# Kaledin obstruction classes & formality criteria

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1. The notion of formality
   - The formality can be addressed as a deformation problem, using the operadic calculus.
2. Kaledin obstruction classes
   - Construction of an obstruction theory to formality over any coefficient ring.
3. Formality criteria
   - Formality descent with torsion coefficients
   - Intrinsic formality criterium
   - Degree twisting & automorphisms lifts

# The notion of formality

# Formal topological spaces

$R$ : commutative ground ring

**Definition**

A topological space $X$ is formal if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C^\bullet_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet_{\text{sing}}(X;R)\ .$$

$\rightarrow$ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

$X$ formal $\Longrightarrow$ The cohomology ring $H^\bullet_{\text{sing}}(X,\mathbb{Q})$ completely determines the rational homotopy type of $X$.

**Examples**

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [DGMS, 1975]

# Formality of an algebraic structure

$A$ : chain complex over $R$

$\mathscr{P}$ : graded algebraic operad

$\phi : \mathscr{P} \to \mathrm{End}_A$ a dg $\mathscr{P}$-algebra structure

## Definition

The dg $\mathscr{P}$-algebra $(A, \phi)$ is formal if

$$\exists\, (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_0)\ ,$$

where $\varphi_0$ is the canonical $\mathscr{P}$-algebra structure on $H(A)$.

## Examples

- $X$ is formal $:= (C^{\bullet}_{\mathrm{sing}}(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

# Operadic homological algebra

$A$ : chain complex over $R$

Assumptions:
- reduced connected weight-graded (co)operads
- $R$ is a $\mathbb{Q}$-algebra in the case of symmetric operads

$\mathscr{P}$ : graded Kozul operad

$\mathscr{P}_\infty \xrightarrow{\sim} \mathscr{P}$ with $\mathscr{P}_\infty := \Omega \mathscr{P}^{\mathrm{i}}$

$\{\mathscr{P}_\infty - \text{algebra structures on } A\} := \mathrm{Hom}_{\mathrm{dgOp}}\left(\Omega \mathscr{P}^{\mathrm{i}}, \mathrm{End}_A\right)$

## Proposition

$$\mathrm{Hom}_{\mathrm{dgOp}}\left(\Omega \mathscr{P}^{i}, \mathrm{End}_A\right) \cong \mathrm{Codiff}\left(\mathscr{P}^{i}(A)\right)$$

## Definition

An $\infty$-morphism $F : (A, \varphi) \rightsquigarrow (B, \psi)$ between $\mathscr{P}_\infty$-algebra structures is a morphism of dg $\mathscr{P}^{\text{i}}$-coalgebras:

$$(\mathscr{P}^{\text{i}}(A), \varphi) \to (\mathscr{P}^{\text{i}}(B), \psi) \ .$$

$F$ is an $\infty$-quasi-isomorphism if $F_0 : A \to B$ is a quasi-isomorphism.

## Proposition ($R$ is a characteristic zero field)

*zig-zag of quasi-isos of $\mathscr{P}$-algebras* — *$\infty$-quasi-iso*

$$\exists\ (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \phi') \quad \Longleftrightarrow \quad \exists\ (A, \phi) \overset{\sim}{\rightsquigarrow} (B, \phi')$$

## Corollary

*A dg $\mathscr{P}$-algebra $(A, \phi)$ is formal* $\Longleftrightarrow$ $\exists\ (A, \phi) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_0)$.

# Homotopy transfer

## Theorem (Homotopy transfer theorem)

*Let $(A, d)$ be a chain complex s.t. $H(A)$ is a homotopy retract:*

$$h \circlearrowright (A, d) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0)$$

*where $\mathrm{id}_A - ip = d_A h + h d_A$ and $i$ is a quasi-isomorphism.*
*For every $\mathscr{P}$-algebra structure $(A, \phi)$, there exists a $\mathscr{P}_\infty$-algebra structure $\varphi$ on $H(A)$ s.t. $p$ extends to an $\infty$-quasi-isomorphism:*

$$\begin{array}{ccc} (A, \phi) & \overset{p_\infty}{\rightsquigarrow} & (H(A), \varphi) \\ & \underset{\textit{Formality}}{\searrow} & \Big\downarrow \exists\,? \\ & & (H(A), \varphi_0) \end{array}$$

