The Vlasov-Monge-Ampère system: a route from pure randomness to ideal incompressible Fluid Mechanics

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EULER'S MODEL OF INCOMPRESSIBLE FLUIDS

One can describe the motion of an incompressible fluid inside a bounded domain D in $\mathbb{R}^d$ by a time-dependent family $t \to \mathcal{X}_t$ of maps belonging to the Hilbert space $H = L^2(D, \mathbb{R}^d)$, valued in the subset $VPM(D) \subset H$ of all Lebesgue measure-preserving maps

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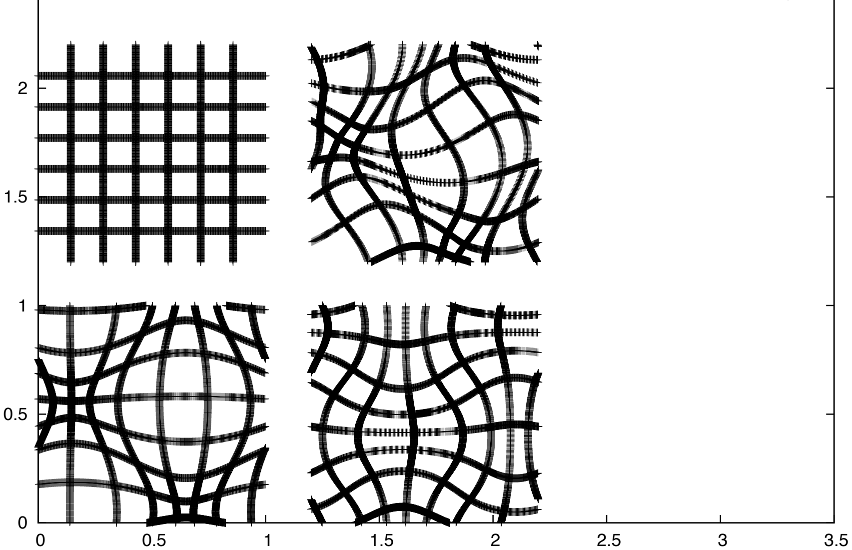
The Euler model, introduced in 1755, correspond to those curves $t \to \mathcal{X}_t \in$ VPM(D) for which there is a "pressure field" $p_t(x)$ s.t.

$$\frac{d^2\mathcal{X}_t}{dt^2} + (\nabla p_t) \circ \mathcal{X}_t = 0$$



three maps of the periodic square: one is area preserving

THE PRINCIPLE OF LEAST ACTION

(easy) THEOREM Let D be convex and $(\mathcal{X}_t, p_t)$ be a solution of the Euler equations, with $D_x^2 p_t \leq C\mathbb{I}$.

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In other words, such a curve is nothing but a (constant speed) minimizing geodesic along VPM(D), with respect to the metric induced by $H = L^2(D, \mathbb{R}^d)$ on VPM(D).

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See Arnold 1966, Ebin-Marsden 1970, Arnold-Khesin book 1998.

VOLUME-PRESERVING MAPS: APPROXIMATION PAR PERMUTATIONS

Fix D $= [0,1]^d$ and consider its dyadic decomposition by N $= 2^{nd}$ sub-cubes D(α), of barycenters A(α), $\alpha = 1, \cdots,$ N.

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For numerical purposes, we approximate the set $VPM(D)$ of all volume-preserving maps by the discrete subset $P_N(D)$ of all rigid rearrangements of the N sub-cubes, namely maps of form:

$$s(a) = a - A(\alpha) + A(\sigma(\alpha)), \quad a \in D(\alpha), \quad \alpha = 1, ..., N, \quad \sigma \in \mathcal{S}_N$$

where $\mathcal{S}_N$ is the set of all permutations of $\{1, \cdots, N\}$.

REARRANGEMENTS OF N=16 SUB-CUBES AS EXAMPLES OF VOLUME PRESERVING MAPS

PENALIZATION OF THE EULER ACTION

Since minimizing geodesics along a discrete set such as the set of rigid permutations $P_N(D)$ do not make much sense, we rather consider a penalized version of the Euler action (*)

$$\int_{t_0}^{t_1} \left(\left|\left| \frac{d\mathcal{X}_t}{dt} \right|\right|_H^2 + \epsilon^{-1} \inf_{s \in P_N(D)} ||\mathcal{X}_t - s||^2_{\frac{1}{2}H} \right) dt, \quad H = L^2(D, \mathbb{R}^d)$$

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(*) For smooth sets, this is a consistent approximation to minimizing geodesics (cf. Rubin-Ungar, CPAM 1957).

