Small defects in mechanics

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Abstract. In this paper, we present a method to compute rapidly the solution of the Navier equation in domains with small inclusions close to each other. The main feature of our method is the use of a coarse description of the geometry. The computation relies on asymptotic expansion and computation of *profiles*, which are solution of a problem posed in unbounded domain. We propose and compare several artificial boundary conditions to compute these profiles efficiently.

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ASYMPTOTIC EXPANSION FOR THE NAVIER EQUATION

We would like to understand the influence of interior small defects on the behavior till rupture of some mechanical structure. Our approach, exposed with more details in [1, 2, 3], is based on two phases: the first step consists in evaluating how the size and shape of the defects influence the perturbation thanks to an asymptotic analysis of the Navier equation. The second step describes the response till rupture by using a strong discontinuity approach. In this paper, we only present results concerning the first phase for the evaluation of stress concentration due to the presence of geometrical defects. We refer to [3, 4] for a more complete presentation of the asymptotic expansion.

We evaluate the influence of geometrical perturbations by a multiscale asymptotic analysis of the equations of linear elasticity. The inclusions are assumed to be located inside the domain. Let Ω_0 be a domain of $\mathbb{R}^2$ such that the regular point 0 belongs to Ω_0. We consider a domain Ω_ε pierced with two perturbations of size ε near 0:

$$\Omega_\varepsilon = \Omega_0 \setminus \overline{\omega_\varepsilon^1 \cup \omega_\varepsilon^2}, \qquad \text{with} \qquad \omega_\varepsilon^j = x_\varepsilon^j + \varepsilon\omega^j, \qquad x_\varepsilon^1 = \varepsilon^\alpha \mathbf{d} \text{ and } x_\varepsilon^2 = -\varepsilon^\alpha \mathbf{d}, \quad \alpha \in (0,1).$$

We assume that ω^j contains 0 and $\mathbf{d}$ is a unit vector. We denote by $\mathbf{H}_\infty^j$ the unbounded domains obtained by a blow-up around each perturbation:

$$\mathbf{H}_\infty^j = \mathbb{R}^2 \setminus \overline{\omega^j}.$$

The problem we focus on is written on the perturbed domain as:

$$\left\{ \begin{array}{rcll} -\mu\Delta\mathbf{u}_\varepsilon - (\lambda+\mu)\nabla \operatorname{div}\mathbf{u}_\varepsilon &=& \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \mathbf{u}_\varepsilon &=& \mathbf{u}^d & \text{on } \Gamma_d, \\ \sigma(\mathbf{u}_\varepsilon)\cdot\mathbf{n} &=& \mathbf{g} & \text{on } \Gamma_n, \end{array} \right. \tag{1}$$

where Γ_d and Γ_n denote the Dirichlet and Neumann boundary of the domain respectively, Γ_n includes the boundary of the perturbations and $\mathbf{g}$ is supposed to be zero in a neighborhood of the perturbations. In the former equation, $\mathbf{u}_\varepsilon$ denotes the displacement and σ stands for the stress tensor:

$$\sigma_{ij}(\mathbf{u}) = \lambda(\partial_1\mathbf{u}_1 + \partial_2\mathbf{u}_2)\delta_{ij} + \mu(\partial_i\mathbf{u}_j + \partial_j\mathbf{u}_i).$$

