Generalized solutions and hydrostatic approximation of the Euler equations

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Solutions to the Euler equations on a 3 dimensional domain D_3 (typically the unit cube or the periodic unit cube) can be formally obtained by minimizing the action of an incompressible fluid moving inside D_3 between two given configurations. When these two configurations are very close to each other, classical solutions do exist, as shown by Ebin and Marsden. However, Shnirelman found a class of data (essentially two dimensional in the sense that they trivially depend on the vertical coordinate) for which there cannot be any classical minimizer. For such data generalized solutions can be shown to exist, as a substitute for classical solutions. These generalized solutions have unusual features that look highly unphysical (in particular different fluid parcels can cross at the same point and the same time), but the pressure field, which does not depend on the vertical coordinate, is well and uniquely defined. In the present paper, we show that these generalized solutions are actually quite conventional in the sense they obey, up to a suitable change of variable, a well-known variant (widely used for geophysical flows) of the 3D Euler equations, for which the vertical acceleration is neglected according to the so-called hydrostatic approximation.

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I. THE EULER EQUATIONS

A fluid moving inside a three dimensional compact domain D_3, such as the unit cube or the periodic unit cube, can be described by a time dependent family $t \to g(t)$ of orientation preserving diffeomorphisms of D_3 giving, at each time t, the position $g(t,a)$ of each fluid parcel of initial position $g(0,a) = a$ in D_3. A fluid is incompressible if and only if, for each t, the map

$$a \in D_3 \to g(t,a) \in D_3$$

has a unit jacobian determinant $|\partial_a g(t,a)| = 1$ or, equivalently,

$$\int_{D_3} f(g(t,a))da = \int_{D_3} f(a)da, \tag{1}$$

for all continuous function f. The fluid obeys the Euler equations if and only if g satisfies:

$$\partial^2_{tt} g(t,a) = -(\nabla p)(t, g(t,a)), \tag{2}$$

for some time dependent scalar field $p(t,x)$ (called the pressure field), that plays the role of a Lagrange multiplier for the incompressibility condition. Introducing the Eulerian velocity field $u(t,x) \in R^3$, defined by:

$$u(t, g(t,a)) = \partial_t g(t,a), \tag{3}$$

we recover from (2) the more familiar Euler equations written in "Eulerian coordinates" [9]:

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \tag{4}$$

together with the divergence free condition $\nabla \cdot u = 0$. The mathematical analysis of this system of non-linear PDEs is one of the most important and challenging problem in modern analysis (see [10–12] for discussions). As Euler said: "s'il reste des difficultés, ce ne sera pas du côté du méchanique, mais uniquement du côté de l'analytique" [9] (first page of the original edition).

II. THE LEAST ACTION PRINCIPLE

The Euler equations, written in "Lagrangian coordinates" (2), have a variational interpretation. For smooth g and p, they *exactly* means that, for each time interval $[t_0, t_1]$, the curve $t \to g(t)$ makes stationary the Action

$$\int_{t_0}^{t_1} \int_{D_3} \frac{1}{2} |\partial_t g(t,a)|^2 \, da \, dt, \tag{5}$$

among all smooth curves valued in $SDiff(D_3)$, the class of volume and orientation preserving diffeomorphisms of D_3, that coincide with g at $t = t_0$ and $t = t_1$. This can be seen immediately by varying with respect to both g and p the Lagrangian:

$$\int_{t_0}^{t_1} \int_{D_3} (\frac{1}{2} |\partial_t g(t,a)|^2 - p(t, g(t,a)) + p(t,a)) \, da \, dt,$$

that takes into account the incompressibility constraint (1) (obtained by varying p only). In addition, the curve is not only a critical point of the Action but also a minimizer if the time interval is small enough. If D_3 is a convex domain, a sufficient condition for that is:

$$(t_1 - t_0)^2 \sum_{i,j=1,3} \frac{\partial^2 p(t,x)}{\partial x_i \partial x_j} \xi_i \xi_j \le \pi^2 |\xi|^2, \tag{6}$$

for all $t \in [t_0, t_1]$, $x \in D_3$ and $\xi \in R^3$. (This can be shown using the one-dimensional Poincaré inequality.) Thus the

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Euler equations are governed by the Least Action Principle, as guessed from the very beginning by Euler himself ([9] p.287 of the original edition): "Cette belle propriété convient admirablement avec le beau principe de la moindre action dont nous devons la découverte à notre Illustre Président, M. de Maupertuis." Through the Least Action Principle, a remarkable geometric interpretation of the Euler equations has been emphasized by Arnold (see [3] for more details): the Euler equations are just the equations of geodesics curves (with constant speed) along the group of all orientation and volume preserving diffeomorphisms $SDiff(D_3)$ for the metric induced by the embedding of the group in the space L^2 of all square integrable maps from D_3 into R^3.

