Peak power in the 3D magnetic Schrödinger equation

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Abstract

This paper is devoted to the spectral analysis of the magnetic Neumann Laplacian on an infinite cone of aperture α. When the magnetic field is constant and parallel to the revolution axis and when the aperture goes to zero, we prove that the first n eigenvalues exist and admit asymptotic expansions in powers of α^2.

Keywords. Magnetic Laplacian, singular 3D domain, spectral asymptotics.

MSC classification. 35P15, 35J10, 81Q10, 81Q15.

1 Introduction

1.1 Presentation of the problem

We are interested in the low-lying eigenvalues of the magnetic Neumann Laplacian with a constant magnetic field applied to a " peak ", i.e. a right circular cone $\mathcal{C}_\alpha$, along to its symmetry axis.
The right circular cone $\mathcal{C}_\alpha$ of angular opening $\alpha \in (0,\pi)$ (see Figure 1) is defined in the cartesian coordinates (x,y,z) by

$$\mathcal{C}_\alpha = \{(x,y,z)\in\mathbb{R}^3,\ z>0,\ x^2+y^2<z^2\tan^2\tfrac{\alpha}{2}\}.$$

Let $\mathbf{B}$ be the constant magnetic field

$$\mathbf{B}(x,y,z) = (0,0,1)^{\mathsf{T}}.$$

We choose the following magnetic potential $\mathbf{A}$:

$$\mathbf{A}(x,y,z) = \frac{1}{2}(-y,x,0)^{\mathsf{T}},$$

which is compatible with the axisymmetry. We consider $\mathfrak{L}_\alpha$ the Friedrichs extension associated with the quadratic form

$$Q_{\mathbf{A}}(\psi) = \|(-i\nabla+\mathbf{A})\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)},$$

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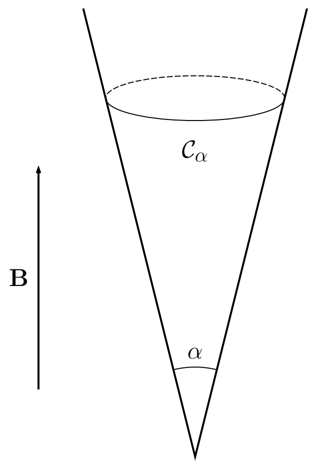


Figure 1: Geometric setting.

defined for $\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha)$ with

$$\mathrm{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha) = \{u \in \mathrm{L}^2(\mathcal{C}_\alpha), (-i\nabla + \mathbf{A})u \in \mathrm{L}^2(\mathcal{C}_\alpha)\}.$$

The operator $\mathfrak{L}_\alpha$ is $(-i\nabla + \mathbf{A})^2$ with domain:

$$\mathrm{H}^2_{\mathbf{A}}(\mathcal{C}_\alpha) = \{u \in \mathrm{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha), (-i\nabla + \mathbf{A})^2 u \in \mathrm{L}^2(\mathcal{C}_\alpha), (-i\nabla + \mathbf{A})u \cdot \nu = 0 \text{ on } \partial\mathcal{C}_\alpha\}.$$

We define the n-th eigenvalue $\lambda_n(\alpha)$ of $\mathfrak{L}_\alpha$ by using Rayleigh quotients:

$$\lambda_n(\alpha) = \sup_{\Psi_1,\dots,\Psi_{n-1} \in \mathrm{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha)} \inf_{\substack{\Psi \in [\Psi_1,\dots,\Psi_{n-1}]^\perp \\ \Psi \in \mathrm{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha),\ \|\Psi\|_{\mathrm{L}^2(\mathcal{C}_\alpha)}=1}} Q_{\mathbf{A}}(\Psi) = \inf_{\Psi_1,\dots,\Psi_n \in \mathrm{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha)} \sup_{\substack{\Psi \in [\Psi_1,\dots,\Psi_n] \\ \|\Psi\|_{\mathrm{L}^2(\mathcal{C}_\alpha)}=1}} Q_{\mathbf{A}}(\Psi). \tag{1.1}$$

Let $\psi_n(\alpha)$ be a normalized associated eigenvector (if it exists).

Remark 1.1 *In the constant magnetic field case, due to the dilation invariance of the cone and to the scaling $\mathsf{x} = b^{-1/2}\mathsf{X}$, the operator $(-i\nabla_{\mathsf{x}} + b\mathbf{A}(\mathsf{x}))^2$ with $b > 0$ is unitarily equivalent to $b(-i\nabla_{\mathsf{X}} + \mathbf{A}(\mathsf{X}))^2$.*

1.2 Motivation

Let us describe the motivation of this paper. The main motivation comes from the theory of superconductivity where the linearization of the Ginzburg-Landau leads to the study of the magnetic Laplacian. It is well-known (see [9]) that an applied magnetic field strong enough makes superconductivity break down. This critical value of the magnetic field[1] above which superconductivity disappears is directly related to the lowest eigenvalue of $(-i\nabla + \mathbf{A})^2$ (see [12], [7, Proposition 1.9], [5, Theorem 1.4] for example). The spectral study of the magnetic Laplacian has given rise to numerous investigations in the last fifteen years, in particular in the strong magnetic field limit i.e. when one considers $(-i\nabla + b\mathbf{A})^2$ with large b (for non smooth domains, see [3, 8, 17]). One of the most interesting results is provided by Helffer and Morame in [10] where they prove that superconductivity, in 2D, concentrates near the points of the boundary where the (algebraic) curvature is maximal.

[1]This critical value, denoted by H_{C_3}, is called "third critical field of the Ginzburg-Landau functional".

This nice property aroused interest in domains with corners, which somehow correspond to points of the boundary where the curvature becomes infinite (see [11, 14] for the quarter plane and [2, 3] for more general domains). Denoting by $\mathcal{S}_\alpha$ the sector in $\mathbb{R}^2$ with angle α and considering the magnetic Neumann Laplacian with constant magnetic field of intensity 1, it is proved in [2] that, as soon as α is small enough, a bound state exists. Its energy is denoted by $\mu(\alpha)$. An asymptotic expansion at any order is even provided (see [2, Theorem 1.1]):

$$\mu(\alpha) \sim \alpha \sum_{j\geq 0} m_j \alpha^{2j}, \qquad \text{with} \quad m_0 = \frac{1}{\sqrt{3}}. \tag{1.2}$$

In particular, this proves that $\mu(\alpha)$ becomes smaller than the lowest eigenvalue, denoted by Θ_0, of the magnetic Neumann Laplacian in the half-plane with constant magnetic field (of intensity 1). An important consequence is that the third critical field is larger when there are corners than in the regular boundary case (see [5]). As already mentioned, this result only concerns dimension 2. Nevertheless the case of the 2D sector can be used to describe the infinite wedge with magnetic field parallel to the edge. This motivates the study of dihedral domains (see [16]). Another possibility of investigation in 3D, with which the present paper is concerned, is the case of a conical singularity of the boundary (and, for sake of simplicity, with a magnetic field parallel to the cone axis). We would especially like to answer the following questions: Can we go below $\mu(\alpha)$ and can we describe the structure of the spectrum when the aperture of the cone goes to zero ?

1.3 The magnetic Laplacian in spherical coordinates

Due to the geometry setting, it is natural to deal with the spherical coordinates which are combined with a dilation:

$$\Phi(t,\theta,\varphi) := (x,y,z) = \alpha^{-1/2}(t\cos\theta\sin\alpha\varphi,\ t\sin\theta\sin\alpha\varphi,\ t\cos\alpha\varphi).$$

We denote by $\mathcal{P}$ the semi-infinite rectangular parallelepiped

$$\mathcal{P} := \{(t,\theta,\varphi)\in\mathbb{R}^3,\ t>0,\ \theta\in[0,2\pi),\ \varphi\in(0,\tfrac{1}{2})\}.$$

Let $\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha)$. We write $\psi(\Phi(t,\theta,\varphi)) = \alpha^{1/4}\tilde{\psi}(t,\theta,\varphi)$ for any $(t,\theta,\varphi)\in\mathcal{P}$ in these new coordinates and then, using Appendix A, we have

$$\|\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)} = \int_{\mathcal{P}} |\tilde{\psi}(t,\theta,\varphi)|^2\, t^2 \sin\alpha\varphi \,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

and:

$$Q_{\mathbf{A}}(\psi) = \alpha Q_\alpha(\tilde{\psi}),$$

where the quadratic form Q_α is defined on the form domain $\mathrm{H}^1_{\tilde{\mathbf{A}}}(\mathcal{P})$ by

$$Q_\alpha(\psi) := \int_{\mathcal{P}} \left(|\partial_t\psi|^2 + \frac{1}{t^2\sin^2\alpha\varphi}\left|\left(-i\partial_\theta + \frac{t^2\sin^2\alpha\varphi}{2\alpha}\right)\psi\right|^2 + \frac{1}{\alpha^2 t^2}|\partial_\varphi\psi|^2 \right) \mathrm{d}\tilde{\mu}, \tag{1.3}$$

with the measure

$$\mathrm{d}\tilde{\mu} = t^2 \sin\alpha\varphi\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

and, using (A.1),

$$\mathrm{H}^1_{\tilde{\mathbf{A}}}(\mathcal{P}) = \{\psi\in\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu}), (-i\nabla+\tilde{\mathbf{A}})\psi\in\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})\}.$$

We consider $\mathcal{L}_\alpha$ the Friedrichs extension associated with the quadratic form Q_α. We define the n-th eigenvalue $\tilde{\lambda}_n(\alpha)$ of $\mathcal{L}_\alpha$ by using the Rayleigh quotients as in (1.1) and $\tilde{\psi}_n(\alpha)$ a normalized associated eigenvector if it exists. We have

$$\lambda_n(\alpha) = \alpha\tilde{\lambda}_n(\alpha), \qquad \psi_n(\alpha)(x,y,z) = \tilde{\psi}_n(\alpha)(t,\theta,\varphi).$$

1.4 Main result

In this paper we aim at estimating the discrete spectrum, if it exists, of $\mathfrak{L}_\alpha$. For that purpose, we shall first determine the bottom of its essential spectrum. From Persson's characterization of the infimum of the essential spectrum, it is enough to consider the behavior at infinity. Far away from the origin, the magnetic field makes an angle $\alpha/2$ with the boundary of the cone so that we can compare with a half-space model and deduce the following proposition (see Section 2).

Proposition 1.2 *Let us denote by $\mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_\alpha)$ the essential spectrum of $\mathfrak{L}_\alpha$. We have:*

$$\mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_\alpha) = \left[\sigma\left(\tfrac{\alpha}{2}\right), +\infty\right),$$

where $\sigma(\theta)$ is the bottom of the spectrum of the Schrödinger operator with constant magnetic field $\mathbf{B} = (0, \cos\theta, \sin\theta)$ which makes the angle θ with the boundary of the half-space $\mathbb{R}^3_+$ (see Section 2.1).

At this stage we have still not proved that discrete spectrum exists. As it is the case in 2D (see [2]) or in the case on infinite wedge (see [16]), there is hope to prove such an existence in the limit $\alpha \to 0$.

Philosophy of the investigation Let us explain the structure of our analysis. The first natural step to perform the investigation of the discrete spectrum is the introduction of appropriate quasi-eigenpairs[2] whose energy is below the essential spectrum. Then we have to prove that the constructed quasi-eigenpairs exactly describe the lowest eigenvalues. This is in fact the most delicate part of the analysis. As often in the study of the magnetic operator, the spectral behavior is deeply related to localization and microlocalization properties of the eigenfunctions. The localization estimates are standardly given by the so-called Agmon estimates, whereas the microlocal behavior is more subtle to investigate. In order to succeed, the key point is to introduce a system of coordinates which is compatible with the geometry of the magnetic field. Here our initial choice of gauge and the spherical coordinates play this role. In the present situation, the phase variable that we should understand is the dual variable of θ given by a Fourier series decomposition and denoted by $m \in \mathbb{Z}$. In other words, we realize a Fourier decomposition of $\mathcal{L}_\alpha$ with respect to θ and we introduce the family of 2D-operators $(\mathcal{L}_{\alpha,m})_{m\in\mathbb{Z}}$ acting on $\mathrm{L}^2(\mathcal{R}, \mathrm{d}\mu)$:

$$\mathcal{L}_{\alpha,m} = -\frac{1}{t^2}\partial_t t^2 \partial_t + \frac{1}{t^2\sin^2(\alpha\varphi)}\left(m + \frac{\sin^2(\alpha\varphi)}{2\alpha}t^2\right)^2 - \frac{1}{\alpha^2\, t^2 \sin(\alpha\varphi)}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi,$$

with

$$\mathcal{R} = \{(t,\varphi) \in \mathbb{R}^2,\ t > 0,\ \varphi \in (0, \tfrac{1}{2})\},$$

and

$$\mathrm{d}\mu = t^2 \sin(\alpha\varphi)\,\mathrm{d}t\,\mathrm{d}\varphi.$$

[2]By "eigenpair" we mean a pair (λ, ψ) where λ is an eigenvalue and ψ a corresponding eigenfunction.

