From Euler to Monge and vice versa.
Y.B. Lecture 2:
From Euler to Vlasov through Monge-Ampère and Kantorovich.

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EULER'S MODEL OF INCOMPRESSIBLE FLUIDS

One can describe the motion of an incompressible fluid inside a bounded domain D in $\mathbb{R}^d$ by a time-dependent family $t \to \mathcal{X}_t$ of maps belonging to the Hilbert space $H = L^2(D, \mathbb{R}^d)$, valued in the subset $VPM(D) \subset H$ of all Lebesgue measure-preserving maps

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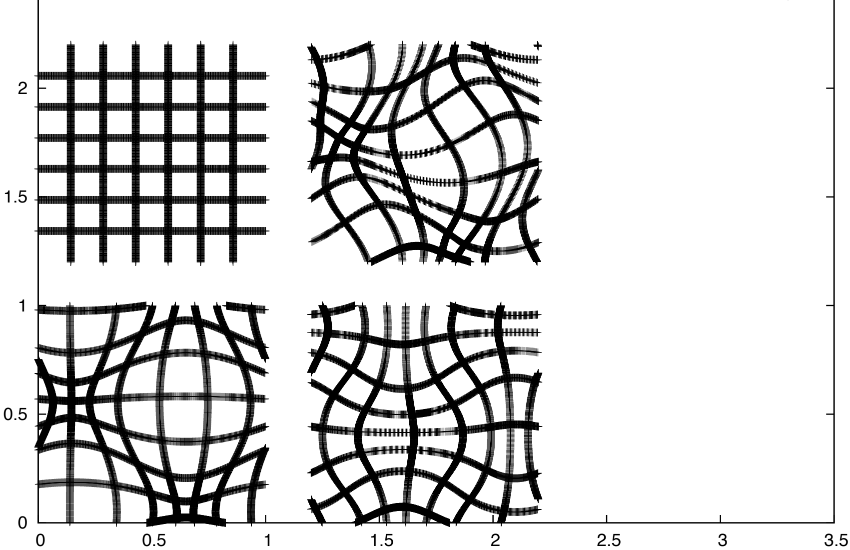
The Euler model, introduced in 1755, correspond to those curves $t \to \mathcal{X}_t \in VPM(D)$ for which there is a "pressure field" $p_t(x)$ s.t.

$$\boxed{\frac{d^2\mathcal{X}_t}{dt^2} + (\nabla p_t) \circ \mathcal{X}_t = 0}$$



three maps of the periodic square: one is area preserving

THE PRINCIPLE OF LEAST ACTION

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$$\boxed{\int_{t_0}^{t_1} \left|\left|\frac{d\mathcal{X}_t}{dt}\right|\right|_H^2 dt, \quad H = L^2(D, \mathbb{R}^d)}$$

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$$\boxed{\int_{t_0}^{t_1} \|\frac{d\mathcal{X}_t}{dt}\|_H^2 \, dt, \quad H = L^2(D, \mathbb{R}^d)}$$

In other words, such a curve is nothing but a (constant speed) minimizing geodesic along $VPM(D)$, with respect to the metric induced by $H = L^2(D, \mathbb{R}^d)$ on $VPM(D)$.

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$$\boxed{\int_{t_0}^{t_1} ||\frac{d\mathcal{X}_t}{dt}||_H^2 \, dt, \quad H = L^2(D, \mathbb{R}^d)}$$

In other words, such a curve is nothing but a (constant speed) minimizing geodesic along $VPM(D)$, with respect to the metric induced by $H = L^2(D, \mathbb{R}^d)$ on $VPM(D)$.

See Arnold 1966, Ebin-Marsden 1970, Arnold-Khesin book 1998.

VOLUME-PRESERVING MAPS: APPROXIMATION PAR PERMUTATIONS

Fix $D = [0, 1]^d$ and consider its dyadic decomposition by $N = 2^{nd}$ sub-cubes $D(\alpha)$, of barycenters $A(\alpha)$, $\alpha = 1, \cdots, N$.

