Geometry and gradient flow structure of the arctangential heat flow

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in association with the CNRS-INRIA team "MOKAPLAN".

Gradient flows: challenges and new directions,
ICMS, 10 to 14 September 2018, Edinburgh.

Outline

We introduce the "arctangential" heat flow

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and show that this degenerate scalar parabolic equation has a hidden gradient flow structure, in parallel with the mean curvature flow for graphs

$$\partial_t \phi = \sqrt{1+|\nabla\phi|^2}\ \nabla\cdot\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi|^2}}\right).$$

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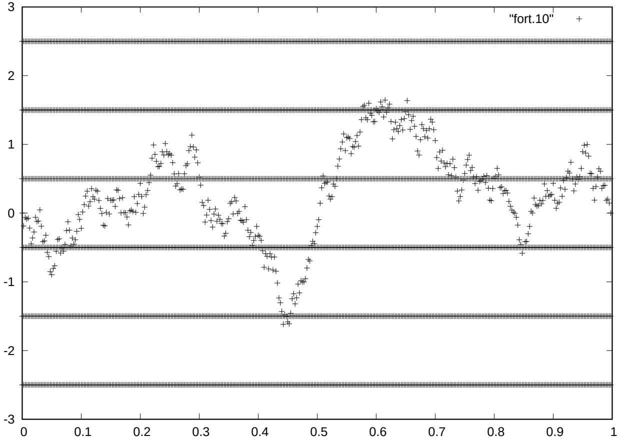
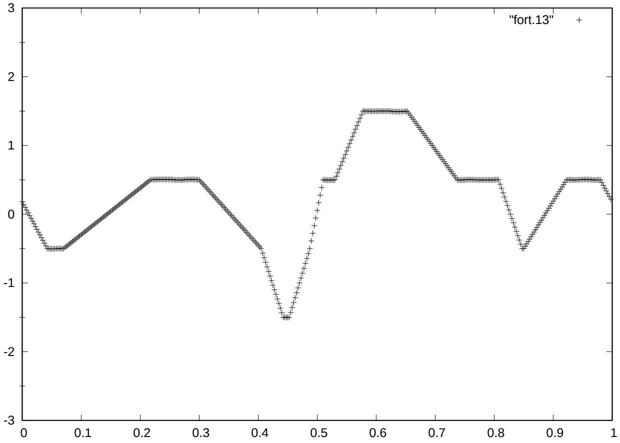
In (saturated) discrete form and 2 space dimensions:

$$\frac{\psi_{i,j}^{n+1} - \psi_{i,j}^{n}}{\cos(\pi\psi_{i,j}^{n})^2} = \frac{\psi_{i+1,j}^{n} + \psi_{i-1,j}^{n} + \psi_{i,j+1}^{n} + \psi_{i,j-1}^{n}}{4} - \psi_{i,j}^{n}.$$

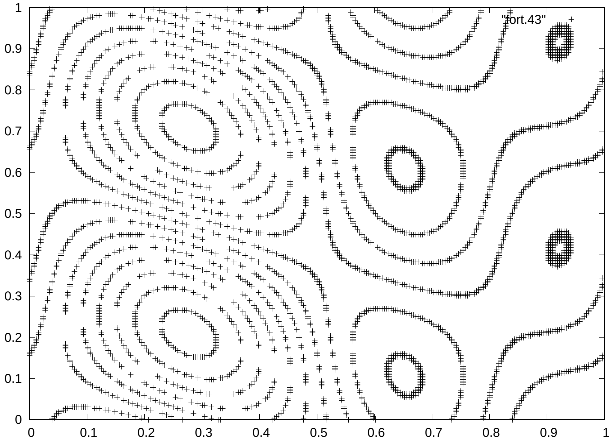


Few numerical results.

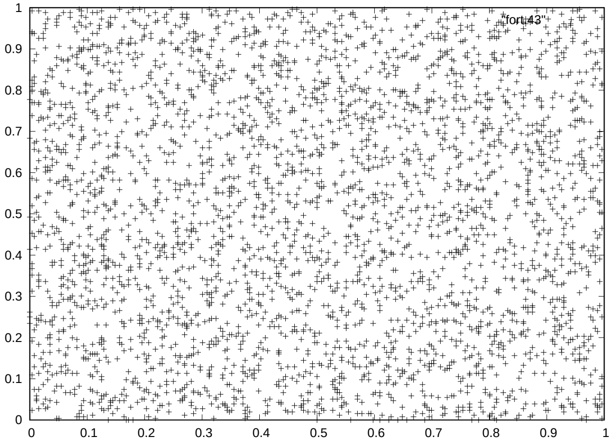
The 1d arctangential heat flow: brownian initial condition, $x \in \mathbb{R}/\mathbb{Z}$.



