

On spectral minimal partitions : the disk revisited

VIRGINIE BONNAILLIE-NOËL AND BERNARD HELFFER

Abstract - In continuation of [14] and in the spirit of [3], we analyze the properties of spectral minimal partitions of the disk using the magnetic characterization of minimal partitions, we recover and improve some of the results which were proved in [14] towards the conjecture that the minimal 3-partition for the disk is the Mercedes Star.

Key words and phrases : spectral theory, minimal partitions, nodal domains, Aharonov-Bohm Hamiltonian, numerical simulations, finite element method.

Mathematics Subject Classification (2010) : 35B05, 35J05, 35P15, 49M25, 65F15, 65N25, 65N30.

1. Introduction

In [17], the properties of spectral minimal partitions of a domain $\Omega \subset \mathbb{R}^2$ have been analyzed and the links between this notion and the partitions obtained as the family of the nodal domains of an eigenfunction associated with an eigenvalue of the corresponding Dirichlet problem in Ω have been clarified. We would like to propose some techniques for determining minimal partitions in concrete cases starting with the first non trivial case $k = 3$. We will concentrate our analysis on minimal 3-partitions in the case of the disk and discuss the conjecture proposed in [14] that a minimal 3-partition for the disk should be the Mercedes Star.

1.1. Minimal partitions

Before presenting this conjecture more explicitly, let us first recall notations, definitions and results extracted essentially of [17]. We consider the Dirichlet-Laplacian $H = H(\Omega)$ on a bounded regular domain $\Omega \subset \mathbb{R}^2$. We are interested in the eigenvalue problem for $H(\Omega)$ and note that $H(\Omega)$ has discrete spectrum $\text{Sp}(H(\Omega))$, i.e. consists of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ with finite multiplicities which tend to infinity, so that

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_k \leq \dots \quad (1.1)$$

and such that the associated eigenfunctions u_k can be chosen to form an orthonormal basis for $L^2(\Omega)$. We shall denote for any open domain D by

$\lambda(D)$ the lowest eigenvalue of H with Dirichlet boundary condition

$$\lambda(D) = \lambda_1(H(D)). \quad (1.2)$$

Without loss we can assume that the u_k are real valued and by elliptic regularity $u_k \in C^\infty(\Omega) \cap C_0^0(\overline{\Omega})$. We know that u_1 can be chosen to be strictly positive in Ω , but the other eigenfunctions u_k must have non empty zerosets. We define for any function $u \in C_0^0(\Omega)$

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}, \quad (1.3)$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u . The number of nodal domains of such a function will be called $\mu(u)$.

We now introduce the notion of partitions.

Definition 1.1. *Let $1 \leq k \in \mathbb{N}$. We call a k -partition of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of pairwise disjoint open regular domains such that*

$$\cup_{i=1}^k D_i \subset \Omega. \quad (1.4)$$

*It is called **strong** if*

$$\text{Int } \overline{\cup_{i=1}^k D_i} \setminus \partial\Omega = \Omega. \quad (1.5)$$

We denote by \mathfrak{D}_k the set of such partitions.

We now introduce spectral minimal partitions.

Definition 1.2. *Let $1 \leq k \in \mathbb{N}$. For $\mathcal{D} = \{D_i\}_{i=1}^k \in \mathfrak{D}_k$, we call energy of the partition the expression*

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i), \quad (1.6)$$

and introduce

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{D}_k} \Lambda(\mathcal{D}). \quad (1.7)$$

We call a k -partition $\mathcal{D} \in \mathfrak{D}_k$ a spectral minimal k -partition if $\mathfrak{L}_k(\Omega) = \Lambda(\mathcal{D})$.

The existence and the regularity of spectral minimal k -partitions was proven in [10, 12, 11, 17].

1.2. The case of the disk

If $k = 2$, the minimal value $\mathfrak{L}_2(\Omega)$ is the second eigenvalue and any minimal 2-partition is represented as the nodal partition associated with some second eigenfunction. For $k > 2$, the analysis of various domains like the rectangle shows that $\mathfrak{L}_k(\Omega)$ is not necessarily an eigenvalue. For example, one can

prove that this is the case, when $k = 3$, each time that the second eigenvalue has multiplicity at least 2. This will be the case for the disk.

We are concerned in this paper by the unit disk $\mathbf{D} = \{|x| < 1\} \subset \mathbb{R}^2$ and we study the corresponding $\mathfrak{L}_3(\mathbf{D})$ and some associated¹ minimal partition $\mathcal{D} = (D_1, D_2, D_3)$. Of course we do not know *a priori* that \mathcal{D} is unique (up to rotations and reflections). But we have observed in [17] that \mathcal{D} is not created by a nodal set. The argument was as follows. By analysis of the tables giving the zeros of Bessel functions [14] (see also Figure 10 where the nodal partitions of the first eigenfunctions of the disk are represented), the first eigenvalue whose eigenfunction has three nodal domains is $\lambda_{15}(\mathbf{D})$. We have $\lambda_{15}(\mathbf{D}) > \lambda_4(\mathbf{D})$ and the eigenfunctions associated with $\lambda_4(\mathbf{D})$ have four nodal domains, so (as a consequence of Theorem 2.3 which will be recalled in Section 2) $\lambda_4(\mathbf{D}) = \mathfrak{L}_4(\mathbf{D})$. Hence

$$\lambda_1(\mathbf{D}) < \mathfrak{L}_2(\mathbf{D}) = \lambda_2(\mathbf{D}) = \lambda_3(\mathbf{D}) < \mathfrak{L}_3(\mathbf{D}) < \lambda_4(\mathbf{D}) = \mathfrak{L}_4(\mathbf{D}). \quad (1.8)$$

The aim of this paper is to rediscuss the following conjecture of [14].

Conjecture. *For the disk \mathbf{D} , there is a unique regular representative (up to rotation) of the spectral minimal 3-partition \mathcal{D} associated with $\mathfrak{L}_3(\mathbf{D})$. It is given (see Figure 1) by three disjoint sectors with opening angle $2\pi/3$, i.e.*

$$D_1 = \{x = \rho e^{i\omega} \in \mathbf{D} \mid \omega \in]0, 2\pi/3[\}, \quad (1.9)$$

and D_2, D_3 are obtained by rotating D_1 by $2\pi/3$, respectively by $4\pi/3$.

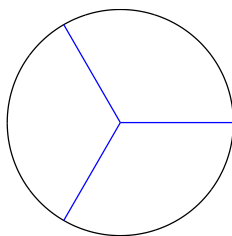


Figure 1. Mercedes Star partition.

For short, we call this partition a Mercedes Star (MS). Despite many efforts, this conjecture has not been proved yet. We note that our numerical simulations (see [5] and [7]) reinforce this conjecture even though there is no definite conclusion. Nevertheless, weaker versions of this conjecture were proved in [14].

For example the following propositions have been proved in [14].

¹We choose always, possibly changing by sets of capacity 0, a regular representative which has been proved in [17] to exist.

Proposition 1.1. *If all the open sets of the minimal 3-partition of \mathbf{D} are simply connected in the punctured disk $\dot{\mathbf{D}} := \mathbf{D} \setminus \{0\}$, then this partition is the Mercedes Star.*

Proposition 1.2. *A minimal 3-partition of \mathbf{D} cannot be invariant by inversion.*

Proposition 1.3. *If a minimal 3-partition \mathcal{D} of \mathbf{D} is such that $(\cap_{i=1}^3 \partial D_i) \cap \mathbf{D}$ consists of just one point and is symmetric with respect to (say) $\{y = 0\}$, then it is the Mercedes Star.*

Our new but not decisive improvement is:

Proposition 1.4. *If the boundary set $(\cap_{i=1}^3 \partial D_i) \cap \mathbf{D}$ of the minimal 3-partition has only one critical point, then the minimal partition is the Mercedes star.*

The novelty of the result is that we have eliminated either the assumption that the minimal partition was symmetric or the assumption that the critical point was at the center. The main idea will be to exploit a magnetic characterization of the minimal partition together with the symmetry of the domain for finding a natural condition under which a minimal partition takes the symmetry of the domain.