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For numerical purposes, we approximate the set $VPM(D)$ of all volume-preserving maps by the discrete subset $P_N(D)$ of all rigid rearrangements of the N sub-cubes, namely maps of form:

$$\boxed{s(a) = a - A(\alpha) + A(\sigma(\alpha)), \quad a \in D(\alpha), \quad \alpha = 1, ..., N, \quad \sigma \in \mathcal{S}_N}$$

where $\mathcal{S}_N$ is the set of all permutations of $\{1, \cdots, N\}$.

REARRANGEMENTS OF N=16 SUB-CUBES AS EXAMPLES OF VOLUME PRESERVING MAPS

PENALIZATION OF THE EULER ACTION

Since minimizing geodesics along a discrete set such as the set of rigid permutations $P_N(D)$ do not make much sense, we rather consider a penalized version of the Euler action (*)

$$\boxed{\int_{t_0}^{t_1} \left(\left\|\frac{d\mathcal{X}_t}{dt}\right\|_H^2 + \epsilon^{-1} \inf_{s \in P_N(D)} \|\mathcal{X}_t - s\|_H^2\right) dt, \quad H = L^2(D, \mathbb{R}^d)}$$

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(*) For smooth sets, this is a consistent approximation to minimizing geodesics (cf. Rubin-Ungar, CPAM 1957).

FINITE-DIMENSIONAL REDUCTION

It is consistent to limit ourself to piecewise affine maps of form

$$\mathcal{X}_{\mathbf{t}}(\mathbf{a}) = \mathbf{a} - \mathbf{A}(\alpha) + \mathbf{X}_{\mathbf{t}}(\alpha), \quad \mathbf{a} \in \mathbf{D}(\alpha),$$

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Here $X_t \in (\mathbb{R}^d)^N$ becomes the new, finite-dimensional, unknown. Accordingly, the penalized action can be easily computed

$$\boxed{\int_{t_0}^{t_1} \left(\left|\left|\frac{dX_t}{dt}\right|\right|^2 + \epsilon^{-1} \inf_{\sigma \in \mathcal{S}_N} ||X_t - A_\sigma||^2\right) dt}$$

Here $||\cdot||$ denotes the euclidean norm in $H = (\mathbb{R}^d)^N$, $\mathcal{S}_N$ is the set of all permutations of $\{1, \cdots, N\}$ and $A_\sigma(\alpha) = A(\sigma(\alpha)), \ \ \alpha = 1, ..., N$.

THE RESULTING (DISCRETE) VLASOV-MONGE-AMPERE SYSTEM

Using the least-action principle, we end up with the following finite-dimensional dynamical system

$$\boxed{\epsilon \frac{d^2 X_t(\alpha)}{dt^2} = X_t(\alpha) - A(\sigma_{opt}(\alpha)), \quad \alpha = 1, ..., N}$$

$$\boxed{\sigma_{opt} = \text{Arginf}\{\sum_{\alpha=1}^{N} |X_t(\alpha) - A(\sigma(\alpha)|^2, \quad \sigma \in \mathcal{S}_N\}}$$

This can be used for numerical purposes! See related work by Mérigot and Mirebeau arXiv:1505.03306, based on Mérigot's fast Monge-Ampère solver.

The explicit time discrete version was introduced in Y.B. CMP 2000 for $\epsilon < 0$, with convergence to the Euler model as $|\epsilon| \to 0, \ \ N \geq C|\epsilon|^{-8d}, \ \ \delta t \leq C|\epsilon|^4$.

