal{I}_{\mathbf{x}_0}$, $\Pi_{\mathbb{X}}$ is a half-space, containing one of the two faces $\partial^\pm\Pi_{\mathbf{x}_0}$ of the wedge $\Pi_{\mathbf{x}_0}$.
  (c) $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ where $\mathbf{x}_1 \in \partial\mathcal{I}_{\mathbf{x}_0}$, $\mathbf{x}_2 = 1$ and $\Pi_{\mathbb{X}} = \mathbb{R}^3$.

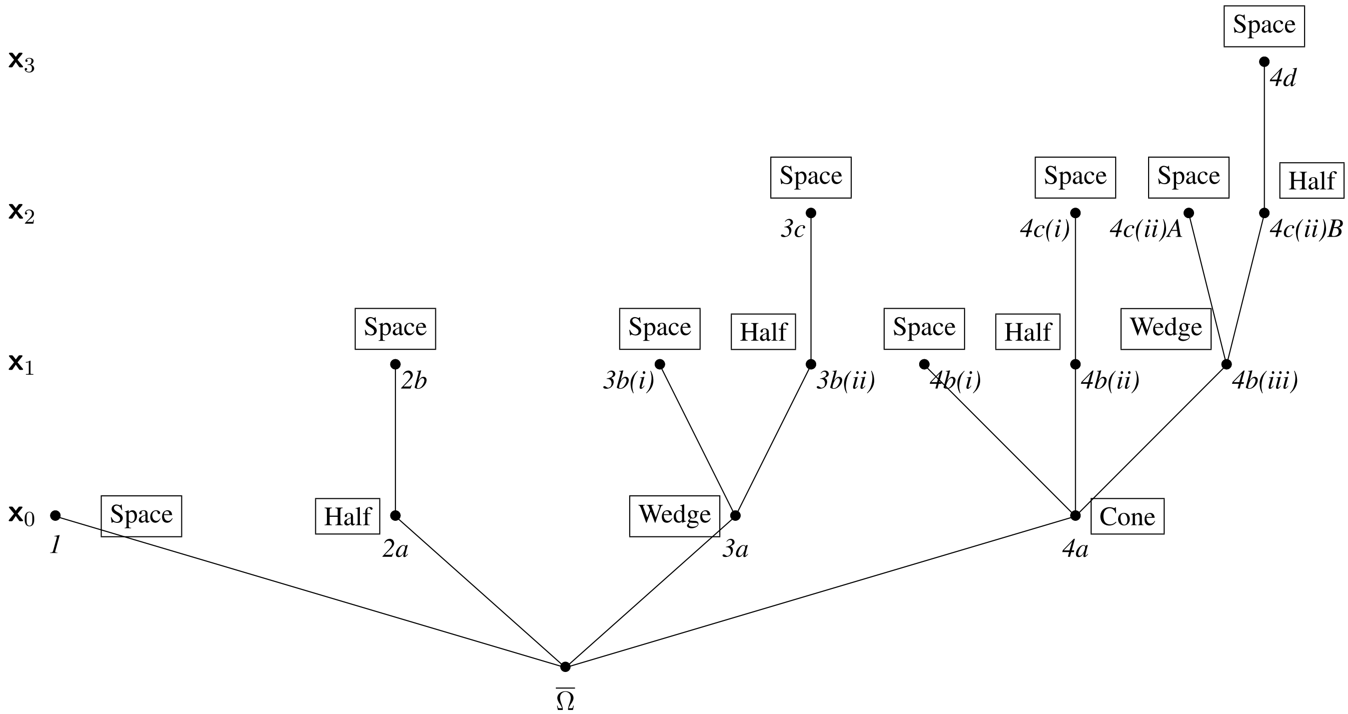

FIGURE 1. The tree of singular chains with numbering according to Description 3.30 (Half is for half-space)

(4) Let x_0 be a corner. There are four possible lengths for chains in $\mathfrak{C}_{\mathsf{x}_0}(\Omega)$:
   (a) $\mathbb{X} = (\mathsf{x}_0)$ with $\Pi_{\mathbb{X}} = \Pi_{\mathsf{x}_0}$, the tangent cone (which is not a wedge). It coincides with its reduced cone. Its section Ω_{x_0} is a polygonal domain in $\mathbb{S}^2$.
   (b) $\mathbb{X} = (\mathsf{x}_0, \mathsf{x}_1)$ where $\mathsf{x}_1 \in \overline{\Omega}_{\mathsf{x}_0}$.
      (i) If x_1 is interior to Ω_{x_0}, $\Pi_{\mathbb{X}} = \mathbb{R}^3$. No further chain.
      (ii) If x_1 is in a side of Ω_{x_0}, $\Pi_{\mathbb{X}}$ is a half-space.
      (iii) If x_1 is a corner of Ω_{x_0}, $\Pi_{\mathbb{X}}$ is a wedge. Its edge contains one of the edges of Π_{x_0}.
   (c) $\mathbb{X} = (\mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_2)$ where $\mathsf{x}_1 \in \partial\Omega_{\mathsf{x}_0}$
      (i) If x_1 is in a side of Ω_{x_0}, $\mathsf{x}_2 = 1$, $\Pi_{\mathbb{X}} = \mathbb{R}^3$. No further chain.
      (ii) If x_1 is a corner of Ω_{x_0}, $C_{\mathsf{x}_0,\mathsf{x}_1}$ is plane sector, and $\mathsf{x}_2 \in \overline{\mathcal{I}}_{\mathsf{x}_0,\mathsf{x}_1}$ where the interval $\mathcal{I}_{\mathsf{x}_0,\mathsf{x}_1}$ is its section.
         (A) If x_2 is an interior point of $\mathcal{I}_{\mathsf{x}_0,\mathsf{x}_1}$, then $\Pi_{\mathbb{X}} = \mathbb{R}^3$.
         (B) If x_2 is a boundary point of $\mathcal{I}_{\mathsf{x}_0,\mathsf{x}_1}$, then $\Pi_{\mathbb{X}}$ is a half-space.
   (d) $\mathbb{X} = (\mathsf{x}_0, \mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3)$ where x_1 is a corner of Ω_{x_0}, $\mathsf{x}_2 \in \partial\mathcal{I}_{\mathsf{x}_0,\mathsf{x}_1}$ and $\mathsf{x}_3 = 1$. Then $\Pi_{\mathbb{X}} = \mathbb{R}^3$.

As a consequence of this description we may identify equivalence classes in $\mathfrak{C}_{\mathsf{x}_0}(\Omega)$:

— If x_0 is an edge point, there are 4 equivalence classes: $\mathbb{X} = (\mathsf{x}_0)$, $\mathbb{X} = (\mathsf{x}_0, \mathsf{x}_1^{\pm})$ with $\mathsf{x}_1^-, \mathsf{x}_1^+$ the ends of $\mathcal{I}_{\mathsf{x}_0}$, and $\mathbb{X} = (\mathsf{x}_0, \mathsf{x}_1^{\circ})$ with x_1° any chosen point in $\mathcal{I}_{\mathsf{x}_0}$.

— If $\mathbf{x}_0$ is a polyhedral corner, the set of the equivalence classes of $\mathfrak{C}_{\mathbf{x}_0}(\Omega)$ is finite according to the following description. Let $\mathbf{x}_1^j$, $1 \le j \le N$, be the corners of $\Omega_{\mathbf{x}_0}$, and $\mathbf{f}_1^j$, $1 \le j \le N$, be its sides (notice that there are as many corners as sides). There are $2N+2$ equivalence classes: $\mathbb{X} = (\mathbf{x}_0)$ (vertex), $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1^j)$ with $1 \le j \le N$ (edge-point limit), $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1^{\circ,j})$ with $\mathbf{x}_1^{\circ,j}$ any chosen point inside $\mathbf{f}_1^j$ (face-point limit), and $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1^\circ)$ with $\mathbf{x}_1^\circ$ any chosen point in $\Omega_{\mathbf{x}_0}$ (interior point limit).

— If $\mathbf{x}_0$ belongs to $\mathfrak{V}^\circ$, the set of chains which are face-point limits is infinite. Moreover, chains $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ obtained by the general above procedure (4)-(c)-(ii)-(B) can be irreducible: Such chains represent the limit of a conical face close to an edge. $\square$

4. Magnetic Laplacians and their tangent operators

Let $\mathbf{A}$ be a magnetic potential associated with the magnetic field $\mathbf{B}$ on a corner domain $\Omega \in \mathfrak{D}(\mathbb{R}^3)$. We recall that the corresponding magnetic Laplacian is $H_h(\mathbf{A}, \Omega) = (-ih\nabla + \mathbf{A})^2$. At each point $\mathbf{x}_0 \in \overline{\Omega}$ is associated a local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ and a tangent cone $\Pi_{\mathbf{x}_0}$, cf. (3.1). We will associate a tangent magnetic potential to $\Pi_{\mathbf{x}_0}$ and provide formulas and estimates for the operator transformed from the magnetic Laplacian $H_h(\mathbf{A}, \Omega)$ by the local map $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$.

4.1. Change of variables.

Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$. We consider a magnetic potential $\mathbf{A} \in \mathscr{C}^1(\overline{\Omega})$. Let $\mathbf{x}_0 \in \overline{\Omega}$. Let us recall that with $\mathbf{x}_0$ are associated the local smooth diffeomorphism $\mathrm{U}^{\mathbf{x}_0}$ (3.1), the Jacobian matrix $\mathrm{J}^{\mathbf{x}_0}$ (3.2) of the inverse of $\mathrm{U}^{\mathbf{x}_0}$ and the associated metric $\mathrm{G}^{\mathbf{x}_0}$ (3.3). According to formulas (A.4)–(A.5), we introduce the magnetic potential $\mathbf{A}^{\mathbf{x}_0}$ and magnetic field $\mathbf{B}^{\mathbf{x}_0} = \operatorname{curl} \mathbf{A}^{\mathbf{x}_0}$ transformed by $\mathrm{U}^{\mathbf{x}_0}$ in $\mathcal{V}_{\mathbf{x}_0} \cap \Pi_{\mathbf{x}_0}$

$$\mathbf{A}^{\mathbf{x}_0} := (\mathrm{J}^{\mathbf{x}_0})^\top \big((\mathbf{A} - \mathbf{A}(\mathbf{x}_0)) \circ (\mathrm{U}^{\mathbf{x}_0})^{-1}\big) \quad \text{and} \quad \mathbf{B}^{\mathbf{x}_0} := |\det \mathrm{J}^{\mathbf{x}_0}|\, (\mathrm{J}^{\mathbf{x}_0})^{-1} \big(\mathbf{B} \circ (\mathrm{U}^{\mathbf{x}_0})^{-1}\big)\,. \tag{4.1}$$

We also introduce the phase shift

$$\zeta_h^{\mathbf{x}_0}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}(\mathbf{x}_0), \mathbf{x}\rangle/h}, \quad \mathbf{x} \in \Omega, \tag{4.2}$$

so that there holds for any f in $H^1(\Omega)$

$$q_h[\mathbf{A}, \Omega](f) = q_h[\mathbf{A} - \mathbf{A}(\mathbf{x}_0), \Omega](\zeta_h^{\mathbf{x}_0} f). \tag{4.3}$$

To $f \in H^1(\Omega)$ with support in $\mathcal{U}_{x_0}$ we associate the function ψ

$$\psi := (\zeta_h^{\mathbf{x}_0} f) \circ (\mathrm{U}^{\mathbf{x}_0})^{-1}, \tag{4.4}$$

defined in $\Pi_{\mathbf{x}_0}$, with support in $\mathcal{V}_{\mathbf{x}_0}$. For any $h > 0$ Lemma A.3 provides the identities

$$q_h[\mathbf{A}, \Omega](f) = q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi) \quad \text{and} \quad \|f\|_{L^2(\Omega)} = \|\psi\|_{L^2_{\mathrm{G}^{\mathbf{x}_0}}(\Pi_{\mathbf{x}_0})}\,, \tag{4.5}$$

where the quadratic forms $q_h[\mathbf{A}, \Omega]$ and $q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}]$ are defined in (1.18) and (1.21), respectively. Using the Rayleigh quotient, we immediately deduce

$$\mathscr{Q}_h[\mathbf{A}, \Omega](f) = \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi). \tag{4.6}$$

4.2. **Model and tangent operators.**

Definition 4.1. We call *model operator* any magnetic Laplacian $H(\mathbf{A},\Pi)$ where $\Pi\in\mathfrak{P}_3$ and $\mathbf{A}$ is a linear potential associated with the constant magnetic field $\mathbf{B}$. We denote by $E(\mathbf{B},\Pi)$ the bottom of the spectrum (ground state energy) of $H(\mathbf{A},\Pi)$ and by $\lambda_{\mathrm{ess}}(\mathbf{B},\Pi)$ the bottom of its essential spectrum.

Let $\Omega\in\mathfrak{D}(\mathbb{R}^3)$ and $\mathbf{A}\in\mathscr{C}^1(\overline{\Omega})$. For each $\mathbf{x}_0\in\overline{\Omega}$ we set

$$\mathbf{B}_{\mathbf{x}_0}=\mathbf{B}(\mathbf{x}_0)\quad\text{and}\quad\mathbf{A}_{\mathbf{x}_0}(\mathbf{v})=\nabla\mathbf{A}(\mathbf{x}_0)\cdot\mathbf{v},\ \ \mathbf{v}\in\Pi_{\mathbf{x}_0},\tag{4.7}$$

so that $\mathbf{B}_{\mathbf{x}_0}$ is the magnetic field frozen at $\mathbf{x}_0$ and $\mathbf{A}_{\mathbf{x}_0}$ the linear part[7] of the potential at $\mathbf{x}_0$.

By extension, for each singular chain $\mathbb{X}=(\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_p)\in\mathfrak{C}(\Omega)$ we set

$$\mathbf{B}_{\mathbb{X}}=\mathbf{B}(\mathbf{x}_0)\quad\text{and}\quad\mathbf{A}_{\mathbb{X}}(\mathbf{x})=\nabla\mathbf{A}(\mathbf{x}_0)\cdot\mathbf{x},\ \ \mathbf{x}\in\Pi_{\mathbb{X}}.\tag{4.8}$$

We have obviously

$$\operatorname{curl}\mathbf{A}_{\mathbb{X}}=\mathbf{B}_{\mathbb{X}}\,.$$

Definition 4.2. Let $\Omega\in\mathfrak{D}(\mathbb{R}^3)$ and $\mathbf{A}\in\mathscr{C}^1(\overline{\Omega})$. Let $\mathbb{X}\in\mathfrak{C}(\Omega)$ be a singular chain of Ω. The model operator $H(\mathbf{A}_{\mathbb{X}},\Pi_{\mathbb{X}})$ is called a *tangent operator*.

Remark 4.3. The notion of equivalence classes between singular chains as introduced in Definition 3.19 is sufficient for the analysis of operators $H_h(\mathbf{A},\Omega)$ in the case of magnetic fields $\mathbf{B}$ smooth in Cartesian variables. Should $\mathbf{B}$ be smooth in polar variables only, the whole hierarchy of singular chains would be needed.

The potential $\mathbf{A}_{\mathbf{x}_0}$ and the field $\mathbf{B}_{\mathbf{x}_0}$ are connected to the potential $\mathbf{A}^{\mathbf{x}_0}$ and field $\mathbf{B}^{\mathbf{x}_0}$ (4.1) obtained through the local map: Since $\mathrm{dU}^{\mathbf{x}_0}(\mathbf{x}_0)=\mathbb{I}_3$ by definition, there holds

$$\mathbf{B}^{\mathbf{x}_0}(\mathbf{0})=\mathbf{B}(\mathbf{x}_0)\,.\tag{4.9}$$

Likewise, let $\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}$ be the linear part of $\mathbf{A}^{\mathbf{x}_0}$ at the vertex $\mathbf{0}$ of Π_{x_0}. Then, there holds

$$\mathbf{A}^{\mathbf{x}_0}(\mathbf{0})=0\quad\text{and}\quad\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}=\mathbf{A}_{\mathbf{x}_0}.\tag{4.10}$$

Local and minimum energies are introduced as follows.

Definition 4.4. Let $\Omega\in\mathfrak{D}(\mathbb{R}^3)$ and $\mathbf{B}\in\mathscr{C}^0(\overline{\Omega})$. The application $\mathbf{x}\mapsto E(\mathbf{B}_{\mathbf{x}},\Pi_{\mathbf{x}})$ is called *local ground energy* (with $E(\mathbf{B},\Pi)$ introduced in Definition 4.1). We define the *lowest local energy* of $\mathbf{B}$ on $\overline{\Omega}$ by

$$\mathscr{E}(\mathbf{B},\Omega):=\inf_{\mathbf{x}\in\overline{\Omega}}E(\mathbf{B}_{\mathbf{x}},\Pi_{\mathbf{x}}).\tag{4.11}$$

■

The relations with singular chains and the question whether $\mathscr{E}(\mathbf{B},\Omega)$ is a minimum are addressed later on Section 8.

[7]In (4.7), $\nabla\mathbf{A}$ is the 3×3 matrix with entries ∂_kA_j, $1\le j,k\le3$, and $\cdot\,\mathbf{v}$ denotes the multiplication by the column vector $\mathbf{v}=(v_1,v_2,v_3)^\top$.

4.3. **Linearization.** Starting from the identity (4.5) $q_h[\mathbf{A},\Omega](f) = q_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0},\mathrm{G}^{\mathbf{x}_0}](\psi)$, we want to compare $q_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0},\mathrm{G}^{\mathbf{x}_0}](\psi)$ with the term $q_h[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}},\Pi_{\mathbf{x}_0}](\psi) = q_h[\mathbf{A}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\psi)$ obtained by linearizing the potential and the metric.

4.3.1. *Change of metric.* Here we compare L^2 norm and quadratic forms associated with the metric $\mathrm{G}^{\mathbf{x}_0}$, with the corresponding quantities associated with the trivial metric $\mathbb{I}_3$. Like in Proposition 3.13 and Corollary 3.27, and for the same reasons, we have essentially two distinct cases, resulting into a uniform approximation in a polyhedral domain, and a controlled blow up close to conical points when they are present.

Lemma 4.5. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and $(\mathcal{U}_\mathbf{x},\mathrm{U}^\mathbf{x})_{\mathbf{x}\in\overline{\Omega}}$ be an admissible atlas. We recall that the set of reference points $\mathfrak{X}$ contains the set of conical vertices $\mathfrak{V}^\circ$. Let $\mathbf{A} \in W^{1,\infty}(\Omega)$ be a magnetic potential and, for $\mathbf{x}_0 \in \overline{\Omega}$, let $\mathbf{A}^{\mathbf{x}_0}$ be the potential (4.1) produced by the local map $\mathrm{U}^{\mathbf{x}_0}$. There exists $c(\Omega)$ such that*

(a) for all $\mathbf{x}_0 \in \mathfrak{X}$ and $r \in (0, R_{\mathbf{x}_0})$, for all $\psi \in H^1(\Pi_{\mathbf{x}_0})$ satisfying $\operatorname{supp}(\psi) \subset \mathcal{B}(\mathbf{0}, r)$, there holds

$$\begin{aligned}\big|q_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0},\mathrm{G}^{\mathbf{x}_0}](\psi) - q_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\psi)\big| &\le c(\Omega)\, r\, q_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0},\mathrm{G}^{\mathbf{x}_0}](\psi),\\ \big|\|\psi\|_{L^2_{\mathrm{G}^{\mathbf{x}_0}}(\Pi_{\mathbf{x}_0})} - \|\psi\|_{L^2(\Pi_{\mathbf{x}_0})}\big| &\le c(\Omega)\, r\, \|\psi\|_{L^2(\Pi_{\mathbf{x}_0})}\,.\end{aligned}\tag{4.12}$$

(b) for all $\mathbf{u}_0 \in \overline{\Omega}\setminus\mathfrak{X}$ and $r \in (0,\rho(\mathbf{u}_0))$ (with $\rho(\mathbf{u}_0)$ given by Proposition 3.13), for all $\psi \in H^1(\Pi_{\mathbf{u}_0})$ satisfying $\operatorname{supp}(\psi) \subset \mathcal{B}(\mathbf{0}, r)$, there holds

$$\begin{aligned}\big|q_h[\mathbf{A}^{\mathbf{u}_0},\Pi_{\mathbf{u}_0},\mathrm{G}^{\mathbf{u}_0}](\psi) - q_h[\mathbf{A}^{\mathbf{u}_0},\Pi_{\mathbf{u}_0}](\psi)\big| &\le c(\Omega)\,\frac{r}{d_{\mathfrak{V}^\circ}(\mathbf{u}_0)}\, q_h[\mathbf{A}^{\mathbf{u}_0},\Pi_{\mathbf{u}_0},\mathrm{G}^{\mathbf{u}_0}](\psi),\\ \big|\|\psi\|_{L^2_{\mathrm{G}^{\mathbf{u}_0}}(\Pi_{\mathbf{u}_0})} - \|\psi\|_{L^2(\Pi_{\mathbf{u}_0})}\big| &\le c(\Omega)\,\frac{r}{d_{\mathfrak{V}^\circ}(\mathbf{u}_0)}\, \|\psi\|_{L^2(\Pi_{\mathbf{u}_0})}\,,\end{aligned}\tag{4.13}$$

with $d_{\mathfrak{V}^\circ}$ defined in (3.38).

Proof. The lemma is a direct consequence of Corollary 3.27 providing estimates for the L^∞ norm of the difference $\mathrm{G}^{\mathbf{x}_0} - \mathbb{I}_3$. Let $\tau_i = \tau_i(\mathbf{x})$ be the eigenvalues of $\mathrm{G}^{\mathbf{x}_0}(\mathbf{x})$. The estimate on $\mathrm{G}^{\mathbf{x}_0} - \mathbb{I}_3$ implies a similar estimate for $\max\{\|\tau_i - 1\|_{L^\infty}, 1 \le i \le 3\}$, which allows to compare the quadratic forms associated with $\mathrm{G}^{\mathbf{x}_0}$ and with $\mathbb{I}_3$. □

Combining the identities (4.5) with Lemma 4.5, we see that it is equivalent to deal with $q_h[\mathbf{A},\Omega](f)$ or $q_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\psi)$ modulo a well-controlled error. This will be useful later on when we will estimate the corresponding Rayleigh quotients (see Sections 5 and 9).

4.3.2. *Linearization of the potential.* We estimate the remainders due to the linearization $\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}$ at the vertex $\mathbf{0}$ of the tangent cone $\Pi_{\mathbf{x}_0}$ of the potential $\mathbf{A}^{\mathbf{x}_0}$ resulting from a local map. For this, we first use a Taylor expansion around $\mathbf{0}$ in $\Pi_{\mathbf{x}_0}$.

Lemma 4.6. *Let $\mathbf{x}_0 \in \overline{\Omega}$. For any $r > 0$ such that $\mathcal{V}_{\mathbf{x}_0} \supset \mathcal{B}(\mathbf{0}, r)$*

$$\forall \mathbf{v} \in \mathcal{B}(\mathbf{0}, r)\cap\Pi_{\mathbf{x}_0},\quad |\mathbf{A}^{\mathbf{x}_0}(\mathbf{v}) - \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}(\mathbf{v})| \le \tfrac{1}{2}\|\mathbf{A}^{\mathbf{x}_0}\|_{W^{2,\infty}(\mathcal{B}(\mathbf{0},r)\cap\Pi_{\mathbf{x}_0})}\,|\mathbf{v}|^2\,.\tag{4.14}$$

So we have to estimate the second derivatives of the mapped potentials $\mathbf{A}^{\mathbf{x}_0}$.

Lemma 4.7. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ with an associated admissible atlas with set of reference points $\mathfrak{X}$. Let $\mathbf{A} \in W^{2,\infty}(\Omega)$ be a magnetic potential. For $\mathbf{x}_0 \in \overline{\Omega}$, let $\mathbf{A}^{\mathbf{x}_0}$ be the potential (4.1). There exists $c(\Omega)$ such that*

(a) for all $\mathbf{x}_0 \in \mathfrak{X}$,

$$\|\mathrm{d}^2\mathbf{A}^{\mathbf{x}_0}(\mathbf{v})\| \le c(\Omega)\|\mathbf{A}\|_{W^{2,\infty}(\Omega)}, \qquad \forall \mathbf{v} \in \mathcal{B}(\mathbf{0}, R_{\mathbf{x}_0}). \tag{4.15}$$

(b) for all $\mathbf{u}_0 \in \overline{\Omega} \setminus \mathfrak{X}$, with $\rho(\mathbf{u}_0)$ given in Proposition 3.13 and $d_{\mathfrak{Y}^\circ}$ defined in (3.38),

$$\|\mathrm{d}^2\mathbf{A}^{\mathbf{u}_0}(\mathbf{v})\| \le c(\Omega)\left(\frac{\|\mathbf{A}\|_{W^{1,\infty}(\Omega)}}{d_{\mathfrak{Y}^\circ}(\mathbf{u}_0)} + \|\mathbf{A}\|_{W^{2,\infty}(\Omega)}\right), \qquad \forall \mathbf{v} \in \mathcal{B}(\mathbf{0}, \rho(\mathbf{u}_0)). \tag{4.16}$$

Proof. Let $\mathbf{u}_0 \in \overline{\Omega}$. Differentiating twice (4.1), we obtain, for $\mathbf{u} \in \mathcal{U}_{\mathbf{x}_0}$ and $\mathbf{v} = \mathrm{U}^{\mathbf{u}_0}(\mathbf{u})$,

$$\|\mathrm{d}^2\mathbf{A}^{\mathbf{u}_0}(\mathbf{v})\| \lesssim \|\mathrm{d}\mathrm{K}^{\mathbf{u}_0}(\mathbf{v})\|\,|\mathbf{A}(\mathbf{u}) - \mathbf{A}(\mathbf{u}_0)| + \|\mathrm{K}^{\mathbf{u}_0}(\mathbf{v})\|\,\|\mathrm{J}^{\mathbf{u}_0}(\mathbf{v})\|\,\|\mathrm{d}\mathbf{A}(\mathbf{u})\| + \|\mathrm{J}^{\mathbf{u}_0}(\mathbf{v})\|^3\,\|\mathrm{d}^2\mathbf{A}(\mathbf{u})\|.$$

(a) When $\mathbf{u}_0 = \mathbf{x}_0 \in \mathfrak{X}$, (4.15) is a consequence of Proposition 3.13 and Remark 3.15 (1).

(b) Let $\mathbf{u}_0 \in \overline{\Omega} \setminus \mathfrak{X}$ and $\mathbf{x}_0 \in \mathfrak{X}$ such that $\mathbf{u}_0 \in \mathcal{U}_{\mathbf{x}_0}$. The above inequality, Proposition 3.13 and Remark 3.15 (2) yield for $\mathbf{v} \in \mathcal{B}(\mathbf{0}, \rho(\mathbf{u}_0))$,

$$\begin{aligned}\|\mathrm{d}^2\mathbf{A}^{\mathbf{u}_0}(\mathbf{v})\| &\lesssim \frac{|\mathbf{u} - \mathbf{u}_0|}{|\mathbf{u}_0 - \mathbf{x}_0|^2}\|\mathbf{A}\|_{W^{1,\infty}} + \frac{1}{|\mathbf{u}_0 - \mathbf{x}_0|}\|\mathbf{A}\|_{W^{1,\infty}} + \|\mathbf{A}\|_{W^{2,\infty}}\\ &\lesssim \frac{1}{|\mathbf{u}_0 - \mathbf{x}_0|}\|\mathbf{A}\|_{W^{1,\infty}} + \|\mathbf{A}\|_{W^{2,\infty}}.\end{aligned}$$

Here we have used the inequality $|\mathbf{u} - \mathbf{u}_0| \le |\mathbf{u}_0 - \mathbf{x}_0|$ which holds by construction of the admissible atlas. $\square$

Estimates between $\mathbf{A}^{\mathbf{x}_0}$ and $\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}$ deduced from the combination of Lemmas 4.6 and 4.7 allow to compare $q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi)$ and $q_h[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}](\psi)$ via identity (A.6) which writes

$$q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi) = q_h[\mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}}, \Pi_{\mathbf{x}_0}](\psi) + 2\,\mathrm{Re}\big\langle(-ih\nabla + \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}})\psi, (\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}})\psi\big\rangle + \|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}})\psi\|^2.$$

This will be extensively used in Sections 5 and 9.

4.4. A general rough upper bound.

As a first consequence of a weaker form of Lemmas 4.5 and 4.7, we are going to prove a very general rough upper bound for the Rayleigh quotients $\mathscr{Q}_h[\mathbf{A}, \Omega]$ (1.19) as $h \to 0$. In fact this reasoning holds in a natural way for n-dimensional corner domains. In the n-dimensional case, the magnetic field is a 2-form and associated magnetic potentials are 1-forms that we write by using their representation as vector fields in a canonical basis of $\mathbb{R}^n$, see (1.2)–(1.3). In dimension n, $E(\mathbf{B}, \Pi)$ and $\mathscr{E}(\mathbf{B}, \Omega)$ are defined as in Definition 4.4.

In this context we prove a rough upper bound on the first eigenvalue of $H_h(\mathbf{A}, \Omega)$ by using only elementary arguments. We need the following Lemma, that will also be useful later:

Lemma 4.8. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^n)$ and let $\mathbf{A} \in W^{2,\infty}(\overline{\Omega})$ be a twice differentiable magnetic potential associated with the magnetic field $\mathbf{B}$. Let $\mathbf{x}_0 \in \overline{\Omega}$ be a chosen point and let $\varepsilon > 0$. Then there exists $h_0 > 0$ such that for all $h \in (0, h_0)$ there exists a function f_h supported near $\mathbf{x}_0$ satisfying*

$$\mathscr{Q}_h[\mathbf{A}, \Omega](f_h) \le h\big(E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \varepsilon\big)\,,$$

where $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ *is the ground state energy of* $H(\mathbf{A}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$.

Proof. Let $(\mathcal{U}_{\mathbf{x}_0}, \mathrm{U}^{\mathbf{x}_0})$ be a local map with $\mathrm{U}^{\mathbf{x}_0} : \mathcal{U}_{\mathbf{x}_0} \mapsto \mathcal{V}_{\mathbf{x}_0} \subset \Pi_{\mathbf{x}_0}$, cf. (3.1). This change of variables transforms the magnetic potential into $\mathbf{A}^{\mathbf{x}_0}$ given by (4.1):

$$\mathbf{A}^{\mathbf{x}_0} = (\mathrm{J}^{\mathbf{x}_0})^\top \big((\mathbf{A} - \mathbf{A}(\mathbf{x}_0)) \circ (\mathrm{U}^{\mathbf{x}_0})^{-1}\big)\,.$$

Denote by $\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}$ its linear part. Recall that $\operatorname{curl} \mathbf{A}_\mathbf{0}^{\mathbf{x}_0} = \mathbf{B}_{\mathbf{x}_0}$. By definition of $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ there exists $\psi \in \mathrm{Dom}(q[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}])$ a L^2-normalized function such that

$$q[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi) \le E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \tfrac{\varepsilon}{4}\,.$$

Let us consider a smooth cut-off function χ with support in $\mathcal{B}(\mathbf{0}, 1)$ and equal to 1 on $\mathcal{B}(\mathbf{0}, \frac12)$. Then the functions with compact support $\mathbf{x} \longmapsto \chi(\frac{\mathbf{x}}{R})\,\psi(\mathbf{x})$ converge to ψ in $\mathrm{Dom}(q[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}])$ as $R \to \infty$. Therefore there exists $R = R(\varepsilon, \mathbf{x}_0) > 0$ and a new function $\psi \in \mathrm{Dom}(q[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}])$ with support in $\mathcal{B}(\mathbf{0}, R)$ which satisfies

$$q[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi) \le E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \tfrac{\varepsilon}{2}\,.$$

For $h > 0$, define the L^2-normalized function $\psi_h(\mathbf{x}) = h^{-n/4}\psi(h^{-1/2}\mathbf{x})$ so that, cf. Lemma A.4,

$$q_h[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h) \le h\big(E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) + \tfrac{\varepsilon}{2}\big)\,.$$

We have the inclusion $\mathrm{supp}(\psi_h) \subset \mathcal{B}(\mathbf{0}, h^{1/2}R)$ and therefore there exists $h_\varepsilon > 0$ such that for all $h \in (0, h_\varepsilon)$, $\mathrm{supp}(\psi_h) \subset \mathcal{V}_{\mathbf{x}_0}$. Combining (A.6) with a Cauchy-Schwarz inequality we find

$$\begin{aligned} q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h) \le{}& q_h[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h) \\ &+ 2\sqrt{q_h[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h)}\, \|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_\mathbf{0}^{\mathbf{x}_0})\psi_h\| + \|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_\mathbf{0}^{\mathbf{x}_0})\psi_h\|^2. \end{aligned} \tag{4.17}$$

Notice now that the estimates *(a)* of Proposition 3.13 are still valid for any chosen $\mathbf{x}_0$ in $\overline{\Omega}$ with constants $c(\mathbf{x}_0)$ and radius $R_{\mathbf{x}_0}$ depending on $\mathbf{x}_0$. Hence estimates *(a)* of Lemma 4.7 holds at $\mathbf{x}_0$ with a constant $c(\mathbf{x}_0)$ replacing the uniform constant $c(\Omega)$. Therefore applying Lemma 4.6 with $r = h^{1/2}R$ we get $c = c(\varepsilon, \mathbf{x}_0) > 0$ such that

$$\|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_\mathbf{0}^{\mathbf{x}_0})\psi_h\| \le cR^2h\|\psi_h\|, \quad \forall h \in (0, h_\varepsilon).$$

Let $\mathrm{G}^{\mathbf{x}_0}$ be the metric associated with the change of variables (see Section 4.1). Again *(a)* of Lemma 4.5 is valid for all $\mathbf{x}_0 \in \overline{\Omega}$ with $c(\mathbf{x}_0)$ instead of $c(\Omega)$. Applying this with $r = h^{1/2}R$ provides another constant $c = c(\varepsilon, \mathbf{x}_0) > 0$ such that

$$\Big|q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi_h) - q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h)\Big| \le c\,Rh^{1/2}\, q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\psi_h), \tag{4.18}$$

$$\Big|\|\psi_h\|_{L^2_{\mathrm{G}^{\mathbf{x}_0}}(\Pi_{\mathbf{x}_0})} - \|\psi_h\|_{L^2(\Pi_{\mathbf{x}_0})}\Big| \le c\,Rh^{1/2}\,\|\psi_h\|_{L^2(\Pi_{\mathbf{x}_0})}\,. \tag{4.19}$$

According to Section 4.1 (4.1)–(4.5), we define for $h \in (0, h_\varepsilon)$:

$$f_h := (\zeta_h^{\mathbf{x}_0})^{-1}\,\psi_h \circ \mathrm{U}^{\mathbf{x}_0} \qquad \text{with} \qquad \zeta_h^{\mathbf{x}_0}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}(\mathbf{x}_0), \mathbf{x}\rangle/h}, \quad \mathbf{x} \in \mathcal{U}_{\mathbf{x}_0} \cap \overline{\Omega}$$

and we have

$$q_h[\mathbf{A}, \Omega](f_h) = q_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}, \mathrm{G}^{\mathbf{x}_0}](\psi_h) \quad \text{and} \quad \|f_h\|_{L^2(\Omega)} = \|\psi_h\|_{L^2_{\mathrm{G}^{\mathbf{x}_0}}(\Pi_{\mathbf{x}_0})}.$$

Thus, combining with (4.17)–(4.19) we deduce

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A},\Omega](f_h) &\leq (1+cRh^{1/2})\Big(\mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\psi_h)+c\left(R^2h^{3/2}+R^4h^2\right)\Big)\\ &\leq (1+cRh^{1/2})\Big(h\big(E(\mathbf{B}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0})+\tfrac{\varepsilon}{2}\big)+c\left(R^2h^{3/2}+R^4h^2\right)\Big).\end{aligned}$$

We can write this in the form

$$\mathscr{Q}_h[\mathbf{A},\Omega](f_h)\leq h\big(E(\mathbf{B}_{\mathbf{x}_0},\Pi_{\mathbf{x}_0})+\tfrac{\varepsilon}{2}+h^{1/2}M_\varepsilon(h)\big),$$

where $M_\varepsilon(h)$ is a bounded function for $h\in[0,h_\varepsilon]$ that depends on $\varepsilon>0$. We deduce the lemma by choosing h so small that $h^{1/2}M_\varepsilon(h)\leq\frac{\varepsilon}{2}$. □

As a consequence of Lemma 4.8 and the min-max principle we obtain:

Proposition 4.9. *Let $\Omega\in\mathfrak{D}(\mathbb{R}^n)$ and let $\mathbf{A}\in W^{2,\infty}(\overline{\Omega})$ be a magnetic potential associated with the magnetic field $\mathbf{B}$. Then the first eigenvalue $\lambda_h(\mathbf{B},\Omega)$ of $H(\mathbf{A},\Omega)$ satisfies*

$$\limsup_{h\to0}\frac{\lambda_h(\mathbf{B},\Omega)}{h}\leq\mathscr{E}(\mathbf{B},\Omega)\,.$$

5. Lower bounds for ground state energy in corner domains

In this section we establish a lower bound for the first eigenvalue $\lambda_h(\mathbf{B},\Omega)$ of the magnetic Laplacian $H_h(\mathbf{A},\Omega)$ with Neumann boundary conditions.

