EXTENDED MONGE-KANTOROVICH THEORY

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1 Abstract

We extend in various ways the Monge-Kantorovich theory (MKT), also known as optimal transportation theory (OTT) [Ka], [KS], [RR], [Su]. This theory has become familiar in the last ten years in the field of nonlinear PDEs, especially because of its connection with the Monge-Ampère equation [Br0], [Br1], [Ca], [CP], [GM], the Eikonal equation [EGn], and the heat equation [JKO], [Ot1], [Ot2]... The first and crucial step of *all* our extensions consists in revisiting the MKT as a theory of generalized geodesics, following [BB]. Then, various generalizations of the MKT can be investigated, including a relativistic heat equations and a variational interpretation of Moser's lemma. Next, we define generalized harmonic functions and open several questions. Then, we consider multiphase MKT with constraints, which includes the relaxed theory of geodesics on groups of volume preserving maps related to incompressible fluid Mechanics [Br3]. Finally, we consider generalized extremal surfaces and we relate them to classical Electrodynamics, namely to the Maxwell equations and to the pressureless Euler-Maxwell equations.

2 Generalized geodesics and the Monge-Kantorovich theory

2.1 Generalized geodesics

Although we could consider the general framework of a Riemannian manifold, we only address the case of a subset D of the Euclidean space $\mathbf{R}^d$, and we

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assume D to be the closure of a convex open bounded set. Given two points X_0 and X_1 in D, the geodesic curve

$$X(s) = (1-s)X_0 + sX_1 \tag{1}$$

achieves

$$\inf_X \int_0^1 k(X'(s))ds, \tag{2}$$

for all continuous convex even function k on $\mathbf{R}^d$, among all smooth paths $s \in [0,1] \to X(s) \in D$ such that $X(1) = X_1$, $X(0) = X_0$. This immediately follows from Jensen's inequality. In the spirit of Young's generalized functions [Yo], [Ta], let us now associate to each admissible path X the following pair of (Borel) measures (ρ, E) defined on the compact set $[0,1] \times D$ by

$$\rho(s,x) = \delta(x - X(s)), \quad E(s,x) = X'(s)\delta(x - X(s)), \quad (s,x) \in [0,1] \times D. \tag{3}$$

They satisfy the following compatibility condition in the sense of distributions

$$\partial_s \rho + \nabla \cdot E = 0, \;\; \rho(0,\cdot) = \rho_0, \;\; \rho(1,\cdot) = \rho_1, \tag{4}$$

where

$$\rho_0(x) = \delta(x - X_0), \quad \rho_1(x) = \delta(x - X_1). \tag{5}$$

Indeed, we have

$$-\int_D \int_0^1 (\partial_s \phi(s,x) d\rho(s,x) + \nabla \phi(s,x) \cdot dE(s,x)) \tag{6}$$

$$+ \int_D (\phi(1,x) d\rho_1(x) - \phi(0,x) d\rho_0(x)) = 0,$$

for all smooth functions $\phi(s,x)$ defined on $[0,1] \times \mathbf{R}^d$. (This also implies, in a weak sense, that E is parallel to the boundary ∂D.) We notice that E is absolutely continuous with respect to ρ and, by Jensen's inequality,

$$\int_0^1 k(X'(s))ds \tag{7}$$

is bounded from below by

$$K(\rho, E) = \int k(e) d\rho, \tag{8}$$

where $e(s,x)$ is the Radon-Nikodym derivative of E with respect to ρ. A more precise definition of K can be given in terms of the Legendre-Fenchel transform of k denoted by k^* and defined by

$$k^*(y) = \sup_{x\in\mathbb{R}^d} x \cdot y - k(x), \tag{9}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^d$. We assume k^* to be continuous on $\mathbb{R}^d$. Typically

$$k(x) = \frac{|x|^p}{p}, \quad k^*(y) = \frac{|y|^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p, q < +\infty,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. We have

$$K(\rho, E) = \sup_{\alpha,\beta} \int_D \int_0^1 \alpha(s,x)d\rho(s,x) + \beta(s,x) \cdot dE(s,x), \tag{10}$$

where the supremum is performed over all pair (α, β) of respectively real and vector valued continuous defined on $[0,1] \times D$ subject to satisfy

$$\alpha(s,x) + k^*(\beta(s,x)) \leq 0 \tag{11}$$

pointwise. (Indeed, it can be easily checked that, with this definition, $K(\rho, E)$ is infinite unless i) ρ is nonnegative, ii) E absolutely continuous with respect to ρ and has a Radon-Nikodym density e, ii) $K(\rho, E)$ is just the ρ integral of $k(e)$.) Notice that K is a convex functional, valued in $[0, +\infty]$.

It is now natural to consider the infimum, denoted by $\inf K$, of functional K defined by (10) among *all* pairs (ρ, E) that satisfy compatibility conditions (4), with data (5), and *not only* among those which are of form (3). This new minimization problem is convex (as the original one). Since the class of admissible solutions has been enlarged, the following upper bound follows

$$\inf K \leq k(X_1 - X_0) \tag{12}$$

(by using (1) and (3) as an admissible pair). It turns out that there is no gap between the original infimum and the relaxed one.

Theorem 2.1 *The infimum of functional K, defined by (10), among all pair $(\rho, E)(s,x)$ of measures on $[0,1] \times D$, satisfying (4), with boundary conditions*

$$\rho(0,x) = \delta(x - X_0), \quad \rho(1,x) = \delta(x - X_1), \tag{13}$$

is achieved by the one associated, through (3), to the straight path between the end points $X(s) = (1-s)X_0 + sX_1$.

Proof

The proof is obtained through the following simple, and typical, duality argument that will be used several times subsequently in these lecture notes. First, we use (6) to relax constraint (4) and write

$$\inf K = \inf_{\rho,E} \sup_{\alpha,\beta,\phi} \int_D \int_0^1 (\alpha(s,x) - \partial_s \phi(s,x)) d\rho(s,x) \tag{14}$$

$$+(\beta(s,x) - \nabla\phi(s,x)) \cdot dE(s,x) + \int_D (\phi(1,x) d\rho_1(x) - \phi(0,x) d\rho_0(x)),$$

where (α, β) are subject to (11), and ϕ should be considered as a Lagrange multiplier for condition (4).

The *formal* optimality conditions for (α, β, ϕ) are

$$\alpha = \partial_s \phi, \quad \beta = \nabla\phi, \quad \alpha + k^*(\beta) = 0, \tag{15}$$

which leads to the Hamilton-Jacobi equation

$$\partial_s \phi + k^*(\nabla\phi) = 0. \tag{16}$$

Thus, a good guess for (α, β, ϕ) is

$$\phi(s,x) = x \cdot y - sk^*(y), \quad \alpha = \partial_s \phi, \quad \beta = \nabla\phi, \tag{17}$$

where $y \in \mathbb{R}^d$ will be chosen later. From definition (14), we deduce, with such a guess,

$$\inf K \geq \phi(1, X_1) - \phi(0, X_0) = (X_1 - X_0) \cdot y - k^*(y),$$

for all $y \in \mathbb{R}^d$. Optimizing in y and using that

$$k(x) = \sup_{y \in \mathbb{R}^d} x \cdot y - k^*(y),$$

we get

$$\inf K \geq k(X_1 - X_0),$$

i.e. the reverse inequality of (12), which concludes the proof.

2.2 Extension to probability measures

The main advantage of the concept of generalized geodesics (ρ, E) as minimizers of $K(\rho, E)$ subject to (4) is that (ρ, E) can achieve boundary data

$$\rho(s=0,\cdot) = \rho_0, \quad \rho(s=1,\cdot) = \rho_1, \tag{18}$$

that are (Borel) probability measures defined on the subset D. Probability measures should be seen in this context, as generalized (or fuzzy) points.