# $\mathscr{P}_\infty$-algebra structures on $H(A)$

The convolution dg Lie algebra associated to $H(A)$:

$$\mathfrak{g} := \left(\mathrm{Hom}(\overline{\mathscr{P}}^{\mathrm{i}}, \mathrm{End}_{H(A)}), [-,-], d\right)$$

→ $\mathrm{Hom}(\overline{\mathscr{P}}^{\mathrm{i}}, \mathrm{End}_{H(A)}) := \prod_{n\geq 0} \mathrm{Hom}(\overline{\mathscr{P}}^{\mathrm{i}}(n), \mathrm{End}_{H(A)}(n))$

→ $d(\varphi) := -(-1)^{|\varphi|}\varphi \circ d_{\overline{\mathscr{P}}^{\mathrm{i}}}$

→ $\varphi \star \psi := \overline{\mathscr{P}}^{\mathrm{i}} \xrightarrow{\Delta_{(1)}} \overline{\mathscr{P}}^{\mathrm{i}} \circ_{(1)} \overline{\mathscr{P}}^{\mathrm{i}} \xrightarrow{\varphi \circ_{(1)} \psi} \mathrm{End}_{H(A)} \circ_{(1)} \mathrm{End}_{H(A)} \xrightarrow{\gamma_{(1)}} \mathrm{End}_{H(A)}$

→ $[\varphi, \psi] := \varphi \star \psi - (-1)^{|\varphi||\psi|}\psi \star \varphi$

Every $\varphi \in \mathrm{Hom}(\overline{\mathscr{P}}^{\mathrm{i}}, \mathrm{End}_{H(A)})$ decomposes as

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \cdots$$

where $\varphi_k$ is the restriction $\varphi_k : \overline{\mathscr{P}}^{\mathrm{i}}(k+2) \Longrightarrow \mathrm{End}_{H(A)}(k+2)$.

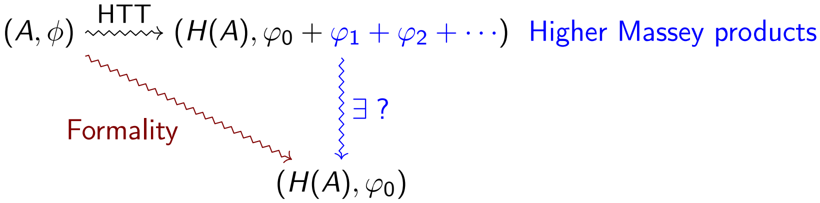
Its set of Maurer–Cartan elements:

$$\mathrm{MC}(\mathfrak{g}) = \{\varphi \in \mathfrak{g}_{-1},\ d(\varphi) + \tfrac{1}{2}[\varphi, \varphi] = 0\}$$

## Proposition

$$\{\mathscr{P}_\infty - \textit{algebra structures on } H(A)\} \cong \mathrm{MC}(\mathfrak{g})$$

## Remark




$\Longrightarrow$ If the higher Massey products vanish, then $(A, \phi)$ is formal.

# The gauge group

The convolution dg Lie algebra:

$$\mathfrak{g} := \left(\mathsf{Hom}(\overline{\mathscr{P}}^{\text{i}}, \mathrm{End}_{H(A)}), [-,-], d\right)$$

Its set of degree zero elements:

$$\mathfrak{g}_0 := \mathsf{Hom}(\overline{\mathscr{P}}^{\text{i}}, \mathrm{End}_{H(A)})_0$$

The Baker–Campbell–Hausdorff formula, with $\mathrm{ad}_\lambda := [\lambda, -]$ :

$$\lambda, \mu \in \mathfrak{g}_0, \qquad e^{\mathrm{ad}_{\mathrm{BCH}(\lambda,\mu)}} = e^{\mathrm{ad}_\lambda} \circ e^{\mathrm{ad}_\mu} \ .$$

$$\mathrm{BCH}(\lambda, \mu) = \lambda + \mu + \frac{1}{2}[\lambda, \mu] + \frac{1}{12}([\lambda, [\lambda, \mu]] + [\mu, [\mu, \lambda]]) + \cdots$$

$$\Gamma := (\mathfrak{g}_0, \mathrm{BCH}, 0)$$

# The gauge action

$\Gamma := (\mathfrak{g}_0, \mathrm{BCH}, 0)$

$\{\mathscr{P}_\infty - \text{algebra structures on } \mathrm{H(A)}\} := \mathrm{MC}(\mathfrak{g})$