2. Previous results

Most of the basic general results are taken from [17]. Attached to a regular partition \mathcal{D} we can associate its boundary set $N = N(\mathcal{D})$ which is the closed set in $\bar{\Omega}$ defined by

$$N(\mathcal{D}) = \bigcup_i \overline{(\Omega \cap \partial D_i)}. \quad (2.1)$$

This leads us to introduce the set $\mathcal{M}(\Omega)$ of the regular closed sets.

Definition 2.1. *A closed set $N \subset \bar{\Omega}$ belongs to $\mathcal{M}(\Omega)$ if N meets the following requirements:*

- (i) *There is a (possibly empty) finite set of distinct points $\mathbf{X}_i \in \Omega \cap N$ and associated positive integers $\nu(\mathbf{X}_i)$ with $\nu(\mathbf{X}_i) \geq 3$ such that, in a sufficiently small neighborhood of each of the \mathbf{X}_i , N is the union of $\nu(\mathbf{X}_i)$ smooth arcs (non self-crossing) with one end at \mathbf{X}_i (and each pair defining at \mathbf{X}_i a positive angle in $(0, 2\pi)$) and such that in the complement of these points in Ω , N is locally diffeomorphic to a regular curve.*

We denote the set of these critical points of N by $X(N)$.

(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points Z_i , such that at each Z_i , $\rho(Z_i)$, with $\rho(Z_i) \geq 1$ arcs hit the boundary. Moreover for each $Z_i \in \partial\Omega$, N is near Z_i the union of $\rho(Z_i)$ distinct smooth arcs which hit at Z_i the (two arcs constituting the) boundary with strictly positive distinct angles.

We denote the set of these critical points of $N \cap \partial\Omega$ by $Y(N)$.

We split the set of interior critical points in two parts:

$$X(N) = X_{\text{even}}(N) \cup X_{\text{odd}}(N), \quad (2.2)$$

where

$$X_{\text{even}}(N) = \{X \in X(N) \mid \nu(X) \text{ is even}\},$$

and

$$X_{\text{odd}}(N) = \{X \in X(N) \mid \nu(X) \text{ is odd}\}.$$

Definition 2.2. We say that a closed regular set satisfies the **equal angle meeting property** if the arcs meet with equal angles at each critical point $X_i \in N \cap \Omega$ and also with equal angles at the $Z_i \in N \cap \partial\Omega$. For the boundary points Z_i , we mean that the two arcs in the boundary are included.

Let us also recall the relations between graphs and partitions. If \mathcal{D} is a strong partition, we say $D_i, D_j \in \mathcal{D}$ are **neighbors** if

$$\text{Int}(\overline{D_i \cup D_j}) \setminus \partial\Omega \text{ is connected} \quad (2.3)$$

and denote this by $D_i \sim D_j$.

We will say that the partition is **bipartite**, if the partition can be colored by two colors, two neighbours having different colors. We recall that a collection of nodal domains of an eigenfunction is always bipartite.

It has been proved by Conti-Terracini-Verzini [10, 12, 11] and Helffer-Hoffmann-Ostenhof-Terracini in [17] the following result.

Theorem 2.1. *For any k , there exists a minimal regular strong k -partition. Furthermore, any minimal spectral k -partition admits a representative which is regular and strong.*

Analogous (somewhat weaker in our particular case) results have been obtained by (or referred in) Bucur-Buttazzo-Henrot [8], Henrot [18] and Caffarelli-Lin [9].

A natural question is whether a minimal partition is the nodal partition induced by an eigenfunction. The next theorem [17] gives a simple criterion:

Theorem 2.2. *If a minimal spectral k -partition (we choose then a regular strong representative) of Ω is bipartite then this partition is associated with the nodal set of an eigenfunction of $H(\Omega)$ corresponding to an eigenvalue equal to $\mathfrak{L}_k(\Omega)$.*

To determine how general is the situation described in the previous theorem, we introduce the notion of Courant-sharp situation as explained below. Suppose that u is an eigenfunction of the Dirichlet-Laplacian in Ω attached to the eigenvalue λ_k :

$$H(\Omega)u = \lambda_k u .$$

Courant's Theorem says that the number of nodal domains $\mu(u)$ satisfies $\mu(u) \leq k$. Then we say, as in [2], that u is **Courant-sharp** if $\mu(u) = k$. For any integer $k \geq 1$, we denote by L_k the smallest eigenvalue such that its eigenspace contains an eigenfunction with k nodal domains. In general we have (by the variational principle)

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega) . \quad (2.4)$$

The last result [17] gives the full picture of the equality cases :

Theorem 2.3. *Suppose $\Omega \subset \mathbb{R}^2$ is regular. If $\mathfrak{L}_k(\Omega) = L_k(\Omega)$ or $\lambda_k(\Omega) = \mathfrak{L}_k(\Omega)$, then*

$$\lambda_k(\Omega) = \mathfrak{L}_k(\Omega) = L_k(\Omega) . \quad (2.5)$$

In addition, if (2.5) holds, any minimal k -partition admits a representative which is the nodal partition of some eigenfunction u associated with $\lambda_k(\Omega)$.

3. Going to the double covering

Given some $\Omega \subset \mathbb{R}^2$ such that $0 \in \Omega$. We consider the double covering² $\dot{\Omega}^{\mathcal{R}}$ of the punctured $\dot{\Omega} := \Omega \setminus \{0\}$ and denote by $\pi^{\mathcal{R}}$ the projection of $\dot{\Omega}^{\mathcal{R}}$ onto $\dot{\Omega}$. Of course $\{0\}$ does not play a specific role and we can do the same thing with any point $X \in \Omega$ leading to some covering $\dot{\Omega}_X^{\mathcal{R}}$ of $\dot{\Omega}_X = \Omega \setminus \{X\}$.

Let \mathcal{D} be a minimal k -partition of Ω . We now make the assumption that this k -partition has the property that all its open sets are simply connected in $\dot{\Omega}$, one can associate with this k -partition a $(2k)$ -partition of $\dot{\Omega}^{\mathcal{R}}$, by considering for each i the two components D_i^{\pm} of $(\pi^{\mathcal{R}})^{-1}(D_i)$. This partition has the symmetry property $G(D_i^+) = D_i^-$ relatively to the deck transformation G which associates with each point x of $\dot{\Omega}^{\mathcal{R}}$ the other point $G(x)$ of $\dot{\Omega}^{\mathcal{R}}$ which has the same projection on $\dot{\Omega}$.

Under this assumption, it is then clear by the definition of $\mathfrak{L}_{2k}(\dot{\Omega}^{\mathcal{R}})$, that

$$\mathfrak{L}_k(\Omega) \geq \mathfrak{L}_{2k}(\dot{\Omega}^{\mathcal{R}}) . \quad (3.1)$$

²with its natural flat metrics

A particular situation occurs when $\mathfrak{L}_{2k}(\dot{\Omega}^{\mathcal{R}})$ is attained by a $(2k)$ -partition which has the G -symmetry that is, which satisfies for k disjoint pairs (D_i^+, D_i^-) ($i = 1, \dots, k$)

$$G(D_i^\pm) = D_i^\mp. \quad (3.2)$$

One gets indeed in this case the reverse inequality

$$\mathfrak{L}_k(\Omega) \leq \mathfrak{L}_{2k}(\dot{\Omega}^{\mathcal{R}}). \quad (3.3)$$

Let us consider the particular case of the disk \mathbf{D} and of $k = 3$. The above argument leads to the proof of Proposition 1.1 in the following way.

Proof. (of Proposition 1.1) We first note that:

$$\mathfrak{L}_3(\mathbf{D}) \leq \Lambda(MS). \quad (3.4)$$

As seen from inspection of the tables for Bessel functions (see the discussion in [17], Section 8), one can verify that the 6-th eigenvalue of the Laplacian on $\dot{\mathbf{D}}^{\mathcal{R}}$ is Courant sharp and corresponds to a minimal 6-partition. The corresponding eigenfunction is symmetric with respect to G . Its projection gives the Mercedes Star and we would obtain under the assumption of the proposition the reverse inequality to (3.4)

$$\mathfrak{L}_3(\mathbf{D}) \geq \mathfrak{L}_6(\dot{\mathbf{D}}^{\mathcal{R}}) = \Lambda(MS). \quad (3.5)$$

The proof of Proposition 1.1 is then immediate. \square

Remark 3.1. The argument does not work for k odd, $k \geq 5$. By using a notion of Courant-sharp for antisymmetric eigenfunctions for the Deck map (or equivalently for the Aharonov-Bohm operator as we shall see later), one can discuss the case $k = 5$ (see Figure 3). Although this is the 11-th eigenfunction on the covering, this is indeed the 5-th G -antisymmetric eigenvalue.