Theorem 2.2 *Let (ρ_0, ρ_1) a pair of probability measures on D. Then* $\inf K$ *is always finite and does not differ from the Monge-Kantorovich generalized distance between ρ_0 and ρ_1 usually defined by*

$$I_k(\rho_0, \rho_1) =: \inf \int_{D^2} k(x-y) d\mu(x,y), \tag{19}$$

where the infimum is performed on all nonnegative measures μ on $D \times D$ with projections ρ_0 and ρ_1 on each copy of D.

The relationship with the MKP has been established in [BB] for numerical purposes.

Proof

The proof requires the following fact, known as Kantorovich duality [RR], [GM] :

$$I_k(\rho_0, \rho_1) = \sup \int_D (\phi_1(x) d\rho_1(x) - \phi_0(x) d\rho_0(x)), \tag{20}$$

where ϕ_1 and ϕ_0 are continuous functions on D subject to

$$\phi_1(y) \le k(x-y) + \phi_0(x), \quad \forall x, \ y \in D. \tag{21}$$

From definition (19), there is always a minimizer μ, so that

$$\int_{D^2} k(b-a) d\mu(a,b) = I_k(\rho_0, \rho_1).$$

Let us introduce, for this μ,

$$\rho(s,x) = \int_{D^2} \delta(x - X(s,a,b)) d\mu(a,b), \tag{22}$$

$$E(s,x) = \int_{D^2} \partial_s X(s,a,b)\delta(x - X(s,a,b))d\mu(a,b), \tag{23}$$

where

$$X(s,a,b) = (1-s)a + sb. \tag{24}$$

Just as in the proof of Theorem 2.1, compatibility condition (4) is satisfied and, by Jensen's inequality,

$$\int_{D^2} k(b-a)d\mu(a,b) \geq K(\rho, E) \tag{25}$$

$$\geq \inf K \geq \sup_{\phi} \int_D (\phi(1,x)d\rho_1(x) - \phi(0,x)d\rho_0(x)),$$

where

$$\partial_s \phi + k^*(\nabla\phi) \leq 0. \tag{26}$$

So we can choose ϕ to be any solution of the Hamilton-Jacobi equation (16) on D. Using the Hopf formula to solve (16) (see [Ba], [Li]), we get

$$\phi(s,x) = \inf_{y\in D}(\phi(0, x + s(y-x)) + sk(y-x)),$$

for all $s \geq 0$. Thus, from (25), we finally get

$$I_k(\rho_0,\rho_1) = \int_{D^2} k(b-a)d\mu(a,b) \geq \inf K \geq \sup_{\phi} \int_D (\phi(1,x)d\rho_1(x) - \phi(0,x)d\rho_0(x)),$$

where

$$\phi(1,x) = \inf_{y\in D}(\phi(0,y) + k(y-x)).$$

Thus, we conclude, using Kantorovich duality (20), that there is no difference between $I_k(\rho_0,\rho_1)$ and $\inf K$, which concludes the proof.

2.3 A decomposition result

From the proof of Theorem 2.2, we immediately get the following *decomposition* result that asserts that generalized geodesics are mixtures of classical geodesics.

Theorem 2.3 *Each pair (ρ_0, ρ_1) of (Borel) probability measures on D admits a generalized geodesic (ρ, E) linking them with the following structure*

$$\rho(s,x) = \int_{D^2} \delta(x - X(s,a,b))d\mu(a,b), \tag{27}$$

$$E(s,x) = \int_{D^2} \partial_s X(s,a,b)\delta(x - X(s,a,b))d\mu(a,b), \tag{28}$$

where $X(\cdot,a,b)$ is the shortest path between a and b in D

$$X(s,a,b) = (1-s)a + sb, \tag{29}$$

and μ is a probability measure on D^2 with projections ρ_0 and ρ_1 on each copy of D.

Remark

Under strict convexity assumptions on k, the structure theorem can be made more precise, because of the well known properties of the Monge-Kantorovich problem. Indeed, in such cases, there is a unique minimizer μ with structure

$$\mu(a,b) = \delta(b - T(a))$$

where $T : D \to D$ is a Borel map [GM], [RR]. In particular, as $k(x) = |x|^2/2$, [Br1], [Ca], T is a map with Lipschitz convex potential Ψ. (An interesting application of this fact to pure analysis can be found in [Bt].) This potential solves the Monge-Ampère equation

$$\det(D^2\Psi(x))\rho_1(\nabla\Psi(x)) = \rho_0(x) \tag{30}$$

in the weak sense that ρ_1 is the image measure of ρ_0 by $\nabla\Psi$. If D is stricltly convex with a smooth boundary and if ρ_0 and ρ_1 are smooth functions bounded away from zero on D, then Ψ inherits the regularity of the data and becomes a classical solution to the Monge-Ampère equation, as shown by Caffarelli [Ca]. Notice that the assumption that D is a convex set, which is convenient but not at all essential for the existence and uniqueness theory, is crucial for the regularity theory, as pointed out by Caffarelli.

2.4 Relativistic MKT

Beyond the most important cost functions, namely $k(v) = |v|$, which corresponds to the original Monge problem, and $k(v) = |v|^2/2$, which corresponds to the Monge-Ampère equations and is related to PDEs as different as the Euler equations of incompressible flows [Br1], [Br2] and the heat equation [JKO], more general cost functions have been considered in the litterature (see for instance [GM]). Surprisingly,two important cost functions have been neglected, in spite of their obvious geometric and relativistic flavour, namely

$$k(v) = (1 - \sqrt{1 - \frac{|v|^2}{c^2}})c^2, \tag{31}$$

(with value $+\infty$ as $|v| > c$) and its dual function

$$\left(\sqrt{1 + \frac{|p|^2}{c^2}} - 1\right) c^2, \tag{32}$$

where $c > 0$ can be interpreted as a maximal speed. Notice that the later interpolates the important cost functions $k(p) = |p|$ and $k(p) = |p|^2/2$ as c varies from 0 to $+\infty$. Certainly, the case of (31) is the more interesting. Indeed, due to the finite maximal propagation speed, the MK problem may have no solution with finite cost, as the support of data ρ_0 and ρ_1 are too far from each other. The most interesting and challenging case is when only a part of the mass can be transported, which seems a very realistic approach to many applications. This case can be easily rephrased as a free boundary optimal transportation problem. From the Analysis point of view, the regularity of the free boundary is certainly a challenging problem.

2.5 A relativistic heat equation

As an application of the relativistic cost (31), let us compute, a relativistic heat equation, defined, in the spirit of [JKO], as a gradient flow of the Boltzmann entropy for the metric corresponding to cost (31). The Boltzmann entropy is given by

$$\eta(\rho) = \int_{\mathbb{R}^d} (\log \rho(x) - 1)\rho(x)dx, \tag{33}$$

where ρ is a density function defined on $\mathbb{R}^d$. To do the computation, we follow the time discrete approach of [JKO], rather than the stricter formalism of [Ot2]. The time dependent solution $\rho(t, x)$ is approximated at each time step $n\delta t$, $n = 1, 2, 3, ...$ by $\rho_n(x)$ subject to achieve

$$\inf_{\rho_n} \left(\nu\eta(\rho_n) + \int_0^1 \int_{\mathbb{R}^d} k(e(s, x))\rho(s, x)dsdx\right),$$

where (ρ, e) are subject to (4) (with $E = \rho e$) and boundary conditions

$$\rho(0, \cdot) = \rho_{n-1}, \quad \rho(1, \cdot) = \rho_n.$$

Here $\nu > 0$ is a parameter. This minimization problem can be easily written as a saddle point problem

$$\inf_{\rho_n} \sup_{\phi} \int_0^1 \int (k(e) - \partial_s \phi - e \cdot \nabla \phi)\, \rho ds dx$$

$$+ \int (\phi(1, x) + \nu(\log \rho_n(x) - 1))\rho_n(x) dx - \int \phi(0, x)\rho_{n-1}(x) dx,$$

which leads to the optimality conditions

$$\nabla k(e) = \nabla \phi, \quad \phi(1, \cdot) + \nu \log \rho_n = 0,$$

and

$$\partial_s \phi + k^*(\nabla \phi) = 0.$$

The first condition is equivalent to

$$e = \nabla k^*(\nabla \phi) = \frac{\nabla \phi}{\sqrt{1 + \frac{|\nabla \phi|^2}{c^2}}},$$

which, as expected, is always bounded by c. Letting formally δt go to zero, leads to the closure relation

$$\phi = -\nu \log \rho,$$

which, combined to (4), gives the desired relativistic heat equation with propagation speed bounded by c :

$$\partial_t \rho = \nu \nabla \cdot \frac{\rho \nabla \rho}{\sqrt{\rho^2 + \frac{\nu^2 |\nabla \rho|^2}{c^2}}}. \tag{34}$$

