A modified least action principle allowing mass concentrations for the early universe reconstruction problem

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Outline

1. The early universe reconstruction EUR problem and the pressure-less Euler-Poisson model

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4. Gradient flow solutions and modified action taking concentrations into account

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2. Zeldovich approximation and Monge-Ampère gravitation
3. The least-action principle for Monge-Ampère gravitation
4. Gradient flow solutions and modified action taking concentrations into account
5. Numerics for the EUR problem in 1D

THE EARLY UNIVERSE RECONSTRUCTION Pb

Following Peebles 1989, Frisch and coauthors (Nature 417) 2002, we want to reconstruct the history of the Universe from the knowledge of the present mass density field. We consider an expanding universe with self-gravitating matter.

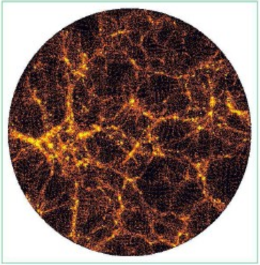

Figure 7. N-body simulation output in the Eulerian space used for testing our reconstruction method (shown is a projection onto the x-y plane of a 10% slice of the simulation box of size $200h^{-1}$ Mpc). Points are highlighted in yellow when reconstruction fails by more than $6.25\,h^{-1}$ Mpc, which happens mostly in high-density regions.

SIMPLIFIED MATHEMATICAL FORMULATION

Given $t_1 > t_0 > 0$, find a time-dependent family of probability measures on the unit 3D periodic box $D = T^3 = R^3/Z^3$

$$\boxed{t \in [t_0, t_1] \to \rho(t, dx) \in \mathrm{Prob}(D)}$$

prescribed at $t = t_0$ and $t = t_1$, that minimizes the "EUR" action

$$\boxed{\int_{t_0}^{t_1} dt \int_D \ t^{3/2}\{2\rho(t, dx)|v(t, x)|^2 + 3|\nabla\varphi(t, x)|^2 dx\ \}}$$

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where $v = v(t, x) \in R^3$, $\varphi = \varphi(t, x) \in R$, are subject to

$$\boxed{\partial_t \rho + \nabla \cdot (\rho v) = 0\ , \quad \rho = 1 + t\ \nabla^2\varphi}$$

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(all coefficients in red come from general relativity)

LEAST ACTION SOLUTIONS versus DYNAMICAL CONCENTRATIONS

The EUR action is strictly convex in $(\rho, \rho\mathbf{v}, \varphi)$ which leads to the existence and uniqueness of least action solutions (Loeper 2006). They satisfy the pressure-less Euler Poisson equations

$$\boxed{\partial_t(t^{3/2}\rho\mathbf{v}) + \nabla\cdot(t^{3/2}\rho\mathbf{v}\otimes\mathbf{v}) = -\frac{3t^{1/2}}{2}\rho\nabla\varphi\ , \quad \nabla\times\mathbf{v} = 0}$$

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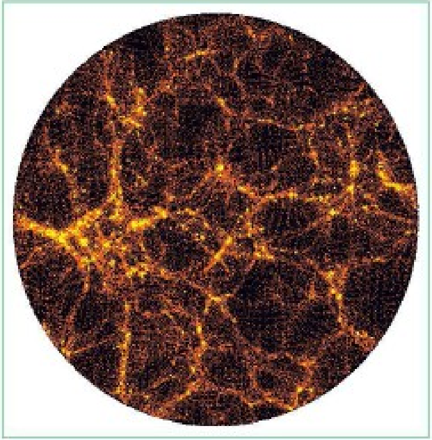
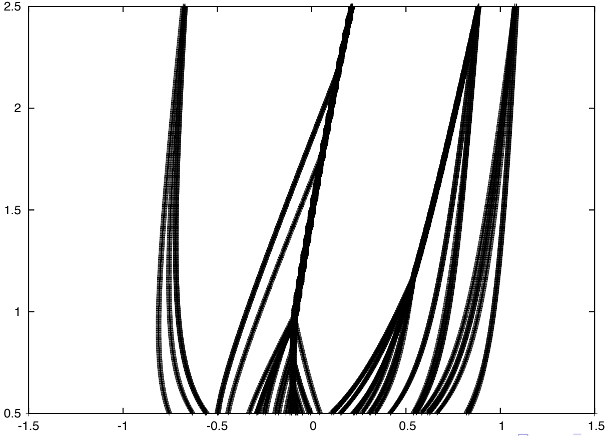
Unfortunately, typical solutions of the corresponding IVP do concentrate in finite time, i.e. $\rho(t, dx)$ becomes singular. This severely diminishes the interest of Loeper's result which cannot handle dynamical concentrations