III. THE ACTION MINIMIZATION PROBLEM

The least Action principe suggests a possible (and of course not unique) way to get solutions to the Euler equations. We minimize (5), where $t_0 < t_1$ are fixed and $g(t_0) = h_0$, $g(t_1) = h$ are prescribed in $SDiff(D_3)$. Due to the homogeneity of the Euler equations, we can normalize $t_0 = 0$, $t_1 = 1$ and assume h_0 to be the identity map, so that the only datum is $h \in SDiff(D_3)$. The Action minimization problem has indeed smooth solutions (which do satisfy the Euler equations) provided that h is sufficiently closed to the identity in some suitable norm (typically for the Sobolev norm $H^s(D_3)$ with $s > 5/2$). This has been shown by Ebin and Marsden in [8]. However, in the large, as shown by Shnirelman [15], in the case when D_3 is the unit cube $[0,1]^3$, there are data h for which the existence of a smooth minimizer is impossible. Shnirelman's data are of form:

$$h(a) = (H(a_1,a_2),a_3), \quad a = (a_1,a_2,a_3) \in [0,1]^3, \tag{7}$$

where H belongs to $SDiff([0,1]^2)$, the set of all area and orientation preserving diffeomorphisms of the unit square $[0,1]^2$.

They are chosen [15] so that whenever a minimizer g exists, it must have a non trivial vertical component (i.e. $g_3(t,a) = a_3$ is impossible). (In other words, the Action can be reduced by using some vertical motion between $t = 0$ and $t = 1$. As a matter of fact, this happens for a lot of maps H, since purely horizontal motions are very rigid in comparison with fully 3D motions.) Then, we easily see that for such data H classical minimizers cannot exist. Indeed, any admissible solution $g(t,a)$ with non trivial vertical component, as well as the corresponding eulerian velocity field $u(t,x)$ defined by (3), can be rescaled in its vertical component by a positive integer factor n which leads to a strictly lower value of the Action. More precisely, let us define the rescaled space coordinate:

$$x^{(n)} = (x_1, x_2, nx_3 \;\; modulo\, 1), \quad x = (x_1,x_2,x_3),$$

the rescaled velocity:

$$u^{(n)}(t,x) = (u_1(t,x^{(n)}), u_2(t,x^{(n)}), n^{-1}u_3(t,x^{(n)})),$$

and recover the corresponding $g^{(n)}$ through (3). The remarkable fact is that $g^{(n)}$ is still admissible, with unit jacobian determinant (because $u^{(n)}$ is still divergence free) and unchanged end point values:

$$g^{(n)}(0,a) = g(0,a) = a,$$

$$g^{(n)}(1,a) = g(1,a) = h(a) = (H(a_1,a_2),a_3)$$

(because h depends trivially on the vertical coordinate), but has a strictly reduced Action, given by:

$$\int \frac{1}{2}\{\partial_t g_1(t,a)^2 + \partial_t g_2(t,a)^2 + n^{-2}\partial_t g_3(t,a)^2\}\, da\, dt.$$

Since there is no end to this rescaling process, we conclude there cannot be a minimizer, at least in a classical sense. (Strictly speaking, there is a flaw in the previous reasoning, since the renormalized flow may loose the smoothness of the original flow. This can be cured in two ways. The first one followed by Shnirelman amounts to slightly mollify the renormalized flow. The second one is to do the construction on the *periodic* unit cube, in which case there is no mollification to do.)

IV. THE HYDROSTATIC APPROXIMATION

In the case of Shnirelman's data, when we try to minimize the Action, we cannot get a classical solution because of the degeneracy of the data in the vertical coordinate, as explained in the previous section. It is therefore natural to minimize instead the renormalized Action obtained by dropping the vertical component of the velocity in definition (5). Then, we expect to get, at least formally, generalized solutions that substitute for the missing classical solutions. More precisely, we are now looking for a solution $t \to g(t)$ still valued in $SDiff([0,1]^3)$, with $g(0,a) = a$, $g(1,a) = h(a) = (H(a_1,a_2),a_3)$, that minimizes:

$$\int_0^1 \int_{[0,1]^3} \{\partial_t g_1(t,a)^2 + \partial_t g_2(t,a)^2\}\, da\, dt, \tag{8}$$

where the vertical component of the velocity has been dropped. The corresponding Lagrangian now reads:

$$\int \{\frac{1}{2}(\partial_t g_1(t,a)^2 + \partial_t g_2(t,a)^2) - p(t,g(t,a)) + p(t,a)\} da\, dt.$$