Notice that ν has the dimensionality of a kinematic viscosity (length^2/time). This equation interpolates the regular heat equation (as $c \to +\infty$) and the following limit equation, where the propagation speed is always c

$$\partial_t \rho = c \nabla \cdot \rho \frac{\nabla \rho}{|\nabla \rho|}. \tag{35}$$

An interesting output of equation (34) is the concept of 'relativistic Fischer information' defined as the entropy production, namely :

$$\frac{d}{dt}\int \rho \log \rho dx = \int \nu \frac{|\nabla \rho|^2}{\sqrt{\rho^2 + \frac{\nu^2 |\nabla \rho|^2}{c^2}}},$$

which interpolates the classical Fischer information (see [OV] for instance)

$$\nu \int \frac{|\nabla \rho|^2}{\rho} dx$$

and the 'total variation'

$$c \int |\nabla \rho| dx.$$

Let us finally mention that this relativistic heat equation can probably be found among the various "flux limited diffusion" equations used in the theory of radiation hydrodynamics [MM]. Indeed, A very similar equation

$$\partial_t \rho = \nu \nabla \cdot \frac{\rho \nabla \rho}{\rho + \frac{\nu |\nabla \rho|}{c}} \tag{36}$$

can be read (in our notations!) in [MM] (p.479). The author is grateful to Bruno Després for this comment.

2.6 Laplace's equation and Moser's lemma revisited

The formulation of the MKT in terms of generalized geodesics allows us to introduce a genuine extension of the MKT by relaxing the constraint for functional $K(\rho, E)$ to be homogeneous of degree one in the pair (ρ, E), as enforced by dual definition (10), (11). In particular, we can consider functionals that *do not* depend on ρ, such as

$$K(\rho, E) = K(E) = \int_0^1 \int_D k(E(s,x)) ds dx, \tag{37}$$

at least when E is absolutely continuous with respect to the Lebesgue measure, where k is a fixed continuous convex function on $\mathbb{R}^d$. We can give a more precise definition of K by setting

$$K(\rho, E) = \sup_{\beta} \int_0^1 \int_D (\beta(s,x) \cdot dE(s,x) - k^*(\beta(s,x)) ds dx), \tag{38}$$

where β is any continuous function on $[0,1]\times D$ and k^* is the Legendre-Fenchel transform of k.

Notice that, with this definition, the finiteness of $K(\rho,E)$ implies that E is absolutely continuous with respect to the Lebesgue measure, provided k and k^* are strictly convex (which rules out $k(x)=|x|$ for example). Given probability measures ρ_0, ρ_1 on D, let us find a generalized geodesics between them for this new type of functionals (inaccessible by the usual MKT). The minimization problem amounts to solve

$$\inf_{\rho,E}\sup_{\phi}\int_0^1\int_D (k(E)-\partial_s\phi\rho-E\cdot\nabla\phi)+\int_D(\phi(1,\cdot)\rho_1-\phi(0,\cdot)\rho_0).$$

The saddle-point conditions obtained by differientating in E and ρ are :

$$(\nabla k)(E)=\nabla\phi,\quad \partial_s\phi=0,$$

which reduces to

$$E(s,x)=E(x)=(\nabla k^*)(\nabla\phi(x)),\quad \rho(s,x)=\rho_0(x)(1-s)+\rho_1(x)s.$$

In a complete contrast with the MKT, here the data ρ_0 and ρ_1 are just linearly interpolated, while potential ϕ does not depend on s. Because of the compatibility condition

$$\partial_s\rho+\nabla\cdot E=0,$$

we deduce that ϕ solves the $k-$ Laplace equation

$$-\nabla.((\nabla k^*)(\nabla\phi))=\rho_1-\rho_0, \tag{39}$$

with $k-$ Neumann boundary condition

$$(\nabla k^*)(\nabla\phi))\cdot n=0,$$

along ∂D, where n denotes the outward normal to ∂D. To get a more rigorous argument, we notice that to each admissible pair $(\overline{\rho},\overline{E})$ we may associate a new admissible pair (ρ,E) such that

$$E(s,x)=E(x)\quad \rho(s,x)=\rho_0(x)(1-s)+\rho_1(x)s,$$

with $K(E)\le K(\overline{E})$, just by setting

$$E(x)=\int_0^1\overline{E}(s,x)ds.$$

Indeed, K is lowered by Jensen's inequality and (6) is enforced as soon as

$$\int_D \nabla\Phi(x)\cdot E(x)dx = \int \Phi(x)(\rho_1(x)-\rho_0(x))dx \tag{40}$$

which means, in weak sense that

$$-\nabla\cdot E = \rho_1 - \rho_0$$

with E parallel to ∂D. It follows that

$$\inf K = \inf_E \int_D k(E(x))dx,$$

where E is subject to (40). This new minimization problem is nothing but the dual formulation of the $p-$ Laplacian equation with homogeneous $p-$ Neumann boundary condition. So, we have obtained an interpretation of $p-$ Laplace problems as generalized Monge-Kantorovich problems to the expense of using a cost K which is not homogeneous of degree one. This interpretation may look artificial, since, after all, the interpolation variable s has disappeared at the end as well as the transportation framework. Nevertheless, thanks to this approach, we get a new interpretation, in terms of generalized MKT, of the Moser lemma in its simplest form (see [DM], [GY] for more sophisticated versions). The purpose of Moser's lemma is to construct a smoth map T transporting two given probability densities ρ_0 to ρ_1 that are assumed to be smooth and bounded away from zero on a nice domain D. The simplest Moser construction consists first in solving the Laplace equation

$$-\Delta\phi = \rho_1 - \rho_0, \tag{41}$$

inside D, with homogeneous Neumann boundary conditions along ∂D, next in introducing a velocity field

$$e(s,x) = \frac{\nabla\phi(x)}{\rho(s,x)},$$

where

$$\rho(s,x) = \rho_0(x)(1-s) + \rho_1(x)s,$$

which automatically satisfies

$$\partial_s\rho + \nabla\cdot(\rho e) = 0, \quad \rho(0,\cdot) = \rho_0, \quad \rho(1,\cdot) = \rho_1. \tag{42}$$

Then T is obtained, after integrating e, as $T(x) = X(1, x)$ where

$$\partial_s X(s,x) = e(s, X(s,x)), \quad X(0,x) = x.$$

This scheme exactly fits our generalized MK problem with cost

$$K(\rho, E) = K(E) = \frac{|E|^2}{2}.$$

Thus, the Moser construction, in its simpler version, can be interpreted as the solution to a generalized MK problem.

3 Generalized Harmonic functions

3.1 Classical harmonic functions

Let U be a smooth bounded open set in $\mathbb{R}^m$ and let D be the closure of a bounded convex open subset of $\mathbb{R}^d$. A map $u \in \overline{U} \to X(u)$ valued in the interior of D is harmonic if it minimizes

$$\int_U \frac{1}{2} |\nabla X(u)|^2 du \tag{43}$$

among all other maps assuming the same values along the boundary ∂U.

