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HIDDEN CONVEXITY IN SOME NONLINEAR PDEs FROM GEOMETRY AND PHYSICS

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HIDDEN CONVEXITY IN SOME NONLINEAR PDEs FROM GEOMETRY AND PHYSICS

1. **THE MONGE-AMPERE EQUATION**
   (solving the Minkowski problem and playing a key role in Optimal Transport Theory since the 90's)
2. **THE EULER EQUATION**
   (describing the motion of inviscid and incompressible fluids, interpreted by Arnold as geodesic curves on infinite dimensional groups of volume preserving diffeomorphisms)
3. **THE BORN-INFELD EQUATION**
   (a non-linear electromagnetic model introduced in 1934, with a strong revival in high energy Physics in the 90's)

I) THE MONGE-AMPERE EQUATION

Given two positive smooth functions α and β of same finite integral over $\mathbf{R^d}$, find a smooth convex function Φ such that:

$$\boxed{\beta(\mathbf{D}\boldsymbol{\Phi}(\mathbf{x}))\mathbf{det}(\mathbf{D^2}\boldsymbol{\Phi}(\mathbf{x})) = \alpha(\mathbf{x})}$$

Assuming $\mathbf{D^2}\boldsymbol{\Phi}(\mathbf{x})$ to be uniformly bounded away from zero and infinity (in the sense of symmetric matrices) makes this PDE well-posed.
The Monge-Ampère equation is usually related to the Minkowski problem, which amounts to find hypersurfaces of prescribed Gaussian curvature.

A WEAK FORM OF THE MONGE-AMPERE EQUATION

When $D^2\Phi(x)$ is uniformly bounded away from zero, then
$x \in R^d \to D\Phi(x) \in R^d$ is one-to-one.
Using the change of variable $y = D\Phi(x)$, we deduce

$$\boxed{\int f(y)\beta(y)dy = \int f(D\Phi(x))\beta(D\Phi(x))\det(D^2\Phi(x))dx = \int f(D\Phi(x))\alpha(x)dx}$$

for all suitable test function f
Thus, a possible **WEAK FORMULATION** of the MA equation is

$$\boxed{\int f(y)\beta(y)dy = \int f(D\Phi(x))\alpha(x)dx, \quad \forall f}$$

In other words $\boxed{\beta(y)dy}$ AS A MEASURE IS $\boxed{\alpha(x)dx}$

"TRANSPORTED" BY $\boxed{x \in R^d \to D\Phi(x) \in R^d}$ AS A MAP.

A CONVEX VARIATIONAL PRINCIPLE FOR THE MONGE-AMPERE EQUATION

Any solution to the Monge-Ampère equation

$$\beta(\mathbf{D\Phi(x)})\mathbf{det}(\mathbf{D^2\Phi(x)}) = \alpha(\mathbf{x})$$

minimizes the CONVEX functional

$$\boxed{\mathbf{\Phi} \quad \Rightarrow \int \mathbf{\Phi(x)}\alpha(\mathbf{x})\mathbf{dx} + \int \mathbf{\Phi^*(y)}\beta(\mathbf{y})\mathbf{dy}}$$

among all suitable convex functions Φ, where Φ^* denotes the Legendre-Fenchel transform

$$\mathbf{\Phi^*(y)} = \sup_{\mathbf{x}\in\mathbf{R^d}} \mathbf{x}\cdot\mathbf{y} - \mathbf{\Phi(x)}.$$

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A CONVEX VARIATIONAL PRINCIPLE FOR THE MONGE-AMPERE EQUATION 3

Proof

$$\int \mathbf{\Psi}(\mathbf{x})\alpha(\mathbf{x})\mathbf{dx} + \int \mathbf{\Psi}^*(\mathbf{y})\beta(\mathbf{y})\mathbf{dy} = \int (\mathbf{\Psi}(\mathbf{x}) + \mathbf{\Psi}^*(\mathbf{D\Phi}(\mathbf{x})))\alpha(\mathbf{x})\mathbf{dx}$$

(since $D\Phi$ transports α toward β)

$$\geq \int \mathbf{x} \cdot \mathbf{D\Phi}(\mathbf{x})\alpha(\mathbf{x})\mathbf{dx}$$

(by definition: $\mathbf{\Psi}^*(\mathbf{y}) = \sup_{\mathbf{x}\in\mathbf{R}^\mathbf{d}} \mathbf{x}\cdot\mathbf{y} - \mathbf{\Psi}(\mathbf{x})$.)