The *formal* optimality equations are just:

$$\partial^2_{tt} g_i(t,a) + (\partial_i p)(t,g(t,a)) = 0, \quad i = 1,2, \tag{9}$$

$$(\partial_3 p)(t,g(t,a)) = 0,$$

in addition to the incompressibility condition (1). Written in Eulerian coordinates, with the velocity field u given by (3), these equations

$$\partial_t u_i(t,a) + (u \cdot \nabla)u_i + \partial_i p = 0, \quad i = 1,2, \tag{10}$$

$$\partial_3 p = 0, \quad \nabla \cdot u = 0, \tag{11}$$

are nothing but the Euler equations where the vertical acceleration is neglected under the so-called "hydrostatic approximation" widely used for the modelling of geophysical flows [14]. In particular, the pressure field does not depend on the vertical coordinate.

V. GENERALIZED FLOWS AND GENERALIZED EULER EQUATIONS

Although the motion described by the hydrostatic approximation (10) to the Euler equations is fully 3 dimensional, the vertical component is actually slaved by the horizontal one. Indeed, we may completely ignore g_3 and still find a self-consistent set of equations for p and the horizontal component (g_1, g_2). To do that, we keep (9), with the boundary conditions at $t = 0$ and $t = 1$:

$$(g_1, g_2)(t = 0, a_1, a_2, a_3) = (a_1, a_2), \tag{12}$$

$$(g_1, g_2)(t = 1, a_1, a_2, a_3) = H(a_1, a_2),$$

corresponding to a Shnirelman data, and we use the incompressibility condition (1) only for continuous functions $f(a_1, a_2)$ that do not depend on a_3, which leads to:

$$\int_{[0,1]^3} f((g_1, g_2)(t, a_1, a_2, a_3))da_1 da_2 da_3 \tag{13}$$

$$= \int_{[0,1]^2} f(a_1, a_2)da_1 da_2.$$

At this point, the full incompressibility condition (1) is not needed to get (p, g_1, g_2) but can be used *a posteriori* to recover the vertical component g_3 from the horizontal component (g_1, g_2). Notice the particular role of a_3 in these equations, which is just an extra parameter without geometric meaning, and that we may decide now to call ω (just as a random variable valued in a probability space Ω). So the horizontal component of the 3D Euler flow obtained through the hydrostatic approximation, $G = (g_1, g_2)$, can also be seen as a non classical 2D flow on the horizontal domain $D = [0, 1]^2$. This flow does not look conventional at all, since each 2D fluid parcel initially located at $A = (a_1, a_2) \in D$ may split and follow different paths (that are allowed to cross each other!), each of them being labelled by $\omega \in \Omega$:

$$t \in [0, 1] \to G(t, A, \omega) \in D, \tag{14}$$

with time boundary conditions:

$$G(t = 0, A, \omega) = A, \;\; G(t = 1, A, \omega) = H(A). \tag{15}$$

This unusual description of a 2D flow becomes natural once it is understood that G actually is the horizontal projection of a conventional 3D incompressible flow. Indeed, each 2D fluid parcel initially located at $A \in D$ actually corresponds to an entire vertical column of 3D fluid parcels. This column ends up at time $t = 1$ as the vertical column above $H(A)$. However, at each intermediary time $0 < t < 1$, the 3D fluid parcels initially above A do not necessarily form a vertical column but rather a curve in $[0, 1]^3$ with horizontal projection given by $\omega \to G(t, A, \omega)$. So the strange behaviour of the 2D 'generalized' flow described by G comes naturally from the projection from 3 to 2 dimensions. Also notice that condition (13) can be understood as a generalized incompressibility condition, meaning that the density of the fluid parcels stays uniform on D:

$$\int_{D\times\Omega} f(G(t, A, \omega))dA\, d\omega = \int_D f(A)dA, . \tag{16}$$

for all function f continuous on D. In this language, the optimality condition, say (9), becomes a generalized version of the 2D Euler equation:

$$\partial^2_{tt} G(t, A, \omega) + (\nabla p)(t, G(t, A, \omega)) = 0, \tag{17}$$

where $p = p(t, x)$ is a time dependent function defined on D. Let us finally observe that the renormalized Action (8) can be easily written as:

$$\frac{1}{2}\int_0^1 dt \int_{D\times\Omega} |\partial_t G(t, A, \omega)|^2 dA\, d\omega. \tag{18}$$

