On the mixing time of the flip walk on triangulations of the sphere

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Planar maps

Definitions

- A *planar map* is a finite, connected graph embedded in the sphere in such a way that no two edges cross (except at a common endpoint), considered up to orientation-preserving homeomorphism.
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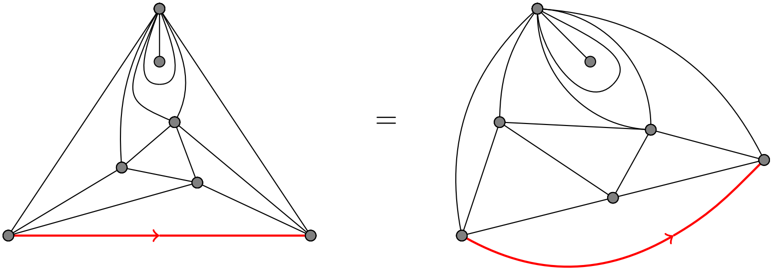

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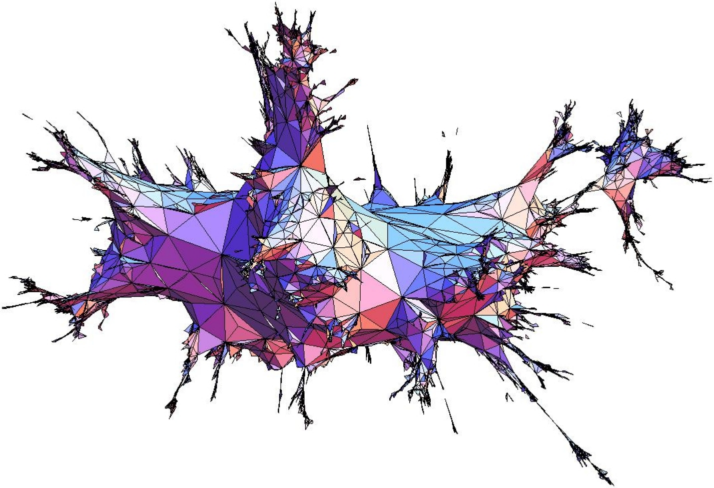
- if we don't renormalize the distances and look at a neighbourhood of the root, convergence to an infinite triangulation of the plane called the UIPT [Angel-Schramm],
- the volume of the ball of radius r in the UIPT grows like r^4 [Angel, Curien-Le Gall].

A uniform triangulation of the sphere with 10 000 vertices



How to sample a large uniform triangulation ?

- "Modern" tools : bijections with trees, peeling process.
- Back in the 80's : Monte Carlo methods : we look for a Markov chain on $\mathscr{T}_n$ for which the uniform measure is stationary.
- A simple local operation on triangulations : flips.

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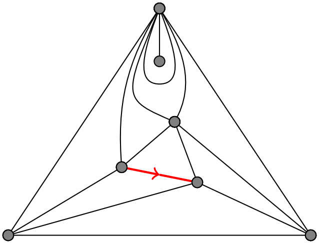

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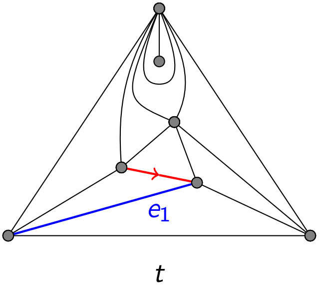

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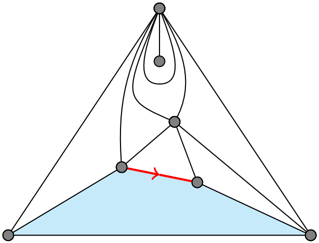

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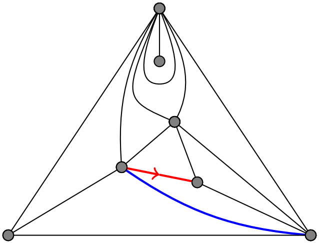

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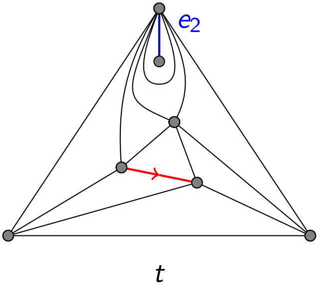

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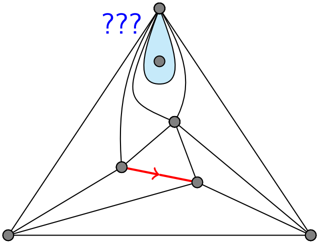

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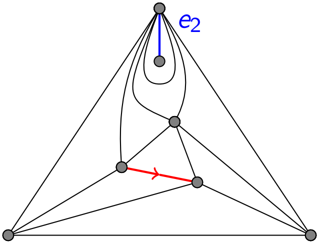

$$\mathfrak{flip}(t, e_2) = t$$

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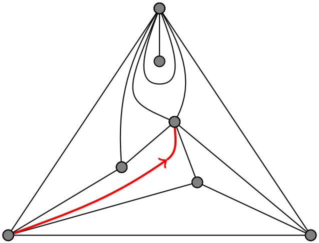

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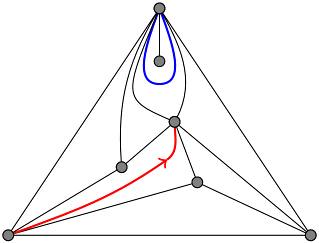

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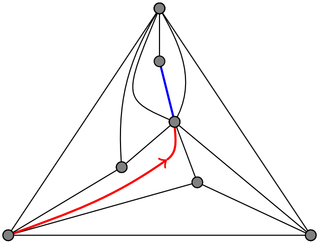

A Markov chain on $\mathscr{T}_n$

- We fix $t_0 \in \mathscr{T}_n$ and take $T_n(0) = t_0$.
- Conditionally on $(T_n(k))_{0 \leq i \leq k}$, let e_k be a uniform edge of $T_n(k)$ and $T_n(k+1) = \mathfrak{flip}\left(T_n(k), e_k\right)$.

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- The chain T_n is irreducible (the flip graph is connected [Wagner 36]) and aperiodic (non flippable edges), so it converges to the uniform measure.
- Question : how quick is the convergence ?

Mixing time of T_n

- For $n \geq 3$ and $0 < \varepsilon < 1$ we define the mixing time $t_{mix}(\varepsilon, n)$ as the smallest k such that

$$\max_{t_0 \in \mathscr{T}_n} \max_{A \subset \mathscr{T}_n} |\mathbb{P}(T_n(k) \in A) - \mathbb{P}(T_n(\infty) \in A)| \leq \varepsilon,$$

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Theorem (B., 2016)

For all $0 < \varepsilon < 1$, *there is a constant* $c > 0$ *such that*

$$t_{mix}(\varepsilon, n) \geq cn^{5/4}.$$

Sketch of proof

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Let $\ell_n = o(n^{1/4})$. Then, with probability going to 1 as $n \to +\infty$, there is no cycle in $T_n(\infty)$ of length at most ℓ_n that separates $T_n(\infty)$ in two parts, each of which contains at least $\frac{n}{4}$ vertices.

