From Euler to Monge and vice versa:
Link between MFG and initial value problems (Y.B.)

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Yesterday, FS discussed the variational MFG

$$\partial_t \mu + \nabla \cdot (\mu \nabla \phi) = \nu \Delta \mu, \quad \partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + \nu \Delta \phi = f'(\mu),$$

$t \in [0, T]$, $x \in D = \mathbb{T}^d$, $\mu(t, x) \geq 0$, $\phi(t, x)$, respectively prescribed at $t = 0$ and $t = T$.

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With $\nu = 0$ and written in terms of $v = \nabla \phi$, these equations read

$$\partial_t \mu + \nabla \cdot (\mu v) = 0, \quad \partial_t v + (v \cdot \nabla) v = \nabla (f'(\mu)),$$

and looks like the equations written by Euler in 1755-57 for compressible fluids.

The Euler equations written in conservation form

$$\partial_t \mu + \nabla \cdot q = 0, \quad q = \mu v,$$

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CONCAVITY of f is needed to get a WELL-POSED INITIAL VALUE PROBLEM, with boundary conditions only at $t = 0$ and none at $t = T$, in contrast with MFG.

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This will be possible through a GENERALIZED MFG, involving vector-potentials (and measures taking values in the cone of semi-definite positive matrices).

The class of "entropic conservation laws"

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$$\partial_t U + \nabla \cdot (F(U)) = 0, \quad U = U(t,x) \in \mathcal{W} \subset \mathbb{R}^m, \quad x \in D$$

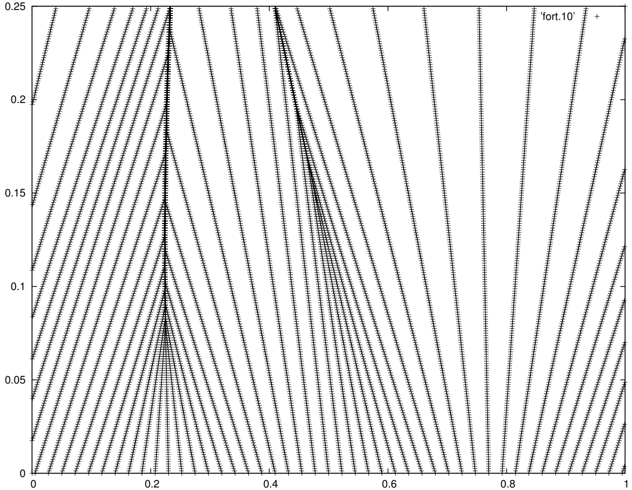
(where F is given so that $\sum_{\beta=1}^{m} \partial_\beta \mathcal{E}(W) \partial_\alpha F^{i\beta}(W) = \partial_\alpha Q^i(W), \ \forall W \in \mathcal{W},$

for some $(\mathcal{E}, Q) : \mathcal{W} \to \mathbb{R}^{1+d}$, with $\mathcal{W}$ open convex and "entropy" $\mathcal{E}$ strictly convex,

which implies: $\partial_t(\mathcal{E}(U)) + \nabla \cdot (Q(U)) = 0$, for all smooth solutions U)

contains the Euler equations,

for which: $\mathcal{E}(\mu, q) = \frac{|q|^2}{2\mu} - f(\mu), \quad \mu > 0$, with f concave.



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t,x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)

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The problem is not trivial since there may be many weak solutions starting from U_0 which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi).

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N.B. The supremum in A exactly encodes that U is a weak solution with initial condition U_0, all test functions A acting like Lagrange multipliers.

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$$= \sup_{A(T,\cdot)=0} \int_0^T \int_D -G(\partial_t A, \nabla A) - \int_D A(0,\cdot) \cdot U_0$$

$$G(E,B) = \sup_{V \in \mathcal{W} \subset \mathbb{R}^m} E \cdot V + B \cdot F(V) - \mathcal{E}(V), \ (E,B) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}.$$

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Notice that G is automatically convex.

Comparison with variational MFG

$$\sup_{\phi} \int_0^T \int_D -G(\partial_t \phi + \nu \Delta \phi, \nabla \phi) - < \mu_0, \phi_0 >$$

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Thus our dual maximization problem to solve the initial value problem can be interpreted as a generalized variational 1st order-MFG with vector-valued potential.

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Theorem 1: If U is a smooth solution to the Cauchy problem and T is not too large (*), then U can be recovered from the concave maximization problem which admits $A(t,x) = (t-T)\mathcal{E}'(U(t,x))$ as solution.

