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Mini-Workshop: Boundary Value Problems and Spectral Geometry

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Abstract. Boundary value problems and spectral geometry is an attractive and rapidly developing area in modern mathematical analysis. The interaction of PDE methods with concepts from operator theory and differential geometry is particularly challenging and leads directly to new insights and applications in various branches of pure and applied mathematics, e.g., analysis on manifolds, global analysis and mathematical physics. Some recent contributions in the field of boundary value problems and spectral geometry concern, e.g., construction of isospectral manifolds with boundary, eigenvalue and resonance distribution for large energies, multidimensional inverse spectral problems, singular perturbations, new regularity techniques, Dirichlet-to-Neumann maps and Titchmarsh-Weyl functions.

Mathematics Subject Classification (2000): Primary 35P05; Secondary 47A10, 58J32.

Introduction by the Organisers

The basic aim of the mini-workshop is to bring together a special selection of world leading experts from different areas, as, e.g., spectral theory, differential geometry, analysis of PDEs, mathematical imaging, to join the efforts in studying several problems arising in the modern mathematical physics.

In various situations one deals with the spectral or scattering analysis of complex objects which are built together from elementary pieces interacting with each other through the boundary. The elementary pieces may have rather simple properties and admit an explicit description, and the properties of the total system come mostly from the global geometry or the interaction conditions. The situation can be modeled in many ways, for example, there is a considerable progress in understanding the properties of systems composed from one-dimensional pieces (differential operators on metric graphs), both the direct and the inverse spectral theory are in active development. A more difficult problem is to study differential operators on coupled domains or manifolds. It is known that domains of some special geometries can be approximated by metric graphs and hence one is interested in the question if the same research philosophy can be transferred from metric graphs to more complicated coupled objects. Even a partial progress in this direction needs combining various techniques like spectral analysis of operator pencils, trace and embedding theorems, pseudodifferential operators, differential geometry and many others. During the last years there were several attempts to fill the gap, in particular, by developing operator-theoretical tools suitable for studying rather general boundary value problems for PDEs.

Nowadays many of the new trends in the field of boundary value problems and spectral geometry develop rapidly and independently into different directions of modern analysis. A special feature of the mini-workshop is to combine the expertise of colleagues from these different areas and to focus on the following topics:

- Boundary value problems with low regularity (rough domains, singular spaces, mixed boundary value problems);
- Decomposition techniques for composed domains and understanding the relation between local and global properties;
- Operator-theoretical tools like abstract boundary triples in spectral geometry,
- Spectral analysis via Dirichlet-to-Neumann maps and Titchmarsh-Weyl functions.

Mini-Workshop: Boundary Value Problems and Spectral Geometry

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Abstracts

Spectral Properties of the Dirchlet-to-Neumann Operator

Wolfgang Arendt
(joint work with Rafe Mazzeo)

We compare the spectra of the Dirichlet-to-Neumann operator, the Dirichlet Laplacian, and the Robin Laplacian. This will allow us to reprove an interesting spectral inequality due to Friedlander [5] (see Eq. (1) below). This interesting inequality between the $(k+1)^{th}$ Neumann and the k^{th} Dirichlet eigenvalue holds in the Euclidean case.

For manifolds the inequality corresponds to geometric properties which are not yet well understood and somehow surprising. Some interesting results are proved by Mazzeo [6] who shows, for example, that Friedlander's inequalities hold on any symmetric space of noncompact type. Here we show that in each compact Riemannian manifold there exists a (large) Lipschitz domain on which the inequality fails, and there exists a (small) domain on which it is valid.

The construction of the Dirichlet-to-Neumann operator itself is interesting too. The operator depends on a real parameter λ. If λ is not a Dirichlet eigenvalue, then form methods can be used for the construction. If λ is a Dirichlet eigenvalue then one is in a less conventional situtation and the Dirichlet-to-Neumann operator is a multi-valued self-adjoint operator. Our point in the talk is a description of the Robin eigenvalue branches (depending on a parameter) whose inverses describe the Dirichlet-to-Neumann eigenvalues. This makes the proof of Friedlander's inequality

$$\lambda_{k+1}^N < \lambda_k^D \tag{1}$$

very transparent and intuitive (and more general than Friedlander's original result [5] where at least C^1-boundary is needed). Here λ_k^D denotes the k^{th} Dirichlet and λ_k^N denotes the k^{th} Neumann eigenvalue. Another short and elegant (but maybe less intuitive proof) is given by Filonov [4], which holds even if Ω has merely continuous boundary.

We also investigate when the semigroup generated by the Dirichlet-to-Neumann operator is positive and irreducible. This surprizing fact is true whenever Ω is connected even if the boundary is not connected. This reflects the fact that the Dirichlet-to-Neumann operator is non-local. The talk is based on joint work with Rafe Mazzeo [1, 2]. It is remarkable that the Dirichlet-to-Neumann operator can also be defined on arbitrary domains without any regularity assumption on the boundary [3].

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Minimal and spectral partitions

Virginie Bonnaillie-Noël

Let Ω be a bounded and regular connected domain in $\mathbb{R}^2$. Let $k \geq 1$ be an integer and $\mathcal{D} = (D_i)_{i=1}^k$ be a k-partition with D_i open, regular, connected, disjoint and satisfying

$$\mathrm{Int}(\overline{\cup_{i=1}^k D_i}) = \Omega.$$

Let $\mathcal{D}_k$ be the set of such k-partitions. The eigenvalues of the Dirichlet-Laplacian $-\Delta$ are denoted by $\lambda_1(D_i) < \lambda_2(D_i) \leq \ldots \leq \lambda_n(D_i)$. We are interested in the properties of the 'minimal' k-partitions of Ω. These partitions are minimal in the sense that they minimize the maximum over $i = 1, \ldots, k$ of the lowest eigenvalues of the Dirichlet-Laplacian in D_i. Thus we define

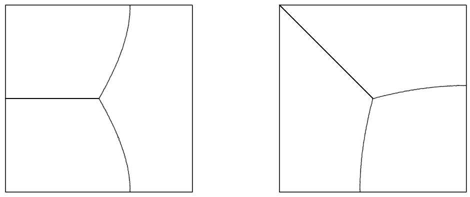
$$\mathfrak{L}_k(\Omega) = \inf\{\max\{\lambda_1(D_1), \ldots, \lambda_1(D_k)\}, \mathcal{D} = (D_i)_{i=1}^k \in \mathcal{D}_k\}.$$

We say that a k-partition $\mathcal{D} = (D_i)_{i=1}^k$ is minimal if it satisfies

$$\Lambda(\mathcal{D}) = \mathfrak{L}_k(\Omega) \qquad \text{with} \qquad \Lambda(\mathcal{D}) := \max\{\lambda_1(D_1), \ldots, \lambda_1(D_k)\}.$$

It is well known that for any k, there exists a regular minimal k-partition, see [5]. We are interested in determining some of them. In particular, we would like to determine in which cases this minimal partition is actually the family of the nodal domains of a given eigenfunction of the Dirichlet-Laplacian in Ω. In the case of 2-partitions, the answer is very simple because a variational characterization of the second eigenvalue of the Dirichlet-Laplacian in Ω shows that a minimal 2-partition is always a nodal partition corresponding to the second eigenvalue and we have $\mathfrak{L}_2(\Omega) = \lambda_2(\Omega)$. So the interesting questions start with $k = 3$. Although general properties of these minimal partitions have been proved by B. Helffer, T. Hoffmann-Ostenhof and S. Terracini, there are very few theoretical results (except for thin domains) for obtaining an explicit determination of minimal partitions. First, we present some results about symmetric minimal 3-partition (see [3]) and then use the Dirichlet-Laplacian on the double covering to propose some new candidates (see [2]).

If the minimal 3-partition for the square has a symmetry axis, wa reduce its determinations to the study of a family of mixed Dirichlet-Neumann problems with



$\Lambda(\mathcal{D}_0) \simeq 66.581124$ $\Lambda(\mathcal{D}_1) \simeq 66.581124$

FIGURE 1. Symmetric candidates for the square.

mixed points on the symmetry axis and find the "best" candidate. By topological arguments, there are three types:

$$\begin{cases} -\Delta\varphi = \lambda\varphi \text{ in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 \text{ on } [X_0, B] \\ \varphi = 0 \text{ elsewhere} \end{cases} \quad \begin{cases} -\Delta\varphi = \lambda\varphi \text{ in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 \text{ on } [A, X_0] \cup [X_1, b] \\ \varphi = 0 \text{ elsewhere} \end{cases} \quad \begin{cases} -\Delta\varphi = \lambda\varphi \text{ in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 \text{ on } [X_0, X_1] \\ \varphi = 0 \text{ elsewhere,} \end{cases}$$

where Ω^+ denotes the half-square $[-1,1] \times [0,1]$, $A = (-1,0)$, $B = (1,0)$ and the points X_0, X_1 are free on the axis $y = 0$. With these three configurations, we obtain just one candidate given in Figure 1. Using the diagonal symmetry axis, we obtain, by the same way, a unique candidate with the same energy (cf. Figure 1). We observe that the critical point is at the center of the square. Using the Aharonov-Bohm operator (see [2]), we can prove that the mixed Dirichlet-Neumann problems on the half-domains Ω^+ and $\Omega^d = \{(x,y) \in \Omega, x \geq -y\}$ are isospectral when the mixed point is at the middle of the symmetry axis. Then we have a new tool to detect some non-symmetric partitions by computing the spectrum of the Aharonov-Bohm operator, or, equivalently the anti-symmetric spectrum of the Dirichlet-Laplacian on the double covering of $\Omega \setminus \{(0,0)\}$. By this way, we can construct a continuous family of 3-partitions with the same energy and generating by the partitions given in Figure 1. The aim is now to prove that these candidates are indeed minimal.