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Let $T_n^1(0)$ and $T_n^2(0)$ be two independent uniform triangulations of a 1-gon with $\frac{n}{2}$ inner vertices each, and $T_n(0)$ the gluing of $T_n^1(0)$ and $T_n^2(0)$ along their boundary.

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Proposition

Let $k_n = o(n^{5/4})$. There is a cycle γ in $T_n(k_n)$ of length $o(n^{1/4})$ in probability that separates $T_n(k_n)$ in two parts, each of which contains at least $\frac{n}{4}$ vertices.

Spatial Markov property

We write $\mathscr{T}_{n,p}$ for the set of triangulations of a p-gon with n inner vertices. If T is uniform in $\mathscr{T}_{n,p}$, we consider an edge $e \in \partial T$ and the face f of T adjacent to e.

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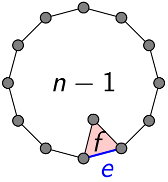
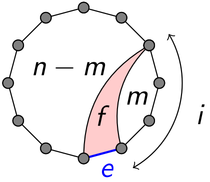


with probability

$$\frac{\#\mathscr{T}_{n-1,p+1}}{\#\mathscr{T}_{n,p}}$$



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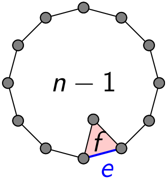

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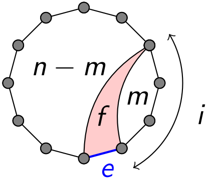

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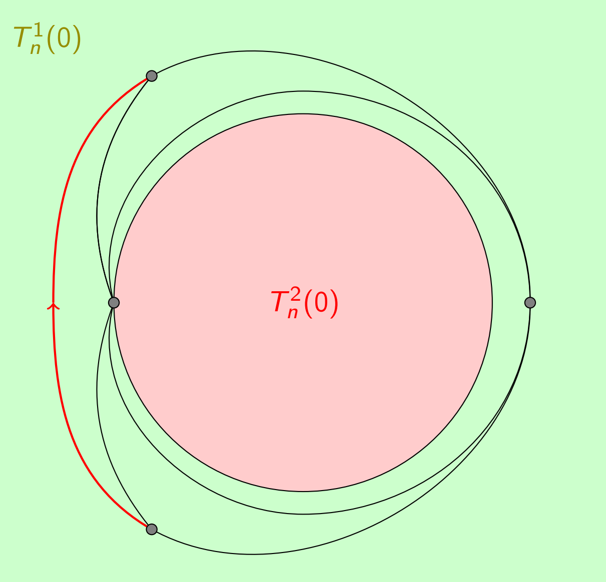
Moreover, in every case, the (one or two) connected components of $T \backslash f$ are independent and uniform given their volume and perimeter.

Exploration of $T_n(k)$



$\widetilde{P}_n(0) = 1$
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$\tau_0 = 0$

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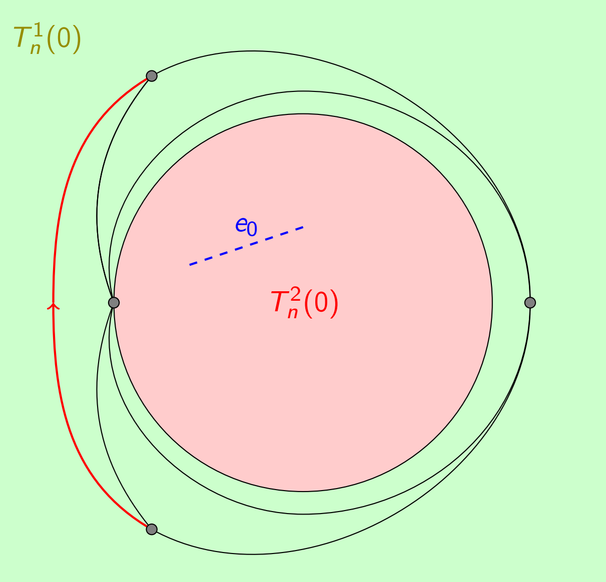

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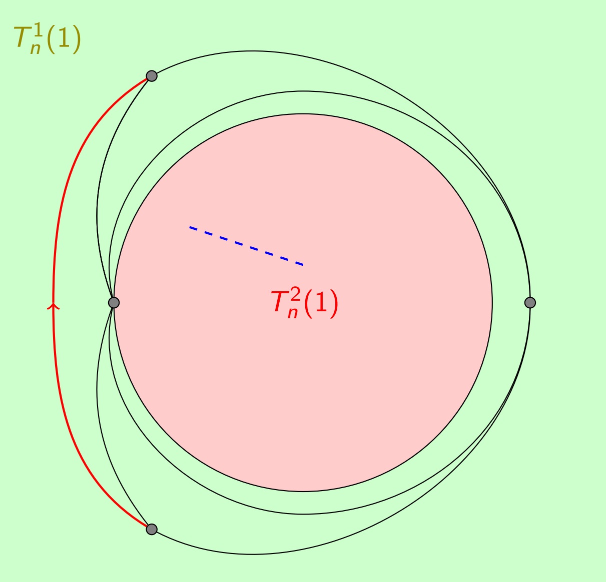

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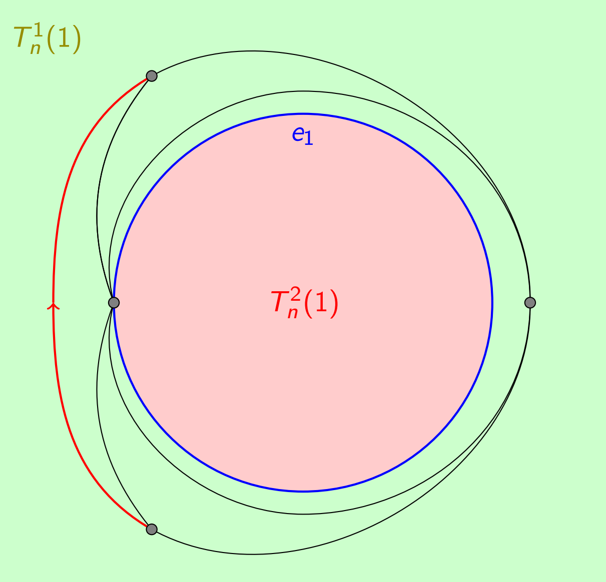

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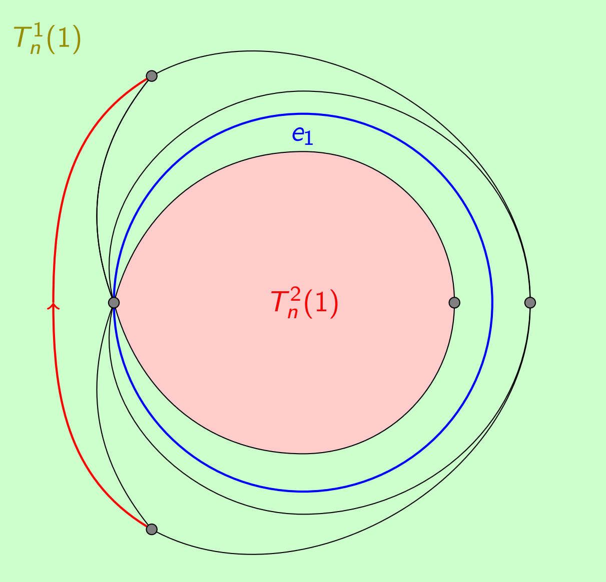