4. On the structure of minimal partitions

4.1. On the topological structure of minimal partitions

We recall now some consequences of Euler's formula in the case of non bipartite minimal 3-partitions. Using the Courant-sharp statement, all the minimal 3-partitions are non bipartite if

$$\lambda_3(\Omega) < \mathfrak{L}_3(\Omega), \quad (4.1)$$

and this condition is in particular true in the case of the disk.

Proposition 4.1. *Consider a non bipartite minimal 3-partition $\mathcal{D} = (D_1, D_2, D_3)$ of Ω associated with $\mathfrak{L}_3(\Omega)$. Let us assume that $\partial\Omega$ has one component. Let $N = N(\mathcal{D})$ and let $X(N)$ and $Y(N)$ be defined as in Definition 2.1. Then there are three possibilities (see Figure 2):*

- (a) $X(N)$ consists of one point X with $\nu(X) = 3$ and $Y(N)$ consists of either three distinct Y_1, Y_2, Y_3 points with $\rho(Y_1) = \rho(Y_2) = \rho(Y_3) = 1$, two distinct points Y_1, Y_2 with $\rho(Y_1) = 2, \rho(Y_2) = 1$ or one point Y with $\rho(Y) = 3$.
- (b) $X(N)$ consists of two distinct points X_1, X_2 so that $\nu(X_1) = \nu(X_2) = 3$. $Y(N)$ consists either of two points Y_1, Y_2 such that $\rho(Y_1) = \rho(Y_2) = 1$ or of one point Y with $\rho(Y) = 2$.
- (c) $X(N)$ consists of two distinct points X_1, X_2 with $\nu(X_1) = \nu(X_2) = 3$, but $Y(N) = \emptyset$.

The proof of Proposition 4.1 is given in [14].

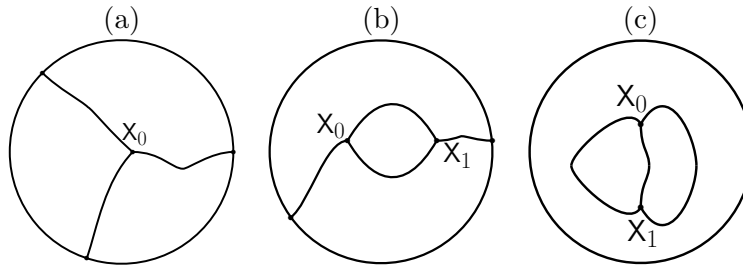


Figure 2. Topological structure of minimal 3-partitions.

4.2. On the size of the subdomains

Let us mention some simple remarks about the size of the subdomains of a minimal 3-partition $\mathcal{D} = (D_1, D_2, D_3)$. A partition cannot be minimal,

- if there exists an isometry i such that $i(D_1) \subsetneq D_2$ (up to a relabelling), (the $\lambda(D_i)$ should indeed be equal),
- or, if there exists an open set O_1 such that $O_1 \subset D_1$ with $\lambda_2(\mathbf{D} \setminus \overline{O_1}) > \lambda_1(O_1)$.

These two remarks have the following applications (Figure 4 compares the first eigenvalue on a subdomain and the second one on the complementary for geometries represented in Figure 3):

1. For $a \in (-1, 1)$, we define $\mathbf{C}_a = \mathbf{D} \cap \{y > a\}$. Let a_* be such that $\lambda_2(\mathbf{C}_{-a_*}) = \lambda_1(\mathbf{C}_{a_*})$. We have $a_* \simeq 0.19$. If (after rotation or relabelling) $\mathbf{C}_a \subset D_1$, then $a \geq a_*$.
2. For $\alpha \in (0, 2\pi)$, we denote S_α the angular sector of aperture α . If there exists α such that (after rotation or relabelling) $S_\alpha \subset D_1$, then $\alpha \leq \frac{2\pi}{3}$.

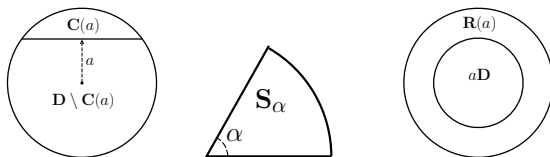


Figure 3. Geometries C_a , S_α , $R(a)$.

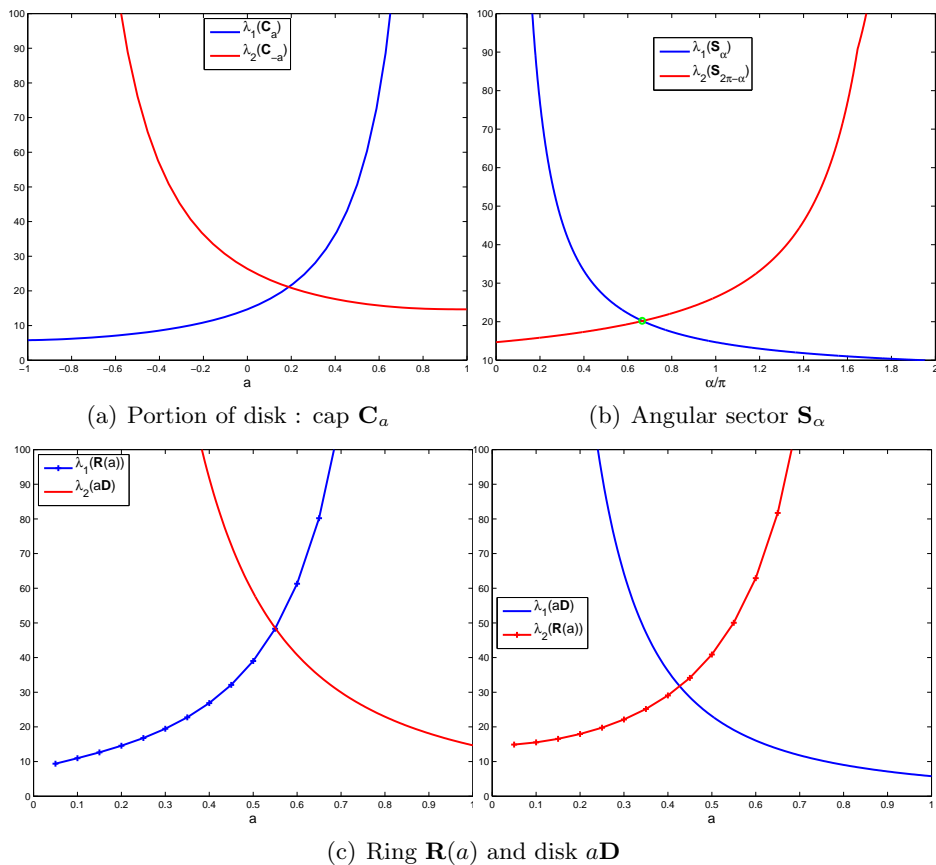


Figure 4: Comparison between $\lambda_1(O_1)$ and $\lambda_2(\mathbf{D} \setminus \overline{O_1})$ when O_1 is a portion of disk C_a , an angular sector S_α , a ring $R(a)$ or a disk of radius a .

3. For $a \in (0, 1)$, we denote R_a the ring of interior radius a : $R(a) = \mathbf{D} \setminus a\mathbf{D}$ with $a\mathbf{D}$ the disk centered at the origin of radius a . Let a_r^1 and a_r^2 be such that

$$\lambda_2(R(a_r^2)) = \lambda_1(a_r^2\mathbf{D}), \quad \lambda_2(a_r^1\mathbf{D}) = \lambda_1(R(a_r^1)).$$

We have $a_r^1 \simeq 0.55$ and $a_r^2 \simeq 0.42$.

If (after rotation or relabeling) $\mathbf{R}(a) \subset D_1$ then $a \geq a_r^1$. If $a\mathbf{D} \subset D_1$ then $a \leq a_r^2$.