$$= \int (\mathbf{\Phi}(\mathbf{x}) + \mathbf{\Phi}^*(\mathbf{D\Phi}(\mathbf{x})))\alpha(\mathbf{x})\mathbf{dx}$$

(indeed, in the definition of $\mathbf{\Phi}^*(\mathbf{y}) = \sup\ \mathbf{x}\cdot\mathbf{y} - \mathbf{\Phi}(\mathbf{x})$, the supremum is achieved whenever $\mathbf{y} = \mathbf{D\Phi}(\mathbf{x})$, which implies $\mathbf{\Phi}^*(\mathbf{D\Phi}(\mathbf{x})) = \mathbf{x}\cdot\mathbf{D\Phi}(\mathbf{x}) - \mathbf{\Phi}(\mathbf{x})$

$$= \int \mathbf{\Phi}(\mathbf{x})\alpha(\mathbf{x})\mathbf{dx} + \int \mathbf{\Phi}^*(\mathbf{y})\beta(\mathbf{y})\mathbf{dy}$$

CONCLUSION: $\mathbf{\Phi}$ IS A MINIMIZER

AN EXISTENCE AND UNIQUENESS RESULT FOR THE WEAK MONGE-AMPERE PROBLEM

YB, C. R. Acad. Sci. Paris Sér. I Math. 305 (1987) and CPAM 44 (1991), Smith and Knott, J. Optim. Theory Appl. 52 (1987), Caffarelli, J. AMS 5 (1992) and Ann. of Math. (2) 144 (1996), W. Gangbo, Arch. Rat. Mech. (1994), R. McCann Duke Math J. (1995), C. Villani, Topics in optimal transportation, AMS, 2003, see also reviews and lecture notes and many other papers and books.

THEOREM

Whenever α and β are Lebesgue integrable, with same integral, and bounded second order moments,

$$\int |\mathbf{x}|^2\alpha(\mathbf{x})\mathbf{dx} < +\infty, \quad \int |\mathbf{y}|^2\beta(\mathbf{y})\mathbf{dy} < +\infty,$$

there is a unique map with convex potential $\mathbf{x} \to \mathbf{D}\Phi(\mathbf{x})$ that solves the Monge-Ampère problem in its weak formulation.

$\mathbf{x} \to \mathbf{D}\Phi(\mathbf{x})$ IS CALLED THE OPTIMAL TRANSPORT MAP BETWEEN α AND β

Using the optimal map to prove the isoperimetric inequality

Let Ω be a smooth bounded open set and B_1 the unit ball in $\mathbb{R}^d$.

THE ISOPERIMETRIC INEQUALITY READS:

$$|\Omega|^{1-1/d}|B_1|^{1/d} \le \frac{1}{d}|\partial\Omega|$$

A PROOF USING THE OPTIMAL MAP:

Let $D\Phi$ the optimal transportation map between

$$\alpha(x) = \frac{1}{|\Omega|}, \quad x \in \Omega\,, \qquad \beta(y) = \frac{1}{|B_1|}, \quad y \in B_1.$$

So that

$$(\Omega, \alpha) \to (B_1, \beta)\,, \qquad \beta(D\Phi(x))\det(D^2\Phi(x)) = \alpha(x)$$

$$\text{i.e.} \quad \det(D^2\Phi(x)) = \frac{|B_1|}{|\Omega|}\,, \quad x \in \Omega.$$

NB: version "quantitative" avec exposant optimal par Figalli-Maggi-Pratelli, CVGMT Pisa 2007

The isoperimetric inequality 2

Proof (adaptated from Gromov): Since the range of $D\Phi$ is the unit ball, we have:

$$|\partial\Omega| = \int_{\partial\Omega} d\sigma(x) \geq \int_{\partial\Omega} D\Phi(x)\cdot n(x)d\sigma(x) = \int_{\Omega} \Delta\Phi(x)dx$$

(using Green's formula)

$$\geq d\int_{\Omega}(\det(D^2\Phi(x))^{1/d}dx$$

(using that $(\det A)^{1/d} \leq 1/d\ \ \mathrm{Trace}(A)$ for any nonnegative symmetric matrix A)

$$= d|\Omega|^{1-1/d}|B_1|^{1/d}$$

since $\det(D^2\Phi(x)) = \frac{|B_1|}{|\Omega|}\,, \quad x\in\Omega.$ So

$$|\Omega|^{1-1/d}|B_1|^{1/d} \leq \frac{1}{d}|\partial\Omega|$$

follows and equality holds *only* when Ω is a ball as it can be easily checked.