This means that X solves the homogeneous Laplace equation

$$\Delta X = 0. \tag{44}$$

To each map X, we can associate a pair of measures (ρ, E), valued in $\mathbb{R}^+ \times \mathbb{R}^{md}$, defined by

$$\rho(u,x) = \delta(x - X(u)), \quad E(u,x) = \nabla X(u) \delta(x - X(u)), \tag{45}$$

or, more precisely,

$$E_{i\alpha}(u,x) = \partial_{u_\alpha} X_i(u) \delta(x - X(u)), \quad \alpha = 1, ..., m, \quad i = 1, ..., d,$$

for all $(u, x) \in \overline{U} \times D$. These measures satisfy the following compatibility condition in the sense of distributions

$$\nabla_u \rho + \nabla_x \cdot E = 0, \tag{46}$$

with boundary conditions

$$\rho(u, \cdot) = \delta(x - X(u)), \quad \forall u \in \partial U. \tag{47}$$

Equation (46) is a compact notation for

$$\partial_{u_\alpha}\rho + \sum_{i=1,d} \partial_{x_i} E_{i\alpha}, \quad \alpha = 1, ..., m.$$

Equation (46) and boundary condition (47) can be expressed in weak form :

$$\int_{U\times D} (\nabla_u \cdot \phi(u,x) d\rho(u,x) + \nabla_x \phi(u,x) \cdot dE(u,x)) \tag{48}$$

$$= \int_{\partial U \times D} \phi(u,x)\rho(u,dx) \cdot dn(u),$$

for all smooth functions $\phi = (\phi_\alpha, \alpha = 1, ..., m)$ defined on in $\overline{U} \times D$ and valued in $\mathbb{R}^m$, where $dn(u) = n(u)d\mathbf{H}^{m-1}(u)$, $n(u)$ is the outward normal to ∂U at u and $\mathbf{H}^{m-1}$ stands for the $m-1$ dimensional Hausdorff measure. This equation is a compact notation for

$$\int_{U\times D} \left(\sum_{\alpha=1}^{m} \partial_{u_\alpha} \phi_\alpha(u,x) d\rho(u,x) + \sum_{\alpha=1}^{m} \sum_{i=1}^{d} \partial_{x_i} \phi_\alpha(u,x) dE_{i\alpha}(u,x) \right)$$

$$= \int_{\partial U \times D} \sum_{\alpha=1}^{m} \phi_\alpha(u,x) n_\alpha(u) \rho(u,dx) d\mathbf{H}^{m-1}(u).$$

We observe that E is absolutely continuous with respect to ρ and has a Radon-Nikodym density $e(u,x)$. By Jensen's inequality,

$$K(\rho, E) =: \frac{1}{2} \int_{U\times D} |e(s,x)|^2 d\rho(s,x) \leq \int_U \frac{1}{2} |\nabla_u X(u)|^2 du.$$

Functional K can be equivalently defined by Legendre duality as :

$$K(\rho, E) = \sup_{\alpha,\beta} \int_{U\times D} \alpha(u,x) d\rho(u,x) + \beta(u,x) \cdot dE(u,x), \tag{49}$$

where the supremum is performed over all pair (α, β) of continuous functions defined on $\overline{U} \times D$ respectively valued in $\mathbb{R}$ and $\mathbb{R}^{md}$, subject to satisfy

$$\alpha(u,x) + \sum_{\alpha=1}^{m} \sum_{i=1}^{d} \frac{1}{2} \beta_{i\alpha}(u,x)^2 \leq 0 \tag{50}$$

pointwise.

Now, we can define a generalized harmonic functions to be a pair of measures (ρ, E) subject to (46) that minimizes $K(\rho, E)$ defined by (49) as the value of ρ along the boundary ∂U is fixed. This is an obvious generalization of the earlier concept of generalized geodesics which corresponds to the special case $m = 1$, $U =]0,1[$. We have again a consistency result

Theorem 3.1 *Let $X : \overline{U} \to \mathbb{R}^d$ be harmonic, with values in the interior of D. Then, the corresponding pair of measures (ρ, E), defined on $\overline{U} \times D$ by (45), achieves the infimum of K, defined by (49), among all pairs of measures (ρ, E) satisfying (46) with boundary values*

$$\rho(u, x) = \delta(x - X(u)), \quad \forall u \in \partial U. \tag{51}$$

Proof

The proof is very similar to the one we had for generalized harmonic functions. Almost identically, we get

$$\int_U \frac{1}{2} |\nabla_u X(u)|^2 du \geq \inf K$$

$$\geq \sup_\phi \int_{\partial U} \phi(u, X(u)) \cdot dn(u),$$

where $\phi = (\phi_\alpha, \alpha = 1, ..., m)$ is subject to

$$\nabla_u \cdot \phi + \frac{1}{2} |\nabla_x \phi|^2 \leq 0. \tag{52}$$

Let us look for a solution ϕ of the generalized Hamilton-Jacobi equation

$$\nabla_u \cdot \phi + \frac{1}{2} |\nabla_x \phi|^2 = 0, \tag{53}$$

which is linear in x (which turns out to be sufficient for our purpose). We set

$$\phi_\alpha(u, x) = \sum_{i=1}^{d} w_{i\alpha}(u) x_i + z_\alpha(u), \quad \alpha = 1, ..., m,$$

where w is a smooth fixed function and z is chosen so that

$$\nabla_u . z(u) + \frac{1}{2} |w(u)|^2 = 0,$$

which is always possible (by solving an inhomogeneous Laplace equation

$$-\Delta \zeta(u) = \frac{1}{2} |w(u)|^2, \quad u \in U,$$

and setting $z(u) = \nabla \zeta(u)$). Thus,

$$\int_{\partial U} \phi(u, X(u)) \cdot dn(u) = \int_U \nabla_u \cdot (\phi(u, X(u)) du$$

(by Green's formula)

$$= \int_U \nabla_u \cdot (w(u) \cdot X(u) + z(u))du = \int_U (\nabla_u \cdot (w(u) \cdot X(u)) - \frac{1}{2}|w(u)|^2)du.$$

So, if we choose

$$w(u) = \nabla_u X(u),$$

we obtain

$$\int_{\partial U} \phi(u, X(u)) \cdot dn(u) = \int_U \left(\nabla_u \cdot (\nabla_u X(u) \cdot X(u)) - \frac{1}{2}|\nabla_u X(u)|^2\right) du,$$

$$= \int_U (\Delta_u X(u) \cdot X(u)) + |\nabla_u X(u)|^2 - \frac{1}{2}|\nabla_u X(u)|^2)du,$$

which is exactly

$$\int_U \frac{1}{2}|\nabla_u X(u)|^2 du,$$

since X is harmonic. Thus, we have obtained the desired reverse inequality

$$\inf K \geq \int_U \frac{1}{2}|\nabla_u X(u)|^2 du,$$

which completes the proof.

3.2 Open problems

Optimality equations

Because of the definition of generalized harmonic functions as minimizers of a (lower semi continuous) convex functional on a compact set of measures, we immediately get the following result.

Proposition 3.2 *Let (A, da) be a probability space. Let $(u, a) \in U \times A \to X(u, a) \in \mathbb{R}^d$ be a (measurable) family of maps such that*

$$\int_U \int_A |\nabla_u X(u, a)|^2 duda < +\infty.$$

Define

$$\overline{\rho}(u, x) = \int_A \delta(x - X(u, a))da.$$

Then there is always a generalized harmonic functions (ρ, E) such that $\rho = \overline{\rho}$ along the boundary of U.

