From Euler to Monge and vice versa. Introduction.

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OUTLINE OF THE INTRODUCTORY LECTURE:

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Combinatorial optimization and discrete optimal transportation.

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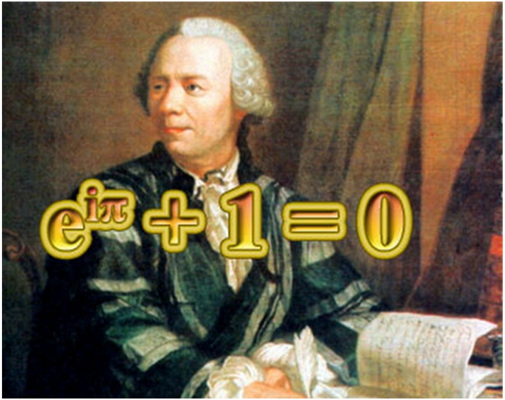
But, he was the first one, in 1755, able to describe fluids in a definite way, by what we can call now a "field theory", with a comprehensive and consistent set of partial differential equations.

This was the prototype of the future field theories in Physics (Maxwell, Einstein, Schrödinger, Dirac).



Euler: portrait, bank note and stamp...

Euler 1757 : the 1st PDEs ever writen...

XXI. Nous n'avons donc qu'à égaler ces forces accélératrices avec les accélerations actuelles que nous venons de trouver, & nous obtiendrons les trois équations suivantes :

$$P - \frac{1}{q}\left(\frac{dp}{dx}\right) = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right)$$

$$Q - \frac{1}{q}\left(\frac{dp}{dy}\right) = \left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right)$$

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Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la confidération de la continuité du fluide :

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Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la conſidération de la continuité du fluide :

...at least in modern style

$$\left(\frac{dq}{dt}\right)+\left(\frac{d.qu}{dx}\right)+\left(\frac{d.qv}{dy}\right)+\left(\frac{d.qw}{dz}\right)=0.$$

Si le fluide n'étoit pas compreſſible, la denſité q ſeroit la même en Z, & en Z′, & pour ce cas on auroit cette équation :

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0.$$

qui eſt auſſi celle ſur laquelle j'ai établi mon Mémoire latin allégué ci-deſſus.

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(Of course, the "source terms" P-Q-R in the Euler equations are usually very difficult to model in numerical codes: they rely on complex thermal exchanges between sun, earth, air and water.)

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The understanding of fluid turbulence.

tombent dans la ſurface même. Or nous voyons par là ſuffiſamment, combien nous ſommes encore éloignés de la connoiſſance complette du mouvement des fluides, & que ce que je viens d'expliquer, n'en contient qu'un foible commencement. Cependant tout ce que la Théorie des fluides renferme, eſt contenu dans les deux équations rapportées cy-deſſus (§. XXXIV.), de ſorte que ce ne ſont pas les principes de Méchanique qui nous manquent dans la pourſuite de ces recherches, mais uniquement l'Analyſe, qui n'eſt pas encore aſſés cultivée, pour ce deſſein: & partant on voit clairement, quelles découvertes nous reſtent encore à faire dans cette Science, avant que nous puiſſions arriver à une Théorie plus parfaite du mouvement des fluides.

Euler's conclusion still correct after 261 years

GEOMETRY OF THE EULER MODEL OF INCOMPRESSIBLE FLOWS.

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As already guessed by Euler himself, the "principle of least action" is behind the Euler equations of incompressible fluids. This has been elaborated by the mathematician Vladimir ARNOLD (1937-2010) in 1966.

According to Arnold, an incompressible fluid, confined in a domain denoted by D and moving according to the Euler equations, just follows a (constant speed) geodesic curve along the manifold of all possible incompressible maps of D.

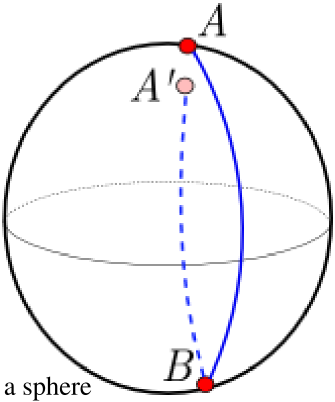
Geometric interpretation of the Euler equations by Arnold, 1966.