Theorem 5.1. *Let $\Omega\in\mathfrak{D}(\mathbb{R}^3)$ be a corner domain, and let $\mathbf{A}\in W^{2,\infty}(\overline{\Omega})$ be a twice differentiable magnetic potential. Then there exist $C_\Omega>0$ and $h_0>0$ such that for all $h\in(0,h_0)$ there holds*

$$\lambda_h(\mathbf{B},\Omega)\geq\begin{cases}h\mathscr{E}(\mathbf{B},\Omega)-C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\,h^{11/10}, & \Omega\text{ general corner domain,}\\ h\mathscr{E}(\mathbf{B},\Omega)-C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\,h^{5/4}, & \Omega\text{ polyhedral domain.}\end{cases}\tag{5.1}$$

We recall that the quantity $\mathscr{E}(\mathbf{B},\Omega)$ is the lowest local energy defined in (4.11).

Remark 5.2. If the magnetic field $\mathbf{B}$ vanishes, then $\mathscr{E}(\mathbf{B},\Omega)=0$ and Theorem 5.1 is obvious. In contrast, if $\mathbf{B}$ does not vanish on $\overline{\Omega}$, we will see in Corollary 8.5 that $\mathscr{E}(\mathbf{B},\Omega)>0$.

• *Structure of the proof.* The proof proceeds from an IMS partition argument coupled with the analysis of remainders due to the cut-off effects, the local maps and the linearization of the potential. The less classical piece of the analysis is our special construction of cut-off functions in regions close to conical points $\mathbf{x}_0\in\mathfrak{V}^\circ$, where a second, smaller, scale is introduced.

We choose first an admissible atlas on $\overline{\Omega}$ according to Definition 3.11 and we recall that the conical points are part of the set $\mathfrak{X}$ of its reference points.

• *Splitting off the conical points.* We start with a (smooth) macro partition of unity on $\overline{\Omega}$, independent of h, $(\Xi_0, (\Xi_{\mathbf{x}})_{\mathbf{x}\in\mathfrak{V}^\circ})$ which aims at separating the conical points, *i.e.*, such that

- $\overline{\operatorname{supp}\Xi_0} \cap \mathfrak{V}^\circ = \emptyset$,
- for any $\mathbf{x}_0 \in \mathfrak{V}^\circ$, $\operatorname{supp}\Xi_{\mathbf{x}_0} \subset \mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$.

Here $R_{\mathbf{x}_0}$ is the radius associated with the reference point $\mathbf{x}_0$ in the admissible atlas. In the polyhedral case, *i.e.*, when $\mathfrak{V}^\circ = \emptyset$, we simply set $\Xi_0 \equiv 1$.

For any $f \in H^1(\Omega)$ IMS formula (see Lemma A.5) gives

$$q_h[\mathbf{A},\Omega](f) = q_h[\mathbf{A},\Omega](\Xi_0 f) + \sum_{\mathbf{x}\in\mathfrak{V}^\circ} q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}} f) - h^2\Big(\|(\nabla\Xi_0)f\|^2 + \sum_{\mathbf{x}\in\mathfrak{V}^\circ}\|(\nabla\Xi_{\mathbf{x}})f\|^2\Big)$$

$$\geq q_h[\mathbf{A},\Omega](\Xi_0 f) + \sum_{\mathbf{x}_0\in\mathfrak{V}^\circ} q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}} f) - Ch^2\|f\|^2. \tag{5.2}$$

In Section 5.1, we give a lower bound of $q_h[\mathbf{A},\Omega](\Xi_0 f)$. In the polyhedral case, this will finish the proof. Section 5.2 is devoted to conical points and estimates of $q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}} f)$.

5.1. **Estimates outside conical points.** Here we prove a lower bound for $q_h[\mathbf{A},\Omega](\Xi_0 f)$.

• *IMS localization.* Let $\delta \in (0, \frac{1}{2})$ be an exponent which will be determined later on. Now, we make a h-dependent partition of $\operatorname{supp}\Xi_0 \cap \overline{\Omega}$ with size h^δ. Relying on Lemma B.1, we can choose for $0 < h \leq h_0$ (h_0 small enough) a finite set $\mathscr{C}(h)$ of points $\mathbf{c} \in \overline{\Omega}$ together with radii $\rho_{\mathbf{c}}$ equivalent to h^δ (with uniformity as $h \to 0$) such that

(1) The union of balls $\mathcal{B}(\mathbf{c}, \rho_{\mathbf{c}})$ covers $\operatorname{supp}\Xi_0 \cap \overline{\Omega}$
(2) Each ball $\mathcal{B}(\mathbf{c}, 2\rho_{\mathbf{c}})$ is contained in a map-neighborhood of the admissible atlas
(3) The finite covering condition holds

Relying on Lemma B.2, we choose an associate partition of unity $\big(\xi_{\mathbf{c}}\big)_{\mathbf{c}\in\mathscr{C}(h)}$ such that

$$\xi_{\mathbf{c}} \in \mathscr{C}_0^\infty(\mathcal{B}(\mathbf{c},\rho_{\mathbf{c}})), \quad \forall \mathbf{c}\in\mathscr{C}(h) \qquad \text{and} \qquad \Xi_0 \sum_{\mathbf{c}\in\mathscr{C}(h)} \xi_{\mathbf{c}}^2 = \Xi_0 \quad \text{on} \quad \overline{\Omega},$$

and satisfying the uniform estimate of gradients

$$\exists C>0, \quad \forall h\in(0,h_0), \quad \forall \mathbf{c}\in\mathscr{C}(h), \quad \|\nabla\xi_{\mathbf{c}}\|_{L^\infty(\Omega)} \leq Ch^{-\delta}. \tag{5.3}$$

The IMS formula (see Lemma A.5) provides for all $f \in H^1(\Omega)$

$$q_h[\mathbf{A},\Omega](\Xi_0 f) = \sum_{\mathbf{c}\in\mathscr{C}(h)} q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\,\Xi_0 f) - h^2 \sum_{\mathbf{c}\in\mathscr{C}(h)} \|\nabla\xi_{\mathbf{c}}\,\Xi_0 f\|^2_{L^2(\Omega)}$$

and using (5.3) we get $C = C(\Omega) > 0$ such that

$$q_h[\mathbf{A},\Omega](\Xi_0 f) \geq \sum_{\mathbf{c}\in\mathscr{C}(h)} q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\Xi_0 f) - C\,h^{2-2\delta}\|\Xi_0 f\|^2_{L^2(\Omega)}. \tag{5.4}$$

• *Local control of the energy.* For each center $\mathbf{c} \in \mathscr{C}(h)$, we are going to bound from below the term $q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\,\Xi_0 f)$ appearing in (5.4). By construction $\operatorname{supp}(\xi_{\mathbf{c}}\,\Xi_0 f)$ is contained in the map-neighborhood $\mathcal{U}_{\mathbf{c}}$. Using (4.2) and (4.4), we set

$$\psi_{\mathbf{c}} := (\zeta_h^{\mathbf{c}}\,\xi_{\mathbf{c}}\,\Xi_0 f)\circ(\mathrm{U}^{\mathbf{c}})^{-1}, \qquad \text{with} \qquad \zeta_h^{\mathbf{c}}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}(\mathbf{c}),\mathbf{x}\rangle/h}. \tag{5.5}$$

According to (4.5) with $\mathbf{x}_0$ replaced by $\mathbf{c}$, we have

$$q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\Xi_0 f) = q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}},\mathrm{G}^{\mathbf{c}}](\psi_{\mathbf{c}}) \quad \text{and} \quad \|\xi_{\mathbf{c}}\,\Xi_0 f\|_{L^2(\Omega)} = \|\psi_{\mathbf{c}}\|_{L^2_{\mathrm{G}^{\mathbf{c}}}(\Pi_{\mathbf{c}})}\,. \tag{5.6}$$

In order to replace the metric $\mathrm{G}^{\mathbf{c}}$ by the identity, we apply Lemma 4.5 with $r \simeq h^\delta$. Using that the distance $d_{\mathfrak{V}^\circ}$ to conical points is bounded from below by a positive number on $\operatorname{supp}\Xi_0$, we obtain the existence of a constant $c(\Omega)>0$ such that for all centers $\mathbf{c}\in\mathscr{C}(h)$

$$\mathscr{Q}_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}},\mathrm{G}^{\mathbf{c}}](\psi_{\mathbf{c}}) \geq (1-c(\Omega)h^\delta)\mathscr{Q}_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}})\,. \tag{5.7}$$

We now want to replace $\mathbf{A}^{\mathbf{c}}$ in the above Rayleigh quotient by its linear part $\mathbf{A}_{\mathbf{0}}^{\mathbf{c}}$ at $\mathbf{0}$. For this we use identity (A.6) with $\psi = \psi_{\mathbf{c}}$ and $\mathcal{O} = \Pi_{\mathbf{c}}$:

$$\begin{aligned} q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) = q_h&[\mathbf{A}_{\mathbf{0}}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \\ &+ 2\,\mathrm{Re}\,\big\langle(-ih\nabla+\mathbf{A}_{\mathbf{0}}^{\mathbf{c}})\psi_{\mathbf{c}},(\mathbf{A}^{\mathbf{c}}-\mathbf{A}_{\mathbf{0}}^{\mathbf{c}})\psi_{\mathbf{c}}\big\rangle + \|(\mathbf{A}^{\mathbf{c}}-\mathbf{A}_{\mathbf{0}}^{\mathbf{c}})\psi_{\mathbf{c}}\|^2. \end{aligned} \tag{5.8}$$

This yields $q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \geq q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) - 2\,(q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}))^{1/2}\,\|(\mathbf{A}^{\mathbf{c}}-\mathbf{A}_{\mathbf{0}}^{\mathbf{c}})\psi_{\mathbf{c}}\|$ by Cauchy-Schwarz inequality, leading to the parametric estimate (based on inequality $2ab\leq \eta a^2+\eta^{-1}b^2$)

$$\forall \eta>0, \quad q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \geq (1-\eta)q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) - \eta^{-1}\|(\mathbf{A}^{\mathbf{c}}-\mathbf{A}_{\mathbf{0}}^{\mathbf{c}})\psi_{\mathbf{c}}\|^2\,. \tag{5.9}$$

Since $\operatorname{curl}\mathbf{A}_{\mathbf{0}}^{\mathbf{c}} = \mathbf{B}_{\mathbf{c}}$, we have the lower bound by the minimum local energy at $\mathbf{c}$:

$$q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \geq h\,E(\mathbf{B}_{\mathbf{c}},\Pi_{\mathbf{c}})\|\psi_{\mathbf{c}}\|^2 \tag{5.10}$$

$$\geq h\mathscr{E}(\mathbf{B},\Omega)\|\psi_{\mathbf{c}}\|^2\,. \tag{5.11}$$

According to Lemmas 4.6 and 4.7 (note that $d_{\mathfrak{V}^\circ}\geq r_0>0$ on $\operatorname{supp}\Xi_0$), we have

$$\|(\mathbf{A}^{\mathbf{c}}-\mathbf{A}_{\mathbf{0}}^{\mathbf{c}})\psi_{\mathbf{c}}\| \leq c(\Omega)\|\mathbf{A}\|_{W^{2,\infty}(\Omega)}h^{2\delta}\|\psi_{\mathbf{c}}\|\,. \tag{5.12}$$

Combining (5.9)–(5.12) we deduce for all $\eta>0$:

$$q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \geq (1-\eta)h\mathscr{E}(\mathbf{B},\Omega)\|\psi_{\mathbf{c}}\|^2 - \eta^{-1}h^{4\delta}c(\Omega)^2\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\|\psi_{\mathbf{c}}\|^2.$$

Choosing $\eta = h^{2\delta-\frac12}$ to equilibrate ηh and $\eta^{-1}h^{4\delta}$, we get the following lower bound

$$q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \geq \Big(h\mathscr{E}(\mathbf{B},\Omega) - C_\Omega\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{2\delta+\frac12}\Big)\,\|\psi_{\mathbf{c}}\|^2, \quad \forall \mathbf{c}\in\mathscr{C}(h). \tag{5.13}$$

• *Conclusion.* Combining the previous localized estimate (5.13) with (5.7) we deduce:

$$q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\,\Xi_0 f) \geq \Big(h\mathscr{E}(\mathbf{B},\Omega) - C_\Omega(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2\delta+\frac12}+h^{1+\delta})\Big)\,\|\xi_{\mathbf{c}}\,\Xi_0 f\|^2. \tag{5.14}$$

Summing up in $\mathbf{c}\in\mathscr{C}(h)$, we obtain

$$\frac{\sum_{\mathbf{c}\in\mathscr{C}(h)} q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\,\Xi_0 f)}{\|\Xi_0 f\|^2_{L^2(\Omega)}} \geq h\mathscr{E}(\mathbf{B},\Omega) - C_\Omega(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2\delta+\frac12}+h^{1+\delta}). \tag{5.15}$$

Using (5.4), we get another constant $C_\Omega > 0$ such that for all $f \in H^1(\Omega)$,

$$\mathscr{Q}_h[\mathbf{A},\Omega](\Xi_0 f) \geq h\mathscr{E}(\mathbf{B},\Omega) - C_\Omega(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})\big(h^{2\delta+\frac12} + h^{1+\delta} + h^{2-2\delta}\big). \tag{5.16}$$

In the polyhedral case, $\Xi_0 \equiv 1$ and the remainders are optimized by taking $\delta = \frac38$ in (5.16), which implies Theorem 5.1 in this case.

5.2. **Estimates near conical points.** Let $\mathbf{x}_0 \in \mathfrak{V}^\circ$. We estimate $q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}_0} f)$ from below.

• *IMS partition.* For $h > 0$ small enough we construct a special covering of the support of $\Xi_{\mathbf{x}_0}$. We recall that this support is included in the ball $\mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0})$. We cover $\mathcal{B}(\mathbf{x}_0, R_{\mathbf{x}_0}) \cap \overline{\Omega}$ by a finite collection of h-dependent balls $\mathcal{B}(\mathbf{c},\rho_\mathbf{c})$:

- The first ball is centered at $\mathbf{x}_0$ itself and its radius is $2h^{\delta_0}$: $\mathcal{B}(\mathbf{c},\rho_\mathbf{c}) = \mathcal{B}(\mathbf{x}_0, 2h^{\delta_0})$. Here the exponent $\delta_0 \in (0,\frac12)$ will be chosen later on.
- The other balls $\mathcal{B}(\mathbf{c},\rho_\mathbf{c})$ cover the annular region $h^{\delta_0} \leq |\mathbf{x}-\mathbf{x}_0| < R_{\mathbf{x}_0}$ and their radii are $\simeq h^{\delta_0+\delta_1}$ where the new exponent $\delta_1 > 0$ is such that $\delta_0+\delta_1 < \frac12$ and will be also chosen later on. Thanks to Lemma B.1 the set $\mathscr{C}(h,\mathbf{x}_0)$ of the centers and the corresponding radii can be taken so that the conditions of this lemma are satisfied (inclusion in map-neighborhoods, finite covering), see previous case § 5.1.

So this covering contains a "large" ball centered at the corner and a whole bunch of smaller ones covering the remaining part.

Relying on Lemma B.2, we choose an associate partition of unity $\big(\xi_\mathbf{c}\big)_{\mathbf{c}\in\{\mathbf{x}_0\}\cup\mathscr{C}(h,\mathbf{x}_0)}$ such that

$$\xi_\mathbf{c} \in \mathscr{C}^\infty_0(\mathcal{B}(\mathbf{c},\rho_\mathbf{c})),\ \ \forall \mathbf{c} \in \{\mathbf{x}_0\}\cup\mathscr{C}(h,\mathbf{x}_0), \qquad \text{and} \qquad \Xi_{\mathbf{x}_0}\sum_{\mathbf{c}\in\{\mathbf{x}_0\}\cup\mathscr{C}(h,\mathbf{x}_0)} \xi^2_\mathbf{c} = \Xi_{\mathbf{x}_0} \quad \text{on} \quad \overline{\Omega},$$

and satisfying the following uniform estimate of gradients for all $h \in (0,h_0)$:

$$\text{for } \ \mathbf{c}=\mathbf{x}_0,\ \ \|\nabla\xi_\mathbf{c}\|_{L^\infty(\Omega)} \leq Ch^{-\delta_0} \quad \text{and} \quad \forall \mathbf{c}\in\mathscr{C}(h,\mathbf{x}_0),\ \ \|\nabla\xi_\mathbf{c}\|_{L^\infty(\Omega)} \leq Ch^{-\delta_0-\delta_1}\,. \tag{5.17}$$

Using the IMS formula (see Lemma A.5), we have like previously in (5.4)

$$q_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}_0}f) \geq q_h[\mathbf{A},\Omega](\xi_{\mathbf{x}_0}\,\Xi_{\mathbf{x}_0}f) + \sum_{\mathbf{c}\in\mathscr{C}(h,\mathbf{x}_0)} q_h[\mathbf{A},\Omega](\xi_\mathbf{c}\,\Xi_{\mathbf{x}_0}f) - Ch^{2-2(\delta_0+\delta_1)}\|\Xi_{\mathbf{x}_0}f\|^2. \tag{5.18}$$

• *Local control of the energy.* When $\mathbf{c}=\mathbf{x}_0$, we can proceed in the same way as in the polyhedral case due to the "good" estimates stated in Lemma 4.5 (a) and Lemma 4.7 (a). So we obtain a similar estimate as in (5.14): There exists a constant $C = C(\Omega)$ such that for any function $f \in H^1(\Omega)$

$$q_h[\mathbf{A},\Omega](\xi_{\mathbf{x}_0}\Xi_{\mathbf{x}_0}f) \geq \Big(h\mathscr{E}(\mathbf{B},\Omega) - C(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{2\delta_0+\frac12}+h^{1+\delta_0})\Big)\,\|\xi_{\mathbf{x}_0}\Xi_{\mathbf{x}_0}f\|^2. \tag{5.19}$$

When $\mathbf{c}\in\mathscr{C}(h,\mathbf{x}_0)$, we have to revisit the arguments leading from (5.5) to the final individual estimate (5.14). First we define $\psi_\mathbf{c}$ like in (5.5), replacing the cut-off Ξ_0 by $\Xi_{\mathbf{x}_0}$. Then we have (5.6) *mutatis mutandis*. Next we have to use Lemma 4.5 (b) with $\mathbf{u}_0 = \mathbf{c}$ to flatten the metric. Here we have to take the distance $d_{\mathfrak{V}^\circ}(\mathbf{c})$ to conical points into account. By construction $d_{\mathfrak{V}^\circ}(\mathbf{c})$

coincides with $|\mathbf{c}-\mathbf{x}_0|$, so is larger than h^{δ_0}, while the quantity r equals $\rho_{\mathbf{c}}$, thus is $\lesssim h^{\delta_0+\delta_1}$: In short

$$\frac{r}{d_{\mathfrak{Y}^\circ}(\mathbf{c})} = \frac{\rho_{\mathbf{c}}}{|\mathbf{c}-\mathbf{x}_0|} \lesssim h^{\delta_1}.$$

Hence, we obtain in place of (5.7):

$$\mathscr{Q}_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}},\mathrm{G}^{\mathbf{c}}](\psi_{\mathbf{c}}) \geq (1-c(\Omega)h^{\delta_1})\mathscr{Q}_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}})\,. \tag{5.20}$$

For the linearization of the potential $\mathbf{A}^{\mathbf{c}}$, the expressions (5.8)–(5.11) are still valid, leading to the parametric estimate

$$\forall\eta>0,\quad q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \geq (1-\eta)h\mathscr{E}(\mathbf{B},\Omega)\|\psi_{\mathbf{c}}\|^2 - \eta^{-1}\|(\mathbf{A}^{\mathbf{c}}-\mathbf{A}^{\mathbf{c}}_{\mathbf{0}})\psi_{\mathbf{c}}\|^2\,. \tag{5.21}$$

Here we use Lemmas 4.6 and 4.7 (b) and obtain, since $\rho_{\mathbf{c}} \lesssim h^{\delta_0+\delta_1}$ and $d_{\mathfrak{Y}^\circ}(\mathbf{c}) \geq h^{\delta_0}$

$$\|(\mathbf{A}^{\mathbf{c}}-\mathbf{A}^{\mathbf{c}}_{\mathbf{0}})\psi_{\mathbf{c}}\| \leq c(\Omega)\frac{\rho_{\mathbf{c}}^2}{d_{\mathfrak{Y}^\circ}(\mathbf{c})}\|\mathbf{A}\|_{W^{2,\infty}(\Omega)}\|\psi_{\mathbf{c}}\| \leq c(\Omega)h^{\delta_0+2\delta_1}\|\mathbf{A}\|_{W^{2,\infty}(\Omega)}\|\psi_{\mathbf{c}}\|\,. \tag{5.22}$$

Combining (5.21) with (5.22) and taking $\eta = h^{\delta_0+2\delta_1-\frac12}$ we deduce

$$q_h[\mathbf{A}^{\mathbf{c}},\Pi_{\mathbf{c}}](\psi_{\mathbf{c}}) \geq \Big(h\mathscr{E}(\mathbf{B},\Omega) - C(\Omega)\big(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{\delta_0+2\delta_1+\frac12}\Big)\|\psi_{\mathbf{c}}\|^2,\quad \forall\mathbf{c}\in\mathscr{C}(h,\mathbf{x}_0), \tag{5.23}$$

and then with (5.20) (and (5.6) with $\Xi_{\mathbf{x}_0}$)

$$q_h[\mathbf{A},\Omega](\xi_{\mathbf{c}}\Xi_{\mathbf{x}_0}f) \geq \Big(h\mathscr{E}(\mathbf{B},\Omega) - C(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})(h^{\delta_0+2\delta_1+\frac12}+h^{1+\delta_1})\Big)\|\xi_{\mathbf{c}}\Xi_{\mathbf{x}_0}f\|^2. \tag{5.24}$$

Summing up (5.19) and (5.24) for $\mathbf{c}\in\mathscr{C}(h,\mathbf{x}_0)$, and combining with the IMS formula, we deduce

$$\mathscr{Q}_h[\mathbf{A},\Omega](\Xi_{\mathbf{x}_0}f) \geq h\mathscr{E}(\mathbf{B},\Omega) - C(h^{2\delta_0+\frac12}+h^{1+\delta_0}+h^{\frac12+\delta_0+2\delta_1}+h^{1+\delta_1}+h^{2-2(\delta_0+\delta_1)}), \tag{5.25}$$

with $C = c(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})$.

• *Conclusion.* Combining (5.2), (5.16) and (5.25), we deduce

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A},\Omega](f) \geq h\mathscr{E}(\mathbf{B},\Omega) - Ch^2 - C\left(h^{2\delta+\frac12}+h^{1+\delta}+h^{2-2\delta}\right)\\ - C\left(h^{2\delta_0+\frac12}+h^{1+\delta_0}+h^{\frac12+\delta_0+2\delta_1}+h^{1+\delta_1}+h^{2-2(\delta_0+\delta_1)}\right),\end{aligned} \tag{5.26}$$

with $C = c(\Omega)(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})$.

Remind that the error with power δ_0 and δ_1 only appears when Ω has conical points. To optimize the remainder, we first choose $\delta = 3/8$. We have now to optimize parameters δ_0,δ_1 under the constraints $0<\delta_0+\delta_1<\frac12$, $\delta_0>0$, $\delta_1>0$. We have

$$\min(1+\delta_0,\tfrac12+2\delta_0) = \tfrac12+2\delta_0,$$

and

$$\min(1+\delta_1,\tfrac12+\delta_0+2\delta_1) = \tfrac12+\delta_0+2\delta_1.$$

We are reduced to solve

$$\begin{cases}\frac12+2\delta_0 = \frac12+\delta_0+2\delta_1\\ \frac12+2\delta_0 = 2-2\delta_0-2\delta_1\end{cases} \iff \begin{cases}2\delta_1=\delta_0\\ \frac32 = 4\delta_0+2\delta_1\end{cases} \iff \delta_0 = \frac{3}{10}\text{ and }\delta_1=\frac{3}{20}.$$

Then we get $C(\Omega) > 0$ such that

$$\forall f \in H^1(\Omega), \quad \mathscr{Q}_h[\mathbf{A},\Omega](f) \geq h\mathscr{E}(\mathbf{B},\Omega) - C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{11/10}. \tag{5.27}$$

For further use we extract the following corollary of the previous proof:

Corollary 5.3. *Let $\mathbf{x}_0 \in \overline{\Omega}$ and $K := \mathcal{B}(\mathbf{x}_0, \delta)$ with $\delta > 0$. We define*

$$\mathscr{E}_K(\mathbf{B},\Omega) := \inf_{\mathbf{x}\in\overline{\Omega}\cap\overline{K}} E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})\,.$$

Then there exists $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$ and for all $f \in \mathrm{Dom}(q_h[\mathbf{A},\Omega])$ with support $\operatorname{supp} f \subset\subset K$, there holds

$$\mathscr{Q}_h[\mathbf{A},\Omega](f) \geq h\mathscr{E}_K(\mathbf{B},\Omega) - Ch^{11/10}\,.$$

Proof. The corollary is obtained by slight modifications in the above proof. First we make a covering of $\overline{\Omega} \cap K$ instead in $\overline{\Omega}$. Therefore in the lower bound (5.10), we only have to consider $\mathbf{c} \in K$, and the energy is bounded below by $\mathscr{E}_K(\mathbf{B},\Omega)$ in (5.11). We finally reached (5.27) and deduce the Corollary. □

5.3. **Generalization.** For the proofs above, we used very few knowledge on the magnetic Laplacians—essentially the change of gauge, the change of variables, and the perturbation identity (A.6). The finest part of the analysis is related to the corner structure. With the same approach and relying on the general estimates presented in Section 3.4.4, we are able to establish lower bounds for the ground state energy of magnetic Laplacians in n-dimensional corner domains.

Let $\Omega \in \mathfrak{D}(\mathbb{R}^n)$, and let us introduce ν as the maximal integer such that there exists a singular chain $(\mathbf{x}_0, \dots, \mathbf{x}_{\nu-1})$ of length ν with a non-polyhedral reduced cone $\Gamma_{\mathbf{x}_0,\dots,\mathbf{x}_{\nu-1}}$. We make the convention that $\nu = 0$ if all tangent cones are polyhedral.

Using an IMS partition on a hierarchy of balls of size h^{δ_0}, $h^{\delta_0+\delta_1}$, …, $h^{\delta_0+\delta_1+\ldots+\delta_\nu}$ according to the position of their centers, and taking advantage of estimates (3.36), we arrive to the following collection of errors

$$\begin{aligned}
&h^{1+\delta_0},\ h^{1+\delta_1},\ \dots,\ h^{1+\delta_\nu}\\
&h^{\frac12+2\delta_0},\ h^{\frac12+\delta_0+2\delta_1},\ \dots,\ h^{\frac12+\delta_0+\ldots+\delta_{\nu-1}+2\delta_\nu}\\
&h^{2-2(\delta_0+\delta_1+\ldots+\delta_\nu)},
\end{aligned}$$

which is optimized choosing

$$\delta_k = 2^{\nu-k}\delta_\nu,\ k = 0,\dots,\nu, \quad \text{with} \quad \delta_\nu = \frac{3}{3\cdot 2^{\nu+2} - 4}\,.$$

The outcome is the following lower bound

$$\lambda_h(\mathbf{B},\Omega) \geq h\mathscr{E}(\mathbf{B},\Omega) - C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{1+1/(3\cdot 2^{\nu+1}-2)}\,.$$

Here $\mathscr{E}(\mathbf{B},\Omega)$ is the natural generalization of (4.11) to n-dimensional domains. The results of Theorem 5.1 correspond to the values $\nu = 1$ and $\nu = 0$. Note that the remainder $\mathcal{O}(h^{5/4})$ is valid in a polyhedral domain in any dimension ($\nu = 0$).

6. TAXONOMY OF MODEL PROBLEMS

Refined estimates for an *upper bound* of the ground state energy $\lambda_h(\mathbf{B},\Omega)$ will be obtained with the help of quasimode constructions. This relies on a better knowledge of tangent model problems $H(\mathbf{A}_{\mathbb{X}},\Pi_{\mathbb{X}})$ for any singular chain $\mathbb{X}$ of Ω. In this section, we review and, when required, complete, essential facts concerning three-dimensional model problems, that is magnetic Laplacians $H(\mathbf{A},\Pi)$ where Π is a cone in $\mathfrak{P}_3$ and $\mathbf{A}$ is a linear potential.

With the aim of constructing quasimodes for our original problem on Ω, we need (bounded) generalized eigenvectors for its tangent problems. To introduce such eigenvectors we make use of the localized domain $\mathrm{Dom}_{\mathrm{loc}}(H(\mathbf{A},\Pi))$ of the model magnetic Laplacian $H(\mathbf{A},\Pi)$ as introduced in (1.23):

Definition 6.1 (Generalized eigenvector)**.** Let $\Pi\in\mathfrak{P}_3$ be a cone and $\mathbf{A}$ a linear magnetic potential. We call *generalized eigenvector* for $H(\mathbf{A},\Pi)$ a nonzero function $\Psi\in\mathrm{Dom}_{\mathrm{loc}}(H(\mathbf{A},\Pi))$ associated with a real number Λ, so that

$$\begin{cases}(-i\nabla+\mathbf{A})^2\Psi=\Lambda\Psi & \text{in } \Pi,\\ (-i\nabla+\mathbf{A})\Psi\cdot\mathbf{n}=0 & \text{on } \partial\Pi.\end{cases} \tag{6.1}$$

Let $\Pi\in\mathfrak{P}_3$ be a 3D cone and let $\mathbf{B}$ be a constant magnetic field associated with a linear potential $\mathbf{A}$. Let d be the reduced dimension of Π and $\Gamma\in\mathfrak{P}_d$ be a minimal reduced cone associated with Π. We recall from Definition 3.16 that this means that $\Pi\equiv\mathbb{R}^{3-d}\times\Gamma$ and that the dimension d is minimal for such an equivalence. By analogy with Definition 3.18, $\mathfrak{C}_{\mathbf{0}}(\Pi)$ denotes the set of singular chains of Π originating at its vertex $\mathbf{0}$ and $\mathfrak{C}^*_{\mathbf{0}}(\Pi)$ is the subset of chains of length ≥ 2. Note that $\mathfrak{C}^*_{\mathbf{0}}(\Pi)$ is empty if and only if $\Pi=\mathbb{R}^3$, *i.e.*, if $d=0$. We introduce the energy on tangent substructures:

Definition 6.2 (Energy on tangent substructures)**.** We define the quantity

$$\mathscr{E}^*(\mathbf{B},\Pi):=\begin{cases}\inf_{\mathbb{X}\in\mathfrak{C}^*_{0}(\Pi)}E(\mathbf{B},\Pi_{\mathbb{X}}) & \text{if } d>0,\\ +\infty & \text{if } d=0,\end{cases} \tag{6.2}$$

which is the infimum of the ground state energy of the magnetic Laplacian over all the singular chains of length ≥ 2.

We will see later in Section 7 that this quantity plays a key role in the existence of generalized eigenvectors that have exponential decay properties in certain directions.

Now, in each of Sections 6.1–6.4 we consider one value of the reduced dimension d, ranging from 0 to 3 and give in each case relations between the ground state energy $E(\mathbf{B},\Pi)$ and the energy on tangent substructures $\mathscr{E}^*(\mathbf{B},\Pi)$, and we provide generalized eigenvectors Ψ if they exist.

On the one hand, thanks to Lemma A.4, we may reduce to deal with unitary magnetic field $|\mathbf{B}|=1$. On the other hand, quantities $E(\mathbf{B},\Pi)$ and $\mathscr{E}^*(\mathbf{B},\Pi)$ are independent of a choice of Cartesian coordinates. Thus, once Π and a constant unitary magnetic field $\mathbf{B}$ are chosen, we exhibit a system of Cartesian coordinates $\mathbf{x}=(x_1,x_2,x_3)$ that allows the simplest possible description of the configuration $(\mathbf{B},\Pi)$. In these coordinates, the magnetic field can be viewed as a reference field, and for convenience, we denote it by $\underline{\mathbf{B}} = (b_0, b_1, b_2)$. We also choose a corresponding reference linear potential $\underline{\mathbf{A}}$, since we have gauge independence by virtue of Lemma A.1.