Trajectories with dynamical concentrations (1D pressure-less Euler Poisson system)

horizontal : space /vertical : time



ZELDOVICH APPROXIMATION

Formally, optimal trajectories $(t, a) \to X(t, a)$ are ruled by:

$$\boxed{\frac{2t}{3}\frac{d^2X}{dt^2} + \frac{dX}{dt} + \nabla\varphi(t, X(t)) = 0}$$

$$\boxed{\rho(t, x) = \int \delta(x - X(t, a))da = 1 + t\nabla^2\varphi(t, x)}$$

where a denotes the particle label. At early times $t \downarrow 0$, "friction" takes over "inertia" (Einstein+Newton go back to Aristoteles!)

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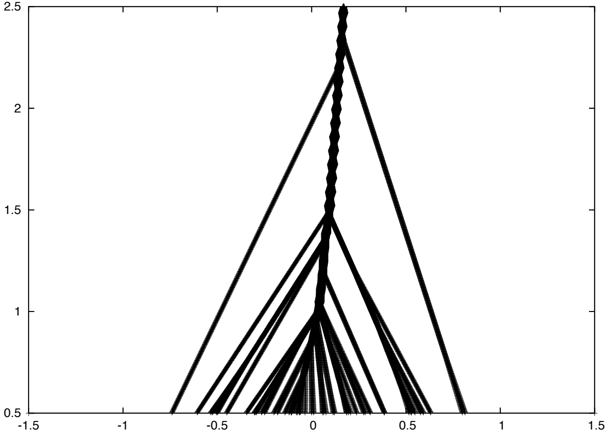
A simple approximation (exact in 1D!) is due to Zeldovich $\sim$ 1970

$$\boxed{X(t,a) = a - t\nabla\varphi_0(a), \quad \nabla^2\varphi_0(x) = \lim_{t\downarrow 0}\frac{\rho(t,x)-1}{t}}$$

1D Zeldovich solutions with concentrations

horizontal : space /vertical : time



"BURGURLENCE" AND ADHESION DYNAMICS

Zeldovich formula

$$\boxed{\mathbf{X}(\mathbf{t},\mathbf{a}) = \mathbf{a} - \mathbf{t}\nabla\varphi_{\mathbf{0}}(\mathbf{a}), \quad \nabla^{\mathbf{2}}\varphi_{\mathbf{0}}(\mathbf{x}) = \lim_{\mathbf{t}\downarrow\mathbf{0}} \frac{\rho(\mathbf{t},\mathbf{x}) - \mathbf{1}}{\mathbf{t}}}$$

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implies concentration in finite time
This is the starting point of "burgurlence" theory (Frisch, Sinai etc...) based on adhesion dynamics ruled by the multidimensional non-viscous Burgers equation

$$\boxed{\partial_{\mathbf{t}}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{0}, \quad \mathbf{u}(\mathbf{t},\mathbf{X}(\mathbf{t},\mathbf{a})) = \frac{\mathbf{X}(\mathbf{t},\mathbf{a}) - \mathbf{a}}{\mathbf{t}}, \quad \mathbf{u}(\mathbf{0},\mathbf{x}) = \nabla\varphi_{\mathbf{0}}(\mathbf{x})}$$

MONGE-AMPERE GRAVITATION

Zeldovich' formula turns out to be exact if the Monge-Ampère equation $\boxed{\rho(\mathbf{t},\mathbf{x}) = \mathbf{det}(\mathbf{I} + \mathbf{t}\mathbf{D}^2\varphi(\mathbf{t},\mathbf{x}))}$ **substitutes for the Poisson equation** $\boxed{\rho(\mathbf{t},\mathbf{x}) = \mathbf{1} + \mathbf{t}\nabla^2\varphi(\mathbf{t},\mathbf{x})}$ **which is the same in 1D.**
This observation relies on optimal transport theory

