Topology-preserving diffusion equations for divergence-free vector fields

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TOPOLOGY PRESERVING DIFFUSION EQUATIONS FOR DIVERGENCE-FREE VECTOR FIELDS

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Warning: there will be only formal derivations and no rigorous analysis.

Loop decomposition of divergence-free vector fields

Every smooth loop $s \in R/Z \rightarrow X(s) \in R^d$ generates, in the sense of distributions, a divergence-free vector field $x \in R^d \rightarrow B(x) \in R^d$

$$B(x) = \int_{s \in R/Z} X'(s)\delta(x - X(s))ds.$$

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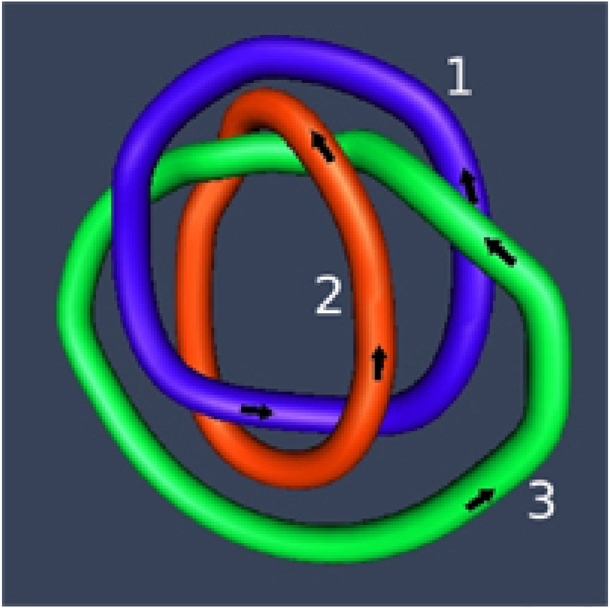
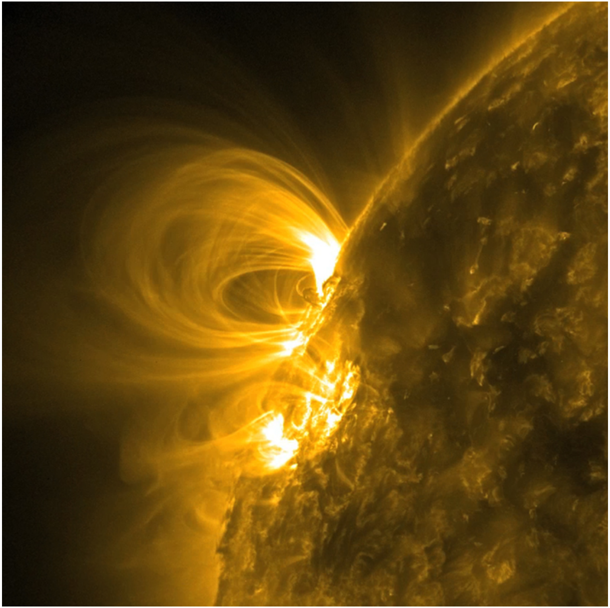
Indeed, for any smooth function q

$$< B, \nabla q >= \int_{s \in R/Z} X'(s) \cdot (\nabla q)(X(s))ds = \int_{s \in R/Z} \frac{d}{ds}(q(X(s))) = 0.$$

Conversely, every divergence-free field can be approximated by a superposition of loops (cf. related work by S. Smirnov).

SUPERPOSITION OF LOOPS





Transport of a loop by a velocity field

Let us consider a time-dependent loop $(t,s) \to X(t,s) \in R^d$. moved by some velocity field $v(t,x) \in R^d$ so that $\boxed{\partial_t X(t,s) = v(t, X(t,s))}$

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the "transport" equation of B by v in the sense of distributions:

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$$\frac{d}{dt} < B, w >= \frac{d}{dt} \int w(X(t,s))\partial_s X(t,s)ds = \int [(Dw)(X)\partial_t X \otimes \partial_s X$$

$$+w(X)\partial^2_{ts}X]ds = \int (Dw)(X)[\partial_t X \otimes \partial_s X - \partial_s X \otimes \partial_t X]ds$$

$$= \int (Dw)(x)[v(t,x) \otimes B(t,x) - B(t,x) \otimes v(t,x)]dx, \quad \forall w \in C_c^\infty.$$

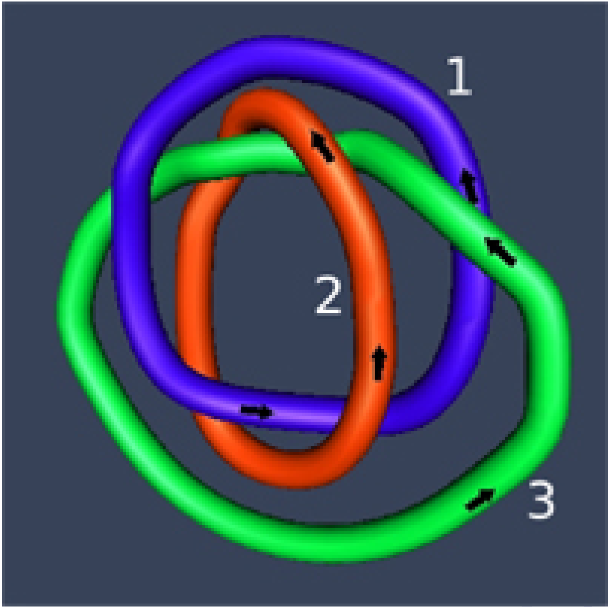
Transport of divergence-free vector fields

By superposition of loops, the transport of a time-dependent divergence-free vector field $\mathbf{B}(t,x) \in \mathbf{R}^d$ by some velocity field $\mathbf{v}(t,x) \in \mathbf{R}^d$, is still described by the "topology-preserving" transport equation

$$\partial_t \mathbf{B} + \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B}) = 0$$

(Of course, this equation can be derived by many other means.)