6.1. **Full space.** $d = 0$. Π is the full space. We take coordinates $\mathbf{x} = (x_1, x_2, x_3)$ so that

$$\Pi = \mathbb{R}^3 \quad \text{and} \quad \underline{\mathbf{B}} = (1, 0, 0),$$

and choose as reference potential $\underline{\mathbf{A}} = (0, -\frac{x_3}{2}, \frac{x_2}{2})$. It is classical (see [35]) that the spectrum of $H(\underline{\mathbf{A}}, \mathbb{R}^3)$ is $[1, +\infty)$. Therefore

$$E(\underline{\mathbf{B}}, \mathbb{R}^3) = 1\,. \tag{6.3}$$

A generalized eigenvector associated with the ground state energy is

$$\Psi(\mathbf{x}) = \mathrm{e}^{-(x_2^2+x_3^2)/4} \quad \text{with} \quad \Lambda = 1. \tag{6.4}$$

6.2. **Half space.** $d = 1$. Π is a half-space. We take coordinates $\mathbf{x} = (x_1, x_2, x_3)$ so that

$$\Pi = \mathbb{R}^2 \times \mathbb{R}_+ := \{(x_1, x_2, x_3) \in \mathbb{R}^3,\ x_3 > 0\} \quad \text{and} \quad \underline{\mathbf{B}} = (0, b_1, b_2) \text{ with } b_1^2 + b_2^2 = 1\,,$$

and choose as reference potential $\underline{\mathbf{A}} = (b_1 x_3 - b_2 x_2, 0, 0)$. We note that

$$\mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^2 \times \mathbb{R}_+) = E(\underline{\mathbf{B}}, \mathbb{R}^3) = 1. \tag{6.5}$$

There exists $\theta \in [0, 2\pi)$ such that $b_1 = \cos\theta$ and $b_2 = \sin\theta$. Due to symmetries we can reduce to $\theta \in [0, \frac{\pi}{2}]$. Denote by $\mathcal{F}_1$ the Fourier transform in x_1-variable and by τ the dual variable. There holds

$$\mathcal{F}_1\ H(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+)\ \mathcal{F}_1^* = \int_{\tau\in\mathbb{R}}^{\oplus} \widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+)\, \mathrm{d}\tau.$$

where $\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+) = (\tau + b_1 x_3 - b_2 x_2)^2 - \partial_2^2 - \partial_3^2$. We discriminate three cases:

- *Tangent field.* $\theta = 0$, then $\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+) = (\tau + x_3)^2 - \partial_2^2 - \partial_3^2$. Let ξ be the partial Fourier variable associated with x_2. Define the operators $\widehat{H}_{\xi,\tau}(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+) = (\tau + x_3)^2 + \xi^2 - \partial_3^2$ and $\mathcal{H}(\tau) = \mathrm{D}_3^2 + (\tau + x_3)^2$, where $\mathcal{H}(\tau)$ (sometimes called the de Gennes operator) acts on $L^2(\mathbb{R}_+)$ with Neumann boundary conditions. Its first eigenvalue is denoted by $\mu(\tau)$. There holds

$$\inf \mathfrak{S}(\widehat{H}_{\tau,\xi}(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+)) = \mu(\tau) + \xi^2.$$

From [19]) we know that μ admits a unique minimum denoted by $\Theta_0 \simeq 0.59$ for the value $\tau_0 = -\sqrt{\Theta_0}$. Hence

$$E(\underline{\mathbf{B}}, \mathbb{R}^2 \times \mathbb{R}_+) = \Theta_0 < \mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^2 \times \mathbb{R}_+). \tag{6.6}$$

If Φ denotes an eigenvector of $\mathcal{H}(\tau_0)$, the corresponding generalized eigenvector for $H(\underline{\mathbf{A}}, \Pi)$ is

$$\Psi(\mathbf{x}) = \mathrm{e}^{-i\sqrt{\Theta_0}\,x_1}\, \Phi(x_3) \quad \text{with} \quad \Lambda = \Theta_0. \tag{6.7}$$

- *Normal field.* $\theta = \frac{\pi}{2}$, then $\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+) = (\tau - x_2)^2 - \partial_2^2 - \partial_3^2$. There holds for all $\tau \in \mathbb{R}$, $\inf \mathfrak{S}(\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+)) = 1$ (see [38, Theorem 3.1]), hence

$$E(\underline{\mathbf{B}}, \mathbb{R}^2 \times \mathbb{R}_+) = 1 = \mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^2 \times \mathbb{R}_+). \tag{6.8}$$

• *Neither tangent nor normal.* $\theta \in (0, \frac{\pi}{2})$. Then for any $\tau \in \mathbb{R}$, $\widehat{H}_\tau(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+)$ is isospectral to $\widehat{H}_0(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+)$ the ground state energy of which is an eigenvalue $\sigma(\theta) < 1$, cf. [30]. We deduce

$$E(\underline{\mathbf{B}}, \mathbb{R}^2 \times \mathbb{R}_+) = \sigma(\theta) < 1 = \mathscr{E}^*(\underline{\mathbf{B}}, \mathbb{R}^2 \times \mathbb{R}_+).. \tag{6.9}$$

This eigenvalue $\sigma(\theta)$ is associated with an exponentially decreasing eigenvector Φ that is a function of $(x_2, x_3) \in \mathbb{R} \times \mathbb{R}_+$. The corresponding generalized eigenvector for $H(\underline{\mathbf{A}}, \Pi)$ is

$$\Psi(\mathbf{x}) = \Phi(x_2, x_3) \quad \text{with} \quad \Lambda = \sigma(\theta). \tag{6.10}$$

We recall from the literature:

Lemma 6.3. *The function $\theta \mapsto \sigma(\theta)$ is continuous and increasing on $(0, \frac{\pi}{2})$ ([30, 38]). Set $\sigma(0) = \Theta_0$ and $\sigma(\frac{\pi}{2}) = 1$. Then the function $\theta \mapsto \sigma(\theta)$ is of class $\mathscr{C}^1$ on $[0, \frac{\pi}{2}]$ ([10]).*

6.3. **Wedges.** $d = 2$. Π is a wedge and let $\alpha \in (0, \pi) \cup (\pi, 2\pi)$ denote its opening. Let us introduce the model sector $\mathcal{S}_\alpha$ and the model wedge $\mathcal{W}_\alpha$

$$\mathcal{S}_\alpha = \begin{cases} \{x = (x_2, x_3),\ x_2 \tan\frac{\alpha}{2} > |x_3|\} & \text{if } \alpha \in (0, \pi) \\ \{x = (x_2, x_3),\ x_2 \tan\frac{\alpha}{2} > -|x_3|\} & \text{if } \alpha \in (\pi, 2\pi) \end{cases} \quad \text{and} \quad \mathcal{W}_\alpha = \mathbb{R} \times \mathcal{S}_\alpha\,. \tag{6.11}$$

We take coordinates $\mathbf{x} = (x_1, x_2, x_3)$ so that

$$\Pi = \mathcal{W}_\alpha \quad \text{and} \quad \underline{\mathbf{B}} = (b_0, b_1, b_2) \text{ with } b_0^2 + b_1^2 + b_2^2 = 1\,,$$

and choose as reference potential $\underline{\mathbf{A}} = (b_1 x_3 - b_2 x_2, 0, b_0 x_2)$. The singular chains of $\mathfrak{C}_\mathbf{0}^*(\mathcal{W}_\alpha)$ have three equivalence classes, cf. Definition 3.19 and Description 3.30 (3): The full space $\mathbb{R}^3$ and the two half-spaces $\Pi_\alpha^\pm$ corresponding to the two faces $\partial^\pm\mathcal{W}_\alpha$ of $\mathcal{W}_\alpha$. Thus

$$\mathscr{E}^*(\underline{\mathbf{B}}, \mathcal{W}_\alpha) = \min\{E(\underline{\mathbf{B}}, \mathbb{R}^3),\ E(\underline{\mathbf{B}}, \Pi_\alpha^+),\ E(\underline{\mathbf{B}}, \Pi_\alpha^-)\}.$$

Let $\theta^\pm \in [0, \frac{\pi}{2}]$ be the angle between $\underline{\mathbf{B}}$ and the face $\partial\Pi_\alpha^\pm$. We have, cf. Lemma 6.3,

$$\mathscr{E}^*(\underline{\mathbf{B}}, \mathcal{W}_\alpha) = \min\{1,\ \sigma(\theta^+), \sigma(\theta^-)\} = \sigma(\min\{\theta^+,\ \theta^-\}). \tag{6.12}$$

With τ the dual variable of x_1 and

$$\widehat{H}_\tau(\underline{\mathbf{A}}, \mathcal{W}_\alpha) = (\tau + b_1 x_3 - b_2 x_2)^2 - \partial_2^2 + (-i\partial_3 + b_0 x_2)^2 \tag{6.13}$$

we have

$$\mathcal{F}_1\ H(\underline{\mathbf{A}}, \mathcal{W}_\alpha)\ \mathcal{F}_1^* = \int_{\tau\in\mathbb{R}}^{\oplus} \widehat{H}_\tau(\underline{\mathbf{A}}, \mathcal{W}_\alpha)\,\mathrm{d}\tau\,.$$

Thus

$$E(\underline{\mathbf{B}}, \mathcal{W}_\alpha) = \inf_{\tau\in\mathbb{R}} s(\underline{\mathbf{B}}, \mathcal{S}_\alpha; \tau) \quad \text{with} \quad s(\underline{\mathbf{B}}, \mathcal{S}_\alpha; \tau) := \inf \mathfrak{S}(\widehat{H}_\tau(\underline{\mathbf{A}}, \mathcal{W}_\alpha))\,. \tag{6.14}$$

We quote from [49, Theorem 3.5]:

Lemma 6.4. *Let $\alpha \in (0, \pi) \cup (\pi, 2\pi)$. There holds the inequality*

$$E(\underline{\mathbf{B}}, \mathcal{W}_\alpha) \le \mathscr{E}^*(\underline{\mathbf{B}}, \mathcal{W}_\alpha). \tag{6.15}$$

Moreover, if $E(\underline{\mathbf{B}}, \mathcal{W}_\alpha) < \mathscr{E}^*(\underline{\mathbf{B}}, \mathcal{W}_\alpha)$*, then the function* $\tau \mapsto s(\underline{\mathbf{B}}, \mathcal{S}_\alpha; \tau)$ *reaches its infimum. Let* τ^* *be a minimizer. Then* $E(\underline{\mathbf{B}}, \mathcal{W}_\alpha)$ *is the first eigenvalue of the operator* $\widehat{H}_{\tau^*}(\underline{\mathbf{A}}, \mathcal{W}_\alpha)$ *and any associated eigenfunction* Φ *has exponential decay. The function*

$$\Psi(\mathbf{x}) = \mathrm{e}^{i\tau^* x_1}\Phi(x_2, x_3) \tag{6.16}$$

is a generalized eigenvector for the operator $H(\underline{\mathbf{A}}, \mathcal{W}_\alpha)$ *associated with* $\Lambda = E(\underline{\mathbf{B}}, \mathcal{W}_\alpha)$*.*

Finally, let us quote now the continuity result on dihedra from [49, Theorem 4.5]:

Lemma 6.5. *The function* $(\mathbf{B}, \alpha) \mapsto E(\mathbf{B}, \mathcal{W}_\alpha)$ *is continuous on* $\mathbb{S}^2 \times ((0, \pi) \cup (\pi, 2\pi))$*.*

6.4. **3D cones.** $d = 3$. Denote by $\lambda_{\text{ess}}(\mathbf{B}, \Pi)$ the bottom of the essential spectrum of $H(\mathbf{A}, \Pi)$.

Theorem 6.6. *Let* $\Pi \in \mathfrak{P}_3$ *be a cone with* $d = 3$*, which means that* Π *is not a wedge, nor a half-space, nor the full space. Let* $\mathbf{B}$ *be a constant magnetic field. With the quantity* $\mathscr{E}^*(\mathbf{B}, \Pi)$ *introduced in* (6.2)*, there holds*

$$\lambda_{\text{ess}}(\mathbf{B}, \Pi) = \mathscr{E}^*(\mathbf{B}, \Pi)\,.$$

Recall Persson's Lemma [46] that gives a characterization of the bottom of the essential spectrum:

Lemma 6.7. *Let* $\Pi \in \mathfrak{P}_3$ *and let* $\mathbf{A}$ *be a linear magnetic potential associated with* $\mathbf{B}$*. For* $R > 0$*, we define* $\mathrm{Dom}_0^R(q[\mathbf{A}, \Pi])$ *as the subspace of functions* Ψ *in* $\mathrm{Dom}(q[\mathbf{A}, \Pi])$ *with compact support, and* $\operatorname{supp}\Psi \cap \mathcal{B}(\mathbf{0}, R) = \emptyset$*. Then we have*

$$\lambda_{\text{ess}}(\mathbf{B}, \Pi) = \lim_{R\to+\infty}\left(\inf_{\Psi\,\in\,\mathrm{Dom}_0^R(q[\mathbf{A},\Pi])\,\setminus\,\{0\}} \mathscr{Q}[\mathbf{A}, \Pi](\Psi)\right).$$

Before proving Theorem 6.6, we show

Lemma 6.8. *Let* $\Pi \in \mathfrak{P}_3$ *be a cone with* $d = 3$*, let* $\Omega_{\mathbf{0}} = \Pi \cap \mathbb{S}^2$ *be its section. Then* $\mathscr{E}^*(\mathbf{B}, \Pi)$ *coincides with the infimum of the local energy over singular chains of length 2:*

$$\mathscr{E}^*(\mathbf{B}, \Pi) = \inf_{\mathbf{x}_1 \in \overline{\Omega}_{\mathbf{0}}} E(\mathbf{B}, \Pi_{\mathbf{0},\mathbf{x}_1})\,. \tag{6.17}$$

Proof. For all singular chains $\mathbb{X}$ and $\mathbb{X}'$ in $\mathfrak{C}^*(\Pi)$ such that $\mathbb{X} \leq \mathbb{X}'$, we have $E(\Pi_{\mathbb{X}}, \mathbf{B}) \leq E(\Pi_{\mathbb{X}'}, \mathbf{B})$ as a consequence of (6.6), (6.8), (6.9), and (6.15). Hence (6.17). ☐

Proof of Theorem 6.6. Combining Lemmas 6.7 and A.4, we get that

$$\lambda_{\text{ess}}(\mathbf{B}, \Pi) = \lim_{h\to 0}\left(h^{-1}\inf_{\Psi\,\in\,\mathrm{Dom}_0^1(q_h[\mathbf{A},\Pi])\,\setminus\,\{0\}} \mathscr{Q}_h[\mathbf{A}, \Pi](\Psi)\right). \tag{6.18}$$

• *Upper bound for* $\lambda_{\text{ess}}(\mathbf{B}, \Pi)$. Let $\varepsilon > 0$. By Lemma 6.8 there exist $\mathbf{x} \in \overline{\Omega}_{\mathbf{0}}$ and an associated chain $\mathbb{X} = (\mathbf{0}, \mathbf{x})$ of length 2 such that

$$E(\mathbf{B}, \Pi_{\mathbb{X}}) < \mathscr{E}^*(\mathbf{B}, \Pi) + \varepsilon\,. \tag{6.19}$$

Let $\mathbf{x}' := 2\mathbf{x}$. Notice that the tangent cone to Π at $\mathbf{x}'$ is $\Pi_{\mathbf{x}'} = \Pi_{\mathbb{X}}$ and therefore $E(\mathbf{B}, \Pi_{\mathbf{x}'}) = E(\mathbf{B}, \Pi_{\mathbb{X}})$. We use Lemma 4.8 (that clearly applies even though Π is unbounded): So there exists $h_0 > 0$ such that for all $h \in (0, h_0)$ we can find f_h normalized and supported near $\mathbf{x}'$ satisfying

$h^{-1}\mathscr{Q}_h[\mathbf{A},\Pi](f_h) \le E(\mathbf{B},\Pi_{\mathbb{X}})+\varepsilon$. Since $|\mathbf{x}'| = 2$, we may assume without restriction that $\operatorname{supp}(f_h)\cap\mathcal{B}(0,1)=\emptyset$. Combining this with (6.19) we get

$$\frac{1}{h}\mathscr{Q}_h[\mathbf{A},\Pi](f_h) \le \mathscr{E}^*(\mathbf{B},\Pi)+2\varepsilon\,,$$

and therefore deduce from (6.18) the upper bound of $\lambda_{\text{ess}}(\mathbf{B},\Pi)$ by $\mathscr{E}^*(\mathbf{B},\Pi)$.

• *Lower bound for* $\lambda_{\text{ess}}(\mathbf{B},\Pi)$. Notice that for all $\mathbf{x}\in\overline{\Pi}\setminus\mathcal{B}(0,1)$, we have $\Pi_{\mathbf{x}}=\Pi_{\mathbb{X}}$ where $\mathbb{X}=(\mathbf{0},\mathbf{x}/|\mathbf{x}|)$. Therefore (see (6.17)):

$$\inf_{\mathbf{x}\in\overline{\Pi}\setminus\mathcal{B}(0,1)} E(\mathbf{B},\Pi_{\mathbf{x}}) = \mathscr{E}^*(\mathbf{B},\Pi)\,.$$

Then we easily deduce the lower bound from Corollary 5.3 and (6.18). □

Corollary 6.9. *Let $\Pi\in\mathfrak{P}_3$ be a cone with $d=3$. Assume that $E(\mathbf{B},\Pi)<\mathscr{E}^*(\mathbf{B},\Pi)$. Then any eigenfunction Ψ of $H(\mathbf{A},\Pi)$ associated with the lowest eigenvalue $E(\mathbf{B},\Pi)$, satisfies the following exponential decay estimates:*

$$\forall c<\sqrt{\mathscr{E}^*(\mathbf{B},\Pi)-E(\mathbf{B},\Pi)},\quad \exists C>0,\quad \|e^{c|\mathbf{x}|}\Psi\|\le C\|\Psi\|.$$

Proof. Recall that Theorem 6.6 states that the bottom of the essential spectrum is $\mathscr{E}^*(\mathbf{B},\Pi)$. Therefore we are in the standard framework for the techniques *à la* Agmon, see [1], and also [6, Section 7] for its application on plane sectors. □

7. Dichotomy and substructures for model problems

Relying on the exhaustive description of model problems provided above, we arrive to one of the main results, the "dichotomy" Theorem 7.3 that states the existence of a generalized eigenvector (called *admissible*) living on a tangent structure of a cone $\Pi\in\mathfrak{P}_3$ and associated with the ground state energy. In this section, the local energies $E(\mathbf{B},\Pi_{\mathbb{X}})$ related to singular chains $\mathbb{X}\in\mathfrak{C}_{\mathbf{0}}(\Pi)$, play for the first time a major role in the analysis.

7.1. Admissible Generalized Eigenvectors.

Definition 7.1 (Admissible Generalized Eigenvector)**.** Let $\Pi\in\mathfrak{P}_3$ be a cone. Recall that $d(\Pi)\in[0,3]$ is the dimension of its minimal reduced cone. Let $\mathbf{A}$ be a linear magnetic potential. A generalized eigenvector Ψ for $H(\mathbf{A},\Pi)$ (cf. Definition 6.1) is said to be *admissible* if there exist an integer $k\ge d(\Pi)$ and a rotation $\underline{\mathrm{U}}:\mathbf{x}\mapsto(\mathbf{y},\mathbf{z})$ that maps Π onto the product $\mathbb{R}^{3-k}\times\Upsilon$ with Υ a cone in $\mathfrak{P}_k$, and such that

(7.1) $$\Psi\circ\underline{\mathrm{U}}^{-1}(\mathbf{y},\mathbf{z}) = \mathrm{e}^{i\vartheta(\mathbf{y},\mathbf{z})}\,\Phi(\mathbf{z})\qquad \forall\mathbf{y}\in\mathbb{R}^{3-k},\ \ \forall\mathbf{z}\in\Upsilon,$$

with some real polynomial function ϑ of degree ≤ 2 and some exponentially decreasing function Φ, namely there exist positive constants c_Ψ and C_Ψ such that

(7.2) $$\|\mathrm{e}^{c_\Phi|\mathbf{z}|}\Phi\|_{L^2(\Upsilon)}\le C_\Phi\|\Phi\|_{L^2(\Upsilon)}\,.$$

"Admissible Generalized Eigenvector" will be shortened as AGE.

The following lemma will be used for going from any tangent operator to one of the reference situations described in Section 6. Its proof is straightforward and relies on Lemmas A.1, A.3, A.4, and A.7.

Lemma 7.2. *Let $\Pi \in \mathfrak{P}_3$ be a cone and $\mathbf{A}$ be a linear potential. Assume that Ψ is an AGE for $H(\mathbf{A},\Pi)$ associated with the energy $E(\mathbf{B},\Pi)$, of the form (7.1).*

a1) *For all $b > 0$, the function*
$$\Psi_b : \mathbf{x} \mapsto \Psi(\frac{\mathbf{x}}{\sqrt{b}}),$$
is an AGE for $H(b^{-1}\mathbf{A},\Pi)$ associated with the energy $E(b^{-1}\mathbf{B},\Pi) = b^{-1}E(\mathbf{B},\Pi)$. This AGE has the form (7.1) with $\underline{\mathrm{U}}_b = \underline{\mathrm{U}}$, $\vartheta_b(\mathbf{y},\mathbf{z}) = \vartheta(b^{-1/2}\mathbf{y}, b^{-1/2}\mathbf{z})$ and $\Phi_b(\mathbf{z}) = \Phi(b^{-1/2}\mathbf{z})$.

a2) *The function*
$$\Psi_- : \mathbf{x} \mapsto \overline{\Psi(\mathbf{x})},$$
is an AGE for $H(-\mathbf{A},\Pi)$ associated with the energy $E(-\mathbf{B},\Pi) = E(\mathbf{B},\Pi)$. This AGE has the form (7.1), with $\underline{\mathrm{U}}^- = \underline{\mathrm{U}}$, $\vartheta^-(\mathbf{y},\mathbf{z}) = -\vartheta(\mathbf{y},\mathbf{z})$ and $\Phi^-(\mathbf{z}) = \overline{\Phi(\mathbf{z})}$.

b) *Let $\mathbf{A}'$ be another linear potential such that* $\operatorname{curl}\mathbf{A}' = \operatorname{curl}\mathbf{A}$. *Then there exists a polynomial ϕ of degree ≤ 2 such that $\mathbf{A}' = \mathbf{A} + \nabla\phi$. The function*
$$\Psi' : \mathbf{x} \mapsto e^{-i\phi(\mathbf{x})}\Psi(\mathbf{x}),$$
is an AGE for $H(\mathbf{A}',\Pi)$ associated with $E(\mathbf{B},\Pi)$. This AGE has the form (7.1), with $\underline{\mathrm{U}}' = \underline{\mathrm{U}}$, $\vartheta' = \vartheta - \phi \circ \underline{\mathrm{U}}^{-1}$ and $\Phi' = \Phi$.

c) *Let $\mathrm{J} \in \mathfrak{O}_3$ be a rotation, $\Pi_\mathrm{J} := \mathrm{J}(\Pi)$ and $\mathbf{A}_\mathrm{J} := \mathrm{J} \circ \mathbf{A} \circ \mathrm{J}^{-1}$. Introduce the constant magnetic field $\mathbf{B}_\mathrm{J} = \mathrm{J}(\mathbf{B})$, so that* $\operatorname{curl}\mathbf{A}_\mathrm{J} = \mathbf{B}_\mathrm{J}$. *Then*
$$\Psi_\mathrm{J} : \mathbf{x} \mapsto \Psi \circ \mathrm{J}^{-1}(\mathbf{x})$$
is an AGE for $H(\mathbf{A}_\mathrm{J},\Pi_\mathrm{J})$ associated with $E(\mathbf{B}_\mathrm{J},\Pi_\mathrm{J}) = E(\mathbf{B},\Pi)$. It has the form (7.1), with $\underline{\mathrm{U}}_\mathrm{J} = \underline{\mathrm{U}} \circ \mathrm{J}^{-1}$, $\vartheta_\mathrm{J} = \vartheta$ and $\Phi_\mathrm{J} = \Phi$.

7.2. **Dichotomy Theorem.**

Theorem 7.3 (Dichotomy Theorem)**.** *Let $\Pi \in \mathfrak{P}_3$ be a cone and $\mathbf{B} \neq 0$ be a constant magnetic field. Let $\mathbf{A}$ be any associated linear magnetic potential. Recall that $E(\mathbf{B},\Pi)$ is the ground state energy of $H(\mathbf{A},\Pi)$ and $\mathscr{E}^*(\mathbf{B},\Pi)$ is the energy on tangent substructures, see Definition 6.2. Then,*

$$E(\mathbf{B},\Pi) \leq \mathscr{E}^*(\mathbf{B},\Pi), \tag{7.3}$$

and we have the dichotomy:

(i) If $E(\mathbf{B},\Pi) < \mathscr{E}^*(\mathbf{B},\Pi)$, then $H(\mathbf{A},\Pi)$ admits an Admissible Generalized Eigenvector associated with the value $E(\mathbf{B},\Pi)$.

(ii) If $E(\mathbf{B},\Pi) = \mathscr{E}^*(\mathbf{B},\Pi)$, then there exists a singular chain $\mathbb{X} \in \mathfrak{C}^*_\mathbf{0}(\Pi)$ such that
$$E(\mathbf{B},\Pi_\mathbb{X}) = E(\mathbf{B},\Pi) \quad \textit{and} \quad E(\mathbf{B},\Pi_\mathbb{X}) < \mathscr{E}^*(\mathbf{B},\Pi_\mathbb{X}).$$

Remark 7.4. In the case (ii), we note that by statement (i) applied to the cone $\Pi_\mathbb{X}$, $H(\mathbf{A},\Pi_\mathbb{X})$ admits an AGE associated with the value $E(\mathbf{B},\Pi)$.

Remark 7.5. If $\mathbf{B} = 0$, there is no magnetic field and $E(\Pi, \mathbf{B}) = 0$. An associated AGE is the constant function $\Psi \equiv 1$.

Proof of Theorem 7.3. The proof relies on an exhaustion of cases based on Section 6 combined with a hierarchical classification of model problems on tangent structures of a cone Π.

• *Geometrical invariance.* Thanks to Lemma 7.2, we may assume that $\mathbf{B}$ is unitary, choose any suitable Cartesian coordinates and any suitable linear potential. Hence, to prove the theorem, we may reduce to the reference configurations investigated in Sections 6.1–6.3.

• *Algorithm of the proof.* We first establish the theorem when $d = 0$, then we apply the following analysis for increasing values of $d = d(\Pi)$ from 1 to 3:

(1) Check inequality (7.3).
(2) Check assertion (i).
(3) Prove that there exists a singular chain $\mathbb{X} \in \mathfrak{C}_{\mathbf{0}}^*(\Pi)$ such that $\mathscr{E}^*(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_{\mathbb{X}})$. Since $d(\Pi_{\mathbb{X}}) < d$, assertion (ii) will be a consequence of the analysis made for lower dimensions.

This procedure applied to reference problems described in Section 6 will provide the theorem.

• $d = 0$. Here $\Pi = \mathbb{R}^3$, see Section 6.1. We have $E(\mathbf{B}, \mathbb{R}^3) = 1$ and $\mathscr{E}^*(\mathbf{B}, \mathbb{R}^3) = +\infty$, moreover there always exists an admissible generalized eigenvector associated with 1, see (6.4). Theorem 7.3 is proved for $d = 0$.

• $d = 1$. The model cone is $\mathbb{R}^2 \times \mathbb{R}_+$, see Section 6.2. Inequality (7.3) has already been proved, see (6.6), (6.8), (6.9). We also know that $E(\mathbf{B}, \mathbb{R}^2 \times \mathbb{R}_+) < \mathscr{E}^*(\mathbf{B}, \mathbb{R}^2 \times \mathbb{R}_+)$ if and only if $\mathbf{B}$ is not normal to the boundary. In this case, AGE have already been written, see (6.7) and (6.10), so point (i) of Theorem 7.3 holds in the non-normal case. When $\mathbf{B}$ is normal, $E(\mathbf{B}, \mathbb{R}^2 \times \mathbb{R}_+) = \mathscr{E}^*(\mathbf{B}, \mathbb{R}^2 \times \mathbb{R}_+)$. The sole tangent substructure is $\mathbb{R}^3$ and we have $\mathscr{E}^*(\mathbf{B}, \mathbb{R}^2 \times \mathbb{R}_+) = E(\mathbf{B}, \mathbb{R}^3) < \mathscr{E}^*(\mathbf{B}, \mathbb{R}^3)$ (see the above paragraph $d = 0$). Therefore Theorem 7.3 is proved for $d = 1$.

• $d = 2$. The model cone is the wedge $\mathcal{W}_\alpha$, see Section 6.3. Inequality (7.3) and assertion (i) come from Lemma 6.4. To deal with case (ii), we define $\circ \in \{-, +\}$ satisfying $\theta^\circ = \min(\theta^-, \theta^+)$ and Π_α° as the corresponding face. Due to (6.12) $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \sigma(\theta^\circ) = E(\mathbf{B}, \Pi_\alpha^\circ)$. Therefore in case (ii) we reduce to the situation $d = 1$ and Theorem 7.3 is proved for $d = 2$.

• $d = 3$. Due to Theorem 6.6, we have $\mathscr{E}^*(\mathbf{B}, \Pi) = \lambda_{\text{ess}}(\mathbf{B}, \Pi)$ and therefore (7.3). Moreover if $E(\mathbf{B}, \Pi) < \mathscr{E}^*(\mathbf{B}, \Pi)$, the existence of an eigenfunction with exponential decay is stated in Corollary 6.9. Therefore (i) is proved.

It remains to find $\mathbb{X} \in \mathfrak{C}_{\mathbf{0}}^*(\Pi)$ such that $\mathscr{E}^*(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_{\mathbb{X}})$. Define on $\mathfrak{C}_{\mathbf{0}}^*(\Pi)$ the function $F(\mathbb{X}) = E(\mathbf{B}, \Pi_{\mathbb{X}})$. Let $\Omega_{\mathbf{0}}$ denotes the section of Π, define the function $F^\star$ on $\mathfrak{C}(\Omega_{\mathbf{0}})$ by the partial application

$$F^\star(\mathbb{Y}) = F((\mathbf{0}, \mathbb{Y})), \quad \mathbb{Y} \in \mathfrak{C}(\Omega_{\mathbf{0}}).$$

Since (7.3) has already been proved for $d \le 2$, we have for all $\mathbb{Y}$ and $\mathbb{Y}'$ in $\mathfrak{C}(\Omega_{\mathbf{0}})$:

$$\mathbb{Y} \le \mathbb{Y}' \implies F^\star(\mathbb{Y}) \le F^\star(\mathbb{Y}'). \tag{7.4}$$

Let us show that $F^\star$ is continuous with respect to the distance $\mathbb{D}$ introduced in Definition 3.22. Since $\Omega_{\mathbf{0}}$ has a finite number of vertices, the chains $\mathbb{Y} \in \mathfrak{C}(\Omega_{\mathbf{0}})$ such that $\Pi_{\mathbb{Y}}$ is a sector (and $\Pi_{\mathbb{X}} = \Pi_{(\mathbf{0},\mathbb{Y})}$ is a wedge) are isolated for the topology associated with the distance $\mathbb{D}$. If $\mathbb{Y}$ is such that $\Pi_{(\mathbf{0},\mathbb{Y})} = \mathbb{R}^3$, then $F^\star(\mathbb{Y}) = 1$ (see (6.3)). Therefore it remains to treat the case where the tangent substructures $\Pi_{(\mathbf{0},\mathbb{Y})}$ are half-spaces. Let $\mathbb{Y}$ and $\mathbb{Y}'$ be such chains. Denote by θ (resp. θ') the unoriented angle in $[0, \frac{\pi}{2})$ between $\mathbf{B}$ and $\Pi_{\mathbb{X}}$ (resp. between $\mathbf{B}$ and $\Pi_{\mathbb{X}'}$). We have $|\theta - \theta'| \to 0$ as $\mathbb{D}(\mathbb{Y}, \mathbb{Y}') \to 0$. Moreover

$$F^\star(\mathbb{Y}) - F^\star(\mathbb{Y}') = E(\mathbf{B}, \Pi_{\mathbb{X}}) - E(\mathbf{B}, \Pi_{\mathbb{X}'}) = \sigma(\theta) - \sigma(\theta') .$$

As a consequence of the continuity of the function σ, see Lemma 6.3, we get that $F^\star(\mathbb{Y}) - F^\star(\mathbb{Y}')$ goes to 0 as $\mathbb{D}(\mathbb{Y}, \mathbb{Y}')$ goes to 0. This shows that $F^\star$ is continuous on $\mathfrak{C}(\Omega_{\mathbf{0}})$. Thanks to (7.4), we can apply Theorem 3.25: the function $\overline{\Omega}_{\mathbf{0}} \ni \mathbf{x} \mapsto F^\star((\mathbf{x})) = E(\mathbf{B}, \Pi_{\mathbf{0},\mathbf{x}})$ is lower semicontinuous on $\overline{\Omega}_{\mathbf{0}}$. Since $\overline{\Omega}_{\mathbf{0}}$ is compact, it reaches its infimum. Combining this with Lemma 6.8, we get:

$$\exists \mathbf{x}_1 \in \overline{\Omega}_{\mathbf{0}}, \quad \mathscr{E}^*(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_{\mathbf{0},\mathbf{x}_1}) .$$

Therefore (ii) follows from the analysis of lower dimensions and Theorem 7.3 is proved. □

Remark 7.6. Any AGE provided by case (i) of Theorem 7.3 satisfies:

$$\forall c_\Phi < \sqrt{\mathscr{E}^*(\mathbf{B}, \Pi) - E(\mathbf{B}, \Pi)}, \quad \exists C_\Phi > 0, \quad \| e^{c_\Phi |\mathbf{z}|} \Phi \|_{L^2(\Upsilon)} \le C_\Phi \| \Phi \|_{L^2(\Upsilon)} .$$

This is a consequence of the exponential decays given by [10, Theorem 1.3] for half-planes, [49, Proposition 4.2] for dihedra, and Corollary 6.9 for 3D cones.

7.3. **Examples.** In the case $d = 1$, it is known whether we are in situation (i) or (ii) of the Dichotomy Theorem. This is not the case in general for model cones Π with $d \ge 2$, and only in few cases it is known whether inequality (7.3) is strict or not. We provide below some examples of wedges and 3D cones where $E(\mathbf{B}, \Pi)$ has been studied. In this whole section $\mathbf{B} \in \mathbb{S}^2$ is a unitary constant magnetic field.

Example 7.7 (Wedges)**.** Let $\alpha \in (0, \pi) \cup (\pi, 2\pi)$.

(a) For α small enough there holds $E(\mathbf{B}, \mathcal{W}_\alpha) < \mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha)$, see [49] and [47, Ch. 7].
(b) Let $\mathbf{B} = (0, 0, 1)$ be tangent to the edge. Then $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \Theta_0$ and $E(\mathbf{B}, \mathcal{W}_\alpha) = E(1, \mathcal{S}_\alpha)$, cf. Section 2.2.2. According to whether the ground state energy $E(1, \mathcal{S}_\alpha)$ of the plane sector $\mathcal{S}_\alpha$ is less than Θ_0 or equal to Θ_0, we are in case (i) or (ii) of the dichotomy.
(c) Let $\mathbf{B}$ be tangent to a face of the wedge and normal to the edge. Then $\mathscr{E}^*(\mathbf{B}, \mathcal{W}_\alpha) = \Theta_0$. It is proved in [48] that $E(\mathbf{B}, \mathcal{W}_\alpha) = \Theta_0$ for $\alpha \in [\frac{\pi}{2}, \pi)$ (case (ii)).

Example 7.8 (Octant)**.** Let $\Pi = (\mathbb{R}_+)^3$ be the model octant. We quote from [44, §8]:

(a) If the magnetic field $\mathbf{B}$ is tangent to a face but not to an edge of Π, there exists an edge $\mathbf{e}$ such that $\mathscr{E}^*(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_{\mathbf{e}})$ and there holds $E(\mathbf{B}, \Pi) < E(\mathbf{B}, \Pi_{\mathbf{e}})$. We are in case (i).
(b) If the magnetic field $\mathbf{B}$ is tangent to an edge $\mathbf{e}$ of Π, $\mathscr{E}^*(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_{\mathbf{e}}) = E(\mathbf{B}, \Pi)$. Moreover by [44, §4], $E(\mathbf{B}, \Pi_{\mathbf{e}}) = E(1, \mathcal{S}_{\pi/2}) < \Theta_0 = \mathscr{E}^*(\mathbf{B}, \Pi_{\mathbf{e}})$. We are in case (ii).

Example 7.9 (Circular cone)**.** Let $\mathcal{C}_\alpha$ be the right circular cone of angular opening $\alpha \in (0, \pi)$. It is proved in [12, 13] that

(a) For α small enough, $E(\mathbf{B},\mathcal{C}_\alpha) < \mathscr{E}^*(\mathbf{B},\mathcal{C}_\alpha)$.

(b) If $\mathbf{B}=(0,0,1)$, then $\mathscr{E}^*(\mathbf{B},\mathcal{C}_\alpha)=\sigma(\alpha/2)$.

7.4. **Scaling and truncating Admissible Generalized Eigenvectors.** AGE's are corner-stones for our construction of quasimodes. Here, as a preparatory step towards final construction, we show a couple of useful properties when suitable scalings and cut-off are performed.

Let $H(\mathbf{A},\Pi)$ be a model operator that has an AGE Ψ associated with the value Λ. Then for any positive h, the scaled function

$$\Psi_h(\mathbf{x}) := \Psi\Big(\frac{\mathbf{x}}{\sqrt{h}}\Big), \quad \text{for} \quad \mathbf{x}\in\Pi, \tag{7.5}$$

defines an AGE for the operator $H_h(\mathbf{A},\Pi)$ associated with $h\Lambda$:

$$\begin{cases}(-ih\nabla+\mathbf{A})^2\Psi_h = h\Lambda\Psi_h & \text{in } \Pi,\\ (-ih\nabla+\mathbf{A})\Psi_h\cdot\mathbf{n}=0 & \text{on } \partial\Pi.\end{cases} \tag{7.6}$$

We will need to localize Ψ_h. For doing this, let us choose, once for all, a model cut-off function $\underline{\chi}\in\mathscr{C}^\infty(\mathbb{R}^+)$ such that

$$\underline{\chi}(r)=1 \ \text{ if } \ r\le 1 \quad \text{and} \quad \underline{\chi}(r)=0 \ \text{ if } \ r\ge 2. \tag{7.7}$$

For any $R>0$, let $\underline{\chi}_R$ be the cut-off function defined by $\underline{\chi}_R(r)=\underline{\chi}\left(\frac{r}{R}\right)$, and, finally

$$\chi_h(\mathbf{x}) = \underline{\chi}_R\left(\frac{|\mathbf{x}|}{h^\delta}\right) = \underline{\chi}\left(\frac{|\mathbf{x}|}{Rh^\delta}\right) \quad \text{with} \quad 0\le\delta\le\frac12\,. \tag{7.8}$$

Here the exponent δ is the decay rate of the cut-off. It will be tuned later to optimize remainders.

Since Ψ_h belongs to $\mathrm{Dom}_{\,\mathrm{loc}}(H_h(\mathbf{A},\Pi))$, we can rely on Lemma A.6 to obtain the following identity for the Rayleigh quotient of $\chi_h\Psi_h$:

$$\mathscr{Q}_h[\mathbf{A},\Pi](\chi_h\Psi_h) = h\Lambda + h^2\rho_h \qquad \text{with} \qquad \rho_h = \frac{\|\,|\nabla\chi_h|\,\Psi_h\|^2}{\|\chi_h\Psi_h\|^2}. \tag{7.9}$$

The following lemma estimates the remainder ρ_h:

Lemma 7.10. *Let Ψ be an AGE for the model operator $H(\mathbf{A},\Pi)$. Let k be the number of independent decaying directions of Ψ, cf. (7.1)–(7.2). Let Ψ_h be the rescaled function given by (7.5) and let χ_h be the cut-off function defined by (7.7)–(7.8) involving parameters $R>0$ and $\delta\in[0,\frac12]$. Then there exist constants $C_0>0$ and $c_0>0$ depending only on $h_0>0$, $R_0>0$ and Ψ such that*

$$\rho_h = \frac{\|\,|\nabla\chi_h|\,\Psi_h\|^2}{\|\chi_h\Psi_h\|^2} \le \begin{cases} C_0\,h^{-2\delta} & \text{if } k<3,\\ C_0\,h^{-2\delta}\,\mathrm{e}^{-c_0h^{\delta-1/2}} & \text{if } k=3,\end{cases} \qquad \forall R\ge R_0,\ \forall h\le h_0,\ \forall\delta\in[0,\tfrac12]\,.$$

Proof. By assumption $\Psi(\mathbf{x}) = \mathrm{e}^{i\vartheta(\mathbf{y},\mathbf{z})}\,\Phi(\mathbf{z})$ for $\underline{\mathrm{U}}\mathbf{x}=(\mathbf{y},\mathbf{z})\in\mathbb{R}^{3-k}\times\Upsilon$, where $\underline{\mathrm{U}}$ is a suitable rotation, and there exist positive constants c_Ψ, C_Ψ controlling the exponential decay of Φ in the cone $\Upsilon\in\mathfrak{P}_k$, cf. (7.2). Let us set $T=Rh^\delta$, so that $\chi_h(\mathbf{x})=\underline{\chi}(|\mathbf{x}|/T)$.