We denote $Q_{\alpha,m}$ the quadratic form associated with $\mathcal{L}_{\alpha,m}$. This normal form is also the suitable form to construct quasimodes. Then an integrability argument proves that the eigenfunctions are microlocalized in $m = 0$, i.e. they are axisymmetric. This allows a reduction of dimension. It remains to notice that the last term in $\mathcal{L}_{\alpha,0}$ is penalized by α^{-2} so that the Feshbach-Grushin projection on the groundstate of $-\alpha^{-2}(\sin(\alpha\varphi))^{-1}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi$ (the constant function) acts as an approximation of the identity on the eigenfunctions. In other words the spectrum of $\mathcal{L}_{\alpha,0}$ is described modulo lower order terms by the spectrum of the average of $\mathcal{L}_{\alpha,0}$ with respect to φ.

Organization of the paper and main result Let us now explain the scheme of our investigation. We will construct quasimodes (independent from θ) for the operator $\mathcal{L}_\alpha$ by using an asymptotic expansion in α^2 of $\mathcal{L}_{\alpha,0}$ as explained in Section 3 (see Proposition 3.1). Using these quasimodes, we will prove that there exist eigenvalues below the essential spectrum for angles small enough.

The main part of the analysis is to prove that the quasi-eigenpairs constructed in the proof of Proposition 3.1 exactly give asymptotic expansion of the eigenpairs. As a first step, using the comparison between the bottom of the essential spectrum given in Proposition 1.2 and the upper-bound of the n-th eigenvalue established in Corollary 3.4, we prove in Section 4 a rough localization of the eigenfunctions with respect to z when α is small enough (see Proposition 4.1). In a second step (see Section 5), we use the rough space estimates to prove that the operators $\mathcal{L}_{\alpha,m}$ with $m \neq 0$ can not contribute for the low-lying eigenvalues i.e. that the first eigenfunctions are axisymmetric. In a last step (see Section 6), we need to establish an accurate estimate of the spectral gap between the eigenvalues through the Feshbach-Grushin method (see Proposition 6.1). Finally, combining Propositions 3.1 and 6.1, we deduce our main result which provides the complete asymptotic expansion for the low-lying eigenpairs of $\mathcal{L}_\alpha$:

Theorem 1.3 *For all $n \geq 1$, there exist $\alpha_0(n) > 0$ and a sequence $(\gamma_{j,n})_{j\geq 0}$ such that, for all $\alpha \in (0, \alpha_0(n))$, the n-th eigenvalue exists and satisfies:*

$$\lambda_n(\alpha) \underset{\alpha\to 0}{\sim} \alpha \sum_{j\geq 0} \gamma_{j,n}\alpha^{2j},$$

with $\gamma_{0,n} = \mathfrak{l}_n = 2^{-5/2}(4n-1)$.

Remark 1.4 *In particular Theorem 1.3 states that $\lambda_1(\alpha) \sim \frac{3}{2^{5/2}}\alpha$. We have $\frac{3}{2^{5/2}} < \frac{1}{\sqrt{3}}$ so that the lowest eigenvalue of the magnetic cone goes below the lowest eigenvalue of the 2D magnetic sector (see (1.2)). In terms of the third critical field H_{C_3} in Ginzburg-Landau theory, this means that H_{C_3} is higher. In other words it is possible to apply a larger magnetic field to the superconducting sample before superconductivity breaks down: This phenomenon motivates our terminology "peak power".*

Remark 1.5 *As a consequence of Theorem 1.3, we deduce that the lowest eigenvalues are simple as soon as α is small enough. Therefore, the spectral theorem implies that the quasimodes (see (3.4)) constructed in the proof of Proposition 3.1 are approximations of the eigenfunctions of $\mathcal{L}_\alpha$. In particular, using the rescaled spherical coordinates, for all $n \geq 1$, there exist $\alpha_n > 0$ and C_n such that, for $\alpha \in (0, \alpha_n)$:*

$$\|\tilde{\psi}_n(\alpha) - \mathfrak{f}_n\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \leq C_n\alpha^2,$$

where $\mathfrak{f}_n$ is defined in Corollary C.2. From the Ginzburg-Landau point of view, this means that superconductivity spreads in the cone at the scale $\alpha^{-1/2}$.

2 Essential spectrum

This section is devoted to the proof of Proposition 1.2.

2.1 Magnetic Laplacian on a half-space

As explained in the introduction, the magnetic field makes an angle $\alpha/2$ with the boundary of the cone. Therefore, it is quite intuitive, using Persson's lemma (see Lemma 2.1), that the Schrödinger operator in $\mathbb{R}^3_+$ with constant magnetic field $\mathbf{B} = (0, \cos\frac{\alpha}{2}, \sin\frac{\alpha}{2})$ will determine the bottom of the essential spectrum. Let us recall some results of [12, 4] concerning this operator. Let $\theta \in (0, \pi/2)$ and P_θ be the Neumann realization on the half-space $\mathbb{R}^3_+ = \{(r,s,t) \in \mathbb{R}^3, t > 0\}$ of

$$D_s^2 + D_t^2 + (D_r - t\cos\theta + s\sin\theta)^2.$$

The bottom of the spectrum, which is essential, is denoted by $\sigma(\theta)$. From [12] (see also [4]), we know that the function $\theta \mapsto \sigma(\theta)$ is analytic and increasing from $(0, \frac{\pi}{2})$ onto $(\Theta_0, 1)$, where the definition of Θ_0 is recalled below Formula (1.2).

2.2 Proof of Proposition 1.2

Let us first recall the Persson's lemma (see [15]) which characterizes the essential spectrum:

Lemma 2.1 *Let Ω be an unbounded domain of $\mathbb{R}^3$ with Lipschitzian boundary. Then the bottom of the essential spectrum of the Neumann realization of the Schrödinger operator $-\Delta_{\mathbf{A}} := (-i\nabla + \mathbf{A})^2$ is given by*

$$\inf \mathsf{sp}_{\mathsf{ess}}(P) = \lim_{R\to\infty} \Sigma(-\Delta_{\mathbf{A}}, R),$$

with

$$\Sigma(-\Delta_{\mathbf{A}}, R) = \inf_{\psi \in \mathcal{C}_0^\infty(\overline{\Omega} \cap \complement\mathcal{B}_R)} \frac{\int_\Omega |(-i\nabla + \mathbf{A})\psi|^2}{\int_\Omega |\psi|^2},$$

where $\mathcal{B}_R$ denotes the ball of radius R centered at the origin and $\complement\mathcal{B}_R = \mathbb{R}^2 \backslash \mathcal{B}_R$.

Lower bound for $\boldsymbol{\Sigma(-\Delta_{\mathbf{A}}, R)}$ Let us first prove a lower bound for $\Sigma(-\Delta_{\mathbf{A}}, R)$ for large R. In order to do this, we introduce a partition of unity $(\chi_j)_j = (\chi_{j,R})_j$ such that:

$$\sum_j \chi_j^2 = 1,$$

and which satisfies, in cartesian coordinates:

$$\mathsf{supp}\,(\chi_j) \subset \mathcal{B}(P_j, R^{-1/4}) \quad \text{and} \quad \sum_j \|\nabla\chi_j\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)} \leq CR^{-1/2}.$$

We can also assume that the balls which intersect the boundary have their centers on it. Let us also fix R such that $R > (\tan\frac{\alpha}{2})^{-4/3}$ (thus any ball centered on the boundary at a point P_j such that $\|P_j\| = R_j > R$ and of radius $R^{1/4}$ does not intersect the cone axis). For $\psi \in \mathcal{C}_0^\infty(\mathcal{C}_\alpha \cap \complement\mathcal{B}_R)$, we want to prove a lower bound for $Q_{\mathbf{A}}(\psi)$. The "IMS" formula gives:

$$Q_{\mathbf{A}}(\psi) = \sum_j Q_{\mathbf{A}}(\chi_j\psi) - \sum_j \|\nabla\chi_j\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}.$$

This implies:

$$Q_{\mathbf{A}}(\psi) \geq \sum_j Q_{\mathbf{A}}(\chi_j \psi) - CR^{-1/2}\|\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}. \tag{2.1}$$

Let us consider j such that $\mathcal{B}(P_j, R^{1/4}) \cap \partial\mathcal{C}_\alpha = \emptyset$. Then, we can extend the function $\chi_j\psi$ by zero to $\mathbb{R}^3$ and by the min-max principle applied to the Schrödinger operator with constant magnetic field equal to 1 in $\mathbb{R}^3$, the following inequality holds:

$$Q_{\mathbf{A}}(\chi_j\psi) \geq \|\chi_j\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}. \tag{2.2}$$

Let us now analyze the other balls and consider j such that $\mathcal{B}(P_j, R^{1/4}) \cap \partial\mathcal{C}_\alpha \neq \emptyset$. For such ball, it is convenient to use the normal coordinates which parametrize $\mathcal{C}_\alpha$ (see Appendix B). Denoting ψ_j the function $\chi_j\psi$ in the normal coordinates $(\rho, \theta, \tau) \in \mathcal{D}_\alpha$, the quadratic form can be written (see (B.1)):

$$Q_{\mathbf{A}}(\chi_j\psi) = \hat{Q}_\alpha(\psi_j) = \int_{\mathcal{D}_\alpha} \left(|\partial_\rho \psi_j|^2 + |\partial_\tau \psi_j|^2 + V_\alpha^{-1} \left| \left(-i\partial_\theta + \frac{V_\alpha}{2} \right) \psi_j \right|^2 \right) \mathrm{d}\hat{\mu},$$

with

$$\mathcal{D}_\alpha = \{(\rho, \theta, \tau) \in \mathbb{R}^3, \rho > 0, \theta \in [0, 2\pi), \tau \in (0, \rho \tan\tfrac{\alpha}{2})\},$$

$$V_\alpha(\rho, \tau) = (\rho \sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2})^2, \qquad \mathrm{d}\hat{\mu} = (\rho\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2})\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\tau.$$

Let us use the translation $\rho = R_j + \tilde{\rho}$ and denote $\tilde{\psi}_j(\tilde{\rho}, \theta, \tau) = \psi_j(\rho, \theta, \tau)$. We first notice that $(\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})$ is close to $R_j \sin\frac{\alpha}{2}$ on the support of ψ_j. Indeed, we have there $|\tilde{\rho}\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2}| \leq 2R^{1/4}$ and thus, since $R < R_j$, there exists $C > 0$ such that for all j and for all $(\tilde{\rho}, \theta, \tau)$ on the support of ψ_j, we have

$$|(\rho\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2}) - R_j\sin\tfrac{\alpha}{2}| = |\tilde{\rho}\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2}| \leq 2R^{1/4} \leq CR^{-3/4}R_j\sin\tfrac{\alpha}{2}.$$

With a possibly larger C, we have:

$$\begin{array}{rcccl} (1 - CR^{-3/4})R_j\sin\frac{\alpha}{2} & \leq & |\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2}| & \leq & (1 + CR^{-3/4})R_j\sin\frac{\alpha}{2}, \\ (1 - CR^{-3/4})R_j^{-2}\sin^{-2}\frac{\alpha}{2} & \leq & V_\alpha(R_j + \tilde{\rho}, \tau)^{-1} & \leq & (1 + CR^{-3/4})R_j^{-2}\sin^{-2}\frac{\alpha}{2}. \end{array} \tag{2.3}$$

Using (2.3), we have

$$\frac{\hat{Q}_\alpha(\psi_j)}{R_j\sin\frac{\alpha}{2}\,(1 - CR^{-3/4})} \geq \int_{\mathcal{D}_\alpha} \left(|\partial_\rho\tilde{\psi}_j|^2 + |\partial_\tau\tilde{\psi}_j|^2 + \tilde{V}_\alpha^{-1} \left| \left(-i\partial_\theta + \frac{\tilde{V}_\alpha}{2} \right) \tilde{\psi}_j \right|^2 \right) \mathrm{d}\tilde{\rho}\,\mathrm{d}\theta\,\mathrm{d}\tau, \tag{2.4}$$

where $\tilde{V}_\alpha(\tilde{\rho}, \theta, \tau) = V_\alpha(R_j + \tilde{\rho}, \tau)$. Let us deal with the third term in (2.4):

$$-i\partial_\theta + \frac{\tilde{V}_\alpha}{2} = -i\partial_\theta + R_j\sin\tfrac{\alpha}{2}\,(\tilde{\rho}\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2}) + C_j + \frac{(\tilde{\rho}\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})^2}{2},$$

where $C_j = \frac{1}{2}\sin^2\frac{\alpha}{2}R_j^2$ can be erased modulo a gauge transform: $\tilde{\psi}_j = e^{-i\theta C_j}v_j$. We write, using (2.3), for all $\eta \in (0,1)$:

$$\tilde{V}_\alpha^{-1}\left| \left(-i\partial_\theta + \frac{\tilde{V}_\alpha}{2} \right)\tilde{\psi}_j \right|^2 \geq (1 - CR^{-3/4})(1 - \eta)\left| \left(-\frac{i}{R_j\sin\frac{\alpha}{2}}\partial_\theta + \tilde{\rho}\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2} \right) v_j \right|^2 - \frac{\eta^{-1}}{4}\frac{(\tilde{\rho}\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})^4}{R_j^2\sin^2\frac{\alpha}{2}}|v_j|^2.$$