## Gauge action

$$\begin{array}{ccc} \Gamma \times \mathrm{MC}(\mathfrak{g}) & \longrightarrow & \mathrm{MC}(\mathfrak{g}) \\ (\lambda, \varphi) & \longmapsto & \lambda \cdot \varphi := e^{\mathrm{ad}_\lambda}(\varphi) - \frac{e^{\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda) \end{array}$$

## Proposition (Dotsenko – Shadrin – Vallette, 2016)

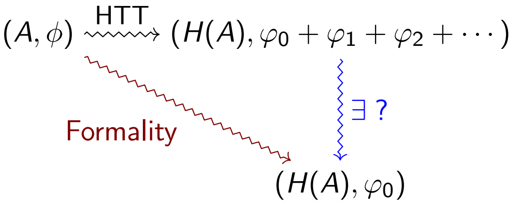
$$\exists\ \infty\text{-quasi-isomorphism } (H(A), \varphi) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_0)$$

$$\Longleftrightarrow$$

$$\exists\ \lambda \in \Gamma \text{ such that } \lambda \cdot \varphi = \varphi_0$$

# An equivalent characterization of formality

$(A, \phi)$: a $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract




## Definition

- $(A, \phi)$ is formal if $\exists\ \lambda \in \Gamma$ such that $\lambda \cdot \varphi = \varphi_0$
- $(A, \phi)$ is $n$-formal if $\exists\ \lambda \in \Gamma$ such that $\lambda \cdot \varphi = \varphi_0 + \psi_{n+1} + \cdots$

# Kaledin obstruction classes

# Formal deformation

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \cdots \in \mathrm{MC}(\mathfrak{g})$$

A formal deformation of $\varphi$:

$$\Phi := \varphi_0 + \varphi_1 h + \varphi_2 h^2 + \cdots + \varphi_k h^k + \cdots$$

in the dg Lie algebra $\mathfrak{g}[[h]] := \mathfrak{g} \otimes R[[h]]$.

## Remark

$\Phi \in \mathrm{MC}(\mathfrak{g}[[h]])$, i.e. $d(\Phi) + \frac{1}{2}[\Phi, \Phi] = 0$.

## Proposition

$d^\Phi := d + [\Phi, -]$ *is a differential on* $\mathfrak{g}[[h]]$

Twisted dg Lie algebra:

$$\mathfrak{g}[[h]]^\Phi := (\mathfrak{g}[[h]], [-,-], d^\Phi)$$

# The Kaledin classes

$$\partial_h \Phi := \varphi_1 + 2\varphi_2 h + \cdots + k\varphi_k h^{k-1} + \cdots \in \mathfrak{g}[[h]]$$

## Lemma

*$\partial_h \Phi$ is a cycle in $\mathfrak{g}[[h]]^\Phi$, i.e. $d^\Phi(\partial_h \Phi) = 0$.*

## Definition (Kaledin class)

The *Kaledin class* of $\varphi \in \mathrm{MC}(\mathfrak{g})$ is the homology class

$$K_\Phi := [\partial_h \Phi] \in H_{-1}\left(\mathfrak{g}[[h]]^\Phi\right) .$$

Its *$n^{th}$-truncated Kaledin class* is

$$K_\Phi^n := \left[\varphi_1 + 2\varphi_2 h + \cdots + n\varphi_n h^{n-1}\right] \in H_{-1}\left(\left(\mathfrak{g}[[h]]/h^{n+1}\right)^{\widetilde{\Phi}}\right) .$$

$R$ : commutative ground ring

$char(R)$ : smallest non-invertible prime in $R$

$n$ : any positive integer s.t. if $char(R) \neq 0$, $n - 1 < char(R)$

### Theorem (E., 2022)

$(A, \phi)$ : *a* $\mathscr{P}$*-algebra s.t.* $H(A)$ *is a homotopy retract*

- $(A, \phi)$ *is n-formal* $\iff K^n_\Phi = 0$.
- *If* $R$ *is a* $\mathbb{Q}$*-algebra, then* $(A, \phi)$ *is formal* $\iff K_\Phi = 0$.