Theorem 2: For the Burgers equation, all entropy solutions can be recovered, for arbitrarily large T.

(*) more precisely if, $\forall\, t, x, V \in \mathcal{W}$, $\mathcal{E}''(V) - (T-t)F''(V) \cdot \nabla(\mathcal{E}'(U(t,x))) > 0$.

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Then, the maximization problem in A simply reads

$$\sup_A \int_{[0,T]\times\mathbb{T}} -\frac{(\partial_t A)^2}{2(1-\partial_x A)} - \int_{\mathbb{T}} A(0,\cdot)u_0.$$

with $A = A(t,x) \in \mathbb{R}$ subject to $A(T,\cdot) = 0, \ \ \partial_x A \leq 1.$

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Introducing $\mu = 1 - \partial_x A \geq 0$, $q = \partial_t A$, we get the MFG

$$\sup_{(\mu,q)}\{\int_{[0,T]\times\mathbb{T}} -\frac{q^2}{2\mu} - qu_0 \mid \partial_t\mu + \partial_x q = 0, \quad \mu(T,\cdot) = 1\}.$$

Generalized MFG for the Euler equations

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Let us compute the generalized MFG in the particular case of the Euler equations of isothermal fluids

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where

$$f(w) = w - \log w, \quad p'(w) = -wf''(w) \longrightarrow p(w) = w.$$

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Then, the generalized MFG amounts to minimizing

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among all fields $u = u(t,x) \in \mathbb{R}$, $Q = Q(t,x) \in \mathbb{R}^d$, $M = M(t,x) = M^t(t,x) \in \mathbb{R}^{d\times d}$, $M \geq 0$,

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$$u = \partial_t\sigma + \partial^i w_i,\ \ Q_i = \partial_t w_i + \partial_i\sigma,\ \ M_{ij} = \delta_{ij} - \partial_i w_j - \partial_j w_i,$$

where σ and w must vanish at $t = T$.

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The same method also applies to the Euler equations of incompressible fluids ("saturated congestion")

$$\partial_t q + \nabla \cdot (q \otimes q) = -\nabla p, \quad \nabla \cdot q = 0,$$

where q is prescribed at $t = 0$ and p is now a Lagrange multiplier ("price") for constraint $\nabla \cdot q = 0$.

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We get again a generalized MFG for measures valued in the cone of semi-definite symmetric matrices:

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Extension to some parabolic equations

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Using the quadratic change of time $t \to \theta = t^2/2$, as in Y.B., X. Duan (Arma 2018), we may derive from the Euler equations, with pressure $p = \rho^2$, the "porous medium" equation $\partial_\theta \rho = \Delta \rho^2$ and, therefore, we get for it a corresponding convex minimization problem:

$$\inf\{\int_{[0,T]\times\mathbb{T}^d} \frac{q^2}{4\sigma} - \rho_0 q, \text{ s.t. } \partial_\theta \sigma + \Delta q = 0, \ \ \sigma(T,\cdot) = 1\}.$$

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This is again similar to an optimal transport problem.

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We have already obtained the (elementary) MFG

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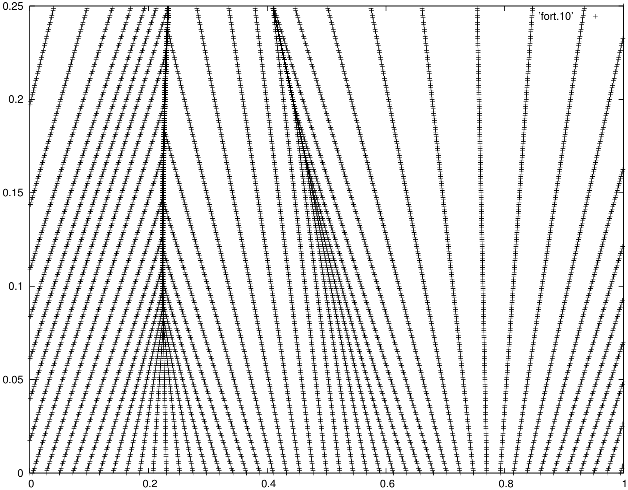
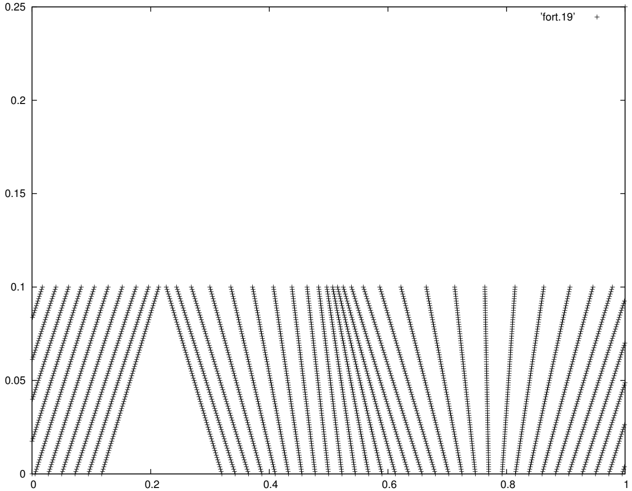
It turns out that, for arbitrarily large T, we may recover, through this problem, the correct "entropy solution" à la Kruzhkov-Panov, but only at time T and (surprisingly enough) not for $t < T$, once shocks have formed).