MONGE-AMPERE GRAVITATION

Zeldovich' formula turns out to be exact if the Monge-Ampère equation $\boxed{\rho(t,x) = \det(I + tD^2\varphi(t,x))}$ **substitutes for the Poisson equation** $\boxed{\rho(t,x) = 1 + t\nabla^2\varphi(t,x)}$ **which is the same in 1D.**

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IDEA: consider the MONGE-AMPERE GRAVITATIONAL MODEL

$$\boxed{\frac{2t}{3}\frac{d^2X}{dt^2} + \frac{dX}{dt} + \nabla\varphi(t, X(t)) = 0}$$

$$\boxed{\rho(t,x) = \int \delta(x - X(t,a))da = \det(1 + t\nabla^2\varphi(t,x))}$$

MONGE-AMPERE GRAVITATION (MAG)

THE ABSTRACT FRAMEWORK

Let $\mathbf{H} = \mathbf{L}^2(\mathbf{D}, \mathbf{R}^\mathbf{d})$, $\mathbf{D} \subset \mathbf{R}^\mathbf{d}$ and

$$\boxed{\mathbf{S} = \{ \ \mathbf{s} \in \mathbf{H} \ , \quad \int_\mathbf{D} \mathbf{f}(\mathbf{s}(\mathbf{a}))\mathbf{da} = \int_\mathbf{D} \mathbf{f}(\mathbf{a})\mathbf{da}, \quad \forall \mathbf{f} \in \mathbf{C}(\mathbf{R}^\mathbf{d}) \ \}}$$

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MONGE-AMPERE GRAVITATION (MAG)

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the subset of all **Lebesgue-measure preserving maps** of D
The **MAG action**
for a curve $t \to X(t) \in H$ is defined by

$$\boxed{\int_{t_0}^{t_1} \frac{2t^{3/2}}{3} \|\frac{dX}{dt}\|^2 + t^{-1/2} Q[X(t)] \quad dt, \qquad Q[X] = \inf\{\|X - s\|^2\ ;\ s \in S\}}$$

MONGE-AMPERE GRAVITATION (MAG)

Using Eulerian coordinates

$$\boxed{\rho(\mathbf{t}, \mathbf{dx})(\mathbf{1}, \mathbf{v}(\mathbf{t}, \mathbf{x})) = \int_{\mathbf{D}} \delta(\mathbf{x} - \mathbf{X}(\mathbf{t}, \mathbf{a}))(\mathbf{1}, \partial_{\mathbf{t}}\mathbf{X}(\mathbf{t}, \mathbf{a}))\mathbf{da}}$$

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we recover the Monge-Ampère gravitation MAG model

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$$\boxed{\partial_{\mathbf{t}}\rho + \nabla \cdot (\rho\mathbf{v}) = \mathbf{0}, \quad \det(\mathbf{I} + \mathbf{t}\ \mathbf{D}^2_{\mathbf{x}}\varphi) = \rho}$$

where the nonlinear Monge-Ampère equation $\boxed{\rho = \det(\mathbf{I} + \mathbf{t}\ \mathbf{D}^2_{\mathbf{x}}\varphi)}$

substitutes for the linear Poisson equation $\boxed{\rho = \mathbf{1} + \mathbf{t}\nabla^2\varphi}$ **and makes Zeldovich' approximation exact**

SPECIAL STRUCTURE OF THE MAG ACTION

In the MAG action, the potential part satisfies:

$$\boxed{Q[X] = \inf\{||X - s||^2 \; ; \; s \in S\} = \frac{1}{4}||\nabla_H Q[X]||^2}$$

and we can rewrite the MAG action:

$$\boxed{\int_{t_0}^{t_1} \frac{8}{3} t^{3/2} ||\frac{dX}{dt}||^2 + t^{-1/2} ||\nabla_H Q[X(t)]||^2 \quad dt}$$

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By reorganizing the squares and integrating by part in time, we find (in the EUR case)

$$\boxed{\frac{8}{3} \int_{t_0}^{t_1} t^{-1/2} ||t\frac{dX}{dt} - \frac{1}{2}\nabla_H Q[X(t)]||^2 \quad dt \quad + \quad \text{time boundary term}}$$