VLADIMIR ARNOLD

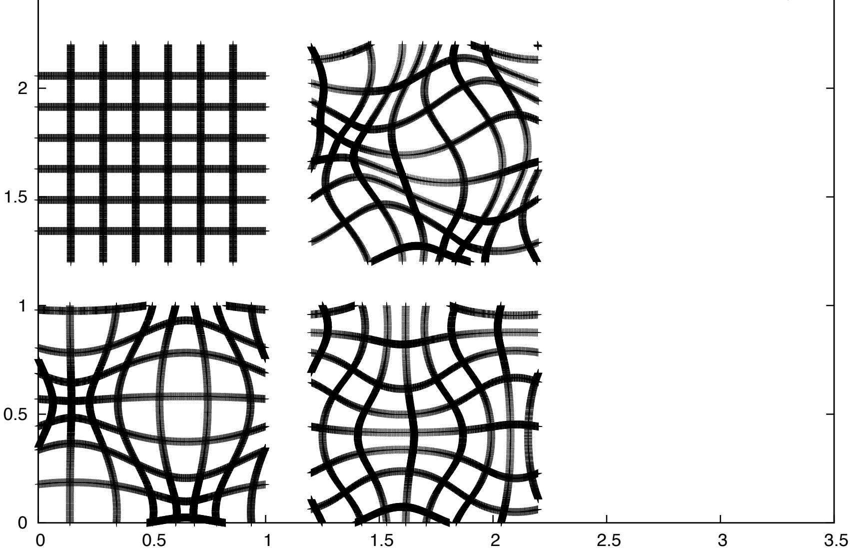
Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits

Annales de l'institut Fourier, tome 16, n^o 1 (1966), p. 319-361.

<http://www.numdam.org/item?id=AIF_1966__16_1_319_0>



Two geodesic curves on a sphere



Three maps of the (periodized) square: only one is incompressible.

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FROM COMBINATORICS TO FLUIDS AND VICE VERSA

THE BIG BANG THEORY!

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Sheldon Cooper and the melting rubik's cube...

Example of a discrete incompressible motion with 7 time steps and 12 sub-cells (in line)

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7 time steps have been performed.

Time is on vertical axis and space on horizontal axis.

The trajectories of 2 selected sub-cells (4 and 5) are drawn in red.

"transportation cost" to reach the final permutation

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The "cost" is obtained by adding up the squares of all displacements at all steps. Here: 12+10+12+42+10+12+10=108.

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The "cost" is obtained by adding up the squares of all displacements at all steps. Here: 12+10+12+42+10+12+10=108. This is the "cost" to reach the final permutation in 7 steps. Notice that step 4 costs a lot!

Obviously, there is at least a solution leading to the final permutation at the lowest possible cost, among the... $(12!)^6 \sim 10^{52}$ **possible candidates!**

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This is what we will do in a future lecture, combining probability and convexity tools.

Exercise: let us try to find a discrete geodesic leading to permutation 12-11-10-9-8-7-6-5-4-3-2-1 using twelve steps

1	2	3	4	5	6	7	8	9	10	11	12
12	11	10	9	8	7	6	5	4	3	2	1

LET US TRY TO MOVE BY EXCHANGING NEIGHBORS...

1	2	3	4	5	6	7	8	9	10	11	12
2	1	4	3	6	5	8	7	10	9	12	11
2	4	1	6	3	8	5	10	7	12	9	11
12	11	10	9	8	7	6	5	4	3	2	1

FINALLY ARRIVED...AFTER 12 STEPS.

1	2	3	4	5	6	7	8	9	10	11	12
2	1	4	3	6	5	8	7	10	9	12	11
2	4	1	6	3	8	5	10	7	12	9	11
4	2	6	1	8	3	10	5	12	7	11	9
4	6	2	8	1	10	3	12	5	11	7	9
6	4	8	2	10	1	12	3	11	5	9	7
6	8	4	10	2	12	1	11	3	9	5	7
8	6	10	4	12	2	11	1	9	3	7	5
8	10	6	12	4	11	2	9	1	7	3	5
10	8	12	6	11	4	9	2	7	1	5	3
10	12	8	11	6	9	4	7	2	5	1	3
12	10	11	8	9	6	7	4	5	2	3	1
12	11	10	9	8	7	6	5	4	3	2	1