TOPOLOGY OF LOOPS



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The usual diffusion equation for a divergence-free vector field $\partial_t \mathbf{B} = \nabla^2 \mathbf{B}$ **cannot be written in transport form**

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and, therefore, is not "topology-preserving" (i.e. is not compatible with the loop decomposition).
This is in sharp contrast with the standard heat equation for density fields $\rho > 0$ $\boxed{\partial_t \rho = \nabla^2 \rho}$ **which can be easily put in "transport" form** $\boxed{\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \mathbf{v} = -\nabla(\log \rho)}$

Topology-preserving diffusion equations and turbulence theory

In the fluid mechanics literature (e.g. T. Nishiyama, 2003), one can find, for 3D divergence-free vector fields, non-linear (degenerate) diffusion equations of form

$$\partial_t \mathbf{B} + \nabla \times (\lambda(|\mathbf{B}|^2 \mathbf{I} - \mathbf{B} \otimes \mathbf{B})\nabla \times \mathbf{B}) = 0$$

(where λ is some positive constant or function). They can be written in "transport-form" and are topology-preserving.

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(where λ is some positive constant or function). They can be written in "transport-form" and are topology-preserving.
Such equations are interesting because they have highly non-trivial equilibrium states, namely all fields B which are colinear to their own curl. These fields are special stationary solutions of the 3D Euler equations and are believed to play a crucial role in turbulence. (They include Beltrami flows and all stationary solutions to the 2D Euler equations.)

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This will require a sort of generalization of the Jordan-Kinderlehrer-Otto (JKO) method, which has been a very successful way of deriving parabolic equations (in particular the regular scalar heat equation) from optimal transport considerations.
Unfortunately, there will be no rigorous analysis, due to the highly non-linear and degenerate structure of this kind of equations.

An example of transportation cost

For every elementary loop moving in the Minkowski space $\mathbf{R} \times \mathbf{R}^d$, we define our cost function as the area spanned by the loop

$$\int \int \sqrt{(1 - |\partial_t X|^2)|\partial_s X|^2 + (\partial_t X \cdot \partial_s X)^2} \, ds dt$$

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In terms of fields (B, v), this generalizes (by superposition) as

$$\int\int \sqrt{(1 - |v|^2)|B|^2 + (v \cdot B)^2} \, dxdt$$

which will be our definition of the transport cost of B by v.

Optimal velocity field

Let us optimize in $\mathbf{v} = \mathbf{v}(\mathbf{t}, \mathbf{x}) \in \mathbf{R}^{\mathbf{d}}$ the convex cost function

$$-\int\int \sqrt{(1 - |\mathbf{v}|^2)|\mathbf{B}|^2 + (\mathbf{v} \cdot \mathbf{B})^2}\, \mathbf{dxdt}$$

under the linear differential constraint

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Introducing Lagrange multiplier $\mathbf{A} = \mathbf{A}(t, x) \in \mathbf{R}^d$ for the differential constraint, we find the structure condition

$$\mathbf{v} = \lambda\, d\mathbf{A} \cdot \mathbf{B}, \quad d\mathbf{A}_{ij} = \partial_j \mathbf{A}_i - \partial_i \mathbf{A}_j$$

where $\lambda = 1/\sqrt{|\mathbf{B}|^2 + |d\mathbf{A} \cdot \mathbf{B}|^2}$.

A generalized (BB-)JKO scheme

Let B $\rightarrow$ E[B] $\in$ R be a given functional on divergence-free vector fields, say, for simplicity, E[B] $= \int |\mathbf{B}|^2/2\mathbf{dx}$.

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We want to define the gradient flow of this functional according to the "optimal transport metric". Let us do that using the JKO (Jordan-Kinderlehrer-Otto) scheme, in BB (Benamou-Brenier) style:

A generalized (BB-)JKO scheme

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Given a time step $h > 0$, $B(t, x)$, supposed to be already known for $0 \le t \le (n-1)h$, is obtained for $nh \ge t \ge (n-1)h$ as a critical point of the functional

$$E[B(nh, \cdot)] - \int_{(n-1)h}^{nh} dt\{\int \sqrt{(1 - |v|^2)|B|^2 + (v \cdot B)^2}\, dx\ \}$$

where v is optimized under the linear differential constraint

$$\partial_t B + \nabla \cdot (B \otimes v - v \otimes B) = 0$$

Resulting diffusion equation

We find, as necessary conditions:

$$\mathbf{A}(\mathbf{nh},\cdot) = \mathbf{E}'[\mathbf{B}(\mathbf{nh},\cdot)], \quad \mathbf{v} = \lambda\, \mathbf{dA}\cdot\mathbf{B}$$

where $\lambda > 0$ is an explicit function of B and $\mathbf{dA}\cdot\mathbf{B}$.

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where $\lambda > 0$ is an explicit function of B and dA $\cdot$ B.
Combined with the transport equation

$$\boxed{\partial_t \mathbf{B} + \nabla\cdot(\mathbf{B}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{B}) = 0}$$

the closure equations formally obtained as h $\to$ 0,

$$\boxed{\mathbf{v} = \lambda\, \mathbf{dA}\cdot\mathbf{B}, \quad \mathbf{A} = \mathbf{E}'[\mathbf{B}]}$$

provide a self-consistent evolution equation for B, which is the desired "topology-preserving" diffusion equation (up to the precise definition of $\lambda > 0$ as a function of B and dA $\cdot$ B).

THANKS FOR YOUR ATTENTION... SOME REFERENCES

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