FINITE-DIMENSIONAL REDUCTION

It is consistent to limit ourself to piecewise affine maps of form

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Here $X_t \in (\mathbb{R}^d)^N$ becomes the new, finite-dimensional, unknown. Accordingly, the penalized action can be easily computed

$$\int_{t_0}^{t_1} \left(\left|\left| \frac{dX_t}{dt} \right|\right|^2 + \epsilon^{-1} \inf_{\sigma \in \mathcal{S}_N} ||X_t - A_\sigma||^2 \right) dt$$

Here $||\cdot||$ denotes the euclidean norm in $H = (\mathbb{R}^d)^N$, $\mathcal{S}_N$ is the set of all permutations of $\{1, \cdots, N\}$ and $A_\sigma(\alpha) = A(\sigma(\alpha)), \quad \alpha = 1, ..., N$.

THE RESULTING (DISCRETE) VLASOV-MONGE-AMPERE SYSTEM

Using the least-action principle, we end up with the following finite-dimensional dynamical system

$$\epsilon \frac{d^2 X_t(\alpha)}{dt^2} = X_t(\alpha) - A(\sigma_{opt}(\alpha)), \quad \alpha = 1, ..., N$$

$$\sigma_{opt} = \mathrm{Arginf}\{\sum_{\alpha=1}^{N} |X_t(\alpha) - A(\sigma(\alpha)|^2, \quad \sigma \in \mathcal{S}_N\}$$

This can be used for numerical purposes! See related work by Mérigot and Mirebeau arXiv:1505.03306, based on Mérigot's fast Monge-Ampère solver.

The explicit time discrete version was introduced in Y.B. CMP 2000 for $\epsilon < 0$, with convergence to the Euler model as $|\epsilon| \to 0, \quad N \geq C|\epsilon|^{-8d}, \quad \delta t \leq C|\epsilon|^4$.

THE VLASOV-MONGE-AMPERE SYSTEM

The continuous version, involving the Monge-Ampère equation, was introduced in B. and Loeper (GAFA 2004), studied by Cullen, Gangbo, Pisante (Arma 2007), Ambrosio-Gangbo (CPAM 2008)...

$$\boxed{\partial_t f(t,x,\xi) + \nabla_x \cdot (\xi\ f(t,x,\xi)) - \nabla_\xi \cdot (\nabla_x \varphi(t,x) f(t,x,\xi)) = 0}$$

$$\boxed{\det(\mathbb{I} + \epsilon D_x^2 \varphi(\theta,x)) = \int_{\mathbb{R}^d} f(t,x,\xi) d\xi, \quad (t,x,\xi) \in \mathbb{R}^{1+d+d}}$$

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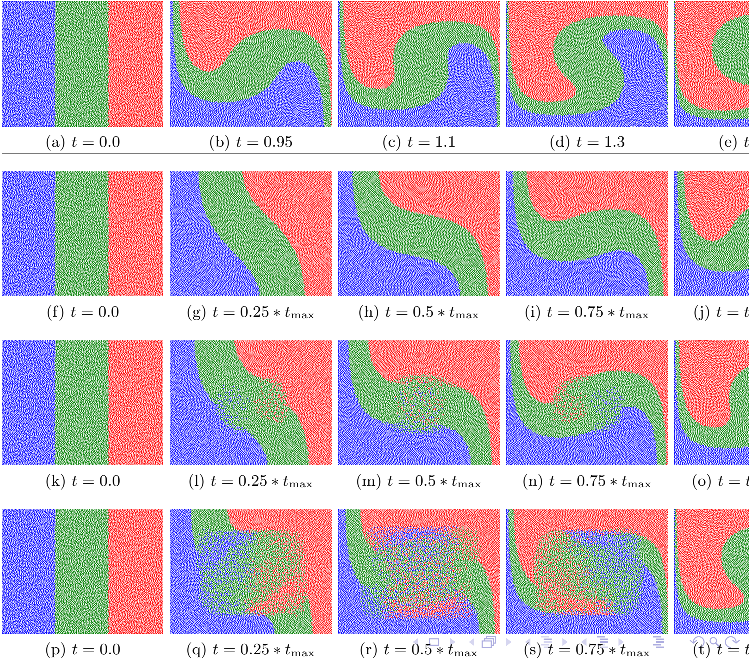
It is a fully nonlinear correction of the well-known Vlasov-Poisson system describing Newtonian gravitation as $d = 3$.