Let us first give an upper bound for $\| \, |\nabla\chi_h| \, \Psi_h\|$:
If $k<3$, then

$$\| \, |\nabla\chi_h| \, \Psi_h\|^2 \le CT^{-2}\int_{|\mathbf{y}|\le 2T}\mathrm{d}\mathbf{y}\int_{\Upsilon\cap\{|\mathbf{z}|\le 2T\}}\left|\Phi\Big(\frac{\mathbf{z}}{\sqrt{h}}\Big)\right|^2\mathrm{d}\mathbf{z} = CT^{-2}\,T^{3-k}\,h^{k/2}\|\Phi\|^2_{L^2(\Upsilon)},$$

else, if $k=3$

$$\begin{aligned}\| \, |\nabla\chi_h| \, \Psi_h\|^2 \;&\le\; CT^{-2}\int_{\Upsilon\cap\{T\le|\mathbf{z}|\le 2T\}}\left|\Phi\Big(\frac{\mathbf{z}}{\sqrt{h}}\Big)\right|^2\mathrm{d}\mathbf{z} \;=\; CT^{-2}\,h^{k/2}\int_{\Upsilon\cap\left\{Th^{-\frac12}\le|\mathbf{z}|\le 2Th^{-\frac12}\right\}}|\Phi(\mathbf{z})|^2\,\mathrm{d}\mathbf{z}\\ &\le\; CT^{-2}\,h^{k/2}\,\mathrm{e}^{-2c_\Psi T/\sqrt{h}}\int_{\Upsilon\cap\left\{Th^{-\frac12}\le|\mathbf{z}|\le 2Th^{-\frac12}\right\}}\mathrm{e}^{2c|\mathbf{z}|}\,|\Phi(\mathbf{z})|^2\,\mathrm{d}\mathbf{z}\\ &\le\; CT^{-2}\,h^{k/2}\,\mathrm{e}^{-2c_\Psi T/\sqrt{h}}\,\|\Phi\|^2_{L^2(\Upsilon)}.\end{aligned}$$

Let us now consider $\|\chi_h\Psi_h\|$ (we use that $2|\mathbf{y}|<R$ and $2|\mathbf{z}|<R$ implies $|\mathbf{x}|<R$):

$$\begin{aligned}\|\chi_h\Psi_h\|^2 \;&\ge\; \int_{2|\mathbf{y}|\le T}\mathrm{d}\mathbf{y}\int_{\Upsilon\cap\{2|\mathbf{z}|\le T\}}\left|\Phi\Big(\frac{\mathbf{z}}{\sqrt{h}}\Big)\right|^2\mathrm{d}\mathbf{z} = CT^{3-k}h^{k/2}\int_{\Upsilon\cap\left\{2|\mathbf{z}|\le Th^{-\frac12}\right\}}|\Phi(\mathbf{z})|^2\,\mathrm{d}\mathbf{z}\\ &\ge\; CT^{3-k}h^{k/2}\,\mathcal{I}(Th^{-\frac12})\,\|\Phi\|^2_{L^2(\Upsilon)}\end{aligned}$$

where we have set for any $S\ge 0$

$$\mathcal{I}(S) := \Big(\int_{\Upsilon\cap\{2|\mathbf{z}|\le S\}}|\Phi(\mathbf{z})|^2\,\mathrm{d}\mathbf{z}\Big)\Big(\int_{\Upsilon}|\Phi(\mathbf{z})|^2\,\mathrm{d}\mathbf{z}\Big)^{-1}.$$

The function $S\mapsto\mathcal{I}(S)$ is continuous, non-negative and non-decreasing on $[0,+\infty)$. It is moreover *increasing and positive* on $(0,\infty)$ since Φ, as a solution of an elliptic equation with polynomial coefficients and null right hand side, is analytic inside Υ. Consequently, $\mathcal{I}(Th^{-\frac12}) = \mathcal{I}(Rh^{\delta-\frac12})$ is uniformly bounded from below for $R\ge R_0$, $h\in(0,h_0)$, $\delta\in[0,\frac12]$ and thus

$$\rho_h \;\le\; \begin{cases} CT^{-2}\left\{\mathcal{I}(Th^{-\frac12})\right\}^{-1}\le C_0h^{-2\delta} & \text{if } k<3,\\ CT^{-2}\,\mathrm{e}^{-2c_\Psi T/\sqrt{h}}\left\{\mathcal{I}(Th^{-\frac12})\right\}^{-1}\le C_0h^{-2\delta}\mathrm{e}^{-c_0h^{\delta-1/2}} & \text{if } k=3,\end{cases}$$

where the constants C_0 and c_0 in the above estimation depend only on the lower bound R_0 on R, the upper bound h_0 on h, and on the model problem associated with $\mathbf{x}_0$, provided $\delta\in[0,\frac12]$. Lemma 7.10 is proved. □

Remark 7.11. The estimate of ρ_h provided by Lemma 7.10 is still true when $k=0$, *i.e.*, when Ψ has no decay direction (but is of modulus 1 everywhere).

8. Properties of the local ground state energy

In this section we describe the regularity properties of the local ground energy. The main result of this section is that the function $\mathbf{x}\mapsto E(\mathbf{B}_\mathbf{x},\Pi_\mathbf{x})$ is lower semicontinuous on a corner domain and therefore it reaches its infimum.

8.1. **Lower semicontinuity.**

Theorem 8.1. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and let $\mathbf{B} \in \mathscr{C}^0(\overline{\Omega})$ be a continuous magnetic field. Then the function $\Lambda_{\overline{\Omega}} : \mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is lower semicontinuous on $\overline{\Omega}$.*

Proof. For $\mathbb{X} = (\mathbf{x}_0, \dots) \in \mathfrak{C}(\Omega)$, define the function $F(\mathbb{X}) := E(\mathbf{B}_{\mathbf{x}_0}, \Pi_\mathbb{X})$, which coincides on the chains of length 1 with the function $\Lambda_{\overline{\Omega}}$: $F((\mathbf{x}_0)) = \Lambda_{\overline{\Omega}}(\mathbf{x}_0)$. Recall that we have introduced a partial order on $\mathfrak{C}(\Omega)$, see Definition 3.24. Then due to (7.3) applied to $\Pi_\mathbb{X}$ for any chain $\mathbb{X}$, the function $F : \mathfrak{C}(\Omega) \mapsto \mathbb{R}_+$ is clearly order preserving.

Let us show that it is continuous with respect to the distance $\mathbb{D}$ (see Definition 3.22). Let $\mathbb{X} \in \mathfrak{C}(\Omega)$ and $\mathbb{X}'$ tending to $\mathbb{X}$. This means that $\mathbf{x}'_0$ tends to $\mathbf{x}_0$ in $\mathbb{R}^3$ and that there exists $J \in \mathsf{BGL}(3)$ tending to the identity $\mathbb{I}_3$ such that $J(\Pi_\mathbb{X}) = \Pi_{\mathbb{X}'}$. In particular for $\mathbb{X}'$ close enough to $\mathbb{X}$, the reduced dimensions of the cones $\Pi_\mathbb{X}$ and $\Pi_{\mathbb{X}'}$ are equal: $d(\Pi_{\mathbb{X}'}) = d(\Pi_\mathbb{X})$.

(1) If $\Pi_\mathbb{X} = \mathbb{R}^3$, then $F(\mathbb{X}) = |\mathbf{B}_{\mathbf{x}_0}|$ and $F(\mathbb{X}') = |\mathbf{B}_{\mathbf{x}'_0}|$, and since $\mathbf{B}$ is continuous, $F(\mathbb{X}')$ converges toward $F(\mathbb{X})$ when $\mathbb{D}(\mathbb{X}', \mathbb{X}) \to 0$.

(2) When $\Pi_\mathbb{X}$ is a half-space, we denote by $\theta(\mathbb{X})$ the angle between $\Pi_\mathbb{X}$ and $\mathbf{B}_{\mathbf{x}_0}$. We have $\theta(\mathbb{X}') \to \theta(\mathbb{X})$ when $\mathbb{D}(\mathbb{X}', \mathbb{X}) \to 0$. Moreover
$$F(\mathbb{X}') - F(\mathbb{X}) = |\mathbf{B}_{\mathbf{x}'_0}|\sigma(\theta(\mathbb{X}')) - |\mathbf{B}_{\mathbf{x}_0}|\sigma(\theta(\mathbb{X})),$$
therefore $F(\mathbb{X}')$ tends to $F(\mathbb{X})$ due to Lemma 6.3 and the continuity of $\mathbf{B}$.

(3) When $\Pi_\mathbb{X}$ is a wedge, there exists $(\underline{\mathrm{U}}, \underline{\mathrm{U}}')$ in $\mathfrak{O}_3$ and (α, α') in $(0, \pi) \cup (\pi, 2\pi)$ such that $\underline{\mathrm{U}}(\Pi_\mathbb{X}) = \mathcal{W}_\alpha$ and $\underline{\mathrm{U}}'(\Pi_{\mathbb{X}'}) = \mathcal{W}_{\alpha'}$. Therefore
$$F(\mathbb{X}') - F(\mathbb{X}) = E(\underline{\mathrm{U}}(\mathbf{B}_{\mathbf{x}_0}), \mathcal{W}_\alpha) - E(\underline{\mathrm{U}}'(\mathbf{B}_{\mathbf{x}'_0}), \mathcal{W}_{\alpha'}),$$
with $\alpha' \to \alpha$ and $\underline{\mathrm{U}}' \to \underline{\mathrm{U}}$ when $\mathbb{D}(\mathbb{X}', \mathbb{X}) \to 0$. Lemma 6.5 and the continuity of $\mathbf{B}$ ensure that $F(\mathbb{X}')$ tends to $F(\mathbb{X})$.

(4) Finally chains $\mathbb{X}$ such that $\Pi_\mathbb{X}$ is a 3D cone are of length 1 and are isolated in $\mathfrak{C}(\Omega)$ for the topology associated with $\mathbb{D}$ (see Proposition 3.20).

Therefore F is continuous on $\mathfrak{C}(\Omega)$. We apply Theorem 3.25: So the function $\mathbf{x} \mapsto F((\mathbf{x})) = \Lambda_{\overline{\Omega}}(\mathbf{x})$ is lower semicontinuous on $\overline{\Omega}$. □

As a consequence of the above theorem, the function $\mathbf{x} \mapsto \Lambda_{\overline{\Omega}}(\mathbf{x})$ reaches its infimum over $\overline{\Omega}$. This fact will be one of the key ingredients to prove an upper bound with remainder for $\lambda_h(\mathbf{B}, \Omega)$ in the semiclassical limit.

Remark 8.2. Recall that any stratum $\mathbf{t} \in \mathfrak{T}$ has a smooth submanifold structure (see Proposition 3.20). Denote by $\Lambda_\mathbf{t}$ the restriction of the local ground energy to $\mathbf{t}$. Then it follows from above that $\Lambda_\mathbf{t}$ is continuous. Moreover if $\Omega \in \overline{\mathfrak{D}}(\mathbb{R}^3)$, one can prove that $\Lambda_\mathbf{t}$ admits a continuous extension to $\overline{\mathbf{t}}$. But this is not true anymore if $\overline{\mathbf{t}}$ contains a conical point.

Remark 8.3. Let $\mathbf{B}$ be a constant magnetic field and Ω be a straight polyhedron. So, its faces are plane polygons and its edges are segments of lines. The following properties hold.

a) For each stratum $\mathbf{t} \in \mathfrak{T}$, the function $\Lambda_\mathbf{t} : \mathbf{t} \ni \mathbf{x} \mapsto E(\mathbf{B}, \Pi_\mathbf{x})$ is constant.

b) As a consequence of (7.3) and of the lower semicontinuity, $\mathscr{E}(\mathbf{B}, \Omega)$ is the minimum of the corner local energies:

$$\mathscr{E}(\mathbf{B}, \Omega) = \min_{\mathbf{v} \in \mathfrak{V}} E(\mathbf{B}, \Pi_{\mathbf{v}}).$$

c) A stratum $\mathbf{t} \in \mathfrak{T}$ being chosen we have

$$\forall \mathbf{x} \in \mathbf{t}, \quad \mathscr{E}^*(\mathbf{B}, \Pi_{\mathbf{x}}) = \min_{\mathbf{t}' \in \mathfrak{N}(\mathbf{t})} \Lambda_{\mathbf{t}'},$$

where $\mathfrak{N}(\mathbf{t}) := \{\mathbf{t}' \in \mathfrak{T},\ \mathbf{t} \subset \partial \mathbf{t}'\} \setminus \{\mathbf{t}\}$ is the set of the strata adjacent to $\mathbf{t}$.

d) As a consequence of a), c) and the Dichotomy Theorem, there exists $\mathbf{x}_0 \in \overline{\Omega}$ such that

$$\mathscr{E}(\mathbf{B}, \Omega) = E(\mathbf{B}, \Pi_{\mathbf{x}_0}) < \mathscr{E}^*(\mathbf{B}, \Pi_{\mathbf{x}_0}).$$

8.2. **Positivity of the ground state energy.** The classical diamagnetic inequality (see [34, 56] for example) implies that the ground state energy is in general larger than the one without magnetic field, that is 0 in our case due to Neumann boundary conditions. Usually it is harder to show that this inequality is strict. A strict diamagnetic inequality has been proved for the Neumann magnetic Laplacian in a bounded regular domain, in [24, Section 2.2]. For our unbounded domains Π with constant magnetic field, we have:

Proposition 8.4. *Let $\Pi \in \mathfrak{P}_3$ and $\mathbf{B} \neq 0$ be a constant magnetic field. Then $E(\mathbf{B}, \Pi) > 0$.*

Proof. It is enough to make the proof for unitary magnetic field, see Lemma A.4. Let $d \in [0, 3]$ be the reduced dimension of the cone Π. If $d = 0$, then $E(\mathbf{B}, \Pi) = 1$ (see (6.3)). If $d = 1$, then $E(\mathbf{B}, \Pi)$ is expressed with the function σ that satisfies $\sigma(\theta) \geq \Theta_0 > 0$ for all $\theta \in [0, \frac{\pi}{2}]$, see Lemma 6.3. When $d = 2$, the strict positivity has been shown in [49, Corollary 3.9].

Assume now that $d = 3$. If we are in case (i) of Theorem 7.3, then there exists an eigenfunction $\Psi \in L^2(\Pi)$ for $H(\mathbf{A}, \Pi)$ associated with $E(\mathbf{B}, \Pi)$. Assume that $E(\mathbf{B}, \Pi) = 0$, then due to the standard diamagnetic inequality (see [34, Lemma A]), we have

$$0 \leq \int_{\Pi} \left|\nabla |\Psi|\right|^2 \leq \int_{\Pi} |(-i\nabla - \mathbf{A})\Psi|^2 = 0,$$

that leads to $\Psi = 0$, which is a contradiction. If we are in case (ii) of Theorem 7.3, then there exists a tangent substructure $\Pi_{\mathbb{X}}$ of Π with $d(\Pi_{\mathbb{X}}) < 3$ such that $E(\mathbf{B}, \Pi) = E(\mathbf{B}, \Pi_{\mathbb{X}})$ that is strictly positive due to the analysis of the cases $d \leq 2$, see above. $\square$

Combining the above proposition with Theorem 8.1, we get:

Corollary 8.5. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ and let $\mathbf{B} \in \mathscr{C}^0(\overline{\Omega})$ be non-vanishing. Then we have $\mathscr{E}(\mathbf{B}, \Omega) > 0$.*

9. Upper bounds for ground state energy in corner domains

In this section, we prove an upper bound involving error estimates that contains the same powers of h than the lower bound in Theorem 5.1.

Theorem 9.1. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ be a general 3D corner domain, and let $\mathbf{A} \in W^{2,\infty}(\overline{\Omega})$ be a twice differentiable magnetic potential.*

(a) Then there exist $C_\Omega > 0$ and $h_0 > 0$ such that

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \le h\mathscr{E}(\mathbf{B}, \Omega) + C_\Omega\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{11/10}\,. \tag{9.1}$$

(b) If Ω is a polyhedral domain, this upper bound is improved:

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \le h\mathscr{E}(\mathbf{B}, \Omega) + C_\Omega\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{5/4}\,. \tag{9.2}$$

(c) If there exists a point $\mathbf{x}_0 \in \overline{\Omega}$ such that $\mathbf{B}(\mathbf{x}_0) = 0$, then $\mathscr{E}(\mathbf{B}, \Omega) = 0$ and we have the optimal upper bound

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \le C_\Omega\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{4/3}\,. \tag{9.3}$$

(d) If there exists a corner $\mathbf{x}_0$ such that $\mathscr{E}(\mathbf{B}, \Omega) = E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) < \mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ then

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \le h\mathscr{E}(\mathbf{B}, \Omega) + C_\Omega(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})\, h^{3/2}|\log h|\,. \tag{9.4}$$

(e) If Ω is a straight polyhedron and $\mathbf{B}$ is constant,

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \le h\mathscr{E}(\mathbf{B}, \Omega) + Ch^2\,. \tag{9.5}$$

We recall the notation $\mathscr{Q}_h[\mathbf{A}, \Omega](\varphi)$ (1.19) for Rayleigh quotients and the min-max principle

$$\lambda_h(\mathbf{B}, \Omega) = \min_{\varphi \in H^1(\Omega) \setminus \{0\}} \mathscr{Q}_h[\mathbf{A}, \Omega](\varphi)\,.$$

9.1. **Principles of construction for quasimodes.** By lower semicontinuity (see Theorem 8.1), the energy $\mathbf{x} \mapsto E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ reaches its infimum over $\overline{\Omega}$. Let $\mathbf{x}_0 \in \overline{\Omega}$ be a point such that

$$E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) = \mathscr{E}(\mathbf{B}, \Omega).$$

By the dichotomy result (Theorem 7.3) there exists a singular chain $\mathbb{X}$ starting at $\mathbf{x}_0$ such that (see also notation (4.8)):

$$E(\mathbf{B}_\mathbb{X}, \Pi_\mathbb{X}) = E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) \quad \text{and} \quad E(\mathbf{B}_\mathbb{X}, \Pi_\mathbb{X}) < \mathscr{E}^*(\mathbf{B}_\mathbb{X}, \Pi_\mathbb{X}).$$

For shortness, we denote $\Lambda_\mathbb{X} = E(\mathbf{B}_\mathbb{X}, \Pi_\mathbb{X})$. Still by Theorem 7.3, there exists an AGE for the tangent model operator $H(\mathbf{A}_\mathbb{X}, \Pi_\mathbb{X})$ denoted by $\Psi^\mathbb{X}$ and associated with $\Lambda_\mathbb{X}$

$$\begin{cases} (-i\nabla + \mathbf{A}_\mathbb{X})^2\Psi^\mathbb{X} = \Lambda_\mathbb{X}\Psi^\mathbb{X} & \text{in } \Pi_\mathbb{X}, \\ (-i\nabla + \mathbf{A}_\mathbb{X})\Psi^\mathbb{X} \cdot \mathbf{n} = 0 & \text{on } \partial\Pi_\mathbb{X}. \end{cases} \tag{9.6}$$

For $h > 0$, we define $\Psi^\mathbb{X}_h$ by using the canonical scaling (7.5). This gives an AGE for the operator $H_h(\mathbf{A}_\mathbb{X}, \Pi_\mathbb{X})$ associated with the value $h\Lambda_\mathbb{X}$. Let χ_h be the cut-off function defined by (7.7)–(7.8) involving the parameter $R > 0$ and the exponent $\delta \in (0, \frac{1}{2})$. Then the function

$$(\chi_h\Psi^\mathbb{X}_h)(\mathbf{x}) = \underline{\chi}\left(\frac{|\mathbf{x}|}{Rh^\delta}\right)\Psi^\mathbb{X}\Big(\frac{\mathbf{x}}{\sqrt{h}}\Big), \quad \text{for} \quad \mathbf{x} \in \Pi_\mathbb{X}\,, \tag{9.7}$$

is a canonical quasimode on the tangent structure $\Pi_\mathbb{X}$ for the model operator $H_h(\mathbf{A}_\mathbb{X}, \Pi_\mathbb{X})$: Indeed the identity (7.9) and Lemma 7.10 yield

$$\mathscr{Q}_h[\mathbf{A}_\mathbb{X}, \Pi_\mathbb{X}](\chi_h\Psi^\mathbb{X}_h) = h\Lambda_\mathbb{X} + \mathcal{O}(h^{2-2\delta}). \tag{9.8}$$

Let us recall that the fact that $\Psi^\mathbb{X}_h$ belongs to $\mathrm{Dom}_{\,\mathrm{loc}}(H_h(\mathbf{A}_\mathbb{X}, \Pi_\mathbb{X}))$ is essential for the validity of the identity above.

In order to prove Theorem 9.1, we are going to construct a family of quasimodes $\varphi_h^{[0]} \in H^1(\Omega)$ satisfying the estimate for $h > 0$ small enough and the suitable power κ

$$\mathscr{Q}_h[\mathbf{A}, \Omega](\varphi_h^{[0]}) \leq h\Lambda_{\mathbb{X}} + C_\Omega(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})h^\kappa. \tag{9.9}$$

The rationale of this construction is to build a link between the canonical quasimode $\chi_h \Psi_h^{\mathbb{X}}$ on the tangent structure $\Pi_{\mathbb{X}}$ with our original operator $H_h(\mathbf{A}, \Omega)$.

Let ν be the length of the chain $\mathbb{X}$. By Proposition 3.29, we can always reduce to $\nu \leq 3$. We write

$$\mathbb{X} = (\mathbf{x}_0, \dots, \mathbf{x}_{\nu-1}) \quad \text{with} \quad \nu \in \{1, 2, 3\}.$$

Our quasimode $\varphi_h^{[0]}$ will have distinct features according to the value of ν: We will need $\nu - 1$ intermediaries $\varphi_h^{[j]}$, $0 < j < \nu$, between $\varphi_h^{[0]}$ and the final object $\varphi_h^{[\nu]}$ defined by the truncated AGE given in (9.7), *i.e.*,

$$\varphi_h^{[\nu]} = \chi_h \Psi_h^{\mathbb{X}}. \tag{9.10}$$

For $j = 1, \dots, \nu$, the function $\varphi_h^{[j]}$ is defined in the tangent structure $\Pi_{\mathbf{x}_0,\dots,\mathbf{x}_{j-1}}$. At a glance

$\nu = 1$ The quasimode $\varphi_h^{[0]}$ is deduced from $\varphi_h^{[1]} = \chi_h \Psi_h^{\mathbb{X}}$ through the local map $\mathrm{U}^{\mathbf{x}_0}$. This is the classical construction: We say that the quasimode is *sitting* because as $h \to 0$ the supports of $\varphi_h^{[0]}$ are included in each other and concentrate to $\mathbf{x}_0$.

$\nu = 2$ The quasimode $\varphi_h^{[0]}$ is deduced from $\varphi_h^{[1]}$ through the local map $\mathrm{U}^{\mathbf{x}_0}$, and $\varphi_h^{[1]}$ is itself deduced from $\varphi_h^{[2]} = \chi_h \Psi_h^{\mathbb{X}}$ through another local map $\mathrm{U}^{\mathbf{v}_1}$ connected to the second element $\mathbf{x}_1$ of the chain. We say that the quasimode is *sliding* because as $h \to 0$ the supports of $\varphi_h^{[0]}$ are shifted along a direction $\widehat{\mathbf{x}}_1$ determined by $\mathbf{x}_1$. At this point, the construction will be very different depending on whether $\mathbf{x}_0$ is a conical point or not, and we say that the quasimodes are respectively *hard sliding* and *soft sliding*.

$\nu = 3$ The quasimode $\varphi_h^{[0]}$ is still deduced from $\varphi_h^{[1]}$ through $\mathrm{U}^{\mathbf{x}_0}$, and $\varphi_h^{[1]}$ from $\varphi_h^{[2]}$ through $\mathrm{U}^{\mathbf{v}_1}$. Finally $\varphi_h^{[2]}$ is itself deduced from $\varphi_h^{[3]} = \chi_h \Psi_h^{\mathbb{X}}$ through a third local map $\mathrm{U}^{\mathbf{v}_2}$ connected to the third element $\mathbf{x}_2$ of the chain. We say that the quasimode is *doubly sliding* because as $h \to 0$ the supports of $\varphi_h^{[0]}$ are shifted along two directions $\widehat{\mathbf{x}}_1$ and $\widehat{\mathbf{x}}_2$ determined by $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively.

At each level of these constructions, different transformations of the quadratic form will be performed. We organize them in 3 steps [a], [b], and [c]:

[a] for a change of variable into a higher tangent substructure,

[b] for a linearization of the metrics,

[c] for a linearization of the potential.

This construction is illustrated in Figure 2.

Let us introduce some notation.

Notation 9.2. (1) If U is a diffeomorphism, let U_* be the operator of composition: $\mathrm{U}_*(f) = f \circ \mathrm{U}$.
(2) If $\zeta_h^{\mathbf{v}}$ is a phase, let $\mathrm{Z}_h^{\mathbf{v}}$ be the operator of multiplication $\mathrm{Z}_h^{\mathbf{v}}(f) = f\,\overline{\zeta_h^{\mathbf{v}}}$.

We are going to define recursively functions $\varphi_h^{[j]}$ assuming that $\varphi_h^{[j+1]}$ is known. Typically, these relations will take the form

$$\varphi_h^{[j]} = \mathrm{Z}_h^{\mathbf{v}_j} \circ \mathrm{U}_*^{\mathbf{v}_j}(\varphi_h^{[j+1]}). \tag{9.11}$$

Remark 9.3. Since $\mathbf{x}_0$ is determined, we can always assume that $\mathbf{x}_0$ belongs to the reference set $\mathfrak{X}$ of an admissible atlas. The error rate that we will obtain in the end will depend on whether $\nu = 1$ or is larger, and on whether $\mathbf{x}_0$ is a conical point or not.

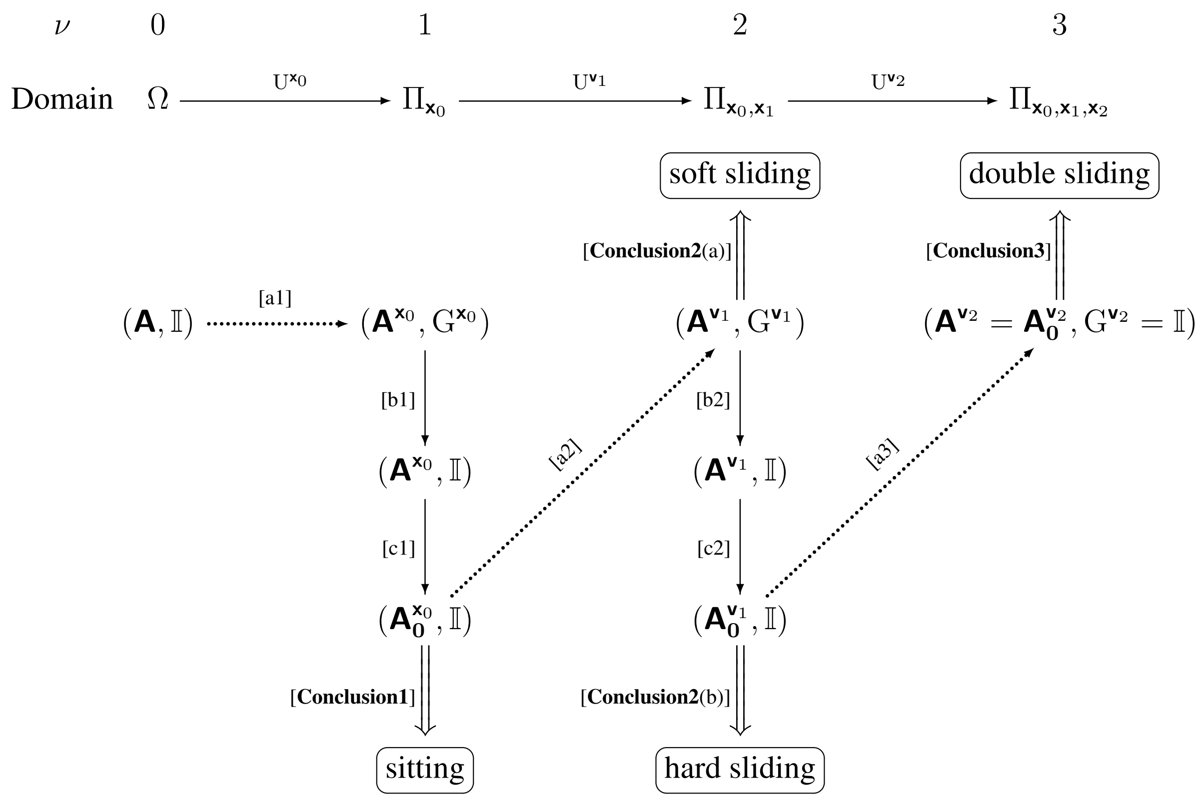

FIGURE 2. Construction of quasimodes

9.2. **First level of construction and sitting quasimodes.** We perform the first change of variables as in Section 4.1: The local diffeomorphism $\mathrm{U}^{\mathbf{x}_0}$ sends (a neighborhood of) $\mathbf{x}_0$ in $\overline{\Omega}$ to (a neighborhood of) $\mathbf{0}$ in $\overline{\Pi}_{\mathbf{x}_0}$.

• [a1]. Let $\mathbf{A}^{\mathbf{x}_0}$ be the new potential (4.1) deduced from $\mathbf{A} - \mathbf{A}(\mathbf{x}_0)$ by the local map $\mathrm{U}^{\mathbf{x}_0}$. Let $\zeta_h^{\mathbf{x}_0}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}(\mathbf{x}_0),\, \mathbf{x}/h\rangle}$, for $\mathbf{x} \in \Omega$. Let us introduce the relation

$$\varphi_h^{[0]} = \mathrm{Z}_h^{\mathbf{x}_0} \circ \mathrm{U}_*^{\mathbf{x}_0}(\varphi_h^{[1]}), \tag{9.12}$$

and let $r_h^{[1]}$ be the radius of the smallest ball centered at $\mathbf{0}$ containing the support of $\varphi_h^{[1]}$ in $\overline{\Pi}_{\mathbf{x}_0}$. The number $r_h^{[1]}$ is intended to converge to 0 as h tends to 0.

Using (4.5), we have

$$\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) = \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0},\mathrm{G}^{\mathbf{x}_0}](\varphi_h^{[1]}). \tag{9.13}$$

• [b1]. We now linearize the metric $\mathrm{G}^{\mathbf{x}_0}$ in (9.13) by using Lemma 4.5, case (a). We find the relation between the Rayleigh quotients

$$\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) = \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\,\big(1+\mathcal{O}(r_h^{[1]})\big), \tag{9.14}$$

which implies

$$\Big|\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) - \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\Big| \le C_\Omega\, r_h^{[1]}\, \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}). \tag{9.15}$$

• [c1]. We recall that $\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}$ is the linear part of $\mathbf{A}^{\mathbf{x}_0}$ at $\mathbf{0}$. Using relation (A.6) with $\mathbf{A} = \mathbf{A}^{\mathbf{x}_0}$ and $\mathbf{A}' = \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}$ and a Cauchy-Schwarz inequality, we obtain

$$\Big|q_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - q_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\Big| \le 2\Big(a_h^{[1]}\sqrt{\mu_h^{[1]}} + \big(a_h^{[1]}\big)^2\Big)\|\varphi_h^{[1]}\|^2, \tag{9.16}$$

where we have set

$$\mu_h^{[1]} = \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) \quad\text{and}\quad a_h^{[1]} = \frac{\|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0})\varphi_h^{[1]}\|}{\|\varphi_h^{[1]}\|}. \tag{9.17}$$

By Lemmas 4.6 and 4.7 (a), and since $\varphi_h^{[1]}$ is supported in the ball $\mathcal{B}(\mathbf{0}, r_h^{[1]})$, we have

$$a_h^{[1]} \le C(\mathbf{A})\,\big(r_h^{[1]}\big)^2 \quad\text{with}\quad C(\mathbf{A}) = C_\Omega\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big). \tag{9.18}$$

Putting together (9.16)–(9.18), we obtain

$$\Big|\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\Big| \le C(\mathbf{A})\Big(\big(r_h^{[1]}\big)^2\sqrt{\mu_h^{[1]}} + \big(r_h^{[1]}\big)^4\Big). \tag{9.19}$$

Using the above estimate (9.19), we have

$$r_h^{[1]}\,\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) \le r_h^{[1]}\,\left(\mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) + C(\mathbf{A})\Big(\big(r_h^{[1]}\big)^2\sqrt{\mu_h^{[1]}} + \big(r_h^{[1]}\big)^4\Big)\right).$$

Combining this last inequality, (9.19) and (9.15), we have for $r_h^{[1]}$ small enough

$$\Big|\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) - \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\Big| \le C(\mathbf{A})\Big(\mu_h^{[1]} r_h^{[1]} + \big(r_h^{[1]}\big)^2\sqrt{\mu_h^{[1]}} + \big(r_h^{[1]}\big)^4\Big). \tag{9.20}$$

• **[Conclusion1].** If $\nu = 1$, we set, as already mentioned, $\varphi_h^{[1]} = \chi_h \Psi_h^{\mathbb{X}}$. Note that $\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}$ coincides with $\mathbf{A}_{\mathbb{X}}$. To tune the cut-off χ_h, we choose the exponent δ as δ_0 and the radius R as 1. Therefore $r_h^{[1]} = \mathcal{O}(h^{\delta_0})$ and by (9.8) $\mu_h^{[1]} = \mathcal{O}(h)$. Using (9.20) and again (9.8), we deduce

$$\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) \le h\Lambda_{\mathbb{X}} + C(\mathbf{A})\big(h^{2-2\delta_0} + h^{1+\delta_0} + h^{\frac12+2\delta_0} + h^{4\delta_0}\big). \tag{9.21}$$

So we can conclude in the sitting case. Choosing $\delta_0 = 3/8$, we optimize remainders and we get the upper bound

$$\lambda_h(\mathbf{B},\Omega) \le h\mathscr{E}(\mathbf{B},\Omega) + C_\Omega\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{5/4}\,.$$

• *Case when* $\mathbf{B}(\mathbf{x}_0) = 0$. If $\mathbf{B}(\mathbf{x}_0) = 0$, the function $\Psi^{\mathbb{X}} \equiv 1$ is an AGE on $\Pi_{\mathbf{x}_0}$ associated with the value $\Lambda_{\mathbb{X}} = 0$. We are in the sitting case $\nu = 1$ and the estimate (9.20) is still valid. But now (9.8) (combined with Remark 7.11) yields

$$\mathscr{Q}_h[\mathbf{A}_{\mathbb{X}}, \Pi_{\mathbb{X}}](\chi_h \Psi^{\mathbb{X}}_h) \le Ch^{2-2\delta}.$$

Choosing δ as δ_0 as above, we deduce $\mu^{[1]}_h = \mathcal{O}(h^{2-2\delta_0})$. Hence

$$\mathscr{Q}_h[\mathbf{A}, \Omega](\varphi^{[0]}_h) \le C\big(h^{2-2\delta_0} + h^{2-2\delta_0+\delta_0} + h^{1-\delta_0+2\delta_0} + h^{4\delta_0}\big). \tag{9.22}$$

Choosing $\delta_0 = 1/3$, we optimize remainders and we get the upper bound

$$\lambda_h(\mathbf{B}, \Omega) \le C_\Omega\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{4/3}\,.$$

• *Case when* $\mathbf{x}_0$ *is a corner and* $\Psi^{\mathbb{X}}$ *is an eigenvector.* Since $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) < \mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ and $\lambda_{\mathrm{ess}}(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) = \mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ by Theorem 6.6, the generalized eigenfunction $\Psi^{\mathbb{X}}$ of $H(\mathbf{A}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ provided by Theorem 7.3 is an eigenfunction and has exponential decay. Here $\mathbb{X} = (\mathbf{x}_0)$ and the quasimode $\varphi^{[0]}_h$ is sitting. Using (4.14) and Lemma 4.7 (a), we get $C_\Omega > 0$ such that

$$\forall \mathbf{x} \in \operatorname{supp}(\varphi^{[1]}_h), \quad |(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}})(\mathbf{x})| \le C_\Omega \|\mathbf{A}^{\mathbf{x}_0}\|_{W^{2,\infty}(\operatorname{supp}(\varphi^{[1]}_h))} |\mathbf{x}|^2\,.$$

Using the change of variable $\mathbf{X} = \mathbf{x}h^{-1/2}$ and the exponential decay of $\Psi^{\mathbb{X}}$ we get

$$a^{[1]}_h = \frac{\|(\mathbf{A}^{\mathbf{x}_0} - \mathbf{A}^{\mathbf{x}_0}_{\mathbf{0}})\varphi^{[1]}_h\|}{\|\varphi^{[1]}_h\|} \le C_\Omega \|\mathbf{A}^{\mathbf{x}_0}\|_{W^{2,\infty}(\operatorname{supp}(\varphi^{[1]}_h))} h. \tag{9.23}$$

Using (9.16) with estimate (9.23) and Lemma 7.10, for any $\delta \in (0, \frac{1}{2}]$, we get

$$\begin{aligned}
\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi^{[1]}_h) &\le h\Lambda_{\mathbb{X}} + C\Big(h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac{1}{2}}} + \|\mathbf{A}\|_{W^{2,\infty}(\Omega)}h^{\frac{3}{2}} + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}h^2\Big) \\
&\le h\Lambda_{\mathbb{X}} + C\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\,\big(h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac{1}{2}}} + h^{\frac{3}{2}}\big).
\end{aligned}$$

Thanks to (9.15), the quasimode $\varphi^{[0]}_h$ satisfies

$$\begin{aligned}
\mathscr{Q}_h[\mathbf{A}, \Omega](\varphi^{[0]}_h) &\le \big(1 + \mathcal{O}(h^\delta)\big)\big\{h\Lambda_{\mathbb{X}} + C\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)(h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac{1}{2}}} + h^{3/2})\big\} \\
&\le h\Lambda_{\mathbb{X}} + C\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)\big\{h^{1+\delta} + h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac{1}{2}}} + h^{3/2}\big\}.
\end{aligned}$$

Here C denotes various constants depending on Ω but independent from $h \le h_0$ and $\delta \le \frac{1}{2}$. We optimize this by taking $\delta = \frac{1}{2} - \varepsilon(h)$ with $\varepsilon(h)$ so that $h^{1+\delta} = h^{2-2\delta}\mathrm{e}^{-ch^{\delta-\frac{1}{2}}}$, *i.e.*,

$$h^{\frac{3}{2}-\varepsilon(h)} = h^{1+2\varepsilon(h)}\mathrm{e}^{-ch^{-\varepsilon(h)}}.$$

We find

$$\mathrm{e}^{ch^{-\varepsilon(h)}} = h^{-\frac{1}{2}+3\varepsilon(h)}, \quad \textit{i.e.}, \quad h^{-\varepsilon(h)} = \tfrac{1}{c}(-\tfrac{1}{2} + 3\varepsilon(h))\log h\,.$$

The latter equation has one solution $\varepsilon(h)$ which tends to 0 as h tends to 0. Replacing $h^{-\varepsilon(h)}$ by the value above in $h^{\frac{3}{2}-\varepsilon(h)}$, we find that the remainder is a $\mathcal{O}(h^{3/2}|\log h|)$.