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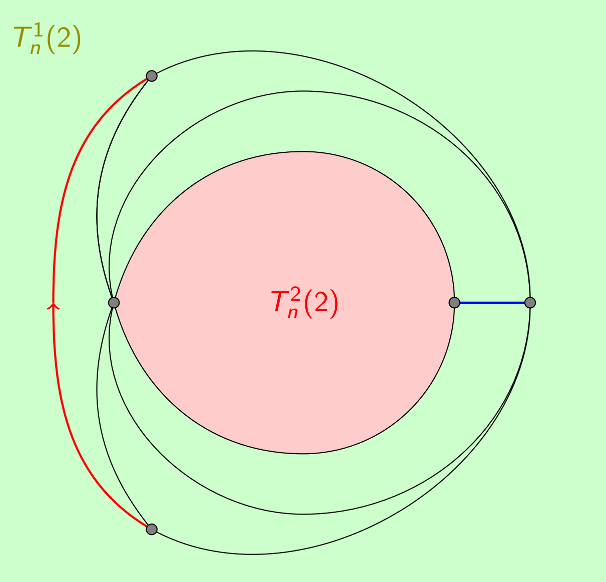

$\widetilde{P}_n(2) = 2$
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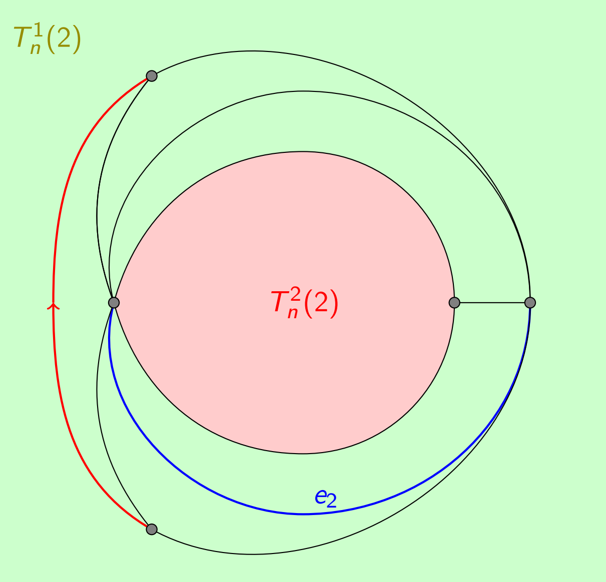

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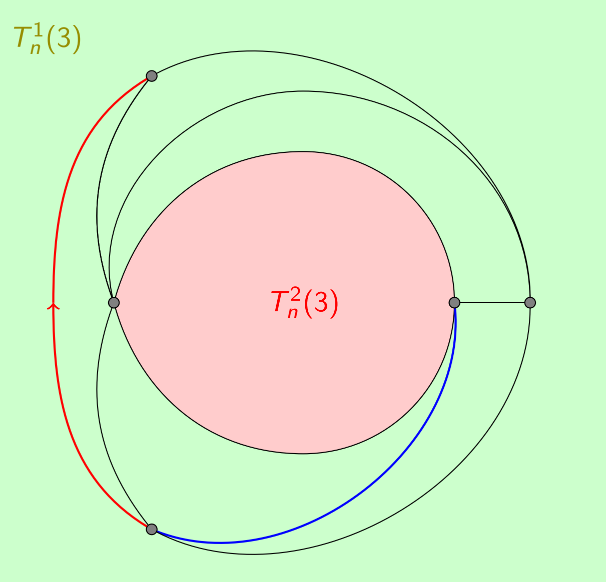

$\widetilde{P}_n(3) = 2$
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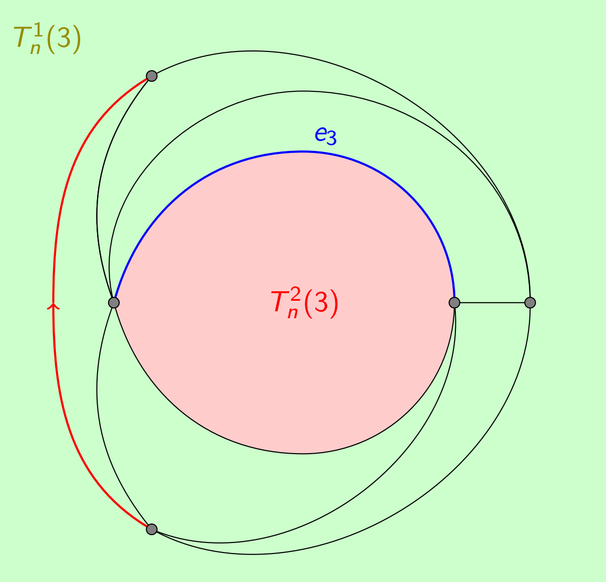

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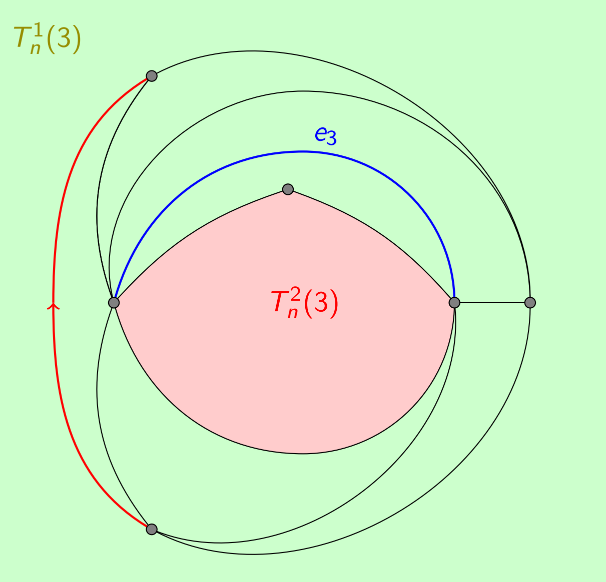

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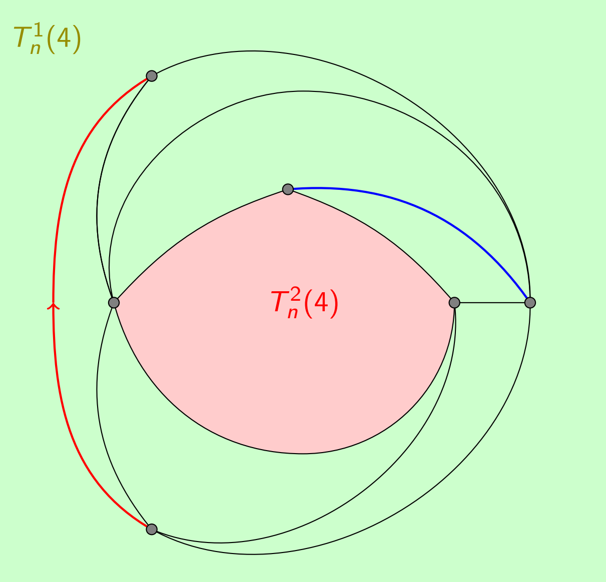

$\widetilde{P}_n(4) = 3$
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$\tau_2 = 4$