LET US FOLLOW THE TRAJECTORIES OF TWO NEIGHBOURS: 4 AND 5

1	2	3	4	5	6	7	8	9	10	11	12
2	1	4	3	6	5	8	7	10	9	12	11
2	4	1	6	3	8	5	10	7	12	9	11
4	2	6	1	8	3	10	5	12	7	11	9
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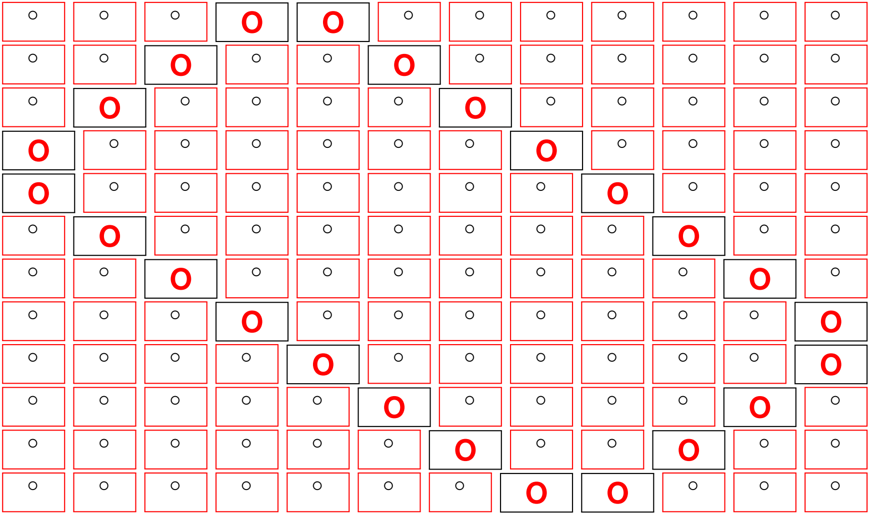
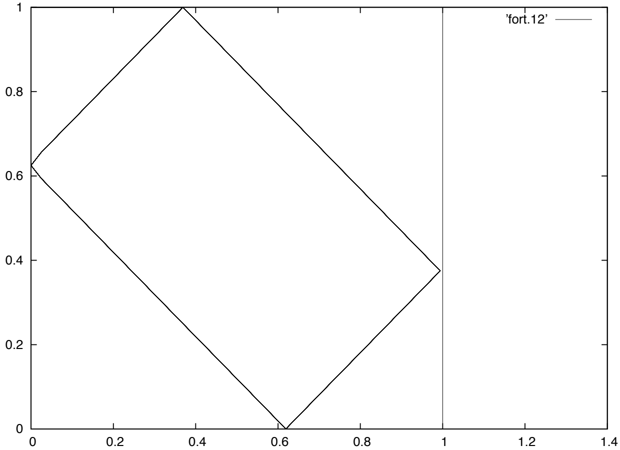
8	6	10	4	12	2	11	1	9	3	7	5
8	10	6	12	4	11	2	9	1	7	3	5
10	8	12	6	11	4	9	2	7	1	5	3
10	12	8	11	6	9	4	7	2	5	1	3
12	10	11	8	9	6	7	4	5	2	3	1

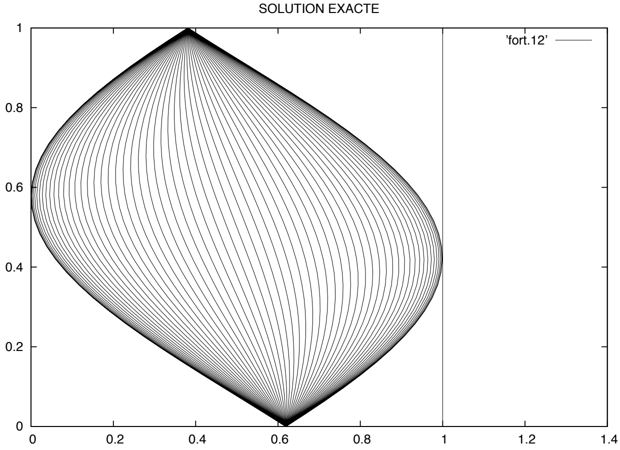
Is it really the lowest possible cost?