(a) $t = 0.0$ (b) $t = 0.95$ (c) $t = 1.1$ (d) $t = 1.3$ (e) t

(f) $t = 0.0$ (g) $t = 0.25 * t_{\max}$ (h) $t = 0.5 * t_{\max}$ (i) $t = 0.75 * t_{\max}$ (j) $t = t$

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PART II: A PURELY STOCHASTIC ORIGIN OF THE (discrete) VLASOV-MONGE-AMPERE MODEL

Using large deviation principles and the concept of "onde pilote" (coming from quantum mechanics), we will recover this discrete dynamical system from the trivial stochastic model of a Brownian point cloud.

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As a consequence and in some sense, the Euler model of incompressible fluids can be obtained out of pure noise!

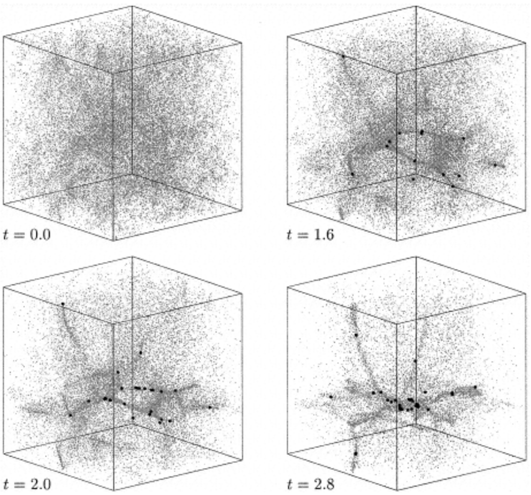
WANDERING OF A BROWNIAN POINT CLOUD

We consider N independent Brownian curves issued from the cubic lattice $\{A(\alpha) \in \mathbb{R}^d,\ \alpha = 1, \cdots, N\}$ and wandering in $\mathbb{R}^d$:

$$\boxed{A(\alpha) + \sqrt{\epsilon} B_t(\alpha), \quad \alpha = 1, \cdots, N}$$

We define a point cloud as a finite set of indistinguishable points, i.e. as a point in the quotient space $(\mathbb{R}^{\mathbf{d}})^{\mathbf{N}}/\mathcal{S}_{\mathbf{N}}$.

Wandering of a cloud in $\mathbb{R}^3$



LET US FOLLOW "L'ONDE PILOTE"

Introducing the heat equation in the space of "clouds" $X \in \mathbb{R}^{Nd}$

$$\frac{\partial \rho}{\partial t}(t,X) = \frac{\epsilon}{2} \triangle \rho(t,X), \quad \rho(t=0,X) = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \delta(X - A_\sigma)$$

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we solve the ODE

$$\frac{dX_t}{dt} = v(t,X_t), \quad v(t,X) = -\frac{\epsilon}{2} \nabla_X \log \rho(t,X)$$

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This is an adaptation of de Broglie's "onde pilote" concept. As a matter of fact, a similar calculation also works for the free Schrödinger equation:

$$(i\partial_t + \triangle)\psi = 0, \quad \psi(0,X) = \sum_\sigma \exp(-||X - A_\sigma||^2/a^2), \quad v = \nabla \mathcal{I}m \log \psi$$

THE "ONDE PILOTE" SYSTEM

We get the "onde pilote" system, setting $t = \exp(2\theta)$,

$$\frac{dX_\theta}{d\theta} = X_\theta - < A > \quad < A > = \frac{\sum_{\sigma \in \mathcal{S}_N} A_\sigma \exp(\frac{-||X_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)})}{\sum_{\sigma \in \mathcal{S}_N} \exp(\frac{-||X_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)})}$$

ZERO-NOISE LIMIT ANALYSIS

As ϵ goes to zero, we get the first order dynamical system

$$\boxed{\frac{dX_\theta}{d\theta} = X_\theta - A_{\sigma_{opt}} \ , \quad \sigma_{opt} = \mathrm{Arginf}_{\ \sigma \in \mathcal{S}_N} \ ||X_\theta - A_\sigma||^2}$$