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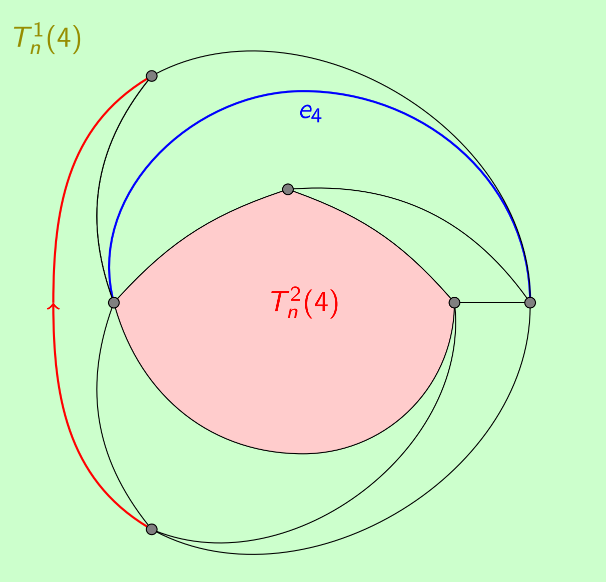

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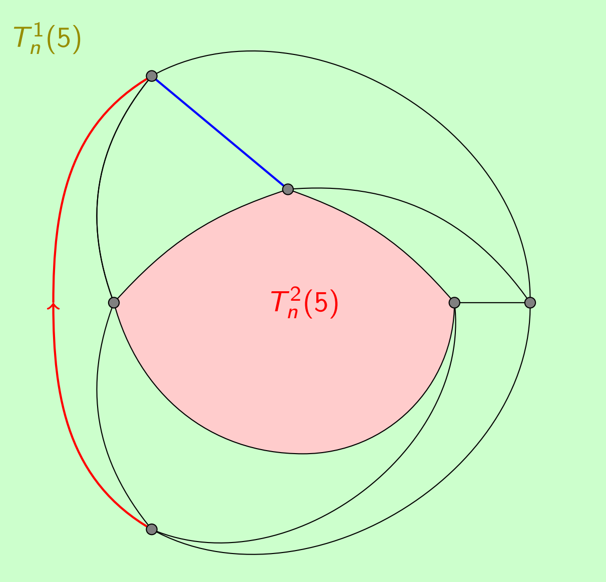

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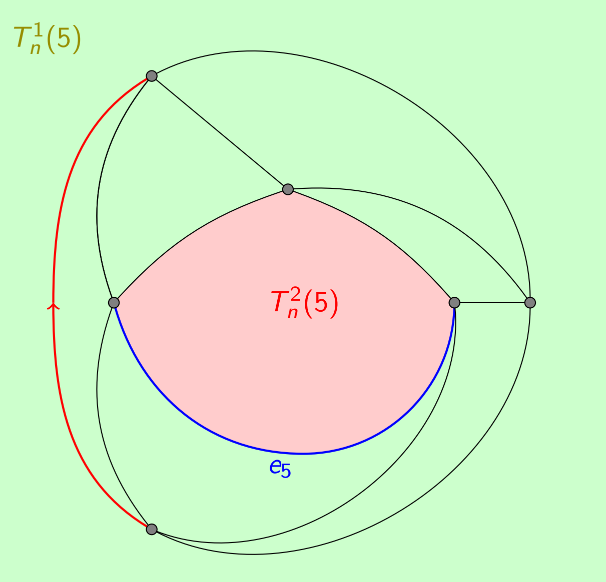

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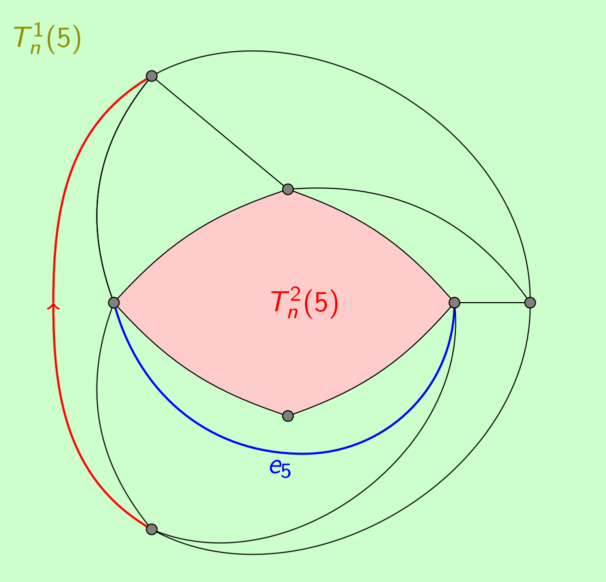

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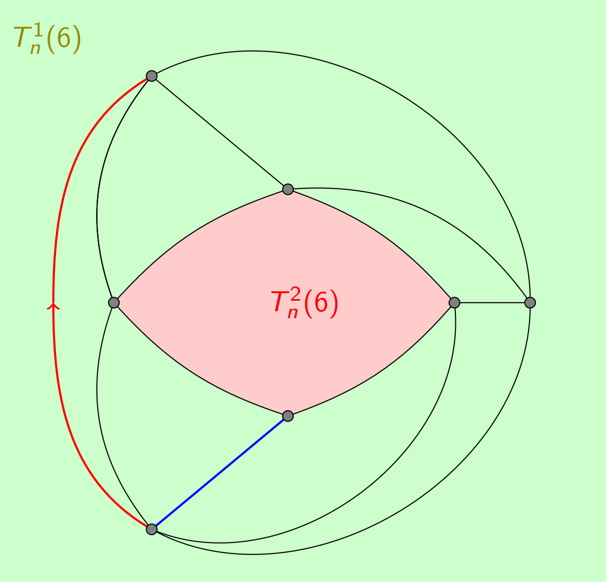

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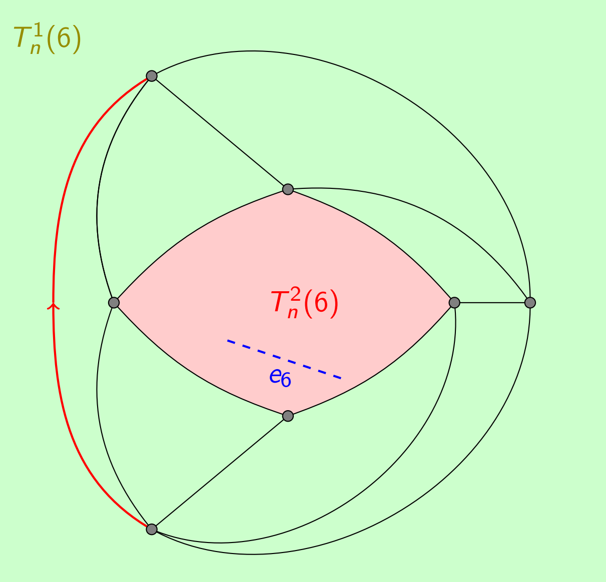