THE VLASOV-MONGE-AMPERE SYSTEM

The continuous version, involving the Monge-Ampère equation, was introduced in B. and Loeper (GAFA 2004), studied by Cullen, Gangbo, Pisante (Arma 2007), Ambrosio-Gangbo (CPAM 2008)...

$$\boxed{\partial_t f(t,x,\xi) + \nabla_x \cdot (\xi\ f(t,x,\xi)) - \nabla_\xi \cdot (\nabla_x \varphi(t,x) f(t,x,\xi)) = 0}$$

$$\boxed{\det(\mathbb{I} + \epsilon D_x^2 \varphi(t,x)) = \int_{\mathbb{R}^d} f(t,x,\xi) d\xi, \quad (t,x,\xi) \in \mathbb{R}^{1+d+d}}$$

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It is a fully nonlinear correction of the well-known Vlasov-Poisson system describing Newtonian gravitation as $d = 3$. As $\epsilon = 0$ we recover the "kinetic" formulation of the Euler equations.

PART II: A PURELY STOCHASTIC ORIGIN OF THE (discrete) VLASOV-MONGE-AMPERE MODEL

Using large deviation principles and the concept of "onde pilote" (coming from quantum mechanics), we will recover this discrete dynamical system from the trivial stochastic model of a Brownian point cloud.

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As a consequence and in some sense, the Euler model of incompressible fluids can be obtained out of pure noise!

LET US RECALL SOME VERY OLD IDEAS ABOUT RANDOMNESS BY LUCRETIUS

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When atoms move straight down through the void by their own weight, they deflect a bit in space at a quite uncertain time and in uncertain places, just enough that you could say that their motion has changed. But if they were not in the habit of swerving, they would all fall straight down through the depths of the void, like drops of rain, and no collision would occur, nor would any blow be produced among the atoms. In that case, nature would never have produced anything. (Lucretius, $\sim$ 99 – 55 BC.)

BROWNIAN VIBRATION OF A CUBIC LATTICE

We consider a cubic lattice $\{A(\alpha) \in \mathbb{R}^d,\ \alpha = 1, \cdots, N\}$ subject to brownian vibrations

$$\boxed{A(\alpha) + \sqrt{\epsilon} B_t(\alpha), \quad \alpha = 1, \cdots, N}$$

We define a point cloud as a finite set of indistinguishable points, i.e. as a point in the quotient space $(\mathbb{R}^{\mathbf{d}})^{\mathbf{N}}/\mathcal{S}_{\mathbf{N}}$.

WHERE IS THE LOCATION OF THE LATTICE AT TIME T?

At a fixed time $T > 0$, the probability for the point cloud

$$\boxed{Y_t(\alpha) = A(\alpha) + \sqrt{\epsilon} B_t(\alpha), \quad \alpha = 1, \cdots, N}$$

to be observed at $X_T = (X_T(\alpha),\ \alpha = 1, \cdots, N) \in \mathbb{R}^{dN}$ has density

$$\frac{1}{Z} \sum_{\sigma \in \mathcal{S}_N} \prod_{\alpha=1}^{N} \exp\left(-\frac{|X_T(\alpha) - A(\sigma(\alpha))|^2}{2\epsilon T}\right)$$

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$\mathcal{S}_N = \{$**permutations**$\}$, $|\cdot|$ **and** $||\cdot|| =$ **euclidean norms in** $\mathbb{R}^d$ **and** $\mathbb{R}^{Nd}$.
$Z = (2\pi\epsilon T)^{-Nd/2} N!$. **We crucially used the indistinguishability of the particles.**

LET US "SURF" THE "HEAT WAVE"

This density is just the solution of the heat equation in $\mathbb{R}^{Nd}$

$$\boxed{\frac{\partial \rho}{\partial t}(t,X) = \frac{\epsilon}{2} \triangle \rho(t,X), \quad \rho(t=0,X) = \frac{1}{N!}\sum_{\sigma \in \mathcal{S}_N} \delta(X - A_\sigma).}$$

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Given $X_{t_0} \in \mathbb{R}^{Nd}$ at $t_0 > 0$ we follow the "heat wave" by solving

$$\boxed{\frac{dX_t}{dt} = v(t,X_t), \quad v(t,X) = -\frac{\epsilon}{2}\nabla_X \log \rho(t,X), \quad t \geq t_0}$$

