Solving the initial value problem for Euler and Burgers equations by convex optimization

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Reference: Y.B. ArXiv Oct. 2017, to appear in CMP

Euler 1755/57: The first PDEs ever written (at least in modern style)!

XXI. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélerations actuelles que nous venons de trouver, & nous obtiendrons les trois équations ſuivantes :

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

$$R - \frac{1}{q}\left(\frac{dp}{dz}\right) = \left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right)$$

Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la conſidération de la continuité du fluide :

$$\left(\frac{dq}{dt}\right)+\left(\frac{d.qu}{dx}\right)+\left(\frac{d.qv}{dy}\right)+\left(\frac{d.qw}{dz}\right)=0.$$

Si le fluide n'étoit pas compreſſible, la denſité q ſeroit la même en Z, & en Z', & pour ce cas on auroit cette équation :

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0.$$

qui eſt auſſi celle ſur laquelle j'ai établi mon Mémoire latin allégué ci-deſſus.

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$$\partial_t(\mathcal{E}(U)) + \nabla \cdot (Q(U)) = 0,$$

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A typical example: the equations of an isothermal gas (Euler, 1755)

$$\partial_t \rho + \nabla \cdot q = 0, \quad \partial_t q + \nabla \cdot (\frac{q \otimes q}{\rho}) + \nabla \rho = 0, \quad \mathcal{E} = \frac{|q|^2}{2\rho} + \rho \log \rho, \quad \rho > 0, \quad q \in \mathbb{R}^d.$$

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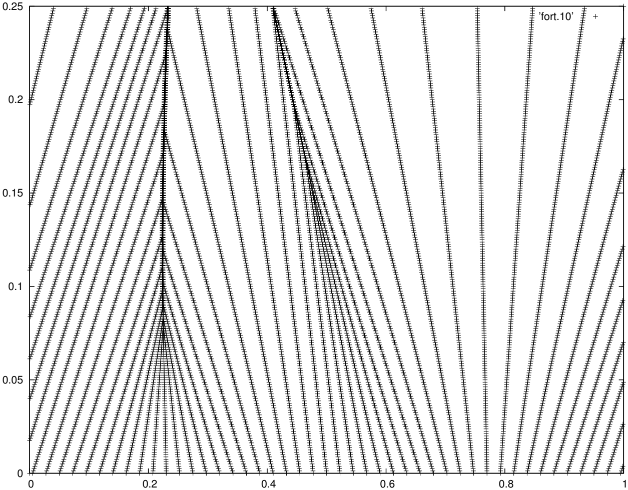
Typically, such systems are locally well-posed, with generic formation of shock waves.



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t,x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)

The least square approach?

Given U_0 on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $T > 0$, if $F(U)$ is linear in U, the least square method obviously leads to a (degenerate) convex problem

$$\inf_{U(t=0,\cdot)=U_0} \int_{[0,T]\times\mathbb{T}^d} |\partial_t U + \nabla \cdot (F(U))|^2$$

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We need a different idea!

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for all smooth $A = A(t,x) \in \mathbb{R}^m$ with $A(T,\cdot) = 0$. A priori, by "convex integration" à la De Lellis-Székelyhidi, there are many non entropy-preserving weak solutions attached to U_0.

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$$\inf_U \sup_A \int_{[0,T]\times\mathbb{T}^d} \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U)$$

$$- \int_{\mathbb{T}^d} A(0,\cdot)\cdot U_0$$

where $A = A(t,x) \in \mathbb{R}^m$ is smooth with $A(T,\cdot) = 0$.
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N.B. The supremum in A exactly encodes that U is a weak solution with initial condition U_0, all test functions A acting like Lagrange multipliers.

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leads to a *concave* maximization problem in A, namely

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$$= \sup_A \int_{[0,T]\times\mathbb{T}^d} K(\partial_t A, \nabla A) - \int_{\mathbb{T}^d} A(0,\cdot)\cdot U_0.$$

$$K(E,B) = \inf_{U\in\mathbb{R}^m} \mathcal{E}(U) - E\cdot U - B\cdot F(U).$$

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might be a duality gap with possibly $\inf\sup > \sup\inf$.