The solution of (1) is given at first order by

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) - \varepsilon\sum_{j=1}^{2}\left[\alpha_1\mathbf{v}_1^j\left(\frac{x-x_\varepsilon^j}{\varepsilon}\right) + \alpha_2\mathbf{v}_2^j\left(\frac{x-x_\varepsilon^j}{\varepsilon}\right)\right] + \mathscr{O}\left(\varepsilon^{\min(1+\alpha,3-2\alpha)}\right), \tag{2}$$

with $\mathbf{u}_0$ the solution on the unperturbed domain, $\alpha_1 = \sigma_{11}(\mathbf{u}_0)(0)$ and $\alpha_2 = \sigma_{12}(\mathbf{u}_0)(0)$. The profiles $\mathbf{v}_1^j$ and $\mathbf{v}_2^j$ are obtained as solution of an homogeneous Navier equation stated on the unbounded domain $\mathbf{H}_\infty^j$ with Neumann boundary conditions on the boundary of the normalized perturbation:

$$\left\{\begin{array}{rcll} -\mu\Delta\mathbf{v}_\ell^j - (\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{v}_\ell^j &=& \mathbf{0} & \text{in } \mathbf{H}_\infty^j, \\ \sigma(\mathbf{v}_\ell^j)\cdot\mathbf{n} &=& \mathrm{G}_\ell^j & \text{on } \partial\omega^j, \\ \mathbf{v}_\ell^j &\to& 0 & \text{at infinity,} \end{array}\right. \tag{3}$$

with $\mathbf{G}_1^j = (\mathbf{n}_1^j, 0)$, $\mathbf{G}_2^j = (0, \mathbf{n}_1^j)$ and $\mathbf{n}_1^j$ the first component of the outer normal to $\partial\mathbf{H}_\infty^j = \partial\omega^j$.

NUMERICAL SIMULATION

Superposition

The computation of the solution $\mathbf{u}_\varepsilon$ of the Navier equation (1) by a finite element method is not straightforward since a very fine mesh is required around the inclusions to catch them accurately.
For small values of ε, we can use the asymptotic expansion (2) and we approximate $\mathbf{u}_\varepsilon$ by

$$\tilde{\mathbf{u}}_\varepsilon(x) = \mathbf{u}_0(x) - \varepsilon \sum_{j=1}^{2} \left[\alpha_1^j \mathbf{v}_1^j \left(\frac{x - x_\varepsilon^j}{\varepsilon} \right) + \alpha_2^j \mathbf{v}_2^j \left(\frac{x - x_\varepsilon^j}{\varepsilon} \right) \right]. \tag{4}$$

Then we have to compute

- the solution $\mathbf{u}_0$ of the unperturbed problem

$$\left\{\begin{array}{rcll} -\mu\Delta\mathbf{u}_0 - (\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{u}_0 &=& \mathbf{0} & \text{in } \Omega_0, \\ \mathbf{u}_0 &=& \mathbf{u}^d & \text{on } \Gamma_d, \\ \sigma(\mathbf{u}_0)\cdot\mathbf{n} &=& \mathbf{g} & \text{on } \Gamma_n. \end{array}\right. \tag{5}$$

  The function $\mathbf{u}_0$ can be easily computed by a finite element method with a coarse mesh for Ω_0 independent of ε.
- the profiles $\mathbf{v}_\ell^j$ which are solution of the problem (3) posed on an infinite domain. The computation of these profiles is the object of the next subsection.

Profiles approximation

To approximate the profile defined by (3), we compute a solution of a problem posed in a bounded domain

$$\mathbf{H}_R^j = \{x \in \mathbf{H}_\infty^j \mid |x| < R\},$$

with adapted condition on the artificial boundary $|x| = R$. The most natural artificial condition is the Dirichlet one. Thus we define

$$\left\{\begin{array}{rcll} -\mu\Delta\mathbf{v}_{\ell,D}^j - (\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{v}_{\ell,D}^j &=& \mathbf{0} & \text{in } \mathbf{H}_R^j, \\ \sigma(\mathbf{v}_{\ell,D}^j)\cdot\mathbf{n} &=& \mathrm{G}_\ell^j & \text{on } \partial\omega^j, \\ \mathbf{v}_{\ell,D}^j &=& 0 & \text{on } \partial\mathbf{H}_R^j \setminus \partial\omega^j. \end{array}\right. \tag{6}$$