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This is an adaptation of de Broglie's "onde pilote" concept. As a matter of fact, a similar calculation also works for the free Schrödinger equation:

$$(i\partial_t + \triangle)\psi = 0, \quad \psi(0,X) = \sum_\sigma \exp(-||X - A_\sigma||^2/a^2), \quad v = \nabla \mathcal{I}m \log \psi$$

THE "HEAT WAVE" ODE

We get the "onde pilote" system, setting $t = \exp(2\theta)$,

$$\boxed{\frac{dX_\theta}{d\theta} = X_\theta - < A > \quad < A > = \frac{\sum_{\sigma \in \mathcal{S}_N} A_\sigma \exp(\frac{-||X_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)})}{\sum_{\sigma \in \mathcal{S}_N} \exp(\frac{-||X_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)})}}$$

ZERO-NOISE LIMIT ANALYSIS

As ϵ goes to zero, we get the first order dynamical system

$$\boxed{\frac{dX_\theta}{d\theta} = X_\theta - A_{\sigma_{opt}}\ , \quad \sigma_{opt} = \mathrm{Arginf}_{\ \sigma \in \mathcal{S}_N}\ ||X_\theta - A_\sigma||^2}$$

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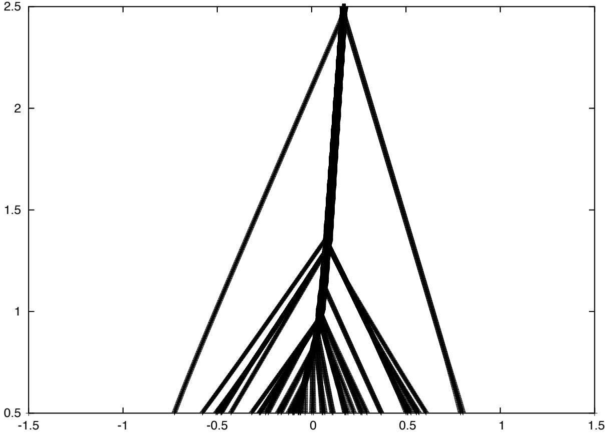
i.e. $\boxed{\frac{d_+ X_\theta}{d\theta} = -\overline{\nabla}\Phi(X_\theta)}$ **which is the "gradient flow" of the semi-convex function** $\boxed{\Phi(X) = -\inf_{\sigma \in \mathcal{S}_N} \ ||X - A_\sigma||^2/2}$

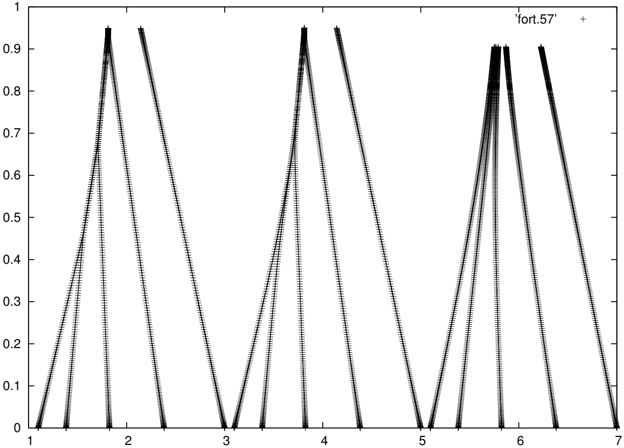
N.B. this formulation automatically include 1D sticky collisions.