We have
$$\frac{(\tilde\rho\sin\frac{\alpha}{2}-\tau\cos\frac{\alpha}{2})^4}{4R_j^2\sin^2\frac{\alpha}{2}}\leq\frac{R}{4R_j^2\sin^2\frac{\alpha}{2}}\leq\frac{C}{R}.$$
After the change of variable $\tilde\theta=R_j\sin\frac{\alpha}{2}\theta$ and denoting $w_j(\tilde\rho,\tilde\theta,\tau)=v_j(\tilde\rho,\theta,\tau)$, the lower bound (2.4) becomes

$$\begin{aligned}\frac{\hat Q_\alpha(\psi_j)}{1-CR^{-3/4}}\geq\int_{\mathbb{R}^3_+}\Big(&|\partial_{\tilde\rho}w_j|^2+|\partial_\tau w_j|^2+(1-CR^{-3/4})(1-\eta)\left|\left(-i\partial_{\tilde\theta}+\tilde\rho\sin\tfrac{\alpha}{2}-\tau\cos\tfrac{\alpha}{2}\right)w_j\right|^2\\&-\frac{C}{\eta R}|w_j|^2\Big)\,\mathrm{d}\tilde\rho\,\mathrm{d}\tilde\theta\,\mathrm{d}\tau\\ \geq(1-CR^{-3/4})(1-\eta)\int_{\mathbb{R}^3_+}&\left(|\partial_{\tilde\rho}w_j|^2+|\partial_\tau w_j|^2+\left|\left(-i\partial_{\tilde\theta}+\tilde\rho\sin\tfrac{\alpha}{2}-\tau\cos\tfrac{\alpha}{2}\right)w_j\right|^2\right)\mathrm{d}\tilde\rho\,\mathrm{d}\tilde\theta\,\mathrm{d}\tau\\&-\frac{C}{\eta R}\int_{\mathbb{R}^3_+}|w_j|^2\,\mathrm{d}\tilde\rho\,\mathrm{d}\tilde\theta\,\mathrm{d}\tau,\end{aligned}$$

where we have extended the function w_j by zero outside its support and then defined a function in the half-space. Thus, applying the min-max principle for the Schrödinger operator with a constant magnetic field which makes an angle $\alpha/2$ with the boundary of the half-space (see Section 2.1), we deduce (returning in (ρ,θ,τ) variables)
$$\hat Q_\alpha(\psi_j)\geq\left(1-CR^{-3/4}\right)R_j\sin\tfrac{\alpha}{2}\left[(1-CR^{-3/4})(1-\eta)\sigma\left(\frac{\alpha}{2}\right)-\frac{C}{\eta R}\right]\int_{\mathcal{D}_\alpha}|\psi_j|^2\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\tau.\tag{2.5}$$
Let us now estimate the norm $\|\psi_j\|_{\mathrm{L}^2(\mathcal{D}_\alpha,\,\mathrm{d}\hat\mu)}$ with (2.3):
$$\begin{aligned}\|\psi_j\|^2_{\mathrm{L}^2(\mathcal{D}_\alpha,\,\mathrm{d}\hat\mu)}&=\int_{\mathcal{D}_\alpha}|\psi_j|^2|\rho\sin\tfrac{\alpha}{2}-\tau\cos\tfrac{\alpha}{2}|\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\tau\\&\leq\left(1+CR^{-3/4}\right)R_j\sin\tfrac{\alpha}{2}\int_{\mathcal{D}_\alpha}|\psi_j|^2\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\tau.\end{aligned}\tag{2.6}$$
Relations (2.5) and (2.6) give
$$Q_{\mathbf{A}}(\chi_j\psi)=\hat Q_\alpha(\psi_j)\geq\frac{1-CR^{-3/4}}{1+CR^{-3/4}}\left[(1-CR^{-3/4})(1-\eta)\sigma\left(\frac{\alpha}{2}\right)-\frac{C}{4\eta R}\right]\|\chi_j\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}.\tag{2.7}$$
Combining (2.1) with (2.7) and (2.2), we deduce for R large enough and η small enough (since $\sigma(\frac{\alpha}{2})\leq1$):
$$Q_{\mathbf{A}}(\psi)\geq\frac{1-CR^{-3/4}}{1+CR^{-3/4}}\left[(1-CR^{-3/4})(1-\eta)\sigma\left(\frac{\alpha}{2}\right)-\frac{C}{4\eta R}\right]\sum_j\|\chi_j\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}-\frac{C}{R^{1/2}}\|\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}.$$
Thus we get:
$$\Sigma(-\Delta_{\mathbf{A}},R)\geq\frac{1-CR^{-3/4}}{1+CR^{-3/4}}\left[(1-CR^{-3/4})(1-\eta)\sigma\left(\frac{\alpha}{2}\right)-\frac{C}{4\eta R}\right]-\frac{C}{R^{1/2}},$$
and
$$\lim_{R\to\infty}\Sigma(-\Delta_{\mathbf{A}},R)\geq(1-\eta)\sigma(\tfrac{\alpha}{2}).$$
This relation is available for any $\eta\in(0,1)$. Therefore, we have with Lemma 2.1
$$\inf\mathsf{sp}_{\mathsf{ess}}(\mathcal{L}_\alpha)\geq\sigma(\tfrac{\alpha}{2}).$$

Upper bound for $\Sigma(-\Delta_{\mathbf{A}}, R)$ We have now to prove the upper-bound. Let α be fixed. By the min-max principle applied to the operator $P_{\frac{\alpha}{2}}$, for any ε, there exists $\psi \in \mathcal{C}_0^\infty(\overline{\mathbb{R}^3_+})$ such that

$$\sigma\left(\frac{\alpha}{2}\right) \leq \frac{\int_{\mathbb{R}^3_+} \left(|\partial_s\psi|^2 + |\partial_t\psi|^2 + |(-i\partial_r + t\cos\frac{\alpha}{2} - s\sin\frac{\alpha}{2})\psi|^2\right)\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t}{\int_{\mathbb{R}^3_+} |\psi|^2\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t} \leq \sigma\left(\frac{\alpha}{2}\right) + \varepsilon. \tag{2.8}$$

We can assume that

$$\mathsf{supp}(\psi) \subset \{(r,s,t) \in \mathbb{R}^3_+,\ r \in (-\ell,\ell), s \in (-\ell,\ell), t \in (0,\ell)\},$$

with $\ell > 0$. For any $R > 0$, let us construct, using ψ, a function $u \in \mathcal{C}_0^\infty(\overline{\mathcal{D}_\alpha} \cap \complement\mathcal{B}_R)$ in normal coordinates such that

$$\hat{Q}_\alpha(u) \leq (\sigma(\tfrac{\alpha}{2}) + o(1))\|u\|^2_{\mathrm{L}^2(\mathcal{D}_\alpha,\,\mathrm{d}\hat{\mu})}.$$

Let us analyze $\hat{Q}_\alpha(u)$ by using the previous computations for the lower bound. We assume that

$$\mathsf{supp}(u) \subset \{(\rho,\theta,\tau) \in \mathbb{R}^3, \rho \in (R-\ell, R+\ell), \theta \in (-\pi,\pi), \tau \in (0,\ell)\}.$$

We use the change of variables $\rho = R + \tilde{\rho}$. Thus, we have on the support of u:

$$|(\rho\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2}) - R\sin\tfrac{\alpha}{2}| = |\tilde{\rho}\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2}| \leq 2\ell. \tag{2.9}$$

Thus there exists $C > 0$ such that:

$$\begin{aligned}
(1 - C\ell R^{-1})R\sin\tfrac{\alpha}{2} \quad &\leq \quad |\rho\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2}| \quad \leq \quad (1 + C\ell R^{-1})R\sin\tfrac{\alpha}{2},\\
(1 - C\ell R^{-1})R^{-2}\sin^{-2}\tfrac{\alpha}{2} \quad &\leq \quad V_\alpha(R+\tilde{\rho},\tau)^{-1} \quad \leq \quad (1 + C\ell R^{-1})R^{-2}\sin^{-2}\tfrac{\alpha}{2}.
\end{aligned} \tag{2.10}$$

Using (2.10) and denoting $\tilde{u}(\tilde{\rho},\theta,\tau) = u(\rho,\theta,\tau)$ and $\tilde{V}_\alpha(\tilde{\rho},\theta,\tau) = V_\alpha(R+\tilde{\rho},\tau)$, we have:

$$\frac{\hat{Q}_\alpha(u)}{R\sin\frac{\alpha}{2}\,(1 + C\ell R^{-1})} \leq \int_{\mathcal{D}_\alpha} \left(|\partial_\rho\tilde{u}|^2 + |\partial_\tau\tilde{u}|^2 + \tilde{V}_\alpha^{-1}\left|\left(-i\partial_\theta + \frac{\tilde{V}_\alpha}{2}\right)\tilde{u}\right|^2\right)\mathrm{d}\tilde{\rho}\,\mathrm{d}\theta\,\mathrm{d}\tau. \tag{2.11}$$

Let us deal with the third term in (2.11):

$$-i\partial_\theta + \frac{\tilde{V}_\alpha}{2} = -i\partial_\theta + R\sin\tfrac{\alpha}{2}\ (\tilde{\rho}\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2}) + \frac{1}{2}\sin^2\tfrac{\alpha}{2}R^2 + \frac{(\tilde{\rho}\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})^2}{2}.$$

We let $\tilde{u} = e^{-\frac{i}{2}\theta\sin^2\alpha R^2}v$ and we have

$$\left|\left(-i\partial_\theta + \frac{\tilde{V}_\alpha}{2}\right)\tilde{u}\right| = \left|\left(-i\partial_\theta + R\sin\tfrac{\alpha}{2}\ (\tilde{\rho}\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2}) + \frac{(\tilde{\rho}\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})^2}{2}\right)v\right|.$$

We write, using (2.9) and (2.10), for all $\eta \in (0,1)$:

$$\begin{aligned}
&\tilde{V}_\alpha^{-1}\left|\left(-i\partial_\theta + \frac{\tilde{V}_\alpha}{2}\right)\tilde{u}\right|^2\\
&\leq (1 + C\ell R^{-1})\left((1+\eta)\left|\left(\frac{-i\partial_\theta}{R\sin\frac{\alpha}{2}} + (\tilde{\rho}\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2})\right)v\right|^2 + \left(1+\eta^{-1}\right)\frac{C\ell^4}{R^2}|v|^2\right).
\end{aligned}$$

Let us now make the change of variable $\tilde{\theta} = \theta R \sin\frac{\alpha}{2}$ (denoting $w(\tilde{\rho}, \tilde{\theta}, \tau) = v(\tilde{\rho}, \theta, \tau)$ extended to $\mathbb{R}^3_+$ by 0 outside its support), the upper bound (2.11) reads

$$\begin{aligned}\frac{\hat{Q}_\alpha(u)}{1+C\ell R^{-1}} \leq \int_{\mathbb{R}^3_+} \Big(|\partial_{\tilde{\rho}} w|^2 + |\partial_\tau w|^2 + (1+C\ell R^{-1})(1+\eta) \left|\left(-i\partial_{\tilde{\theta}} + (\tilde{\rho}\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2})\right) w\right|^2 \\ + (1+C\ell R^{-1})\left(1+\eta^{-1}\right) \frac{C\ell^4}{R^2}|w|^2 \Big) \,\mathrm{d}\tilde{\rho}\,\mathrm{d}\tilde{\theta}\,\mathrm{d}\tau \\ \leq (1+C\ell R^{-1})(1+\eta) \int_{\mathbb{R}^3_+} \Big(|\partial_{\tilde{\rho}} w|^2 + |\partial_\tau w|^2 + \left|\left(-i\partial_{\tilde{\theta}} + (\tilde{\rho}\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2})\right) w\right|^2 \Big) \,\mathrm{d}\tilde{\rho}\,\mathrm{d}\tilde{\theta}\,\mathrm{d}\tau \\ + (1+C\ell R^{-1})\left(1+\eta^{-1}\right) \frac{C\ell^4}{R^2} \int_{\mathbb{R}^3_+} |w|^2 \,\mathrm{d}\tilde{\rho}\,\mathrm{d}\tilde{\theta}\,\mathrm{d}\tau.\end{aligned}$$

Let us now estimate the norm $\|u\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\hat{\mu})}$ with (2.10):

$$\begin{aligned}\|u\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\hat{\mu})} &= \int_{\mathcal{D}_\alpha} |u|^2 |\rho\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2}| \,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\tau \\ &\geq \left(1 - C\ell R^{-1}\right) R\sin\tfrac{\alpha}{2} \int_{\mathbb{R}^3_+} |u|^2 \,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\tau \\ &= \left(1 - C\ell R^{-1}\right) \int_{\mathbb{R}^3_+} |w|^2 \,\mathrm{d}\tilde{\rho}\,\mathrm{d}\tilde{\theta}\,\mathrm{d}\tau. \qquad (2.12)\end{aligned}$$

We use for w the function ψ satisfying (2.8). Therefore, we get:

$$\Sigma(-\Delta_{\mathbf{A}}, R-\ell) \leq \frac{\hat{Q}_\alpha(u)}{\|u\|^2_{\mathrm{L}^2(\mathcal{D}_\alpha,\,\mathrm{d}\hat{\mu})}} \leq \frac{(1+C\ell R^{-1})^2}{1 - C\ell R^{-1}} \left((1+\eta)\left(\sigma\left(\frac{\alpha}{2}\right) + \varepsilon\right) + \left(1+\eta^{-1}\right)\frac{C\ell^4}{R^2} \right),$$

and

$$\lim_{R\to+\infty} \Sigma(-\Delta_{\mathbf{A}}, R) \leq (1+\eta)(\sigma(\tfrac{\alpha}{2}) + \varepsilon).$$

Since this relation is available for any $\eta \in (0,1)$, we deduce

$$\inf \mathsf{sp}_{\mathsf{ess}}(\mathcal{L}_\alpha) \leq \sigma(\tfrac{\alpha}{2}) + \varepsilon, \qquad \forall \varepsilon > 0.$$

3 Construction of quasimodes

This section deals with the proof of the following proposition.