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t,x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)



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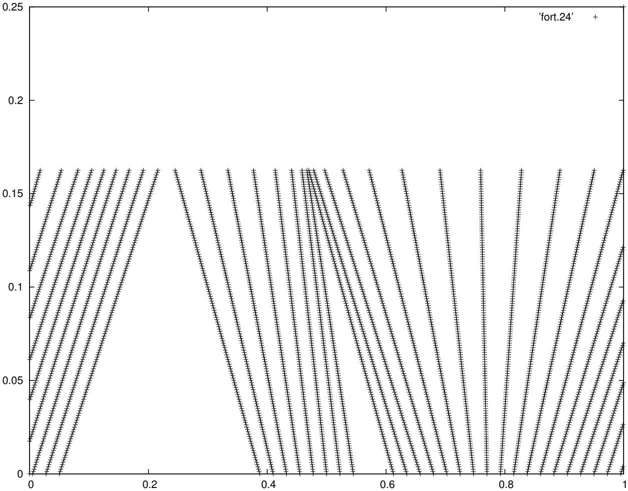
Recovery of the solution at time T=0.1 by convex optimization.
Observe the formation of a first vacuum zone as the first shock has formed.



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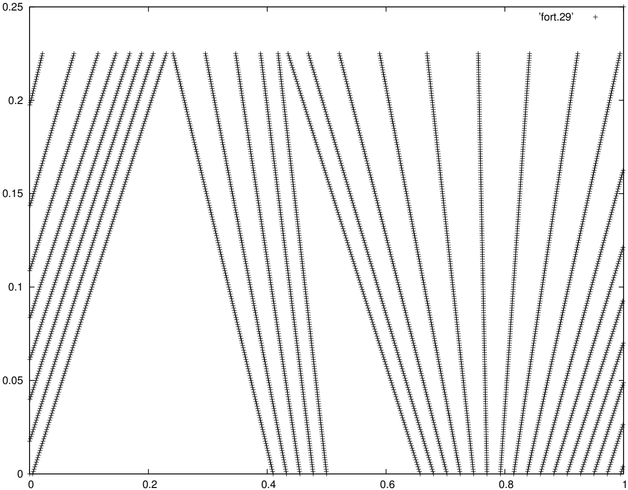
Recovery of the solution at time T=0.16 by convex optimisation.
Observe the formation of a second vacuum zone as the second shock has formed.



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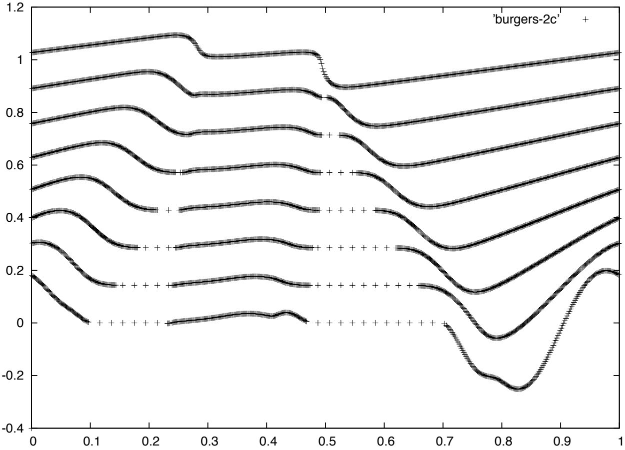
Recovery of the solution at time T=0.225 by convex optimisation.
Observe the extension of the two vacuum zones.



Numerics: 2 lines of code differ from a standard (Benamou-B.) OT solver!

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Thanks for your attention! For more details, voir Y.B. ArXiv Oct. 2017, to appear in CMP.