A last remark is that by Faber-Krahn's inequality, the area of any D_i is larger than $\pi \frac{\lambda_1(\mathbf{D})}{\Lambda(MS)} \simeq 0.899$.

Unfortunately all these remarks do not permit to exclude type (b) or type (c) configurations.

5. On the minimal partitions symmetric with respect to one axis of reflection

In this section we recall some partial results of [14]. In particular we assume that the 3-partition \mathcal{D}_3 satisfies some symmetry condition and then show that this implies or excludes one of the topological types described in Proposition 4.1. Suppose that associated with $\mathfrak{L}_3(\mathbf{D})$ for the unit disk \mathbf{D} there is (up to rotation and reflection) a minimal 3-partition which is now invariant by the map

$$(x, y) \mapsto \sigma(x, y) = (x, -y). \quad (5.1)$$

We observe that the **Mercedes Star** has this symmetry. We would like to analyze a weak form of the main conjecture.

Conjecture. *Suppose that there exists some minimal 3-partition for the disk which is symmetric with respect to Σ . Then, possibly after a rotation, this partition corresponds to the **Mercedes Star**.* This weak form of the conjecture was also presented in [14] and important steps towards the proof were proposed. First remember that Proposition 4.1 tells us that there are just three topologically different candidates for a minimal 3-partition. We have of course (3.4). Let us now analyze the different configurations.

5.1. Symmetric configuration of type (a)

Proposition 1.3 will be an immediate consequence of Proposition 4.1 and of the following statement.

Lemma 5.1. *Suppose we are considering a minimal 3-partition \mathcal{D} which is σ -symmetric and has the property (a) in Proposition 4.1. Then $(0, 0)$ is the critical point of the partition.*

Proof. First we observe that $X = X_{\text{odd}}$ consists of one point $X_0 = (x_0, 0)$ and we pick the x -axis as the symmetry axis.

We also know that $\mathfrak{L}_3(\mathbf{D})$ is the ground state energy of $H(D_j)$ and the second eigenvalue of the Dirichlet problem in $D_{i,j} = \text{Int}(\overline{D_i} \cup D_j)$.

For any $x \in [-1, +1]$, let us introduce the segment $\ell(x)$ defined by

$$\ell(\mathbf{x}) := \{x \in [-1, \mathbf{x}], y = 0\}.$$

We can assume w.l.o.g. (possibly after a rotation by π) that the segment $\ell(\mathbf{x}_0)$ belongs to N , so that $\mathbf{x}_0 \in]-1, 1[$.

The corresponding domains D_1, D_2, D_3 satisfy

$$\sigma D_1 = D_2, \sigma D_2 = D_1, \sigma D_3 = D_3.$$

We label the partition such that $D_1 \subset \{y > 0\}$.

Another consequence of [17] (see also [5] for an extensive use of this property) is that

$$\mathfrak{L}_3(\mathbf{D}) = \lambda_3(\mathbf{D}(\mathbf{x}_0)),$$

with $\mathbf{D}(\mathbf{x}) = \mathbf{D} \setminus \ell(\mathbf{x})$. We can indeed show that \mathcal{D} is a minimal 3-partition of $\mathbf{D}(\mathbf{x}_0)$ and that $\mathfrak{L}_3(\mathbf{D})$ is an eigenvalue of $H(\mathbf{D}(\mathbf{x}_0))$ corresponding to an eigenfunction with \mathcal{D} as nodal partition. We can then use the results recalled in Section 2.

We now consider different cases depending on \mathbf{x}_0 :

Case 1. $\mathbf{x}_0 \geq 0$: In this case, we can indeed apply Proposition 1.1.

Case 2. $\mathbf{x}_0 < 0$: This is proved in [14] by Verzini’s trick. We emphasize that this trick supposes the invariance by rotation of the domain.

□

Numerically, we consider the mixed Dirichlet-Neumann Laplacian in the half-disk $\mathbf{D}_+ = \mathbf{D} \cap \{y > 0\}$:

Neumann-Dirichlet Laplacian	Dirichlet-Neumann Laplacian
$\begin{cases} -\Delta u = \lambda u \text{ in } \mathbf{D}_+, \\ \partial_{\mathbf{n}} u = 0 \text{ on } (0, \mathbf{x}) \times \{0\}, \\ u = 0 \text{ elsewhere,} \end{cases}$	$\begin{cases} -\Delta u = \lambda u \text{ in } \mathbf{D}_+, \\ \partial_{\mathbf{n}} u = 0 \text{ on } (\mathbf{x}, 1) \times \{0\}, \\ u = 0 \text{ elsewhere.} \end{cases}$

We denote by $\lambda_2^{ND}(\mathbf{x})$ and $\lambda_2^{DN}(\mathbf{x})$ respectively the second eigenvalue. We use the finite element library MÉLINA (see [20]) and compute these second eigenvalues for $\mathbf{x} \in \{k/40, 0 \leq k \leq 40\}$. Figure 5(a) gives the numerical simulations. We notice that the two curves are symmetric according to $\mathbf{x} = 0$ due to the symmetry of the disk. Figure 5(b) gives the position of the abscissa $y^{DN}(\mathbf{x})$ of the intersection point between the nodal line and the axis $\{y = 0\}$ according to the position of the critical point \mathbf{x} of the mixed Dirichlet-Neumann Laplacian. We notice that $y^{DN}(\mathbf{x}) = \mathbf{x}$ only for $\mathbf{x} = 0$.

5.2. Exclusion of symmetric configuration of types (b)-(c)

Next we consider the cases (b) and (c). There are two possibilities to place the two points in X_{odd} once we have fixed the x -axis as the reflection line. These two possibilities are:

- (i) $X_{\text{odd}} = \{(\mathbf{x}_0, y_0), (\mathbf{x}_0, -y_0)\}$.
- (ii) $X_{\text{odd}} = \{(\mathbf{x}_1, 0), (\mathbf{x}_2, 0)\}$.

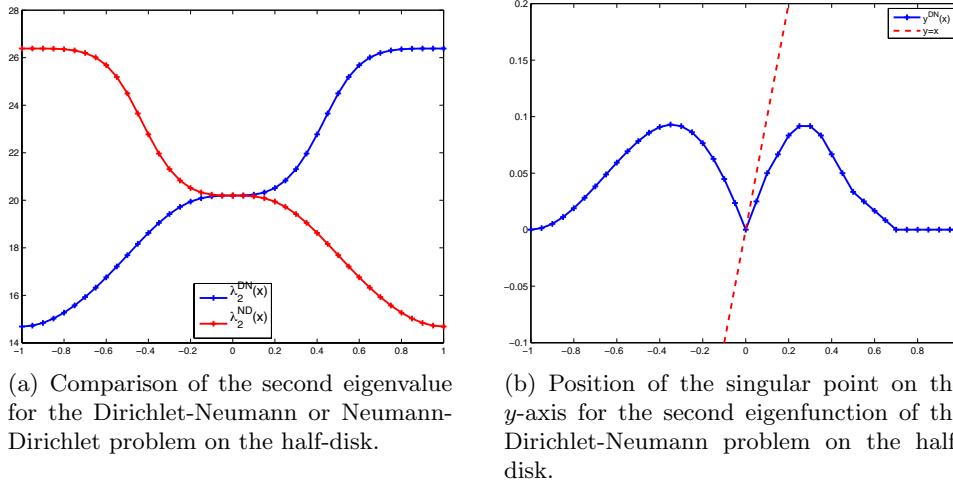


Figure 5. Mixed Dirichlet-Neumann problem on the half-disk.