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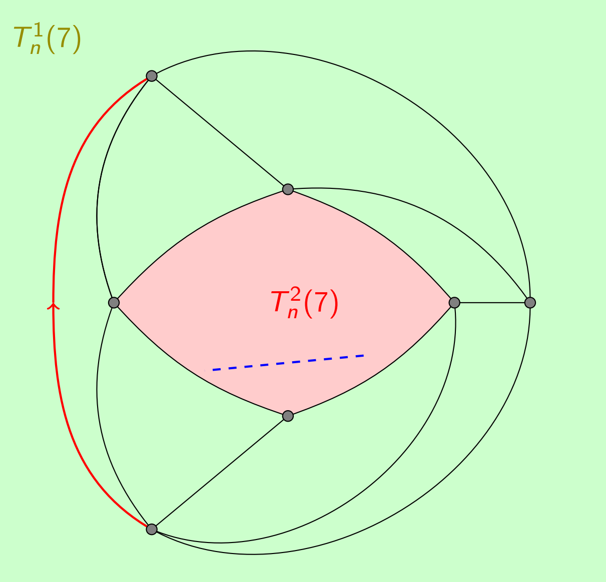
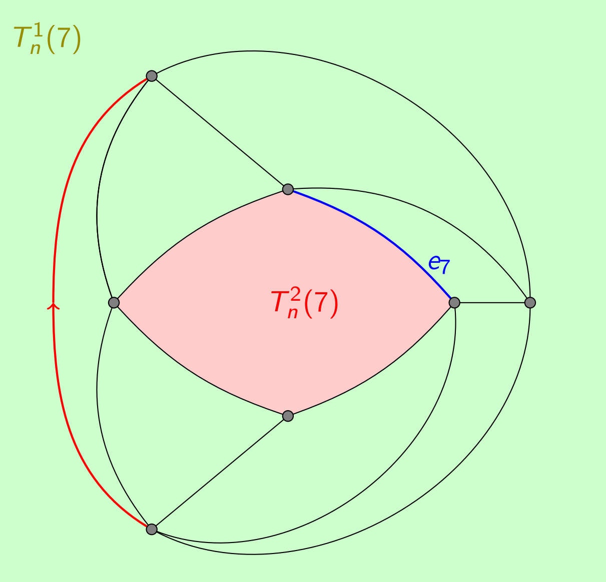
$\tau_3 = 6$

Exploration of $T_n(k)$



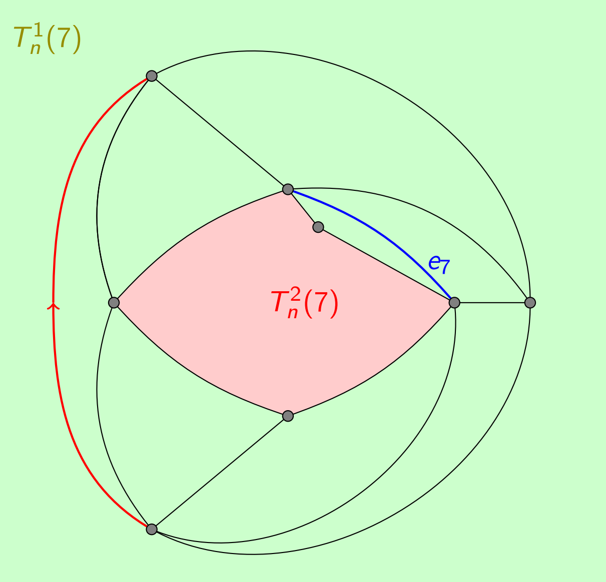
$\widetilde{P}_n(7) = 4$
$\widetilde{V}_n(7) = 4$