• *Case when Ω is a straight polyhedron and* $\mathbf{B}$ *constant.* According to Remark 8.3 d), we may assume that $(\mathbf{B}, \Pi_{\mathbf{x}_0})$ is in case (i) of the Dichotomy Theorem. We construct a sitting quasimode near $\mathbf{x}_0$. Since the magnetic field is constant, we may associate a linear magnetic potential $\mathbf{A}$. Define now $\varphi_h^{[0]}$ from $\varphi_h^{[1]}$ as in (9.12) and tune the cut-off by choosing $\delta = 0$ and $R > 0$ large enough such that the support of χ_h is contained in a map-neighborhood $\mathcal{V}_{\mathbf{x}_0}$ of $\mathbf{0}$ in $\Pi_{\mathbf{x}_0}$.

Notice that $\mathrm{U}^{\mathbf{x}_0}$ is the translation $\mathbf{x} \mapsto \mathbf{x} - \mathbf{x}_0$ and that the linear part of the potential satisfies $\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0} = \mathbf{A}^{\mathbf{x}_0}$. Therefore the error terms due to the change of variables and the linearization of the potential appearing in step [b1] are zero, and (9.20) is improved in

$$\mathscr{Q}_h[\mathbf{A}, \Omega](\varphi_h^{[0]}) = \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]})\,.$$

Estimate (9.5) is then a direct consequence of identity (7.9) combined with Lemma 7.10.

9.3. **Second level of construction and sliding quasimodes.** We have now to deal with the case $\nu \geq 2$. So $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$ or $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$.

Here we use the same notation as the introduction of singular chains in Section 3.4. Let $\underline{\mathrm{U}}^0 \in \mathfrak{O}_3$ such that $\Pi_{\mathbf{x}_0} = \underline{\mathrm{U}}^0(\mathbb{R}^{3-d_0} \times \Gamma_{\mathbf{x}_0})$ where $\Gamma_{\mathbf{x}_0}$ is the reduced cone of $\Pi_{\mathbf{x}_0}$. Let $\Omega_{\mathbf{x}_0} = \Gamma_{\mathbf{x}_0} \cap \mathbb{S}^{d_0-1}$ be the section of $\Gamma_{\mathbf{x}_0}$. By definition of chains, $\mathbf{x}_1$ belongs to $\overline{\Omega}_{\mathbf{x}_0}$ and let $C_{\mathbf{x}_0,\mathbf{x}_1}$ be the tangent cone to $\Omega_{\mathbf{x}_0}$ at $\mathbf{x}_1$. Then the tangent substructure $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ is determined by the formula

$$\Pi_{\mathbf{x}_0,\mathbf{x}_1} = \underline{\mathrm{U}}^0\big(\mathbb{R}^{3-d_0} \times \langle \mathbf{x}_1 \rangle \times C_{\mathbf{x}_0,\mathbf{x}_1}\big)\,.$$

Let us define the unitary vector $\widehat{\mathbf{x}}_1$ by the formulas

$$\underline{\widehat{\mathbf{x}}_1} := (0, \mathbf{x}_1) \in \mathbb{R}^{3-d_0} \times \Gamma_{\mathbf{x}_0} \quad \text{and} \quad \widehat{\mathbf{x}}_1 = \underline{\mathrm{U}}^0\, \underline{\widehat{\mathbf{x}}_1} \in \overline{\Pi}_{\mathbf{x}_0} \cap \mathbb{S}^2. \tag{9.24}$$

With this definition, the substructure $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ is the tangent cone to $\Pi_{\mathbf{x}_0}$ at the point $\widehat{\mathbf{x}}_1$. Note that in the case when $\mathbf{x}_0$ is a vertex of Ω, the above formulas simplify: $\Pi_{\mathbf{x}_0}$ is its own reduced cone, $\Pi_{\mathbf{x}_0,\mathbf{x}_1} = \langle \mathbf{x}_1 \rangle \times C_{\mathbf{x}_0,\mathbf{x}_1}$, and $\widehat{\mathbf{x}}_1$ coincides with $\mathbf{x}_1$.

Note also that the cone $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ can be the full space, a half-space or a wedge, and that $\widehat{\mathbf{x}}_1$ gives a direction associated with $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ starting from the origin $\mathbf{0}$ of $\Pi_{\mathbf{x}_0}$:

(1) If $\Pi_{\mathbf{x}_0,\mathbf{x}_1} \equiv \mathbb{R}^3$, then $\widehat{\mathbf{x}}_1$ belongs to the interior of $\Pi_{\mathbf{x}_0}$.
(2) If $\Pi_{\mathbf{x}_0,\mathbf{x}_1} \equiv \mathbb{R}^2 \times \mathbb{R}_+$, then $\widehat{\mathbf{x}}_1$ belongs to a face of $\Pi_{\mathbf{x}_0}$.
(3) If $\Pi_{\mathbf{x}_0,\mathbf{x}_1} \equiv \mathcal{W}_\alpha$, then $\widehat{\mathbf{x}}_1$ belongs to an edge of $\Pi_{\mathbf{x}_0}$.

Unless we are in the latter case ($\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ is a wedge), the choice of $\widehat{\mathbf{x}}_1$ is not unique.

Set $\mathbf{v}_1 = d_h^{[1]}\widehat{\mathbf{x}}_1$ where $d_h^{[1]}$ is a positive quantity intended to converge to 0 with h. The vector $\mathbf{v}_1$ is a shift that allows to pass from the cone $\Pi_{\mathbf{x}_0}$ to the substructure $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$, which is also the tangent cone to $\Pi_{\mathbf{x}_0}$ at the point $\mathbf{v}_1$. Let $\mathrm{U}^{\mathbf{v}_1}$ be a local diffeomorphism that sends (a neighborhood $\mathcal{U}_{\mathbf{v}_1}$ of) $\mathbf{v}_1$ in $\Pi_{\mathbf{x}_0}$ to (a neighborhood $\mathcal{V}_{\mathbf{v}_1}$ of) $\mathbf{0}$ in $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$. We can assume without restriction that $\mathrm{U}^{\mathbf{v}_1}$ is part of an admissible atlas on $\Pi_{\mathbf{x}_0}$.

• [a2]. By the change of variable $\mathrm{U}^{\mathbf{v}_1}$, the potential $\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}(\mathbf{v}_1)$ becomes $\mathbf{A}^{\mathbf{v}_1}$ (cf. (4.1))

$$\mathbf{A}^{\mathbf{v}_1} = (\mathrm{J}^{\mathbf{v}_1})^\top \Big(\big(\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0} - \mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}(\mathbf{v}_1)\big) \circ (\mathrm{U}^{\mathbf{v}_1})^{-1}\Big) \quad \text{with} \quad \mathrm{J}^{\mathbf{v}_1} = \mathrm{d}(\mathrm{U}^{\mathbf{v}_1})^{-1}.$$

Let $\zeta_h^{\mathbf{v}_1}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}_\mathbf{0}^{\mathbf{x}_0}(\mathbf{v}_1), \mathbf{x}/h\rangle}$, for $\mathbf{x} \in \Pi_{\mathbf{x}_0}$. We introduce the relation

$$\varphi_h^{[1]} = \mathrm{Z}_h^{\mathbf{v}_1} \circ \mathrm{U}_*^{\mathbf{v}_1}(\varphi_h^{[2]}), \tag{9.25}$$

and let $r_h^{[2]}$ be the radius of the smallest ball centered at $\mathbf{0}$ containing the support of $\varphi_h^{[2]}$ in $\overline{\Pi}_{\mathbf{x}_0,\mathbf{x}_1}$. This new quantity is also intended to converge to 0 with h.

We now have a turning point of the algorithm: if $\mathbf{x}_0$ is not a conical point, we use the fact that $\mathrm{U}^{\mathbf{v}_1}$ is a translation. Then $\mathrm{G}^{\mathbf{v}_1} = \mathbb{I}$ and $\mathbf{A}^{\mathbf{v}_1}$ coincides with its linear part $\mathbf{A}_\mathbf{0}^{\mathbf{v}_1}$. Steps [b] and [c] are replaced by the following identity:

$$\mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) = \mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{v}_1}, \Pi_{\mathbb{X}}](\varphi_h^{[2]}), \tag{9.26}$$

and we are able to make a direct estimation of the quasimodes, see the [Conclusion2(a)] below. We will called them *soft sliding quasimodes*.

If $\mathbf{x}_0$ is a conical point, we continue the algorithm as described below:

- [b2]. Using (4.5) and (4.13) in Lemma 4.5, we find a relation between Rayleigh quotients of the same form as (9.14), with $\mathcal{O}(r_h^{[1]})$ replaced by $\mathcal{O}(r_h^{[2]}/d_h^{[1]})$. Like for (9.15), we deduce

$$\Big|\mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - \mathscr{Q}_h[\mathbf{A}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]})\Big| \lesssim \frac{r_h^{[2]}}{d_h^{[1]}} \mathscr{Q}_h[\mathbf{A}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}). \tag{9.27}$$

- [c2]. Let $\mathbf{A}_\mathbf{0}^{\mathbf{v}_1}$ be the linear part of $\mathbf{A}^{\mathbf{v}_1}$ at $\mathbf{0} \in \Pi_{\mathbf{x}_0,\mathbf{x}_1}$. Thus, by relation (A.6) and a Cauchy-Schwarz inequality, we have

$$\Big|q_h[\mathbf{A}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}) - q_h[\mathbf{A}_\mathbf{0}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]})\Big| \le C\Big(a_h^{[2]}\sqrt{\mu_h^{[2]}} + \big(a_h^{[2]}\big)^2\Big)\|\varphi_h^{[2]}\|^2, \tag{9.28}$$

with

$$\mu_h^{[2]} = \mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}) \quad \text{and} \quad a_h^{[2]} = \frac{\|(\mathbf{A}^{\mathbf{v}_1} - \mathbf{A}_\mathbf{0}^{\mathbf{v}_1})\varphi_h^{[2]}\|}{\|\varphi_h^{[2]}\|}. \tag{9.29}$$

By Lemmas 4.6–4.7, case (b), and since $\varphi_h^{[2]}$ is supported in the ball $\mathcal{B}(\mathbf{0}, r_h^{[2]})$, we have

$$a_h^{[2]} \lesssim \frac{\big(r_h^{[2]}\big)^2}{d_h^{[1]}}. \tag{9.30}$$

Using (9.27)–(9.30), we find, if $r_h^{[2]}/d_h^{[1]}$ is small enough,

$$\Big|\mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - \mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]})\Big| \lesssim \mu_h^{[2]}\frac{r_h^{[2]}}{d_h^{[1]}} + \frac{\big(r_h^{[2]}\big)^2}{d_h^{[1]}}\sqrt{\mu_h^{[2]}} + \frac{\big(r_h^{[2]}\big)^4}{\big(d_h^{[1]}\big)^2}. \tag{9.31}$$

- **[Conclusion2].** If $\nu = 2$, we set, as already mentioned, $\varphi_h^{[2]} = \chi_h \Psi_h^{\mathbb{X}}$. Note that $\mathbf{A}_\mathbf{0}^{\mathbf{v}_1}$ coincides with $\mathbf{A}_{\mathbb{X}}$. We have now to distinguish two cases, according as $\mathbf{x}_0$ is or not a conical point.

(a) *Soft sliding*. If $\mathbf{x}_0$ is not a conical point, *i.e.*, $\mathbf{x}_0 \notin \mathfrak{V}^\circ$, the local map $\mathrm{U}^{\mathbf{v}_1}$ is the translation $\mathbf{x} \mapsto \mathbf{x} - \mathbf{v}_1$. To tune the cut-off χ_h, we choose the exponent δ as δ_0 and the shift $d_h^{[1]}$ as h^{δ_0}.

We choose the radius R for the cut-off χ_h (7.8) so that the support of $\underline{\chi}_R$ is contained in a map neighborhood $\mathcal{V}_{\mathbf{v}_1}$ of $\mathbf{0}$ in $\overline{\Pi}_{\mathbf{x}_0,\mathbf{x}_1}$, *i.e.*, a neighborhood such that:

$$\mathrm{U}^{\mathbf{v}_1}(\mathcal{U}_{\mathbf{v}_1} \cap \Pi_{\mathbf{x}_0}) = \mathcal{V}_{\mathbf{v}_1} \cap \Pi_{\mathbf{x}_0,\mathbf{x}_1},$$

where $\mathrm{U}^{\mathbf{v}_1}(\mathbf{x}) = \mathbf{x} - \mathbf{v}_1$ and $\mathcal{U}_{\mathbf{v}_1} = \mathcal{V}_{\mathbf{v}_1} + \mathbf{v}_1$. Then the quantities $r_h^{[1]}$ and $r_h^{[2]}$ are both $\mathcal{O}(h^{\delta_0})$ and we can combine (9.26) with (9.20) and the cut-off estimate (9.8). Moreover for h small enough, the quantities $\mu_h^{[1]}$ is $\mathcal{O}(h)$, and we deduce the estimate (9.21) as in the case $\nu = 1$, which leads, like in the sitting case, to the upper bound (9.2) with $h^{5/4}$. The latter step ends in particular the handling of the polyhedral case since we can always reduce to chains of length $\nu \leq 2$ in polyhedral domains, cf. Proposition 3.29.

(b) *Hard sliding*. If $\mathbf{x}_0$ is a conical point, to tune the cut-off χ_h, we choose the exponent δ as $\delta_0 + \delta_1$ and the shift $d_h^{[1]}$ as h^{δ_0}, with $\delta_0, \delta_1 > 0$ such that $\delta_0 + \delta_1 < \frac{1}{2}$. We choose the radius R equal to 1. Therefore $r_h^{[2]} = \mathcal{O}(h^{\delta_0+\delta_1})$ and $r_h^{[1]} = \mathcal{O}(h^{\delta_0})$. By (9.8) $\mu_h^{[2]} = \mathcal{O}(h)$ and, since for h small enough, $r_h^{[2]}/d_h^{[1]}$ is arbitrarily small, we also deduce with the help of (9.31) that $\mu_h^{[1]} = \mathcal{O}(h)$. Putting this together with (9.20) and (9.31), and using (9.8) once more, we deduce the estimate

$$\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) \leq h\Lambda_{\mathbb{X}} + C\left(h^{1+\delta_0} + h^{\frac{1}{2}+2\delta_0} + h^{4\delta_0}\right) + C\left(h^{2-2\delta_0-2\delta_1} + h^{1+\delta_1} + h^{\frac{1}{2}+\delta_0+2\delta_1} + h^{2\delta_0+4\delta_1}\right). \tag{9.32}$$

The exponents that appear here are the same as for the lower bound (5.26). Thus taking $\delta_0 = 3/10$ and $\delta_1 = 3/20$, we optimize remainders and deduce

$$\lambda_h(\mathbf{B},\Omega) \leq h\mathscr{E}(\mathbf{B},\Omega) + C_\Omega\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{11/10}\,.$$

9.4. **Third level of construction and doubly sliding quasimodes.** It remains to deal the case $\nu = 3$. In that case, the chain $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ is such that

- $\mathbf{x}_0$ is a conical point,
- $\mathbf{x}_1$ is a vertex of $\Omega_{\mathbf{x}_0}$, $\widehat{\mathbf{x}}_1$ coincides with $\mathbf{x}_1$, the corresponding edge of $\Pi_{\mathbf{x}_0}$ is generated by $\mathbf{x}_1$, and $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ is a wedge,
- $\mathbf{x}_2$ is an end of the interval $\Omega_{\mathbf{x}_0,\mathbf{x}_1}$, it corresponds to a point $\widehat{\mathbf{x}}_2$ on a face of $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$, defined as in (9.24). Finally $\Pi_{\mathbf{x}_1,\mathbf{x}_1,\mathbf{x}_2} = \Pi_{\mathbb{X}}$ is a half-space.

Set $\mathbf{v}_2 = d_h^{[2]}\widehat{\mathbf{x}}_2$ where $d_h^{[2]}$ is a positive quantity intended to converge to 0 with h. Let $\mathrm{U}^{\mathbf{v}_2}$ be the translation that sends (a neighborhood of) $\mathbf{v}_2$ in $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ to (a neighborhood of) $\mathbf{0}$ in $\Pi_{\mathbb{X}} = \Pi_{\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2}$.

- [a3]. By the change of variable $\mathrm{U}^{\mathbf{v}_2}$, since $\mathrm{J}^{\mathbf{v}_2} = \mathbb{I}_3$, the potential $\mathbf{A}_\mathbf{0}^{\mathbf{v}_1} - \mathbf{A}_\mathbf{0}^{\mathbf{v}_1}(\mathbf{v}_2)$ becomes

$$\mathbf{A}^{\mathbf{v}_2} = \big(\mathbf{A}_\mathbf{0}^{\mathbf{v}_1} - \mathbf{A}_\mathbf{0}^{\mathbf{v}_1}(\mathbf{v}_2)\big) \circ (\mathrm{U}^{\mathbf{v}_2})^{-1},$$

and it coincides with its linear part $\mathbf{A}_\mathbf{0}^{\mathbf{v}_2}$. Let $\zeta_h^{\mathbf{v}_2}(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}_\mathbf{0}^{\mathbf{v}_1}(\mathbf{v}_2),\, \mathbf{x}/h\rangle}$, for $\mathbf{x} \in \Pi_{\mathbf{x}_0,\mathbf{x}_1}$. We define

$$\varphi_h^{[2]} = Z_h^{\mathbf{v}_2} \circ \mathrm{U}_*^{\mathbf{v}_2}(\varphi_h^{[3]}). \tag{9.33}$$

Since $\mathrm{G}^{\mathbf{v}_2} = \mathbb{I}_3$, we have

$$\mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}) = \mathscr{Q}_h[\mathbf{A}_\mathbf{0}^{\mathbf{v}_2}, \Pi_{\mathbb{X}}](\varphi_h^{[3]}). \tag{9.34}$$

• **[Conclusion3].** We set, as already mentioned $\varphi_h^{[3]} = \chi_h \Psi_h^{\mathbb{X}}$. We have $\mathbf{A}_{\mathbf{0}}^{\mathbf{v}_2} = \mathbf{A}_{\mathbb{X}}$. We choose the exponent δ as $\delta_0 + \delta_1$, the shifts $d_h^{[2]}$ as $h^{\delta_0+\delta_1}$ and $d_h^{[1]}$ as h^{δ_0}, with $\delta_0, \delta_1 > 0$ such that $\delta_0 + \delta_1 < \frac{1}{2}$. We conclude as the conical case at level 2 and obtain again (9.32). We deduce

$$\lambda_h(\mathbf{B}, \Omega) \leq h\mathscr{E}(\mathbf{B}, \Omega) + C_\Omega\big(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\big)h^{11/10}\,.$$

9.5. **Conclusion.** The outcome of the last four sections is the achievement of the proof of Theorem 9.1. We may notice that there is only one configuration where we cannot prove the convergence rate $h^{5/4}$: This is the case when all points with minimal local energy $\mathbf{x}_0$ satisfy all the following conditions

(1) $\mathbf{x}_0$ is a conical point ($\mathbf{x}_0 \in \mathfrak{V}^\circ$),
(2) The model operator $H(\mathbf{A}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0})$ has no eigenvalue below its essential spectrum,
(3) The geometry around $\mathbf{x}_0$ is not trivial i.e., the derivative $\mathrm{K}^{\mathbf{x}_0}(\mathbf{0})$ of the Jacobian is not zero.

10. Stability of Admissible Generalized Eigenvectors

In order to meet our claim for the improved upper bounds (1.12), we need to revisit AGE's (Admissible Generalized Eigenvectors) of model problems $H(\mathbf{A}, \Pi)$. In particular we want to know what are their stability properties under perturbation of the constant magnetic field $\mathbf{B} = \operatorname{curl} \mathbf{A}$.

10.1. **Structure of AGE's.** In this section we recall from Section 6 the model reference configurations $(\mathbf{B}, \Pi)$ owning an AGE and give a comprehensive overview of their structure in a table.

Let $\mathbf{B}$ be a constant magnetic field and Π a cone in $\mathfrak{P}_3$. Remind that $d = d(\Pi)$ is the reduced dimension of Π, cf. Definition 3.16. Let us assume that $E(\mathbf{B}, \Pi) < \mathscr{E}^*(\mathbf{B}, \Pi)$. Therefore by Theorem 7.3 there exists an AGE Ψ that has the form (7.1). We recall the discriminant parameter $k \in \{1, 2, 3\}$ that is the number of directions in which the generalized eigenvector has an exponential decay. For further use we call (G1), (G2), and (G3) the situation where $k = 1$, 2, and 3, respectively. As a consequence of Lemma 7.2, it is enough to concentrate on *reference configurations* for the magnetic field $\mathbf{B}$, its potential $\mathbf{A}$ and the cone Π. In such a reference configuration the AGE writes as

$$\Psi(\mathbf{y}, \mathbf{z}) = \mathrm{e}^{i\vartheta(\mathbf{y},\mathbf{z})}\, \Phi(\mathbf{z}) \qquad \forall \mathbf{y} \in \mathbb{R}^{3-k}, \ \ \forall \mathbf{z} \in \Upsilon.$$

In Table 1 we gather all possible situations for the couple of dimensions (k, d). We provide the explicit form of an admissible generalized eigenfunction Ψ of $H(\mathbf{A}, \Pi)$ in variables $(\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{3-k} \times \Upsilon$ where $\mathbf{A}$ is a reference linear potential associated with $\mathbf{B}$. Note that the cone Υ on which Ψ has exponential decay does not always coincide with the reduced cone Γ of Π.

Remark 10.1. Table 1 provides all reference situations where condition (i) of the Dichotomy Theorem holds. This condition guaranties the existence of an AGE. However there exist cases where this condition does not hold and, nevertheless, there exists an AGE. An example of this is the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^2$ with coordinates (y, z_1, z_2), and $\mathbf{B}$ the field $(1, 0, 0)$ normal to the boundary. We take the same reference potential as in the case $\Pi = \mathbb{R}^3$ and we find, as described in [38, Lemma 4.3], that the same function $\Psi : (y, \mathbf{z}) \mapsto e^{-|\mathbf{z}|^2/4}$ displayed in Row 2 of Table 1 is also

(k,d) $(\mathbf{y},\mathbf{z})$	Reference field $\mathbf{B}$ and cone Π	Reference potential $\mathbf{A}$	Υ	Explicit Ψ	Φ eigenvector of
$(1,1)$ (y_1,y_2,z)	$(0,1,0)$ $\Pi=\mathbb{R}^2\times\mathbb{R}_+$	$(z,0,0)$	$\mathbb{R}_+=\Gamma$	$\mathrm{e}^{-i\sqrt{\Theta_0}y_1}\Phi(z)$	$-\partial_z^2+(z-\sqrt{\Theta_0})^2$
$(2,0)$ (y,z_1,z_2)	$(1,0,0)$ $\Pi=\mathbb{R}^3$	$(0,-\frac12 z_2,\frac12 z_1)$	$\mathbb{R}^2$	$\mathrm{e}^{-\lvert\mathbf{z}\rvert^2/4}$	$-\Delta_{\mathbf{z}}+i\mathbf{z}\times\nabla_{\mathbf{z}}+\frac{\lvert\mathbf{z}\rvert^2}{4}$
$(2,1)$ (y,z_1,z_2)	$(0,b_1,b_2)$, $b_2\neq0$ $\Pi=\mathbb{R}^2\times\mathbb{R}_+$	$(b_1z_2-b_2z_1,0,0)$	$\mathbb{R}\times\mathbb{R}_+$	$\Phi(\mathbf{z})$	$-\Delta_{\mathbf{z}}+(b_1z_2-b_2z_1)^2$
$(2,2)$ (y,z_1,z_2)	(b_0,b_1,b_2) $\Pi=\mathbb{R}\times\mathcal{S}_\alpha$	$(b_1z_2-b_2z_1,0,b_0z_1)$	$\mathcal{S}_\alpha=\Gamma$	$\mathrm{e}^{i\tau^* y}\Phi(\mathbf{z})$	$\widehat{H}_\tau(\underline{\mathbf{A}},\mathcal{W}_\alpha)$, cf. (6.13)
$(3,3)$	———	———	$\Pi=\Gamma$	$\Phi(\mathbf{z})$	$H(\mathbf{A},\Pi)$

TABLE 1. AGE of $H(\mathbf{A},\Pi)$ when $E(\mathbf{B},\Pi)<\mathscr{E}^*(\mathbf{B},\Pi)$, written in variables $(\mathbf{y},\mathbf{z})$.

an AGE for $H(\mathbf{A},\mathbb{R}_+\times\mathbb{R}^2)$, since it satisfies the Neumann boundary conditions at the boundary $y=0$.

10.2. **Stability under perturbation.** Here we describe stability properties of AGE's under perturbations of the magnetic field $\mathbf{B}$.

Assume that we are in case (i) of the dichotomy (Theorem 7.3). We recall that the notations (G1), (G2) and (G3) refer to the number $k=1,2,3$, of independent decaying directions for the AGE, cf. Section 10.1. We first note that we do not need any stability analysis in situation (G3) since the points $\mathbf{x}$ in $\Omega\in\mathfrak{D}(\mathbb{R}^3)$ for which $d(\Pi_{\mathbf{x}})=3$ are but corners, so they are isolated. In contrast, points in situation (G1) or (G2) are not isolated, in general. A perturbation of the magnetic field has distinct effects in each case. The geometrical situation leading to (G1) is clearly not stable. However, we prove in the following lemma the local stability of case (i) of the dichotomy, together with local uniform estimates for exponential decay in situation (G2).

Lemma 10.2. *Let $\mathbf{B}_0$ be a nonzero constant magnetic field and Π be a cone in $\mathfrak{P}_3$ with reduced dimension $d\le 2$. Assume that $E(\mathbf{B}_0,\Pi)<\mathscr{E}^*(\mathbf{B}_0,\Pi)$.*

(a) *There exists a positive ε_0 such that in the ball $\mathcal{B}(\mathbf{B}_0,\varepsilon_0)$, the function $\mathbf{B}\mapsto E(\mathbf{B},\Pi)$ is Lipschitz-continuous and*
$$E(\mathbf{B},\Pi)<\mathscr{E}^*(\mathbf{B},\Pi)\quad\forall\mathbf{B}\in\mathcal{B}(\mathbf{B}_0,\varepsilon_0).$$

(b) *We suppose moreover that $(\mathbf{B}_0,\Pi)$ is in situation (G2). For $\mathbf{B}\in\mathcal{B}(\mathbf{B}_0,\varepsilon_0)$, we denote by $\Psi^{\mathbf{B}}$ an AGE given by Theorem 7.3. Then there exists $\varepsilon_1\in(0,\varepsilon_0]$ such that $(\mathbf{B},\Pi)$ is still in situation (G2) if $\mathbf{B}\in\mathcal{B}(\mathbf{B}_0,\varepsilon)$ and $\Psi^{\mathbf{B}}$ has the form*
$$\Psi^{\mathbf{B}}(\mathbf{x})=\mathrm{e}^{i\varphi^{\mathbf{B}}(\mathbf{y},\mathbf{z})}\Phi^{\mathbf{B}}(\mathbf{z})\quad\textit{for}\quad\underline{\mathrm{U}}^{\mathbf{B}}\mathbf{x}=(\mathbf{y},\mathbf{z})\in\mathbb{R}\times\Upsilon,$$
with $\underline{\mathrm{U}}^{\mathbf{B}}$ a suitable rotation, and there exist constants $c_{\mathrm{e}}>0$ and $C_{\mathrm{e}}>0$ such that there hold the uniform exponential decay estimates

$$\forall\mathbf{B}\in\mathcal{B}(\mathbf{B}_0,\varepsilon_1),\quad\|\Phi^{\mathbf{B}}\mathrm{e}^{c_{\mathrm{e}}\lvert\mathbf{z}\rvert}\|_{L^2(\Upsilon)}\le C_{\mathrm{e}}\|\Phi^{\mathbf{B}}\|_{L^2(\Upsilon)}\,.\tag{10.1}$$

Proof. Let us distinguish the three possible situations according to the value of d:

$d=0$: When $\Pi=\mathbb{R}^3$, we have $E(\mathbf{B},\Pi)=|\mathbf{B}|$ and $\mathscr{E}^*(\mathbf{B},\Pi)=+\infty$. Combining Row 2 of Table 1 and Lemma 7.2, the admissible generalized eigenvector $\Psi^{\mathbf{B}}$ is explicit. Thus (a) and (b) are established in this case.

$d=1$: When Π is a half-space, we denote by $\theta(\mathbf{B})$ the unoriented angle in $[0,\frac{\pi}{2}]$ between $\mathbf{B}$ and the boundary. Then $E(\mathbf{B},\Pi)=|\mathbf{B}|\,\sigma(\theta(\mathbf{B}))$. The function $\mathbf{B}\mapsto\theta(\mathbf{B})$ is Lipschitz outside $\{\mathbf{0}\}$ and, moreover, the function σ is $\mathscr{C}^1$ on $[0,\pi/2]$ (see Lemma 6.3). We deduce that the function $\mathbf{B}\mapsto\sigma(\theta(\mathbf{B}))$ is Lipschitz outside $\{\mathbf{0}\}$. Thus point (a) is proved. Assuming furthermore that $(\Pi,\mathbf{B}_0)$ is in situation (G2), we have $\theta(\mathbf{B}_0)\in(0,\frac{\pi}{2})$ and there exist $\varepsilon>0$, $\theta_{\min}$ and $\theta_{\max}$ such that

$$\forall\mathbf{B}\in\mathcal{B}(\mathbf{B}_0,\varepsilon),\quad \theta(\mathbf{B})\in[\theta_{\min},\theta_{\max}]\subset(0,\tfrac{\pi}{2})\,.$$

The admissible generalized eigenvector is constructed above. The uniform exponential estimate is proved in [10, §2].

$d=2$: When Π is a wedge, point (a) comes from [49, Proposition 4.6]. Due to the continuity of $\mathbf{B}\mapsto E(\mathbf{B},\Pi)$ there exist $c>0$ and $\varepsilon>0$ such that

$$\forall\mathbf{B}\in\mathcal{B}(\mathbf{B}_0,\varepsilon),\quad \mathscr{E}^*(\mathbf{B},\Pi)-E(\mathbf{B},\Pi)>c.$$

Point (b) is then a direct consequence of [49, Proposition 4.2].

The proof of Lemma 10.2 is complete. □

Remark 10.3. The latter lemma can be generalized in several directions.

a) Lemma 10.2 (a) is still valid when $d=3$. This can be proved by arguments similar to those employed in [49, Section 4] for dihedra.

b) When $d=2$, it is proved in [49] that the ground state energy is also Lipschitz with respect to the aperture angle of the wedge in case (i) of the Dichotomy Theorem, whereas one can prove only $\frac{1}{3}$-Hölder regularity under perturbations in the general case (i.e., without the condition $E(\mathbf{B}_0,\Pi)<\mathscr{E}^*(\mathbf{B}_0,\Pi)$).

Remark 10.4. A constant magnetic field enters the family of long range magnetic fields. So Lemma 10.2 can be related to some spectral analyses of Schrödinger operators in $\mathbb{R}^n$ under long range magnetic perturbations. Such perturbations do not pertain to the usual Kato theory. When the spectrum has a band structure, the question of the stability of, e.g., its lower bound with respect to the strength of the perturbation has been addressed by many authors, see for example [3, 4] for the continuity, then [43, 14] for Hölder properties, and [15] for Lipschitz continuity in the case of constant magnetic fields.

As a consequence of the local uniform estimate (10.1), we obtain the following local uniform version of Lemma 7.10 for situation (G2).

Lemma 10.5. *Let $\mathbf{B}_0$ be a nonzero constant magnetic field and Π a cone in $\mathfrak{P}_3$. Assume that $E(\mathbf{B}_0,\Pi)<\mathscr{E}^*(\mathbf{B}_0,\Pi)$ and that $k=2$. With ε_1 given in Lemma 10.2 (b), for any $\mathbf{B}\in\mathcal{B}(\mathbf{B}_0,\varepsilon_1)$ let $\Psi^{\mathbf{B}}$ be an AGE for $(\mathbf{B},\Pi)$. Let $\delta_0<\frac{1}{2}$ be a positive number. Let $\Psi_h^{\mathbf{B}}$ be the rescaled function given by (7.5) and let χ_h be the cut-off function defined by (7.7)–(7.8) involving parameters $R>0$*

and $\delta \in [0, \delta_0]$. *Let* $R_0 > 0$. *Then there exist constants* $h_0 > 0$, $C_1 > 0$ *depending only on* R_0, δ_0 *and on the constants* c_e, C_e *in* (10.1) *such that*

$$\rho_h = \frac{\| \, |\nabla \chi_h| \, \Psi_h^{\mathbf{B}} \|^2}{\|\chi_h \Psi_h^{\mathbf{B}}\|^2} \leq C_1 \, h^{-2\delta} \qquad \forall R \geq R_0, \; \forall h \leq h_0, \; \forall \delta \in [0, \delta_0] \, .$$

Proof. We obtain an upper bound of $\| \, |\nabla \chi_h| \, \Psi_h^{\mathbf{B}} \|^2$ as in the proof of Lemma 7.10. Let us now deal with the lower-bound of $\|\chi_h \Psi_h^{\mathbf{B}}\|^2$. With $T = Rh^{\delta}$ and $k = 2$, we have

$$\begin{aligned} \|\chi_h \Psi_h^{\mathbf{B}}\|^2 \;&\geq\; C T^{3-k} h^{k/2} \int_{\Upsilon \cap \left\{ 2|\mathbf{z}| \leq T h^{-\frac{1}{2}} \right\}} \left| \Phi^{\mathbf{B}}(\mathbf{z}) \right|^2 \mathrm{d}\mathbf{z} \\ &\geq\; C T^{3-k} h^{k/2} \left(1 - C_e \mathrm{e}^{-c_e R h^{\delta - 1/2}} \right) \|\Phi^{\mathbf{B}}\|^2_{L^2(\Upsilon)} . \end{aligned} \tag{10.2}$$

Since $0 \leq \delta \leq \delta_0 < \frac{1}{2}$, there holds $C_e \mathrm{e}^{-c_e R h^{\delta - 1/2}} < \frac{1}{2}$ for h small enough or R large enough. Thus we deduce the lemma. □

11. Improvement of upper bounds for more regular magnetic fields

For our improvement of remainders, in comparison with Theorem 9.1 our sole additional assumption is a supplementary regularity on the magnetic potential (or equivalently on the magnetic field). Our result is also general, in the sense that it addresses general corner domains.

Theorem 11.1. *Let* $\Omega \in \mathfrak{D}(\mathbb{R}^3)$ *be a general corner domain,* $\mathbf{A} \in W^{3,\infty}(\overline{\Omega})$ *be a magnetic potential such that the associated magnetic field does not vanish.*

(i) Then there exist $C(\Omega) > 0$ *and* $h_0 > 0$ *such that*

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \leq h \mathscr{E}(\mathbf{B}, \Omega) + C(\Omega) \big(1 + \|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)} \big) h^{9/8} \, . \tag{11.1}$$

(ii) If Ω *is a polyhedral domain, this upper bound is improved:*

$$\forall h \in (0, h_0), \quad \lambda_h(\mathbf{B}, \Omega) \leq h \mathscr{E}(\mathbf{B}, \Omega) + C(\Omega) \big(1 + \|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)} \big) h^{4/3} \, . \tag{11.2}$$

The strategy is to optimize the construction of adapted sitting or sliding quasimodes by taking actually advantage of the decaying properties of AGE's $\Psi^{\mathbb{X}}$ associated with the minimal energy $\mathscr{E}(\mathbf{B}, \Omega)$. In fact, our proof of the $h^{11/10}$ or $h^{5/4}$ upper bounds as done in Section 9 weakly uses the exponential decay of generalized eigenfunctions in some directions. It would also work with purely oscillating generalized eigenfunctions. Now the proof of the $h^{9/8}$ or $h^{4/3}$ upper bound makes a more extensive use of fine properties of the model problems: First, the decay properties of admissible generalized eigenvectors, and second, the Lipschitz regularity of the ground state energy depending on the magnetic field, cf. Lemma 10.2.

The method depends on the number k of directions in which $\Psi^{\mathbb{X}}$ has exponential decay, namely whether we are in situation (G1), (G2) or (G3). Indeed, situation (G3) is already handled in Theorem 9.1 *(d)* and we have already obtained a better estimate in this case. So it remains situations (G1) and (G2) which are considered in Section 11.1 and 11.2, respectively.