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II) THE EULER EQUATION

GEOMETRIC DEFINITION OF THE EULER EQUATIONS :

The Euler equations, introduced in 1755, describe the motion of inviscid incompressible fluids. They have a very simple geometric interpretation. For a fluid moving inside a bounded convex domain D in R^d, we get:

$$\boxed{\frac{\mathbf{d^2 g_t}}{\mathbf{dt^2}} \circ \mathbf{g_t^{-1}} + \nabla \mathbf{p_t} = \mathbf{0},}$$

where $t \to g_t$ is a curve valued in the (formal) Lie group $SDiff(D)$ of all volume preserving diffeomorphisms of D and p_t is a time dependent scalar function defined on D and called the 'pressure field'.
Such a curve is just a geodesic, with respect to the L^2 metric on the Lie Algebra of $SDiff(D)$.

cf. Arnold Ann. Inst. Fourier 1966, Ebin-Marsden Ann. Maths 1970, Abraham-Marsden-Ratiu, Springer 1988, Arnold-Khesin, Topological methods in hydrodynamics, Springer 1998.

A CONVEX PRINCIPLE FOR THE EULER EQUATION

A MAXIMIZATION PRINCIPLE FOR THE PRESSURE FIELD

For each time interval $[t_0, t_1]$ small enough, the pressure field p maximizes the CONCAVE functional

$$\boxed{\mathbf{p} \;\Rightarrow\; \int_{t_0}^{t_1}\int_D \mathbf{p_t(x)dtdx} + \int_D \mathbf{J_p[g_{t_0}(x), g_{t_1}(x)]dx},}$$

with

$$\boxed{\mathbf{J_p[x, y]} = \inf \int_{t_0}^{t_1} (-\mathbf{p_t(z(t))} + \frac{|\mathbf{z'(t)}|^2}{2})\mathbf{dt},}$$

where the infimum is taken over all curves $t \to z(t) \in D$ such that $z(t_0) = x \in D$, $z(t_1) = y \in D$.

A CONVEX PRINCIPLE FOR THE EULER EQUATION...

Proof: For all test function q, by definition of J_q:

$$\int_D J_q[g_{t_0}(x), g_{t_1}(x)]dx \le \int_{t_0}^{t_1}\int_D (\frac{1}{2}|\frac{dg_t}{dt}|^2 - q_t(g_t(x)))dtdx.$$

Because $[t_0, t_1]$ is short and g_t solves the Euler equation:

$$\int_D J_p[g_{t_0}(x), g_{t_1}(x)]dx = \int_{t_0}^{t_1}\int_D (\frac{1}{2}|\frac{dg_t}{dt}|^2 - p_t(g_t(x)))dtdx.$$

Since $g_t \in SDiff(D)$ is volume preserving, we have:

$$\int_D (q_t(x) - q_t(g_t(x))dx = \int_D (p_t(x) - p_t(g_t(x))dx = 0.$$

Thus

$$\int_{t_0}^{t_1}\int_D q_t(x)dtdx + \int_D J_q[g_{t_0}(x), g_{t_1}(x)]dx$$

$$\le \int_{t_0}^{t_1}\int_D p_t(x)dtdx + \int_D J_p[g_{t_0}(x), g_{t_1}(x)]dx$$

CONCLUSION: p IS A MAXIMIZER

EULER EQUATIONS AND MINIMIZING GEODESICS

A. Shnirelman, Math Sb. 128 (1985), GAFA 4 (1994), YB, J. AMS 2 (1989), Arch. Rational Mech. Anal. 138 (1997), Comm. Pure Appl. Math. 52 (1999), L. Ambrosio, A. Figalli, CVGMT preprint 2007, Pisa, to appear in ARMA

THEOREM

Whenever g_0 and g_1 are given volume preserving Borel maps of D (not necessarily diffeomorphisms),

1) There is a unique pressure field (up to an additive constant) that solves the Maximization problem (in a suitable weak sense) and $\mathbf{p \in L^2(]t_0, t_1[, BV_{loc}(D))}$.