$\tau_3 = 6$



$\widetilde{P}_n(7) = 4$
$\widetilde{V}_n(7) = 4$

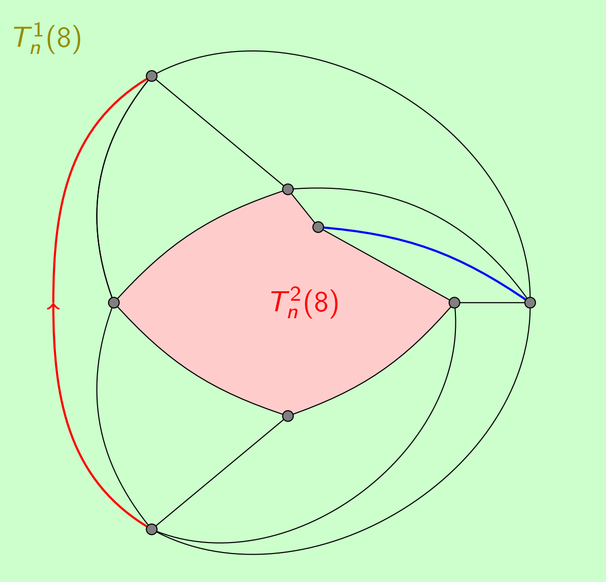
$\tau_3 = 6$



$\widetilde{P}_n(7) = 4$
$\widetilde{V}_n(7) = 4$

$\tau_3 = 6$

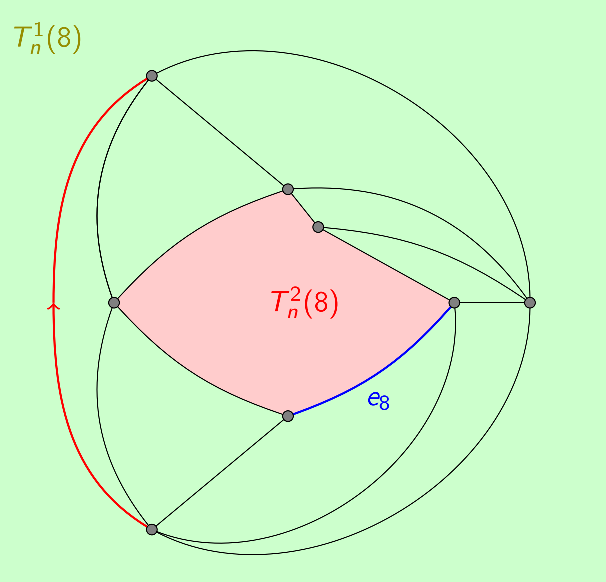
Exploration of $T_n(k)$



$\widetilde{P}_n(8) = 5$
$\widetilde{V}_n(8) = 5$

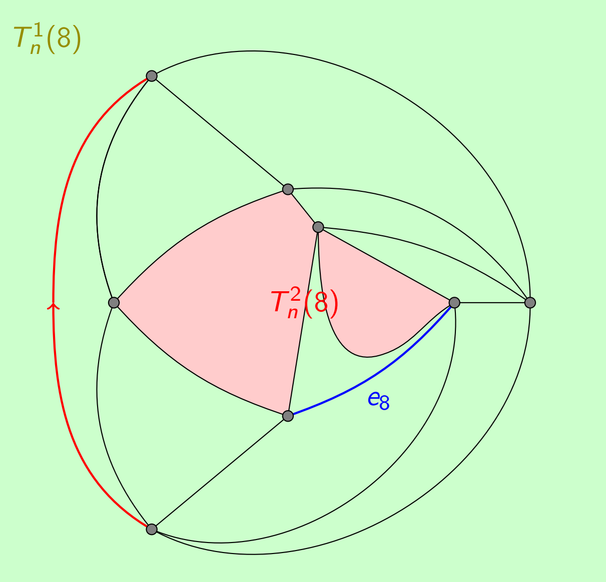
$\tau_4 = 8$

Exploration of $T_n(k)$



$\widetilde{P}_n(8) = 5$
$\widetilde{V}_n(8) = 5$

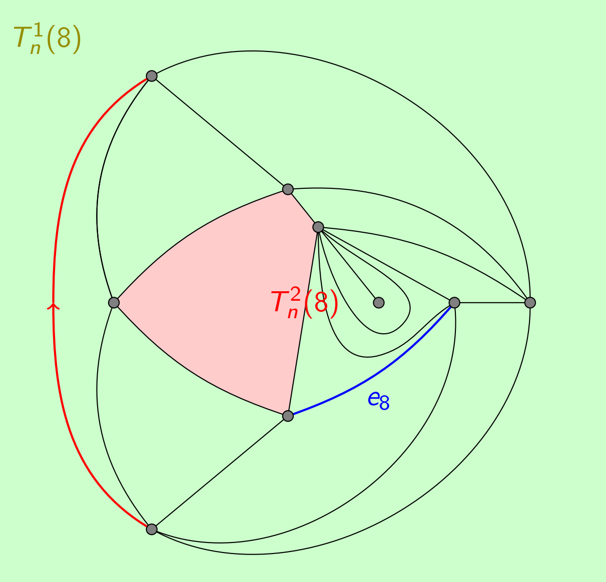
$\tau_4 = 8$



$\widetilde{P}_n(8) = 5$
$\widetilde{V}_n(8) = 5$

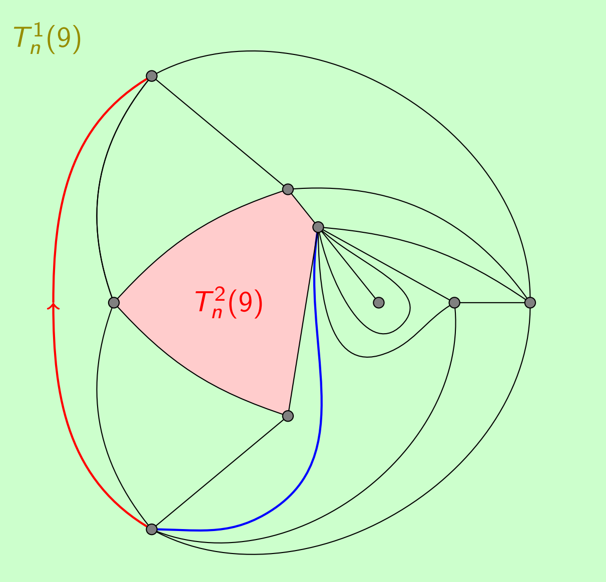
$\tau_4 = 8$

Exploration of $T_n(k)$



$\widetilde{P}_n(8) = 5$
$\widetilde{V}_n(8) = 5$

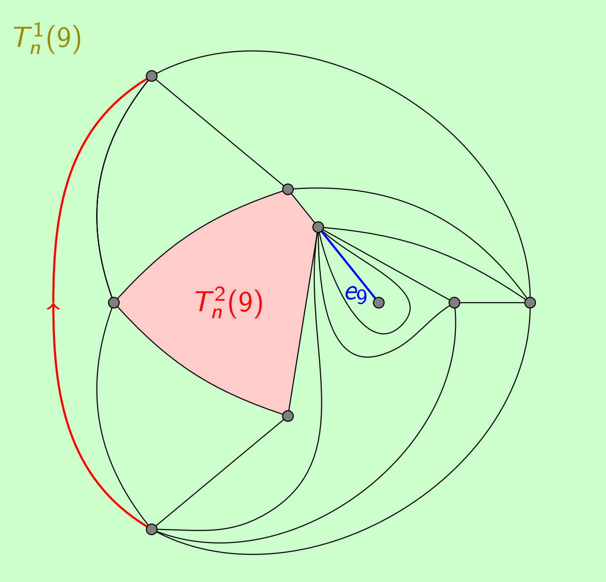
$\tau_4 = 8$



$\widetilde{P}_n(9) = 4$
$\widetilde{V}_n(9) = 6$

$\tau_5 = 9$

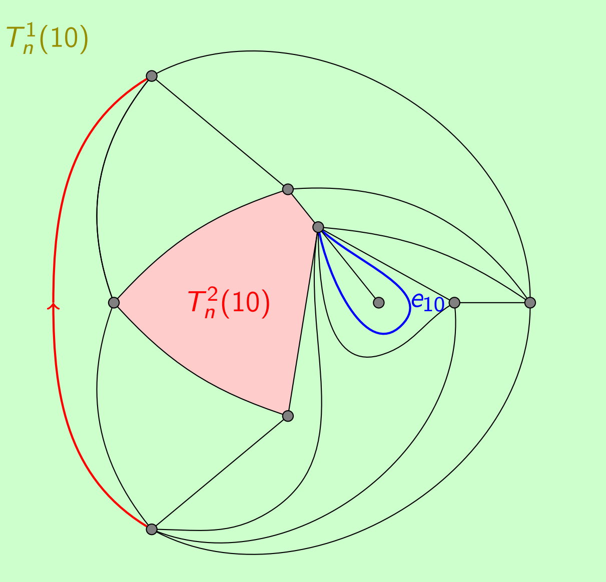
Exploration of $T_n(k)$



$\widetilde{P}_n(9) = 4$
$\widetilde{V}_n(9) = 6$

$\tau_5 = 9$

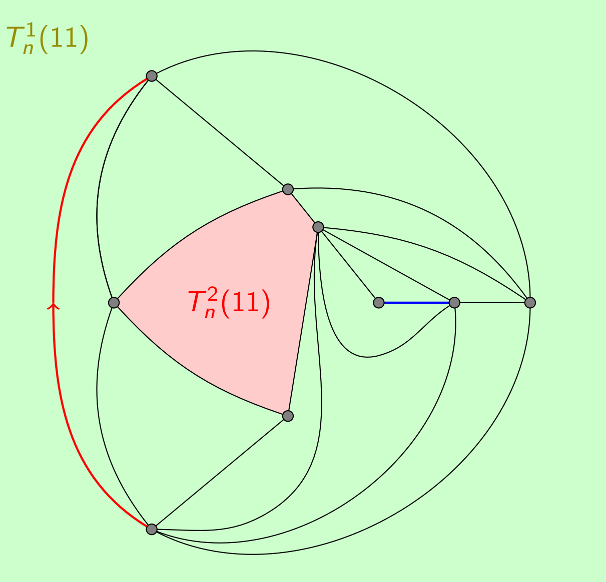
Exploration of $T_n(k)$



$\widetilde{P}_n(10) = 4$
$\widetilde{V}_n(10) = 6$

$\tau_5 = 9$

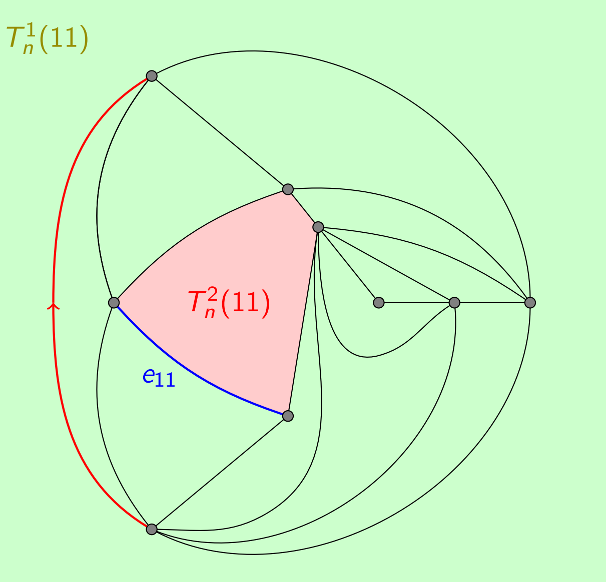
Exploration of $T_n(k)$



$\widetilde{P}_n(11) = 4$
$\widetilde{V}_n(11) = 6$

$\tau_5 = 9$

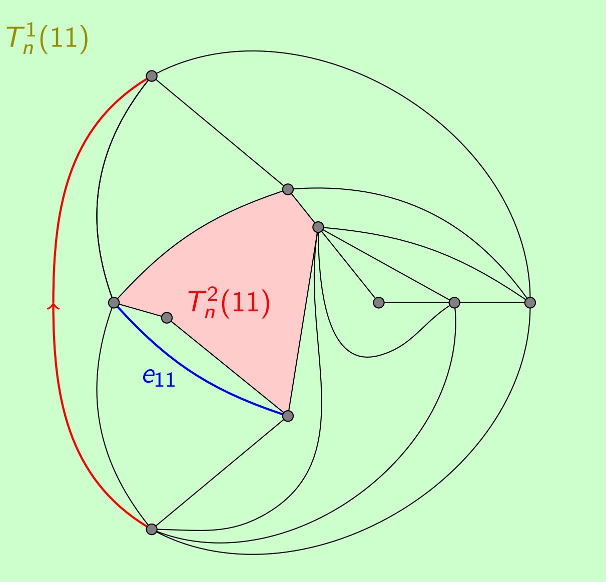
Exploration of $T_n(k)$



$\widetilde{P}_n(11) = 4$
$\widetilde{V}_n(11) = 6$

$\tau_5 = 9$

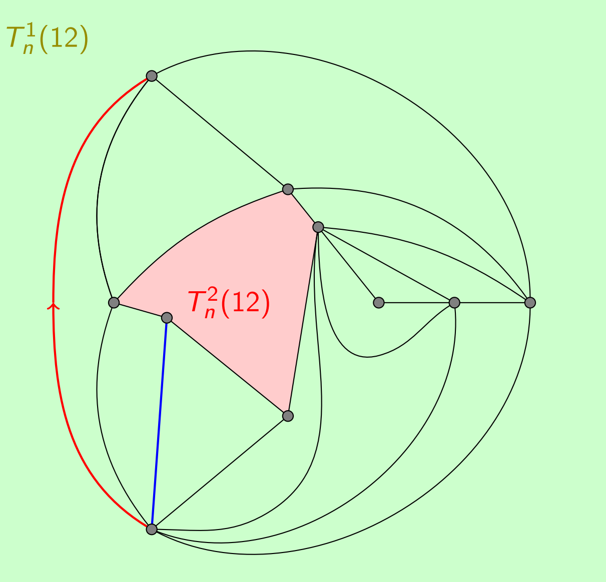
Exploration of $T_n(k)$



$\widetilde{P}_n(11) = 4$
$\widetilde{V}_n(11) = 6$

$\tau_5 = 9$

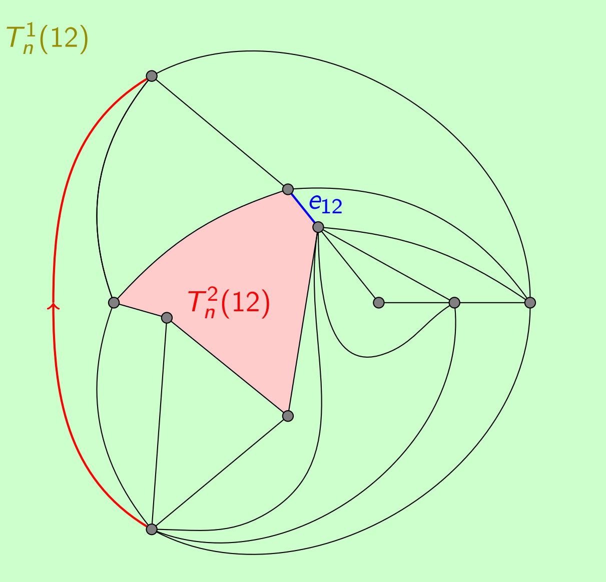
Exploration of $T_n(k)$



$\widetilde{P}_n(12) = 5$
$\widetilde{V}_n(12) = 7$

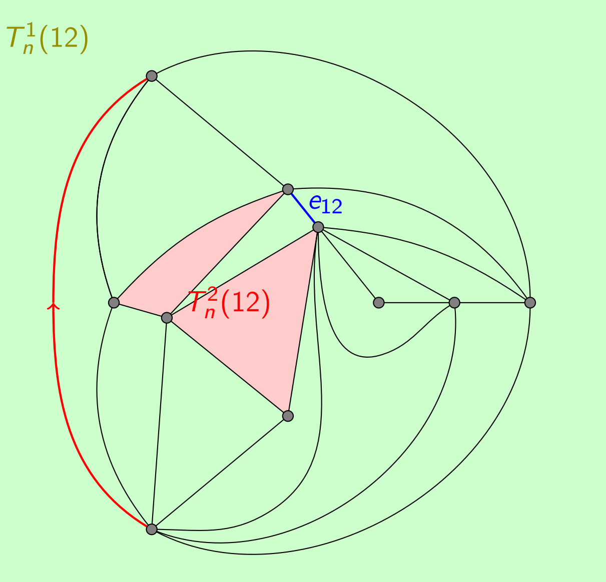
$\tau_6 = 12$

Exploration of $T_n(k)$



$\widetilde{P}_n(12) = 5$
$\widetilde{V}_n(12) = 7$

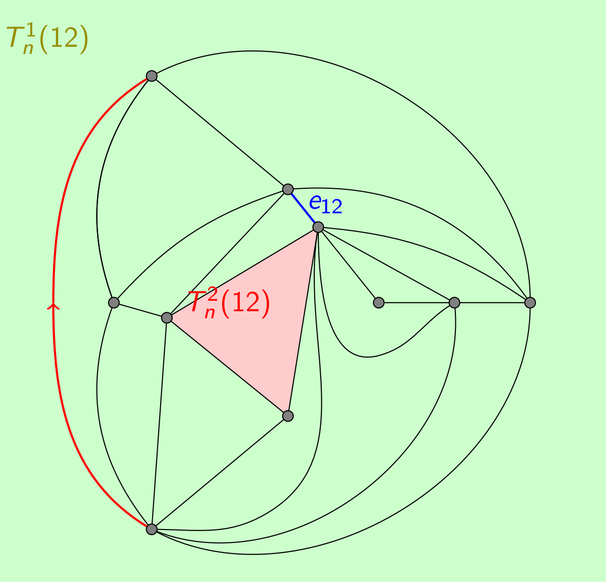
$\tau_6 = 12$



$\widetilde{P}_n(12) = 5$
$\widetilde{V}_n(12) = 7$