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Then, the maximization problem in A simply reads

$$\sup_A \int_{[0,T]\times\mathbb{T}} -\frac{(\partial_t A)^2}{2(1-\partial_x A)} - \int_{\mathbb{T}} A(0,\cdot)u_0.$$

with $A = A(t,x) \in \mathbb{R}$ subject to $A(T,\cdot) = 0, \;\; \partial_x A \leq 1.$

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Introducing $\rho = 1 - \partial_x A \geq 0$, $q = \partial_t A \in \mathbb{R}$, we get

$$\sup\{\int_{[0,T]\times\mathbb{T}} -\frac{q^2}{2\rho} + qu_0, \ \text{s.t.}\ \partial_t\rho + \partial_x q = 0, \ \ \rho(T,\cdot) = 1\}$$

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looking very similar to an optimal transport problem!

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We maximize in $u = u(t,x) \in \mathbb{R}$, $Q = Q(t,x) \in \mathbb{R}^d$,

$M = M(t,x) = M^t(t,x) \in \mathbb{R}^{d \times d}, \quad M \geq 0,$

$$\int_{[0,T]\times D} -\exp(u)\exp(\frac{1}{2} Q \cdot M^{-1} \cdot Q) - \int_D \sigma_0 \rho_0 - w_0 \cdot q_0,$$

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$$u = \partial_t \sigma + \partial^i w_i, \ Q_i = \partial_t w_i + \partial_i \sigma, \ M_{ij} = \delta_{ij} - \partial_i w_j - \partial_j w_i,$$

where σ and w must vanish at $t = T$.

Example 3:
the Euler equations of incompressible fluids

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In this limit case, the resulting problem reads

$$\sup_{(M,Q)} \; -\int_{[0,T]\times D} q_0\cdot Q+\frac{1}{2}\,Q\cdot M^{-1}\cdot Q,$$

where, again, Q is a vector field and $M=M^t\geq 0$ is a field of semi-definite symmetric matrices, subject to

$$M_{ij}(T,\cdot)=\delta_{ij},\quad \partial_t M_{ij}=\partial_j Q_i+\partial_i Q_j+2\partial_i\partial_j(-\triangle)^{-1}\partial_k Q^k.$$

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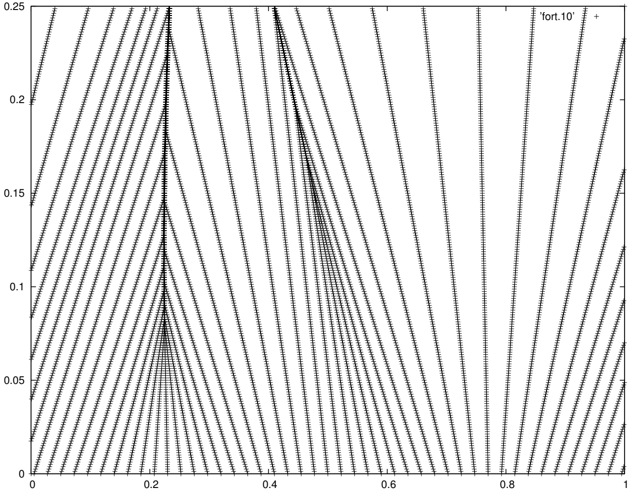
(*) more precisely if
$\mathcal{E}"(V) - (T-t)F"(V) \cdot \nabla(\mathcal{E}'(U(t,x))) > 0, \ \forall t, x, V.$



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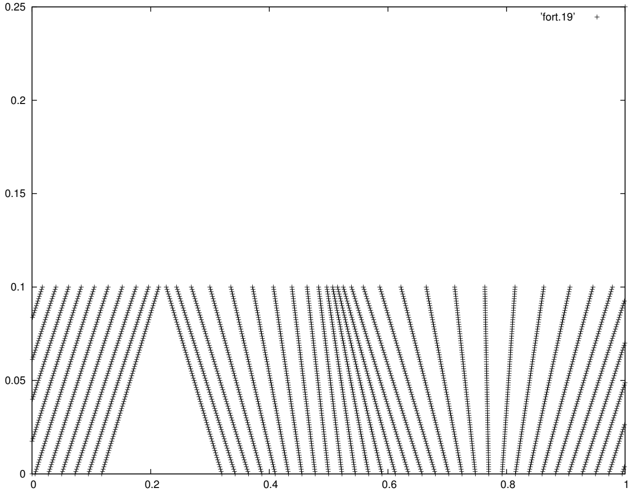
Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)