We can do the same analysis for any geometries like a disk, ellipses, rectangles, angular sectors. Numerical simulations are presented in [4].

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Index calculations for Dirac operators and spectral theory

Jochen Brüning

(joint work with Werner Ballmann, Gilles Carron)

Introduction. We consider Dirac operators on complete noncompact Riemannian manifolds; for the purpose of this talk, we will assume that we are given a complete manifold M with pinched negative curvature and finite volume which has only one end, and a Dirac operator defined on the smooth sections of a Dirac bundle over M, cf. [4]. This operator is essentially self-adjoint in the relevant L^2 -space, but may produce a nonzero index after restriction to the eigenspaces of an involution that anticommutes with D. We describe here some techniques originating in spectral theoretic considerations which are decisive in carrying out the index calculation. One of the main difficulties arises from the fact that Fredholmness may not be available for the Dirac operators in question. Using an operator model derived from the cylindrical case which was thorouhgly studied in [1], we explain how to overcome this obstacle.

The extended operator. Consider the cylindrical operator

$$D = \gamma(\frac{d}{dt} + A), \tag{1}$$

acting in $\mathcal{H} := L^2(\mathbb{R}_+, H)$ where H is a separable Hilbert space, A is a self-adjoint operator in H with compactly imbedded domain H_A, and γ is an antiinvolution anticommuting with A, such that D is symmetric with domain $C_c^1(\mathbb{R}_+, H)$. It is easy to see that the maximal operator $D_{\max}$ is an isomorphism from its domain onto $\mathcal{H}$ iff $\ker A = 0$. If $\ker A \neq 0$, then we meet the problem that the map

$$\mathbb{C} \oplus L^2(\mathbb{R}_+) \ni (x, \tau) \mapsto x + \int_0^t \tau(s) ds \tag{2}$$

does not land in $L^2(\mathbb{R}_+)$. Writing $\tau = D_{\max}\sigma, x = \sigma(0)$, the obvious norm on the image of $D_{\max}$ becomes

$$||\sigma||_W^2 := ||\sigma(0)||_H^2 + ||D_{\max}\sigma||_{L^2}^2, \tag{3}$$

which is weaker than the graph norm of $D_{\max}$. Thus, if we denote by W the closure of $\mathrm{im}D_{\max}$ under the W-norm then $D_{\max}$ extends to W by continuity; this extension we denote by D_{ext}. Now D_{ext} has closed image in $\mathcal{H}$, $\mathrm{im}D_{\mathrm{ext}} = \overline{\mathrm{im}D_{\max}}$, and its kernel equals the constant functions with values in $\ker A$, hence it is a Fredholm operator with kernel not contained in $\mathcal{H}$; the elements of the kernel are the extended solutions introduced in [1].

This model can be extended to the geometric situation described above by replacing A with A_t, an operator family with common domain H_A but with a t-dependent equivalent Hilbert metric; the corresponding family of Hilbert spaces will be denoted by $(H_t)_{t\geq 0}$. A serious technical difficulty arises from the fact that the family A_t will be only Lipschitz in general, which is dealt with in [3]; for the purpose of this exposition we will assume it to be C^1. More serious is the fact that the W-norm is not a norm any more in general; we call D *non-parabolic at infinity* if this is the case.

The spectral gap. The geometric assumptions imply that

$$||A'_t\sigma|| \leq C(||\sigma|| + ||A_t\sigma||), \tag{4}$$

hence non-parabolicity would follow from the existence of a sufficiently large spectral gap around 0, for all A_t. This can be weakened to the condition

$$\operatorname{spec} A_t \subset \{\mu \in \mathbb{R} : |\mu| \leq \lambda \ or \ |\mu| \geq \Lambda\}, t \geq 0, \ \lambda \ll \Lambda, \tag{5}$$

by introducing suitable weights modifying the Hilbert norm of $\mathcal{H}$. To prove (5), we need a geometric tool: our assumptions allow to introduce a flat connection on the Dirac bundle, restricted to the end of the manifold under consideration. Its constant sections then will span a subspace of $\mathcal{H}$ which induces, by restriction to H_t, an invariant subspace of each A_t, with eigenvalues uniformly bounded in t. On the orthogonal complement of these spaces, we use an argument involving the holonomy of the flat connection and [2, Theorem 5] to show that the smallest eigenvalue of A_t tends to infinity with t on the complementary space. This shows that D is non-parabolic at infinity and, hence, that the extended operator D_{ext} exists and is a Fredholm operator.

The index calculation. To compute the index of D^+_{ext}, after reduction by an anticommuting involution, we now translate the above arguments in t, by cutting the end at t_0 and introducing complementary boundary conditions at t_0, such that the index becomes the sum of the indices of the interior and the exterior problem. The interior index is computed by the formula of [1] while the exterior index is localized at t_0; in the resulting formula, we take the limit as $t_0 \to \infty$. Under some additional assumptions which imply that A_t splits orthogonally into the low energy and the high energy part, $A_t = A_t^{\text{le}} \oplus A_t^{\text{he}}$ we arrive at the following index formula, where we denote by ω_{D^+} the Atiyah-Singer local integrand and by $\eta(B)$ the eta-invariant of a discrete operator B (this invariant exists for all Dirac operators, cf. [1]):

$$\operatorname{ind} D^+_{\text{ext}} = \int_M \omega_{D^+} + \frac{1}{2}\big(\eta(A_0^{\text{le},+}) + \dim\ker A_0^{\text{le},+} + \lim_{t\to\infty} \eta(A_t^{\text{he},+})\big). \tag{6}$$

This general result applies in particular to locally symmetric spaces of $\mathbb{Q}$-rank 1. E.g. for cofinite quotients of complex hyperbolic space and the signature operator, we arrive at formulas which are more transparent and somewhat more accessible than the known computations; full details will appear soon.

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Geometric properties of point-interaction Hamiltonians ground state

Pavel Exner

In this talk I am going to present several results on relations between the principal eigenvalue of various "atractive" point-interaction Hamiltonians and the underlying geometry. The first question concerns polymer rings, that is, point interactions on a loop with an upper bound to distance to the neighbours; following the papers [1, 2] it is shown that the ground-state energy is minimized by a regular polygon; we also mention the continuous analogue to this problem solved in [3].

Then we will consider a point interaction in a region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and derive a condition under which the eigenvalue increases as the interaction site moves [4]. Finally, we will discuss behaviour of the principle eigenvalue w.r.t. increasing distances between the interaction sites both in $\mathbb{R}^d$, $d = 1, 2, 3$, and on quantum graphs [5].

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Curvature and spectrum for graphs

Matthias Keller

Classically discreteness of spectrum for Schrödinger operators can result from the potential or from the geometry. The case of a Schrödinger operator $H = \Delta + v$ on $\mathbb{R}^d$ with a non-negative potential v was considered by Friedrichs in 1934, [Fr]. He showed that if v tends uniformly to ∞ in every direction, then H has purely discrete spectrum. A complete characterization for the phenomena of discreteness of the spectrum due to a unbounded potential was given by Mazya/Shubin in 2005 [MS]. See also Lenz/Stollmann/Wingert [LSW] for a corresponding result in the context of Dirichlet forms. On the other hand, Donnelly/Li [DL] showed in 1979 that if the sectional curvatures of a complete, simply connected, negatively curved Riemannian manifold tend uniformly to $-\infty$, then the Laplace Beltrami operator has purely discrete spectrum.

Our first aim is to give a discrete analogue of the Donnelly/Li theorem in the framework of graphs. Secondly, we aim for a unified treatment of decreasing curvature and increasing potential.

Let $G = (V, E)$ be an infinite, simple, connected, locally finite graph. For a finitely supported function $u : V \to \mathbb{R}$, let the quadratic form Q be given by

$$Q(u) = \sum_{\{x,y\}\in E} \big(u(x) - u(y)\big)^2.$$

There are two natural choices of a measure on V. One is constantly 1 and the other one is the vertex degree function deg. So, firstly the positive selfadjoint operator Δ associated to the completion of Q in $\ell^2(V)$ acts as

$$\Delta u(x) = \sum_{x\sim y} \big(u(x) - u(y)\big)$$

on $D(\Delta) = \{u \in \ell^2(V) \mid \Delta u \in \ell^2(V)\}$. It is not hard to see that Δ is bounded iff the vertex degree function is bounded. Secondly, taking the completion of Q in $\ell^2(V, \deg)$ yields the bounded positive selfadjoint operator $\widetilde{\Delta}$ acting as

$$\widetilde{\Delta} u(x) = \frac{1}{\deg(x)} \sum_{x\sim y} \big(u(x) - u(y)\big)$$

which satisfies $0 \le \widetilde{\Delta} \le 2$.

The Cheeger constant α_U for $U \subseteq V$ is defined as

$$\alpha_U := \inf_{W\subseteq U \text{ finite}} \frac{|\partial_E W|}{\mathrm{vol}(W)},$$

where $\partial_E W := \{\{x, y\} \in E \mid x \in W, y \in V \setminus W\}$ and $\mathrm{vol}(W) = \sum_{x\in W} \deg(x)$. Clearly, $0 \le \alpha_U < 1$ and $\alpha_U \le \alpha_{U'}$ for $U' \subseteq U \subseteq V$. We define following [Fu]

$$\alpha_\infty := \lim_{K\subset V \text{ finite}} \alpha_{V\setminus K},$$

where the limit is taking along the net of finite subsets of V. In [Fu], Fujiwara proved the following remarkable theorem:

Theorem 1 (Fujiwara '96). *The essential spectrum of $\widetilde{\Delta}$ is $\{1\}$ iff $\alpha_\infty = 1$.*

The key ingredient of the proof are the Cheeger estimates $1 - \sqrt{1 - \alpha_U^2} \le \widetilde{\Delta}_U \le \min\{1 + \sqrt{1 - \alpha_U^2}, \alpha_U\}$, $U \subseteq V$ due to Mohar, Dodziuk/Karp and Dodziuk/Kendall. This estimate implies that $\alpha_\infty = 1$ is equivalent to compactness of the operator $P = \widetilde{\Delta} - I$.