GRADIENT FLOW SOLUTIONS AS SPECIAL LEAST-ACTION SOLUTIONS

Due to the special structure of the MAG action, we find as particular least action solutions any solution to the GRADIENT FLOW EQUATION

$$\boxed{\mathbf{t}\frac{\mathbf{dX}}{\mathbf{dt}} = \frac{\mathbf{1}}{\mathbf{2}}\nabla_{\mathbf{H}}\mathbf{Q}[\mathbf{X}(\mathbf{t})]\ ,\quad \mathbf{Q}[\mathbf{X}] = \inf\{||\mathbf{X}-\mathbf{s}||^{\mathbf{2}}\ ;\ \mathbf{s}\in\mathbf{S}\}}$$

(just like "instantons" in YANG-MILLS theory)

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(just like "instantons" in YANG-MILLS theory)
It turns out that Zeldovich solutions are just exact solutions of this gradient flow equation!

GLOBAL SOLUTIONS OF THE GRADIENT FLOW

Let us introduce the Lipschitz convex function R defined by

$$\boxed{\mathbf{R}[\mathbf{X}] = \frac{||\mathbf{X}||^2 - \mathbf{Q}[\mathbf{X}]}{2} = \sup\{((\mathbf{X}, \mathbf{s})) - \frac{1}{2}||\mathbf{s}||^2, \quad \mathbf{s} \in \mathbf{S}\}}$$

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For each $\mathbf{X} \in \mathbf{H}$, we use the key concept of "lazy" gradient $\mathbf{d^0R}[\mathbf{X}]$, which is the unique vector $\mathbf{w} \in \mathbf{H}$ with lowest norm $||\mathbf{w}||$ such that

$$\boxed{\mathbf{R}[\mathbf{Z}] \geq \mathbf{R}[\mathbf{X}] + ((\mathbf{w}, \mathbf{Z} - \mathbf{X})), \quad \forall \mathbf{Z} \in \mathbf{H}}$$

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By the classical maximal monotone operator theory (going back to the 70'), the initial value problem for the gradient-flow equation has a unique global solution $\mathbf{X} \in \mathbf{C}^0([\mathbf{t}_0, +\infty[, \mathbf{H})$ in the sense

$$\boxed{\mathbf{t}\frac{\mathbf{dX}(\mathbf{t}+0)}{\mathbf{dt}} = \mathbf{X}(\mathbf{t}) - \frac{1}{2}\mathbf{d}^0\mathbf{R}[\mathbf{X}(\mathbf{t})]\,, \quad \forall \mathbf{t} \geq \mathbf{t}_0}$$

THE MODIFIED MAG ACTION TAKING CONCENTRATIONS INTO ACCOUNT

It turns out that the "lazy" gradient favors dynamical concentrations!
Thus, we take concentration into account by introducing a modified MAG action, by substituting the "lazy" gradient for the regular gradient

$$\boxed{\int_{t_0}^{t_1} t^{-1/2}||t\frac{dX}{dt} - X(t) + d^0R[X(t)]||^2 \quad dt}$$

where we recall that

$$\boxed{R[X] = \sup\{((X, s)) - \frac{1}{2}||s||^2, \quad s \in S\}}$$

NUMERICS

In the next slides,
we show samples of 1D simulations, directly based on the minimization of the fully space and time discrete version of the action.

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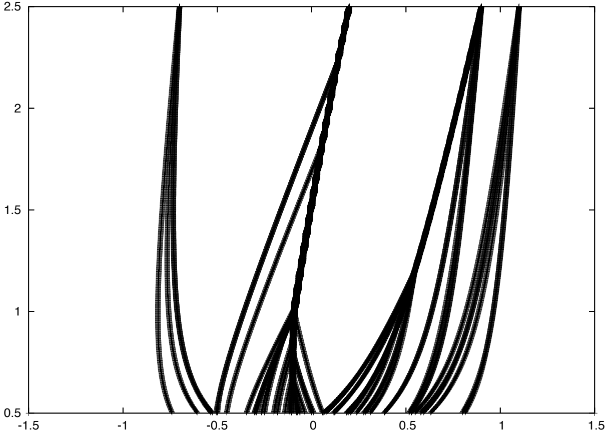
**In the next slides,
we show samples of 1D simulations, directly based on the minimization of the fully space and time discrete version of the action.
As a matter of fact, the discrete scheme does not even rely on the computation of "lazy" gradients. The calculation entirely relies on many ($\sim 10^5$) iterations of an elementary sorting algorithm.**