### Remark

This construction generalizes to other types of algebraic structures: algebras over colored operads, gebras over properads, etc.

Previous works: [Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019]

# Heuristic behind these obstruction classes

Gauge action: $\lambda \cdot \varphi := e^{\mathrm{ad}_\lambda}(\varphi) - \frac{e^{\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda)$

Vector field $\Upsilon_\lambda$ on $\mathrm{MC}(\mathfrak{g})$: $\forall\, \lambda \in \Gamma, \quad \Upsilon_\lambda(\varphi) := -d\lambda - [\varphi, \lambda]$

Associated flow:

$$\frac{d}{dt}\gamma_\lambda(t) = \Upsilon_\lambda\left(\gamma_\lambda(t)\right)$$

Integration of the flow starting at $\gamma_\lambda(0) = \varphi$:

$$\gamma_\lambda(t) = e^{t\mathrm{ad}_\lambda}(\varphi) - \frac{e^{t\mathrm{ad}_\lambda} - \mathrm{id}}{t\mathrm{ad}_\lambda}(td\lambda)$$

$$\boxed{\lambda \cdot \varphi = \psi \Longleftrightarrow \gamma_\lambda(1) = \psi}$$

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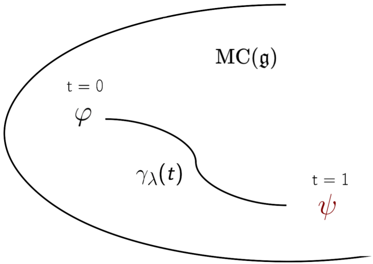
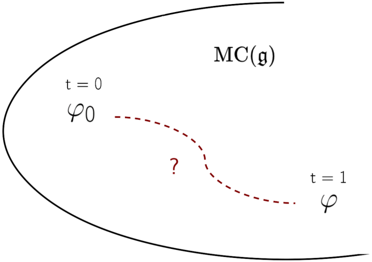


# The case of formality




Does there exists $\lambda \in \Gamma$, such that $\gamma_\lambda(0) = \varphi_0$ and $\gamma_\lambda(1) = \varphi$ ?

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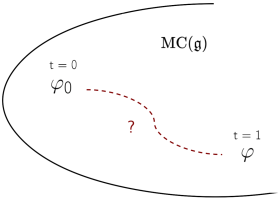


Attempt: "$\Phi(h) = \varphi_0 + \varphi_1 h + \varphi_2 h^2 + \cdots$"

Does there exists $\lambda \in \Gamma$, such that $\Phi = \gamma_\lambda$ ?

$$\Phi = \gamma_\lambda \iff \partial_h \Phi = \Upsilon_\lambda(\Phi) := -d\lambda - [\Phi, \lambda]$$

$$\iff \partial_h \Phi = d^\Phi(-\lambda)$$

$$\iff \partial_h \Phi \text{ is a boundary in } \mathfrak{g}[[h]]^\Phi$$

$$\iff K_\Phi := [\partial_h \Phi] = 0$$

# Formality criteria

# Formality descent

$(A,\phi)$ : a dg $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract

$H_i(A)$ : projective, finitely generated for all $i$.

$Q$ : faithfully flat commutative $R$-algebra.