These two possibilities correspond to the cases when the associated permutation τ (defined by $\sigma D_i = D_{\tau(i)}$) is either the identity (case (i)) or exchanging 1 and 2 (case (ii)). Let us prove the following result:

Proposition 5.1. *A symmetric 3-partition of type (b) or (c) cannot be minimal if it satisfies (i).*

Proof. Consider as before $\mathbf{D}_+ = \{(x, y) \in \mathbf{D} \mid y > 0\}$. Define also H_+ as $-\Delta$ on \mathbf{D}_+ with Dirichlet boundary condition on $\partial\mathbf{D}_+ \cap \{y > 0\}$ and Neumann on $\partial\mathbf{D}_+ \cap \{y = 0\}$. We can associate with H_+ also minimal spectral partitions. This point of view is explained in detail in [15]. Looking up the tables of eigenvalues of the disk and the corresponding eigenfunctions (see [5], [14] or Appendix A) reveals that

$$\lambda_1(\mathbf{D}) = \lambda_1^{DN}(\mathbf{D}_+) < \lambda_2(\mathbf{D}) = \lambda_2^{DN}(\mathbf{D}_+) < \lambda_3^{DN}(\mathbf{D}_+) = \lambda_4(\mathbf{D}).$$

According to Theorem 2.3, we have

$$\lambda_3^{DN}(\mathbf{D}_+) = \mathfrak{L}_3^{DN}(\mathbf{D}_+) = L_3^{DN}(\mathbf{D}_+).$$

But by inspection cases (b) and (c) lead to three domains in \mathbf{D}_+ and then

$$\lambda_3^{DN}(\mathbf{D}_+) \geq \mathfrak{L}_3(\mathbf{D}).$$

Since $\lambda_3^{DN}(\mathbf{D}_+) = \lambda_4(\mathbf{D})$, we have hence a contradiction. □

Remark 5.1. Notice that Proposition 5.1 holds true for more general symmetric domains like the square (see [5]) or angular sectors (see [6]).

Assume that $\sigma\Omega = \Omega$ and there exists a spectral minimal 3-partition \mathcal{D} of type (b) or (c) such that $\sigma\mathcal{D} = \mathcal{D}$ and

$$\lambda_3^{DN}(\Omega_+) > \Lambda_3(\mathcal{D}),$$

where $\Omega_+ = \Omega \cap \{y > 0\}$ and $\lambda_3^{DN}(\Omega_+)$ is the third eigenvalue of the Laplacian with Dirichlet condition on $(\partial\Omega_+) \cap \{y > 0\}$ and Neumann condition on $(\partial\Omega_+) \cap \{y = 0\}$. Then the singular points are on the symmetry axis.

Another result is

Lemma 5.2. *The origin cannot belong to $N(\mathcal{D})$ for a minimal 3-partition \mathcal{D} of type (b).*

Proof. If $0 \in N(\mathcal{D})$, then the double covering $\dot{\mathbf{D}}^{\mathcal{R}}$ have a nodal pattern with six domains. This is excluded by the proof of Proposition 1.1 and the "Courant-sharp" result. \square

Remark 5.2. A natural question is what we obtain when $\{0\}$ is replaced by X ? We get a G -symmetric 6-partition on the double-covering. This leads to $\Lambda_3(\mathcal{D}) \geq \lambda_6(\dot{\Omega}_X^{\mathcal{R}})$. As we can see from the numerics, this does not lead to any conclusion except a weak lower bound for the energy of a partition of type (b).

6. Aharonov-Bohm operators and symmetry

6.1. Aharonov-Bohm operators

This material appears already in [13] in the case with holes (see [1] for the punctured case) and then is used intensively in [3, 4, 16, 6]. If Ω is an open set such that $0 \in \Omega$, a possibility is to consider the Aharonov-Bohm Laplacian in the punctured $\dot{\Omega} = \Omega \setminus \{0\}$, with the singularity of the potential at the center and normalized³ flux Φ . The magnetic potential with flux Φ takes the form

$$\mathbf{A}(x, y) = (A_1(x, y), A_2(x, y)) = \Phi \left(-\frac{y}{r^2}, \frac{x}{r^2} \right). \tag{6.1}$$

We know that the magnetic field vanishes identically in $\dot{\Omega}$ and, in any cut domain (such that it becomes simply connected), one has

$$A_1 dx + A_2 dy = \Phi d\theta, \tag{6.2}$$

where

$$z = x + iy = r e^{i\theta}. \tag{6.3}$$

³This is the flux divided by 2π , i.e. $\frac{1}{2\pi} \int_{\gamma} A$ for a simple path γ turning anticlockwise around 0.

So the Aharonov-Bohm operator in any open set $\dot{\Omega} \subset \mathbb{R}^2 \setminus \{0\}$ will always be defined by considering the Friedrichs extension starting from $C_0^\infty(\dot{\Omega})$ and the associated differential operator is

$$H_{\{0\}}^{AB} := (D_x - A_1)^2 + (D_y - A_2)^2. \quad (6.4)$$

From now on, we will assume that $\Phi = \frac{1}{2}$.

In polar coordinates, the Aharonov-Bohm Laplacian reads :

$$H_{\{0\}}^{AB} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(i\partial_\theta + \frac{1}{2} \right)^2. \quad (6.5)$$

This operator is preserving “real” functions in some modified sense. Following [13], we will say that a function u is K -real, if it satisfies

$$Ku = u, \quad (6.6)$$

where K is an antilinear operator in the form

$$K = e^{i\theta} \Gamma, \quad (6.7)$$

and where Γ is the complex conjugation

$$\Gamma u = \bar{u}. \quad (6.8)$$

The fact that $H_{\{0\}}^{AB}$ preserves K -real eigenfunctions is an immediate consequence of

$$K \circ (H_{\{0\}}^{AB}) = (H_{\{0\}}^{AB}) \circ K. \quad (6.9)$$

As observed in [13], it is easy to find a basis of K -real eigenfunctions. These eigenfunctions (which can be identified with real antisymmetric eigenfunctions of the Laplacian on the double covering $\dot{\Omega}^{\mathcal{R}}$ of $\dot{\Omega}$) have a nice nodal structure (which is locally in the covering the same as the nodal set of real eigenfunctions of the Laplacian), with the specific property that the number of lines in $\dot{\Omega}$ ending at the origin is odd. More generally a path of index one around the origin should always meets an odd number of nodal lines.

All what we have done at the center can be done at any point $X \in \Omega$. We can define H_X^{AB} and K_X .

6.2. Quantification of the symmetry of the disk

Here we follow [4] (see also [3]). We consider now (more generally than the disk) a domain Ω which has the symmetry σ , defined in (5.1), with respect to $\{y = 0\}$. For simplicity, we assume that Ω is convex and write

$$\Omega \cap \{y = 0\} =]a, b[\times \{0\}.$$

We define

$$\Sigma u(x, y) = u(x, -y), \tag{6.10}$$

For any $X \in]a, b[\times \{0\}$ ($X = (x, 0)$), we observe that the Aharonov-Bohm operator H_X^{AB} does not commute with Σ but with the antilinear operator

$$\Sigma^c := \Gamma \Sigma. \tag{6.11}$$

So if u is an eigenfunction of H_X^{AB} , $\Sigma^c u$ is also an eigenfunction. Moreover, due to the commutation

$$K_X \circ \Sigma^c = \Sigma^c \circ K_X, \tag{6.12}$$

$\Sigma^c u$ is also a K_X -real eigenfunction if u is a K_X -real eigenfunction. We now show as observed in [13], we can reduce the analysis of the Aharonov-Bohm Hamiltonian to the K -real space L_K^2 where

$$L_K^2(\dot{\Omega}_X) = \{u \in L^2(\dot{\Omega}_X), K_X u = u\}.$$

The scalar product on L_K^2 , making of L_K^2 a real Hilbert space, is obtained by restricting the scalar product on $L^2(\Omega)$ to L_K^2 and it is immediate to verify that $\langle u, v \rangle$ is indeed real for u and v in L_K^2 .

Observing now that

$$\Sigma^c \circ \Sigma^c = I, \tag{6.13}$$

we obtain by writing

$$u = \frac{1}{2}(I + \Sigma^c)u + \frac{1}{2}(I - \Sigma^c)u,$$

an orthogonal decomposition of L_K^2 into

$$L_K^2 = L_{K,\Sigma}^2 \oplus L_{K,a\Sigma}^2, \tag{6.14}$$

where

$$L_{K,\Sigma}^2 = \{u \in L_K^2, \Sigma^c u = u\}, \quad \text{and} \quad L_{K,a\Sigma}^2 = \{u \in L_K^2, \Sigma^c u = -u\}.$$

We have just to show that the restriction Π of $\frac{1}{2}(I + \Sigma^c)$ to L_K^2

$$\Pi := \frac{1}{2}(I + \Sigma^c)_{/L_K^2}, \tag{6.15}$$

is a projector. It is indeed clear that Π is (\mathbb{R} -)linear and that $\Pi^2 = \Pi$.