To approximate more accurately the profile, we cancel the leading singular parts at infinity of the solution in polar coordinates. We then obtain Ventcel type problem:

$$\left\{\begin{array}{rcll} -\mu\Delta\mathbf{v}_{\ell,V}^j - (\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{v}_{\ell,V}^j &=& \mathbf{0} & \text{in } \mathbf{H}_R^j, \\ \sigma(\mathbf{v}_{\ell,V}^j)\cdot\mathbf{n} &=& \mathrm{G}_\ell^j & \text{on } \partial\omega^j, \\ \sigma(\mathbf{v}_{\ell,V}^j)\cdot\mathbf{n} + \frac{1}{2R}\begin{bmatrix} -\frac{\lambda\mu}{\lambda+2\mu} & 0 \\ 0 & \lambda+2\mu \end{bmatrix}\Delta_\tau\mathbf{v}_{\ell,V}^j + \frac{2\mu}{R}\mathbf{v}_{\ell,V}^j &=& 0 & \text{on } \partial\mathbf{H}_R^j \setminus \partial\omega^j. \end{array}\right. \tag{7}$$

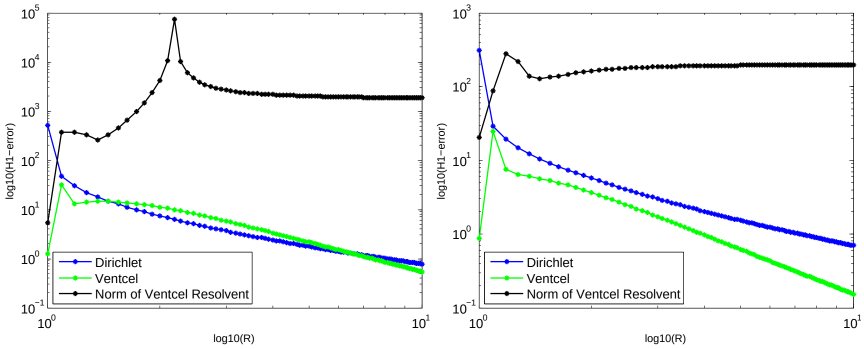

FIGURE 1. Convergence for Dirichlet or Ventcel conditions, norm of the resolvent for Ventcel operator; $\nu < 0$ on left, and $\nu > 0$ on right.

Lamé's coefficients are linked to the physical parameters (Young's modulus and Poisson's coefficient) throught the following relations:

$$\left\{ \begin{array}{lcl} \lambda & = & \dfrac{\nu E}{(1+\nu)(1-2\nu)} \\ \mu & = & \dfrac{E}{2(1+\nu)} \end{array} \right. \qquad \text{with} \qquad E>0,\ -1<\nu<0.5.$$

So that, this boundary condition of Ventcel's type rewrites in polar coordinates on $\partial \mathbf{H}_R^j \setminus \partial \omega^j$:

$$\sigma(\mathbf{v}_{\ell,V}^j)\cdot\mathbf{n} + \frac{E}{2R}\begin{bmatrix} \frac{-\nu}{2(1-\nu^2)} & 0 \\ 0 & \frac{1-\nu}{(1+\nu)(1-2\nu)} \end{bmatrix} \Delta_\tau \mathbf{v}_{\ell,V}^j + \frac{E}{R(1+\nu)}\mathbf{v}_{\ell,V}^j = 0.$$

We notice that

$$\frac{1-\nu}{(1+\nu)(1-2\nu)} > 0, \qquad \text{and} \qquad \left\{ \begin{array}{lll} \dfrac{-\nu}{2(1-\nu^2)} < 0 & \text{if} & \nu \in (0,0.5), \\ \dfrac{-\nu}{2(1-\nu^2)} > 0 & \text{if} & \nu \in (-1,0). \end{array} \right.$$

Consequently, we can not apply the Lax-Milgram theory to prove existence of solution for the problem (7). In [5], we deal with Ventcel condition for the Laplace equation and prove that the problem is generically well-posed without a sign condition and solution exists except for a countable set of real numbers R. We hope to prove similar results for problem (7). Figure 1 gives the convergence according to R for Dirichlet and Ventcel conditions and for a positive and negative real number ν. We observe that the order of convergence is respectively 1 and 2 for Dirichlet and Ventcel conditions. Furthermore, these figures represent also the norm of the resolvent of the Ventcel operator and we observe that the resolvent can become very large, corresponding certainly to the "forbidden" radius R. We use the finite element library MÉLINA, [6] for these computations.