So, in this section, we have derived from the hydrostatic approximation of the Euler equations (that comes up in a natural way to deal with Shnirelman's data for the Action Minimization problem), a generalized framework (14,16, 17, 18), that can be used outside of the hydrostatic context and still makes sense for a general $d-$dimensional domain D, not only the unit square $[0, 1]^2$, and without refering to any additional dimension. In particular, D can be taken to be D_3 itself. In addition, time boundary data can be taken in a much more general class than Shnirelman's data as in (15). As a matter of fact, $G(t = 0, A, \omega) \in D$ and $G(t = 1, A, \omega) \in D$ can be chosen arbitrarily provided they are compatible with the generalized incompressibility condition (16). In particular, we can consider boundary data of type (15), where H is chosen in the class $M(D)$ of all measure preserving map of D, which means that H is just a (Borel) measurable maps that satisfy

$$\int_D f(H(A))dA = \int_D f(A)dA, \tag{19}$$

for all function f continuous on D.

VI. MATHEMATICAL ANALYSIS OF THE ACTION MINIMIZATION PROBLEM

So far, we have just made a formal analysis of the Action Minimization problem for Shnirelman's data (7) leading in a natural way to the hydrostatic approximation to the Euler equations, that can rephrased in terms of 2D generalized incompressible flows and generalized 2D Euler equations. A rigorous justification of this formal analysis has been provided in [1, 4–6, 16]. Let us summarize the results obtained in this series of papers. The results are stated either for $D = T^d$ or $D = [0,1]^d$ and $d \geq 1$.

1) For all generalized data $G(0, A, \omega)$, $G(1, A, \omega)$, there is at least one generalized incompressible flow $G(t, A, \omega)$ that minimizes the generalized Action (18) [4], [1].

2) There is a *unique* pressure gradient $\nabla p(t, x)$ depending only on the data such that the generalized Euler equation (17) is satisfied by G (which is not necessarily unique), in a suitable sense [5]. More precisely, an Eulerian version of the generalized Euler equations has been established in [6]. More recently, Ambrosio and Figalli [1] have shown that (almost surely) each individual trajectories, $t \to \gamma(t) = G(t, A, \omega)$, A and ω being fixed, is a minimizer of the localized Action

$$\int_0^1 (\frac{1}{2}|\gamma'(t)|^2 - p(t, \gamma(t)))dt, \tag{20}$$

γ being fixed at time $t = 0$ and $t = 1$. (A key point being that the known regularity of p is sufficient to give sense to this localized Least Action principle.)

3) In the case $d = 3$, with "deterministic" time boundary data

$$G(t = 0, A, \omega) = A, \quad G(t = 1, A, \omega) = H(A),$$

where H is a given in $M(D)$ (the class of all measure preserving maps of D, which includes Shnirelman's data), for each generalized solution G and each $\epsilon > 0$, there is a classical incompressible flow $g(t, A)$ such that i) $g(0, A) = A$, ii) $g(1, A) - H(A)$ has an L^2 norm less than ϵ, iii) the classical Action of g (5) differs from the generalized Action of G (18) by less than ϵ. Moreover, the acceleration field $\partial^2_{tt} g \circ g^{-1}$ approachs $-\nabla p$ in the distributional sense as ϵ tends to zero.

The last statement shows that generalized solutions can be approximated by nearly classical solutions to the 3D Euler solutions. In our opinion, all these results provide a full legitimacy to the generalized framework in the mathematical study of the Action Minimization problem for general data H given in $M([0,1]^3)$. In addition, as discussed in the previous sections, in the case of Shnirelman's data, generalized solutions have a clear physical interpretation in terms of hydrostatic approximation to the Euler equations.