Sticky collisions

horizontal : 51 grid points in x /vertical : 60 grid points in t



From free (Bosonic) Schrödinger to sticky particles



LARGE DEVIATIONS OF THE "HEAT WAVE ODE"

Let us surf the heat wave with some additional noise $\eta > 0$

$$\boxed{\frac{dX^\epsilon_\theta}{d\theta} = X^\epsilon_\theta - < A > + \eta \frac{dB_\theta}{d\theta}\ , \quad < A > = \frac{\sum_{\sigma \in \mathcal{S}_N} A_\sigma \exp\left(\frac{-||X^\epsilon_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)}\right)}{\sum_{\sigma \in \mathcal{S}_N} \exp\left(\frac{-||X^\epsilon_\theta - A_\sigma||^2}{2\epsilon \exp(2\theta)}\right)}}$$

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For ϵ fixed, we first use the Freidlin-Vencel theory to get the "good rate function" for the large deviations of the system as $\eta \to 0$.

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For ϵ fixed, we first use the Freidlin-Vencel theory to get the "good rate function" for the large deviations of the system as $\eta \to 0$. Then, we may pass to the limit $\epsilon \to 0$ (*) and obtain as "$\Gamma-$limit"

$$\boxed{\int ||\frac{dX_\theta}{d\theta}||^2 + ||\overline{\nabla}\Phi(X_\theta)||^2 d\theta, \quad \Phi(X) = -\inf_{\sigma \in \mathcal{S}_N} ||X - A_\sigma||^2/2}$$

(*) thanks to L. Ambrosio, private communication.

LEAST ACTION PRINCIPLE

The least action principle applied to

$$\boxed{\int ||\frac{dX_\theta}{d\theta}||^2 + ||\nabla\Phi(X_\theta)||^2 d\theta, \quad \Phi(X) = -\inf_{\sigma\in\mathcal{S}_N} \; ||X - A_\sigma||^2/2}$$

(formally) leads to the following dynamical system

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(formally) leads to the following dynamical system

$$\boxed{\frac{d^2X_\theta}{d\theta^2} = \nabla(\frac{||\nabla\Phi||^2}{2})(X_\theta) = -(\nabla\Phi)(X_\theta)}$$

Indeed $||\nabla\Phi||^2 = -2\Phi$ because -2Φ is a squared distance function.

THE RESULTING DYNAMICAL SYSTEM

So, we have finally obtained

$$\boxed{\frac{d^2 X_\theta(\alpha)}{d\theta^2} = X_\theta(\alpha) - A(\sigma_{opt}(\alpha))\ , \quad X_\theta(\alpha) \in \mathbb{R}^d, \ \ \alpha = 1, \cdots, N}$$

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references: Y.B. arXiv:1504.07583, L. Ambrosio, A. Baradat, Y.B., in preparation.

Some references

Stochastics: Y. B., "A double LD principle for MA gravitation", arXiv 2015, Bull. of the Inst. Math. Acad. Sinica 2016. L. Ambrosio, A. Baradat, Y.B., in preparation.

The Monge-Ampère equation in optimal transport theory:

Y.B. CRAS 1987, CPAM 1991, L. Caffarelli 1990, books: C. Villani and F. Santambrogio.

Zeldovich approximation in Cosmology:

Y. Zeldovich, Astron. Astrophys. 1970,

Y. B., U. Frisch, M. Hénon, G. Loeper, S. Matarrese, R. Mohayaee, A. Sobolevskii, Mon. Not. R. Astron. Soc. 2003 (and references included)

The Vlasov-Monge-Ampère system:

Y.B. CMP 2000,

Y. B., G. Loeper, GAFA 2004,

M. Cullen, W. Gangbo, L. Pisante, ARMA 2007,

L. Ambrosio, W. Gangbo, CPAM 2008,

Y. B., Confluentes Mathematici 2011.

Sticky particles:

Y.B. SINUM 1984, WASCOM XI 2001, Meth. Appl. Anal. 2004,

Y.B., E. Grenier, SINUM 1998,

Y.B., W. Gangbo, G. Savaré, M. Westdickenberg, JMPA. 2013.

Berman's papers (related to Kählerian Geometry)

ArXiv (2008/2013/2015).