2) There is a sequence $\mathbf{g_t^n}$ valued in SDiff(D) such that

$$\mathbf{\frac{d^2 g_t^n}{dt^2} \circ (g_t^n)^{-1} + \nabla p_t \to 0,}$$

in the sense of distributions and $\mathbf{g_0^n \to g_0, \quad g_1^n \to g_1}$ in $\mathbf{L^2}$.

3) When g_0 and g_1 are diffeomorphisms and $d \geq 3$, all minimizing geodesics behave as in 2). This is not true when $d = 2$.

III) THE BORN-INFELD SYSTEM

(cf.G. Boillat, CIME 1994-Springer lecture notes 1640)

$$\partial_t \mathbf{B} + \nabla \times \left(\frac{\mathbf{B} \times (\mathbf{D} \times \mathbf{B}) + \mathbf{D}}{\sqrt{1 + \mathbf{D}^2 + \mathbf{B}^2 + (\mathbf{D} \times \mathbf{B})^2}}\right) = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\partial_t \mathbf{D} + \nabla \times \left(\frac{\mathbf{D} \times (\mathbf{D} \times \mathbf{B}) - \mathbf{B}}{\sqrt{1 + \mathbf{D}^2 + \mathbf{B}^2 + (\mathbf{D} \times \mathbf{B})^2}}\right) = 0, \quad \nabla \cdot \mathbf{D} = 0,$$

This system is a nonlinear correction to the Maxwell equations, which can describe strings and branes in high energy Physics. Global smooth solutions have been proven to exist for small localized initial conditions (Chae and Huh, J. Math. Phys. 2003, using Klainerman's null forms). The additional conservation law

$$\partial_t \mathbf{h} + \nabla \cdot \mathbf{Q} = 0,$$

where

$$\mathbf{h} = \sqrt{1 + \mathbf{D}^2 + \mathbf{B}^2 + (\mathbf{D} \times \mathbf{B})^2}, \quad \mathbf{Q} = \mathbf{D} \times \mathbf{B}.$$

provides an 'entropy function' h which is a convex function of the unknown $(\mathbf{D}, \mathbf{B})$ **ONLY** in a neighborhood of $(0, 0)$.

THE AUGMENTED BORN-INFELD (ABI) SYSTEM

The 10×10 augmented Born-Infeld system (ABI) is made of the original BI system augmented by adding the 4 'energy-momentum' conservation laws (provided by Noether's theorem):

$$\boxed{\partial_t \mathbf{Q} + \nabla \cdot \left(\frac{\mathbf{Q} \otimes \mathbf{Q} - \mathbf{B} \otimes \mathbf{B} - \mathbf{D} \otimes \mathbf{D}}{\mathbf{h}}\right) = \nabla\left(\frac{\mathbf{1}}{\mathbf{h}}\right), \quad \partial_t \mathbf{h} + \nabla \cdot \mathbf{Q} = \mathbf{0}}$$

to the 6 original BI evolution equations

$$\boxed{\partial_t \mathbf{B} + \nabla \times \left(\frac{\mathbf{B} \times \mathbf{Q} + \mathbf{D}}{\mathbf{h}}\right) = \partial_t \mathbf{D} + \nabla \times \left(\frac{\mathbf{D} \times \mathbf{Q} - \mathbf{B}}{\mathbf{h}}\right) = \mathbf{0}\,, \quad \nabla \cdot \mathbf{B} = \mathbf{0}, \quad \nabla \cdot \mathbf{D} = \mathbf{0}}$$

while DISREGARDING THE ALGEBRAIC CONSTRAINTS

$$\boxed{\mathbf{h} = \sqrt{\mathbf{1} + \mathbf{D}^2 + \mathbf{B}^2 + (\mathbf{D} \times \mathbf{B})^2}, \quad \mathbf{Q} = \mathbf{D} \times \mathbf{B},}$$

which define the 6 dimensional BI MANIFOLD in the space $(\mathbf{h}, \mathbf{Q}, \mathbf{D}, \mathbf{B}) \in \mathbf{R}^{10}$.