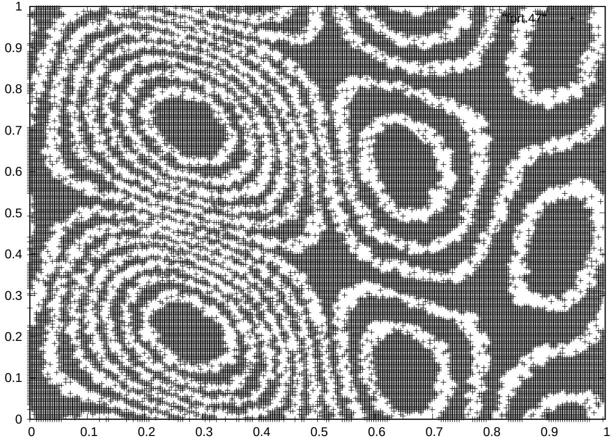
Numerical solution for 256 grid points, 4096 time steps.



The 2d arctangential heat flow: level sets of the data, $x \in \mathbb{R}^2/\mathbb{Z}^2$.

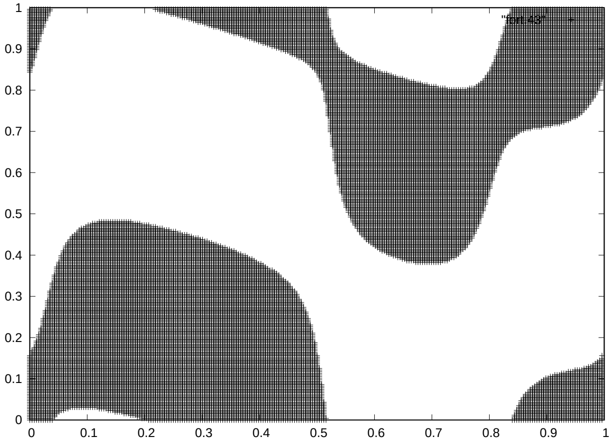


The 2d arctangential heat flow: noisy initial condition, $x \in \mathbb{R}^2/\mathbb{Z}^2$.

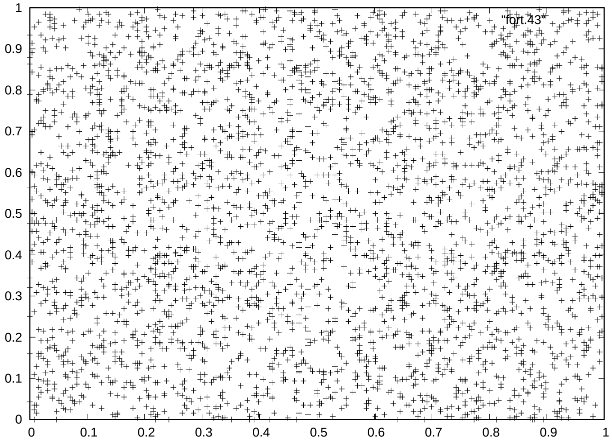


The 2d arctangential heat flow: recovery of the level sets

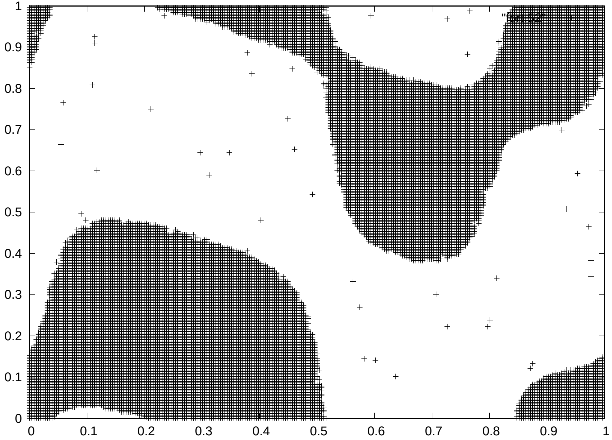
$x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.



The 2d arctangential heat flow: other choice of data (with binary values), $x \in \mathbb{R}^2/\mathbb{Z}^2$.