Since $\widetilde{\Delta}$ is a bounded infinite dimensional operator it cannot have purely discrete spectrum. Thus, in order to get a better analogy to the theorem of Donnelly/Li, one has to consider the operator Δ in the case of unbounded vertex degree. In [K1] such a theorem is proven. To this end, let

$$d_\infty := \lim_{K\subset V \text{ finite}} \min_{x\in V\setminus K} \deg(x).$$

Theorem 2 (K. '10). *Let $\alpha_\infty > 0$. Then the essential spectrum of Δ is empty iff $d_\infty = \infty$.*

The proof uses an estimate on the bottom of the spectrum of Δ by the minimum of the vertex degree times the bottom of the spectrum of $\widetilde{\Delta}$. Since such an estimate holds on all complements of finite sets one gets an estimate for the bottoms of the essential spectra.

The theorem gives an analogue to the Donnelly/Li theorem as $\alpha_\infty > 0$ can be understood as a negative curvature assumption and $d_\infty = \infty$ as the curvature tending to $-\infty$. In the framework of planar tessellations the analogue is even clearer.

Assume G is a planar tessellation which is embedded into a topological surface $S \cong \mathbb{R}^2$. Let the set of faces F be given by the closures of the connected components of $S \setminus \bigcup E$. The face degree $\deg(f)$, $f \in F$, is defined as the number of vertices contained in f. The curvature $\kappa : V \to \mathbb{R}$ is given by

$$\kappa(x) := 1 - \frac{\deg(x)}{2} + \sum_{f \in F,\, x \in f} \frac{1}{\deg(f)},$$

which can be understood as an angle defect. Let

$$\kappa_\infty := \lim_{K \subset V \text{ finite}} \sup_{x \in V \setminus K} \kappa(x).$$

Theorem 3 (K. '10)**.** *Let G be a planar tessellation. Then, the essential spectrum of Δ is empty iff $\kappa_\infty = -\infty$. Moreover, $\kappa_\infty = -\infty$ implies that the essential spectrum of $\widetilde{\Delta}$ is equal to $\{1\}$.*

The proof given in [K1] uses that $\kappa_\infty = -\infty$ iff $d_\infty = \infty$ and a Cheeger estimate of the form $\alpha_\infty \geq 1 - 6/\inf_x \deg(x)$. More subtle estimates using curvature can be found in [K1, K2, KP].

Finally, to consider Schrödinger operators $H = \Delta + v$ with $v : V \to [0, \infty)$, let

$$\gamma_\infty := \lim_{K \subset V \text{ finite}} \inf_{W \subseteq V \setminus K \text{finite}} \frac{|\partial_E W| + v(W)}{\operatorname{vol}(W)},$$

where $v(W) = \sum_{x \in W} v(x)$,

$$g_\infty := \lim_{K \subset V \text{ finite}} \inf_{x \in V \setminus K} (\deg + v)(x).$$

and if G is a planar tessellation

$$k_\infty := \lim_{K \subset V \text{ finite}} \inf_{x \in V \setminus K} (-\kappa + v)(x).$$

Our second aim was to give a unified treatment of the relation of uniform growth of curvature and the potential and discreteness of H. From [KL] we can deduce

Theorem 4 (K./Lenz '10)**.** (1) *Let $\gamma_\infty > 0$. Then, the essential spectrum of H is empty iff $g_\infty = \infty$.*
(2) *Let G be a planar tessellation. Then, the essential spectrum of H is empty iff $k_\infty = \infty$.*

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Inverse Spectral Problems and Extrapolation of Stationary Stochastic Processes from a Finite Time Interval

Heinz Langer

Let $X = (X_t)_{t\in\mathbb{R}}$ be a second order weakly continuous and weakly stationary stochastic process on the probability space $(\Omega, \mathcal{A}, P)$ with $EX_t = 0, \quad t \in \mathbb{R}$, and covariance function

$$f(t-s) := EX_t\overline{X_s} = (X_t, X_s)_{L^2(\Omega,\mathcal{A},P)}, \quad s,t \in \mathbb{R}. \tag{1}$$

According to Bochner's theorem there exists a bounded measure σ on $\mathbb{R}$, such that

$$f(t) = \int_{\mathbb{R}} e^{i\lambda t}\, d\sigma(\lambda), \quad t \in \mathbb{R}. \tag{2}$$

We consider the following extrapolation problem. Suppose that $a > 0$ and that the process X has been observed for $-a \le t \le a$, find the best mean square prediction for $X_s,\ s \notin [-a, a]$. In analytical terms, find the orthogonal projection of

$$X_s \in \mathcal{H}(X) := \text{c.l.s.}\, \{X_t :\ t \in \mathbb{R}\} \subset L^2(\Omega, \mathcal{A}, P)$$

onto the subspace

$$\mathcal{H}(X; a) := \text{c.l.s.}\, \{X_t :\ t \in [-a, a]\} \subset \mathcal{H}(X).$$

Here we assume that

$$\mathcal{H}(X) \ne \mathcal{H}(X; a). \tag{3}$$

For a real valued process X this problem was solved by M.G. Krein using his spectral theory of a string (see [2]). The corresponding problem with observation during the infinite past $t \le 0$ was solved in the 1940-ies by A.N. Kolmogorov for discrete time and by M.G. Krein for continuous time (see, e.g. [4]).

The relation

$$(X_t, X_s)_{L^2(\Omega,\mathcal{A},P)} = EX_t\overline{X_s} = f(t-s) = \int_{\mathbb{R}} e^{\mathrm{i}\lambda t}\,\overline{e^{\mathrm{i}\lambda s}}\,d\sigma(\lambda)$$

implies the following isomorphisms:

$$\mathcal{H}(X) \ni X_t \quad \longleftrightarrow \quad e^{\mathrm{i}\cdot t} \in L^2(\sigma),$$

$$\mathcal{H}(X;a) = \text{c.l.s.}\,\{X_t : \ |t| \le a\} \quad \longleftrightarrow \quad \text{c.l.s.}\,\left\{e^{\mathrm{i}\cdot t} : |t| \le a\right\} =: L^2(\sigma;a).$$

Therefore the above problem can be formulated as follows: Given a measure σ on $\mathbb{R}$ and $a > 0$, such that $L^2(\sigma;a) \ne L^2(\sigma)$. Find the orthogonal projection of $e^{\mathrm{i}\cdot s} \in L^2(\sigma)$ onto $L^2(\sigma;a)$, or, more generally, find the orthogonal projection from $L^2(\sigma)$ onto $L^2(\sigma;a)$.

The assumption (3) means for the positive definite function f from (1) that the restriction $f\big|_{[-2a,2a]}$, which is a positive definite function on $[-a,a]$, has infinitely many positive definite continuations $\widetilde{f}$ to $\mathbb{R}$. According to a result of M.G. Krein these continuations can be described as follows: There exists a 2×2–matrix function $W(a;z)$, such that the relation

$$-i\int_0^\infty e^{-itz}\widetilde{f}(t)\,dt \;=\; \frac{w_{11}(a;z)\,\omega(z) + w_{12}(a;z)}{w_{21}(a;z)\,\omega(z) + w_{22}(a;z)}, \quad z \in \mathbb{C}^-,$$

establishes a bijective correspondence between all such positive definite continuations $\widetilde{f}$ and all $\omega \in \mathbf{N}\cup\{\infty\}$; here $\mathbf{N}$ is the set of all Nevanlinna functions (these are the complex valued functions which are holomorphic in the upper half plane $\mathbb{C}^+$ and map $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$), and the functions $w_{ij}(a;\cdot)$, $i,j = 1,2$, are entire functions of exponential type a.

A canonical systems of differential equations is a system of the form

$$\text{(4)} \qquad -J\mathbf{y}'(\xi) = zH(\xi)\mathbf{y}(\xi), \quad 0 \le \xi < \infty, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where

$$H(\xi) = \begin{pmatrix} h_{11}(\xi) & h_{12}(\xi) \\ h_{12}(\xi) & h_{22}(\xi) \end{pmatrix}, \quad 0 \le \xi < \infty,$$

is a real symmetric locally summable matrix function on $[0,\infty)$ of positive type ($\mathbf{e} \in \mathbb{C}^2$, $\int_0^\infty (H(\xi)\mathbf{e},\mathbf{e})\,d\xi = 0 \Longrightarrow \mathbf{e} = 0$) which is trace normed ($\operatorname{tr} H(\xi) = 1$, $0 \le \xi < \infty$). With (4) the matrix differential equation

$$\frac{dV(\xi,z)}{d\xi}J = zV(\xi,z)H(\xi), \quad 0 \le \xi < \infty, \quad V(0,z) = I_2.$$

is considered. The elements $v_{ij}(\xi,z)$ of the matrix function $V(\xi,z) = \big(v_{ij}(\xi,z)\big)_{i,j=1}^2$ are entire functions of z of exponential type $a = \int_0^\xi \sqrt{\det H(\eta)}\,d\eta$. With (4) the Hilbert space

$$\mathcal{L}^2_H(0,\infty) = \left\{\mathbf{u} = \begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} : \int_0^\infty \big(H(\xi)\mathbf{u}(\xi),\mathbf{u}(\xi)\big)_2\,d\xi < \infty\right\}$$

and the Fourier transformation:

$$\mathcal{F}(\mathbf{u})(z) := \int_0^\infty \big(v_{21}(\xi,z) \quad v_{22}(\xi,z)\big)\, H(\xi)\,\mathbf{u}(\xi)\,d\xi \tag{5}$$

are associated. The measure σ on $\mathbb{R}$ is a spectral measure of the canonical system if $\int_{\mathbb{R}} \dfrac{d\sigma(\lambda)}{1+\lambda^2} < \infty$ and $\mathbf{u} \mapsto \mathcal{F}(\mathbf{u})$ is an isometry from $\mathcal{L}^2_H$ onto $L^2(\sigma)$:

$$\int_0^\infty \big(H(\xi)\mathbf{u}(\xi), \mathbf{u}(\xi)\big)_2\, d\xi = \int_{\mathbb{R}} |\mathcal{F}(\mathbf{u})(\lambda)|^2 d\sigma(\lambda).$$

According to a theorem of L. deBranges [1] each measure σ with $\int_{\mathbb{R}} \dfrac{d\sigma(\lambda)}{1+\lambda^2} < \infty$ is the spectral measure of a uniquely determined canonical system on $[0,\infty)$.