Now we would like to analyze the nodal patterns of eigenfunctions in the various symmetry spaces.

Lemma 6.1. *If $u \in C^\infty(\dot{\Omega}_X) \cap L_{K,\Sigma}^2$ is an eigenfunction of H_X^{AB} with $X = (x, 0)$, then its nodal set contains $[a, x] \times \{0\}$. Similarly, if $u \in C^\infty(\dot{\Omega}_X) \cap L_{K,a\Sigma}^2$ is an eigenfunction of H_X^{AB} , then its nodal set contains $[x, b] \times \{0\}$.*

6.3. Two critical points

We consider configuration of types (b) or (c) such that the interior critical points X_0 and X_1 are on a circle. We consider coordinates such that $X_0 = (x_0, y_0)$ and $X_1 = (x_0, -y_0)$. Then the y -axis is a symmetry axis. Let us consider the magnetic potential attached to each singular point:

$$\begin{aligned} \mathbf{A}^+(x, y) &= (A_1^+(x, y), A_2^+(x, y)) \\ &= \Phi \left(-\frac{y - y_0}{r_0^{+2}}, \frac{x - x_0}{r_0^{+2}} \right) \quad \text{with } r_0^{+2} = (y - y_0)^2 + (x - x_0)^2, \\ \mathbf{A}^-(x, y) &= (A_1^-(x, y), A_2^-(x, y)) \\ &= \Phi \left(-\frac{y + y_0}{r_0^{-2}}, \frac{x - x_0}{r_0^{-2}} \right) \quad \text{with } r_0^{-2} = (y + y_0)^2 + (x - x_0)^2. \end{aligned}$$

We have

$$A_1^-(x, -y) = -A_1^+(x, y), \quad A_2^-(x, -y) = A_2^+(x, y).$$

We define $\tilde{\Omega} = \Omega \setminus \{X_0, X_1\}$ and consider the Aharonov-Bohm operator Laplacian in $\tilde{\Omega}$ with the magnetic potential

$$\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^- = (A_1, A_2),$$

with

$$A_1(x, -y) = -A_1(x, y), \quad A_2(x, -y) = A_2(x, y).$$

Proposition 6.1. *Let \mathcal{D} be a 3-partition of type (b) or (c) such that the two interior critical points on a circle, but not diametrically opposite. If the third eigenvalue of the Aharonov-Bohm Laplacian on $\tilde{\Omega}$ is simple, then \mathcal{D} can not be minimal.*

Proof. After a rotation, we are reduced to the two situations considered in Proposition 5.1. The assumption of simplicity implies that the eigenfunction is either Σ^c -symmetric or Σ^c -antisymmetric. \square

7. Proof of Proposition 1.4

Under the assumption that the minimal partition is of type (a), we can use the magnetic characterization of a minimal-partition [16] and obtain that this minimal 3-partition is the nodal partition of some K -real eigenfunction of the Aharonov-Bohm Hamiltonian with a pole as the critical point. More concretely in our specific case, if $\mathcal{D} = (D_1, D_2, D_3)$ denotes the 3-partition and if u_1, u_2, u_3 denote the normalized positive groundstates of the Dirichlet-Laplacian in D_1, D_2, D_3 respectively, it is shown in [17] that one can find

positive α_i such that $\alpha_i u_i - \alpha_j u_j$ becomes a second eigenfunction on $D_{ij} = \text{Int}(\overline{D_i \cup D_j})$ for any (i, j) with $i \neq j$. We can now see that the function defined in each D_i by $(-1)^i \alpha_i u_i e^{i\frac{\theta}{2}}$ (where for defining $e^{i\frac{\theta}{2}}$ we take a cut say on $\partial D_1 \cap \partial D_2$) extends to an eigenfunction of the Aharonov-Bohm operator. Moreover, extending the proof of [17], one can show that it is actually the third eigenvalue of this Aharonov-Bohm operator. There is an alternative version using the double covering, and we find in this case that it is the nodal partition (with 6 nodal domains) of a real eigenfunction of the Laplacian which is antisymmetric with respect to the desk transformation.

Using the invariance by rotation of the disk we can without loss of generality assume that the critical point is on $\{y = 0\}$.

We now exploit the symmetry with respect to σ of the disk and the existence of a quantization Σ^c of this symmetry by Σ which commutes with the Aharonov-Bohm Laplacian and respects the K -real eigenfunctions. For each eigenvalue, we can decompose (see Subsection 6.2 and (6.14) for the definition of the invariant spaces) the corresponding K -real eigenspace in a orthogonal direct sum of one Σ^c -invariant space and of one $a\Sigma^c$ -invariant space. We have discussed in [3] (see also [4]) the implication of this Σ^c -symmetry or Σ^c -antisymmetry on the nodal sets of the eigenfunction.

By identification to a Dirichlet-Neumann problem, it can be observed that the Σ^c -symmetric eigenvalues (resp. antisymmetric) are monotonic increasing (resp. decreasing) as function of the pole X on $[0, 1] \times \{0\}$ (see Figures 7-9).

Moreover by Hillairet-Lena-Norris-Terracini result [19], we know that the k -th eigenvalue tends to the k -th eigenvalue of the Dirichlet-Laplacian of the disk as the pole tends to $(x = 1, y = 0)$.

In addition, if the eigenvalue is simple, then a K_X -real eigenfunction is either Σ^c -symmetric or Σ^c -antisymmetric.

Many of these considerations appear in [3] in the case of the square but the situation is finally much simpler in the case of the disk due to this reduction to the case where the pole is on one axis, which makes the problem (1D) for the position of the pole.

We now use some numerical simulations realized with the finite element Library MÉLINA (see [20]) : we consider a critical point $X = (x, 0)$ and mesh the double covering $\dot{\mathbf{D}}_X^{\mathcal{R}}$ in this way : we mesh the first sheet \mathbf{D} such that X is a vertex and the segment $[x, 1] \times \{0\}$ is at the boundary of some elements of the mesh. We duplicate this mesh to obtain a mesh for the second sheet. We have then to exchange the vertex along the segment $[x, 1] \times \{0\}$ between the first and second sheet to obtain a mesh of the double covering $\dot{\mathbf{D}}_X^{\mathcal{R}}$ (see Figure 6 and [3] for more details). Moving the pole, we compute the eigenmodes of the Dirichlet-Laplacian on the double covering $\dot{\mathbf{D}}_X^{\mathcal{R}}$. Figure 7(a) gives the eigenvalues on the double covering whereas we have extracted the Aharonov-Bohm eigenvalues in Figure 7(b). Figures 8-9

represent the first eigenfunctions of the Dirichlet-Laplacian and their nodal lines for $\mathbf{X} = (k/5, 0)$, $0 \leq k < 5$.

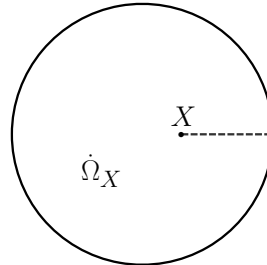
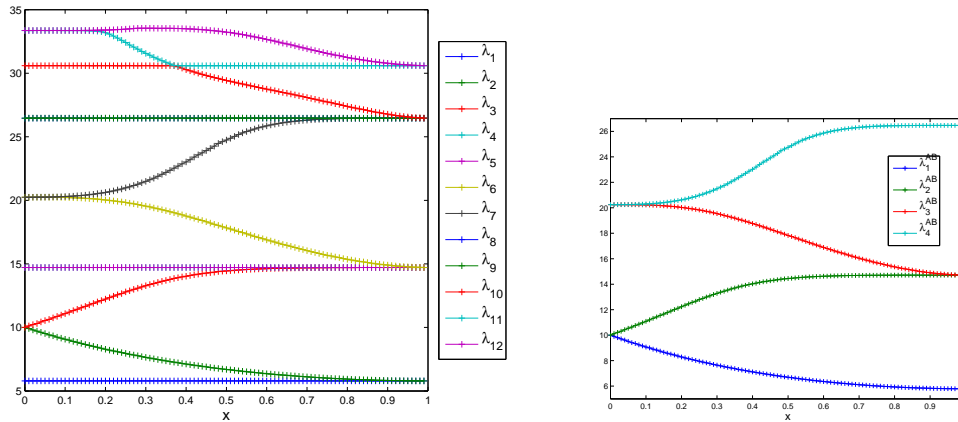


Figure 6. $\dot{\mathbf{D}}_X^{\mathcal{R}}$, $\mathbf{X} = (x, 0)$.