## Proposition (E., 2022)

*$(A,\phi)$ is $n$-formal $\Longleftrightarrow$ $(A\otimes_R Q, \phi\otimes 1)$ is $n$-formal.*

## Proof.

$$\begin{array}{ccc} H_{-1}\left(\mathfrak{g}_{H(A)}[[h]]^{\Phi}\right)\otimes_{R[[h]]} Q[[h]] & \cong & H_{-1}\left(\mathfrak{g}_{H(A\otimes_R Q)}[[h]]^{\Phi\otimes 1}\right) \\ \cup & & \cup \\ K_\Phi\otimes 1 = 0 & \Longleftrightarrow & K_{\Phi\otimes 1} = 0 \end{array}$$

□

## Examples

- $C(\mathcal{D}_k;\mathbb{R})$ is formal $\Longleftrightarrow$ $C(\mathcal{D}_k;\mathbb{Q})$ is formal [GSNPR, 2005]
- $\mathbb{Z}_{(l)}\subset\mathbb{Z}_l$

# Intrinsic formality

A graded $\mathscr{P}$-algebra $(H, \varphi_0)$ is intrinsically formal if every $\mathscr{P}$-algebra $(A, \phi)$ such that $(H(A), \varphi_0) = (H, \varphi_0)$ is itself formal.

$\mathfrak{g}^{\varphi_0}$ : $(\mathfrak{g}, [-,-], d + [\varphi_0, -])$

## Proposition (E., 2022)

$$H_{-1}(\mathfrak{g}^{\varphi_0}) = 0 \implies (H, \varphi_0) \textit{ intrinsically formal.}$$

## Proof.

For all $(A, \phi)$ such that $(H(A), \varphi_0) = (H, \varphi_0)$, then

$$K_\Phi = 0 \in H_{-1}\left(\mathfrak{g}[[h]]^\Phi\right) .$$

□

Previous works: [Hinich, 2003]

# Tamarkin's proof of Kontsevich formality

$k$ : a characteristic zero field

$A$ : a polynomial algebra over $k$

## Theorem (Hinich, 2003)

*The shifted cohomological Hochschild complex $C(A;A)[1]$ is formal as a dg Lie algebra.*

## Proof.

- $\mathrm{Lie}[1] \subset Gerst$;
- $(HH^\bullet(A), \varphi_0)$ has a *Gerst*-algebra structure;
- $C(A;A)$ has a $Gerst_\infty$-algebra structure inducing $\varphi_0$;
- $(HH^\bullet(A), \varphi_0)$ is intrinsically formal as a *Gerst*-algebra;

$\rightarrow H_{-1}(\mathfrak{g}^{\varphi_0}) = 0$, where $\mathfrak{g} = \mathrm{Hom}(\overline{Gerst}^{\text{i}}, \mathrm{End}_{HH^\bullet(A)})$ .

# The degree twisting

$(A,\phi)$ : a dg $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract

$\alpha$ : a unit in $R$.

$\sigma_\alpha$ : the degree twisting by $\alpha$

$\rightarrow$ linear automorphism of $H(A)$ which acts via $\alpha^k \times$ on $H_k(A)$.

**Theorem (Drummond-Cole – Horel, 2021)**

*Suppose that $\sigma_\alpha$ admits a lift, i.e. $\exists\ f \in \mathrm{End}(A,\phi)$ s.t. $H(f) = \sigma_\alpha$.*

- $\forall k,\ \alpha^k - 1 \in R^\times \implies (A,\phi)$ *is formal.*
- $\forall k \leq n,\ \alpha^k - 1 \in R^\times \implies (A,\phi)$ *is n-formal.*

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- $\forall k,\ \alpha^k - 1 \in R^\times \implies (A, \phi)$ *is formal.*
- $\forall k \leq n,\ \alpha^k - 1 \in R^\times \implies (A, \phi)$ *is n-formal.*

Heuristic :

→ Higher Massey products have to be compatible with the lift.

→ They intertwine multiplication by $\alpha^l$ with multiplication by $\alpha^k$ with $l \neq k$.

→ They have to vanish

Previous works: [DGMS, 1975], [Sullivan, 1977], [GSNPR, 2005]

# Complement of subspace arrangements

$X$ : a complement of a hyperplane arrangement over $\mathbb{C}$
$\rightarrow$ complement of a finite collection of affine hyperplanes in $\mathbb{A}^n_{\mathbb{C}}$.