Proposition 3.1 *There exists a sequence $(\gamma_{j,n})_{j\geq 0, n\geq 1}$ such that for all $N \geq 1$ and $J \geq 0$, there exist $C_{N,J}$ and α_0 such that for all $0 < \alpha < \alpha_0$ and $1 \leq n \leq N$, we have:*

$$\mathrm{dist}\left(\mathsf{sp}_{\mathsf{dis}}(\mathfrak{L}_\alpha), \sum_{j=0}^{J} \gamma_{j,n} \alpha^{2j+1} \right) \leq C_{N,J}\ \alpha^{2J+3},$$

where $\gamma_{0,n} = \mathfrak{l}_n = 2^{-5/2}(4n-1)$.

Proof: We construct quasimodes which do not depend on θ. In other words, we look for quasimodes for:

$$\mathcal{L}_{\alpha,0} = -\frac{1}{t^2}\partial_t t^2 \partial_t + \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2 - \frac{1}{\alpha^2\ t^2 \sin(\alpha\varphi)} \partial_\varphi \sin(\alpha\varphi) \partial_\varphi.$$

We write a formal Taylor expansion of $\mathcal{L}_{\alpha,0}$ in powers of α^2:

$$\mathcal{L}_{\alpha,0} \sim \alpha^{-2}L_{-1} + L_0 + \sum_{j\geq 1}\alpha^{2j}L_j,$$

where:

$$L_{-1} = -\frac{1}{t^2\varphi}\partial_\varphi\varphi\partial_\varphi, \qquad L_0 = -\frac{1}{t^2}\partial_t t^2\partial_t + \frac{\varphi^2t^2}{4} + \frac{1}{3t^2}\varphi\partial_\varphi.$$

We look for quasi-eigenpairs expressed as formal series:

$$\psi \sim \sum_{j\geq 0}\alpha^{2j}\psi_j, \qquad \lambda \sim \alpha^{-2}\lambda_{-1} + \lambda_0 + \sum_{j\geq 1}\alpha^{2j}\lambda_j,$$

so that, formally, we have:

$$\mathcal{L}_{\alpha,0}\psi \sim \lambda\psi.$$

Term in $\boldsymbol{\alpha^{-2}}$. We are led to solve the equation:

$$L_{-1}\psi_0 = -\frac{1}{t^2\varphi}\partial_\varphi\varphi\partial_\varphi\psi_0 = \lambda_{-1}\psi_0.$$

We choose $\lambda_{-1} = 0$ and $\psi_0(t,\varphi) = f_0(t)$, with f_0 to be chosen in the next step.

Term in $\boldsymbol{\alpha^{0}}$. We shall now solve the equation:

$$L_{-1}\psi_1 = (\lambda_0 - L_0)\psi_0.$$

We look for ψ_1 in the form: $\psi_1(t,\varphi) = t^2\tilde{\psi}_1(t,\varphi) + f_1(t)$. The equation provides:

$$-\frac{1}{\varphi}\partial_\varphi\varphi\partial_\varphi\tilde{\psi}_1 = (\lambda_0 - L_0)\psi_0. \tag{3.1}$$

For each $t > 0$, the Fredholm condition is $\langle(\lambda_0 - L_0)\psi_0, 1\rangle_{\mathrm{L}^2((0,\frac{1}{2}),\varphi\,\mathrm{d}\varphi)} = 0$, that reads:

$$\int_0^{\frac{1}{2}}(L_0\psi_0)(t,\varphi)\,\varphi\,\mathrm{d}\varphi = \frac{\lambda_0}{2^3}f_0(t).$$

Moreover we have:

$$\int_0^{\frac{1}{2}}(L_0\psi_0)(t,\varphi)\,\varphi\,\mathrm{d}\varphi = -\frac{1}{2^3t^2}\partial_t t^2\partial_t f_0(t) + \frac{1}{2^8}t^2f_0(t),$$

so that we get:

$$\left(-\frac{1}{t^2}\partial_t t^2\partial_t + \frac{1}{2^5}t^2\right)f_0 = \lambda_0 f_0.$$

Using Corollary C.2, we are led to take:

$$\lambda_0 = \mathfrak{l}_n \qquad \text{and} \qquad f_0(t) = \mathfrak{f}_n(t).$$

For this choice of f_0, we infer the existence of a unique function denoted by $\tilde{\psi}_1^{\perp}$ (in the Schwartz class with respect to t) orthogonal to 1 in $\mathrm{L}^2((0,\frac{1}{2}),\varphi\,\mathrm{d}\varphi)$ which satisfies (3.1). Using the decomposition of ψ_1, we have:

$$\psi_1(t,\varphi) = t^2\tilde{\psi}_1^{\perp}(t,\varphi) + f_1(t),$$

where f_1 has to be determined in the next step.

Further terms. Let us consider $k \geq 1$ and assume that we have already constructed $(\lambda_j)_{j=-1,\ldots,k-1}$, $(f_j)_{j=0,\ldots,k-1}$, $(\psi_j)_{j=0,\ldots,k-1}$ (which are in the Schwartz class with respect to t) and that, for $j = 0, \ldots, k$, we can write:

$$\psi_j(t,\varphi) = t^2 \tilde{\psi}_j^{\perp}(t,\varphi) + f_j(t),$$

where $(\tilde{\psi}_k^{\perp}(t,\varphi))_{j=0,\ldots k}$ are determined functions in the Schwartz class (and orthogonal to 1 in $\mathrm{L}^2((0,\frac{1}{2}),\varphi\,\mathrm{d}\varphi))$ and where f_k has to be determined.
We write the equation corresponding to α^{2k}, $L_{-1}\psi_{k+1} = \sum_{j=0}^{k}(\lambda_j - L_j)\psi_{k-j}$ that reads:

$$L_{-1}\psi_{k+1} = \lambda_k\psi_0 + \lambda_0\psi_k - L_0\psi_k + R_k,$$

where R_k is a determined function in the Schwartz class with respect to t:

$$R_k = \sum_{j=1}^{k-1}(\lambda_j - L_j)\psi_{k-j} - L_k\psi_0.$$

We look for ψ_{k+1} in the form $\psi_{k+1}(t,\varphi) = t^2\tilde{\psi}_{k+1}(t,\varphi) + f_{k+1}(t)$ and we can write:

$$-\frac{1}{\varphi}\partial_\varphi\varphi\partial_\varphi\tilde{\psi}_{k+1} = \lambda_k\psi_0 + \lambda_0\psi_k - L_0\psi_k + R_k. \tag{3.2}$$

The Fredholm alternative provides $\langle\lambda_k\psi_0 + R_k + (L_0 - \lambda_0)\psi_k, 1\rangle_{\mathrm{L}^2((0,\frac{1}{2}),\varphi\,\mathrm{d}\varphi)} = 0$ and thus:

$$\left(-\frac{1}{t^2}\partial_t t^2\partial_t + \frac{1}{2^5}t^2 - \lambda_0\right) f_k = \lambda_k f_0 + r_k, \tag{3.3}$$

where r_k is a determined function in the Schwartz class. The Fredholm alternative implies that:

$$\lambda_k = -\langle r_k, f_0\rangle_t.$$

For this choice, we can find a unique normalized f_k in the Schwartz class such that it satisfies (3.3) and $\langle f_k, f_0\rangle_t = 0$. Then, we obtain the existence of a unique function denoted by $\tilde{\psi}_{k+1}^{\perp}$, in the Schwartz class with respect to t and orthogonal to 1 in $\mathrm{L}^2((0,\frac{1}{2}),\varphi\,\mathrm{d}\varphi)$ which satisfies (3.2).

We define:

$$\Psi_n^J(\alpha)(t,\theta,\varphi) = \sum_{j=0}^{J}\alpha^{2j}\psi_j(t,\varphi), \quad \forall(t,\theta,\varphi)\in\mathcal{P}, \tag{3.4}$$

$$\Lambda_n^J(\alpha) = \sum_{j=0}^{J}\alpha^{2j}\lambda_j. \tag{3.5}$$

Due to the exponential decay of the ψ_j and thanks to Taylor expansions, there exists $C_{n,J}$ such that:

$$\|\left(\mathcal{L}_\alpha - \Lambda_n^J(\alpha)\right)\Psi_n^J(\alpha)\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \leq C_{n,J}\alpha^{2J+2}\|\Psi_n^J(\alpha)\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

Using the spectral theorem and going back to the operator $\mathfrak{L}_\alpha$ by change of variables, we conclude the proof of Proposition 3.1 with $\gamma_{j,n} = \lambda_j$. ∎

Considering the main term of the asymptotic expansion, we deduce the three following corollaries.

Corollary 3.2 *For all $N \geq 1$, there exist C and α_0 and for all $1 \leq n \leq N$ and $0 \leq \alpha \leq \alpha_0$, there exists an eigenvalue $\tilde{\lambda}_{k(n,\alpha)}(\alpha)$ of $\mathcal{L}_\alpha$ such that*

$$|\tilde{\lambda}_{k(n,\alpha)}(\alpha) - \mathfrak{l}_n| \leq C\alpha^2.$$

Corollary 3.3 *We observe that for $1 \leq n \leq N$ and $\alpha \in (0, \alpha_0)$:*

$$0 \leq \tilde{\lambda}_n(\alpha) \leq \tilde{\lambda}_{k(n,\alpha)}(\alpha) \leq \mathfrak{l}_n + C\alpha^2.$$

This last corollary implies:

Corollary 3.4 *For all $n \geq 1$, there exist $\alpha_0(n) > 0$ and $C_n > 0$ such that, for all $\alpha \in (0, \alpha_0(n))$, the n-th eigenvalue exists and satisfies:*

$$\lambda_n(\alpha) \leq C_n\alpha,$$

or equivalently $\tilde{\lambda}_n(\alpha) \leq C_n$.

4 Rough Agmon estimates

In our way to prove Theorem 1.3 we will need rough localization estimates “à la Agmon” satisfied by the first eigenfunctions. We shall prove the following proposition.

Proposition 4.1 *Let $C_0 > 0$. There exist $\alpha_0 > 0$ and $C > 0$ such that for any $\alpha \in (0, \alpha_0)$ and for all eigenpair (λ, ψ) of $\mathfrak{L}_\alpha$ satisfying $\lambda \leq C_0\alpha$:*

$$\int_{\mathcal{C}_\alpha} e^{\alpha|z|}|\psi|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}, \tag{4.1}$$

and

$$Q_{\mathbf{A}}(e^{\alpha|z|}\psi) \leq C\alpha\|\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}. \tag{4.2}$$

Proof: Let $(\lambda_n(\alpha), \psi_{n,\alpha}) = (\lambda, \psi)$ an eigenpair for $\mathfrak{L}_\alpha$. Let us introduce a smooth cut-off function $0 \leq \chi \leq 1$ such that $\chi = 1$ on $[-1, 1]$ and $\chi(z) = 0$ for $|z| \geq 2$ and let us also consider, for $R \geq 1$:

$$\Phi_R(z) = \alpha\chi\left(\frac{z}{R}\right)|z|.$$

We can write the Agmon identity:

$$Q_{\mathbf{A}}(e^{\Phi_R}\psi) = \lambda\|e^{\Phi_R}\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)} - \|\nabla\Phi_R e^{\Phi_R}\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}.$$

We have $\lambda \leq C_0\alpha$. Moreover we have $\|\nabla\Phi_R\| \leq \alpha + 2\alpha\|\chi'\|_\infty$. There exists $\alpha_0 > 0$ such that for $\alpha \in (0, \alpha_0)$ and all $R \geq 1$, we have:

$$Q_{\mathbf{A}}(e^{\Phi_R}\psi) \leq C\alpha\|e^{\Phi_R}\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}.$$