(a) First eigenvalues of the Dirichlet-Laplacian on $\dot{\mathbf{D}}_X^{\mathcal{R}}$, $\mathbf{X} = (x, 0)$ vs. $x \in \{\frac{k}{100}, 0 \leq k \leq 99\}$.

(b) First four eigenvalues of H_X^{AB} , $\mathbf{X} = (x, 0)$ according to x .

Figure 7: First eigenvalues of the H_X^{AB} in $\dot{\mathbf{D}}_X$ or Laplacian on $\dot{\mathbf{D}}_X^{\mathcal{R}}$, $\mathbf{X} = (x, 0)$.

The numerics is used now for two points.

The first point is to verify that the four first eigenvalues of the Aharonov-Bohm operator (see Figure 7(b), corresponding to λ_2 , λ_3 , λ_6 and λ_7 on the covering, see Figure 7(a)) are simple except if the pole is at the center of the disk. Figure 7(a) gives the first eigenvalues on the double covering of the disk $\dot{\mathbf{D}}_X^{\mathcal{R}}$ according to abscissa of the puncturing point $\mathbf{X} = (x, 0)$ (it is enough to look at $x \leq 0$ by the rotation invariance) and Figures 8-9 represent the nodal partitions of the first eigenfunctions for several poles $\mathbf{X} = (k/5, 0)$, $0 \leq k < 5$. We also see that these Aharonov-Bohm eigenfunctions belong alternately to the Σ^c -symmetric or the Σ^c -antisymmetric spectrum, crossing only when the pole is at the center. Hence the third eigenfunction should be

Σ^c -symmetric (if $x < 0$). This implies the symmetry of the partition except at the center. Moreover the third eigenvalue has multiplicity 2 at the center. Then we can follow the proof of Lemma 5.1 to have the conclusion in Case 2 and we can eliminate Case 1 using the Σ^c -symmetry of the eigenfunction, without referring to Proposition 1.1.

The second point is to observe numerically that the cardinal of the nodal domains of the third eigenfunction is two except at the center. We recover the previous conclusion in a less rigorous way but obtain a description of the deformation of the nodal lines.

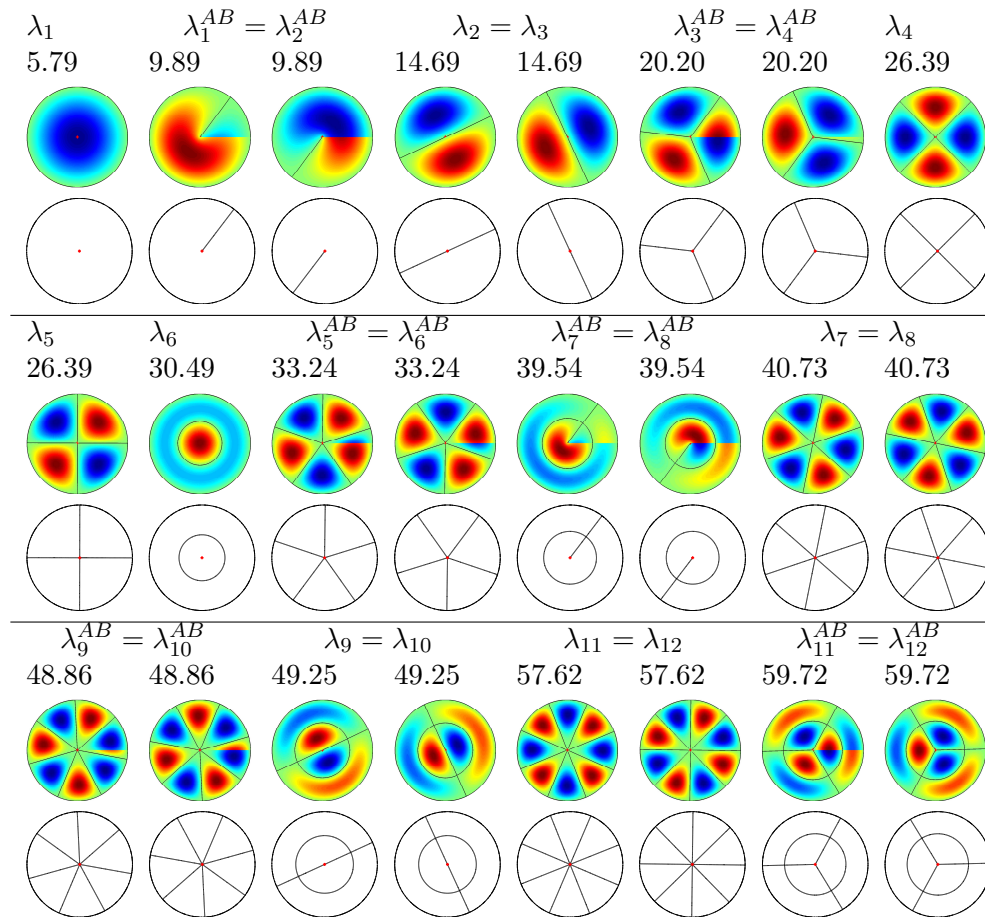


Figure 8: Nodal sets, on the first sheet, of the first 24 Laplacian eigenfunctions in $\dot{\mathbf{D}}_0^{\mathcal{R}}$.

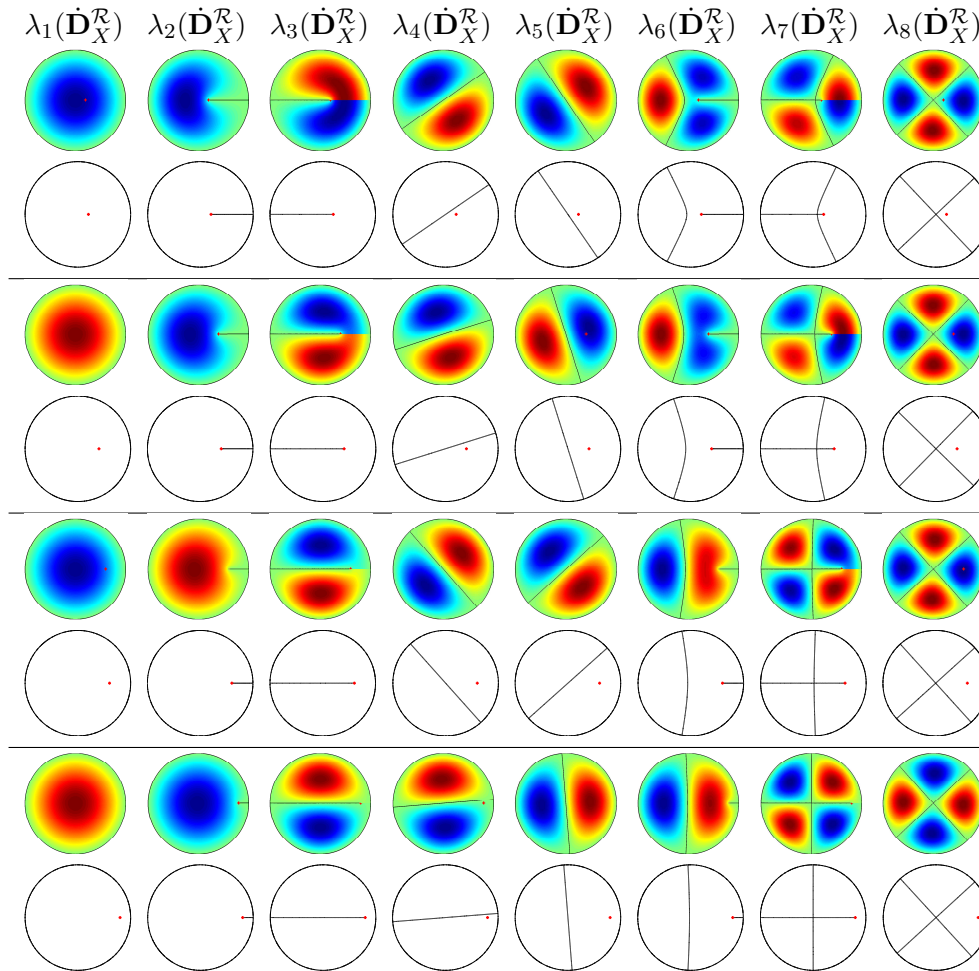


Figure 9: Nodal sets, on the first sheet, of the first eight Dirichlet-Laplacian eigenfunctions on the double covering $\dot{\mathbf{D}}_X^{\mathcal{R}}$, $\mathbf{X} = (x, 0)$, $x = 0.2, 0.4, 0.6, 0.8$.