$K$ : a finite extension of $\mathbb{Q}_p$

$q$ : order of the residue field of the ring of integers of $K$

$l$ : a prime number different from $p$

$h$ : order of $q$ in $\mathbb{F}_l^\times$

**Proposition (Cirici – Horel, 2022)**

*If $X$ is defined over $K$, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a complement of a hyperplane arrangement over $K$ s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_l)$ is $(h-1)$-formal.*

## Proposition (Cirici – Horel, 2022)

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Heuristic :

- $\rightarrow$ $C^\bullet(X_{an}, \mathbb{Z}_l) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_l)$.
- $\rightarrow$ The action of a Frobenius on $H_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_l)$ is $\sigma_q$, [Kim, 1994].

Formality descent $\implies$ $C^\bullet(X_{an}, \mathbb{Z}_{(l)})$ is $(h-1)$-formal.

# Automorphism lifts

$(A, \phi)$ : a dg $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract

## Theorem (E., 2022)

*Suppose that* $\exists\, u \in \mathrm{Aut}(H(A), \varphi_0)$ *admitting a chain lift. Let* $\mathrm{Ad}_u : \mathrm{End}_{H(A)} \to \mathrm{End}_{H(A)}$ *s.t.* $\forall\, p \in \mathbb{N}$, *and* $\forall\, \psi \in \mathrm{End}_{H(A)}(p)$

$$\mathrm{Ad}_u(\psi) = u \circ \psi \circ (u^{-1})^{\otimes p} \ .$$

1. *If* $\mathrm{Ad}_u - \mathrm{id}$ *is invertible on the elements of degree* $k$ *for all* $k < n$, *then* $(A, \phi)$ *is* $n$*-formal.*
2. *If* $R$ *is a* $\mathbb{Q}$*-algebra and* $\mathrm{Ad}_u - \mathrm{id}$ *is invertible, then* $(A, \phi)$ *is formal.*

# Automorphism lifts

$R$ : a characteristic zero field

$(A, \phi)$ : a dg $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract and finite dimensional.

## Corollary

*Suppose that there exists $u \in \mathrm{Aut}(H(A), \varphi_0)$ such that for all $k < n$, and all p-tuples $(k_1, \ldots, k_p)$,*

$$\mathrm{Spec}(u_{k_1+\cdots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing \ ,$$

*where $u_i := u_{|H_i(A)}$. If $u$ admits a lift at the level of chains then $(A, \phi)$ is $n$-formal.*

## Corollary

*For every smooth projective $K$-scheme $X$, $C^\bullet(X_{an}, \mathbb{Q}_l)$ is formal.*

# Frobenius & Weil numbers

$K$ : a finite extension of $\mathbb{Q}_p$

$q$ : order of the residue field of the ring of integers of $K$

$l$ : a prime number different from $p$

$X$ : a smooth projective $K$-scheme

**Definition**

$\alpha \in \overline{\mathbb{Q}}_l$ is a Weil number of weight $n$ if

$$\forall\, \iota : \overline{\mathbb{Q}}_l \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2} .$$

**Theorem (Deligne, 1974)**

*For all $n$, the eigenvalues of a Frobenius action on $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_l)$ are Weil numbers of weight $n$.*

## Corollary

*For every smooth projective $K$-scheme $X$, $C^\bullet(X_{an}, \mathbb{Q}_l)$ is formal.*

## Proof.

- $C^\bullet(X_{an}, \mathbb{Q}_l) \xrightarrow{\sim} C^\bullet_{et}(X_{\overline{K}}, \mathbb{Q}_l)$
- Let $u$ be the Frobenius action on $H^\bullet_{et}(X_{\overline{K}}, \mathbb{Q}_l)$ and fix $\iota : \overline{\mathbb{Q}}_l \hookrightarrow \mathbb{C}$.
- For all $k \geq 1$, $(k_1, \dots, k_p)$ and $s := k_1 + \cdots + k_p$,

$$\begin{array}{ccccc} \mathrm{Spec}(u_{s+k}) & \cap & \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) & = \varnothing \, . \\ \Cup & & \Cup & \\ \alpha & & \beta & \\ |\iota(\alpha)| = q^{\frac{s+k}{2}} & > & |\iota(\beta)| = q^{\frac{s}{2}} & \end{array}$$

□

Previous works: [Deligne, 1980], [GSNPR, 2005]

Thank you for your attention!