From Euler to Monge and vice versa: Geophysical and convection models involving OT (Y.B.)

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Outline

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2. Hydrostatic Boussinesq equations and Cullen-Purser convexity condition.
3. Derivation from the Euler-Boussinesq equation with the "relative-entropy" method.
4. Global existence of "entropy" solutions for the hydrostatic Boussinesq model.

Geophysical flows: a "global change" model :-)

Let D be a smooth bounded domain $D \subset R^3$ in which moves an incompressible fluid of velocity $v(t, x)$ at $x \in D, \ \ t \geq 0$, subject to the Euler equations

$$\mathbf{EB} \quad (\partial_t v + v \cdot \nabla)v + \nabla p = y, \qquad (\partial_t + v \cdot \nabla)y = \epsilon G(\epsilon t, x)$$

with $\nabla \cdot v = 0$ and $v // \partial D$.

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The field $y = y(t,x) \in R^3$ is a vector-valued force, taking into account Coriolis and convection effects, with a small, slowly evolving, "global change"-type, source term, where G is a given smooth function with bounded derivatives.
We want to describe the evolution of this system at large times $t \sim \epsilon^{-1}$

The rescaled EB model and its formal HB limit

Through $(\mathbf{t}, \mathbf{v}, \mathbf{p}, \mathbf{y}) \to (\epsilon \mathbf{t}, \epsilon \mathbf{v}, \mathbf{p}, \mathbf{y})$, we get the rescaled EB model

$$\mathbf{EB}: \quad \mathbf{y} = \nabla \mathbf{p} + \epsilon^{\mathbf{2}}(\partial_{\mathbf{t}} \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}), \qquad \nabla \cdot \mathbf{v} = \mathbf{0}$$

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$$\partial_{\mathbf{t}} \mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{t}, \mathbf{x})$$

We call the formal limit "HYDROSTATIC BOUSSINESQ" HB

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Notice that, $(v \cdot \nabla)y = (D_x^2 p \cdot v)$ and $v = \nabla \times A$, for some divergence-free vector potential $A = A(t,x) \in R^3$, when $d = 3$.

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Notice that, $(\mathbf{v}\cdot\nabla)\mathbf{y} = (\mathbf{D_x^2 p}\cdot\mathbf{v})$ and $\mathbf{v} = \nabla\times\mathbf{A}$, for some divergence-free vector potential $\mathbf{A} = \mathbf{A}(\mathbf{t},\mathbf{x}) \in \mathbf{R}^3$, when $\mathbf{d} = 3$. Taking the curl of the evolution equation, we get

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This linear 'magnetostatic' system in A is elliptic whenever p is strongly convex $0 < \mathbf{cst}\ \mathbf{Id} < \mathbf{D_x^2 p}(\mathbf{t},\mathbf{x}) < \mathbf{cst}'\ \mathbf{Id}$

Derivation of the HB model under strong convexity condition

Theorem

Let (y, p, v) be a smooth solution of HB s.t.

$0 < \text{cst Id} < D_x^2 p(t, x) < \text{cst}' \text{ Id}$ Then, any solution $(y^\epsilon, p^\epsilon, v^\epsilon)$ to the rescaled EB Euler-Boussinesq equations, with same initial condition, converges to (y, p, v).

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The "weak-strong uniqueness principle" easily follows in the case $0 < r \le \mathcal{E}'' \le r^{-1}$ since, then, $|\zeta[u,v]| \le \mathrm{Lip}(F')|u-v|^2 \sim \eta[u,v]$.

Idea of the proof

Do not try to estimate plain L^2 distances (which completely fails) but rather use

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where H is the "relative entropy"

$$\mathbf{H}[\mathbf{y}, \mathbf{y}^\epsilon] = \int_{\mathbf{D}} [\mathbf{p}^*(\mathbf{t}, \mathbf{y}^\epsilon) - \mathbf{p}^*(\mathbf{t}, \mathbf{y}) - \nabla \mathbf{p}^*(\mathbf{t}, \mathbf{y}) \cdot (\mathbf{y}^\epsilon - \mathbf{y})] \mathbf{dx} \sim \int |\mathbf{y} - \mathbf{y}^\epsilon|^2 \mathbf{dx}$$

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$$H[y, y^\epsilon] = \int_D [p^*(t, y^\epsilon) - p^*(t, y) - \nabla p^*(t, y) \cdot (y^\epsilon - y)] dx \sim \int |y - y^\epsilon|^2 dx$$

built on the Legendre-Fenchel transform
$p^*(t, z) = \sup_{x \in D} x \cdot z - p(t, x)$ **of the limit convex potential p.**

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$$\mathbf{v}(\mathbf{t}, \mathbf{x}) = \mathbf{0}, \quad \mathbf{y}(\mathbf{t}, \mathbf{x}) = \nabla \mathbf{p}(\mathbf{t}, \mathbf{x}), \;\; \mathbf{p}(\mathbf{t}, \mathbf{x}) = \mathbf{p_0}(\mathbf{x}) + \mathbf{t}\mathbf{g}(\mathbf{x})$$

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in which case, the force field $y(t,x) = \nabla p(t,x)$ is completely determined by the knowledge of all 'observables'

$$f \rightarrow \int_D f(y(t,x))dx$$ **by OPTIMAL TRANSPORT THEORY**

A concept of "entropy" solutions for the HB system

By analogy with hyperbolic conservation laws, we introduce the concept of "entropy" solution, formally self-consistent, for the HB system

DEFINITION

We say that $(t \to y(t,\cdot)) \in C^0(R_+, L^2(D, R^3))$ is a solution with convex potential to the HB system, if

$$\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(t,x)dx, \quad \forall f$$

with $y(t,x) = \nabla p(t,x)$ for some CONVEX function p.

Global existence of "entropy" solutions

Theorem

For each initial condition in L^2, there is an "entropy solution" y that belongs to the space $C^0(R_+,\ L^2(D,R^d))$ and has a convex potential: $y(t,\cdot) = \nabla p(t,\cdot)$ for each $t \geq 0$.

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See YB, JNLS 2009. Notice that the system is self-consistent, thanks to optimal transport theory. However, our global existence result does not imply stability with respect to initial conditions, except for $d = 1$, where we can use the theory of scalar conservation laws, or $d > 1$ and $G = G(x) = -x$, where we can use maximal monotone operator theory

Open problems

Stability and singularities

Global "entropy" solutions are known to be stable with respect to initial conditions only in some special cases, such as $d = 1$ or $G(x) = -x$. Clearly, this needs to be extended to all cases. Moreover, strict convexity clearly breaks down in finite time for some data, but is it generically true? This is known only for $d = 1$ thanks to scalar conservation law theory.

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Convergence beyond singularities

It is much more challenging to prove, after strict convexity breaks down, that the "extended" solutions which obey the convexity principle, correctly describe the limit of the EB solutions in the HB regime. They may be just crude (but relevant) approximations, in some suitable sense for which a right mathematical framework has to be found. A similar situation occurs in shallow water theory when shock waves ("hydraulic jumps") appear.

Some references

a) General discussion and global existence: YB, JNLS 2009,
b) Local smooth solutions: G. Loeper 2008 (for SG equations)
c) Derivation from the EB equations:
YB and M. Cullen, CMS 2010, YB, Philos. Trans. R. Soc. Lond. Ser. A 2013.