Now let σ be the measure from (2). It can be shown that for the canonical system with spectral measure σ the Fourier transformation establishes also an isomorphism between $L^2(\sigma;a)$ and $\mathcal{L}^2_H(0,L_a)$, where

$$L_a := \min\left\{ L : \quad \int_0^L \sqrt{\det\, \widehat{H}(\xi)}\, d\xi = a \right\}.$$

Clearly, the orthogonal projection in $\mathcal{L}^2_H(0,\infty)$ onto $\mathcal{L}^2_H(0,L_a)$ is multiplication by the characteristic function $\chi_{[0,L_a]}$. Thus, the orthogonal projection $\mathbf{P}_a$ in $L^2(\sigma)$ onto $L^2(\sigma;a)$ is given by

$$\mathbf{P}_a\, F = \mathcal{F}\big(\chi_{[0,L_a]}\mathcal{F}^{-1}(F)\big), \quad F \in L^2(\sigma),$$

where $\mathcal{F}^{-1}$ denotes the inverse to the Fourier transformation (5):

$$\mathcal{F}^{-1}(F)(\xi) = \int_{\mathbb{R}} F(\lambda) \begin{pmatrix} v_{21}(\xi,\lambda) \\ v_{22}(\xi,\lambda)) \end{pmatrix} d\sigma(\lambda) \in \mathcal{L}^2_H, \quad F \in L^2(\sigma).$$

These results go back to unpublished joint work of M.G. Krein with the author in the 1980-ies, comp. [3].

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A differential operator with an interior singularity

Matthias Langer
(joint work with B. Malcolm Brown, Heinz Langer)

We consider two Sturm–Liouville equations that contain an interior singularity, namely an equation in impedance form,

$$-\left(\frac{1}{|x-1|^\alpha}\,y'(x)\right)' = \lambda\,\frac{1}{|x-1|^\alpha}\,y(x), \tag{1}$$

and a Bessel-type equation,

$$-v''(x) + \frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)\frac{v(x)}{(x-1)^2} = \lambda v(x), \tag{2}$$

both on an interval $[0,a]$ with $a>1$ and where α is some positive constant. The equation in (1) with $\alpha=2$ arose in connection with an extension problem of the positive definite function $f(t)=1-|t|$ in [13]. Equation (2) is obtained from (1) by applying the Liouville transform $y(x)=|x-1|^{\alpha/2}v(x)$.

Equations with a weaker singularity than in (2), e.g.

$$-v''(x) + \frac{1}{x}v(x) = \lambda v(x) \tag{3}$$

on an interval that contains 0 as an interior point have been studied by many authors; see, e.g. [2, 3, 4, 9, 10, 11]. In this case the equation is in limit circle case from both sides at 0, and regularized boundary values can be used to describe self-adjoint operators connected with (3).

If $\alpha<1$, then equation (1) is regular at 1 and equation (2) is singular at 1 but still in limit circle case from both sides. Therefore self-adjoint operators can be obtained using similar ideas as were used for (3). However, if $\alpha\geq 1$, then both equations (1) and (2) are in limit point case from both sides at 1, and hence classical Hilbert space theory yields only self-adjoint operators that are a direct sum of operators associated with the differential expression on $[0,1)$ and $(1,a]$, respectively.

Initially, some physicists and later some mathematicians studied abstract constructions for equations like in (1) and (2) that involve operators in a Pontryagin space extension of a given Hilbert space; see, e.g. [14, 5, 7, 6] (a Pontryagin space is a direct sum of a Hilbert space and a finite-dimensional anti-Hilbert space so that the space has an indefinite inner product). In most cases these models have been applied to situations where the singularity is at an endpoint of the interval; see, e.g. [8] for the Bessel equation. Equations with a singular potential like $1/x^2$ has also been viewed as supersingular perturbations; see, e.g. [1, 12].

The main aim is to construct a Pontryagin space and self-adjoint operators that are more naturally associated with the differential equations (1) and (2). When $\alpha<1$, then for equation (1) one works in the weighted space $L^2([0,a],|x-1|^{-\alpha}dx)$ where the weight is the coefficient on the right-hand side of (1). For $\alpha\geq 1$ we replace the measure $|x-1|^{-\alpha}dx$ by a distribution μ_α that is connected with a regularization of $|x-1|^{-\alpha}$, namely (for simplicity, we restrict ourselves to the case when $1<\alpha<3$)

$$\langle \mu_\alpha, f\rangle := a_0 f(1) + \int_0^a \Big(f(x) - f(1) - (x-1)f'(1)\Big)\frac{dx}{|x-1|^\alpha}, \quad f\in C^2([0,a]),$$

where $a_0\in\mathbb{R}$. We can use this distribution to define an inner product

$$[f,g] := \langle \mu_\alpha, f\,\overline{g}\rangle$$

for $f,g\in C^2([0,a])$. It turns out that this inner product is indefinite, and one can complete $C^2([0,a])$ to a Pontryagin space with one negative square, which is isomorphic to $L^2(0,a)\oplus\mathbb{C}^2$. In this space we construct a boundary triple, which is used to describe self-adjoint operators with boundary conditions at 0 and a and interface conditions at 1. For $\alpha=2$ these operators act like

$$u\mapsto -u''(x) + 2\frac{u(x)}{(x-1)^2} - 2\frac{u'(1)}{x-1}$$

in the L^2 component of the representation in $L^2(0,a)\oplus\mathbb{C}^2$. Corresponding Weyl–Titchmarsh functions belong to the class $\mathcal{N}_1$ of generalized Nevanlinna functions with one negative square.

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Geometry and spectral theory of discrete Dirichlet spaces

Daniel Lenz

Spectral and probabilistic properties of Laplacians and their semigroups are intimately related to the geometry of the underlying space. The theory of Dirichlet spaces provides a convenient framework to corresponding investigations. Here, a Dirichlet space is a locally compact metric measure space together with a regular Dirichlet form. This Dirichlet form gives rise to a Laplacian and its semigroup. The characteristic property of a Dirichlet form is that this semigroup is Markovian.

Two most prominent examples of Dirichlet spaces are Riemannian manifolds (RM) together with the Dirichlet form induced by the Laplace-Beltrami operator and discrete spaces (DS) together with a graph structure and the canonical Dirichlet form and the associated graph Laplacian. In fact, (RM) is the typical example of a local Dirichlet space and (DS) is a typical example of a non-local Dirichlet space.

Starting with seminal work of Sturm [14] substantial parts of global spectral geometry of manifolds were extended to all local Dirichlet spaces. Here, we restrict ourselves to mentioning the following results:

- Volume growth criteria for stochastic completeness [14]. This gives a generalization to local Dirichlet spaces of results of Grigor'yan' [5, 6] for manifolds. These results show that the threshold between for stochastic completeness lies along some superexponential volume growth.
- Upper bounds for the the infimum of the spectrum in terms of volume growth [14]. These bounds show in particular that the infimum of the spectrum is at zero if the volume is sub exponentially growing. This is related to 'Brooks Theorem' [3] for manifolds, which deals with the infimum of the essential spectrum.

A key ingredient of corresponding considerations is the intrinsic metric. In fact, a crucial insight of [14] is that this metric belongs locally to the form domain and satisfies a suitable estimate (if it generates the topology). This is then used to construct suitable cut-off functions.

We note in passing that similar cut-off functions can also be used in the study of spectral theory via generalized eigenfunctions. In fact, recent results for strongly local form provide an 'expansion in generalized eigenfunctions' [1], a 'Shnol Theorem' [2] and an 'Allegretto-Piepenbrink Theorem' [12].

It is natural to expect that similar results should hold for discrete Dirichlet spaces as well. Quite surprisingly this turns out to be wrong in general: Wojciechowski [15] gives examples, dubed 'antitrees', of stochastically incomplete graphs, whose volume growth is polynomially bounded (when measured with respect to the natural graph metric). These examples were then also shown to have a spectral gap at zero by Keller/Lenz/Wojciechowski [11]. As, by the results discussed above, polynomial volume growth bounds imply both stochastic completeness and non-existence of a spectral gap for strongly local forms these results provide major discrepancies between local and discrete Dirichlet spaces.

In the final analysis these discrepancies rely on structural geometric differences between local and non-local situations. As a result of them and due to a general interest in non-local situation there has been a lot of recent interest in (DS) and its geometry .