We introduce a partition of unity with respect to z:

$$\chi_1^2(z) + \chi_2^2(z) = 1,$$

where $\chi_1(z) = 1$ for $0 \leq z \leq 1$ and $\chi_1(z) = 0$ for $z \geq 2$. For $j = 1, 2$ and $\gamma > 0$, we let:

$$\chi_{j,\gamma}(z) = \chi_j(\gamma^{-1}z),$$

so that:
$$\|\chi'_{j,\gamma}\|_{\mathrm{L}^\infty(\mathcal{C}_\alpha)} \leq C\gamma^{-1}.$$
The "IMS" formula provides:
$$Q_{\mathbf{A}}(e^{\Phi_R}\chi_{1,\gamma}\psi) + Q_{\mathbf{A}}(e^{\Phi_R}\chi_{2,\gamma}\psi) - C^2\gamma^{-2}\|e^{\Phi_R}\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)} \leq C\alpha\|e^{\Phi_R}\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}. \tag{4.3}$$
We want to write a lower bound for $Q_{\mathbf{A}}(e^{\Phi_R}\chi_{2,\gamma}\psi)$. In order to do that we integrate by slices and neglect the z-derivative part:
$$Q_{\mathbf{A}}(e^{\Phi_R}\chi_{2,\gamma}\psi) \geq \int_{z>0}\left(\int_{x^2+y^2\leq z^2\tan^2\frac{\alpha}{2}} |(-i\nabla + \mathbf{A})(e^{\Phi_R}\chi_{2,\gamma}\psi)|^2\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}z$$
$$\geq \int_{z>0}\left(\int_{x^2+y^2\leq z^2\tan^2\frac{\alpha}{2}} |(D_x - \tfrac{y}{2})(e^{\Phi_R}\chi_{2,\gamma}\psi)|^2 + |(D_y + \tfrac{x}{2})(e^{\Phi_R}\chi_{2,\gamma}\psi)|^2\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}z.$$
Let us denote by $\mu_1(\rho)$ the lowest eigenvalue of $\left(D_x - \frac{y}{2}\right)^2 + \left(D_y + \frac{x}{2}\right)^2$ on the disk $\mathcal{D}(0,\rho)$ with Neumann condition. From the min-max principle, we infer:
$$Q_{\mathbf{A}}(e^{\Phi_R}\chi_{2,\gamma}\psi) \geq \int_{\mathcal{C}_\alpha} \mu_1\left(z\tan\tfrac{\alpha}{2}\right)|e^{\Phi_R}\chi_{2,\gamma}\psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{4.4}$$
We choose $\gamma = \frac{2R_0}{\alpha}$. On the support of $\chi_{2,\gamma}$, we have:
$$z\tan\tfrac{\alpha}{2} \geq z\tfrac{\alpha}{2} \geq R_0.$$
We recall (see [1, 6]) that there exists $\rho_0 > 0$ such that, for $\tilde{\rho}_0 \geq \rho_0$:
$$\mu_1(\tilde{\rho}_0) \geq \frac{\Theta_0}{2}. \tag{4.5}$$
We choose $R_0 = \rho_0$. With (4.3), (4.4) and (4.5), we infer:
$$\int_{\mathcal{C}_\alpha}\left(\frac{\Theta_0}{2} - C(R_0)(\alpha+\alpha^2)\right)|e^{\Phi_R}\chi_{2,\gamma}\psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C(R_0)\|e^{\Phi_R}\chi_{1,\gamma}\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}.$$
We deduce that there exist $\alpha_0 > 0$, $C > 0$ such that for $\alpha \in (0,\alpha_0)$ and $R > 0$:
$$\int_{\mathcal{C}_\alpha}|e^{\Phi_R}\chi_{2,\gamma}\psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C\|e^{\Phi_R}\chi_{1,\gamma}\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}.$$
With our choice of $\gamma(= 2R_0\alpha^{-1})$, we infer:
$$\int_{\mathcal{C}_\alpha}|e^{\Phi_R}\chi_{2,\gamma}\psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)},$$
and
$$\int_{\mathcal{C}_\alpha}|e^{\Phi_R}\psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}.$$
Taking the limit $R \to +\infty$ and using the Fatou lemma, it follows:
$$\int_{\mathcal{C}_\alpha} e^{\alpha|z|}|\psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)},$$
which is (4.1). Using again (4.3), we infer (4.2). ■

Remark 4.2 *We can guess with the construction of quasimodes given in Section 3 that the decay of Proposition 4.1 is not optimal. Nevertheless this rough Agmon estimate is enough to establish the optimal length scale ($z \sim \alpha^{-1/2}$ or $t \sim 1$) in Proposition 6.3. In addition, once Theorem 1.3 will be proved we will know that the quasimodes of Section 3 actually approximate the eigenfunctions which inherit the same decay.*

5 Axisymmetry of the first eigenfunctions

In this section, we prove that the first eigenfunctions of $\mathcal{L}_\alpha$ are axisymmetric as soon as α is small anough.

Notation 5.1 *From Propositions 1.2 and 3.1, we infer that, for all $n \geq 1$, there exists $\alpha_n > 0$ such that if $\alpha \in (0, \alpha_n)$, the n-th eigenvalue $\tilde{\lambda}_n(\alpha)$ of $\mathcal{L}_\alpha$ exists. Due to the fact that $-i\partial_\theta$ commutes with the operator, one deduces that for each $n \geq 1$, we can consider a basis $(\psi_{n,j}(\alpha))_{j=1,\cdots J(n,\alpha)}$ of the eigenspace of $\mathcal{L}_\alpha$ associated with $\tilde{\lambda}_n(\alpha)$ such that*

$$\psi_{n,j}(\alpha)(t,\theta,\varphi) = e^{im_{n,j}(\alpha)\theta}\Psi_{n,j}(t,\varphi).$$

As an application of the localization estimates of Section 4, we prove the following proposition.

Proposition 5.2 *For all $n \geq 1$, there exists $\alpha_n > 0$ such that if $\alpha \in (0, \alpha_n)$, we have:*

$$m_{n,j}(\alpha) = 0, \quad \forall j = 1, \ldots, J(n,\alpha).$$

In other words, the functions of the n-th eigenspace are independent from θ as soon as α is small enough.

In order to succeed, we use a contradiction argument: We consider an L^2-normalized eigenfunction of $\mathcal{L}_\alpha$ associated to $\tilde{\lambda}_n(\alpha)$ in the form $e^{im(\alpha)\theta}\Psi_\alpha(t,\varphi)$ and we assume that there exists $\alpha > 0$ as small as we want such that $m(\alpha) \neq 0$ or equivalently $|m(\alpha)| \geq 1$.

We introduce a smooth cut-off function $\chi_{\alpha,\eta}(t) = \chi\left(\alpha^{\frac{1}{2}+\eta}t\right)$ where χ is 1 near 0, and $\eta \in \left(0, \frac{1}{100}\right)$. For short, we let:

$$\Psi^{\mathsf{cut}}(t,\varphi) = \chi_{\alpha,\eta}(t)\Psi_\alpha(t,\varphi).$$

5.1 Dirichlet condition on the axis $\varphi = 0$

Let us prove the following lemma.

Lemma 5.3 *For all $t > 0$, we have $\Psi^{\mathsf{cut}}(t,0) = 0$.*

Proof: We recall the eigenvalue equation:

$$\mathcal{L}_{\alpha,m(\alpha)}\Psi_\alpha = \tilde{\lambda}_n(\alpha)\Psi_\alpha,$$

so that:

$$\mathcal{L}_{\alpha,m(\alpha)}\Psi^{\mathsf{cut}} = \tilde{\lambda}_n(\alpha)\Psi^{\mathsf{cut}} + [\mathcal{L}_{\alpha,m(\alpha)}, \chi_{\alpha,\eta}]\Psi_\alpha. \tag{5.1}$$

Thanks to Agmon's estimates and to Corollary 3.4, we deduce:

$$Q_{\alpha,m(\alpha)}(\Psi^{\mathsf{cut}}) \leq C\|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)}.$$

This implies:

$$\int_{\mathcal{R}} \frac{1}{t^2\sin^2(\alpha\varphi)}\left(m(\alpha) + \frac{\sin^2(\alpha\varphi)}{2\alpha}t^2\right)^2 |\Psi^{\mathsf{cut}}(t,\varphi)|^2\,\mathrm{d}\mu \leq C\|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} < +\infty.$$

Using the inequality $(a+b)^2 \geq \frac{1}{2}a^2 - 2b^2$, it follows:

$$\frac{m(\alpha)^2}{2}\int_{\mathcal{R}} \frac{1}{t^2\sin^2(\alpha\varphi)}|\Psi^{\mathsf{cut}}(t,\varphi)|^2\,\mathrm{d}\mu - 2\int_{\mathcal{R}} \frac{t^2\sin^2(\alpha\varphi)}{4\alpha^2}|\Psi^{\mathsf{cut}}(t,\varphi)|^2\,\mathrm{d}\mu < +\infty,$$

so that:

$$m(\alpha)^2 \int_{\mathcal{R}} \frac{1}{t^2 \sin^2(\alpha\varphi)} |\Psi^{\mathsf{cut}}(t,\varphi)|^2 \, \mathrm{d}\mu < +\infty,$$

and:

$$\int_{\mathcal{R}} \frac{1}{t^2 \sin^2(\alpha\varphi)} |\Psi^{\mathsf{cut}}(t,\varphi)|^2 \, \mathrm{d}\mu < +\infty. \tag{5.2}$$

Therefore, for almost all $t > 0$, we have:

$$\int_0^{\frac{1}{2}} \frac{1}{\sin^2(\alpha\varphi)} |\Psi^{\mathsf{cut}}(t,\varphi)|^2 \, \sin(\alpha\varphi) \, \mathrm{d}\varphi < +\infty. \tag{5.3}$$

The function $\mathcal{R} \ni (t,\varphi) \mapsto \Psi^{\mathsf{cut}}(t,\varphi)$ is smooth by elliptic regularity inside $\mathcal{C}_\alpha$ (thus $\mathcal{R}$). In particular, it is continuous at $\varphi = 0$. By the integrability property (5.3), this imposes that, for all $t > 0$, we have $\Psi^{\mathsf{cut}}(t,0) = 0$. ■

5.2 The operator $-(\sin(\alpha\varphi))^{-1}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi$

Notation 5.4 *For $\alpha \in (0,\pi)$, let us consider the operator on $\mathrm{L}^2\left((0,\frac{1}{2}), \sin(\alpha\varphi)\,\mathrm{d}\varphi\right)$ defined by:*

$$\mathfrak{P}_\alpha = -\frac{1}{\sin(\alpha\varphi)} \partial_\varphi \sin(\alpha\varphi)\partial_\varphi,$$

with domain:

$$\begin{aligned}\mathsf{Dom}\,(\mathfrak{P}_\alpha) = \Big\{ &\psi \in \mathrm{L}^2\left((0,\tfrac{1}{2}), \sin(\alpha\varphi)\,\mathrm{d}\varphi\right), \\ &\frac{1}{\sin(\alpha\varphi)} \partial_\varphi \sin(\alpha\varphi)\partial_\varphi \psi \in \mathrm{L}^2\left((0,\tfrac{1}{2}), \sin(\alpha\varphi)\,\mathrm{d}\varphi\right), \partial_\varphi\psi\left(\tfrac{1}{2}\right) = 0, \ \psi(0) = 0 \Big\}.\end{aligned}$$

We denote by $\nu_1(\alpha)$ its first eigenvalue.

The aim of this subsection is to establish the following lemma:

Lemma 5.5 *There exists $c_0 > 0$ such that for all $\alpha \in (0,\pi)$:*

$$\nu_1(\alpha) \geq c_0.$$

Proof: We consider the associated quadratic form $\mathfrak{p}_\alpha$:

$$\mathfrak{p}_\alpha(\psi) = \int_0^{\frac{1}{2}} \sin(\alpha\varphi) |\partial_\varphi \psi|^2 \, \mathrm{d}\varphi.$$

We have the elementary lower bound:

$$\mathfrak{p}_\alpha(\psi) \geq \int_0^{\frac{1}{2}} \alpha\varphi \left(1 - \frac{(\alpha\varphi)^2}{6}\right) |\partial_\varphi\psi|^2 \, \mathrm{d}\varphi \geq \frac{1}{2} \int_0^{\frac{1}{2}} \alpha\varphi |\partial_\varphi\psi|^2 \, \mathrm{d}\varphi,$$

since $0 \leq \alpha\varphi \leq \frac{\pi}{2}$. We are led to analyze the lowest eigenvalue $\gamma \geq 0$ of the operator on $\mathrm{L}^2\left((0,\frac{1}{2}), \varphi\,\mathrm{d}\varphi\right)$ defined by $-\frac{1}{\varphi}\partial_\varphi \varphi \partial_\varphi$ with Dirichlet condition at $\varphi = 0$ and Neumann condition at $\varphi = \frac{1}{2}$. Let us prove that $\gamma > 0$. If it were not the case, the corresponding eigenvector ψ would satisfy:

$$-\frac{1}{\varphi}\partial_\varphi \varphi \partial_\varphi \psi = 0,$$

so that:
$$\psi(\varphi) = c\ln\varphi + d, \qquad \text{with } c, d \in \mathbb{R}.$$
The boundary conditions provide $c = d = 0$ and thus $\psi = 0$. By contradiction, we infer that $\gamma > 0$.
We deduce that:
$$\mathfrak{p}_\alpha(\psi) \geq \frac{\gamma}{2}\int_0^{\frac{1}{2}} \alpha\varphi|\psi|^2\,\mathrm{d}\varphi \geq \frac{\gamma}{2}\int_0^{\frac{1}{2}} \sin(\alpha\varphi)|\psi|^2\,\mathrm{d}\varphi.$$
By the min-max principle, we conclude that, for all $\alpha \in (0, \pi)$:
$$\nu_1(\alpha) \geq \frac{\gamma}{2} =: c_0 > 0.$$
∎