Like for Theorem 9.1 we start from suitable AGE's $\Psi^{\mathbb{X}}$ and construct sitting or sliding quasimodes adapted to the geometry. In comparison with the proof of Theorem 9.1, the strategy is to improve step [c] that consists in the linearization of the magnetic potential, see Section 9.1 and Figure 2: We take more precisely advantage of the decaying property of the AGE $\Psi^{\mathbb{X}}$, choosing coordinates in which $\Psi^{\mathbb{X}}$ takes the form of reference, as listed in Table 1. Then we adopt different strategies depending on whether we are in situation (G1) or (G2): The improvement relies on a Feynman-Hellmann formula for (G1), and a refined Taylor expansion of the potential for (G2)

We recall that $\mathbf{x}_0 \in \overline{\Omega}$ is a point such that $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) = \mathscr{E}(\mathbf{B}, \Omega)$. Theorem 7.3 and Remark 7.4 provide the existence of a singular chain $\mathbb{X}$ that satisfies

$$\mathscr{E}(\mathbf{B}, \Omega) = E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}) = E(\mathbf{B}_{\mathbb{X}}, \Pi_{\mathbb{X}}) < \mathscr{E}^*(\mathbf{B}_{\mathbb{X}}, \Pi_{\mathbb{X}}).$$

We now split our analysis according to the two geometric configurations (G1) and (G2):

(G1) $\Pi_{\mathbb{X}}$ is a half-space and $\mathbf{B}_{\mathbf{x}_0}$ is tangent to the boundary, *cf.* Row 1 of Table 1.

(G2) We are in one of the following situations:
- $\Pi_{\mathbb{X}} = \mathbb{R}^3$, *cf.* Row 2 of Table 1,
- $\Pi_{\mathbb{X}}$ is a half-space, $\mathbf{B}_{\mathbf{x}_0}$ is neither tangent nor normal to $\partial\Pi_{\mathbb{X}}$, *cf.* Row 3 of Table 1,
- $\Pi_{\mathbb{X}}$ is a wedge, *cf.* Row 4 of Table 1.

In each configuration, the estimates concerning the constructed quasimodes depend on the length ν of the chain $\mathbb{X}$ and on whether $\mathbf{x}_0$ is a conical point or not. The relevant categories of quasimodes are qualified as *sitting* ($\nu = 1$), *hard sliding* ($\nu = 2$, $\mathbf{x}_0$ conical point), *soft sliding* ($\nu = 2$, $\mathbf{x}_0$ not a conical point), and *doubly sliding* ($\nu = 3$), see Section 9.1.

11.1. **(G1) One direction of exponential decay.** In situation (G1) the generalized eigenfunction has exponential decay in one variable z. The upper bounds (9.16) and (9.28) are obtained by a Cauchy-Schwarz inequality. We are going to improve them, going back to the identity (A.6) and using a Feynman-Hellmann formula to simplify the cross term in (A.6).

In situation (G1) $\Pi_{\mathbb{X}}$ is a half-space and $\mathbf{B}_{\mathbb{X}}$ is tangent to its boundary. Denote by $(\mathbf{y}, z) = (y_1, y_2, z) \in \mathbb{R}^2 \times \mathbb{R}_+$ a system of coordinates of $\Pi_{\mathbb{X}}$ such that $\mathbf{B}_{\mathbb{X}}$ is tangent to the y_2-axis. In these coordinates, the magnetic field $\mathbf{B}_{\mathbb{X}}$ writes $(0, b, 0)$.

In the rest of this proof, we will assume without restriction that $b = 1$. Indeed, once quasimodes are constructed for $b = 1$, Lemmas A.4 and A.7 allow to convert them into quasimodes for any b. Thus we have $\Lambda_{\mathbb{X}} = \Theta_0$, *cf.* Row 1 of Table 1.

The principle of the quasimode construction is to replace the last relation (9.10) $\varphi_h^{[\nu]} = \chi_h \Psi_h^{\mathbb{X}}$ with the new relation

$$\varphi_h^{[\nu]} = \underline{\mathrm{U}}_* \circ \mathrm{Z}_h^F (\chi_h^{\square}\, \underline{\Psi}_h) \tag{11.3}$$

where $\underline{\mathrm{U}}$ is the rotation $\mathbf{x} \mapsto \mathbf{x}^{\natural} := (\mathbf{y}, z)$ that maps $\Pi_{\mathbb{X}}$ onto the reference half-space $\mathbb{R}^2 \times \mathbb{R}_+$, the function $\chi_h^{\square}$ is the cut-off in tensor product form (here for simplicity we denote $\underline{\chi}_R$ by χ) defined as

$$\chi_h^{\square}(\mathbf{y}, z) = \chi\Big(\frac{|\mathbf{y}|}{h^{\delta}}\Big)\, \chi\Big(\frac{z}{h^{\delta}}\Big) \tag{11.4}$$

Z_h^F is a change of gauge and $\underline{\Psi}_h$ a canonical generalized eigenvector defined as follows.

The canonical reference potential (see Row 1 of Table 1)

$$\underline{\mathbf{A}}(\mathbf{y}, z) = (z, 0, 0), \tag{11.5}$$

is such that $\operatorname{curl} \underline{\mathbf{A}} = (0, 1, 0)$. We know (see Section 10.1) that the function

$$\underline{\Psi}_h(\mathbf{y}, z) := \mathrm{e}^{-i\sqrt{\Theta_0}\, y_1/\sqrt{h}}\, \Phi\Big(\frac{z}{\sqrt{h}}\Big) \tag{11.6}$$

is a generalized eigenvector of $H_h(\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+)$ for the value $h\Theta_0$. Here Φ is a normalized eigenvector associated with the first eigenvalue of the de Gennes operator $-\partial_z^2 + (z - \sqrt{\Theta_0})^2$. By identity (7.9) and Lemma 7.10 we obtain the cut-off estimate

$$\mathscr{Q}_h[\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+](\chi_h^{\square}\, \underline{\Psi}_h) = h\Theta_0 + \mathcal{O}(h^{2-2\delta}) = h\Lambda_{\mathbb{X}} + \mathcal{O}(h^{2-2\delta}). \tag{11.7}$$

Let J be the matrix associated with $\underline{\mathrm{U}}$. In variables $\mathbf{x}^{\natural}$, the tangent potential $\mathbf{A}_{\mathbb{X}}$ is transformed into the potential $\mathbf{A}_{\mathbf{0}}^{\natural}$

$$\mathbf{A}_{\mathbf{0}}^{\natural}(\mathbf{x}^{\natural}) = \mathrm{J}^{\top}(\mathbf{A}_{\mathbb{X}}(\mathbf{x})), \tag{11.8}$$

that satisfies

$$\operatorname{curl} \mathbf{A}_{\mathbf{0}}^{\natural} = \operatorname{curl} \underline{\mathbf{A}}\,.$$

Since $\underline{\mathbf{A}}$ and $\mathbf{A}_{\mathbf{0}}^{\natural}$ are both linear, there exists a homogenous polynomial function of degree two $F^{\natural}$ such that

$$\mathbf{A}_{\mathbf{0}}^{\natural} - \nabla_{\natural} F^{\natural} = \underline{\mathbf{A}}. \tag{11.9}$$

Therefore, $\mathrm{e}^{-iF^{\natural}/h}\underline{\Psi}_h$ is an admissible generalized eigenvector for $H_h(\mathbf{A}_{\mathbf{0}}^{\natural}, \mathbb{R}^2 \times \mathbb{R}_+)$ associated with the value $h\Lambda_{\mathbb{X}}$.

11.1.1. *Sitting quasimodes.* This is the case when $\nu = 1$ and $\mathbb{X} = (\mathbf{x}_0)$. Thus $\Pi_{\mathbf{x}_0}$ coincides with $\Pi_{\mathbb{X}}$. We keep relation (9.12) linking $\varphi_h^{[0]}$ to $\varphi_h^{[1]}$ and $\varphi_h^{[1]}$ is now defined by the formula

$$\varphi_h^{[1]}(\mathbf{x}) = \mathrm{e}^{-iF^{\natural}(\mathbf{x}^{\natural})/h}\chi_h^{\square}(\mathbf{y}, z)\underline{\Psi}_h(\mathbf{y}, z) = \mathrm{e}^{-iF^{\natural}(\mathbf{x}^{\natural})/h}\underline{\psi}_h(\mathbf{y}, z), \qquad \forall \mathbf{x} \in \Pi_{\mathbb{X}}\,, \tag{11.10}$$

Here we set for shortness

$$\underline{\psi}_h := \chi_h^{\square}\, \underline{\Psi}_h \quad \text{and} \quad \mathcal{V}_h^{\square} := \operatorname{supp}(\chi_h^{\square}).$$

Let J be the matrix associated with $\underline{\mathrm{U}}$. Let $\mathbf{A}^{\natural}$ be the magnetic potential associated with $\mathbf{A}^{\mathbf{x}_0}$ in variables $\mathbf{x}^{\natural}$:

$$\mathbf{A}^{\natural}(\mathbf{x}^{\natural}) = \mathrm{J}^{\top}\big(\mathbf{A}^{\mathbf{x}_0}(\mathbf{x})\big) \quad \forall \mathbf{x} \in \mathcal{V}_{\mathbf{x}_0}\,. \tag{11.11}$$

Then $\mathbf{A}_{\mathbf{0}}^{\natural}$ (11.8) is its linear part at $\mathbf{0}$.

We have

$$\begin{aligned} \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) &= \mathscr{Q}_h[\mathbf{A}^{\natural}, \mathbb{R}^2 \times \mathbb{R}_+](\mathrm{e}^{-iF^{\natural}/h}\underline{\psi}_h) \\ &= \mathscr{Q}_h[\mathbf{A}^{\natural} - \nabla F^{\natural}, \mathbb{R}^2 \times \mathbb{R}_+](\underline{\psi}_h). \end{aligned} \tag{11.12}$$

Now we apply (A.6) with $\mathbf{A} = \mathbf{A}^\natural - \nabla F^\natural$ and $\mathbf{A}' = \underline{\mathbf{A}}$. Using (11.9) we find $\mathbf{A} - \mathbf{A}' = \mathbf{A}^\natural - \mathbf{A}^\natural_{\mathbf{0}}$, and write, instead of (9.16)

$$q_h[\mathbf{A}^\natural - \nabla F^\natural, \mathbb{R}^2 \times \mathbb{R}_+](\underline{\psi}_h) = q_h[\underline{\mathbf{A}}, \mathbb{R}^2 \times \mathbb{R}_+](\underline{\psi}_h) \tag{11.13}$$

$$+ 2\,\mathrm{Re} \int_{\mathbb{R}^2\times\mathbb{R}_+} (-ih\nabla + \underline{\mathbf{A}})\underline{\psi}_h(\mathbf{x}^\natural) \cdot (\mathbf{A}^\natural - \mathbf{A}^\natural_{\mathbf{0}})(\mathbf{x}^\natural)\, \overline{\underline{\psi}_h(\mathbf{x}^\natural)}\, \mathrm{d}\mathbf{x}^\natural \tag{11.14}$$

$$+ \|(\mathbf{A}^\natural - \mathbf{A}^\natural_{\mathbf{0}})\underline{\psi}_h\|^2. \tag{11.15}$$

As in Section 9.2 [e1], we bound from above the term (11.15) using Lemma 4.6

$$\|(\mathbf{A}^\natural - \mathbf{A}^\natural_{\mathbf{0}})\underline{\psi}_h\|^2 \le C(\Omega)\|\mathbf{A}^\natural\|^2_{W^{2,\infty}(\mathcal{V}_h^\square)}\, h^{4\delta}\, \|\underline{\psi}_h\|^2. \tag{11.16}$$

Let us now deal with the term (11.14). We calculate $(-ih\nabla + \underline{\mathbf{A}})\underline{\psi}_h$ using (11.6):

$$(-ih\nabla + \underline{\mathbf{A}})\underline{\psi}_h(\mathbf{x}^\natural) = \mathrm{e}^{-i\sqrt{\Theta_0}\,y_1/\sqrt{h}} \times$$
$$\left\{ \chi\big(\tfrac{|\mathbf{y}|}{h^\delta}\big)\,\chi\big(\tfrac{z}{h^\delta}\big) \begin{bmatrix} (z - \sqrt{h\Theta_0})\,\Phi\big(\tfrac{z}{\sqrt{h}}\big) \\ 0 \\ -i\sqrt{h}\,\Phi'\big(\tfrac{z}{\sqrt{h}}\big) \end{bmatrix} - ih^{1-\delta} \begin{bmatrix} \frac{y_1}{|\mathbf{y}|}\chi'\big(\frac{|\mathbf{y}|}{h^\delta}\big)\,\chi\big(\frac{z}{h^\delta}\big) \\ \frac{y_2}{|\mathbf{y}|}\chi'\big(\frac{|\mathbf{y}|}{h^\delta}\big)\,\chi\big(\frac{z}{h^\delta}\big) \\ \chi\big(\frac{|\mathbf{y}|}{h^\delta}\big)\,\chi'\big(\frac{z}{h^\delta}\big) \end{bmatrix} \Phi\big(\tfrac{z}{\sqrt{h}}\big) \right\}.$$

Since Φ and χ are real valued functions, the term (11.14) reduces to a single term:

(11.17)
$$\begin{aligned}
\mathrm{Re} \int_{\mathbb{R}^2\times\mathbb{R}_+} &(-ih\nabla + \underline{\mathbf{A}})\underline{\psi}_h(\mathbf{x}^\natural) \cdot (\mathbf{A}^\natural - \mathbf{A}^\natural_{\mathbf{0}})(\mathbf{x}^\natural)\overline{\underline{\psi}_h(\mathbf{x}^\natural)}\, \mathrm{d}\mathbf{x}^\natural \\
&= \int_{\mathbb{R}^2\times\mathbb{R}_+} (z - \sqrt{h\Theta_0})\, |\underline{\psi}_h(\mathbf{x}^\natural)|^2 A_1^{(\mathrm{rem},2)}(\mathbf{x}^\natural)\, \mathrm{d}\mathbf{x}^\natural \\
&= \int_{\mathbb{R}^2\times\mathbb{R}_+} (z - \sqrt{h\Theta_0})\, \Big|\Phi\big(\tfrac{z}{\sqrt{h}}\big)\Big|^2\, \Big|\chi\big(\tfrac{|\mathbf{y}|}{h^\delta}\big)\Big|^2\, \big|\chi\big(\tfrac{z}{h^\delta}\big)\big|^2\, A_1^{(\mathrm{rem},2)}(\mathbf{x}^\natural)\, \mathrm{d}\mathbf{x}^\natural,
\end{aligned}$$

where $A_1^{(\mathrm{rem},2)}$ denotes the first component of $\mathbf{A}^\natural - \mathbf{A}^\natural_{\mathbf{0}}$. We write

$$A_1^{(\mathrm{rem},2)}(\mathbf{x}^\natural) = P_1^{(2)}(\mathbf{y}) + R_1^{(2)}(\mathbf{x}^\natural) + A_1^{(\mathrm{rem},3)}(\mathbf{x}^\natural), \tag{11.18}$$

where $A_1^{(\mathrm{rem},3)}$ is the Taylor remainder of degree 3 of the first component of $\mathbf{A}^\natural$ at $\mathbf{0}$, whereas $P_1^{(2)}(\mathbf{y}) + R_1^{(2)}(\mathbf{x}^\natural)$ is a representation of its quadratic part in the form

$$P_1^{(2)}(\mathbf{y}) = a_1 y_1^2 + a_2 y_2^2 + a_3 y_1 y_2 \quad \text{and} \quad R_1^{(2)}(\mathbf{x}^\natural) = b_1 z^2 + b_2 z y_1 + b_3 z y_2.$$

As in (A.2) there holds

$$\|A_1^{(\mathrm{rem},3)}\|_{L^\infty(\mathcal{V}_h^\square)} \le C\|\mathbf{A}^\natural\|_{W^{3,\infty}(\mathcal{V}_h^\square)}\, h^{3\delta},$$

leading to, with the help of the variable change $Z = z/\sqrt{h}$ and the exponential decay of Φ:

$$\left|\int_{\mathbb{R}^2\times\mathbb{R}_+} (z - \sqrt{h\Theta_0})\, |\underline{\psi}_h(\mathbf{x}^\natural)|^2\, A_1^{(\mathrm{rem},3)}(\mathbf{x}^\natural)\, \mathrm{d}\mathbf{x}^\natural\right| \le C\|\mathbf{A}^\natural\|_{W^{3,\infty}(\mathcal{V}_h^\square)} h^{\frac12+3\delta}\|\underline{\psi}_h\|^2\,. \tag{11.19}$$

Likewise, combining the exponential decay of Φ, the change of variable $Z = z/\sqrt{h}$ and the localization of the support in balls of size Ch^{δ}, we deduce

$$\left|\int_{\mathbb{R}^2\times\mathbb{R}_+}(z-\sqrt{h\Theta_0})\,|\underline{\psi}_h(\mathbf{x}^\natural)|^2\,R_1^{(2)}(\mathbf{x}^\natural)\,\mathrm{d}\mathbf{x}^\natural\right| \le C\|\mathbf{A}^\natural\|_{W^{2,\infty}(\mathcal{V}_h^\square)}h^{\min(\frac32,1+\delta)}\|\underline{\psi}_h\|^2\,. \tag{11.20}$$

Let us now deal with the term involving $\mathbf{y}\mapsto P_1^{(2)}(\mathbf{y})$. Due to a Feynman-Hellmann formula applied to the de Gennes operator $\mathcal{H}(\tau)$ at $\tau = -\sqrt{\Theta_0}$ (cf. [29, Lemma A.1]) we find by the scaling $z\mapsto z/\sqrt{h}$ the identity

$$\int_{\mathbb{R}_+}(z-\sqrt{h\Theta_0})\,\left|\Phi\big(\tfrac{z}{\sqrt{h}}\big)\right|^2\,\mathrm{d}z = 0\,.$$

Thus we can write

$$\begin{aligned}\int_{\mathbb{R}^2\times\mathbb{R}_+}&(z-\sqrt{h\Theta_0})\,|\underline{\psi}_h(\mathbf{x}^\natural)|^2\,P_1^{(2)}(\mathbf{y})\,\mathrm{d}\mathbf{x}^\natural\\ &= \int_{\mathbb{R}^2}P_1^{(2)}(\mathbf{y})\,\left|\chi\big(\tfrac{|\mathbf{y}|}{h^\delta}\big)\right|^2\mathrm{d}\mathbf{y}\,\int_{z\in\mathbb{R}_+}(z-\sqrt{h\Theta_0})\,\left|\Phi\big(\tfrac{z}{\sqrt{h}}\big)\right|^2\chi\big(\tfrac{z}{h^\delta}\big)^2\,\mathrm{d}z\\ &= \int_{\mathbb{R}^2}P_1^{(2)}(\mathbf{y})\,\left|\chi\big(\tfrac{|\mathbf{y}|}{h^\delta}\big)\right|^2\mathrm{d}\mathbf{y}\,\int_{z\in\mathbb{R}_+}(z-\sqrt{h\Theta_0})\,\left|\Phi\big(\tfrac{z}{\sqrt{h}}\big)\right|^2\Big(\chi\big(\tfrac{z}{h^\delta}\big)^2-1\Big)\,\mathrm{d}z.\end{aligned}$$

The support of the integral in z is contained in $z\ge Rh^\delta$ with $\delta<\frac12$. Therefore, using once more the changes of variables $\mathbf{Y}=\mathbf{y}/h^\delta$ and $Z=z/\sqrt{h}$, we find:

$$\left|\int_{\mathbb{R}^2\times\mathbb{R}_+}(z-\sqrt{h\Theta_0})\;|\underline{\psi}_h(\mathbf{x}^\natural)|^2P_1^{(2)}(\mathbf{y})\,\mathrm{d}\mathbf{x}^\natural\right| \le C\|\mathbf{A}^\natural\|_{W^{2,\infty}(\mathcal{V}_h^\square)}h^{\frac12+4\delta}\mathrm{e}^{-ch^{\delta-1/2}}.$$

Since $\|\underline{\psi}_h\|^2\ge Ch^{\frac12+2\delta}$ (see (10.2)), this leads to:

$$\left|\int_{\mathbb{R}^2\times\mathbb{R}_+}(z-\sqrt{h\Theta_0})\;|\underline{\psi}_h(\mathbf{x}^\natural)|^2P_1^{(2)}(\mathbf{y})\,\mathrm{d}\mathbf{x}^\natural\right| \le C\|\mathbf{A}^\natural\|_{W^{2,\infty}(\mathcal{V}_h^\square)}\mathrm{e}^{-ch^{\delta-1/2}}\,\|\underline{\psi}_h\|^2\,. \tag{11.21}$$

Collecting (11.19), (11.20), and (11.21) in (11.14), we find the upper bound

$$\begin{aligned}\left|\mathrm{Re}\int_{\mathbb{R}^2\times\mathbb{R}_+}(-ih\nabla+\mathbf{A}_0^\natural)\underline{\psi}_h(\mathbf{x}^\natural)\cdot(\mathbf{A}^\natural-\mathbf{A}_0^\natural)\overline{\underline{\psi}_h(\mathbf{x}^\natural)}\,\mathrm{d}\mathbf{x}^\natural\right|&\\ \le C\Big(\|\mathbf{A}^\natural\|_{W^{3,\infty}(\mathcal{V}_h^\square)}\,h^{\frac12+3\delta}+\|\mathbf{A}^\natural\|_{W^{2,\infty}(\mathcal{V}_h^\square)}\,h^{1+\delta}\Big)\|\underline{\psi}_h\|^2\,.&\end{aligned} \tag{11.22}$$

Returning to (11.12) via (11.13) and combining (11.22) with (11.16), we deduce

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) &\le \mathscr{Q}_h[\underline{\mathbf{A}},\mathbb{R}^2\times\mathbb{R}_+](\underline{\psi}_h)\\ &+C\Big(\|\mathbf{A}^\natural\|_{W^{3,\infty}(\mathcal{V}_h^\square)}\,h^{\frac12+3\delta}+\|\mathbf{A}^\natural\|_{W^{2,\infty}(\mathcal{V}_h^\square)}\,h^{1+\delta}+\|\mathbf{A}^\natural\|^2_{W^{2,\infty}(\mathcal{V}_h^\square)}\,h^{4\delta}\Big).\end{aligned}$$

Inserting the cut-off error (11.7) for $q_h[\underline{\mathbf{A}},\mathbb{R}^2\times\mathbb{R}_+](\underline{\psi}_h)$ we obtain

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) &\le h\Lambda_{\mathbb{X}}+C\,h^{2-2\delta}\\ &+C\Big(\|\mathbf{A}^\natural\|_{W^{3,\infty}(\mathcal{V}_h^\square)}\,h^{\frac12+3\delta}+\|\mathbf{A}^\natural\|_{W^{2,\infty}(\mathcal{V}_h^\square)}\,h^{1+\delta}+\|\mathbf{A}^\natural\|^2_{W^{2,\infty}(\mathcal{V}_h^\square)}\,h^{4\delta}\Big).\end{aligned} \tag{11.23}$$

Using Lemma 4.7 for case (i) we deduce the uniform bound for the derivatives of the potential

$$\|\mathbf{A}^\natural\|_{W^{3,\infty}(\mathcal{V}_h^\square)} \le C\|\mathbf{A}^{\mathbf{x}_0}\|_{W^{3,\infty}(\mathcal{V}_{\mathbf{x}_0})} \le C'\|\mathbf{A}\|_{W^{3,\infty}(\Omega)}.$$

Thus, we deduce from (11.23)

$$\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) \le h\Lambda_{\mathbb{X}} + C(\Omega)(1+\|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)})(h^{2-2\delta}+h^{1+\delta}+h^{\frac12+3\delta}+h^{4\delta}).$$

The quasimode $\varphi_h^{[0]}$ on Ω being still defined by (9.12), we deduce from (9.15) with $r_h^{[1]} = \mathcal{O}(h^\delta)$ the final estimate

$$\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) \le h\Lambda_{\mathbb{X}} + C(\Omega)(1+\|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)})(h^{2-2\delta}+h^{1+\delta}+h^{\frac12+3\delta}+h^{4\delta})\,. \tag{11.24}$$

Choosing $\delta = \frac13$ we optimize remainders and deduce the upper bound (11.2) in situation (G1)–sitting.

11.1.2. *Hard sliding.* This is the case when $\nu = 2$ and $\mathbf{x}_0 \in \mathfrak{V}^\circ$ (*i.e.*, $\mathbf{x}_0$ is a conical point). So $\mathbb{X} = (\mathbf{x}_0,\mathbf{x}_1)$ and $\Pi_{\mathbf{x}_0,\mathbf{x}_1}$ coincides with $\Pi_{\mathbb{X}}$. We keep relations (9.12) and (9.25) linking $\varphi_h^{[0]}$ to $\varphi_h^{[1]}$ and $\varphi_h^{[1]}$ to $\varphi_h^{[2]}$, respectively, and $\varphi_h^{[2]}$ is now defined by the formula

$$\varphi_h^{[2]}(\mathbf{x}) = \mathrm{e}^{-iF^\natural(\mathbf{x}^\natural)/h}\chi_h^\square(\mathbf{y},z)\underline{\Psi}_h(\mathbf{y},z) = \mathrm{e}^{-iF^\natural(\mathbf{x}^\natural)/h}\underline{\psi}_h(\mathbf{y},z), \qquad \forall \mathbf{x}\in\Pi_{\mathbb{X}}\,, \tag{11.25}$$

and $\mathbf{A}^\natural$ is the magnetic potential associated with $\mathbf{A}^{\mathbf{v}_1}$ (step [a2]) in variables $\mathbf{x}^\natural$,

$$\mathbf{A}^\natural(\mathbf{x}^\natural) = \mathrm{J}^\top\big(\mathbf{A}^{\mathbf{v}_1}(\mathbf{x})\big) \quad \forall \mathbf{x}\in\mathcal{V}_{\mathbf{v}_1}\,. \tag{11.26}$$

We recall that $\Pi_{\mathbf{v}_1} = \Pi_{\mathbb{X}}$. We have, instead of (11.12):

$$\mathscr{Q}_h[\mathbf{A}^{\mathbf{v}_1},\Pi_{\mathbf{v}_1}](\varphi_h^{[2]}) = \mathscr{Q}_h[\mathbf{A}^\natural - \nabla F^\natural, \mathbb{R}^2\times\mathbb{R}_+](\underline{\psi}_h), \tag{11.27}$$

and (9.28) is replaced by the analysis of (11.13)–(11.15) which goes along the same lines as before, ending up at, instead of (11.23)

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A}^{\mathbf{v}_1},\Pi_{\mathbf{v}_1}](\varphi_h^{[2]}) &\le h\Lambda_{\mathbb{X}} + C\,h^{2-2\delta}\\ &\quad + C\Big(\|\mathbf{A}^\natural\|_{W^{3,\infty}(\mathcal{V}_h^\square)}\,h^{\frac12+3\delta} + \|\mathbf{A}^\natural\|_{W^{2,\infty}(\mathcal{V}_h^\square)}\,h^{1+\delta} + \|\mathbf{A}^\natural\|^2_{W^{2,\infty}(\mathcal{V}_h^\square)}\,h^{4\delta}\Big).\end{aligned} \tag{11.28}$$

But now we have to use Lemma 4.7 for case (ii) after specifying the different scales: As in Section 9.3 step [e2] (b) we take $|\mathbf{v}_1| = d_h^{[1]} = \mathcal{O}(h^{\delta_0})$ and $\delta = \delta_0+\delta_1$, so the support of $\underline{\psi}_h$ is contained in a ball of radius $r_h^{[2]} = \mathcal{O}(h^{\delta_0+\delta_1})$. The radius $r_h^{[1]}$ is a $\mathcal{O}(h^{\delta_0})$. By using Remark 3.15, we can see that (4.16) generalizes to higher derivative of $\mathbf{A}^{\mathbf{v}_1}$, and thus we may estimate the derivatives of the potential after change of variables:

$$\|\mathbf{A}^\natural\|_{W^{\ell,\infty}(\mathcal{V}_h^\square)} \le C\|\mathbf{A}^{\mathbf{v}_1}\|_{W^{\ell,\infty}(\mathcal{B}(\mathbf{0},r_h^{[2]}))} \le C'h^{-(\ell-1)\delta_0}\|\mathbf{A}\|_{W^{\ell,\infty}(\Omega)}, \quad \ell = 2,3, \tag{11.29}$$

and (11.28) provides

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A}^{\mathbf{v}_1},\Pi_{\mathbf{v}_1}](\varphi_h^{[2]}) &\le h\Lambda_{\mathbb{X}}\\ &\quad + C(1+\|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)})\Big(h^{2-2\delta_0-2\delta_1} + h^{-2\delta_0}h^{\frac12+3\delta_0+3\delta_1} + h^{-\delta_0}h^{1+\delta_0+\delta_1} + h^{-2\delta_0}h^{4\delta_0+4\delta_1}\Big).\end{aligned}$$

Combining the above inequality with (9.20) that bounds $\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) - \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]})$ and (9.27) that bounds $\mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) - \mathscr{Q}_h[\mathbf{A}^{\mathbf{v}_1},\Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]})$ we find

$$\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) \le h\Lambda_{\mathbb{X}} + C(1+\|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)})\Big(h^{1+\delta_0}+h^{\frac12+2\delta_0}+h^{4\delta_0}+h^{1+\delta_1}\Big) + C(1+\|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)})\Big(h^{2-2\delta_0-2\delta_1}+h^{\frac12+\delta_0+3\delta_1}+h^{1+\delta_1}+h^{2\delta_0+4\delta_1}\Big). \tag{11.30}$$

Choosing $\delta_0 = \frac{5}{16}$ and $\delta_1 = \frac18$, we deduce the upper bound (11.1) in situation (G1)–hard sliding.

11.1.3. *Soft sliding.* This is the case when $\nu = 2$ and $\mathbf{x}_0$ is *not a conical point*. We keep relations (9.12) and (9.25) linking $\varphi_h^{[0]}$ to $\varphi_h^{[1]}$ and $\varphi_h^{[1]}$ to $\varphi_h^{[2]}$, respectively, and $\varphi_h^{[2]}$ is defined by formula (11.25) as in the hard sliding case. But now the analysis is different because we can take advantage of the fact that the change of variables $\mathrm{U}^{\mathbf{v}_1}$ is the translation $\mathbf{x}\mapsto\mathbf{x}-\mathbf{v}_1$. Concatenating formulas (9.25) and (11.25), we obtain (recall that $\underline{\mathrm{U}}$ is the rotation $\mathbf{x}\mapsto\mathbf{x}^\natural$)

$$\varphi_h^{[1]} = \mathrm{Z}_h^{\mathbf{v}_1}\circ \mathrm{U}_*^{\mathbf{v}_1}\circ \underline{\mathrm{U}}_*\Big(\mathrm{e}^{-iF^\natural/h}\underline{\psi}_h\Big). \tag{11.31}$$

Our aim is a direct evaluation of $\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]})$, based on the above representation. Here we take the potential $\mathbf{A}^\natural$ in the canonical half-space $\mathbb{R}^2\times\mathbb{R}_+$ as (11.11). Let us set $\mathbf{v}_1^\natural := \underline{\mathrm{U}}\mathbf{v}_1$. Then there holds the following sequence of identities, cf. (11.12) for the last one,

$$\begin{aligned}
\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) &= \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}-\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}(\mathbf{v}_1),\Pi_{\mathbb{X}}]\Big(\mathrm{U}_*^{\mathbf{v}_1}\circ\underline{\mathrm{U}}_*\big(\mathrm{e}^{-iF^\natural/h}\underline{\psi}_h\big)\Big)\\
&= \mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}(\cdot+\mathbf{v}_1)-\mathbf{A}_{\mathbf{0}}^{\mathbf{x}_0}(\mathbf{v}_1),\Pi_{\mathbb{X}}]\Big(\underline{\mathrm{U}}_*\big(\mathrm{e}^{-iF^\natural/h}\underline{\psi}_h\big)\Big)\\
&= \mathscr{Q}_h[\mathbf{A}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\mathbf{v}_1^\natural),\mathbb{R}^2\times\mathbb{R}_+]\big(\mathrm{e}^{-iF^\natural/h}\underline{\psi}_h\big)\\
&= \mathscr{Q}_h[\mathbf{A}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\mathbf{v}_1^\natural)-\nabla F^\natural,\mathbb{R}^2\times\mathbb{R}_+](\underline{\psi}_h).
\end{aligned}$$

For the calculation of the potential, we check that

$$\begin{aligned}
\mathbf{A}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\mathbf{v}_1^\natural)-\nabla F^\natural &= \mathbf{A}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\cdot+\mathbf{v}_1^\natural)+\mathbf{A}_{\mathbf{0}}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\mathbf{v}_1^\natural)-\nabla F^\natural\\
&= \mathbf{A}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\cdot+\mathbf{v}_1^\natural)+\mathbf{A}_{\mathbf{0}}^\natural-\nabla F^\natural\\
&= \mathbf{A}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\cdot+\mathbf{v}_1^\natural)+\underline{\mathbf{A}}\,.
\end{aligned}$$

Then, instead of (11.13)-(11.15) we obtain that $q_h[\mathbf{A}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\mathbf{v}_1^\natural)-\nabla F^\natural,\mathbb{R}^2\times\mathbb{R}_+](\underline{\psi}_h)$ is now the sum of the three following terms:

$$\begin{aligned}
&q_h[\underline{\mathbf{A}},\mathbb{R}^2\times\mathbb{R}_+](\underline{\psi}_h)\\
&\quad+2\,\mathrm{Re}\int_{\mathbb{R}^2\times\mathbb{R}_+}(-ih\nabla+\underline{\mathbf{A}})\underline{\psi}_h(\mathbf{x}^\natural)\cdot\big(\mathbf{A}^\natural(\mathbf{x}^\natural+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\mathbf{x}^\natural+\mathbf{v}_1^\natural)\big)\,\overline{\underline{\psi}_h(\mathbf{x}^\natural)}\,\mathrm{d}\mathbf{x}^\natural\\
&\quad+\|\big(\mathbf{A}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\cdot+\mathbf{v}_1^\natural)\big)\underline{\psi}_h\|^2.
\end{aligned}$$

Since $|\mathbf{v}_1| = h^\delta$, the estimate (11.16) obviously becomes

$$\|(\mathbf{A}^\natural(\cdot+\mathbf{v}_1^\natural)-\mathbf{A}_{\mathbf{0}}^\natural(\cdot+\mathbf{v}_1^\natural))\underline{\psi}_h\|^2 \le C(\Omega)\|\mathbf{A}^\natural\|^2_{W^{2,\infty}(\mathbf{v}_1^\natural+\mathcal{V}_h^\square)}\,h^{4\delta}\,\|\underline{\psi}_h\|^2.$$

As for estimates (11.17)-(11.22) of the crossed term, we may use the fact that the vector $\widehat{\mathbf{x}}_1$ introduced in (9.24) belongs to a face of $\Pi_{\mathbf{x}_0}$ (see the prologue of Section 9.3). It is the same for $\mathbf{v}_1 = h^\delta \widehat{\mathbf{x}}_1$. Therefore $\mathbf{v}_1^\natural$ is tangent to the boundary of $\mathbb{R}^2 \times \mathbb{R}_+$, it has no component in the z direction and can be written $\mathbf{v}_1^\natural = h^\delta \widehat{\mathbf{x}}_1^\natural = (h^\delta \mathbf{p}, 0)$ in coordinates $\mathbf{x}^\natural$. We use the same splitting (11.18) of the potential, at the point $\mathbf{x}^\natural + \mathbf{v}_1^\natural$

$$A_1^{(\mathrm{rem},2)}(\mathbf{x}^\natural + \mathbf{v}_1^\natural) = P_1(\mathbf{y} + h^\delta \mathbf{p}) + R_1(\mathbf{x}^\natural + h^\delta \widehat{\mathbf{x}}_1^\natural) + A_1^{(\mathrm{rem},3)}(\mathbf{x}^\natural + h^\delta \widehat{\mathbf{x}}_1^\natural).$$

Then all estimates (11.17)-(11.22) of the crossed term are still valid now, replacing the norm in $W^{\ell,\infty}(\mathrm{supp}(\underline{\psi}_h))$ by the norm in $W^{\ell,\infty}(\mathbf{v}_1^\natural + \mathrm{supp}(\underline{\psi}_h))$ (for $\ell = 2, 3$). As before we arrive to the upper bound (11.24) for the Rayleigh quotient of our quasimode and conclude as in the sitting case.

11.1.4. *Double sliding.* This is the case when $\nu = 3$. So $\mathbf{x}_0$ is a conical point. We keep relations (9.12) and (9.25) linking $\varphi_h^{[0]}$ to $\varphi_h^{[1]}$ and $\varphi_h^{[1]}$ to $\varphi_h^{[2]}$, respectively, and $\varphi_h^{[2]}$ is now defined by the formula

$$\varphi_h^{[2]}(\mathbf{x}) = \mathrm{Z}_h^{\mathbf{v}_2} \circ \mathrm{U}_*^{\mathbf{v}_2} \circ \underline{\mathrm{U}}_* \Big(\mathrm{e}^{-iF^\natural/h} \underline{\psi}_h \Big). \tag{11.32}$$

and $\mathbf{A}^\natural$ is the magnetic potential (11.26) associated with $\mathbf{A}^{\mathbf{v}_1}$ (step [a2]) in variables $\mathbf{x}^\natural$. A reasoning similar to the soft sliding case yields the same conclusion (11.30) like in the hard sliding case.