$\tau_6 = 12$

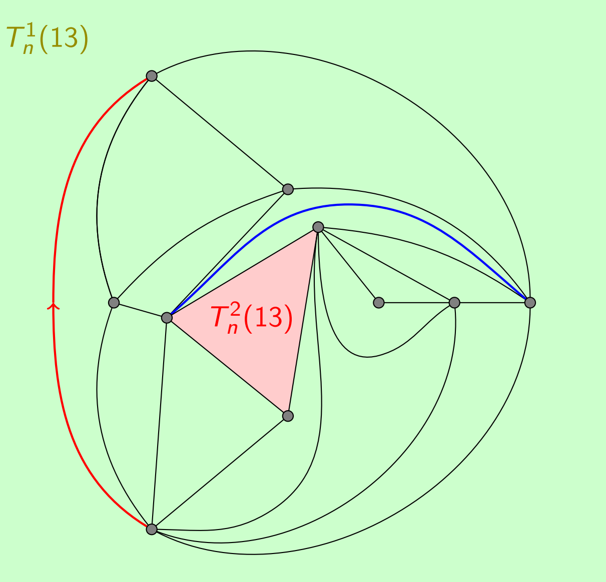
Exploration of $T_n(k)$



$\widetilde{P}_n(12) = 5$
$\widetilde{V}_n(12) = 7$

$\tau_6 = 12$

Exploration of $T_n(k)$



$\widetilde{P}_n(13) = 3$
$\widetilde{V}_n(13) = 7$

$\tau_7 = 13$

$T_n^2(k)$ stays uniform

Lemma

For all $k \geq 0$, conditionally on $(T_n^1(i))_{0 \leq i \leq k}$, the triangulation $T_n^2(k)$ is a uniform triangulation with a boundary of length $|\partial T_n^1(k)|$ and $n - |T_n^1(k)|$ inner vertices.

$T_n^2(k)$ stays uniform

Lemma

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Proof : induction on k :

- if e_k lies in the interior of $T_n^1(k)$, then $T_n^2(k+1) = T_n^2(k)$,
- if e_k lies in the interior of $T_n^2(k)$, stationarity of the uniform measure on triangulations with a boundary,
- if $e_k \in \partial T_n^1(k)$, it follows from the spatial Markov property.

Peeling estimates

- We write τ_j for the times k such that $e_{k-1} \in \partial T_n^1(k-1)$. Let $P_n(j) = \widetilde{P}_n(\tau_j)$ and $V_n(j) = \widetilde{V}_n(\tau_j)$.
- Then (P_n, V_n) has the same distribution as the perimeter and volume processes associated to the peeling of a uniform triangulation of the sphere with $\frac{n}{2}$ vertices.

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Lemma

If $j_n = o(n^{3/4})$ then $\max_{0 \leq j \leq j_n} P_n(j) = o(\sqrt{n})$ and $V_n(j_n) = o(n)$ in probability.

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Proof :

- estimates by Curien–Le Gall for the UIPT : the lemma holds if we replace $P_n(j)$ and $V_n(j)$ by $P_\infty(j)$ and $V_\infty(j)$,
- coupling between the UIPT and uniform finite triangulations.

Time change estimates

Conditionally on (P_n, V_n), the $\tau_{i+1} - \tau_i$ are independent and geometric with parameters $\frac{P_n(i)}{3n-6}$, so for $\varepsilon > 0$ small, w.h.p.