In this context, a general concept of intrinsic metric for arbitrary regular Dirichlet forms is proposed by Frank/Lenz/Wingert in [4]. This work provides a Rademacher type theorem as well as application to spectral theory via generalized eigenfunctions. It also shows that in the local situation the intrinsic metric mentioned above is canonical in a suitable sense whereas in the non-local situation there is no canonical intrinsic metric. It does not contain applications to stochastic completeness but has influenced such studies by Huang [9].

For Dirichlet forms consisting only of a jump part, a somewhat more general class of intrinsic type metrics, called adapted metrics, was featured in the investigations of Masamune/Uemura [13]. This work provides volume growth criteria for stochastic completeness. A rather refined version of such a result generalizing also corresponding results of [9] is given by Grigor'yan/Huang/Masamune in [7]. This work partly reduces the discrepancy between manifolds and graphs in terms of volume growth criteria for stochastic completeness by giving a threshold (in terms of the natural graph distance). It also provides an sufficient exponential volume growth criterion for stochastic completeness in terms of adapted metrics. It is not known whether this criterion is optimal.

As for the disparity concerning spectral gaps there are no results available so far.

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Schrödinger operators with δ and δ'-potentials supported on hypersurfaces

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(joint work with Jussi Behrndt and Matthias Langer)

Preliminaries. We study self-adjoint realizations of the Schrödinger differential expression $-\Delta + V$ in the Hilbert space $L^2(\mathbb{R}^n)$ with certain coupling (transmission) boundary conditions on a compact C^∞-smooth closed hypersurface. In order to keep the abstract relatively short we present the essence of our results in the case $V \equiv 0$.

Let us define a compact C^∞-smooth closed hypersurface Σ which separates the Euclidean space $\mathbb{R}^n, n \geq 2$, into an (interior) bounded domain Ω_{i} and an (exterior) unbounded domain Ω_{e}. Further we denote by $\dim \Sigma\,(= n-1)$ the Euclidean dimension of Σ. We agree to denote by $-\Delta_{\mathrm{D,i}}$, $-\Delta_{\mathrm{N,i}}$ the self-adjoint Dirichlet and Neumann Laplacians on the interior domain, and we denote by $-\Delta_{\mathrm{D,e}}$, $-\Delta_{\mathrm{N,e}}$ the self-adjoint Dirichlet and Neumann Laplacians on the exterior domain. The coupled Laplacians

$$-\Delta_{\mathrm{D,i,e}} := (-\Delta_{\mathrm{D,i}}) \oplus (-\Delta_{\mathrm{D,e}}) \quad \text{and} \quad -\Delta_{\mathrm{N,i,e}} := (-\Delta_{\mathrm{N,i}}) \oplus (-\Delta_{\mathrm{N,e}})$$

and the free Laplacian $-\Delta_{\mathrm{free}}$ are self-adjoint in $L^2(\mathbb{R}^n)$. From physical point of view the hypersurface Σ is absolutely transparent in the case of the operator $-\Delta_{\mathrm{free}}$ and absolutely non-transparent in the case of the operators $-\Delta_{\mathrm{D,i,e}}$ and $-\Delta_{\mathrm{N,i,e}}$.

Recall that a compact operator T belongs to the class $\mathfrak{S}_{p,\infty}$ if its singular values satisfy $s_k(T) = O(k^{-1/p})$ as $k \to \infty$.

δ-interactions supported on hypersurfaces. In papers on mathematical physics one finds sometimes the differential expression $-\Delta + \alpha\delta_\Sigma(\cdot)$, where δ_Σ is the delta-distribution supported on Σ and $\alpha\colon \Sigma \to \mathbb{R}$ is a real-valued bounded function defined on Σ. Behind this expression it used to presume the operator defined via closed semi-bounded quadratic form

$$\mathfrak{q}_{\delta,\alpha}[u] := \big\|\nabla u\big\|^2_{L^2(\mathbb{R}^n;\mathbb{C}^n)} - \int_\Sigma \alpha\big|u|_\Sigma\big|^2, \quad \mathrm{dom}\,\mathfrak{q}_{\delta,\alpha} := H^1(\mathbb{R}^n).$$

This operator is called the Schrödinger operator with δ-interaction supported on Σ.

The quadratic form method has been used in various papers for the definition of Schrödinger operators with δ-interactions supported on curves and hypersurfaces. We refer the reader to Brasche, Exner, Kuperin and Seba [BEKS94] and the review paper [E08] for more details and further references. The motivation to study these operators comes from physics. Such operators describe semi-transparency in optics and interaction of a charged particle with a charged hypersurface in quantum mechanics.

The quadratic form definition is quite natural when dealing with non-smooth Σ while in the case of a smooth hypersurface we suggest the explicit definition of the Schrödinger operator with δ-interactions via action and domain. Within the space

$$H^{3/2}_\Delta(\mathbb{R}^n\setminus\Sigma) := \big\{f = f_\mathrm{i}\oplus f_\mathrm{e} \in H^{3/2}(\Omega_\mathrm{i})\oplus H^{3/2}(\Omega_\mathrm{e})\colon \Delta f\in L^2(\mathbb{R}^n)\big\}$$

we define the Laplace operator $-\Delta_{\delta,\alpha}$ on the domain

$$\mathrm{dom}\,(-\Delta_{\delta,\alpha}) := \Big\{f = f_\mathrm{i}\oplus f_\mathrm{e} \in H^{3/2}_\Delta(\mathbb{R}^n\setminus\Sigma)\colon \substack{f_\mathrm{i}|_\Sigma = f_\mathrm{e}|_\Sigma =: f|_\Sigma \\ \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Sigma + \partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Sigma = \alpha f|_\Sigma}\Big\},$$

where $f_\mathrm{i}|_\Sigma$, $f_\mathrm{e}|_\Sigma$ are the traces of $f = f_\mathrm{i}\oplus f_\mathrm{e}$ from both sides of Σ and $\partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Sigma$, $\partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Sigma$ are the traces of normal derivative of f from both sides of Σ with normals pointing outwards Ω_i and Ω_e, respectively. In words: the domain of the operator $-\Delta_{\delta,\alpha}$ consists of functions which are continuous on Σ, but their normal derivative has a jump which is connected with the usual trace via function α. The machinery based on quasi boundary triples [BL07, BLL12-1] allows to prove that for any real-valued $\alpha\in L^\infty(\Sigma)$ the operator $-\Delta_{\delta,\alpha}$ is self-adjoint in the Hilbert space $L^2(\mathbb{R}^n)$. It comes out after some calculations that the operator $-\Delta_{\delta,\alpha}$ and the operator corresponding to the form $\mathfrak{q}_{\delta,\alpha}$ coincide.

It is known that $\sigma_\mathrm{ess}(-\Delta_{\delta,\alpha}) = \mathbb{R}_+$ an $\sharp\,\sigma_\mathrm{d}(-\Delta_{\delta,\alpha}) < \infty$. The proof of finiteness of negative spectra given in [BEKS94] is quite involved. As we show this question can be simply reduced to an old result by Birman [B62].

As proved in [G84, G84a], for all $m\in\mathbb{N}$ one has

$$(-\Delta_\mathrm{free} - \lambda)^{-m} - (-\Delta_{\mathrm{N,i,e}} - \lambda)^{-m} \in \mathfrak{S}_{\frac{\dim\Sigma}{2m},\infty} \tag{1}$$

and this estimate is sharp. A natural question arises to obtain estimates like (1) for the pairs $\{-\Delta_\mathrm{free}, -\Delta_{\delta,\alpha}\}$ and $\{-\Delta_{\mathrm{N,i,e}}, -\Delta_{\delta,\alpha}\}$. We found out that for all

$m \in \mathbb{N}$

$$(2)\qquad \begin{aligned}(-\Delta_{\delta,\alpha} - \lambda)^{-m} - (-\Delta_{\mathrm{free}} - \lambda)^{-m} &\in \mathfrak{S}_{\frac{\dim \Sigma}{2m+1},\infty},\\ (-\Delta_{\delta,\alpha} - \lambda)^{-m} - (-\Delta_{\mathrm{N,i,e}} - \lambda)^{-m} &\in \mathfrak{S}_{\frac{\dim \Sigma}{2m},\infty}.\end{aligned}$$

In some sense the operator $-\Delta_{\delta,\alpha}$ is closer to the free Laplacian than to the decoupled Neumann Laplacian. In particular, as a consequence of these results wave operators $W_\pm(-\Delta_{\delta,\alpha}, -\Delta_{\mathrm{free}}\}$ exist and are complete in all space dimensions.

Replacing the differential expression $-\Delta$ by $-\Delta+V$ one needs to tackle smoothness of the potential V carefully. Certain smoothness of V in the neighborhood of Σ must be assumed. The higher powers of resolvents we consider – the stronger assumptions on the smoothness of V we have to impose.