Indeed, it is enough to notice that $\overline{\rho}$, together with

$$\overline{E}(u,x) = \int_A \nabla_u X(u,a)\delta(x - X(u,a))da,$$

defines an admissible solution with finite energy (by Jensen's inequality). Thus, there is an optimal solution by a standard compactness argument. It is harder to establish optimality equations. The formal equations are easily derived as saddle point equations for

$$\inf_{(\rho,E)} \sup_{\phi} \int_{U\times D} \left(\frac{1}{2}|e|^2 - \nabla_u \cdot \phi - \nabla_x \phi \cdot e\right) d\rho$$

(where the boundary terms have been dropped since they do not affect the local equations), namely

$$e = \nabla_x \phi, \quad \nabla_u \cdot \phi + \frac{1}{2}|\nabla_x \phi|^2 = 0. \tag{54}$$

This can equivalently expressed by

$$\partial_j e_{i\alpha} = \partial_i e_{j\alpha} \tag{55}$$

(which means that $e(u,x)$ is curl free in x) and

$$\sum_{\alpha=1}^{m} \partial_\alpha e_{i\alpha} + \sum_{\alpha=1}^{m}\sum_{j=1}^{d} e_{j\alpha}\partial_j e_{i\alpha} = 0 \tag{56}$$

(which is obtained by differentiating in x the second optimality condition). Equation (56) can be written in conservation form, using (46),

$$\sum_{\alpha=1}^{m} \partial_\alpha(\rho e_{i\alpha}) + \sum_{\alpha=1}^{m}\sum_{j=1}^{d} \partial_j(\rho e_{j\alpha} e_{i\alpha}) = 0. \tag{57}$$

Following the techniques of [Br3], it seems possible to establish rigorously the later equation for all generalized harmonic functions. However, it seems difficult to justify the curl-free condition (55).

Superharmonicity of the Boltzmann entropy

An interesting output of the optimality equations is the (formal) superharmonicity in $u \in U$ of the Boltzmann entropy. More precisely, given a smooth generalized harmonic functions (ρ, E) satisfying the optimality conditions, the entropy of ρ

$$\eta(u) =: \int_D (\log \rho(x,u) - 1)\rho(x,u)dx, \tag{58}$$

satisfies $\Delta_u \eta \geq 0$. This property is already known for generalized geodesics, as $m = 1$, $U =]0,1[$ (corresponding to the MKT with quadratic cost), as the 'displacement convexity' of the entropy, following McCann's [Mc1], [OV]. So, the entropy is superharmonic. Thus, by the maximum principle, the maximum of the entropy must be achieved along the boundary of U. If this result (that we are going to establish only for smooth generalized harmonic functions) is correct in full generality, we can expect the following result :

Conjecture 3.3 *Let (ρ, E) be a generalized harmonic function. If $\rho(u, \cdot)$ is absolutely continuous with respect to the Lebesgue measure in D for all u along the boundary ∂U, then this property also holds true for all u inside U.*

For the proof, (ρ, E) is assumed to be smooth, $\rho > 0$. For simplicity, implicit summation will be performed on repeated indices and notations

$$\partial_\alpha = \frac{\partial}{\partial u_\alpha}, \quad \alpha = 1, ..., m, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, ..., d,$$

will be used. We have

$$\Delta_u \eta(u) = \partial_\alpha \int \log \rho \; \partial_\alpha \rho \; dx$$

$$= \partial_\alpha \int \partial_i \rho \; e_{i\alpha} \; dx$$

(using (46) and integrating by part in x)

$$= \int \partial_\alpha \partial_i \rho \; e_{i\alpha} \; dx + \int \partial_i \rho \; \partial_\alpha e_{i\alpha} \; dx$$

$$= -\int \rho \; e_{j\alpha} \; \partial_i \partial_j e_{i\alpha} \; dx - \int \partial_i \rho \; e_{j\alpha} \partial_j \; e_{i\alpha} \; dx$$

(using (56) and again (46)

$$= \int \rho \, \partial_i e_{j\alpha} \, \partial_j e_{i\alpha} \, dx$$

$$= \int \rho \, \partial_j e_{i\alpha} \, \partial_j e_{i\alpha} \, dx \geq \ 0$$

(using (55)), which shows that η is superharmonic, as announced.

Decomposition of generalized harmonic functions

A natural question concerns the possibility of decomposing a generalized harmonic functions as a mixture of classical harmonic functions.

Problem 3.4 *Let (ρ, E) be a generalized harmonic function. Is there a probability space (A, da) and a family $(u, a) \in U \times A \to X(u, a) \in \mathbb{R}^d$ of harmonic functions, i.e.*

$$\Delta_u X(u, a) = 0,$$

such that

$$\rho(u, x) = \int_A \delta(x - X(u, a)) da, \quad E(u, x) = \int_A \nabla_u X(u, a) \delta(x - X(u, a)) da$$

holds true?

We already know that the answer is positive as $U =]0, 1[$, $m = 1$, which corresponds to the case of generalized geodesics. It is fairly clear that the answer is also positive as the target is one-dimensional, i.e. as $d = 1$, because of the maximum principle and the irrelevance of the curl free condition (55). Let us sketch a tentative proof, which is far from being complete.

A tentative proof

The idea of the proof, in case $d = 1$, is based on the fact that, given a generalized harmonic function (ρ, E), we can always write

$$\rho(u, x) = \int_A \delta(x - X(u, a)) da, \tag{59}$$

where $A = [0, 1]$ equipped with the Lebesgue measure da and X is nondecreasing in a for each values of $u \in \partial U$. Now, we claim that, because of the

monotonicity of X and because (46) is just an ODE in x,
1) we can write

$$E(s,x) = \int_A \nabla_u X(u,a)\delta(x - X(u,a))da,$$

2) the convexity inequality

$$K(\rho, E) \leq \frac{1}{2}\int_{U\times A} |\nabla_u X(u,a)|^2 da$$

actually is an equality. (This has to be proven with some care.) Then we conclude that, for each a, $X(\cdot, a)$ must be harmonic. Otherwise, we could introduce, for each fixed a, $\overline{X}(\cdot, a)$ to be the harmonic extension in U of the values of $X(\cdot, a)$ along ∂U. Because of the maximum principle, $\overline{X}(u,a)$ is nondecreasing in a. Defining the corresponding $(\overline{\rho}, \overline{E})$, and using that $\overline{X}$ is harmonic, we would get

$$K(\overline{\rho}, \overline{E}) < K(\rho, E),$$

which is a contradiction since (ρ, E) is supposed to be a generalized harmonic function.

4 Multiphasic MKT

As seen earlier, the MKT on a subset D of $\mathbb{R}^d$ (still assumed to be the closure of a bounded convex open set), is just a theory of generalized geodesics, or, equivalently, following Otto's point of view [Ot2], a theory of geodesics on the "manifold" $P\text{rob}(D)$ of all probability measures on D. It is therefore natural to extend this idea to more complex (convex) "manifolds". The most interesting case, in our opinion, is the the set $DS(D)$ of all doubly stochastic probability measures on D, namely the set of all Borel measures μ on $D \times D$ having as projection on each copy of D the (normalized) Lebesgue measure on D, which means

$$\int_{D\times D} f(x)d\mu(x,y) = \int_{D\times D} f(y)d\mu(x,y) = \int_D f(x)dx,$$

for all continuous functions f on D. As $d > 1$, this compact convex set turns out to be just the weak closure of the group of orientation and volume preserving diffeomorphisms of D, usually denoted by $SDiff(D)$ [AK], [Ne], through the following embedding

$$g \in SDiff(D) \to \mu_g \in DS(D), \quad \mu_g(x,y) = \delta(y - g(x)).$$

This group is of particular importance because it is the configuration space of incompressible fluids. $SDiff(D)$ is naturally embedded in the space $L^2(D,\mathbb{R}^d)$ of all square Lebesgue integrable maps from D to $\mathbb{R}^d$. Therefore, $SDiff(D)$ inherits the L^2 metric. Then, as pointed out by Arnold [AK], the equations of geodesic curves along $SDiff(D)$ exactly are the Euler equations of incompressible inviscid fluids (see also [MP] and [Br2]).
In our framework, it is very easy to define generalized geodesic curves (and even harmonic maps!) on $DS(D)$.