5.3 End of the proof of Proposition 5.2

Let us recall that (5.1) holds so that:
$$\mathcal{L}_{\alpha,m(\alpha)}(t\Psi^{\mathsf{cut}}) = \tilde{\lambda}_n(\alpha)t\Psi^{\mathsf{cut}} + t[\mathcal{L}_{\alpha,m(\alpha)}, \chi_{\alpha,\eta}]\Psi_\alpha + [\mathcal{L}_{\alpha,m(\alpha)}, t]\Psi^{\mathsf{cut}}. \tag{5.4}$$
We have:
$$[\mathcal{L}_{\alpha,m(\alpha)}, t] = [-t^{-2}\partial_t t^2\partial_t, t] = -2\partial_t - \frac{2}{t}.$$
We take the scalar product of the equation (5.4) with $t\Psi^{\mathsf{cut}}$. We notice that:
$$\langle[\mathcal{L}_{\alpha,m(\alpha)}, t]\Psi^{\mathsf{cut}}, t\Psi^{\mathsf{cut}}\rangle_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} = -2\|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} + 3\|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} = \|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)}.$$
The Agmon estimates provide:
$$|\langle t[\mathcal{L}_{\alpha,m(\alpha)}, \chi_{\alpha,\eta}]\Psi_\alpha, t\Psi^{\mathsf{cut}}\rangle_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)}| = \mathcal{O}(\alpha^\infty)\|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)}.$$
We infer:
$$Q_{\alpha,m(\alpha)}(t\Psi^{\mathsf{cut}}) \leq C(\|t\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} + \|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)}),$$
and especially:
$$\alpha^{-2}\int_{\mathcal{R}} |\partial_\varphi\Psi^{\mathsf{cut}}|^2\,\mathrm{d}\mu \leq C\left(\|t\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} + \|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)}\right).$$
Lemmas 5.3 and 5.5 imply that:
$$c_0\alpha^{-2}\int_{\mathcal{R}} |\Psi^{\mathsf{cut}}|^2\,\mathrm{d}\mu \leq C\left(\|t\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} + \|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)}\right).$$
Due to support considerations, we have:
$$c_0\alpha^{-2}\|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} \leq \tilde{C}\left(\alpha^{-1-2\eta}\|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} + \|\Psi^{\mathsf{cut}}\|^2_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)}\right).$$
We infer that, for α small enough, $\Psi^{\mathsf{cut}} = 0$. With the Agmon estimates, this implies that $\Psi_\alpha = 0$, and this is a contradiction.
This ends the proof of Proposition 5.2.

6 Accurate estimate of the spectral gap

This section is devoted to the proof of the following proposition.

Proposition 6.1 *For all $n \geq 1$, there exists $\alpha_0(n) > 0$ such that, for all $\alpha \in (0, \alpha_0(n))$, the n-th eigenvalue exists and satisfies:*

$$\lambda_n(\alpha) \geq \gamma_{0,n}\alpha + o(\alpha),$$

or equivalently $\tilde{\lambda}_n(\alpha) \geq \gamma_{0,n} + o(1)$.

We first establish approximation results satisfied by the eigenfunctions in order to investigate their behavior with respect to the t-variable. Then, we can apply a reduction of dimension and we are reduced to a family of 1D model operators which is studied in Appendix C.

6.1 Approximation of the eigenfunctions

Let us consider $N \geq 1$ and let us introduce:

$$\mathfrak{E}_N(\alpha) = \mathsf{span}\{\psi_{n,1}^{\mathsf{cut}}(\alpha), 1 \leq n \leq N\},$$

where $\psi_{n,1}^{\mathsf{cut}}(\alpha)(t,\theta,\varphi) = \chi_{\alpha,\eta}(t)\Psi_{n,1}(t,\varphi)$ are considered as functions defined in $\mathcal{P}$ (see Notation 5.1).

Proposition 6.2 *For all $N \geq 1$, there exist $\alpha_0(N) > 0$ and $C_N > 0$ such that, for all $\psi \in \mathfrak{E}_N(\alpha)$:*

$$\begin{aligned}
\|t^{-1}(\psi - \underline{\psi})\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} &\leq C_N\alpha^2\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}, && (6.1)\\
\|\psi - \underline{\psi}\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} &\leq C_N\alpha^2\left(\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} + \|t\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}\right), && (6.2)\\
\|t(\psi - \underline{\psi})\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} &\leq C_N\alpha^2\left(\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} + \|t\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} + \|t^2\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}\right), && (6.3)
\end{aligned}$$

where:

$$\underline{\psi}(t) = \frac{1}{\int_0^{\frac{1}{2}}\varphi\,\mathrm{d}\varphi}\int_0^{\frac{1}{2}}\psi(t,\varphi)\,\varphi\,\mathrm{d}\varphi. \tag{6.4}$$

Proof: It is sufficient to prove the proposition for $\psi = \psi_{n,1}^{\mathsf{cut}}(\alpha)$ and $1 \leq n \leq N$. We have:

$$\mathcal{L}_\alpha\Psi_{n,1}(\alpha) = \tilde{\lambda}_n(\alpha)\Psi_{n,1}(\alpha).$$

It follows:

$$\mathcal{L}_\alpha\chi_{\alpha,\eta}\Psi_{n,1}(\alpha) = \tilde{\lambda}_n(\alpha)\chi_{\alpha,\eta}\Psi_{n,1}(\alpha) + [\mathcal{L}_\alpha, \chi_{\alpha,\eta}]\Psi_{n,1}(\alpha). \tag{6.5}$$

Due to Agmon's estimates (see Proposition 4.1), we can write:

$$Q_\alpha(\psi) \leq (\tilde{\lambda}_N(\alpha) + \mathcal{O}(\alpha^\infty))\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

In particular, this provides:

$$Q_\alpha(\psi) \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})},$$

and thus, seeing ψ as a function on $\mathcal{P}$:

$$\frac{1}{\alpha^2}\int_{\mathcal{P}} t^{-2}|\partial_\varphi\psi|^2\,\mathrm{d}\tilde{\mu} \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

We get:

$$\int_{\mathcal{P}} |\partial_\varphi \psi|^2 \sin \alpha\varphi \, \mathrm{d}t \, \mathrm{d}\theta \, \mathrm{d}\varphi \leq C\alpha^2 \|\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})},$$

so that (using the inequality $\sin(\alpha\varphi) \geq \frac{\alpha\varphi}{2}$):

$$\int_{\mathcal{P}} \frac{\alpha\varphi}{2} |\partial_\varphi \psi|^2 \, \mathrm{d}t \, \mathrm{d}\theta \, \mathrm{d}\varphi \leq C\alpha^2 \|\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

We infer:

$$\int_{\mathcal{P}} \alpha\varphi |\partial_\varphi (\psi - \underline{\psi})|^2 \, \mathrm{d}t \, \mathrm{d}\theta \, \mathrm{d}\varphi \leq C\alpha^2 \|\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

Let us consider the Neumann realization of the operator $-\frac{1}{\varphi}\partial_\varphi \varphi \partial_\varphi$ on $\mathrm{L}^2((0, \frac{1}{2}), \varphi \, \mathrm{d}\varphi)$. The first eigenvalue is simple, equal to 0 and associated to constant functions. Let $\delta > 0$ be the second eigenvalue. The function $\psi - \underline{\psi}$ is orthogonal to constant functions in $\mathrm{L}^2((0, \frac{1}{2})\varphi \, \mathrm{d}\varphi)$ by definition (6.4). Then, we apply the min-max principle to $\psi - \underline{\psi}$ and deduce:

$$\int_{\mathcal{P}} \delta\alpha\varphi |\psi - \underline{\psi}|^2 \, \mathrm{d}t \, \mathrm{d}\theta \, \mathrm{d}\varphi \leq C\alpha^2 \|\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})},$$

and:

$$\int_{\mathcal{P}} t^{-2} |\psi - \underline{\psi}|^2 \, \mathrm{d}\tilde{\mu} \leq \tilde{C}\alpha^2 \|\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})},$$

which ends the proof of (6.1). We multiply (6.5) by t and we take the scalar product with $t\psi$ to get:

$$Q_\alpha(t\psi) \leq \tilde{\lambda}_N(\alpha) \|t\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})} + \left| \langle [-t^{-2}\partial_t t^2 \partial_t, t]\psi, t\psi \rangle_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})} \right| + \mathcal{O}(\alpha^\infty) \|\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

We recall that:

$$[-t^{-2}\partial_t t^2 \partial_t, t] = -2\partial_t - \frac{2}{t}.$$

We get:

$$Q_\alpha(t\psi) \leq C\|t\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})} + C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

We deduce (6.2) in the same way as (6.1).
Finally, we multiply (6.5) by t^2 and take the scalar product with $t^2\psi$ to get:

$$Q_\alpha(t^2\psi) \leq \tilde{\lambda}_N(\alpha) \|t^2\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})} + \left| \langle [-t^{-2}\partial_t t^2 \partial_t, t^2]\psi, t^2\psi \rangle_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})} \right| + \mathcal{O}(\alpha^\infty) \|\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

The commutator is:

$$[-t^{-2}\partial_t t^2 \partial_t, t^2] = -6 - 4t\partial_t.$$

This implies:

$$Q_\alpha(t^2\psi) \leq C(\|\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})} + \|t\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})} + \|t^2\psi\|^2_{\mathrm{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}).$$

The approximation (6.3) follows. ■

6.2 Control of the eigenfunctions with respect to t

Proposition 6.3 *For all $N \geq 1$, there exist $\alpha_0(N) > 0$ and $C > 0$ such that, for all $\alpha \in (0, \alpha_0(N))$ and $\psi \in \mathfrak{E}_N(\alpha)$, we have:*

$$\|t\psi\|_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})} \leq C\|\psi\|_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})}. \tag{6.6}$$

Proof: It is again enough to prove the proposition for $\psi = \psi_{n,1}^{\mathsf{cut}}(\alpha)$ and $1 \leq n \leq N$. We have:

$$Q_\alpha(\psi) \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})},$$

and this implies in particular that:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\psi|^2 \,\mathrm{d}\tilde{\mu} \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})}. \tag{6.7}$$

We now want to replace ψ by $\underline{\psi}$. For that purpose, we write:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\psi|^2 \,\mathrm{d}\tilde{\mu} - \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\underline{\psi}|^2 \,\mathrm{d}\tilde{\mu} = \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} \left(|t\psi|^2 - |t\underline{\psi}|^2\right) \mathrm{d}\tilde{\mu}.$$

We infer:

$$\begin{aligned} &\left|\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\psi|^2 \,\mathrm{d}\tilde{\mu} - \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\underline{\psi}|^2 \,\mathrm{d}\tilde{\mu}\right| \\ &\qquad\qquad \leq C\|t\psi - t\underline{\psi}\|_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})} \left(\|t\psi\|_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})} + \|t\underline{\psi}\|_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})}\right). \end{aligned} \tag{6.8}$$

Let us compare $\|t^k\underline{\psi}\|_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})}$ and $\|t^k\psi\|_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})}$ for $k = 0, 1$. Using the Jensen inequality and the comparison $\frac{\alpha\varphi}{2} \leq \sin\alpha\varphi \leq \alpha\varphi$ available for any $\varphi \in (0, \frac{1}{2})$, $\alpha \in (0, \pi)$, we have (denoting $c^{-1} = \int_0^{\frac{1}{2}} \varphi \,\mathrm{d}\varphi$):

$$\begin{aligned} \|\underline{\psi}\|^2_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})} &= c^2 \int_{t>0} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{1/2} \left(\int_{\varphi=0}^{1/2} \psi(t,\varphi)\varphi \,\mathrm{d}\varphi\right)^2 t^2 \sin\alpha\phi \,\mathrm{d}t \,\mathrm{d}\theta \,\mathrm{d}\phi \\ &\leq C \int_{t>0} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{1/2} \frac{\sin\alpha\phi}{\alpha} \left(\int_{\varphi=0}^{1/2} |\psi(t,\varphi)|^2 \alpha\varphi \,\mathrm{d}\varphi\right) t^2 \,\mathrm{d}t \,\mathrm{d}\theta \,\mathrm{d}\phi \\ &\leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})}, \end{aligned}$$

and similarly:

$$\begin{aligned} \|t\underline{\psi}\|^2_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})} &= c^2 \int_{t>0} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{1/2} t^2 \left(\int_{\varphi=0}^{1/2} \psi(t,\varphi)\varphi d\varphi\right)^2 t^2 \sin\alpha\phi \,\mathrm{d}t \,\mathrm{d}\theta \,\mathrm{d}\phi \\ &\leq C \int_{\phi=0}^{1/2} \frac{\sin\alpha\phi}{\alpha} \,\mathrm{d}\phi \int_{t>0} \int_{\theta=0}^{2\pi} t^2 \int_{\varphi=0}^{1/2} |\psi(t,\varphi)|^2 \, t^2\alpha\varphi \,\mathrm{d}t \,\mathrm{d}\theta \,\mathrm{d}\varphi \\ &\leq C\|t\psi\|^2_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})}. \end{aligned} \tag{6.9}$$

Then, putting together (6.7), (6.8) and (6.9), we deduce:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\underline{\psi}|^2 \,\mathrm{d}\tilde{\mu} \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})} + C\|t\psi\|_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})}\|t\psi - t\underline{\psi}\|_{\mathrm{L}^2(\mathcal{P},\, \mathrm{d}\tilde{\mu})}.$$