$$\mathbb{E}\left[\tau_{\varepsilon n^{3/4}} | P_n\right] = \sum_{i=1}^{\varepsilon n^{3/4}} \frac{3n-6}{P_n(i)} > \frac{n \times \varepsilon n^{3/4}}{\sqrt{n}} = \varepsilon n^{5/4},$$

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so after $o(n^{5/4})$, the number of peeling steps performed is $o(n^{3/4})$ in probability. Hence,

$$\widetilde{P}_n(o(n^{5/4})) = P_n(o(n^{3/4})) = o(\sqrt{n}),$$

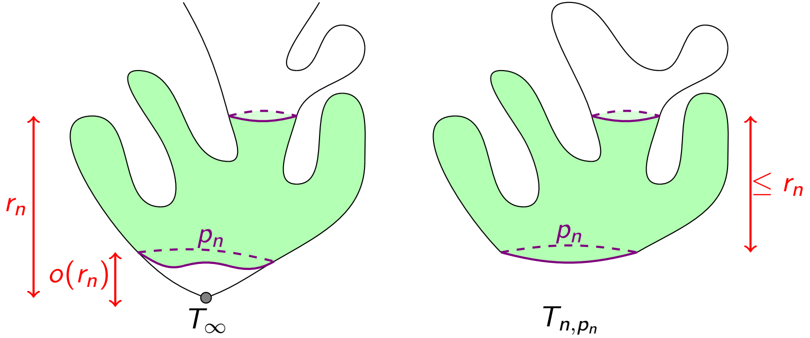
$$\widetilde{V}_n(o(n^{5/4})) = V_n(o(n^{3/4})) = o(n).$$

We know that $T_n^2\left(o(n^{5/4})\right)$ is a uniform triangulation with a boundary of length $o(\sqrt{n})$ and $\left(\frac{1}{2} - o(1)\right) n$ inner vertices.

Theorem (Krikun, 2005)

In the UIPT, there is a cycle of length $O(r)$ in probability surrounding the ball of radius r.

Coupling lemma : for $p_n << r_n^2 << \sqrt{n}$, w.h.p.



Small cycles in triangulations with a small perimeter

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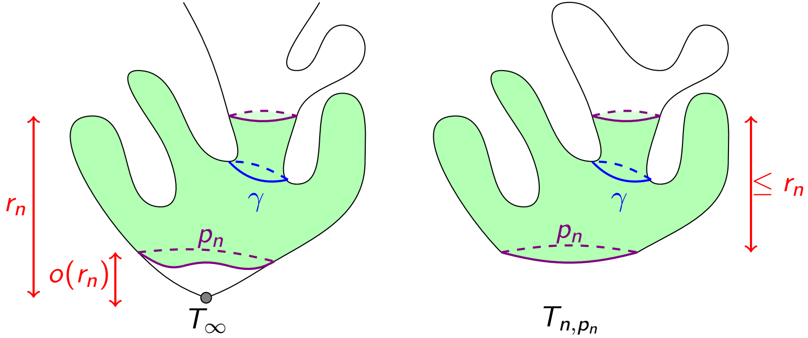

Is the lower bound sharp ?

- Back-of-the-enveloppe computation :
  - in a typical triangulation, the distance between two typical vertices x and y is $\approx n^{1/4}$.
  - The probability that a flip hits a geodesic is $\approx n^{-3/4}$.
  - The distance between x and y changes $\approx kn^{-3/4}$ times before time k.
  - If $d(x, y)$ evolves roughly like a random walk, it varies of $\approx \sqrt{kn^{-3/4}} = n^{1/4}$ for $k = n^{5/4}$.

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  - If $d(x, y)$ evolves roughly like a random walk, it varies of $\approx \sqrt{kn^{-3/4}} = n^{1/4}$ for $k = n^{5/4}$.
- For triangulations of a convex polygon (no inner vertices), the lower bound $n^{3/2}$ is believed to be sharp but the best known upper bound is n^5 [McShine–Tetali].
- Prove that the mixing time is polynomial ?

THANK YOU !