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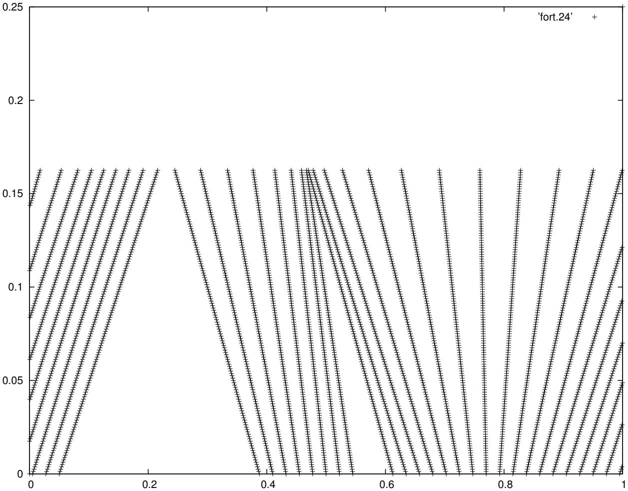
Recovery of the solution at time T=0.1 by convex optimization.
Observe the formation of a first vacuum zone as the first shock has formed.



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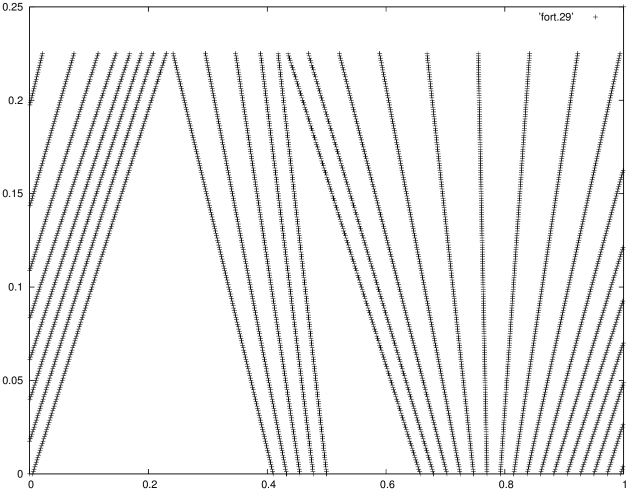
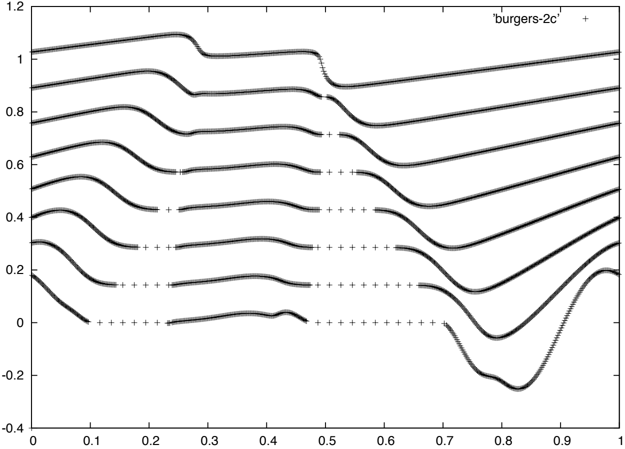
Recovery of the solution at time T=0.16 by convex optimization.
Observe the formation of a second vacuum zone as the second shock has formed.



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t,x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Recovery of the solution at time T=0.225 by convex optimization.
Observe the extension of the two vacuum zones.



Numerics: we recover the entropy solutions (only) at the final time.
Notice that the code differ by only two lines (of fortran code) from a standard (Benamou-B.) optimal transport solver!

Analogy with mountain climbing: going from Everest to Lhotse without following the crest! (Partial credit to Thomas Gallouët for this analogy.)

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