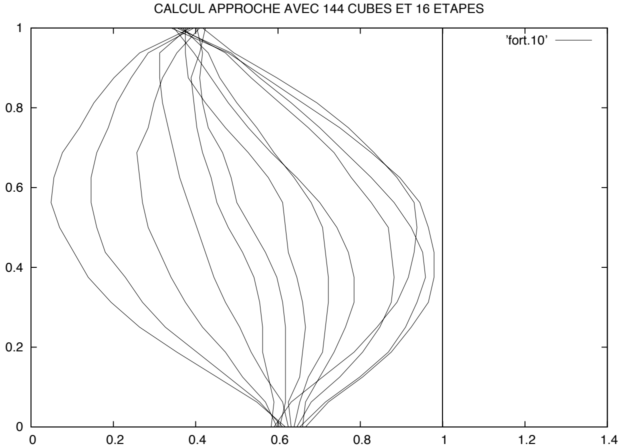


ANYWAY, IT IS EASY TO "PASS TO THE LIMIT"

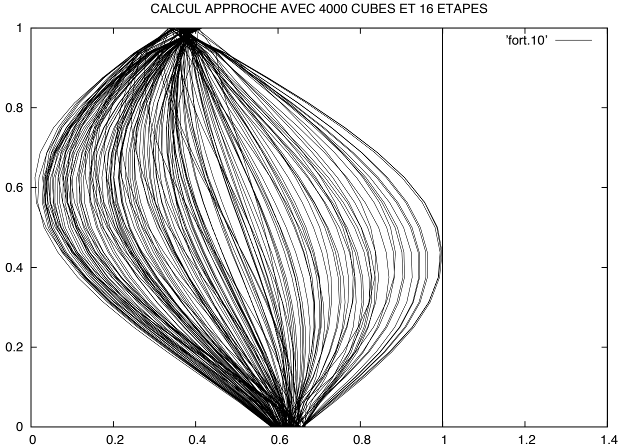
AS A MATTER OF FACT, THIS IS NOT THE BEST SOLUTION. THE COST CAN BE REDUCED BY FACTOR $\pi^2/12 \sim 0.8225$



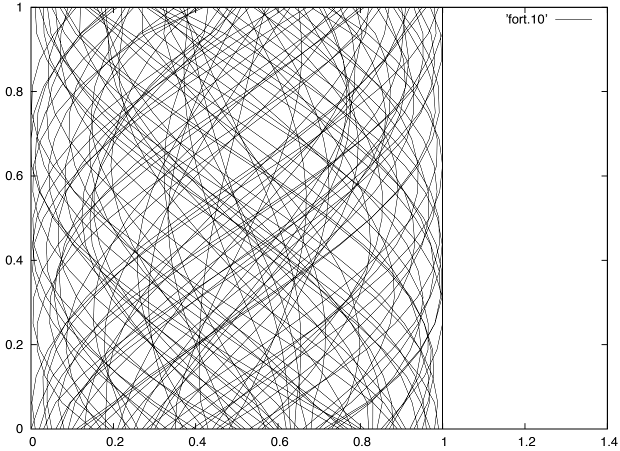
EXACT SOLUTION



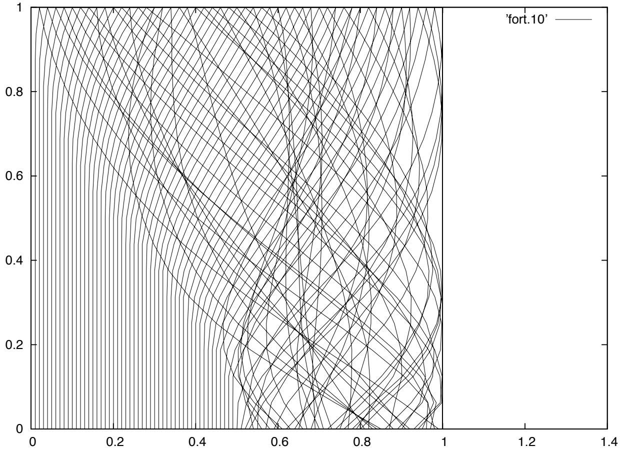
NUMERICS WITH 144 CUBES AND 16 STEPS



NUMERICS WITH 4000 CUBES AND 16 STEPS



SOME OF THE 4000 TRAJECTORIES (1 out of 40)



ANOTHER MINIMIZING GEODESIC for 1-3-5-7-9-11-12-10-8-6-4-2

Let us go back to the combinatorial setting. We define a "discrete geodesic with L steps" as a sequence of L+1 permutations σ^0, σ^1,σ^2,...,σ^{L-1},σ^L

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$$\sum_{i=1,N} \mathrm{dist}(A_{\sigma_i^{(0)}}, A_{\sigma_i^{(1)}})^2 + \sum_{i=1,N} \mathrm{dist}(A_{\sigma_i^{(1)}}, A_{\sigma_i^{(2)}})^2$$

$$+ \cdots + \sum_{i=1,N} \mathrm{dist}(A_{\sigma_i^{(L-1)}}, A_{\sigma_i^{(L)}})^2$$

where we denote by $A_1, \cdots, A_N$ the centers of the N sub-cells and by dist the Euclidean distance.