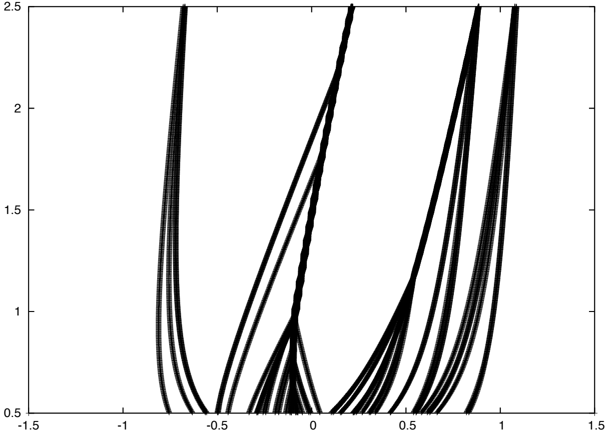
EUR-case 1: reconstructed trajectories

horizontal : 51 grid points in x /vertical : 60 grid points in t



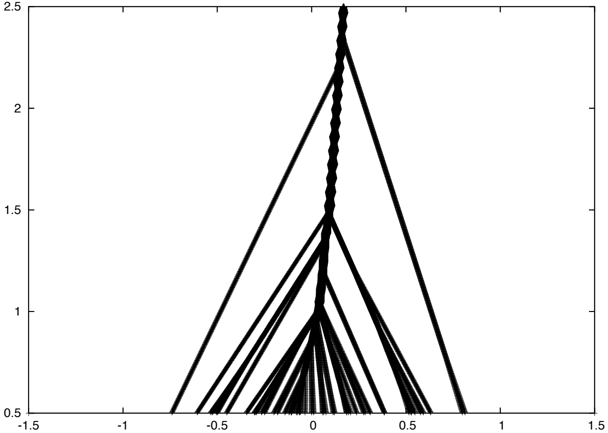
EUR-case 1: IVP with reconstructed velocities

horizontal : 51 grid points in x /vertical : 60 grid points in t



EUR-case 2: solution of the gradient flow equation

horizontal : 51 grid points in x /vertical : 60 grid points in t



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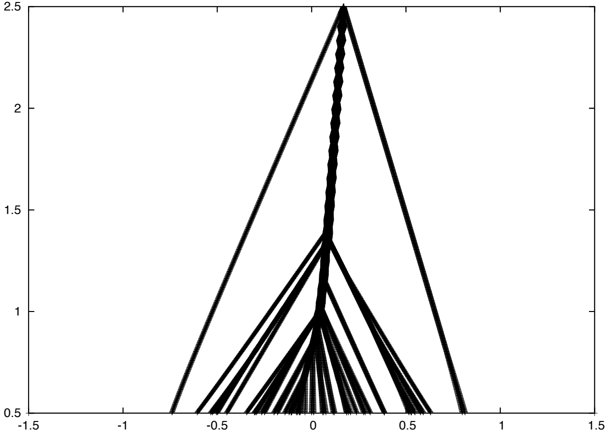

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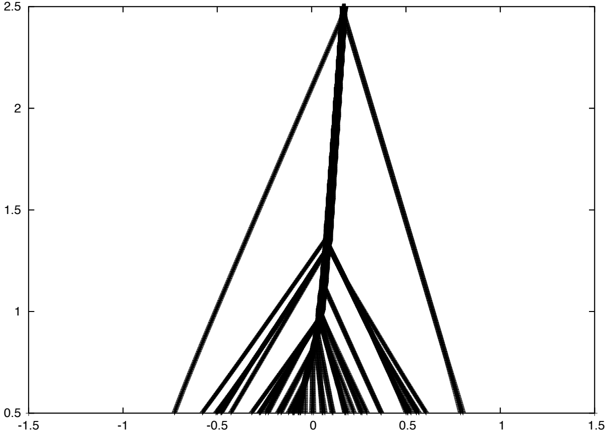

DISCUSSION

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This leads,
in our opinion, to a much better handling of the EUR problem, with a drawback: the substitution of the Monge-Ampère gravitation for the Newton gravitation, which is, of course, questionable from the physical viewpoint.
Numerics are easy to do in 1D but are challenging in higher dimensions.

THANK YOU FOR YOUR ATTENTION!

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Abridged bibliography:

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