δ'-interactions supported on hypersurfaces. Development of a framework for treatment of Schrödinger operators with δ'-interactions supported on hypersurfaces has been posed by Pavel Exner as a non-solved task in [E08]. As we show one can define for a boundedly invertible function $\beta\colon \Sigma \to \mathbb{R}$ the Schrödinger operator with δ'-interactions via closed semi-bounded quadratic form

$$\begin{aligned}\mathfrak{q}_{\delta',\beta}[u] &:= \big\|\nabla u\big\|_{L^2(\mathbb{R}^n;\mathbb{C}^n)} - \int_\Sigma \beta^{-1}\big|u_{\mathrm{e}}|_\Sigma - u_{\mathrm{i}}|_\Sigma\big|^2,\\ \mathrm{dom}\,\mathfrak{q}_{\delta',\beta} &:= H^1(\Omega_{\mathrm{i}}) \oplus H^1(\Omega_{\mathrm{e}}).\end{aligned}$$

Another possibility is to define the Laplace operator $-\Delta_{\delta',\beta}$ on the domain

$$\mathrm{dom}\,(-\Delta_{\delta',\beta}) := \Big\{ f = f_{\mathrm{i}} \oplus f_{\mathrm{e}} \in H^{3/2}_\Delta(\mathbb{R}^n \setminus \Sigma)\colon \begin{smallmatrix} f_{\mathrm{e}}|_\Sigma - f_{\mathrm{i}}|_\Sigma = \beta\partial_{\nu_{\mathrm{e}}} f_{\mathrm{e}}|_\Sigma \\ \partial_{\nu_{\mathrm{e}}} f_{\mathrm{e}}|_\Sigma + \partial_{\nu_{\mathrm{i}}} f_{\mathrm{i}}|_\Sigma = 0\end{smallmatrix} \Big\},$$

In words: the domain of the operator $-\Delta_{\delta',\beta}$ consists of functions which have continuous normal derivative on Σ, but their traces from both sides of Σ are different and the jump of the trace is connected with the normal derivative via function β. The operator $-\Delta_{\delta',\beta}$ is self-adjoint in $L^2(\mathbb{R}^n)$. To see this we employ our quasi boundary triples machinery. It comes out after some calculations that the self-adjoint operator $-\Delta_{\delta',\beta}$ coincides with the self-adjoint operator corresponding to the form $\mathfrak{q}_{\delta',\beta}$.

Spectral theory of Schrödinger operators with δ'-potentials supported on hypersurfaces is weakly developed until now. Some particular results in the case of Σ being a sphere are known [AGS87, S88], where separation of variables is the main tool of analysis. We show that as in the δ-case the essential spectrum is $\sigma_{\mathrm{ess}}(-\Delta_{\delta',\beta}) = \mathbb{R}_+$ and the discrete spectrum satisfies $\sharp\,\sigma_{\mathrm{d}}(-\Delta_{\delta',\beta}) < \infty$. Next question is to obtain estimates like (2) for the pairs $\{-\Delta_{\mathrm{free}}, -\Delta_{\delta',\beta}\}$ and $\{-\Delta_{\mathrm{N,i,e}}, -\Delta_{\delta',\beta}\}$. We found out that for all $m \in \mathbb{N}$

$$(3)\qquad \begin{aligned}(-\Delta_{\delta',\beta} - \lambda)^{-m} - (-\Delta_{\mathrm{free}} - \lambda)^{-m} &\in \mathfrak{S}_{\frac{\dim \Sigma}{2m},\infty},\\ (-\Delta_{\delta',\beta} - \lambda)^{-m} - (-\Delta_{\mathrm{N,i,e}} - \lambda)^{-m} &\in \mathfrak{S}_{\frac{\dim \Sigma}{2m+1},\infty}.\end{aligned}$$

In some sense the operator $-\Delta_{\delta',\beta}$ is closer to the decoupled Neumann Laplacian than to the free Laplacian. In particular, as a consequence of these results wave operators $W_\pm(-\Delta_{\delta',\beta}, -\Delta_{\mathrm{free}}\}$ exist and are complete in all space dimensions.

Replacing the differential expression $-\Delta$ by $-\Delta + V$ one needs again to tackle smoothness of the potential V. Certain smoothness of V in the neighborhood of Σ must be assumed, but what is interesting that one can assume less smoothness of V for the same estimates than in the case of δ-interactions.

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Gap opening and split band edges in couples waveguides

Konstantin Pankrashkin
(joint work with Denis Borisov)

The presence of a band spectrum is a common feature of periodic operators. If, for example, H is the Schrödinger operator on a periodic domain of $\mathbb{R}^d$ and is invariant with respect to the shifts by linearly independent vectors a_j, $j = 1, \dots, d$, then the spectrum of H is the union of the ranges of the band functions $(-\pi, \pi]^d \ni \theta \equiv (\theta_1, \dots, \theta_d) \mapsto E_k(\theta) \in \mathbb{R}$, where $E_k(\theta)$ are the energy values E for which there exists a non-trivial solution ψ to the eigenvalue problem $H\psi = E\psi$ satisfying the Bloch quasiperiodicity condition $\psi(x + a_j) = e^{i\theta_j}\psi(x)$, $j = 1, \dots, d$, and these values are taken for each θ in the non-decreasing order, $E_k(\theta) \le E_{k+1}(\theta)$; the parameter θ is usually referred to as the quasimomentum.

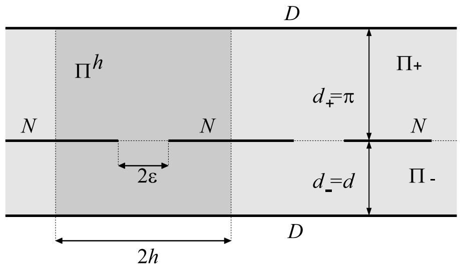

FIGURE 1. Waveguides $\Pi_+ = \mathbb{R}\times(0,\pi)$ and $\Pi_- = \mathbb{R}\times(-d,0)$ coupled by a periodic system of windows. The parts of the boundary with the Dirichlet (respectively, Neumann) boundary conditions are indicated by the symbol D (respectively, N)

If for some j one has $\alpha_j := \max E_j < \min E_{j+1} =: \beta_j$, then the open interval (α_j, β_j) is called a gap of the operator H. The existence of the gaps is one of the central questions e.g. in the theory of photon crystals [3]. One is also interested in the values of θ for which the above extremal values α_j and β_j are realized. In the most studied case, the one-dimensional periodic Schrödinger operator, these values can only be attained at $\theta = 0$ or $\theta = \pm\pi$, while this is not necessarily true for more complicated situations. The papers [1, 2] discussed these questions and provided a number of respective examples using certain combinatorial constructions. Our aim was to provide a rather general mechanism of gap opening at arbitrary values of the quasimomentum in systems with one-dimensional periodicity. As a model example we use the Laplacian in the domain consisting of two strips of width $d_+ = \pi$ and $d_- := d$ coupled by a $2h$-periodic system of windows of width 2ε. At the exterior boundary we impose the Dirichlet boundary conditions, while the Neumann boundary conditions are imposed at the rest of the interior boundary, see figure 1.

Theorem 1. *Assume*

$$h > \frac{\pi}{\sqrt{2}} \quad \text{and} \quad \frac{\pi}{\sqrt{\left(\frac{2\pi}{h}\right)^2 + 1}} < d < \pi,$$

then there exists $\theta_0 \in (0,\pi)$ such that the Laplacian in the above domain has, for ε small enough, a gap whose ends are attained by the respective band functions in a $O(1/\ln\varepsilon)$-neighborhood of θ_0.

A more detailed analysis is possible, which allows one to control the gap opening for some other combinations of the parameters. Note that, due to the real-valuedness of the operators involved, attaining a gap edge at a certain value of θ automatically implies attaining the same extremal value at the value $-\theta$, hence the respective band function appears to have multiple global extrema; in the applications, this situation is usually called split band edge in the physics literature.

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Boundary pairs and Dirichlet-to-Neumann operators

Olaf Post

In this talk I presented a concept for defining the Dirichlet-to-Neumann (DtN) operator in a purely funcional-analytic framework starting with a closed form $\mathfrak{d} \geq 0$ in a Hilbert space $\mathcal{H}$ and a boundary operator $\Gamma\colon \mathcal{H}^1 = \operatorname{dom} \mathfrak{d} \to \mathcal{G}$ into another Hilbert space. We say that $(\Gamma, \mathcal{G})$ is a *boundary pair* associated with $\mathfrak{d}$ if $\mathcal{H}^{1,\mathrm{D}} := \ker \Gamma$ and $\mathcal{G}^{1/2} := \operatorname{ran} \Gamma$ are dense in $\mathcal{H}$ and $\mathcal{G}$, respectively.

The main example is a manifold X with compact boundary $Y = \partial X$ (possibly with certain singularties) in which $\mathcal{H} = \mathsf{L}_2(X)$, $\mathcal{H}^1 = \mathsf{H}^1(X)$, $\mathcal{G} = \mathsf{L}_2(Y)$ and $\Gamma u := u|_Y$. It is now easy to see that $(\Gamma, \mathcal{G})$ is a boundary pair with $\mathcal{G}^{1/2} = \mathsf{H}^{1/2}(\partial Y)$.

Going back to the abstract setting, one can define the *Neumann* operator Δ^{N} as the operator associated with the closed quadratic form $\mathfrak{d}$, and the *Dirichlet* operator Δ^{D} as the operator associated with $\mathfrak{d}|_{\ker \Gamma}$. Moreover, for $z \in \mathbb{C} \setminus \sigma(\Delta^{\mathrm{D}})$ we define the *(weak) solution of the Dirichlet problem* $(\Delta - z)h = 0$, $\Gamma h = \varphi$ in the sense that $h \in \mathcal{H}^1$ is the unique element such that

$$\mathfrak{d}(h, f) - z\langle h, f\rangle = 0$$

for all $f \in \ker \Gamma$. We denote this element by $S(z)\varphi := h$ and call $S(z)\colon \mathcal{G}^{1/2} \to \mathcal{H}^1$ the *Dirichlet solution operator* (in the above example, it is actually the *Poisson operator*).