The proof of Theorem 11.1 is over in situation (G1).

11.2. **(G2) Two directions of exponential decay.** In situation (G2) the generalized eigenfunction $\Psi^{\mathbb{X}}$ has two directions of decay, z_1 and z_2, leaving one direction y with a purely oscillating character. In this case, we are going to improve the linearization error, namely estimates (9.18) and (9.30): Until now we have used that $\mathbf{A}^{\mathbf{x}_0}(\mathbf{x}) - \mathbf{A}_0^{\mathbf{x}_0}(\mathbf{x})$ is a $O(|\mathbf{x}|^2)$. Here, by a suitable phase shift (which corresponds to a change of gauge), we can eliminate from this error the term in $O(|y|^2)$, replacing it by a $O(|y|^3)$. The other terms containing at least one power of $|\mathbf{z}|$, we can take advantage of the decay of $\Psi^{\mathbb{X}}$. This phase shift is done by a change of gauge on the last level of construction, that is on the function $\varphi_h^{[\nu]}$, as in the (G1)-case. The sitting modes will be constructed following exactly this strategy, whereas concerning sliding modes, we have to linearize the potential at a moving point $\mathbf{v} := h^\delta \widehat{\mathbf{x}}$, instead of $\mathbf{0}$ as previously. Let us develop details now. The quasimode $\varphi_h^{[0]}$ is still defined on Ω by formula (9.12) $\varphi_h^{[0]} = \mathrm{Z}_h^{\mathbf{x}_0} \circ \mathrm{U}_*^{\mathbf{x}_0}(\varphi_h^{[1]})$, and relations (9.13)–(9.15) are still valid.

11.2.1. *Sitting quasimodes.* Here we make an improvement of step [c1], see Figure 2. Let $\underline{\mathrm{U}}$ be the rotation $\mathbf{x} \mapsto \mathbf{x}^\natural := (\mathbf{y}, z)$ that maps $\Pi_{\mathbf{x}_0}$ onto the model domain $\mathbb{R} \times \Upsilon$ which equals $\mathbb{R} \times \mathcal{S}_\alpha$, $\mathbb{R}^2 \times \mathbb{R}_+$ or $\mathbb{R}^3$. Let $\mathbf{A}^\natural$ be the magnetic potential associated with $\mathbf{A}^{\mathbf{x}_0}$ in variables $\mathbf{x}^\natural$ given by (11.11) and $\mathbf{A}_0^\natural$, $\mathbf{A}_0^{\mathbf{x}_0} (= \mathbf{A}_{\mathbb{X}})$ be their linear parts at $\mathbf{0}$. Applying Lemma A.2 in variables $(u_1, u_2, u_3) = (y, z_1, z_2)$ with $\ell = 1$ gives us a function F such that $\partial_y^2(\mathbf{A}^\natural - \nabla F)(\mathbf{0}) = 0$ leading to the estimates

$$\big| \big(\mathbf{A}^\natural - \mathbf{A}_0^\natural - \nabla F \big)(\mathbf{x}^\natural) \big| \le C(\mathcal{V}_{\mathbf{x}_0}) \left(\|\mathbf{A}^{\mathbf{x}_0}\|_{W^{2,\infty}(\mathcal{V}_{\mathbf{x}_0})} \big(|y||\mathbf{z}| + |\mathbf{z}|^2 \big) + \|\mathbf{A}^{\mathbf{x}_0}\|_{W^{3,\infty}(\mathcal{V}_{\mathbf{x}_0})} |y|^3 \right). \tag{11.33}$$

We define our new quasimode by

$$\varphi_h^{[1]} = \underline{\mathrm{U}}(\mathrm{e}^{-iF/h}\underline{\psi}_h), \quad \text{in} \quad \Pi_{\mathbf{x}_0}, \tag{11.34}$$

with $\underline{\psi}_h$ a given function in $\mathbb{R}\times\Upsilon$. Using (A.3) and (A.6), we have

$$\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0},\Pi_{\mathbb{X}}](\varphi_h^{[1]}) = \mathscr{Q}_h[\mathbf{A}^{\natural}-\nabla F,\mathbb{R}\times\Upsilon](\underline{\psi}_h) \le \mu_h^{[1]} + 2\hat{a}_h^{[1]}\sqrt{\mu_h^{[1]}} + (\hat{a}_h^{[1]})^2, \tag{11.35}$$

where we have set, by analogy with (9.17),

$$\mu_h^{[1]} = \mathscr{Q}_h[\mathbf{A}_{\mathbf{0}}^{\natural},\mathbb{R}\times\Upsilon](\underline{\psi}_h) \quad \text{and} \quad \hat{a}_h^{[1]} = \frac{\|(\mathbf{A}^{\natural}-\mathbf{A}_{\mathbf{0}}^{\natural}-\nabla F)\underline{\psi}_h\|}{\|\underline{\psi}_h\|}. \tag{11.36}$$

We set $\underline{\psi}_h = \chi_h\,\underline{\Psi}_h$ where $\underline{\Psi}$ is the admissible generalized eigenvector of $H(\mathbf{A}_{\mathbf{0}}^{\natural},\mathbb{R}\times\Upsilon)$ in natural variables as introduced in (7.1) and $\underline{\Psi}_h$ its scaled version.

The following Lemma provides an improvement when compared to Lemmas 4.6–4.7, due to estimates (11.33) which replace (4.14).

Lemma 11.2. *With the previous notation, there exist constants $C(\Omega)>0$ and $h_0>0$ such that for all $h\in(0,h_0)$*

$$\hat{a}_h^{[1]} = \frac{\|(\mathbf{A}^{\natural}-\mathbf{A}_{\mathbf{0}}^{\natural}-\nabla F)\underline{\psi}_h\|}{\|\underline{\psi}_h\|} \le C(\Omega)(\|\mathbf{A}^{\natural}\|_{W^{2,\infty}(\mathcal{V}_h^{\square})}(h+h^{\frac12+\delta_0})+\|\mathbf{A}^{\natural}\|_{W^{3,\infty}(\mathcal{V}_h^{\square})}h^{3\delta_0}). \tag{11.37}$$

Proof. Using the form of the admissible generalized eigenvector $\underline{\Psi}$:

$$\underline{\Psi}(\mathbf{x}^{\natural}) = \mathrm{e}^{i\vartheta(\mathbf{x}^{\natural})}\Phi(\mathbf{z}) \quad \text{with} \quad \mathbf{x}^{\natural}=(y,\mathbf{z}),$$

we obtain by definition of $\underline{\psi}_h$

$$|\underline{\psi}_h(\mathbf{x}^{\natural})| = \underline{\chi}_R\left(\frac{|\mathbf{x}^{\natural}|}{h^{\delta_0}}\right)\left|\Phi\left(\frac{\mathbf{z}}{h^{1/2}}\right)\right|.$$

Using the changes of variables $\mathbf{Z}=\mathbf{z}h^{-1/2}$ and $Y=yh^{-\delta_0}$, we find the bounds

$$\begin{aligned}
\left\||y|^3\,\underline{\chi}_R\left(\frac{|\mathbf{x}^{\natural}|}{h^{\delta_0}}\right)\Phi\left(\frac{\mathbf{z}}{h^{1/2}}\right)\right\| &\le h^{3\delta_0}\,\|\underline{\psi}_h\|\\
\left\||y|\,|\mathbf{z}|\,\underline{\chi}_R\left(\frac{|\mathbf{x}^{\natural}|}{h^{\delta_0}}\right)\Phi\left(\frac{\mathbf{z}}{h^{1/2}}\right)\right\| &\le h^{\delta_0+\frac12}\,\|\underline{\psi}_h\|\\
\left\||\mathbf{z}|^2\,\underline{\chi}_R\left(\frac{|\mathbf{x}^{\natural}|}{h^{\delta_0}}\right)\Phi\left(\frac{\mathbf{z}}{h^{1/2}}\right)\right\| &\le h\,\|\underline{\psi}_h\|.
\end{aligned}$$

Summing up the latter three estimates and using (11.33) lead to the lemma. □

Now, since Remark 3.15 allows to generalize Lemma 4.7 to higher derivatives of the potential as in (11.29), we use (9.8) and Lemmas 11.2, 4.6 and 4.7 for case (i) in (11.35) and combine this with (9.15) to deduce

$$\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) \le h\Lambda_{\mathbb{X}} + C(\Omega)(1+\|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)})(h^{2-2\delta_0}+h^{\frac32}+h^{1+\delta_0}+h^{\frac12+3\delta_0}+h^{6\delta_0}). \tag{11.38}$$

We optimize this upper bound by taking $\delta_0 = \frac{1}{3}$. The min-max principle provides Theorem 11.1 with a remainder in $\mathcal{O}(h^{4/3})$ in the case (G2) with $\mathbb{X} = (\mathbf{x}_0)$.

11.2.2. *Sliding quasimodes.* We assume now $\nu \geq 2$, so $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1)$ or $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$. We use the notation of Section 9.3. The main difference with Section 9.3 is that we deal with the linear part of $\mathbf{A}^{\mathbf{x}_0}$ at $\mathbf{v}_1$ instead of $\mathbf{0}$, that is:

$$\mathbf{A}^{\mathbf{x}_0}_{\mathbf{v}_1}(\mathbf{x}) := \nabla \mathbf{A}^{\mathbf{x}_0}(\mathbf{v}_1) \cdot \mathbf{x}, \quad \mathbf{x} \in \Pi_{\mathbf{x}_0}\,.$$

By the change of variable $\mathrm{U}^{\mathbf{v}_1}$, the potential $\mathbf{A}^{\mathbf{x}_0}_{\mathbf{v}_1}$ becomes $\widehat{\mathbf{A}}^{\mathbf{v}_1}$ (cf. (4.1))

$$\widehat{\mathbf{A}}^{\mathbf{v}_1} = (\mathrm{J}^{\mathbf{v}_1})^\top \Big(\big(\mathbf{A}^{\mathbf{x}_0}_{\mathbf{v}_1} - \mathbf{A}^{\mathbf{x}_0}_{\mathbf{v}_1}(\mathbf{v}_1)\big) \circ (\mathrm{U}^{\mathbf{v}_1})^{-1} \Big) \quad \text{with} \quad \mathrm{J}^{\mathbf{v}_1} = \mathrm{d}(\mathrm{U}^{\mathbf{v}_1})^{-1}.$$

Let $\widehat{\zeta}^{\mathbf{v}_1}_h(\mathbf{x}) = \mathrm{e}^{i\langle \mathbf{A}^{\mathbf{x}_0}_{\mathbf{v}_1}(\mathbf{v}_1), \mathbf{x}/h\rangle}$, for $\mathbf{x} \in \Pi_{\mathbf{x}_0}$ and $\widehat{\mathrm{Z}}^{\mathbf{v}_1}_h$ be the operator of multiplication by $\overline{\widehat{\zeta}^{\mathbf{v}_1}_h}$. By analogy with (9.25), we introduce the relation

$$\varphi^{[1]}_h = \widehat{\mathrm{Z}}^{\mathbf{v}_1}_h \circ \mathrm{U}^{\mathbf{v}_1}_*(\varphi^{[2]}_h). \tag{11.39}$$

Let us assume for the end of this section that $\nu = 2$. Let $\widehat{\mathbf{A}}^{\mathbf{v}_1}_{\mathbf{0}}$ be the linear part of $\widehat{\mathbf{A}}^{\mathbf{v}_1}$ at $\mathbf{0} \in \Pi_{\mathbf{x}_0,\mathbf{x}_1}$. We have $\operatorname{curl} \widehat{\mathbf{A}}^{\mathbf{v}_1}_{\mathbf{0}} = \mathbf{B}^{\mathbf{x}_0}_{\mathbf{v}_1}$ where the constant $\mathbf{B}^{\mathbf{x}_0}_{\mathbf{v}_1}$ is the magnetic field $\mathbf{B}^{\mathbf{x}_0}$ frozen at $\mathbf{v}_1$.

We have $E(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbb{X}}) < \mathscr{E}^*(\mathbf{B}_{\mathbf{x}_0}, \Pi_{\mathbb{X}})$. Due to Lemma 10.2, we have

$$\exists \varepsilon > 0, \ \ \forall \mathbf{v}_1 \in \mathcal{B}(0,\varepsilon) \cap \overline{\Pi}_{\mathbf{x}_0}, \quad E(\mathbf{B}^{\mathbf{x}_0}_{\mathbf{v}_1}, \Pi_{\mathbb{X}}) < \mathscr{E}^*(\mathbf{B}^{\mathbf{x}_0}_{\mathbf{v}_1}, \Pi_{\mathbb{X}})\,, \tag{11.40}$$

and $(\mathbf{B}^{\mathbf{x}_0}_{\mathbf{v}_1}, \Pi_{\mathbb{X}})$ is still in situation (G2). Let $\underline{\mathrm{U}}^{\mathbf{v}_1}$ (J the associated matrix) be the rotation $\mathbf{x} \mapsto \mathbf{x}^\natural := (\mathbf{y}, z)$ that maps $\Pi_{\mathbb{X}}$ onto the model domain $\mathbb{R} \times \Upsilon$. Let $\mathbf{A}^{\natural,\mathbf{v}_1}$ be the magnetic potential associated with $\widehat{\mathbf{A}}^{\mathbf{v}_1}$ in variables $\mathbf{x}^\natural$ and $\mathbf{A}^{\natural,\mathbf{v}_1}_{\mathbf{0}}$ be its linear part at $\mathbf{0}$. Due to (11.40), we are still in case (i) of the Dichotomy Theorem 7.3. We use now the admissible generalized eigenvector $\underline{\Psi}^{\mathbf{v}_1}$ of $H(\mathbf{A}^{\natural,\mathbf{v}_1}_{\mathbf{0}}, \mathbb{R} \times \Upsilon)$ in natural variables as introduced in (7.1) and its scaled version $\underline{\Psi}^{\mathbf{v}_1}_h$. The associated ground state energy is denoted by

$$\Lambda_{\mathbf{v}_1} = E(\mathbf{B}^{\mathbf{x}_0}_{\mathbf{v}_1}, \Pi_{\mathbb{X}}). \tag{11.41}$$

An important point is that, choosing $\varepsilon > 0$ small enough, we may assume that, in virtue of Lemma 10.2 (b), the functions $\underline{\Psi}^{\mathbf{v}_1}$ are uniformly exponentially decreasing

$$\exists c > 0, \ C > 0, \quad \forall \mathbf{v}_1 \in \mathcal{B}(\mathbf{0}, \varepsilon), \quad \|\underline{\Psi}^{\mathbf{v}_1} \mathrm{e}^{c|\mathbf{z}|}\|_{L^2(\Upsilon)} \leq C \|\underline{\Psi}^{\mathbf{v}_1}\|_{L^2(\Upsilon)}\,. \tag{11.42}$$

We are arrived at point where the situation is similar as in the sitting case, with the new feature that the generalized eigenvectors $\underline{\Psi}^{\mathbf{v}_1}_h$ depend (in some smooth way) on the parameter $\mathbf{v}_1$. We define the new function on $\Pi_{\mathbb{X}}$ by

$$\varphi^{[2]}_h = \underline{\mathrm{U}}^{\mathbf{v}_1}(\mathrm{e}^{-iF^{\mathbf{v}_1}/h} \underline{\psi}^{\mathbf{v}_1}_h), \tag{11.43}$$

where $\underline{\psi}^{\mathbf{v}_1}_h = \chi_h \underline{\Psi}^{\mathbf{v}_1}_h$ has a support of size $r^{[2]}_h = \mathcal{O}(h^{\delta_0+\delta_1})$ and the phase shift $F^{\mathbf{v}_1}$ will be chosen later. As always we denote by $\mu^{[2]}_h = \mathscr{Q}_h[\mathbf{A}^{\natural,\mathbf{v}_1}_{\mathbf{0}}, \Pi_{\mathbb{X}}](\underline{\psi}^{\mathbf{v}_1}_h)$.

The function $\mathbf{v} \mapsto \Lambda_{\mathbf{v}}$ is Lipschitz-continuous by Lemma 10.2 (a) and thus $|\Lambda_{\mathbf{v}_1} - \Lambda_{\mathbf{0}}| \leq C|\mathbf{v}_1|$. Combining this with Lemma 10.5, we have

$$\mu^{[2]}_h \leq h\Lambda_{\mathbf{v}_1} + Ch^{2-2\delta_0} \leq h\Lambda_{\mathbb{X}} + C(h^{1+\delta_0} + h^{2-2\delta_0}).$$

Now we distinguish whether our quasimode is soft or hard sliding ($\mathbf{x}_0$ is not, or is, a conical point).

• *Soft sliding.* If $\mathbf{x}_0$ is not a conical point, we recall as mentioned in Section 9.3 that $\mathrm{U}^{\mathbf{v}_1}$ is a translation. As in Section 11.1.3 we have

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A}^{\mathbf{x}_0}, \Pi_{\mathbf{x}_0}](\varphi_h^{[1]}) &= \mathscr{Q}_h[\mathbf{A}^\natural(\cdot + \mathbf{v}_1^\natural) - \mathbf{A}^\natural_{\mathbf{v}_1^\natural}(\mathbf{v}_1^\natural) - \nabla F^{\mathbf{v}_1}, \mathbb{R}\times\Upsilon](\underline{\psi}_h^{\mathbf{v}_1})\\ &\le \mu_h^{[2]} + 2\sqrt{\mu_h^{[2]}}\|\big(\mathbf{A}^\natural(\cdot + \mathbf{v}_1^\natural) - \mathbf{A}^\natural_{\mathbf{v}_1^\natural}(\cdot + \mathbf{v}_1^\natural) - \nabla F^{\mathbf{v}_1}\big)\underline{\psi}_h^{\mathbf{v}_1}\|\\ &\quad + \|\big(\mathbf{A}^\natural(\cdot + \mathbf{v}_1^\natural) - \mathbf{A}^\natural_{\mathbf{v}_1^\natural}(\cdot + \mathbf{v}_1^\natural) - \nabla F^{\mathbf{v}_1}\big)\underline{\psi}_h^{\mathbf{v}_1}\|^2,\end{aligned}$$

where we have used the relation $\mathbf{A}^\natural(\cdot + \mathbf{v}_1^\natural) - \mathbf{A}^\natural_{\mathbf{v}_1^\natural}(\mathbf{v}_1^\natural) - \mathbf{A}_{\mathbf{0}}^{\natural,\mathbf{v}_1} = \mathbf{A}^\natural(\cdot + \mathbf{v}_1^\natural) - \mathbf{A}^\natural_{\mathbf{v}_1^\natural}(\cdot + \mathbf{v}_1^\natural)$. We now use Lemma A.2 to choose $F^{\mathbf{v}_1}$ such that $\mathbf{A}^\natural(\cdot + \mathbf{v}_1^\natural) - \mathbf{A}^\natural_{\mathbf{v}_1^\natural}(\cdot + \mathbf{v}_1^\natural) - \nabla F^{\mathbf{v}_1}$ is still controlled by the r.h.s. of (11.33). The proof of Lemma 11.2 is still valid due to the uniform control (11.42), and provides:

$$\begin{aligned}&\|\big(\mathbf{A}^\natural(\cdot + \mathbf{v}_1^\natural) - \mathbf{A}^\natural_{\mathbf{v}_1^\natural}(\cdot + \mathbf{v}_1^\natural) - \nabla F^{\mathbf{v}_1}\big)\underline{\psi}_h^{\mathbf{v}_1}\|\\ &\qquad\le C(\Omega)(\|\mathbf{A}^\natural\|_{W^{2,\infty}(\mathcal{V}_h^\square)}(h + h^{\frac12+\delta_0}) + \|\mathbf{A}^\natural\|_{W^{3,\infty}(\mathcal{V}_h^\square)}h^{3\delta_0})\ \|\underline{\psi}_h^{\mathbf{v}_1}\|.\end{aligned}$$

The proof goes along as in the sitting case and we deduce the same estimate (11.38) with a remainder in $\mathcal{O}(h^{4/3})$.

• *Hard sliding.* If $\mathbf{x}_0$ is a conical point, using formulas (A.3) and (A.6), we have

$$\mathscr{Q}_h[\widehat{\mathbf{A}}^{\mathbf{v}_1}, \Pi_{\mathbb{X}}](\varphi_h^{[1]}) = \mathscr{Q}_h[\mathbf{A}^{\natural,\mathbf{v}_1} - \nabla F^{\mathbf{v}_1}, \mathbb{R}\times\Upsilon](\underline{\psi}_h^{\mathbf{v}_1}) \le \mu_h^{[2]} + 2\hat{a}_h^{[2]}\sqrt{\mu_h^{[2]}} + (\hat{a}_h^{[2]})^2, \tag{11.44}$$

where we have set

$$\hat{a}_h^{[2]} = \frac{\|(\mathbf{A}^{\natural,\mathbf{v}_1} - \mathbf{A}_{\mathbf{0}}^{\natural,\mathbf{v}_1} - \nabla F^{\mathbf{v}_1})\underline{\psi}_h^{\mathbf{v}_1}\|}{\|\underline{\psi}_h^{\mathbf{v}_1}\|}. \tag{11.45}$$

Like previously, Lemma A.2 gives a function $F^{\mathbf{v}_1}$ satisfying

$$\big|\big(\mathbf{A}^{\natural,\mathbf{v}_1} - \mathbf{A}_{\mathbf{0}}^{\natural,\mathbf{v}_1} - \nabla F^{\mathbf{v}_1}\big)(\mathbf{x}^\natural)\big| \le C(\mathcal{V}_{\mathbf{x}_0})\left(\|\mathbf{A}^{\natural,\mathbf{v}_1}\|_{W^{2,\infty}}(|y||\mathbf{z}| + |\mathbf{z}|^2) + \|\mathbf{A}^{\natural,\mathbf{v}_1}\|_{W^{3,\infty}}|y|^3\right). \tag{11.46}$$

Due to the uniform estimate (11.42), the proof of Lemma 11.2 still applied. Combine this with (11.29) gives

$$\begin{aligned}\hat{a}_h^{[2]} &\le C(\|\mathbf{A}^{\natural,\mathbf{v}_1}\|_{W^{2,\infty}(\mathrm{supp}(\underline{\psi}_h^{\mathbf{v}_1}))}\ (h + h^{\frac12+\delta_0+\delta_1}) + \|\mathbf{A}^{\natural,\mathbf{v}_1}\|_{W^{3,\infty}(\mathrm{supp}(\underline{\psi}_h^{\mathbf{v}_1}))}\ h^{3\delta_0+3\delta_1})\\ &\le C(\|\mathbf{A}\|_{W^{2,\infty}(\Omega)}\ (h^{1-\delta_0} + h^{\frac12+\delta_1}) + \|\mathbf{A}\|_{W^{3,\infty}(\Omega)}\ h^{\delta_0+3\delta_1}).\end{aligned}$$

Then Relation (9.32) becomes

$$\begin{aligned}\mathscr{Q}_h[\mathbf{A},\Omega](\varphi_h^{[0]}) \le h\Lambda_{\mathbb{X}} + C\,(h^{2-2\delta_0} + h^{1+\delta_0}) + C(h^{2-2\delta_0-2\delta_1} + h^{1+\delta_0} + h^{1+\delta_1})\\ + C\left(h^{\frac32-\delta_0} + h^{1+\delta_1} + h^{\frac12+\delta_0+3\delta_1} + h^{2\delta_0+6\delta_1}\right).\end{aligned} \tag{11.47}$$

Choosing $\delta_0 = \frac{5}{16}$ and $\delta_1 = \frac18$ gives the upper-bound (11.1) in situation (G2) for hard sliding quasimodes.

11.2.3. *Doubly sliding quasimode.* In that case, as mentioned in Section 9.4, $\nu = 3$, $\mathbb{X} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_0$ is a conical point and $\mathrm{U}^{\mathbf{v}_2}$ is a translation. We define

$$\varphi_h^{[2]} = \widehat{Z}_h^{\mathbf{v}_2} \circ \mathrm{U}_*^{\mathbf{v}_2}(\varphi_h^{[3]}), \tag{11.48}$$

where $\widehat{Z}_h^{\mathbf{v}_2}$ is the operator of multiplication by $\overline{\widehat{\zeta}^{\mathbf{v}_2}}$ with $\widehat{\zeta}_h^{\mathbf{v}_2}(\mathbf{x}) = \mathrm{e}^{i\langle \widehat{\mathbf{A}}_{\mathbf{v}_2}^{\mathbf{v}_1}(\mathbf{v}_2), \mathbf{x}/h\rangle}$ and

$$\widehat{\mathbf{A}}^{\mathbf{v}_2} = \big(\widehat{\mathbf{A}}_{\mathbf{v}_2}^{\mathbf{v}_1} - \widehat{\mathbf{A}}_{\mathbf{v}_2}^{\mathbf{v}_1}(\mathbf{v}_2)\big) \circ (\mathrm{U}^{\mathbf{v}_2})^{-1},$$

with coincides with its linear part $\widehat{\mathbf{A}}_{\mathbf{0}}^{\mathbf{v}_2}$. Since $\mathrm{G}^{\mathbf{v}_2} = \mathbb{I}_3$, we have

$$\mathscr{Q}_h[\widehat{\mathbf{A}}_{\mathbf{0}}^{\mathbf{v}_1}, \Pi_{\mathbf{x}_0,\mathbf{x}_1}](\varphi_h^{[2]}) = \mathscr{Q}_h[\widehat{\mathbf{A}}_{\mathbf{0}}^{\mathbf{v}_2}, \Pi_{\mathbb{X}}](\varphi_h^{[3]}). \tag{11.49}$$

We set in the same spirit as above, $\varphi_h^{[3]} = \underline{\mathrm{U}}^{\mathbf{v}_2}(\mathrm{e}^{-iF^{\mathbf{v}_2}/h}\chi_h\Psi_h^{\mathbf{v}_2})$. The constant magnetic field $\mathbf{B}_{\mathbf{0}}^{\mathbf{v}_1,\mathbf{v}_2} = \operatorname{curl}\widehat{\mathbf{A}}_{\mathbf{0}}^{\mathbf{v}_2}$ is the magnetic field $\mathbf{B}^{\mathbf{x}_0}$ frozen at $\mathbf{v}_1$, transformed by $\mathrm{U}^{\mathbf{v}_1}$ and then frozen at $\mathbf{v}_2$. Once again, $(\mathbf{B}_{\mathbf{0}}^{\mathbf{v}_1,\mathbf{v}_2}, \Pi_{\mathbb{X}})$ is still in situation (G2) for h small enough and we may use Lipschitz estimates for the associated ground state energy and uniform decay estimates for the associated AGE. As in the soft sliding case described above, we take advantage of the translation $\mathrm{U}^{\mathbf{v}_2}$ and get a better estimate for the last linearization (that is step [c2], see Figure 2) by a suitable choice of $F^{\mathbf{v}_2}$. We can conclude as the conical case at level 2 and obtain again (11.47). We deduce

$$\lambda_h(\mathbf{B}, \Omega) \le h\mathscr{E}(\mathbf{B}, \Omega) + C(\Omega)\big(1 + \|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)}\big)h^{9/8}\,.$$

The proof of Theorem 11.1 is now complete in case (G2).

12. Conclusion: Improvements and extensions

In this work we have shown how a recursive structure of corner domains allows to analyze the Neumann magnetic Laplacian and its ground state energy $\lambda_h(\mathbf{B}, \Omega)$. To conclude, we discuss some standard consequences in the situation of corner concentration. We also address the issues of generalizing our results to any dimension. We finally mention the adaptation of our methods to different boundary value problems, namely the Dirichlet magnetic Laplacian and the Robin Laplacian in the attractive limit.

12.1. **Corner concentration and standard consequences.** Let Ω be a 3D corner domain and $\mathbf{B}$ be a magnetic field. For each corner $\mathbf{v} \in \mathfrak{V}$ of Ω, let us denote by $K_{\mathbf{v}}$ the number of eigenvalues of the tangent model operator $H(\mathbf{A}_{\mathbf{v}}, \Pi_{\mathbf{v}})$ less than the minimal local energy outside the corners $\inf_{\mathbf{x}\in\overline{\Omega}\setminus\mathfrak{V}} E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$. If no such eigenvalue exists, we set $K_{\mathbf{v}} = 0$. If they do exist, we denote them by $\lambda^{(k)}(\mathbf{B}_{\mathbf{v}}, \Pi_{\mathbf{v}})$, $k = 1, \ldots, K_{\mathbf{v}}$, so that

$$\forall \mathbf{v} \in \mathfrak{V}, \quad \forall 1 \le k \le K_{\mathbf{v}}, \quad \lambda^{(k)}(\mathbf{B}_{\mathbf{v}}, \Pi_{\mathbf{v}}) < \inf_{\mathbf{x}\in\overline{\Omega}\setminus\mathfrak{V}} E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}}).$$

Setting $K(\mathbf{B}, \Omega) = \sum_{\mathbf{v}\in\mathfrak{V}} K_{\mathbf{v}}$, we assume that we are in the case of corner concentration, *i.e.*,

$$K(\mathbf{B}, \Omega) > 0\,.$$

Then several standard consequences hold for the eigenvalue asymptotics of the first $K(\mathbf{B}, \Omega)$ eigenvalues $\lambda_h^{(k)}(\mathbf{B}, \Omega)$ of the magnetic Laplacian $H_h(\mathbf{A}, \Omega)$. Indeed, for $1 \le k \le K(\mathbf{B}, \Omega)$, we denote by $\mathscr{E}^{(k)}(\mathbf{B},\Omega)$ the k-th element (repeated with multiplicity) of the collection of eigenvalues $\lambda^{(j)}(\mathbf{A}_{\mathbf{v}},\Pi_{\mathbf{v}})$ of the model operators, for $\mathbf{v}\in\mathfrak{V}$ and $1\leq j\leq K_{\mathbf{v}}$. Then we have

$$\left|\lambda_h^{(k)}(\mathbf{B},\Omega)-h\mathscr{E}^{(k)}(\mathbf{B},\Omega)\right|\leq Ch^{3/2},\quad \forall 1\leq k\leq K(\mathbf{B},\Omega). \tag{12.1}$$

In fact, we can prove like in [7, Section 7] a complete asymptotics expansion in power of $h^{1/2}$ for the eigenvalues $\lambda_h^{(k)}(\mathbf{B},\Omega)$, $1\leq k\leq K(\mathbf{B},\Omega)$ and (12.1) is a consequence. Furthermore, we have corner localization of the eigenvectors. Another consequence of the complete expansion of the low-lying eigenvalues is the monotonicity of the ground state energy $B\mapsto\lambda(\breve{\mathbf{B}},\Omega)$ (1.13) in the point of view of large magnetic field. This can be seen as a strong diamagnetic inequality and relies on the same arguments as in [11, Section 2.1].

12.2. **The necessity of a taxonomy.** Let us emphasize the role of the taxonomy of model problems played in the analysis. The proof of upper bounds with remainder for λ_h strongly relies on the existence of generalized eigenfunctions for model operators associated with the minimum of local energies. Our Dichotomy Theorem provides a positive answer and is based on an exhaustive description of the ground states of model operators depending on the dimension $d\in\{0,\ldots,3\}$ of reduced cones, *i.e.*, on spaces, half-spaces, wedges and 3D cones, respectively. In cases $d\leq 2$, the analysis is made through a fibration (*i.e.*, a partial Fourier transform), leading to a new operator that is not a standard magnetic Laplacian. As consequence, the analysis of the key quantity $\mathscr{E}^*$ seems to be specific to each dimension.

Besides, in higher dimensions, a magnetic field $\mathbf{B}$ can be identified in each point $\mathbf{x}\in\overline{\Omega}$ with a $n\times n$ antisymmetric matrix, thus determines $\frac{n}{2}$ or $\frac{n-1}{2}$ two-dimensional invariant subspaces $P_{\mathbf{x}}^j$ when n is even or odd, respectively (for instance, in dimension $n=3$, the space $P_{\mathbf{x}}^1$ is the orthogonal space to the vector $\mathbf{B}_{\mathbf{x}}$). Given a cone $\mathbb{R}^\nu\times\Gamma$ with $\nu>0$, its interaction with the planes $P_{\mathbf{x}}^j$ can be highly non-trivial and there is no reason that there exists a magnetic potential which depends on less variables than n. Thus the fibration process we have used does not seem available in general in the n dimensional case. At this stage, a recursive analysis of the ground state of the magnetic Laplacian does not seem possible without a deeper analysis of tangent model operators, namely a complete taxonomy valid for all dimension.

12.3. **Continuity of local energies.** A standard procedure to investigate the stability of the ground state energy of a self-adjoint operator consists in constructing quasimodes issued from the spectrum of the unperturbed problem, using them for the perturbed operator, and concluding with the min-max principle. This procedure applied to the ground state energy of model problems associated with $H(\mathbf{A},\Omega)$ would provide upper semicontinuity under perturbation and, therefore, upper semicontinuity for the local energy $\mathbf{x}\mapsto E(\mathbf{B}_{\mathbf{x}},\Pi_{\mathbf{x}})$ on each stratum $\mathbf{t}$ of $\overline{\Omega}$.

In the case of Neumann boundary conditions, we have proved the continuity on each stratum by using once more the taxonomy of model problems. In particular Lemma 6.5 uses intensively the structure of the magnetic Laplacian on wedges and is linked to our Dichotomy Theorem, see [49]. The lower semicontinuity of the local energy between strata is a consequence of Theorem 3.25, and relies on the continuity on each stratum. In contrast with Dirichlet conditions, Neumann boundary conditions imply a decrease of the local ground energy on strata of higher codimensions, including possible discontinuities between strata.

In the general n dimensional case, the sole known result is the continuity of the local energy on the interior stratum, *i.e.*, Ω itself. Indeed, for any $\mathbf{x} \in \Omega$, we have $E(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x}) = b(\mathbf{x})$ with $b(\mathbf{x})$ defined in (2.1). The generic regularity is in fact Hölder of exponent $\frac{1}{2n}$ as mentioned in [32, Lemma 5.4]).

12.4. **Dirichlet boundary conditions.** If one considers now the magnetic Laplacian with Dirichlet boundary conditions, the situation of the local energies denoted now $E^{\mathrm{D}}(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x})$ is far simpler than in the Neumann case. For any interior point $\mathbf{x} \in \Omega$, $E^{\mathrm{D}}(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x}) = E(\mathbf{B}_\mathbf{x}, \mathbb{R}^n)$ is equal to the intensity $b_\mathbf{x}$ of $\mathbf{B}_\mathbf{x}$ (with $b_\mathbf{x} = b(\mathbf{x})$ defined in (2.1)). If $\mathbf{x}$ lies in the boundary of Ω, by Dirichlet monotonicity, $E^{\mathrm{D}}(\mathbf{B}_\mathbf{x}, \Pi_\mathbf{x}) \geq E(\mathbf{B}_\mathbf{x}, \mathbb{R}^n)$, and the converse inequality is the consequence of a standard argument of type Persson Lemma, cf. Theorem 6.6. Thus, like in the case without boundary, the sole ingredient in local energies is the intensity of the magnetic field in each point $\mathbf{x} \in \partial\Omega$. At this point, we could generalize the estimates of [28]

$$-C^{-}h^{5/4} \leq \lambda_h(\mathbf{B}, \Omega) - h\mathscr{E}(\mathbf{B}, \Omega) \leq C^{+}h^{4/3}$$

to any domain Ω with Lipschitz boundary and $W^{3,\infty}(\overline{\Omega})$ magnetic potential with nonvanishing magnetic field $\mathbf{B}$, including the case when the minimum is attained on the boundary. The key arguments are the following:

LOWER BOUND: One uses a IMS partition technique in order to *linearize* the potential on each piece of the partition, but *without local maps*. Then, when a local support crosses the boundary of Ω, one simply uses the lower bound $\lambda_h(\mathbf{B}_{\mathbf{x}_0}, \Omega) \geq \lambda_h(\mathbf{B}_{\mathbf{x}_0}, \mathbb{R}^n)$ for the "central point" $\mathbf{x}_0$ of this local support.

UPPER BOUND: For $\mathbf{x}_0 \in \partial\Omega$, one constructs interior sliding quasimodes with support in a cone interior to Ω and with vertex $\mathbf{x}_0$. In order to obtain the refined convergence rate $h^{4/3}$ instead of $h^{5/4}$, one has to use a gauge transform similar to that in [28, p. 54-55].

12.5. **Robin boundary conditions with a large parameter for the Laplacian.** The spectral behavior of the Neumann magnetic Laplacian has some analogy with the following Robin boundary eigenvalue problem that consists in solving

$$\begin{cases} -\Delta\psi = \lambda\psi & \text{in } \ \Omega, \\ \nabla\psi\cdot\mathbf{n} - \beta\psi = 0 & \text{on } \ \partial\Omega, \end{cases} \tag{12.2}$$

where $\beta \in \mathbb{R}$ is a parameter. We denote by $H^{\mathrm{R}}_\beta(\Omega)$ the associated operator. This problem also arises from a linearization of the Ginzburg-Landau equation, in the zero field regime ([25]). The asymptotics of the ground state energy $\lambda^{\mathrm{R}}_\beta(\Omega)$ in the attractive limit $\beta \to +\infty$ has been studied in [36, 45, 21] and presents several similarities with the semiclassical Neumann magnetic Laplacian. It is still relevant to define the local energies $E(\Pi_\mathbf{x})$ as the ground state energies of tangent operators (with $\beta = 1$) and $\mathscr{E}(\Omega) := \inf_{x\in\overline{\Omega}} E(\Pi_\mathbf{x})$. It is proved in [36] that

$$\lambda^{\mathrm{R}}_\beta(\Omega) \underset{\beta\to+\infty}{\sim} \mathscr{E}(\Omega)\beta^2.$$

If Ω is a general n-dimensional corner domain belonging to the class $\mathfrak{D}(\mathbb{R}^n)$, we expect that the method presented in our paper can yield an improved estimate for the Robin ground state energy $\lambda^{\mathrm{R}}_\beta(\Omega)$ when $\beta \to +\infty$. We may already notice that we have a convenient separation of variables on a tangent cone $\mathbb{R}^\nu \times \Gamma$ and that the tangent operator is unitarily equivalent to $-\Delta_{|\mathbb{R}^\nu} \times H^{\mathrm{R}}_1(\Gamma)$.