For smooth solutions, THE BI SYSTEM IS JUST EQUIVALENT TO THE AUGMENTED SYSTEM RESTRICTED TO THE BI MANIFOLD.

cf. YB, Arch. Rat. Mech. Analysis 2004

FIRST OCCURENCE OF CONVEXITY IN THE BI SYSTEM

Surprisingly enough, the 10×10 augmented ABI system has an extra conservation law:

$$\boxed{\partial_{\mathbf{t}} \eta + \nabla \cdot \mathbf{\Omega} = \mathbf{0},}$$

where

$$\boxed{\eta(\mathbf{h}, \mathbf{Q}, \mathbf{D}, \mathbf{B}) = \frac{\mathbf{1} + \mathbf{D^2} + \mathbf{B^2} + \mathbf{Q^2}}{\mathbf{h}},}$$

is CONVEX, which leads to the GLOBAL hyperbolicity of the system. The ABI system looks like classical MHD equations and enjoys classical Galilean invariance:

$$\boxed{(\mathbf{t}, \mathbf{x}) \rightarrow (\mathbf{t}, \mathbf{x} + \mathbf{U}\ \mathbf{t}), \quad (\mathbf{h}, \mathbf{Q}, \mathbf{D}, \mathbf{B}) \rightarrow (\mathbf{h}, \mathbf{Q} - \mathbf{h}\mathbf{U}, \mathbf{D}, \mathbf{B}),}$$

for any constant speed $\mathbf{U} \in \mathbf{R^3}$!

SECOND OCCURENCE OF CONVEXITY IN THE BI SYSTEM

The 10×10 **ABI (augmented Born-Infeld) system is** *linearly degenerate* **(in the sense of Lax) and stable under weak-* convergence: weak limits of uniformly bounded sequences in** L^∞ **of smooth solutions depending on one space variable only are still solutions. (This can be proven by using the Murat-Tartar 'div-curl' lemma.)**

⇒ CONJECTURE: THE CONVEX HULL OF THE BI MANIFOLD IS THE NATURAL CONFIGURATION SPACE OF THE BI THEORY
(As a matter of fact, the differential constraints $\nabla \cdot D = \nabla \cdot B = 0$ **must be taken into account.)** **(YB, Arch. Rat. Mech. Analysis 2004)**

The convex hull is entirely defined by the following inequality:

$$\boxed{\mathbf{h} \geq \sqrt{\mathbf{1} + \mathbf{D}^2 + \mathbf{B}^2 + \mathbf{Q}^2 + \mathbf{2}|\mathbf{D} \times \mathbf{B} - \mathbf{Q}|}.}$$

D. Serre, A remark on Y. Brenier's approach to Born-Infeld electro-magnetic fields, Contemp. Math., 371, AMS 2005.

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ON THE CONVEXIFIED BI MANIFOLD...

1) The electromagnetic field $(\mathbf{D},\mathbf{B})$ and the 'density and momentum' fields $(\mathbf{h},\mathbf{Q})$ can be chosen *independently* of each other, as long as they satisfy the required *inequality* $\mathbf{h} \geq \sqrt{\mathbf{1}+\mathbf{D^2}+\mathbf{B^2}+\mathbf{Q^2}+\mathbf{2}|\mathbf{D}\times\mathbf{B}-\mathbf{Q}|}$.

The AUGMENTED system describes a field/matter coupling while the original Born-Infeld model is purely electromagnetic.

2) 'Matter' may exist without electromagnetic field: $\mathbf{B}=\mathbf{D}=\mathbf{0}$, which leads to the Chaplygin gas (a possible model for 'dark energy' or 'vacuum energy')

$$\boxed{\partial_t \mathbf{Q}+\nabla\cdot\left(\frac{\mathbf{Q}\otimes\mathbf{Q}}{\mathbf{h}}\right)=\nabla\left(\frac{\mathbf{1}}{\mathbf{h}}\right), \qquad \partial_t \mathbf{h}+\nabla\cdot\mathbf{Q}=\mathbf{0},}$$

3) 'Moderate' Galilean transforms are allowed

$$\boxed{(\mathbf{t},\mathbf{x})\rightarrow(\mathbf{t},\mathbf{x}+\mathbf{U}\,\mathbf{t}), \quad (\mathbf{h},\mathbf{Q},\mathbf{D},\mathbf{B})\rightarrow(\mathbf{h},\mathbf{Q}-\mathbf{h}\mathbf{U},\mathbf{D},\mathbf{B})}$$

(which is impossible on the original BI manifold). This is left from special relativity under weak completion ('subrelativistic' conditions.)

cf. YB, Methods Appl. Anal. 12 (2005)