The DtN operator is now defined via the sesquilinear form $\mathfrak{l}_z$ by

$$\mathfrak{l}_z(\varphi, \psi) := (\mathfrak{d} - z)(S(z)\varphi, g)$$

for $g \in \mathcal{H}^1$ with $\Gamma g = \psi$ (this definition is actually well-defined). The associated operator in the above manifold example turns out to be the usual DtN operator $\Lambda(z)$, i.e., if $\varphi \in \mathsf{H}^{1/2}(\partial X)$ is smooth enough and if h is the solution of the Dirichlet problem then

$$\mathfrak{l}_z(\varphi, \psi) = \langle \Lambda(z)\varphi, \psi\rangle = \langle \Gamma' h, \psi\rangle,$$

where $\Gamma' h = \partial_{\mathrm{n}} h|_{\partial X}$ denotes the (outer) normal derivative at the boundary. Under the stronger assumption that $S(z)$ extends to a bounded operator $\mathcal{G} \to \mathcal{H}$ we can show that $\mathfrak{l}_z$ is sectorial and $\Lambda(z)$ is the associated (closed) sectorial operator. We call such a boundary pair *elliptic regular* since in the manifold example, this condition turns out to be related to an elliptic estimate.

A main result is that for elliptic regular boundary pairs, we have a spectral relation of the form

$$\lambda \in \sigma(\Delta^{\mathrm{N}}) \quad \Leftrightarrow \quad 0 \in \sigma(\Lambda(\lambda))$$

provided $\lambda \in \mathbb{C} \setminus \sigma(\Delta^{\mathrm{D}})$. One can apply this (weak) setting to "boundaries" Y like a hexagonal lattice embedded in $X = \mathbb{R}^2$, and also to a Zaremba problem (a mixed Dirichlet and Neumann problem, where Y is only a (smooth) subset of ∂X. Our work is closely related to other abstract concepts such as quasi-boundary triples [Ar00, AtE08, BBB11, BGP08, BGW09, BL07, DM91, G11, PcR09]. Details of the above ideas will be given in [P11].

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Spectral properties of selfadjoint Schrödinger operators and Dirichlet-to-Neumann maps

Jonathan Rohleder

(joint work with Jussi Behrndt)

The Titchmarsh–Weyl coefficient plays an important role in the spectral theory of ordinary differential operators. If $V : (0, \infty) \to \mathbb{R}$ is a bounded function and λ does not belong to the spectrum of the selfadjoint Schrödinger operator

$$T_{\mathrm{D}} f = -f'' + Vf, \quad \operatorname{dom} T_D = \big\{ f \in H^2(0, \infty) : f(0) = 0 \big\}$$

in $L^2(0, \infty)$, the initial value problem

$$-f'' + Vf = \lambda f, \quad f(0) = f_0$$

has a unique solution f_λ in $L^2(0,\infty)$ for all $f_0 \in \mathbb{C}$, and the Titchmarsh–Weyl coefficient is given by the complex number $m(\lambda)$ which satisfies

$$m(\lambda) f_0 = f'_\lambda(0). \tag{1}$$

The mapping $\lambda \mapsto m(\lambda)$ is known to be holomorphic, and, moreover, to be a Nevanlinna function, that is, $m(\cdot)$ satisfies $m(\overline{\lambda}) = \overline{m(\lambda)}$ and $\frac{\Im m(\lambda)}{\Im \lambda} \geq 0$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. It is well known that the spectrum of the operator $T_{\rm D}$ can be recovered from the limits of $m(\cdot)$ towards the real line. For example, λ_0 is an eigenvalue of $T_{\rm D}$ which is isolated in $\sigma(T_{\rm D})$ if and only if λ_0 is a pole of $m(\cdot)$, and λ_0 belongs to the continuous spectrum of $T_{\rm D}$ if and only if $m(\cdot)$ is not holomorphic at λ_0 and $\lim_{\nu\to 0} \nu m(\lambda_0 + i\nu) = 0$ holds; see, e.g., [2, 5].

Let us consider an open, connected set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, having a compact Lipschitz boundary, and a bounded function $V : \Omega \to \mathbb{R}$. Then the operator

$$A_{\rm D} u = -\Delta u + Vu, \quad \operatorname{dom} A_{\rm D} = \left\{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega), u|_{\partial\Omega} = 0 \right\}$$

is selfadjoint in $L^2(\Omega)$; here $u|_{\partial\Omega}$ denotes the trace of a function u in the Sobolev space $H^1(\Omega)$ at the boundary $\partial\Omega$ of Ω. The spectrum $\sigma(A_{\rm D})$ of $A_{\rm D}$ is bounded from below by the infimum of V and accumulates to $+\infty$. If Ω is bounded, $\sigma(A_{\rm D})$ is purely discrete, that is, it consists of isolated eigenvalues having finite multiplicities; for further details see [3]. For each $\lambda \in \mathbb{C} \setminus \sigma(A_{\rm D})$ the boundary value problem

$$-\Delta u + Vu = \lambda u, \quad u|_{\partial\Omega} = u_0$$

has a unique solution u_λ in $H^1(\Omega)$ for each u_0 in the Sobolev space $H^{1/2}(\partial\Omega)$. Thus the *Dirichlet-to-Neumann map*

$$M(\lambda) u_0 = \partial_\nu u_\lambda|_{\partial\Omega}$$

is well-defined, where $\partial_\nu u_\lambda|_{\partial\Omega}$ denotes the derivative of u_λ with respect to the outer unit normal at $\partial\Omega$; this derivative has to be understood in a weak sense as an element of the Sobolev space $H^{-1/2}(\partial\Omega)$; cf. [4].

It turns out that $M(\lambda)$ is a bounded operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$ for each $\lambda \in \mathbb{C}\setminus\sigma(A_{\rm D})$, and that the function $-M(\cdot)$ is an operator-valued Nevanlinna function; in particular, $M(\cdot)$ is holomorphic. In view of (1), $M(\lambda)$ can be regarded as a generalization of the Titchmarsh–Weyl coefficient. It is our aim to demonstrate that, as in the case of a one-dimensional Schrödinger operator discussed above, the whole spectrum of $A_{\rm D}$ can be characterized by means of the limits of the function $M(\cdot)$. Let us first consider the case that Ω is bounded. Then $\sigma(A_{\rm D})$ is purely discrete and the following result holds, see [1] for a proof.

Theorem 1. *Let Ω be bounded. Then $\lambda_0 \in \sigma(A_{\rm D})$ if and only if λ_0 is a pole of $M(\cdot)$. In this case,*

$$\Gamma_{\lambda_0} : \ker(A_{\rm D} - \lambda_0) \to \operatorname{ran} \operatorname{Res}(M, \lambda_0), \quad u \mapsto \partial_\nu u|_{\partial\Omega}$$

is an isomorphism between the corresponding eigenspace and the range of the residual of $M(\cdot)$ at λ_0.

If Ω is unbounded, a similar result can be proved. We present a statement which characterizes all eigenvalues of A_{D} and the corresponding eigenspaces.

Theorem 2. *A point λ_0 is an eigenvalue of A_{D} if and only if the nontangential limit $\lim_{\lambda \hat{\to} \lambda_0}(\lambda - \lambda_0)M(\lambda)$ is nontrivial. In this case the mapping*

$$\Gamma_{\lambda_0} : \ker(A_{\mathrm{D}} - \lambda_0) \to H^{-1/2}(\partial\Omega), \quad u \mapsto \partial_\nu u|_{\partial\Omega}$$

is injective and the identity

$$\operatorname{cl} \operatorname{ran} \Gamma_{\lambda_0} = \operatorname{cl} \left\{ \lim_{\lambda \hat{\to} \lambda_0} (\lambda - \lambda_0) M(\lambda) u_0 : u_0 \in H^{1/2}(\partial\Omega) \right\}$$

holds, where cl *denotes the closure in $H^{-1/2}(\partial\Omega)$.*

An article which contains a proof of Theorem 2 is in preparation. It will also contain characterizations of the absolutely continuous and singular continuous spectrum of A_{D}. Moreover, it will be shown that even knowledge of the Dirichlet-to-Neumann map on some open, nonempty subset of $\partial\Omega$ suffices to recover the spectrum of A_{D}; cf. [1]. Also selfadjoint operators subject to Robin boundary conditions as well as more general elliptic differential expressions of order two with variable second order coefficients will be treated.

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Computations of Eigenvalues and Spectral Determinants on hyperbolic surfaces

Alexander Strohmaier

(joint work with Ville Uski)

Let (M, g) be an oriented closed Riemannian manifold and Γ a piecewise smooth compact co-dimension one hypersurface in M such that $M \backslash \Gamma$ is the interior of a manifold $\tilde{M}$ with corners. Smooth functions f on $\tilde{M}$ in general have a discontinuity along Γ as left and right boundary values may differ. Let us denote by Df the difference of the left and right boundary values and by $D_n f$ the difference of left and right normal derivatives along Γ. Thus, Df and $D_n f$ are well defined L^∞ functions on Γ. Let now Δ be the Laplace operator acting on functions on M.

Then Δ has compact resolvent as an unbounded operator in $L^2(M)$ and therefore its spectrum consists of countably many eigenvalues λ_i with ∞ as the only possible accumulation point.

We have proved the following theorem (see [1]).

Theorem 1. *There exists a constant $C_{M,\Gamma}$ such that the following holds. Suppose $g \in C^\infty(\tilde{M})$ is a smooth function such that $(\Delta - \lambda)g = h$ as a function on $\tilde{M}$ and suppose*

$$\epsilon = C_{M,\Gamma} \left(\|D_n g\|^2_{H^{-3/2}(\Gamma)} + \|D_n g\|^2_{H^{-1/2}(\Gamma)} \right)^{1/2} < 1.$$

Then there exists an eigenvalue λ_j such that

$$|\lambda - \lambda_j| \leq (1+\lambda)\frac{\epsilon}{1-\epsilon} + \|h\|_{L^2(\tilde{M})}\frac{1}{1-\epsilon}.$$

The constant depends only on the geometry of M and Γ and on the way the Sobolev Norms are defined on Γ and it can be explicitly computed in concrete situations. This estimate allows to prove eigenvalue inclusions rigorously by taking g as a polynomial spline that is an approximate solution to the eigenvalue equation. We give an explicit value for $C_{M,\Gamma}$ for the case when M is a hyperbolic surface and Γ consists of geodesic segments.