Acknowledgments

The second author would like to thank Thomas Hoffmann-Ostenhof for enlightening discussions. This work was partially supported by the ANR (Agence Nationale de la Recherche), projects GAOS n° ANR-09-BLAN-0037-03 and OPTIFORM n° ANR-12-BS01-0007-02. The authors are grateful to the Mittag-Leffler Institute for the very good working conditions and also for the fruitful discussions with other participants during the semester research program *Hamiltonians in Magnetic Fields*.

A. Nodal lines of the first eigenfunctions on the disk

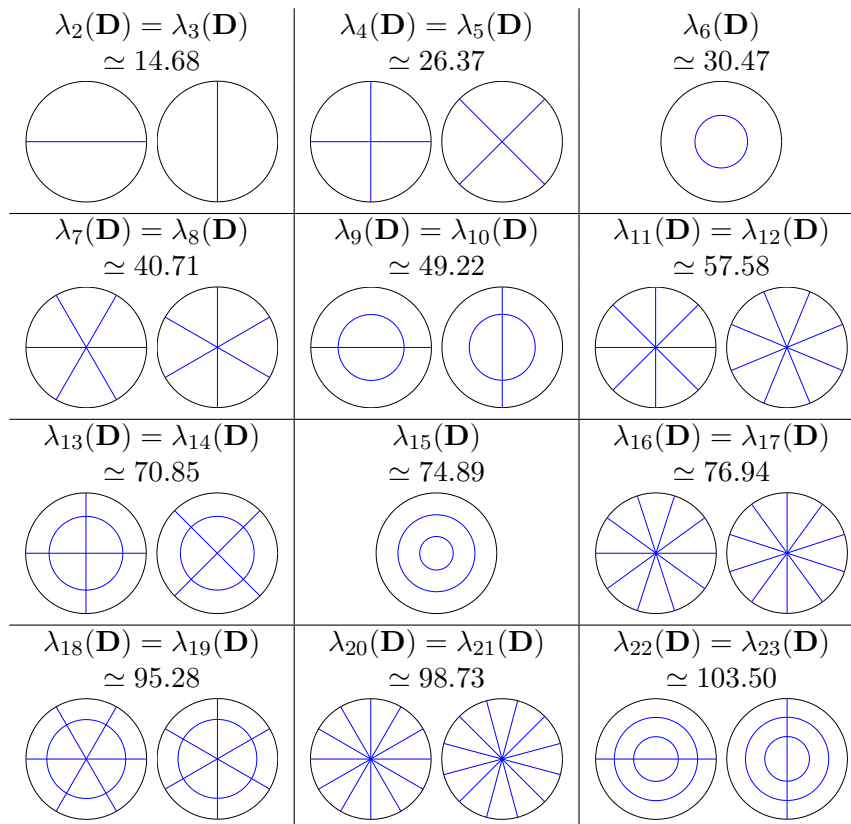


Figure 10: Nodal partitions associated with the k -th eigenvalue of the Dirichlet-Laplacian on the disk, $k = 2, \dots, 23$.

References

- [1] B. ALZIARY, J. FLECKINGER and P. TAKAC, Eigenfunctions and Hardy inequalities for a magnetic Schrödinger operator in \mathbb{R}^2 , *Math. Methods Appl. Sci.*, **26**, 13 (2003), 1093-1136.
- [2] A. ANCONA, B. HELFFER and T. HOFFMANN-OSTENHOF, Nodal domain theorems à la Courant, *Doc. Math.*, **9**, (2004), 283-299 (electronic).
- [3] V. BONNAILLIE-NOËL and B. HELFFER, Numerical analysis of nodal sets for eigenvalues of Aharonov-Bohm Hamiltonians on the square and application to minimal partitions, *Exp. Math.*, **20**, 3 (2011), 304-322.
- [4] V. BONNAILLIE-NOËL, B. HELFFER and T. HOFFMANN-OSTENHOF, Aharonov-Bohm Hamiltonians, isospectrality and minimal partitions, *J. Phys. A*, **42**, 18 (2009), 185203, 20.

- [5] V. BONNAILLIE-NOËL, B. HELFFER and G. VIAL, Numerical simulations for nodal domains and spectral minimal partitions, *ESAIM Control Optim. Calc. Var.*, **16**, 1 (2010), 221-246.
- [6] V. BONNAILLIE-NOËL and C. LÉNA, Spectral minimal partitions of a sector, *Preprint Mittag-Leffler*, **16**, (2012) 26 pages.
- [7] V. BONNAILLIE-NOËL and G. VIAL, Computations for nodal domains and spectral minimal partitions,
<http://w3.bretagne.ens-cachan.fr/math/simulations/MinimalPartitions> (2007).
- [8] D. BUCUR, G. BUTTAZZO and A. HENROT, Existence results for some optimal partition problems, *Adv. Math. Sci. Appl.*, **8**, 2 (1998), 571-579.
- [9] L. A. CAFFERELLI and F. H. LIN, An optimal partition problem for eigenvalues, *J. Sci. Comput.*, **31**, 1-2 (2007), 5-18.
- [10] M. CONTI, S. TERRACINI and G. VERZINI, An optimal partition problem related to nonlinear eigenvalues, *Journal of Functional Analysis*, **198** (2003), 160-196.
- [11] M. CONTI, S. TERRACINI and G. VERZINI, On a class of optimal partition problems related to the Fucik spectrum and to the monotonicity formula, *Calculus of Variations*, **22** (2005), 45-72.
- [12] M. CONTI, S. TERRACINI and G. VERZINI, A variational problem for the spatial segregation of reaction-diffusion systems, *Indiana University Mathematics Journal*, **54** (2005), 779-816.
- [13] B. HELFFER, M. HOFFMANN-OSTENHOF, T. HOFFMANN-OSTENHOF and M. P. OWEN, Nodal sets for groundstates of Schrödinger operators with zero magnetic field in non-simply connected domains, *Comm. Math. Phys.*, **202**, 3 (1999), 629-649.
- [14] B. HELFFER and T. HOFFMANN-OSTENHOF, On spectral minimal partitions : new properties and applications to the disk, *Proceedings of the conference in Montreal. CRM Lectures*, **52**, pp. 119-116, 2010.
- [15] B. HELFFER and T. HOFFMANN-OSTENHOF, *Minimal 3-partitions for a thin strip on a cylinder or a thin annulus like domain with Neumann condition*, Proceedings OTAMP 2010 in Operator Theory: Advances and Applications, **227**, 107-115, 2012.
- [16] B. HELFFER and T. HOFFMANN-OSTENHOF, On a magnetic characterization of spectral minimal partitions, *Journal of the European Math. Society*, **To appear** (2013).
- [17] B. HELFFER, T. HOFFMANN-OSTENHOF and S. TERRACINI, Nodal domains and spectral minimal partitions, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **26** (2009), 101-138.
- [18] A. HENROT, *Extremum Problems for Eigenvalues of Elliptic Operators*, Birkhäuser, Basel, Boston, Berlin, 2006.
- [19] C. LÉNA, *In preparation*, PhD thesis, Université Paris Sud 11, 2013.
- [20] D. MARTIN, MÉLINA, bibliothèque de calculs éléments finis,
<http://perso.univ-rennes1.fr/daniel.martin/melina> (2007).

Virginie Bonnaillie-Noël

IRMAR, ENS Cachan Bretagne, Univ. Rennes 1, CNRS, UEB
 av. Robert Schuman, F-35170 Bruz, France
 E-mail: virginie.bonnaillie@bretagne.ens-cachan.fr

Bernard Helffer

Département de Mathématiques, Université Paris-Sud
 F-91405 Orsay Cedex, France
 E-mail: bernard.helffer@math.u-psud.fr