Definition 4.1 *Given μ_0, μ_1 in $DS(D)$, we define a (minimizing) generalized geodesic curve joining μ_0 and μ_1 to be a pair (μ, E) of (Borel) measures defined on $Q = [0,1]\times D\times D$ and valued in $\mathbb{R}_+ \times \mathbb{R}^d$ such that*

$$\int_Q \partial_s f(s,x,y)d\mu(s,x,y) + \nabla_x f(s,x,y)\cdot dE(s,x,y) \tag{60}$$

$$= \int_{D^2} f(1,x,y)d\mu_1(x,y) - \int_{D^2} f(0,x,y)d\mu_0(x,y),$$

for all smooth function f on $[0,1]\times D^2$, and

$$\int_Q f(s,x)d\mu(s,x,y) = \int_0^1 \int_D f(s,x)dxds, \tag{61}$$

for all continuous function f on $[0,1]\times D$, that minimizes

$$K(\mu,E) = \sup_{\alpha,\beta} \int_Q \alpha(s,x,y)d\mu(s,x,y) + \beta(s,x,y)\cdot dE(s,x,y) \tag{62}$$

where the supremum is performed over all pair (α,β) of continuous functions defined on Q respectively valued in $\mathbb{R}$ and $\mathbb{R}^{md}$, subject to satisfy

$$\alpha(s,x,y) + \frac{1}{2}|\beta(s,x,y)|^2 \le 0. \tag{63}$$

pointwise.

This can be seen as a multiphasic MK problem, where to each point $y\in D$ is attached a "phase" described by $\mu(\cdot,\cdot,y)$ and $E(\cdot,\cdot,y)$. These phases are coupled by constraint (61) which forces the different phases to share the volume available in D during their motion. Not surprisingly, this makes the optimality equations more subtle than in the classical MKT. Indeed, there is a Lagrange multiplier corresponding to constraint (61) that physically speaking is the pressure $p(s,x)$ of the fluid at each point $x \in D$ and each $s \in [0,1]$. The formal optimality conditions read

$$E(s,x,y)/\mu(s,x,y) = e(s,x,y), \quad e(s,x,y) = \nabla_x \phi(s,x,y), \tag{64}$$
$$\partial_s \phi(s,x,y) + \frac{1}{2}|\nabla_x \phi(s,x,y)|^2 + p(t,x) = 0.$$

This multiphasic MK problem has been studied in details in [Br3] and related to the classical Euler equations. (In some cases it is shown that generalized geodesics can be approximated by classical solutions to the Euler equations with vanishing forcing.) To motivate further researches, let us just quote two results that are true for any pair of data (μ_0, μ_1) in $DS(D)$. First, ∇p is uniquely defined (although there may be several generalized minimizing geodesic between μ_0 and μ_1). This fact follows easily from convex duality, but is rather surprising from the classical fluid mechanics point of view. Next, ∇p has a (very) limited regularity. It is a locally bounded measures in the interior of $[0,1] \times D$, which is not obvious and follows from the minimization principle. Further regularity can therefore be expected (maybe the second derivatives in space of p are also measures?), in particular as the data μ_0 and μ_1 are absolutely continuous with respect to the Lebesgue measure on D^2 with smooth positive density, as in Caffarelli's regularity theory of the (quadratic) MKT [Ca]. It is amusing to notice that (at least formally) the Boltzmann entropy, here defined by

$$\eta(\mu) = \int_{D^2} (\log \mu(x,y) - 1)\mu(x,y)dxdy, \tag{65}$$

is again "displacement convex" along generalized geodesics. The formal calculation is almost identical to those previously performed in these notes. But this has not been rigorously proven so far.

5 Generalized extremal surfaces

In this section, we first consider a (hyper)surface Σ of dimension m embedded in $\mathbb{R}^{\mathbf{d}}$. We assume that Σ is the image $\Sigma = X(U)$ of a nice domain U in $\mathbb{R}^m$ by a smooth map X with values in a convex subdomain $\mathbf{D}$ of $\mathbb{R}^{\mathbf{d}}$.
For each sequence $i = (i_1, ..., i_m)$ of integers such that $1 \leq i_1 < ... < i_m \leq \mathbf{d}$, we associate to X a measure $\rho_i(x)$ defined by

$$\rho_i(x) = \int_U \delta(x - X(u)) \sum_\sigma \epsilon(\sigma) \partial_{\sigma_1} X_{i_1}(u) ... \partial_{\sigma_m} X_{i_m}(u) du, \tag{66}$$

where σ is any permutation of the m first integers and $\epsilon(\sigma)$ denotes its signature. For each σ and each $i=(i_1,...,i_m)$ in $\{1,...,\mathbf{d}\}^m$, we set

$$\rho_{\sigma_i}=\epsilon(\sigma)\rho_i,$$

so that from now on ρ_i is antisymmetric in i. If

$$f=\sum_{1\le i_1<...<i_m\le \mathbf{d}} f_i(x)dx_{i_1}\wedge....\wedge dx_{i_{\mathbf{d}}}$$

is a differential form of degree m on $\mathbb{R}^{\mathbf{d}}$, $\rho=(\rho_i,\ i_1<...<i_m)$ acts as a current (i.e. a linear form on differential forms, see [GMS] for instance) on f by the duality bracket

$$<\rho,f>=:\sum_i<\rho_i,f_i> \tag{67}$$

$$=\sum_i\sum_\sigma \epsilon(\sigma)\int_U (f_{i_1,...i_m})(X(u))\partial_{\sigma_1}X_{i_1}(u)...\partial_{\sigma_m}X_{i_m}(u)du,$$

which (by the area formula) is nothing but the integral of f on Σ. If f is the derivative of a $m-1$ differential form ϕ, we get from the Stokes theorem that

$$\int_\Sigma d\phi=\int_{\partial\Sigma}\phi,$$

i.e.

$$\sum_i\sum_\sigma \epsilon(\sigma)\int_U(\partial_{i_1}\phi_{i_2,...i_m})(X(u))\partial_{\sigma_1}X_{i_1}(u)...\partial_{\sigma_m}X_{i_m}(u)du=\int_{\partial\Sigma}\phi, \tag{68}$$

which implies, in the distributional sense,

$$\sum_{i_1=1}^{\mathbf{d}}\partial_{i_1}\rho_{i_1,...i_m}=0, \tag{69}$$

for all $1\le i_2<...<i_m\le \mathbf{d}$. The Euclidean area of the embedded surface Σ is given [EGr] by

$$\int_U\sqrt{\sum_i\left(\sum_\sigma \epsilon(\sigma)\partial_{\sigma_1}X_{i_1}(u)...\partial_{\sigma_m}X_{i_m}(u)\right)^2}du, \tag{70}$$

which can be written in terms of ρ as

$$\int_{\mathbf{D}} k(\rho), \quad k(\rho) = \sqrt{\sum_i \rho_i^2} \tag{71}$$

or, by duality,

$$\sup_f < \rho, f > \tag{72}$$

where the supremum is performed over all compactly supported differential form $f = (f_i)$ of degree m in $\mathbb{R}^{\mathbf{d}}$ such that

$$\sum_i f_i(x)^2 \leq 1, \quad \forall x \in \mathbb{R}^{\mathbf{d}}.$$

We can now define a generalized surface and a generalized minimal surface :

Definition 5.1 *Let $\rho = (\rho_i)$ be a family, antisymmetric in $i \in \{1, ..., \mathbf{d}\}^m$, of measures ρ_i defined on a subset of $\mathbb{R}^{\mathbf{d}}$, denoted by $\mathbf{D}$ (and assumed to be the closure of a bounded open convex set). We say that ρ is a generalized surface lying in $\mathbf{D}$ if*

$$\sum_i \int_{\mathbf{D}} \partial_{i_1} \phi_{i_2,...i_m}(x) d\rho_i(x) = 0, \tag{73}$$

for each family of smooth function ϕ_i, antisymmetric in i and compactly supported in the interior of $\mathbf{D}$.