VII. A NUMERICAL SCHEME

It has been known for a long while that permutations are suitable to approximate volume preserving maps. (See [7, 13, 15], for example.) This suggest the following strategy to compute approximate solutions to the Action Minimization problem. Hereafter, the computational domain will be $D = [0,1]^d$ (and more specifically $d = 1$ for actual computations). First, we fix two integers N and M. Then, we introduce a uniform time step $1/M$ and we split the unit cube D (up to a set of zero Lebesgue measure) into N^d subcubes, denoted by $D_{N,i}$, for $i = 1, N^d$. The center of mass of each $D_{N,i}$ will be denoted by $x_{N,i}$. To each permutation σ of the N^d first integer, we associate the map H that rigidly moves the subcube $D_{N,i}$ to the subcube $D_{N,\sigma(i)}$, for each $i = 1, N^d$. This map is measure-preserving in the sense of definition(19). We call $P(D)$ the collection of all "permutation maps" obtained this way, for all integers N. The class $M(D)$ of all measure preserving maps of D in the sense of definition(19) can be shown to be the L^2 completion of $P(D)$ for all $d \geq 1$. When $d \geq 2$, $M(D)$ is also the L^2 completion of $SDiff(D)$. (See [7, 13, 15].) To each sequence of $M+1$ permutations $\sigma_0, \cdots, \sigma_M$, we may associate a "discrete flow" made of the $M+1$ corresponding permutation maps and define a "discrete Action" defined by:

$$\sum_{k=1,M} \sum_{i=1,N^d} |x_{N,\sigma_m(i)} - x_{N,\sigma_{m-1}(i)}|^2. \tag{21}$$

The discrete Action Minimization problem amounts to fix the initial and final permutations and to minimize the discrete Action. Typically the initial permutation is just $\sigma_0(i) = i$ and the final one is chosen so that the corresponding permutation map is an accurate approximation in L^2 of a given measure-preserving map $H \in M(D)$.

VIII. NUMERICAL RESULTS

Let us consider three maps H of the unit cube $D = [0,1]^3$ of the following form:

$$H(a_1, a_2, a_3) = (T(a_1), a_2, a_3) \tag{22}$$

with, successively,

$$T(s) = \min(2s, 2 - 2s), \quad s \in [0,1],$$

$$T(s) = s + \frac{1}{2} \, mod \, 1, \quad s \in [0,1],$$

$$T(s) = 1 - s, \quad s \in [0,1].$$

These three maps H clearly belong to the class of measure preserving maps $M(D)$ but certainly not to the class of diffeomorphisms $SDiff(D)$. However, as mentioned earlier, they do belong to the L^2 *closure* of $SDiff(D)$ and we know that the corresponding generalized solution for the generalized Action Minimization problem describes limit of nearly solutions to the 3D Euler equations. Therefore, looking for approximate numerical solutions is not meaningless. Since the corresponding T actually belong to $M([0,1])$, the discrete minimization problem can be reduced to one space dimension. Thus, the minimization can be performed very efficiently by using Gauss-Seidel iterations and sorting algorithms. To approximate the first map T, we use the permutation $\sigma(i) = 2i$, for i between 1 and $N/2$, $\sigma(i) = 2N - 2i + 1$, for i between $1 + N/2$ and N. For the second one, $\sigma(i) = N/2 + i$, for i between 1 and $N/2$, $\sigma(i) = i - N/2$, for i between $1+N/2$ and N. For the third one, $\sigma(i) = N-i+1$, for i between 1 and $N-2$, with $\sigma(N) = 2$ and $\sigma(N-1) = 1$. (These two last values are introduced in order to break the symmetry of the algorithm.) Some numerical results are shown at the end of the paper. First, we show (figures 1-2-3), for the three different maps, the successive permutation maps computed at $t = 0$, $t = 1/8$, $t = 1/4$, $t = 3/8$, $t = 1/2$, $t = 5/8$, $t = 3/4$, $t = 7/8$ and $t = 1$. The value of M is 16 and $N = 400$, $N = 100$ and $N = 4000$ respectively. Next, we draw (figures 4-5-6), for the three different maps, a collection of trajectories (the time axis being vertical, and the space axis being horizontal), obtained by linear interpolation of the discrete trajectories $m \to x_{N,\sigma_{m-1}(i)}$, for a fixed proportion of grid points $i = 1, \cdot\cdot, N$. These pictures give a good feeling of the missing dimension(s) encoded by the one-dimensional computation. In particular, for the first map, we see that the particles issued from the right part $[1/2, 1]$ of the unit interval manage to cover the whole unit interval in reverse order through a kind of vortical flow, meanwhile the particles coming from the left also cover the whole interval, but in an order-preserving way through a potential flow. For the second map, we see a kind of two-phase flow, without vorticity. Concerning the third map, for the sake of clarity, we draw trajectories only for the particles initially located in a neighborhood of $x = 3/4$. We see that they form a bundle of trajectories very close at $t = 0$, then diverging and meeting again in a neighborhood of $x = 1/4$ at $t = 1$. At the moment, there is no rigorous convergence analysis of the numerical method. However, for the third map, the exact unique generalized solution is known (see [4]):