By doing so, we have expressed the Euler model for incompressible flows as a "combinatorial optimization problem":

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$$\inf_{\sigma^{(1)},\cdots,\sigma^{(L-1)}} \sum_{k=1,L} \sum_{i=1,N} \mathrm{dist}(A_{\sigma_i^{(k-1)}}, A_{\sigma_i^{(k)}})^2$$

By doing so, we have expressed the Euler model for incompressible flows as a "combinatorial optimization problem":

$$\inf_{\sigma^{(1)},\cdots,\sigma^{(L-1)}} \sum_{k=1,L} \sum_{i=1,N} \mathrm{dist}(A_{\sigma_i^{(k-1)}}, A_{\sigma_i^{(k)}})^2$$

which, up to the discretization, is fully consistent with the differential equations written by Euler!

A necessary optimality condition:

for each k fixed from 1 to L-1,

σ^k must minimize among all permutations σ

$$\sum_{i=1,N} \mathrm{dist}^2(A_{\sigma_i^{(k-1)}}, A_{\sigma_i}) + \sum_{i=1,N} \mathrm{dist}^2(A_{\sigma_i}, A_{\sigma_i^{(k+1)}})$$

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or, equivalently,

$$\mathbf{c}(1,\sigma_1) + \mathbf{c}(2,\sigma_2) + \mathbf{c}(3,\sigma_3) + \cdots + \mathbf{c}(N,\sigma_N) \ ,$$

$$\mathbf{c}(i,j) = \mathrm{dist}^2(B_i, A_j), \quad B_i = (A_{\sigma_i^{(k+1)}} + A_{\sigma_i^{(k-1)}})/2$$

This exactly means that $\sigma^{(k)}$ solves the so-called "linear assignment problem" (well known in both combinatorial optimization theory and Economics): minimize, among all permutations σ,

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where $c(i, j)$ is the "assignment cost matrix".

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where $c(i,j)$ is the "assignment cost matrix".

(Interpretation in Economics: we want to assign agents $i = 1, \cdots, N$ to tasks $j = 1, \cdots, N$ with cost $c(i,j)$ in an optimal way.)

The assignment problem (as well as its continuous limit) was analyzed in 1942 by Leonid KANTOROVICH (1912-1986) (who got the unique Nobel prize of Economy obtained by former Soviet Union!) and shown to be equivalent to a much simpler convex optimization problem.

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Kantorovich' method is a good example.

(Reduction to convexity is rather simple once we observe that permutations matrices are just the extreme points of the convex set of all matrices with nonnegative coefficients such that each line and each column add up to one.)

Leonid Kantorovich (1912-1986)

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where both $c(i,j)$ and $\lambda(i,j)$ are given matrices.
Interestingly enough, this more difficult problem appears to be a discrete version of the problem of finding *stationary* (i.e. time independent) solution to the Euler equations.

The continuous version of the linear assignment problem goes back to 1780 with Gaspard MONGE (1746-1818) and his "mémoire sur les déblais et les remblais".

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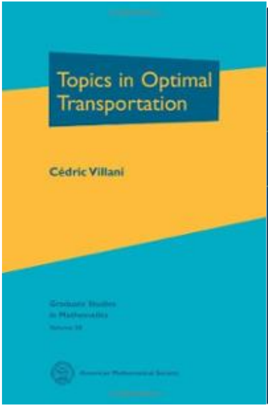
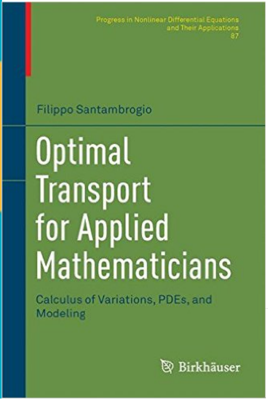
This was the prototype of what is nowadays known as "optimal transport theory", a very active field of mathematics with many connections (analysis, probability, geometry, partial differential equations) and applications (image processing, economics, cosmology...).