Since $\underline{\psi}$ does not depend on the φ-variable, we can write:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2 |\underline{\psi}|^2 \, \mathrm{d}\tilde{\mu} = \frac{\int_0^{\frac{1}{2}} \sin^3(\alpha\varphi)\, \mathrm{d}\varphi}{4\alpha^2 \int_0^{\frac{1}{2}} \sin(\alpha\varphi)\, \mathrm{d}\varphi} \int_{\mathcal{P}} |t\underline{\psi}|^2 \, \mathrm{d}\tilde{\mu}. \tag{6.10}$$

We notice that:

$$\frac{\int_0^{\frac{1}{2}} \sin^3(\alpha\varphi)\, \mathrm{d}\varphi}{4\alpha^2 \int_0^{\frac{1}{2}} \sin(\alpha\varphi)\, \mathrm{d}\varphi} = 2^{-5} + \mathcal{O}(\alpha^2). \tag{6.11}$$

Therefore, we get:

$$\int_{\mathcal{P}} |t\underline{\psi}|^2 \, \mathrm{d}\tilde{\mu} \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} + C\|t\psi\|_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} \, \|t\psi - t\underline{\psi}\|_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})}.$$

With Proposition 6.2, we infer:

$$\begin{aligned} \|t\underline{\psi}\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} &\leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} \\ &+ C\alpha\|t\psi\|_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} \left(\|\psi\|_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} + \|t\psi\|_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} + \|t^2\psi\|_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})}\right). \end{aligned} \tag{6.12}$$

We use the elementary inequality:

$$\|t\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} \leq 2(\|t\underline{\psi}\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} + \|t(\psi - \underline{\psi})\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})}). \tag{6.13}$$

We notice that, thanks to the support of ψ, we can write:

$$\|t^2\psi\|_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} \leq C\alpha^{-1/2-\eta}\|t\psi\|_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})}. \tag{6.14}$$

Combining (6.12), (6.3) of Proposition 6.2 and (6.13), we deduce:

$$\|t\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} \leq C\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} + C\alpha^{1/2-\eta}\|t\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})},$$

and (6.6) follows. ■

6.3 Proof of Proposition 6.1

We have now the elements to prove Proposition 6.1. The main idea is to apply the min-max principle to the quadratic form Q_α and to the space $\mathfrak{E}_N(\alpha)$.

Lemma 6.4 *For all $N \geq 1$, there exist $\alpha_N > 0$ and $C_N > 0$ such that, for all $\alpha \in (0, \alpha_N)$ and for all $\psi \in \mathfrak{E}_N(\alpha)$:*

$$\int_{\mathcal{P}} \left(|\partial_t\psi|^2 + 2^{-5}|t\psi|^2 + \frac{1}{\alpha^2 t^2}|\partial_\varphi\psi|^2\right) \mathrm{d}\tilde{\mu} \leq \tilde{\lambda}_N(\alpha)\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} + C_N\alpha^{1/2-\eta}\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})}.$$

Proof: We recall that, for all $\psi \in \mathfrak{E}_N(\alpha)$, we have:

$$Q_\alpha(\psi) \leq \tilde{\lambda}_N(\alpha)\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})} + \mathcal{O}(\alpha^\infty)\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})}.$$

We infer that:

$$\int_{\mathcal{P}} \left(|\partial_t\psi|^2 + \frac{\sin^2(\alpha\varphi)}{4\alpha^2}|t\psi|^2 + \frac{1}{\alpha^2 t^2}|\partial_\varphi\psi|^2\right) \mathrm{d}\tilde{\mu} \leq \left(\tilde{\lambda}_N(\alpha) + \mathcal{O}(\alpha^\infty)\right) \|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\mathrm{d}\tilde{\mu})}.$$

We shall analyze the term $\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}|t\psi|^2\,\mathrm{d}\tilde{\mu}$. We recall that (6.8) and (6.9) still hold. With Proposition 6.3, we get:

$$\left|\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\psi|^2\,\mathrm{d}\tilde{\mu} - \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\underline{\psi}|^2\,\mathrm{d}\tilde{\mu}\right| \leq C\|t\psi - t\underline{\psi}\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}\|\psi\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})},$$

and thus:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\psi|^2\,\mathrm{d}\tilde{\mu} \geq \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\underline{\psi}|^2\,\mathrm{d}\tilde{\mu} - C\|t\psi - t\underline{\psi}\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}\|\psi\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

Relation (6.14) and Propositions 6.2 and 6.3 provide:

$$\|t\psi - t\underline{\psi}\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \leq C\alpha^{1/2-\eta}\|\psi\|_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}, \tag{6.15}$$

so that:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\psi|^2\,\mathrm{d}\tilde{\mu} \geq \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\underline{\psi}|^2\,\mathrm{d}\tilde{\mu} - C\alpha^{1/2-\eta}\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

We recall that (6.10) and (6.11) hold. We deduce:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\psi|^2\,\mathrm{d}\tilde{\mu} \geq (2^{-5} - C\alpha^2)\int_{\mathcal{P}} |t\underline{\psi}|^2\,\mathrm{d}\tilde{\mu} - C\alpha^{1/2-\eta}\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}. \tag{6.16}$$

We have, for all $\varepsilon \in (0,1)$:

$$\|t\underline{\psi}\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \geq (1-\varepsilon)\|t\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} - \varepsilon^{-1}\|t(\psi - \underline{\psi})\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

Due to (6.15), it follows:

$$\|t\underline{\psi}\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \geq (1-\varepsilon)\|t\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} - C\varepsilon^{-1}\alpha^{1-2\eta}\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

We choose $\varepsilon = \alpha^{1/2-\eta}$ and get:

$$\|t\underline{\psi}\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \geq (1-\alpha^{1/2-\eta})\|t\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} - C\alpha^{1/2-\eta}\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

Proposition 6.3 and (6.16) provide:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2} t^2|\psi|^2\,\mathrm{d}\tilde{\mu} \geq 2^{-5}\int_{\mathcal{P}} |t\psi|^2\,\mathrm{d}\tilde{\mu} - C\alpha^{1/2-\eta}\|\psi\|^2_{\mathrm{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

■

A straightforward consequence of Lemma 6.4 is:

Lemma 6.5 *For all $N \geq 1$, there exist $\alpha_N > 0$ and $C_N > 0$ such that, for all $\alpha \in (0, \alpha_N)$ and for all $\psi \in \mathfrak{E}_N(\alpha)$:*

$$\int_{\mathcal{P}} \left(|\partial_t\psi|^2 + 2^{-5}|t\psi|^2 + \frac{1}{\alpha^2 t^2}|\partial_\varphi\psi|^2\right)\mathrm{d}\breve{\mu} \leq \left(\tilde{\lambda}_N(\alpha) + C_N\alpha^{1/2-\eta}\right)\|\psi\|^2_{L^2(\mathcal{P},\,\mathrm{d}\breve{\mu})},$$

with $\mathrm{d}\breve{\mu} = t^2\varphi\,\mathrm{d}t\,\mathrm{d}\varphi\,\mathrm{d}\theta$.

Proof: It is sufficient to write for any $\varphi \in (0, \frac{1}{2})$:

$$\varphi = \frac{1}{\alpha}\sin(\alpha\varphi)\frac{\alpha\varphi}{\sin(\alpha\varphi)} = \frac{1}{\alpha}\sin(\alpha\varphi)(1 + \mathcal{O}(\alpha^2)) \qquad \text{as } \alpha \to 0.$$

■

Combining Lemma 6.5 and Corollary C.3, we deduce (from the min-max principle) that there exists α_N such that

$$\forall \alpha \in (0, \alpha_N), \qquad \tilde{\lambda}_N(\alpha) \geq \mathfrak{l}_N - C\alpha^{1/2-\eta}.$$

This achieves the proof of Proposition 6.1.

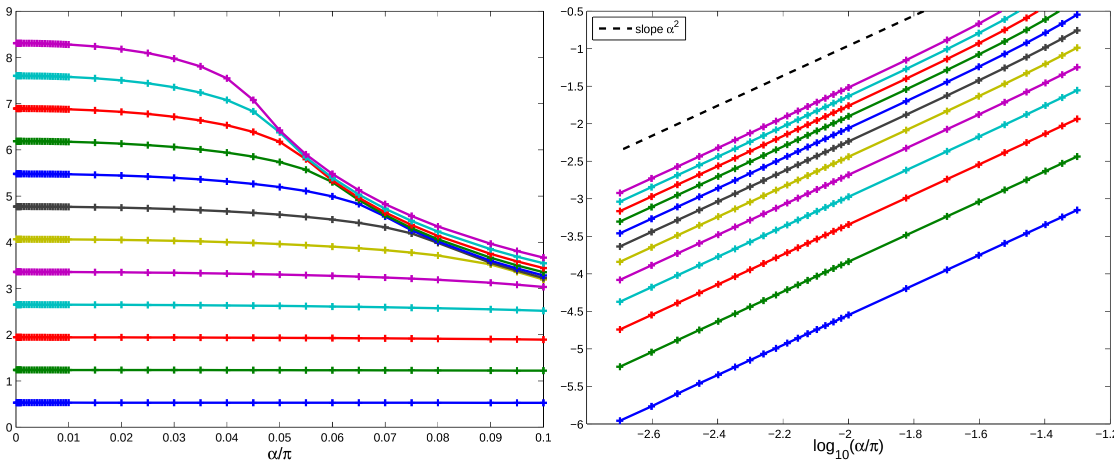
7 Numerical simulations

In this section, we illustrate the asymptotics expansion for the low-lying eigenpairs of $L_{\alpha,0}$ denoted by $(\mathfrak{l}_n(\alpha), \mathfrak{u}_n(\alpha))_{n\geq 1}$. According to Section 3 and Theorem 1.3, we have for α small enough

$$\mathfrak{l}_n(\alpha) = \mathfrak{l}_n + \mathcal{O}(\alpha^2), \qquad \text{and} \qquad \|\mathfrak{u}_n(\alpha) - \mathfrak{f}_n\|_{\mathrm{L}^2(\mathcal{R},\,\mathrm{d}\mu)} \leq C\alpha^2.$$

Numerically, we compute the eigenpairs of the operator $L_{\alpha,0}$ on the rectangle $\mathcal{R}_\ell = (0,\ell) \times (0, \frac{1}{2})$ with a Dirichlet condition on the artificial boundary $t = \ell$. We denote then by $L^\ell_{\alpha,0}$ this operator and $(\mathfrak{l}_n(\alpha,\ell), \mathfrak{u}_n(\alpha,\ell))$ its eigenpairs. We use the finite elements library MÉLINA ([13]) with 40×10 square elements of degree $\mathbb{Q}_{10}$ and $\ell = 40$. Figures 2 illustrate the convergence of the low-lying eigenvalues as α tends to 0. In particular, Figure 2(a) displays the first term of the asymptotic expansion of the eigenvalues, whereas Figure 2(b) confirms the asymptotic expansion in powers of α^2: we represent on Figure 2(b) the function

$$\log_{10}\left(\frac{\alpha}{\pi}\right) \mapsto \rho_{n,1}(\alpha) = \log_{10}(\mathfrak{l}_n(\alpha,\ell) - \mathfrak{l}_n).$$



(a) $\mathfrak{l}_n(\alpha,\ell)$ vs. $\frac{\alpha}{\pi} \in \{\frac{k}{200}, 1 \leq k \leq 20$ et $k = \frac{j}{5}, 1 \leq j \leq 5\}$, $\ell = 40$

(b) $\log_{10}(\mathfrak{l}_n(\alpha,\ell) - \mathfrak{l}_n)$ versus $\log_{10}\left(\frac{\alpha}{\pi}\right)$, $\ell = 40$, $\frac{\alpha}{\pi} \in \{\frac{k}{200}, 1 \leq k \leq 10$ et $k = \frac{j}{5}, 2 \leq j \leq 5\}$

Figure 2: Approximations of $\mathfrak{l}_n(\alpha)$ for $0 \leq \alpha \leq 0.1$, $1 \leq n \leq 12$, $\ell = 40$.