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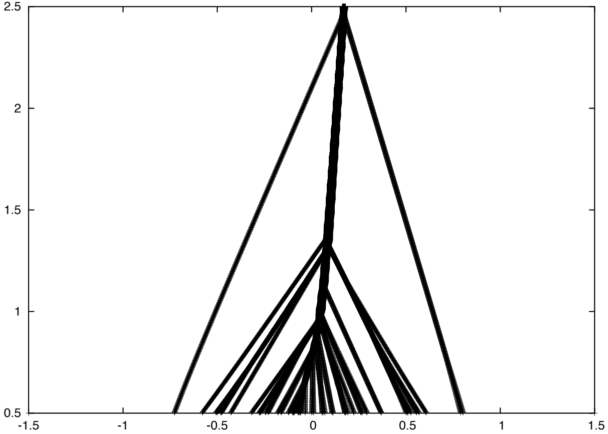
i.e. $\frac{d_+ X_\theta}{d\theta} = -\overline{\nabla}\Phi(X_\theta)$ **which is the "gradient flow" of the semi-convex function** $\Phi(X) = -\inf_{\sigma \in \mathcal{S}_N} ||X - A_\sigma||^2/2$

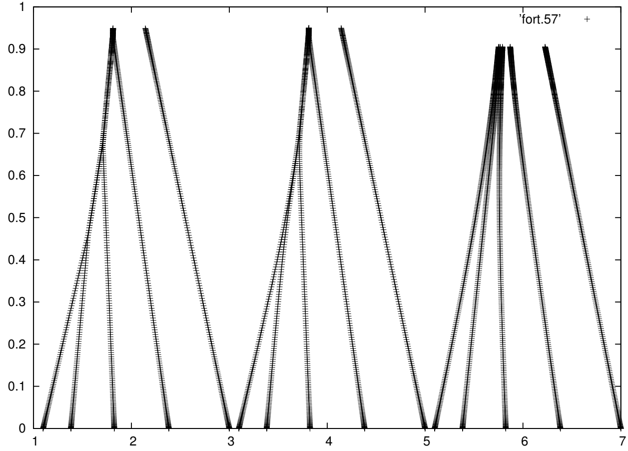
N.B. this formulation automatically include 1D sticky collisions.

Sticky collisions

horizontal : 51 grid points in x /vertical : 60 grid points in t



From free (Bosonic) Schrödinger to sticky particles



LARGE DEVIATIONS OF THE "ONDE PILOTE"

Back to the "onde pilote" trajectories, let us add some noise η

$$\frac{dX^\epsilon_\theta}{d\theta} = X^\epsilon_\theta - < A > + \eta \frac{dB_\theta}{d\theta} , \quad < A > = \frac{\sum_{\sigma \in \mathcal{S}_N} A_\sigma \exp(\frac{-||X^\epsilon_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)})}{\sum_{\sigma \in \mathcal{S}_N} \exp(\frac{-||X^\epsilon_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)})}$$

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For ϵ fixed, we first use the Freidlin-Vencel theory to get the "good rate function" for the large deviations of the system as $\eta \to 0$. Then, we may pass to the limit $\epsilon \to 0$ (*) and obtain as "$\Gamma-$limit"

$$\int ||\frac{dX_\theta}{d\theta}||^2 + ||\overline{\nabla}\Phi(X_\theta)||^2 d\theta, \quad \Phi(X) = -\inf_{\sigma \in \mathcal{S}_N} ||X - A_\sigma||^2/2$$

(*) thanks to L. Ambrosio, private communication.

LEAST ACTION PRINCIPLE

The least action principle applied to

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(formally) leads to the following dynamical system

$$\frac{d^2X_\theta}{d\theta^2} = \nabla(\frac{||\nabla\Phi||^2}{2})(X_\theta) = -(\nabla\Phi)(X_\theta)$$

Indeed $||\nabla\Phi||^2 = -2\Phi$ because -2Φ is a squared distance function.

THE RESULTING DYNAMICAL SYSTEM

So, we have finally obtained

$$\frac{d^2 X_\theta(\alpha)}{d\theta^2} = X_\theta(\alpha) - A(\sigma_{opt}(\alpha)) \ , \quad X_\theta(\alpha) \in \mathbb{R}^d, \ \ \alpha = 1, \cdots, N$$

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THANKS!

see Y.B. arXiv:1504.07583