Definition 5.2 *Let k be a nonnegative continuous convex function, homogeneous of degree one. We say that a generalized surface ρ in $\mathbf{D}$ has a finite $k-$ area if*

$$K(\rho) = \int_{\mathbf{D}} k(\rho) < +\infty.$$

We say that ρ is a $k-$ generalized minimal surface if

$$K(\rho) \ \leq \ K(\overline{\rho})$$

for every generalized surface $\overline{\rho}$ lying in $\mathbf{D}$ such that

$$\sum_i \int_{\mathbf{D}} \partial_{i_1} \phi_{i_2,...i_m}(x)(d\rho_i(x) - d\overline{\rho}_i(x)), \tag{74}$$

for each family of smooth function ϕ_i defined on $\mathbb{R}^d$, antisymmetric in i.

Notice that (74) plays the role of a boundary condition along $\partial\mathbf{D}$ (this is why the ϕ_i are not supposed to vanish along the boundary!).

5.1 MKT revisited as a subset of generalized surface theory

The MKT corresponds to the particular case when

$$m = 1, \quad \mathbf{D} = D \times [0,1], \quad \mathbf{d} = d + 1.$$

A current point of $\mathbf{D}$ is denoted by

$$\mathbf{x} = (x_1, ..., x_d, s) = (x, s),$$

and, accordingly, $\rho_i(\mathbf{x})$ and $\rho_s(\mathbf{x})$ respectively correspond to $E_i(s,x)$ and $\rho(s,x)$, for $i = 1,...,d$ in our previous notations. Similarly $k(\rho)$ is now $k(\rho, E)$. The boundary values $\overline{\rho}$ are null along the $[0,1] \times \partial D$ and, with the previous notations, they are given by ρ_0 on $\{0\} \times D$ and ρ_1 on $\{1\} \times D$.

5.2 Degenerate quadratic cost functions

Since the MKT has been extended to cost functions $k(\rho, E)$ that are not homogeneous of degree one, we may extend the generalized surface theory in the same way. Not surprisingly, in the special case

$$\mathbf{D} = D \times [0,1], \quad \mathbf{d} = d + 1,$$

where current points of $\mathbf{D}$ are denoted by

$$\mathbf{x} = (x_1, ..., x_d, s) = (x, s),$$

the (degenerate) quadratic cost

$$k(\rho) = \sum_{1 \le i_1 < ... < i_m < s} \rho_i^2$$

(where the ρ_i for which $i_m = s$ are absent) will directly lead to a Hodge-Laplace problem for differential forms of degree $m-1$ in D.
For example, if

$$m = 2, \quad \mathbf{D} = D \times [0,1], \quad \mathbf{d} = 5 + 1,$$

and a point is denoted by (t, x_1, x_2, x_3, s) we recover the (elliptic) Maxwell equations in $\mathbb{R}^4$: In classical notations

$$-\nabla \cdot E = \rho_1 - \rho_0, \quad \partial_t E + \nabla \times B = J_1 - J_0, \tag{75}$$

$$\nabla \cdot B = 0, \quad \partial_t B + \nabla \times E = 0. \tag{76}$$

Because of the degeneracy of the cost function, the "electromagnetic field" (E, B) depends only the classical time space variable (t, x), while the "charge and current" densities (ρ, J) are linear interpolation in s of the "boundary" data (ρ_0, J_0) given at $s = 0$ and (ρ_0, J_0) at $s = 1$:

$$\rho(t, s, x) = (1 - s)\rho_0(t, x) + s\rho_1(t, x), \quad J(t, s, x) = (1 - s)J_0(t, x) + sJ_1(t, x). \tag{77}$$

(This will be checked in more details in a subsequent section.)

6 Generalized extremal surfaces in $\mathbb{R}^5$ and Electrodynamics

In order to address the framework of Electrodynamics, we substitute for the Euclidean metric of $\mathbb{R}^{\mathbf{d}}$ the Minkowski metric with signature $(-, +, ..., +)$. Assume that $m = 2$ and $\mathbf{d} = 5$. A point in $\mathbb{R}^{\mathbf{d}}$ will be denoted (t, x_1, x_2, x_3, s) and the partial derivative will be denoted accordingly (i.e. ∂_t, ∂_1, ..., ∂_s). The components of ρ are denoted

$$\rho_{st} = \rho,$$
$$\rho_{it} = E_i,$$
$$\rho_{is} = -J_i,$$
$$\rho_{12} = B_3, \quad \rho_{23} = B_1, \quad \rho_{31} = B_2.$$

Compatibility conditions (69) become

$$\partial_t \rho + \nabla \cdot J = 0 \tag{78}$$

(here $\cdot$ is the inner product in $\mathbb{R}^3$),

$$\partial_s \rho + \nabla \cdot E = 0, \tag{79}$$

$$\partial_t E - \partial_s J + \nabla \times B = 0. \tag{80}$$

We consider a functional K defined by

$$K(\rho, J, E, B) = \int k(\rho, J, E, B), \tag{81}$$

where k is a given function defined on $\mathbb{R}^{10}$. For example, the Euclidean area corresponds to

$$k(\rho, J, E, B) = \sqrt{\rho^2 + J^2 + E^2 + B^2}, \tag{82}$$

while the Minkowski area is given by

$$k(\rho, J, E, B) = \sqrt{\rho^2 - J^2 + E^2 - B^2}. \tag{83}$$

6.1 Recovery of the Maxwell equations

In the same way as the Laplace equation can be recovered from the MKT by using a degenerate quadratic functional, the Maxwell equations can be easily recovered by using a simplified functional such as

$$k(\rho, J, E, B) = \frac{E^2 - B^2}{2}. \tag{84}$$

To check this statement, we introduce two Lagrange multipliers $\phi(t, s, x) \in \mathbf{R}$, $A(t, s, x) \in \mathbf{R}^3$ to enforce the compatibility conditions and we define the corresponding Lagrangian $L(\rho, J, E, B, \phi, A)$:

$$\int \{ \frac{E^2 - B^2}{2} - \rho \partial_s \phi - E.\nabla \phi - E.\partial_t A + J.\partial_s A - B.\nabla \times A \} dtdsdx \tag{85}$$

where the boundary terms have been skipped since they do not affect the local equations. Varying the Lagrangian yields

$$\partial_s \phi = 0, \quad \partial_s A = 0, \tag{86}$$

$$E = \partial_t A + \nabla \phi, \quad B = -\nabla \times A. \tag{87}$$

We see that ϕ, A, E, B depend only on (t, x) and not on s. By using the compatibility conditions (79), (80) and eliminating ϕ and A, we get

$$\partial_s \rho + \nabla \cdot E = 0, \tag{88}$$

$$\nabla \cdot B = 0, \tag{89}$$

$$\partial_t E - \partial_s J - \nabla \times B = 0, \tag{90}$$

$$\partial_t B + \nabla \times E = 0. \tag{91}$$

We deduce that ρ and J depend linearly on s, namely

$$\rho(t, s, x) = (1 - s)\rho_0(t, x) + s\rho_1(t, x), \tag{92}$$

$$J(t, s, x) = (1 - s)J_0(t, x) + sJ_1(t, x), \tag{93}$$

and (E, B) satisfy the Maxwell equations

$$-\nabla \cdot E = \rho_1 - \rho_0, \quad \partial_t E - \nabla \times B = J_1 - J_0, \tag{94}$$

$$\nabla \cdot B = 0, \quad \partial_t B + \nabla \times E = 0. \tag{95}$$

6.2 Derivation of a set of nonlinear Maxwell equations

Let us now get the variational equations corresponding to the original, non quadratic, Action (83), with compatibility conditions (79), (80). The corresponding Lagrangian is given by

$$\int \{R - \rho\partial_s\phi - E.\nabla\phi - E.\partial_t A + J.\partial_s A - B.\nabla\times A\} dtdsdx, \tag{96}$$

where

$$R = \sqrt{\rho^2 + E^2 - J^2 - B^2} \tag{97}$$

and boundary terms have been disregarded since they do not affect the local equations we are looking for, although they play an important role to get correct boundary conditions. Varying the Lagrangian leads to

$$\rho = R\partial_s\phi, \quad J = R\partial_s A, \tag{98}$$

$$E = R(\partial_t A + \nabla\phi), \quad B = -R\nabla\times A. \tag{99}$$

By using compatibility conditions (79), (80) and eliminating the Lagrange multipliers in (98), (99), ϕ and A, we deduce, after elementary calculations,

$$\partial_t(JR^{-1}) - \partial_s(ER^{-1}) + \nabla(\rho R^{-1}) = 0, \tag{100}$$

$$\partial_t(BR^{-1}) + \nabla\times(ER^{-1}) = 0, \quad \nabla.(BR^{-1}) = 0. \tag{101}$$