$$p(x) = \frac{\pi^2}{2}(x - \frac{1}{2})^2, \quad x \in [0, 1],$$

$$G(t, A, \omega) = 1/2 + (A - 1/2)\cos(\pi t) + v(A, \omega)\sin(\pi t),$$

$$v(A, \omega) = \pi\sqrt{\frac{A(1-A)}{2}}\cos(\pi\omega),$$

for $t, A, \omega \in [0, 1]$. We can see that the solution is correctly recovered by the computation. It is striking that a good resolution requires a much more refined mesh in space ($N = 1000$) than in time ($M = 16$).

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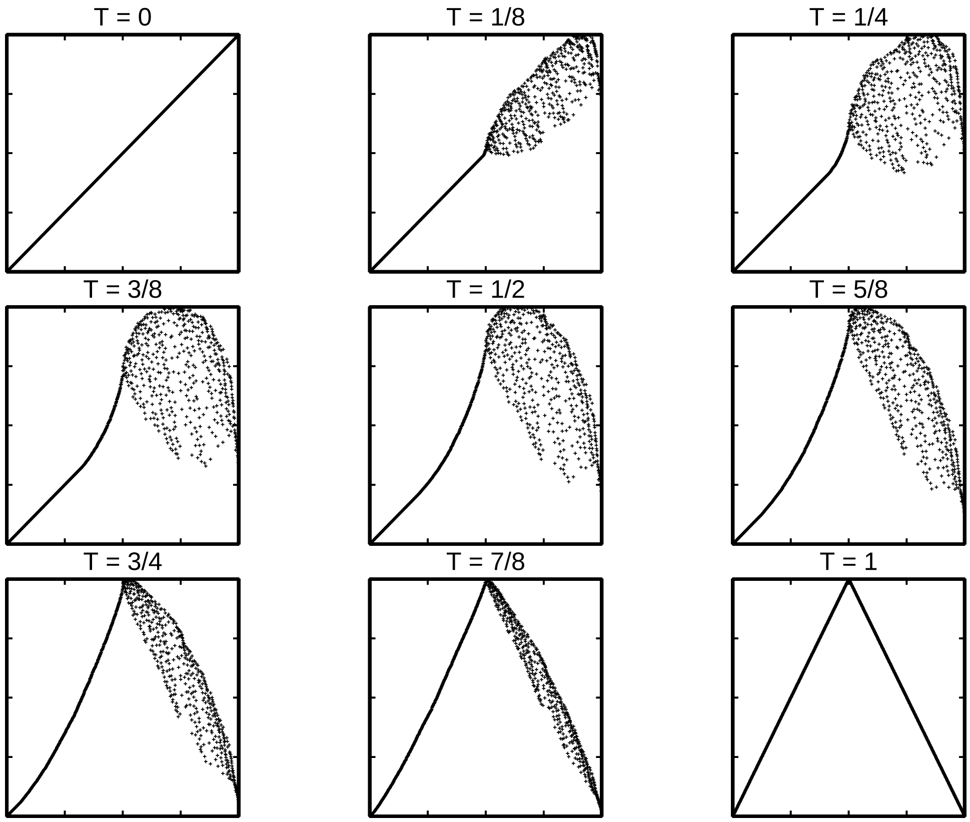