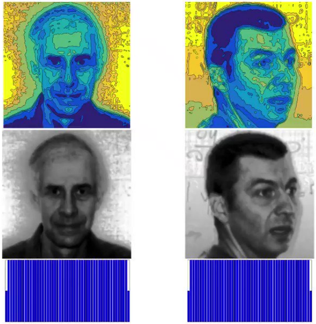


The 2d arctangential heat flow: noisy initial condition, $x \in \mathbb{R}^2/\mathbb{Z}^2$.



The 2d arctangential heat flow: recovery of the level sets $x \in \mathbb{R}^2/\mathbb{Z}^2$, $t \geq 0$.

Yann Brenier's heat equation: $\frac{\partial \psi}{\partial t} = \cos(\psi)^2 \Delta \psi$



twitter.com
Gabriel Peyré on Twitter
"Yann Brenier just introduced and studied the arctangential heat equation, related to Born-Infeld theory of Electromagnetism. This equation quantizes the values of the functions. https://t.co/NDdtI1QC9C https://t.co/kQ1W0uHfwn"

Publier sur Tumblr
Add a caption, if you like

The main result (Y.B. 2018, HAL preprint: hal-01740320):

The equation of extremal graphs in Minkowski's space

$$\partial_t\left(\frac{\partial_t\phi}{R}\right) = \nabla\cdot\left(\frac{\nabla\phi}{R}\right), \quad R = \sqrt{1 - \partial_t\phi^2 + |\nabla\phi|^2}$$

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(which is a nonlinear wave equation) generates two twin gradient flows. The first one is the arctangential heat flow $\partial_t D = \Delta(\arctan D)$, while the second one is just the well known mean curvature flow for graphs

$$\partial_t \phi = \sqrt{1 + |\nabla \phi|^2} \ \nabla \cdot \left(\frac{\nabla \phi}{\sqrt{1 + |\nabla \phi|^2}}\right).$$

A "naive" but interesting gradient flow structure

The arctangential flow $\partial_t D = \lambda\Delta(\arctan(D\lambda^{-1})$ can be easily written in optimal transport style (à la Otto):

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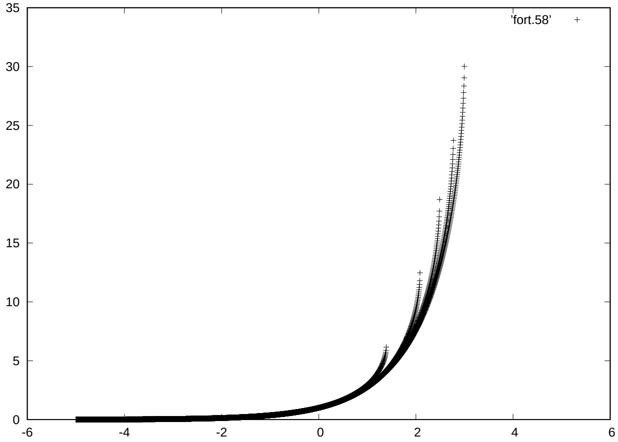
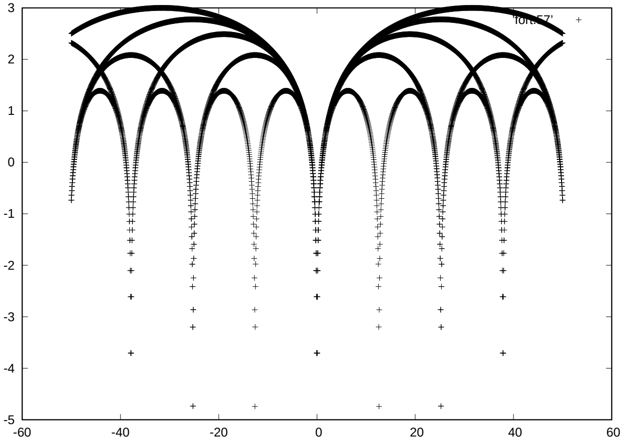


The “catastrophic” exponential for different values of parameter λ.
$u \to \lambda \arcsin(\lambda^{-1} \exp(u))$ (extended by $+\infty$ for $u > \log \lambda$).



The inverse of the "catastrophic" exponential
(after symmetrization and periodization), for different values of λ

$$v \to \frac{1}{2}\log(\lambda^2 \sin^2(v\lambda^{-1}))$$

is also used in "optimal unbalanced transport theory"!