To get an evolution equation for ρ, we can use compatibility condition (78) and disregard (79). Notice that an additional compatibility condition can also be derived from (98), (99), namely

$$\nabla\times(JR^{-1}) + \partial_s(BR^{-1}) = 0. \tag{102}$$

Of course, all these compatibility conditions are not independent from each other and we can select those which lead to a self-consistent system of time-evolution equations.

So, we retain for the set of variables

$$\rho,\ j = JR^{-1},\ E, b = BR^{-1}. \tag{103}$$

the following evolution equations

$$\partial_t\rho + \nabla.(Rj) = 0. \tag{104}$$

$$\partial_t j - \partial_s(ER^{-1}) + \nabla(\rho R^{-1}) = 0, \tag{105}$$

$$\partial_t E - \partial_s(Rj) - \nabla \times (Rb) = 0, \tag{106}$$

$$\partial_t b + \nabla \times (ER^{-1}) = 0, \tag{107}$$

where R is now expressed by

$$R = \sqrt{\frac{\rho^2 + E^2}{1 + j^2 + b^2}}. \tag{108}$$

After introducing

$$Z = \sqrt{(\rho^2 + E^2)(1 + j^2 + b^2)} \tag{109}$$

and noticing that

$$\frac{\partial Z}{\partial \rho} = \rho R^{-1}, \quad \frac{\partial Z}{\partial j} = Rj, \quad \frac{\partial Z}{\partial E} = ER^{-1}, \quad \frac{\partial Z}{\partial b} = Rb, \tag{110}$$

we finally get such a system, namely

$$\partial_t \rho = -\nabla.(\frac{\partial Z}{\partial j}), \qquad \partial_t E = \partial_s(\frac{\partial Z}{\partial j}) + \nabla \times (\frac{\partial Z}{\partial b}), \tag{111}$$

$$\partial_t j = \partial_s(\frac{\partial Z}{\partial E}) - \nabla(\frac{\partial Z}{\partial \rho}), \qquad \partial_t b = -\nabla \times (\frac{\partial Z}{\partial E}). \tag{112}$$

From now on, we call these equations $MKMEs$ (Monge-Kantorovich Maxwell equations).

6.3 An Euler-Maxwell-type system

Since the $MKMEs$ are time evolution equations, it is natural to supplement them with initial value conditions. We also need boundary conditions, at least for the interpolation variable $s \in [0,1]$. As a matter of fact, the most interesting boundary conditions are

$$(E, B)(t, s = 0, x) = (E, B)(t, s = 1, x) = 0. \tag{113}$$

Observe that these conditions are natural since, in the special case when (ρ, J, E, B) is generated from a surface X they correspond to

$$\partial_s X(t, s = 0) = \partial_s X(t, s = 1) = 0, \tag{114}$$

which is the right free boundary condition for an extremal surface.

It is now interesting to focus on the fields (ρ, J) at the end points $s = 0$ and $s = 1$, since they are not prescribed any longer. Let us introduce notations

$$(\rho_-, J_-)(t,x) = (\rho, J)(t, s=0, x), \quad (\rho_+, J_+)(t,x) = (\rho, J)(t, s=1, x). \tag{115}$$

Remarkable simplifications occur in the $MKMEs$ restricted at $s = 0$ and $s = 1$ for such boundary conditions. Indeed, we get, at $s = 0$ and $s = 1$, since $E = B = 0$, from (97), (108),

$$R = \sqrt{\rho^2 - J^2} = \frac{\rho}{\sqrt{1+j^2}}, \tag{116}$$

$$\frac{\rho}{R} = \sqrt{1+j^2}, \quad Rj = \frac{\rho j}{\sqrt{1+j^2}}, \quad \partial_s(Rj) = \partial_s(\frac{\rho j}{\sqrt{1+j^2}}). \tag{117}$$

Thus, using (104), (105), (106), we get at $s = 0$ and $s = 1$

$$\partial_t \rho_- + \nabla.(\rho_- v_-) = \partial_t \rho_+ + \nabla.(\rho_+ v_+) = 0, \tag{118}$$

where we introduce notation

$$v = \frac{j}{\sqrt{1+j^2}}, \tag{119}$$

$$\partial_t j_- + \nabla\sqrt{1+j_-^2} = \partial_s e_{|s=0}, \quad \partial_t j_+ + \nabla\sqrt{1+j_+^2} = \partial_s e_{|s=1}, \tag{120}$$

$$\partial_s(\frac{\rho j}{\sqrt{1+j^2}})_{|s=0} = \partial_s(\frac{\rho j}{\sqrt{1+j^2}})_{|s=1} = 0. \tag{121}$$

Using the standard identity

$$\nabla\sqrt{1+j^2} = (\frac{j}{\sqrt{1+j^2}}.\nabla)j + \frac{j}{\sqrt{1+j^2}} \times (\nabla \times j), \tag{122}$$

and (102) we can rewrite (120),

$$(\partial_t + v_-.\nabla)j_- = E_- + v_- \times B_-, \tag{123}$$

$$(\partial_t + v_+.\nabla)j_+ = E_+ + v_+ \times B_+, \tag{124}$$

where the 'electromagnetic' fields $(E_-, B_-)(t,x)$, $(E_+, B_+)(t,x)$ are defined by

$$E_-(t,x) = \partial_s e(t, s=0, x), \quad E_+(t,x) = \partial_s e(t, s=1, x), \tag{125}$$

$$B_-(t,x) = \partial_s b(t, s=0, x), \qquad B_+(t,x) = \partial_s b(t, s=1, x). \tag{126}$$

These fields are linked by (107)

$$\partial_t B_- + \nabla \times E_- = \partial_t B_+ + \nabla \times E_+ = 0, \qquad \nabla.B_- = \nabla.B_+ = 0, \tag{127}$$

which is one half of the usual Maxwell equations. Thus, with equations (118), (123), (124) and (127), we are not far from the standard (relativistic pressureless) Euler-Maxwell system, for which the very same equations would be supplemented by

$$E_- = -\frac{E_0}{m_-}, \quad E_+ = \frac{E_0}{m_+}, \quad B_- = -\frac{B_0}{m_-}, \quad B_+ = \frac{B_0}{m_+}, \tag{128}$$

$$\partial_t E_0 - \nabla \times B_0 = \rho_+ j_+ - \rho_- j_-, \qquad \nabla.E_0 = \rho_+ - \rho_-, \tag{129}$$

where m_- (resp. m_+) denote the mass of negatively (resp. positively) charged particles and $(E_0, B_0)(t,x)$ the electromagnetic field. Of course, in the resulting Euler-Maxwell system, the s variable has completely disappeared. For the $MKMEs$, however, such a simple closure is not possible since we cannot eliminate the interpolation variable s and the coupling is much subtler.

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