FIG. 1: APPROXIMATE GEODESIC FOR MAP 1

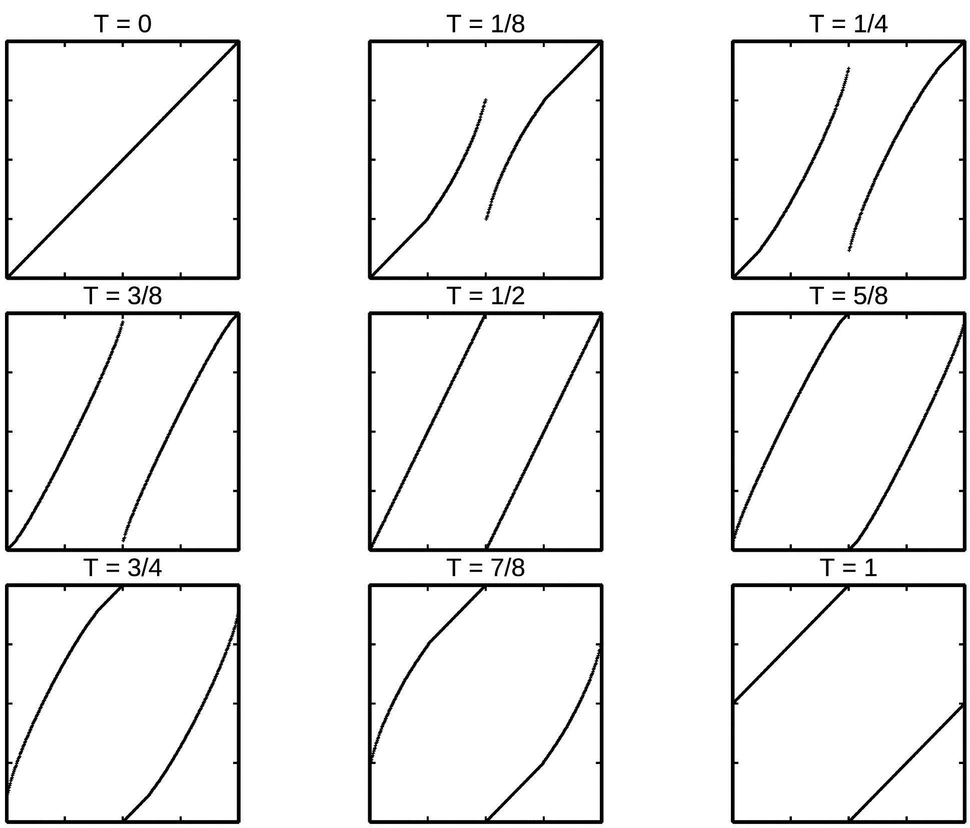

FIG. 2: APPROXIMATE GEODESIC FOR MAP 2

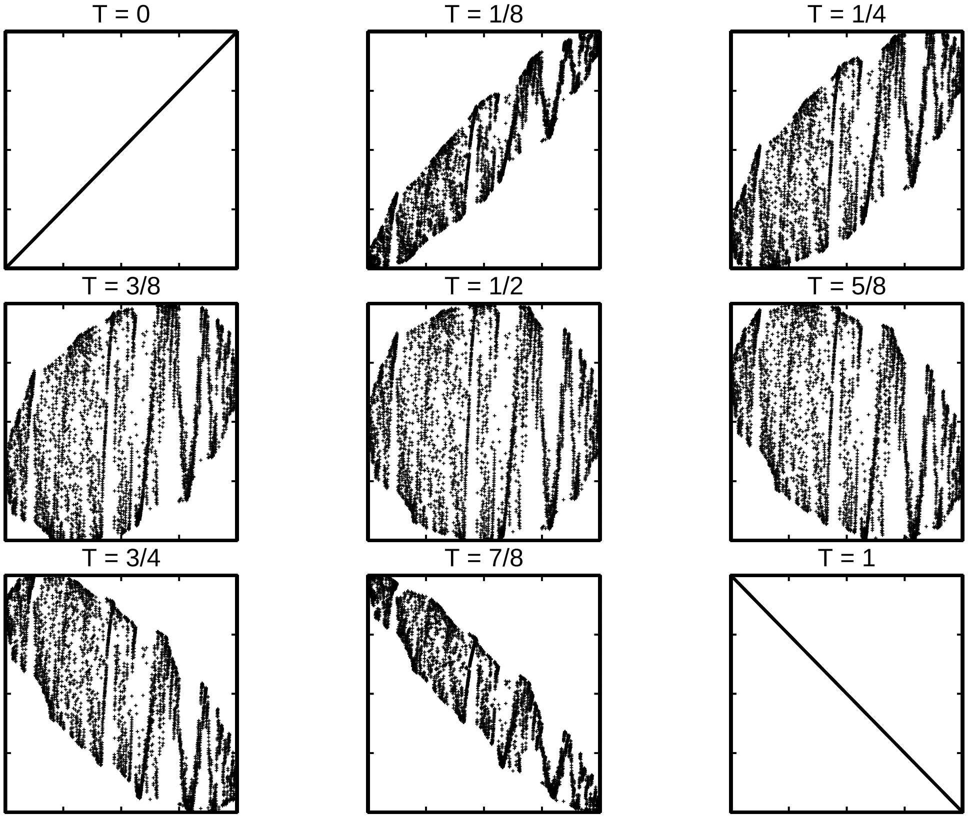

FIG. 3: APPROXIMATE GEODESIC FOR MAP 3