Gradient flow structure of the arctangential flow

Alternately (and more geometrically), the arctangential flow is the gradient flow of functional $D \to \int_{\mathbb{T}^d} \sqrt{1+D^2}$ with metric $||v||_D^2 = \int_{\mathbb{T}^d} (1+D^2)^{-1/2} |vD|^2$, tangent vectors at point D being written as $\dot{D} = -\nabla \cdot (vD)$.

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It turns out that this structure is inherited from the nonlinear wave equation

$$\partial_t\left(\frac{\partial_t \phi}{R}\right) = \nabla \cdot \left(\frac{\nabla \phi}{R}\right), \quad R = \sqrt{1 - \partial_t \phi^2 + |\nabla \phi|^2}.$$

Gradient flow structure of the MC flow for graphs

In a parallel way, the mean curvature flow for graphs can be interpreted as the gradient flow of functional $B \to \int_{\mathbb{T}^d} \sqrt{1+|B|^2}$ with metric

$||v||_B^2 = \int_{\mathbb{T}^d} (1+|B|^2)^{-1/2}(B\cdot v)^2$ where tangent vectors at point $B = \nabla\phi$ are written as $\dot{B} = -\nabla(B\cdot v)$.

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Final part: proof of our main result.

We want to derive from the nonlinear wave equation

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at once, both the arctangential heat flow

$$\partial_t D = \Delta(\arctan D)$$

and the mean curvature flow for graphs

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$$(D,B,P)=\frac{1}{\sqrt{1-\partial_t\phi^2+|\nabla\phi|^2}}(\partial_t\phi,\nabla\phi,-\partial_t\phi\ \nabla\phi)$$

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$$(D,B,P) = \frac{1}{\sqrt{1-\partial_t\phi^2+|\nabla\phi|^2}}(\partial_t\phi, \nabla\phi, -\partial_t\phi\ \nabla\phi)$$

solves the "entropic" system of conservation laws:

$$\partial_t B + \nabla\left(\frac{P\cdot B - D}{h}\right) = 0, \quad \partial_t D + \nabla\cdot\left(\frac{PD-B}{h}\right) = 0,$$

$$\partial_t P + \nabla\cdot\left(\frac{P\otimes P + B\otimes B}{h}\right) = \nabla\left(\frac{1+B^2}{h}\right),$$

with $h=\sqrt{1+D^2+B^2+P^2}$ as convex entropy.

Proof/Second step. To get a gradient flow, we follow Y.B. and Xianglong Duan (ARMA 2018) by performing the simple singular change of time $t \to \theta = t^2/2$:

$$\mathcal{B}(\theta, x) = B(\sqrt{2\theta}, x),$$

$$\mathcal{D}(\theta, x) = \frac{D(\sqrt{2\theta}, x)}{\sqrt{2\theta}}, \quad \mathcal{P}(\theta, x) = \frac{P(\sqrt{2\theta}, x)}{\sqrt{2\theta}},$$

requiring initial condition $D = P = 0$ at $t = 0$ (which corresponds to $\partial_t \phi(0, x) = 0$ in terms of ϕ).

In a somewhat dual way, a second natural change is

$$\mathcal{D}(\theta, x) = D(\sqrt{2\theta}, x),$$

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requiring initial condition $B = P = 0$ at $t = 0$ (which corresponds to $\nabla\phi = 0$ at $t = 0$ for ϕ).

After performing the change of time $t \to \theta = t^2/2$, After performing the change of time $t \to \theta = t^2/2$, we get, in the 1st case, the non automous system:

$$\partial_\theta \mathcal{B} = \nabla \left(\mathcal{D} - \frac{\mathcal{P} \cdot \mathcal{B}}{\mathcal{H}} \right), \; \mathcal{H} = \sqrt{1 + \mathcal{B}^2 + 2\theta(\mathcal{D}^2 + \mathcal{P}^2)},$$

$$\mathcal{D} - \nabla \cdot \left(\frac{\mathcal{B}}{\mathcal{H}} \right) = -2\theta \left(\partial_\theta \mathcal{D} + \nabla \cdot \left(\frac{\mathcal{P}\mathcal{D}}{\mathcal{H}} \right) \right),$$

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Neglecting the red terms leads to the mean curvature flow (for graphs), in form:

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End of proof.

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End of proof. **THANKS !**