A Spherical coordinates

In dilated spherical coordinates $(t, \theta, \varphi) \in \mathcal{P}$ such that

$$(x, y, z) = \Phi(t, \theta, \varphi) = \alpha^{-1/2}(t\cos\theta\sin\alpha\varphi,\ t\sin\theta\sin\alpha\varphi,\ t\cos\alpha\varphi),$$

the magnetic potential reads

$$\mathbf{A}(t, \theta, \varphi) = \frac{\alpha^{-1/2}}{2}(-t\sin\theta\ \sin\alpha\varphi, t\cos\theta\ \sin\alpha\varphi, 0)^{\mathsf{T}}.$$

The Jacobian matrix associated with Φ is

$$D\Phi(t,\theta,\varphi)=\alpha^{-1/2}\begin{pmatrix}\cos\theta\sin\alpha\varphi & -t\sin\theta\sin\alpha\varphi & \alpha\ t\cos\theta\cos\alpha\varphi\\ \sin\theta\sin\alpha\varphi & t\cos\theta\sin\alpha\varphi & \alpha\ t\sin\theta\cos\alpha\varphi\\ \cos\alpha\varphi & 0 & -\alpha\ t\sin\alpha\varphi\end{pmatrix}.$$

We can compute

$$(D\Phi)^{-1}(t,\theta,\varphi)=\alpha^{1/2}t^{-1}\begin{pmatrix}t\cos\theta\sin\alpha\varphi & t\sin\theta\sin\alpha\varphi & t\cos\alpha\varphi\\ -\sin\theta(\sin\alpha\varphi)^{-1} & \cos\theta(\sin\alpha\varphi)^{-1} & 0\\ \frac{1}{\alpha}\cos\theta\cos\alpha\varphi & \frac{1}{\alpha}\sin\theta\cos\alpha\varphi & -\frac{1}{\alpha}\sin\alpha\varphi\end{pmatrix}.$$

Consequently, the metric becomes

$$G=(D\Phi)^{-1}\ {}^{\mathsf{T}}(D\Phi)^{-1}=\alpha\begin{pmatrix}1 & 0 & 0\\ 0 & t^{-2}(\sin\alpha\varphi)^{-2} & 0\\ 0 & 0 & (\alpha t)^{-2}\end{pmatrix}.$$

The change of variables leads to define the new magnetic potential

$$\tilde{\mathbf{A}}(t,\theta,\varphi)=\ {}^{\mathsf{T}}D\Phi\ \mathbf{A}(t,\theta,\varphi)=\alpha^{-1}\left(0,\frac{t^2\sin^2\alpha\varphi}{2},0\right). \tag{A.1}$$

Let ψ be a function in the form domain $\mathrm{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha)$ of the Schrödinger operator $(-i\nabla+\mathbf{A})^2$ and $\tilde{\psi}(t,\theta,\varphi)=\alpha^{-1/4}\psi(x,y,z)$ (where $\alpha^{-1/4}$ is a normalization coefficient). The change of variables on the norm and quadratic form reads

$$\|\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)}=\int_{\mathcal{P}}|\tilde{\psi}(t,\theta,\varphi)|^2\,t^2\sin\alpha\varphi\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

$$\begin{aligned}\int_{\mathcal{C}_\alpha}&|(-i\nabla+\mathbf{A})\psi(x,y,z)|^2\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &=\int_{\mathcal{P}}\langle G(-i\nabla_{t,\theta,\varphi}+\tilde{\mathbf{A}})\tilde{\psi},(-i\nabla_{t,\theta,\varphi}+\tilde{\mathbf{A}})\tilde{\psi}\rangle\ t^2\sin\alpha\varphi\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi\\ &=\alpha\int_{\mathcal{P}}\left(|\partial_t\tilde{\psi}|^2+\frac{1}{t^2\sin^2\alpha\varphi}\left|\left(-i\partial_\theta+\frac{t^2\sin^2\alpha\varphi}{2\alpha}\right)\tilde{\psi}\right|^2+\frac{1}{\alpha^2t^2}|\partial_\varphi\tilde{\psi}|^2\right)t^2\sin\alpha\varphi\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi.\end{aligned}$$

B Normal coordinates

Let us introduce the system of coordinates associated with the exponential map of the cone (see Figure 3). We can use the new coordinates (ρ,θ,τ) in the orthonormal basis

$$\mathbf{e}_\theta=\begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix},\qquad \mathbf{e}_\rho=\begin{pmatrix}\sin\frac{\alpha}{2}\cos\theta\\ \sin\frac{\alpha}{2}\sin\theta\\ \cos\frac{\alpha}{2}\end{pmatrix},\qquad \mathbf{e}_\tau=\begin{pmatrix}-\cos\frac{\alpha}{2}\cos\theta\\ -\cos\frac{\alpha}{2}\sin\theta\\ \sin\frac{\alpha}{2}\end{pmatrix}.$$

We consider the following change of variables

$$\Phi(\rho,\theta,\tau)=(x,y,z)=\begin{pmatrix}(\rho\sin\frac{\alpha}{2}-\tau\cos\frac{\alpha}{2})\sin\theta\\ (\rho\sin\frac{\alpha}{2}-\tau\cos\frac{\alpha}{2})\cos\theta\\ \rho\cos\frac{\alpha}{2}+\tau\sin\frac{\alpha}{2}\end{pmatrix}^{\mathsf{T}},$$

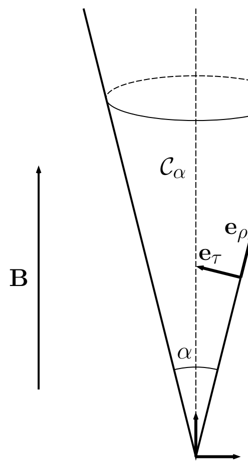


Figure 3: Normal coordinates.

whose differential is given by

$$D\Phi(\rho,\theta,\tau) = \begin{pmatrix} \sin\frac{\alpha}{2}\cos\theta & -(\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})\sin\theta & -\cos\frac{\alpha}{2}\cos\theta \\ \sin\frac{\alpha}{2}\sin\theta & (\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})\cos\theta & -\cos\frac{\alpha}{2}\sin\theta \\ \cos\frac{\alpha}{2} & 0 & \sin\frac{\alpha}{2} \end{pmatrix}.$$

Thus

$$^{\mathsf{T}}D\Phi(\rho,\theta,\tau) = \begin{pmatrix} \sin\frac{\alpha}{2}\cos\theta & \sin\frac{\alpha}{2}\sin\theta & \cos\frac{\alpha}{2} \\ -(\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})\sin\theta & (\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})\cos\theta & 0 \\ -\cos\frac{\alpha}{2}\cos\theta & -\cos\frac{\alpha}{2}\sin\theta & \sin\frac{\alpha}{2} \end{pmatrix},$$

$$(\,^{\mathsf{T}}D\Phi\ D\Phi)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $\det D\Phi = \rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2}$. The considered magnetic potential is

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -(\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})\sin\theta \\ (\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})\cos\theta \\ 0 \end{pmatrix}.$$

The change of variables leads to consider the new magnetic potential :

$$\hat{\mathbf{A}} = \,^{\mathsf{T}}D\Phi\ \mathbf{A} = \frac{1}{2}\begin{pmatrix} 0 \\ (\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})^2 \\ 0 \end{pmatrix}.$$

Thus the quadratic form $Q_{\mathbf{A}}$ becomes in the new coordinates

$$\begin{aligned} Q_{\mathbf{A}}(\psi) &= \|(-i\nabla + \mathbf{A})\psi\|^2_{\mathrm{L}^2(\mathcal{C}_\alpha)} = \hat{Q}_\alpha(\hat{\psi}) \\ &= \int_{\mathcal{D}_\alpha} \left(|\partial_\rho\hat{\psi}|^2 + \frac{1}{(\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})^2} \left| \left(i\partial_\theta + \tfrac{1}{2}(\rho\sin\tfrac{\alpha}{2} - \tau\cos\tfrac{\alpha}{2})^2\right)\hat{\psi} \right|^2 + |\partial_\tau\hat{\psi}|^2 \right) \mathrm{d}\hat{\mu}, \end{aligned} \tag{B.1}$$

with $\mathrm{d}\hat{\mu} = (\rho\sin\frac{\alpha}{2} - \tau\cos\frac{\alpha}{2})\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\tau$ and

$$\mathcal{D}_\alpha = \{(\rho,\theta,\tau) \in \mathbb{R}^3, \rho > 0, \theta \in [0,2\pi), \tau \in (0, \rho\tan\tfrac{\alpha}{2})\}.$$

C Model operators

Proposition C.1 *Let $\mathfrak{H}_\omega$ be defined on $\mathrm{L}^2(\mathbb{R}_+, t^2\,\mathrm{d}t)$ by*

$$\mathfrak{H}_\omega = -\frac{1}{t^2}\partial_t t^2 \partial_t + t^2 + \frac{\omega^2}{t^2}.$$

The eigenmodes of $\mathfrak{H}_\omega$ are $(\mathfrak{l}_n^\omega, \mathfrak{f}_n^\omega)_{n\geq 1}$ given by

$$\mathfrak{l}_n^\omega = 4n - 2 + \sqrt{1+4\omega^2}, \qquad \mathfrak{f}_n(t) = P_n^\omega(t^2)\ e^{-t^2/2},$$

with P_n^ω a polynomial function of degree $n-1$.

Proof: We recognize partially in $\mathfrak{H}_\omega$ the radial part of the harmonic oscillator. We first conjugate the operator $\mathfrak{H}_\omega$ by $t^\gamma e^{t^2/2}$ with a good choice for γ. We have

$$t^{-\gamma}e^{t^2/2}\mathfrak{H}_\omega(t^\gamma e^{-t^2/2}u) = -\partial_t^2 u + 2\left(t - (\gamma+1)\frac{1}{t}\right)\partial_t u + (2\gamma+3)u + (\omega^2 - \gamma(\gamma+1))\frac{1}{t^2}.$$

We cancel the term in t^{-2} by choosing

$$\gamma = \frac{-1+\sqrt{1+4\omega^2}}{2}.$$

This choice leads to deal with the following operator acting on $\mathrm{L}^2(t^{2(1+\gamma)}e^{-t^2}\,\mathrm{d}t)$:

$$t^{-\gamma}e^{t^2/2}\mathfrak{H}_\omega t^\gamma e^{-t^2/2} = -\partial_t^2 + \left(2t - (1+\sqrt{1+4\omega^2})\frac{1}{t}\right)\partial_t + 2 + \sqrt{1+4\omega^2}.$$

The change of variables $r = t^2$ transforms the operator on

$$-4r\partial_r^2 + 4\left(r - 1 - \frac{\sqrt{1+4\omega^2}}{2}\right)\partial_r + 2 + \sqrt{1+4\omega^2},$$

acting on $\mathrm{L}^2(\mathbb{R}_+, r^{1+\gamma}e^{-r}\,\mathrm{d}r)$. This operator is symmetric and stabilizes the polynomial functions of degree at most $n-1$. Therefore it can be diagonalized on $\mathbb{R}_{n-1}[X]$ and by identification, we determine a sequence of eigenpairs $(4n-2+\sqrt{1+4\omega^2}, P_n^\omega)$, with P_n^ω a polynomial function of degree $n-1$. Since the family (P_n^ω) is total, the spectrum is completely determined. ∎

Corollary C.2 *The eigenmodes of the operator*

$$\tilde{\mathfrak{H}} = -\frac{1}{t^2}\partial_t t^2\partial_t + \frac{1}{2^5}t^2,$$

defined on $\mathrm{L}^2(\mathbb{R}_+, t^2\,\mathrm{d}t)$ are given by

$$\mathfrak{l}_n = 2^{-5/2}(4n-1), \qquad \mathfrak{f}_n(t) = 2^{-5/4}\mathfrak{f}_n^0(2^{-5/4}t) = 2^{-5/4}P_n^0(2^{-5/4}t)e^{-t^2/2^{7/2}}.$$

Proof: It is enough to apply Proposition C.1 with $\omega = 0$ and make the change of variable $\tilde{t} = 2^{-5/4}t$. ∎

Corollary C.3 *Let $\mathfrak{T}(\alpha)$ be the Neumann realization on* $\mathrm{L}^2(\mathcal{P}, \mathrm{d}\breve{\mu})$ *(*$\mathrm{d}\breve{\mu} = t^2\varphi\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi$*):*

$$\mathfrak{T}(\alpha) = -\frac{1}{t^2}\partial_t t^2 \partial_t + 2^{-5}t^2 - \frac{1}{\alpha^2 t^2 \varphi}\partial_\varphi \varphi \partial_\varphi.$$

We denote by $\breve{\mathfrak{l}}_n(\alpha)$ the n-th eigenvalue of $\mathfrak{T}(\alpha)$. Then, for all $N \geq 1$, there exists α_N such that

$$\forall 1 \leq n \leq N, \quad \forall \alpha \in (0, \alpha_N), \qquad \breve{\mathfrak{l}}_n(\alpha) = \mathfrak{l}_n.$$

Proof: Let us first realize the change of variable $\tilde{t} = 2^{-5/4}t$, the operator $\mathfrak{T}(\alpha)$ reads

$$\mathfrak{T}(\alpha) = 2^{-5/2}\left(-\frac{1}{t^2}\partial_t t^2 \partial_t + t^2 - \frac{1}{\alpha^2 t^2 \varphi}\partial_\varphi \varphi \partial_\varphi\right).$$

Let us denote by $(c_k)_{k\geq 1}$ the increasing sequence of the eigenvalues of $-\frac{1}{\varphi}\partial_\varphi\varphi\partial_\varphi$ on $\mathrm{L}^2((0, \frac{1}{2}), \varphi\,\mathrm{d}\varphi)$. We notice that $c_1 = 0$ and that $c_k > 0$ for $k \geq 2$. The spectrum of $\mathfrak{T}(\alpha)$ is then given by

$$\mathsf{sp}(\mathfrak{T}(\alpha)) = 2^{-5/2}\bigcup_{k=1}^{\infty} \mathsf{sp}\left(-\frac{1}{t^2}\partial_t t^2 \partial_t + t^2 + \frac{c_k}{\alpha^2 t^2}\right).$$

Applying Proposition C.1 with $\omega = \sqrt{c_k}/\alpha$, we deduce

$$\mathsf{sp}(\mathfrak{T}(\alpha)) = \left\{2^{-5/2}\left(4n - 2 + \sqrt{1 + \frac{4c_k}{\alpha^2}}\right), n \geq 1, k \geq 1\right\}.$$

This implies that the lowest eigenvalues of $\mathfrak{T}(\alpha)$ are the lowest eigenvalues of the operator $-t^{-2}\partial_t t^2 \partial_t + 2^{-5}t^2$, that is to say $\mathfrak{l}_n$, as soon as α is small enough. ■

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