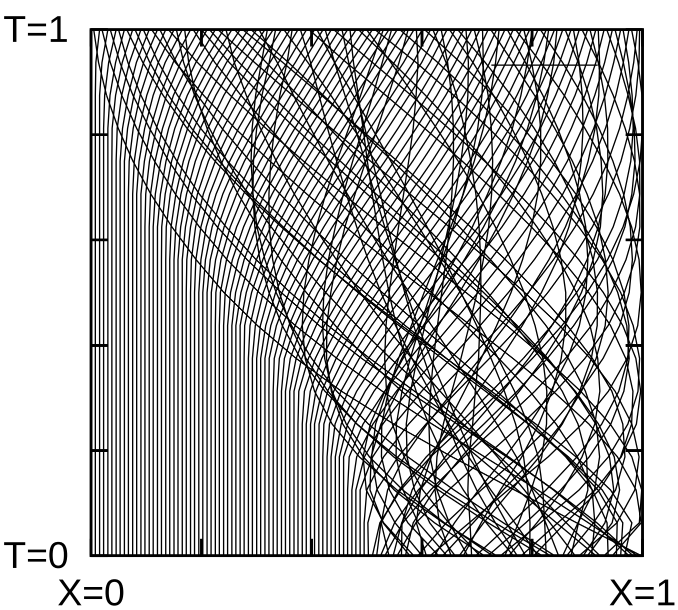

FIG. 4: TRAJECTORIES FOR MAP 1

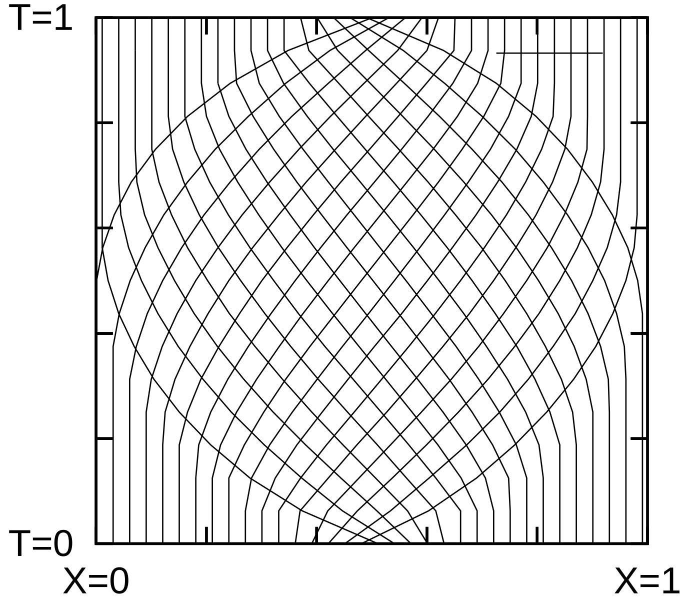

FIG. 5: TRAJECTORIES FOR MAP 2

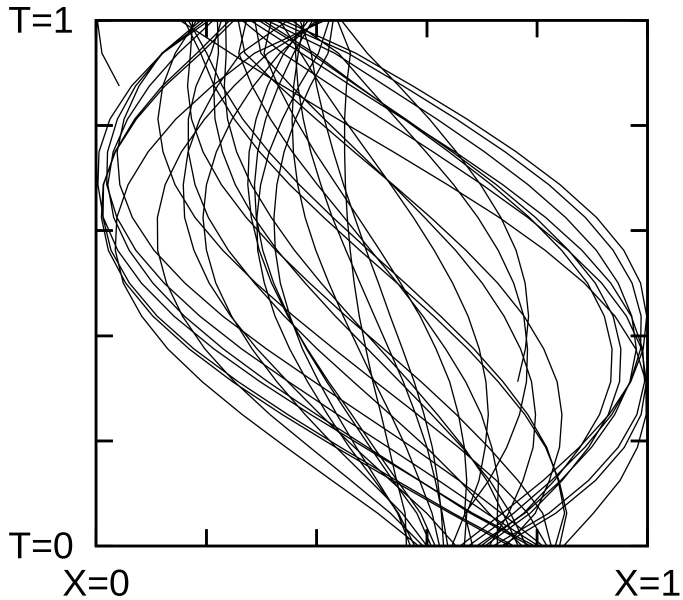

FIG. 6: SELECTED TRAJECTORIES FOR MAP 3