If M is a genus g > 1 hyperbolic surface we can use a pants decomposition to decompose M as a union of truncated hyperbolic cylinders glued along geodesic segments. In this case we proved for a certain basis of functions on these cylinders that eigenfunctions can be approximated quantitively by linear combinations of basis functions with exponential rate of convergence in the number of basis functions.

Both estimates together allow the construction of $m \times k$ matrices $\mathbf{B}^0_\lambda$, $\mathbf{B}_\lambda$ and $\mathbf{C}_\lambda$ with the following properties.

(1) The distance of λ to the spectrum can be bounded from above in terms of the first singular value $\sigma_1(\mathbf{B}^0_\lambda)$ of $\mathbf{B}^0_\lambda$ and its singular vector.
(2) The smallest relative singular value $\sigma_1(\mathbf{B}_\lambda, \mathbf{C}_\lambda)$ is bounded from above by
$$c_1(k,\lambda) + c_2(\lambda)\text{dist}(\text{spec}(\Delta), \lambda),$$
where $c_1(k,\lambda)$ and $c_2(\lambda)$ are explicitly computable constants and c_1 is exponentially decaying in k.

Whereas the first property allows to prove that an eigenvalue is in a certain interval, the second property allows to determine intervals in which there are no eigenvalues. Both estimates together can be used to find all eigenvalues in a specified interval.

As a proof of concept we implemented our method in Fortran and in Mathematica. This resulted in programs that allow to compute eigenvalues rather accurately for a surface of genus g with given Fenchel-Nielsen coordinates. The surface in genus 2 maximizing the order of the symmetry group is the so-called Bolza surface. A Mathematica program was used to compute the first eigenvalues of the Bolza surface with extremely high accuracy. We conjecture that the first eigenvalue in the Teichmüller space of genus 2 is maximized by the Bolza surface. Its value is roughly

$$\lambda_1 \approx 3.8388872588421995185866224504354645970819150$$

where all digits correct. The rigorous bound of our method allows us to prove an eigenvalue inclusion here. Our conjecture was numerically verified in large regions of Teichmüller space

The list of eigenvalues with error bounds together with the Selberg trace formula allows us to compute the spectral determinant as well as the Casimir energy (and values of the spectral Zeta function) quite accurately and with error bounds.

Again for the Bolza surface we obtained

$$\det_\zeta(\Delta) \approx 4.72273280444557, \quad \zeta_\Delta(-1/2) \approx -0.65000636917383,$$

Our data suggests that the spectral determinant attains its global maximum at the Bolza surface.

Our method is the first that permits the computation of spectral determinants of the hyperbolic metric with good accuracy. This is important as the Polyakov formula then allows to compute the spectral determinant for all two dimensional manifolds. Our Fortran code that computes eigenvalues of hyperbolic surfaces will be available under the GPLv3 on our websites as a service to the community. Details about the algorithm can be found in our paper [1].

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Spectral inclusions for operators with spectral gaps

Christiane Tretter

Analytical information about the spectra and resolvents of non-selfadjoint operators is of great importance for numerical analysis and applications. However, even for perturbations of selfadjoint operators there are only a few classical results. The most well-known ones may be found in the monograph of T. Kato; they concern bounded perturbations of selfadjoint operators and relatively bounded perturbations of semi-bounded selfadjoint operators.

Theorem A [1, Theorems V.4.10/11] *Let T be a selfadjoint operator in a Hilbert space H and S a linear operator in H.*

i) *If S is bounded, then* $\operatorname{dist}\big(\sigma(T), \sigma(T+S)\big) \le \|S\|$.

ii) *If T is semi-bounded, $T \ge \gamma$, and S is T-bounded with T-bound < 1, i.e. $D(T) \subset D(S)$ and*

$$\|Sx\| \le a'\|x\| + b'\|Tx\|, \quad x \in D(T), \tag{1}$$

with constants a', $b' \geq 0$ and $b' < 1$, then $T + S$ is semi-bounded with

$$T \geq \gamma - \max\left\{\frac{a'}{1-b'}, a' + b'|\gamma|\right\}.$$

A maybe less well-known perturbation result on relatively compact perturbations of selfadjoint operators may be found in the monograph by I.C. Gohberg and M.G. Kreĭn.

Theorem B [2, Lemma V.10.1] *Let T be a selfadjoint operator in a Hilbert space H and S a T-compact operator in H, i.e. $D(T) \subset D(S)$ and $S(T-\lambda)^{-1}$ is compact for one (hence all) $\lambda \in \rho(T)$. Then for every $\varepsilon > 0$ there is $r_\varepsilon > 0$ so that*

$$\sigma(T+S) \subset K_{r_\varepsilon}(0) \cup \Sigma_\varepsilon \cup (-\Sigma_\varepsilon)$$

where $K_{r_\varepsilon}(0)$ is the closed ball of radius r_ε around 0 and $\Sigma_\varepsilon := \{z \in \mathbb{C} : |\arg z| \leq \varepsilon\}$ is a symmetric sector around the real axis of opening 2ε.

In this talk we generalize the above results to relatively bounded perturbations of selfadjoint operators which do not need to be semi-bounded, and we study the behaviour of spectral gaps. To this end, we use another characterization of relative boundedness equivalent to (1):

$$\|Sx\|^2 \leq a^2\|x\|^2 + b^2\|Tx\|^2, \quad x \in D(T), \tag{2}$$

with constants a, $b \geq 0$; moreover, the T-bound of S, defined as the infimum of all $b' \geq 0$ such that there is $a' \geq 0$ with (1), can equivalently be defined as the infimum of all $b \geq 0$ such that there is $a \geq 0$ with (2).

Theorem 1. *Let T be a selfadjoint operator in a Hilbert space H and S a T-bounded operator in H with T-bound < 1 and a, $b \geq 0$, $b < 1$ as in (2). Then*

$$\sigma(T+S) \subset \left\{z \in \mathbb{C} : |\mathrm{Im}\, z|^2 \leq \frac{a^2 + b^2|\mathrm{Re}\, z|^2}{1-b^2}\right\}.$$

This means that the spectrum of the perturbed operator is bounded by two hyperbolas, symmetric to the real axis and with asymptotes $|\mathrm{Im}\, z| = \pm \arcsin b\, |\mathrm{Re}\, z|$.

If S is bounded, we can choose $a = \|S\|$, $b = 0$ and the hyperbolas degenerate into lines; then the above spectral inclusion coincides with the classical result by Kato (see Theorem A i)). If S is T-compact, it is T-bounded with T-bound 0 (note that H is reflexive and T is closed); then the above spectral inclusion can be applied for every $b = \varepsilon > 0$ and yields the result by Gohberg/Krein (see Theorem B).

Theorem 2. *Let T be a selfadjoint operator in a Hilbert space H with spectral gap, $\sigma(T) \cap (-\beta, \beta) = \emptyset$ with $\beta > 0$, and let S be a T-bounded operator in H with T-bound < 1 and a, $b \geq 0$, $b < 1$ as in (2). If $\mu_\beta := a + b|\beta| < \beta$, then the spectrum of $T + S$ also splits into two parts,*

$$\sigma(T+S) \cap \{z \in \mathbb{C} : -\beta + \mu_\beta < \mathrm{Re}\, z < \beta - \mu_\beta\} = \emptyset.$$

An analogous result holds for spectral gaps (α, β) that are not symmetric with respect to the origin. However, it should be noted that the relative boundedness constants in (1) or (2) are not shift-invariant; if $T-\mu$ is considered instead of T, then a' has to be replaced by $a'+b'|\mu|$. In this case, if

$$\mu_\alpha + \mu_\beta < \beta - \alpha, \tag{3}$$

then

$$\sigma(T+S) \cap \big\{z \in \mathbb{C} : \alpha + \mu_\alpha < \operatorname{Re} z < \beta - \mu_\beta\big\} = \emptyset.$$

This shifted version of the stability theorem for spectral gaps can also be used to formulate conditions under which infinitely many spectral gaps do not close.

Theorem 3. *Let T be a selfadjoint operator in a Hilbert space H with infinitely many spectral gaps, $\sigma(T) \cap (\alpha_n, \beta_n) = \emptyset$, $\alpha_n \to \infty$, and let S be a T-bounded operator in H with T-bound < 1. If there exist a_n, $b_n \geq 0$, $b_n < 1$ as in (2) so that*

$$\limsup_{n\to\infty} \frac{2(a_n + b_n\alpha_n)}{(1-b_n)(\beta_n - \alpha_n)} < 1, \tag{4}$$

then $T+S$ still has infinitely many spectral gaps.

If the T-bound of S is not 0, a necessary condition for (4) is that the gap widths $\beta_n - \alpha_n$ diverge exponentially; if the T-bound is 0, polynomial divergence suffices.

All the above results are accompanied by resolvent estimates for the non-selfadjoint perturbed operator $T+S$. Similar results are proved for perturbations of sectorial or bisectorial operators as well as for gaps in the essential spectrum. Applications include Dirac operatos with complex potentials, unbounded Jacobi operators, and two-channel Hamiltonians; for details and further references see [3].

References

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