The Book





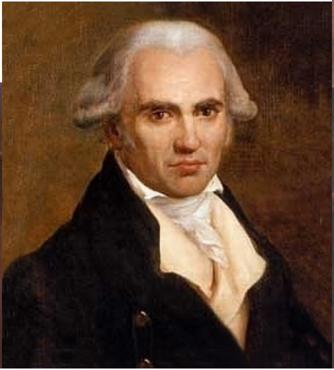
Computational Optimal Transport

You can retrieve the book:

Gabriel Peyré (http://www.gpeyre.com/) and Marco Cuturi (http://marcocuturi.net/), *Computational Optimal Transport (https://arxiv.org/abs/1803.00567)*, ArXiv:1803.00567, 2018.

This book reviews OT with a bias toward numerical methods and their applications in data sciences, and sheds lights on the theoretical properties of OT that make it particularly useful for some of these applications. Our focus is on the recent wave of efficient algorithms that have helped translate attractive theoretical properties onto elegant and scalable tools for a wide variety of applications. We also give a prominent place to the many generalizations of OT that have been proposed in but a few years, and connect them with related approaches originating from statistical inference, kernel methods and information theory.

(/feed.xml) (https://github.com/optimaltransport)

The link is now established between Euler, Monge and Kantorovich

A concept of generalized incompressible flow

1) Consider probability measures μ on paths $t \in [0, T] \to \omega_t \in D$, where $D \subset \mathbb{R}^d$ is a compact domain in which the fluid is confined.

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2) Define the action of μ by $\mathbb{E}_\mu\{\int_0^T \frac{1}{2}|\frac{d\omega_t}{dt}|^2 dt\} \in [0, +\infty]$.

3) When the action is finite, write incompressibility as: $\mathbb{E}_\mu\{\int_0^T f(t, \omega_t)dt\} = \int_0^T \int_D f(t, x)dtdx$, for all smooth f.

Generalized solutions to the Euler model

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We say that a generalized incompressible flow (GIF) μ solves the Euler model if there is a scalar field p defined on $]0, T[\times D$, sufficiently smooth, such that, $\mu-$a.s., every path ω satisfies

$$\frac{d^2\omega_t}{dt^2} = -(\nabla p)(t, \omega_t), \quad \forall t \in]0, T[$$

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$$\frac{d^2\omega_t}{dt^2} = -(\nabla p)(t, \omega_t), \quad \forall t \in]0, T[$$

(which is consistent with Euler's concept of solutions).

"Kinetic" formulation of the Euler equations

We may attach to each generalized solution μ the measure f acting on each $\phi \in C_c([0,T] \times D \times \mathbb{R}^d)$ by

$$< f, \phi >= \mathbb{E}_\mu\{\int_0^T \phi(t, \omega_t, \frac{d\omega_t}{dt})dt\}.$$

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In this framework, classical solutions of the Euler equations just correspond to the special case:
$f = \delta(\xi - v(t,x))$ where v is the "Eulerian" velocity.

SOME REFERENCES...

1) Euler equations for incompressible fluids

L. Euler, opera omnia, seria secunda 12, p. 274,
Arnold, Ann. Fourier 1966, Ebin-Marsden, Ann. Maths 1970,
Shnirelman Math USSR Sb. 1987. Books:
Marchioro-Pulvirenti 1994, Chemin 1995, Lions 1996,
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2) Generalized incompressible flows

Y.B.: JAMS 1990, ARMA 1993, CPAM 1999, CVPDE 2012, CMP 2018+, Shnirelman: GAFA 1994, Ambrosio-Figalli: Arma 2008, CVPDE 2008, Bernot-Figalli-Santambrogio, IHP NL 2009, Baradat, CVPDE 2018+.

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3) Entropic regularization à la Schrödinger :

Arnaudon, Cruzeiro, Léonard, Zambrini...

4) Computational issues

Y. B.: Comp. Mech. 1989, Physica D 2008 (permutations), Mérigot-Mirebeau arXiv-2015, Gallouët-Mérigot arXiv-2016 (Monge-Ampère solver), Nenna's PhD 2016 (entropic reg.).