Computations

We choose

$$\lambda = 0.5769230769 \qquad \text{and} \qquad \mu = 0.3846153396.$$

Figure 2 gives a reference computation on Ω_ε with a mesh refined around the inclusions. We compare this reference computation with result obtained by the superposition method based on the asymptotic expansion (4). We consider two approximations of $\mathbf{u}_\varepsilon$ at the second order : $\tilde{\mathbf{u}}_\varepsilon^D(R)$ and $\tilde{\mathbf{u}}_\varepsilon^V(R)$ are defined by relation (4) where we replace the exact profiles $\mathbf{v}_\ell^j$ by their approximation $\mathbf{v}_{\ell,D}^j$ and $\mathbf{v}_{\ell,V}^j$ defined by (6) and (7) with Dirichlet and Ventcel condition respectively on the boundary $|x| = R$. We observe that the errors $\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon^D(R)$ and $\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon^V(R)$ are equivalent between an approximation of the profiles with Dirichlet condition

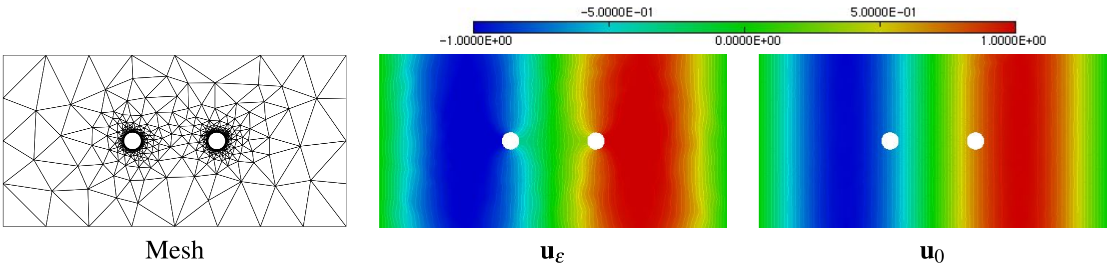

FIGURE 2. Reference computation.

on the a domain of radius $R = 10$ and Ventcel condition on a domain of radius $R = 5$. This corroborates the better accuracy of the Ventcel condition than the Dirichlet ones.

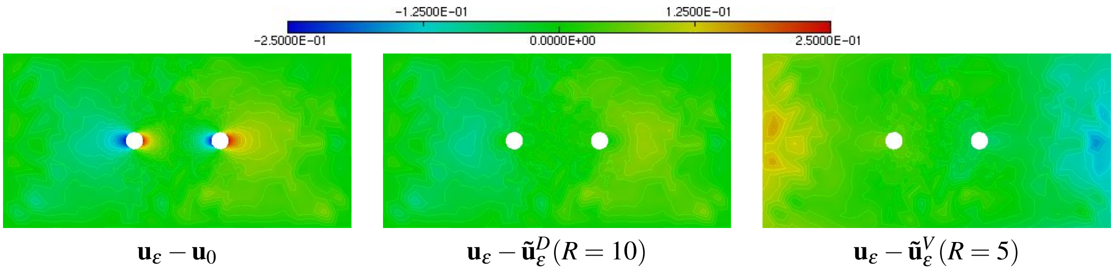

FIGURE 3. Error for $\varepsilon = 0.05$, $\alpha = 0.7$.

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