Therefore the problematics linked to the taxonomy mentioned above are elementary in this case, and one should be able first to prove global lower semicontinuity of the local energies, then to construct quasimodes based on a n-scale procedure, combined with a chain of atlantes (cf. Section 3.4.4). We may conjecture an estimate of the kind $|\lambda_\beta^{\mathrm{R}}(\Omega) - \mathscr{E}(\Omega)\beta^2| \le C\beta^{\kappa(n)}$ with $\kappa(n) < 2$, valid for large β.

Appendix A. Magnetic identities

A.1. Gauge transform.

Lemma A.1. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a domain and let ϑ be a regular function on $\overline{\mathcal{O}}$. Let $\mathbf{A}$ be a regular potential. Then*

$$\forall\psi \in \mathrm{Dom}(q_h[\mathbf{A},\mathcal{O}]), \quad q_h[\mathbf{A}+\nabla\vartheta,\mathcal{O}](\mathrm{e}^{-i\vartheta/h}\psi) = q_h[\mathbf{A},\mathcal{O}](\psi).$$

This well-known result is a consequence of the commutation formula

$$(-ih\nabla + \mathbf{A} + \nabla\vartheta)\left(\mathrm{e}^{-i\vartheta/h}\psi\right) = \mathrm{e}^{-i\vartheta/h}(-ih\nabla + \mathbf{A})\psi\,.$$

Lemma A.2. *Let $\mathcal{O}$ be a bounded domain such that $\mathbf{0} \in \overline{\mathcal{O}}$. Let $\mathbf{u} = (u_1, u_2, u_3)$ denote Cartesian coordinates in $\mathcal{O}$. Let $\mathbf{A} \in W^{3,\infty}(\mathcal{O})$ be a magnetic potential such that $\mathbf{A}(\mathbf{0}) = 0$. Let $\mathbf{A_0}$ denote the linear part of $\mathbf{A}$ at $\mathbf{0}$. Let ℓ be an index in $\{1, 2, 3\}$.*

(a) There exists a change of gauge ∇F where F is a polynomial function of degree 3, so that

(1) The linear part of $\mathbf{A} - \nabla F$ at $\mathbf{0}$ is still $\mathbf{A_0}$,

(2) The second derivative of $\mathbf{A} - \nabla F$ with respect to u_ℓ cancels at $\mathbf{0}$:

$$\partial^2_{u_\ell}(\mathbf{A} - \nabla F)(\mathbf{0}) = 0.$$

(3) The coefficients of F are bounded by $\|\mathbf{A}\|_{W^{2,\infty}(\mathcal{O})}$.

(b) Let us choose $\ell = 1$ for instance. We have the estimate

$$\begin{aligned}(\text{A.1})\quad &|\mathbf{A}(\mathbf{u}) - \mathbf{A_0}(\mathbf{u}) - \nabla F(\mathbf{u})| \\ &\le C(\mathcal{O})\ \left(\|\mathbf{A}\|_{W^{2,\infty}(\mathcal{O})}\big(|u_1u_2| + |u_1u_3| + |u_2|^2 + |u_3|^2\big) + \|\mathbf{A}\|_{W^{3,\infty}(\mathcal{O})}|u_1|^3\right)\,,\end{aligned}$$

where the constant $C(\mathcal{O})$ depends only on the outer diameter of $\mathcal{O}$.

Proof. The Taylor expansion of $\mathbf{A}$ at $\mathbf{0}$ takes the form

$$\mathbf{A} = \mathbf{A_0} + \mathbf{A}^{(2)} + \mathbf{A}^{(\mathrm{rem},3)},$$

where $\mathbf{A}^{(2)}$ is a homogeneous polynomial of degree 2 with 3 components and $\mathbf{A}^{(\mathrm{rem},3)}$ is a remainder:

$$(\text{A.2})\qquad |\mathbf{A}^{(\mathrm{rem},3)}(\mathbf{u})| \le \|\mathbf{A}\|_{W^{3,\infty}(\mathcal{O})}|\mathbf{u}|^3 \quad \text{for} \quad \mathbf{u} \in \mathcal{O}.$$

Let us write the m-th component $A^{(2)}_m$ of $\mathbf{A}^{(2)}$ as

$$A^{(2)}_m(\mathbf{u}) = \sum_{|\alpha|=2} a_{m,\alpha} u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} \quad \text{for} \quad \mathbf{u} = (u_1, u_2, u_3) \in \mathcal{O}.$$

(a) Now, the polynomial F can be explicitly determined. It suffices to take

$$F(\mathbf{u}) = u_\ell^2\big(a_{1,\alpha^*}u_1 + a_{2,\alpha^*}u_2 + a_{3,\alpha^*}u_3 - \tfrac{2}{3}a_{\ell,\alpha^*}u_\ell\big),$$

where α^* is such that $\alpha^*_\ell = 2$ (and the other components are 0). Then

$$\nabla F(\mathbf{u}) = u_\ell^2 \begin{pmatrix} a_{1,\alpha^*} \\ a_{2,\alpha^*} \\ a_{3,\alpha^*} \end{pmatrix}$$

and point (a) of the lemma is proved.

(b) Choosing $\ell = 1$, we see that the m-th components of $\mathbf{A}^{(2)} - \nabla F$ is

$$A^{(2)}_m(\mathbf{u}) - (\nabla F)_m(\mathbf{u}) \\ = a_{m,(1,1,0)}u_1u_2 + a_{m,(1,0,1)}u_1u_3 + a_{m,(0,1,1)}u_2u_3 + a_{m,(0,2,0)}u_2^2 + a_{m,(0,0,2)}u_3^2\,.$$

Hence $\mathbf{A}^{(2)} - \nabla F$ satisfies the estimate

$$|(\mathbf{A}^{(2)}(\mathbf{u}) - \nabla F(\mathbf{u})| \le \|\mathbf{A}\|_{W^{2,\infty}(\mathcal{O})}\big(|u_1u_2| + |u_1u_3| + |u_2|^2 + |u_3|^2\big).$$

But

$$\mathbf{A} - \mathbf{A_0} - \nabla F = \mathbf{A}^{(2)} - \nabla F + \mathbf{A}^{(\mathrm{rem},3)}.$$

Therefore, with (A.2)

$$|\mathbf{A}(\mathbf{u}) - \mathbf{A_0}(\mathbf{u}) - \nabla F(\mathbf{u})| \le \|\mathbf{A}\|_{W^{2,\infty}(\mathcal{O})}\big(|u_1u_2| + |u_1u_3| + |u_2|^2 + |u_3|^2\big) + \|\mathbf{A}\|_{W^{3,\infty}(\mathcal{O})}|\mathbf{u}|^3.$$

Using finally that $|\mathbf{u}|^3 \le 12(|u_1|^3 + |u_2|^3 + |u_3|^3) \le C(\mathcal{O})(|u_1|^3 + |u_2|^2 + |u_3|^2)$, we conclude the proof of estimate (A.1). □

A.2. **Change of variables.** Let G be a metric of $\mathbb{R}^3$, that is a 3×3 positive symmetric matrix with regular coefficients. For a smooth magnetic potential, the quadratic form of the associated magnetic Laplacian with the metric G is denoted by $q_h[\mathbf{A},\mathcal{O},\mathrm{G}]$ and is defined in (1.21). The following lemma describes how this quadratic form is involved when using a change of variables:

Lemma A.3. *Let $\mathrm{U} : \mathcal{O} \to \mathcal{O}'$, $\mathbf{u} \mapsto \mathbf{v}$ be a diffeomorphism with $\mathcal{O}$, $\mathcal{O}'$ domains in $\mathbb{R}^3$. We denote by $\mathrm{J} := \mathrm{d}(\mathrm{U}^{-1})$ the jacobian matrix of the inverse of U. Let $\mathbf{A}$ be a magnetic potential and $\mathbf{B} = \operatorname{curl}\mathbf{A}$ the associated magnetic field. Let f be a function of $\mathrm{Dom}(q_h[\mathbf{A},\mathcal{O}])$ and $\psi := f\circ\mathrm{U}^{-1}$ defined in $\mathcal{O}'$. For any $h>0$ we have*

$$q_h[\mathbf{A},\mathcal{O}](f) = q_h[\widetilde{\mathbf{A}},\mathcal{O}',\mathrm{G}](\psi) \quad \textit{and} \quad \|f\|_{L^2(\mathcal{O})} = \|\psi\|_{L^2_{\mathrm{G}}(\mathcal{O}')} \tag{A.3}$$

where the new magnetic potential and the metric are respectively given by

$$\widetilde{\mathbf{A}} := \mathrm{J}^\top\big(\mathbf{A}\circ\mathrm{U}^{-1}\big) \quad \textit{and} \quad \mathrm{G} := \mathrm{J}^{-1}(\mathrm{J}^{-1})^\top\,. \tag{A.4}$$

The magnetic field $\widetilde{\mathbf{B}} = \operatorname{curl}\widetilde{\mathbf{A}}$ in the new variables is given by

$$\widetilde{\mathbf{B}} := |\det\mathrm{J}|\,\mathrm{J}^{-1}\big(\mathbf{B}\circ\mathrm{U}^{-1}\big). \tag{A.5}$$

Let $\rho>0$, using the previous lemma with the scaling $\mathrm{U}^\rho := \mathbf{x}\mapsto\sqrt{\rho}\,\mathbf{x}$ we get

Lemma A.4. *Let $\mathcal{O}$ be a domain in $\mathbb{R}^n$ and set $r\mathcal{O} := \{\mathbf{x} \in \mathbb{R}^n,\ \mathbf{x} = r\mathbf{x}' \text{ with } \mathbf{x}' \in \mathcal{O}\}$ for a chosen positive r. Let $\mathbf{B}$ be a constant magnetic field and $\mathbf{A}$ be an associated linear potential. For any $\psi \in \mathrm{Dom}(q[\mathbf{A},\mathcal{O}])$ unitary in $L^2(\mathcal{O})$, we define for any positive ρ*

$$\psi_\rho(\mathbf{x}) := \rho^{-n/4}\psi\Big(\frac{\mathbf{x}}{\sqrt{\rho}}\Big), \qquad \forall \mathbf{x} \in \mathcal{O}.$$

Then ψ_ρ belongs to $\mathrm{Dom}(q_\rho[\mathbf{A},\sqrt{\rho}\,\mathcal{O}])$, is unitary in $L^2(\sqrt{\rho}\,\mathcal{O})$ and we have

(1) $q[\mathbf{A},\mathcal{O}](\psi) = \rho\, q[\rho^{-1}\mathbf{A},\sqrt{\rho}\,\mathcal{O}](\psi_\rho) = \rho^{-1}q_\rho[\mathbf{A},\sqrt{\rho}\,\mathcal{O}](\psi_\rho)$.

(2) $E(\mathbf{B},\mathcal{O}) = \rho\, E\big(\rho^{-1}\mathbf{B},\sqrt{\rho}\,\mathcal{O}\big)$.

A.3. **Comparison formula.** Let $\mathcal{O}$ be a domain and let $\mathbf{A}$ and $\mathbf{A}'$ be two magnetic potentials. Then, for any function ψ of $\mathrm{Dom}(q_h[\mathbf{A},\mathcal{O}]) \cap \mathrm{Dom}(q_h[\mathbf{A}',\mathcal{O}])$, we have:

$$q_h[\mathbf{A},\mathcal{O}](\psi) = q_h[\mathbf{A}',\mathcal{O}](\psi) + 2\,\mathrm{Re}\big\langle(-ih\nabla+\mathbf{A}')\psi,(\mathbf{A}-\mathbf{A}')\psi\big\rangle_{\mathcal{O}} + \|(\mathbf{A}-\mathbf{A}')\psi\|^2\,. \tag{A.6}$$

A.4. **Cut-off effect.** In this section we recall standard IMS formulas. This kind of formulas appear for Schrödinger operators in [17], but they can also be found in older works like [41]. In this section $\mathbf{A}$ denotes a regular magnetic potential and notations are those introduced in § 1.5.

The first formula describes the effect of a partition of the unity on the energy of a function which is in the form domain, see for example [57, Lemma 3.1]:

Lemma A.5 (IMS formula)**.** *Assume that $\chi_1,\dots,\chi_L \in \mathscr{C}^\infty(\overline{\mathcal{O}})$ are such that $\sum_{\ell=1}^L \chi_\ell^2 \equiv 1$ on $\mathcal{O}$. Then, for any $\psi \in \mathrm{Dom}(q_h[\mathbf{A},\mathcal{O}])$*

$$q_h[\mathbf{A},\mathcal{O}](\psi) = \sum_{\ell=1}^L q_h[\mathbf{A},\mathcal{O}](\chi_\ell\psi) - h^2\sum_{\ell=1}^L \|\psi\nabla\chi_\ell\|^2_{L^2(\mathcal{O})}$$

The second formula describes the energy of a function satisfying locally the Neumann boundary conditions when applying a cut-off function, see for example [29, (6.11)]:

Lemma A.6. *Let $\chi \in \mathscr{C}_0^\infty(\overline{\mathcal{O}})$ a real smooth function. Then for any $\psi \in \mathrm{Dom}_{\mathrm{loc}}(H_h(\mathbf{A},\mathcal{O}))$*

$$q_h[\mathbf{A},\mathcal{O}](\chi\psi) = \mathrm{Re}\,\big\langle \chi^2 H_h(\mathbf{A},\mathcal{O})\psi,\psi\big\rangle_{\mathcal{O}} + h^2\|\,|\nabla\chi|\,\psi\|^2_{L^2(\mathcal{O})}\,.$$

• *Orientation of the magnetic field.* Let $\mathbf{B}$ be a magnetic field. It is known that changing $\mathbf{B}$ into $-\mathbf{B}$ does not affect the spectrum of the associated magnetic Laplacian. More precisely we have:

Lemma A.7. *Let $\mathcal{O} \subset \mathbb{R}^3$ be a domain, $\mathbf{B}$ be a magnetic field and $\mathbf{A}$ an associated potential. Then $H_h(-\mathbf{A},\mathcal{O})$ and $H_h(\mathbf{A},\mathcal{O})$ are unitary equivalent. We have*

$$\forall \psi \in \mathrm{Dom}(q_h[\mathbf{A},\mathcal{O}]), \quad q_h[-\mathbf{A},\mathcal{O}](\overline{\psi}) = q_h[\mathbf{A},\mathcal{O}](\psi)$$

and ψ is an eigenfunction of $H_h(\mathbf{A},\mathcal{O})$ if and only if $\overline{\psi}$ is an eigenfunction of $H_h(-\mathbf{A},\mathcal{O})$.

APPENDIX B. PARTITION OF UNITY SUITABLE FOR IMS TYPE FORMULAS

Our partitions of unity on general corner domains have to be compatible with an admissible atlas (Definition 3.11).

Lemma B.1. *Let $n \geq 1$ be the space dimension. M denotes $\mathbb{R}^n$ or $\mathbb{S}^n$. Let $\Omega \in \mathfrak{D}(M)$ be a corner domain with an admissible atlas $(\mathcal{U}_{\mathbf{x}}, \mathrm{U}^{\mathbf{x}})_{\mathbf{x}\in\overline{\Omega}}$. Let $K > 1$ be a coefficient. Then there exist a positive integer L and two positive constants $\rho_{\max}$ and $\kappa \leq 1$ (depending on Ω and K) such that for all $\rho \in (0, \rho_{\max}]$, there exists a (finite) set $\mathscr{Z} \subset \overline{\Omega} \times [\kappa\rho, \rho]$ satisfying the following three properties*

1. *We have the inclusion $\overline{\Omega} \subset \cup_{(\mathbf{x},r)\in\mathscr{Z}} \overline{\mathcal{B}}(\mathbf{x}, r)$*
2. *For any $(\mathbf{x}, r) \in \mathscr{Z}$, the ball $\mathcal{B}(\mathbf{x}, Kr)$ is contained in the map-neighborhood $\mathcal{U}_{\mathbf{x}}$,*
3. *Each point $\mathbf{x}_0$ of $\overline{\Omega}$ belongs to at most L different balls $\mathcal{B}(\mathbf{x}, Kr)$.*

Before performing the proof of this lemma, let us draw some easy consequence on the existence of suitable IMS type partitions of unity in corner domains.

Lemma B.2. *Let $\Omega \in \mathfrak{D}(\mathbb{R}^n)$ and choose $K = 2$. Let $(L, \rho_{\max}, \kappa)$ be the parameters provided by Lemma B.1. For any $\rho \in (0, \rho_{\max}]$ let $\mathscr{Z} \subset \overline{\Omega} \times [\kappa\rho, \rho]$ be an associate set of pairs (center, radius). Then there exists a collection of smooth functions $(\chi_{(\mathbf{x},r)})_{(\mathbf{x},r)\in\mathscr{Z}}$ with $\chi_{(\mathbf{x},r)} \in \mathscr{C}_0^\infty(\mathcal{B}(\mathbf{x}, 2r))$ satisfying the identity (partition of unity)*

$$\sum_{(\mathbf{x},r)\in\mathscr{Z}} \chi^2_{(\mathbf{x},r)} = 1 \quad \text{on} \quad \overline{\Omega}$$

and the uniform estimate of gradients

$$\exists C > 0, \quad \forall(\mathbf{x}, r) \in \mathscr{Z}, \quad \|\nabla\chi_{(\mathbf{x},r)}\|_{L^\infty(\Omega)} \leq C\rho^{-1}\,,$$

where C only depends on Ω. By construction any ball $\mathcal{B}(\mathbf{x}, 2r)$ is a map-neighborhood of $\mathbf{x}$ included the maps of an admissible atlas.

Proof. Let $\xi_{(\mathbf{x},r)} \in \mathscr{C}_0^\infty(\mathcal{B}(\mathbf{x}, 2r))$, with the property that $\xi_{(\mathbf{x},r)} \equiv 1$ in $\mathcal{B}(\mathbf{x}, r)$, and satisfying the gradient bound $\|\nabla\xi_{(\mathbf{x},r)}\|_{L^\infty(\mathbb{R}^3)} \leq Cr^{-1}$ where C is a universal constant. Then we set for each $(\mathbf{x}_0, r_0) \in \mathscr{Z}$

$$\chi_{(\mathbf{x}_0,r_0)} = \frac{\xi_{(\mathbf{x}_0,r_0)}}{(\sum_{(\mathbf{x},r)\in\mathscr{Z}} \xi^2_{(\mathbf{x},r)})^{1/2}}\,.$$

Due to property (1) in Lemma B.1, $\sum_{(\mathbf{x},r)\in\mathscr{Z}} \xi^2_{(\mathbf{x},r)} \geq 1$ and due to property (3),

$$\|\sum_{(\mathbf{x},r)\in\mathscr{Z}} \nabla\xi^2_{(\mathbf{x},r)}\|_{L^\infty(\mathbb{R}^3)} \leq CL_\Omega\,.$$

We deduce the lemma. □

Here are preparatory notations and lemmas for the proof of Lemma B.1.

Let $\Omega \in \mathfrak{D}(M)$ and $K > 1$. If the assertions of Lemma B.1 are true for this Ω and this K, we say that Property $\mathscr{P}(\Omega, K)$ holds. We may also specify that the assertion by the sentence

Property $\mathscr{P}(\Omega, K)$ holds with parameters $(L, \rho_{\max}, \kappa)$.

Let $\mathcal{U}^* \subset\subset \mathcal{U}$ be two nested open sets. We say that the property $\mathscr{P}(\Omega, K; \mathcal{U}^*, \mathcal{U})$ holds[8] if the assertions of Lemma B.1 are true for this Ω and this K, with discrete sets $\mathscr{X} \subset (\mathcal{U}^* \cap \overline{\Omega}) \times [\kappa_\Omega \rho, \rho]$ and with (1)-(3) replaced by

(1) We have the inclusion $\mathcal{U}^* \cap \overline{\Omega} \subset \cup_{(\mathbf{x},r)\in\mathscr{X}} \overline{\mathcal{B}}(\mathbf{x}, r)$
(2) For any $(\mathbf{x}, r) \in \mathscr{X}$, the ball $\mathcal{B}(\mathbf{x}, Kr)$ is included in $\mathcal{U}$ and is a map-neighborhood of $\mathbf{x}$ for Ω
(3) Each point $\mathbf{x}_0$ of $\mathcal{U} \cap \overline{\Omega}$ belongs to at most L different balls $\mathcal{B}(\mathbf{x}, Kr)$.

Like above the specification is

$$\text{Property } \mathscr{P}(\Omega, K; \mathcal{U}^*, \mathcal{U}) \text{ holds with parameters } (L, \rho_{\max}, \kappa).$$

In the process of proof, we will construct coverings which are not exactly balls, but domains uniformly comparable to balls. Let us introduce the local version of this new assertion. For $0 < a \le a'$ we say that

$$\text{Property } \mathscr{P}[a, a'](\Omega, K; \mathcal{U}^*, \mathcal{U}) \text{ holds with parameters } (L, \rho_{\max}, \kappa)$$

if for all $\rho \in (0, \rho_{\max}]$, there exists a finite set $\mathscr{X} \subset (\mathcal{U}^* \cap \overline{\Omega}) \times [\kappa_\Omega \rho, \rho]$ and open sets $\mathcal{D}(\mathbf{x}, r)$ satisfying the following four properties

(1) We have the inclusion $\mathcal{U}^* \cap \overline{\Omega} \subset \cup_{(\mathbf{x},r)\in\mathscr{X}} \overline{\mathcal{D}}(\mathbf{x}, r)$
(2) For any $(\mathbf{x}, r) \in \mathscr{X}$, the set[9] $\mathcal{D}(\mathbf{x}, Kr)$ is included in $\mathcal{U}$ and is a map-neighborhood of $\mathbf{x}$ for Ω
(3) Each point $\mathbf{x}_0$ of $\mathcal{U} \cap \overline{\Omega}$ belongs to at most L different sets $\mathcal{D}(\mathbf{x}, Kr)$
(4) For any $(\mathbf{x}, r) \in \mathscr{X}$, we have the inclusions $\mathcal{B}(\mathbf{x}, ar) \subset \mathcal{D}(\mathbf{x}, r) \subset \mathcal{B}(\mathbf{x}, a'r)$.

Note that $\mathscr{P}[1, 1](\Omega, K; \mathcal{U}^*, \mathcal{U}) = \mathscr{P}(\Omega, K; \mathcal{U}^*, \mathcal{U})$.

Lemma B.3. *If Property $\mathscr{P}[a, a'](\Omega, K; \mathcal{U}^*, \mathcal{U})$ holds with parameters $(L, \rho_{\max}, \kappa)$, then*

$$\textit{Property } \mathscr{P}(\Omega, \tfrac{a}{a'}K; \mathcal{U}^*, \mathcal{U}) \textit{ holds with parameters } (L, a'\rho_{\max}, \kappa).$$

Proof. Starting from the covering of $\mathcal{U}^* \cap \overline{\Omega}$ by the sets $\overline{\mathcal{D}}(\mathbf{x}, r)$ and using condition (4), we can consider the covering of $\mathcal{U}^* \cap \overline{\Omega}$ by the balls $\mathcal{B}(\mathbf{x}, a'r)$. Then $r' := a'r \in [\kappa a'\rho, a'\rho] = [\kappa\rho', \rho']$ with $\rho' < a'\rho_{\max}$.

Concerning conditions (2) and (3), it suffices to note the inclusions

$$\mathcal{B}(\mathbf{x}, \frac{a}{a'}Kr') \subset \mathcal{D}(\mathbf{x}, \frac{1}{a'}r'K) = \mathcal{D}(\mathbf{x}, rK)\,.$$

The lemma is proved. □

Proof. of Lemma B.1. The principle of the proof is a recursion on the dimension n.

Step 1. Explicit construction when $n = 1$.
The domain Ω and the localizing open sets $\mathcal{U}^*$ and $\mathcal{U}$ are then open intervals. Let us assume for

[8]This is the localized version of property $\mathscr{P}(\Omega, K)$.
[9]Here $\mathcal{D}(\mathbf{x}, Kr)$ is the set of $\mathbf{y}$ such that $\mathbf{x} + (\mathbf{y} - \mathbf{x})/K \in \mathcal{D}(\mathbf{x}, r)$.

example that $\mathcal{U}^* = (-\ell, \ell), \mathcal{U} = (-\ell-\delta, \ell+\delta)$ and $\Omega = (0, \ell+\delta')$ with $\ell, \delta > 0$ and $\delta' > \delta$. Let $K \geq 1$. We can take

$$\rho_{\max} = \min\left\{\frac{\ell}{K}, \delta\right\}$$

and for any $\rho \leq \rho_{\max}$ the following set of couples $(\mathbf{x}_j, r_j)$, $j = 0, 1, \ldots, J$

$$\mathbf{x}_0 = 0,\ r_0 = \rho \quad \text{and} \quad \mathbf{x}_j = \rho + \frac{2j-1}{K}\rho,\ r_j = \frac{\rho}{K} \quad \text{for} \quad j = 1, \ldots, J$$

with J such that $\mathbf{x}_J < \ell$ and $\rho + \frac{2J+1}{K}\rho \geq \ell$. If $\mathbf{x}_J < \ell - \frac{\rho}{K}$, we add the point $\mathbf{x}_{J+1} = \rho + \frac{2J}{K}\rho$. The covering condition (1) is obvious.

Concerning condition (2), we note that the bound $\rho_{\max} \leq \frac{\ell}{K}$ implies that $[0, Kr_0) = [0, K\rho)$ is a map-neighborhood for the boundary of Ω, and the bound $\rho_{\max} \leq \delta$ implies that when $j \geq 1$, the "balls" $(\mathbf{x}_j - Kr_j, \mathbf{x}_j + Kr_j) = (\mathbf{x}_j - \rho, \mathbf{x}_j + \rho)$ are map-neighborhoods for the interior of Ω.

Concerning condition (3), we can check that $L = K + 2$ is suitable.

Step 2. Localization.
Let $\Omega \in \mathfrak{D}(\mathbb{R}^n)$ or $\Omega \in \mathfrak{D}(\mathbb{S}^n)$. For any $\mathbf{x} \in \overline{\Omega}$, there exists a ball $\mathcal{B}(\mathbf{x}, r_x)$ with positive radius $r_{\mathbf{x}}$ that is a map-neighborhood for Ω. We extract a finite covering of $\overline{\Omega}$ by open sets $\mathcal{B}(\mathbf{x}^{(\ell)}, \frac{1}{2}r^{(\ell)})$. We set

$$\mathcal{U}^*_\ell = \mathcal{B}(\mathbf{x}^{(\ell)}, \frac{1}{2}r^{(\ell)}) \quad \text{and} \quad \mathcal{U}_\ell = \mathcal{B}(\mathbf{x}^{(\ell)}, r^{(\ell)}).$$

The map $\mathrm{U}^\ell := \mathrm{U}^{\mathbf{x}^{(\ell)}}$ transforms $\mathcal{U}^*_\ell$ and $\mathcal{U}_\ell$ into neighborhoods $\mathcal{V}^*_\ell$ and $\mathcal{V}_\ell$ of 0 in the tangent cone $\Pi_\ell := \Pi_{\mathbf{x}^{(\ell)}}$. Thus we are reduced to prove the local property $\mathscr{P}(\Pi_\ell, K; \mathcal{V}^*_\ell, \mathcal{V}_\ell)$ for any ℓ. Indeed

- The local diffeomorphism U^ℓ allows to deduce Property $\mathscr{P}(\Omega, K; \mathcal{U}^*_\ell, \mathcal{U}_\ell)$ from Property $\mathscr{P}(\Pi_\ell, K'; \mathcal{V}^*_\ell, \mathcal{V}_\ell)$ for a ratio K'/K that only depends on U^ℓ (this relies on Lemma B.3).
- Properties $\mathscr{P}(\Omega, K; \mathcal{U}^*_\ell, \mathcal{U}_\ell)$ imply Property $\mathscr{P}(\Omega, K; \cup_\ell \mathcal{U}^*_\ell, \cup_\ell \mathcal{U}_\ell) = \mathscr{P}(\Omega, K)$ (it suffices to merge the (finite) union of the sets $\mathscr{X}$ corresponding to each $\mathcal{U}_\ell$).

Step 3. Core recursive argument: If Ω_0 is the section of the cone Π, Property $\mathscr{P}(\Omega_0, K)$ implies Property $\mathscr{P}(\Pi, K'; \mathcal{B}(0,1), \mathcal{B}(0,2))$ for a suitable ratio K'/K. We are going to prove this separately in several lemmas (B.4 to B.6). Then the proof Lemma B.1 will be complete. □

Lemma B.4. *Let Γ be a cone in $\mathfrak{P}^{n-1}$. For $\ell = 1, 2$, let $\mathcal{B}_\ell$ and $\mathcal{I}_\ell$ be the ball $\mathcal{B}(0, \ell)$ of $\mathbb{R}^{n-1}$ and the interval $(-\ell, \ell)$, respectively. We assume that Property $\mathscr{P}(\Gamma, K; \mathcal{B}_1, \mathcal{B}_2)$ holds (with parameters $(L, \rho_{\max}, \kappa)$). Then Property $\mathscr{P}[1, \sqrt{2}](\Gamma \times \mathbb{R}, K; \mathcal{B}_1 \times \mathcal{I}_1, \mathcal{B}_2 \times \mathcal{I}_2)$ holds.*

Proof. Let us denote by $\mathbf{y}$ and z coordinates in Γ and $\mathbb{R}$, respectively. For $\rho \leq \rho_{\max}$, let $\mathscr{X}_\Gamma$ be an associate set of couples $(\mathbf{y}, r_{\mathbf{y}})$. For each $\mathbf{y}$ we consider the unique set of equidistant points $\mathscr{Z}_{\mathbf{y}} = \{z_j \in [-1, 1],\ j = 1, \ldots, J_{\mathbf{y}}\}$ such that

$$z_j - z_{j-1} = 2r_{\mathbf{y}} \quad \text{and} \quad z_1 + 1 = 1 - z_{J_{\mathbf{y}}} < r_{\mathbf{y}}\,.$$

Then we define

$$\text{(B.1)} \qquad \mathscr{X}^{(\rho)} = \big\{(\mathbf{x}, r_{\mathbf{x}}), \quad \text{for } \mathbf{x} = (\mathbf{y}, z) \text{ with } (\mathbf{y}, r_{\mathbf{y}}) \in \mathscr{X}_\Gamma,\ z \in \mathscr{Z}_{\mathbf{y}} \text{ and } r_{\mathbf{x}} = r_{\mathbf{y}}\big\}.$$

The associate open set $\mathcal{D}(\mathbf{x}, r_{\mathbf{x}})$ is the product

$$\mathcal{D}(\mathbf{x}, r_{\mathbf{x}}) = \mathcal{B}(\mathbf{y}, r_{\mathbf{y}}) \times (z - r_{\mathbf{y}}, z + r_{\mathbf{y}})\,.$$

We have the inclusions $\mathcal{B}(\mathbf{x}, r_\mathbf{x}) \subset \mathcal{D}(\mathbf{x}, r_\mathbf{x}) \subset \mathcal{B}(\mathbf{x}, \sqrt{2}\, r_\mathbf{x})$ and it is easy to check that Property $\mathscr{P}[1,\sqrt{2}](\Gamma\times\mathbb{R}, K; \mathcal{B}(0,1)\times\mathcal{I}_1, \mathcal{B}(0,2)\times\mathcal{I}_2)$ holds with parameters $(L', \rho_{\max}, \kappa)$ with $L' = LK$. ☐

Lemma B.5. *Let Ω be a section in $\mathfrak{D}(\mathbb{S}^{n-1})$, let Π be the corresponding cone, and let $\mathcal{I}_\ell$ be the interval $(2^{-\ell}, 2^\ell)$ for $\ell = 1,2$. We define the annuli*

$$\mathcal{A}_\ell = \Big\{\mathbf{x}\in\Pi, \quad |\mathbf{x}|\in\mathcal{I}_\ell \ \textit{and}\ \frac{\mathbf{x}}{|\mathbf{x}|}\in\Omega\Big\}.$$

We assume that Property $\mathscr{P}(\Omega, K)$ holds (with parameters $(L, \rho_{\max}, \kappa)$). Then, for suitable constants a and a' (independent of Ω and K), Property $\mathscr{P}[a,a'](\Pi, K; \mathcal{A}_1, \mathcal{A}_2)$ holds.

Proof. Let us consider the diffeomorphism

$$\begin{aligned} \mathrm{T}:\ \Omega\times(-2,2) &\longrightarrow \mathcal{A}_2\\ \mathbf{x} = (\mathbf{y}, z) &\longmapsto \breve{\mathbf{x}} = 2^z\mathbf{y} \end{aligned} \tag{B.2}$$

in view of proving Property $\mathscr{P}[a,a'](\Pi, K; \mathcal{A}_1, \mathcal{A}_2)$, for a given $\rho\le\rho_{\max}$, we define a suitable set $\breve{\mathscr{Z}}^{(\rho)}$ using the set $\mathscr{Z}^{(\rho)}$ introduced in (B.1)

$$\breve{\mathscr{Z}}^{(\rho)} = \big\{(\breve{\mathbf{x}}, r_\mathbf{x}), \quad \text{for}\ \breve{\mathbf{x}} = \mathrm{T}\mathbf{x}\ \text{ with }\ (\mathbf{x}, r_\mathbf{x})\in\mathscr{Z}^{(\rho)}\big\}, \tag{B.3}$$

and the associated open sets

$$\breve{\mathcal{D}}(\breve{\mathbf{x}}, r_\mathbf{x}) = \mathrm{T}\big(\mathcal{D}(\mathbf{x}, r_\mathbf{x})\big).$$

We can check that

$$\mathcal{B}(\breve{\mathbf{x}}, a r_\mathbf{x}) \subset \breve{\mathcal{D}}(\breve{\mathbf{x}}, r_\mathbf{x}) \subset \mathcal{B}(\breve{\mathbf{x}}, a' r_\mathbf{x})$$

with $a = \frac18\log 2$ and $a' = 8\sqrt{2}\log 2$ and that Property $\mathscr{P}[a,a'](\Pi, K; \mathcal{A}_1, \mathcal{A}_2)$ holds with parameters $(L', \rho_{\max}, \kappa)$ for $L' = NLK$ with an integer N independent of L and K. ☐

Lemma B.6. *Let Ω be a section in $\mathfrak{D}(\mathbb{S}^{n-1})$, let Π be the corresponding cone, and let $\mathcal{B}_\ell$ be the balls $\mathcal{B}(0,\ell)$ of $\mathbb{R}^n$ for $\ell = 1,2$. We assume that Property $\mathscr{P}(\Omega, K)$ holds with parameters $(L, \rho_{\max}, \kappa)$ for a $\rho_{\max}\le 1$. Then Property $\mathscr{P}[a,a'](\Pi, K; \mathcal{B}_1, \mathcal{B}_2)$ holds for suitable constants a and a' (independent of Ω and K) and with parameters $(L', 1, \kappa\rho_{\max})$.*

Proof. Let $\rho\le 1$ and let M be the natural number such that

$$2^{-M-1} < \rho \le 2^{-M}.$$

On the model of (B.2)-(B.3), we set

$$\breve{\mathscr{Z}}^m = \big\{(2^{-m}\mathrm{T}\mathbf{x}, 2^{-m}r_\mathbf{x}),\ \text{ with }\ (\mathbf{x}, r_\mathbf{x})\in\mathscr{Z}^{(2^m\rho_{\max}\rho)}\big\}, \quad m = 0,\dots,M,$$

and the associated open sets are

$$2^{-m}\mathrm{T}\big(\mathcal{D}(\mathbf{x}, r_\mathbf{x})\big)\ \text{ with }\ (\mathbf{x}, r_\mathbf{x})\in\mathscr{Z}^{(2^m\rho_{\max}\rho)}. \tag{B.4}$$

The set $\breve{\mathscr{Z}}$ associated with the cone Π in the ball $\mathcal{B}_1$ is

$$\{(0,\rho)\}\cup\bigcup_{m=0}^{M}\breve{\mathscr{Z}}^m$$

and the associated open sets are the reunion of the sets (B.4) for $m = 0, \dots, M$ and of the ball $\mathcal{B}(0, \rho)$. As the radii $r_{\mathbf{x}}$ belong to $[\kappa 2^m \rho_{\max}\rho, 2^m\rho_{\max}\rho]$, we have $2^{-m} r_{\mathbf{x}} \in [\kappa\rho_{\max}\rho, \rho_{\max}\rho]$. Since ρ itself belongs to the full collection of radii r, we finally find $r \in [\kappa\rho_{\max}\rho, \rho]$. The finite covering holds with $L' = 3NLK + 1$ for the same integer N appearing at the end of the proof of Lemma B.5. □

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