



Kaledin classes & formality criteria

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December 9, 2024 | Algebra/Topology Seminar

arXiv : 2404.17529



The notion of formality



Formal topological spaces

R : commutative ground ring

Definition

A topological space X is **formal** if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C_{\text{sing}}^{\bullet}(X; R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H_{\text{sing}}^{\bullet}(X; R) .$$

→ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal $\implies$ The cohomology ring $H_{\text{sing}}^{\bullet}(X, \mathbb{Q})$ completely determines the rational homotopy type of X .

Examples

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [Deligne, Griffiths, Morgan & Sullivan, 1975]

Formality of an algebraic structure

A : chain complex over R

$\mathcal{P}$: colored operad or properad

$\phi : \mathcal{P} \rightarrow \text{End}_A$: a dg $\mathcal{P}$ -algebra structure

Definition

The dg $\mathcal{P}$ -algebra (A, ϕ) is **formal** if

$$\exists (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \dots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_*) ,$$

where φ_* is the canonical $\mathcal{P}$ -algebra structure on $H(A)$.

Examples

- X is formal $= (C_{\text{sing}}^\bullet(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Purity implies formality

(A, ϕ) : dg $\mathcal{P}$ -algebra encoded by an operad $\mathcal{P}$

α : unit of infinite order in R

σ_α : the degree twisting by $\alpha =$ automorphism of $(H(A), \varphi_*)$
which acts via $\alpha^k \times$ on $H_k(A)$.

Theorem

If σ_α admits a chain-level lift, i.e. $\exists f \in \text{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$,
then (A, ϕ) is formal.

- Deligne, Griffiths, Morgan, Sullivan [1975]
- Sullivan [1977]
- Guillén Santos, Navarro, Pascual, Roig [2005]
- Drummond-Cole and Horel [2021]

Examples

- Petersen [2014], Boavida de Brito and Horel [2021]
The little disks operad $\mathcal{D}_k$ & Grothendieck-Teichmüller group
- Riche, Soergel, Williamson [2014]
The extensions of parity sheaves on the flag variety.
- Drummond-Cole and Horel [2021]
 X : a complement of a hyperplane arrangement over $\mathbb{C}$
 defined over a finite extension K of $\mathbb{Q}_p$.
 ℓ : a prime number different from p
 $\rightarrow C^\bullet(X_{an}, \mathbb{Z}_\ell) \cong C_{et}^\bullet(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ [Artin].
 $\rightarrow$ A Frobenius action on $H_{et}^\bullet(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is σ_q [Kim, 1994].

Kaledin classes & formality criteria

1. Higher structures

- Formality can be addressed as a deformation problem, using the operadic calculus.

2. Kaledin classes

- An obstruction theory to the formality over any ring

3. Formality criteria

- Formality descent with torsion coefficient, Automorphism lifts

4. Beyond formality

- Generalizing Kaledin classes to study homotopy equivalences



Higher structures



Homotopy retracts

Definition

(W, d_W) is a **homotopy retract** of (V, d_V) if there are maps

$$h \circlearrowleft (V, d_V) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (W, d_W)$$

where $\text{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism .

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Proposition

If R is a field, the cohomology of any cochain complex is a homotopy retract:

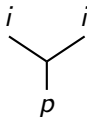
$$h \circlearrowleft (A, d_A) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (H(A), 0) .$$

Transfer of algebraic structure

(A, d_A, ϕ) : a dga algebra and a homotopy retraction:

$$h \circlearrowleft (A, d_A, \phi) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (H, d_H)$$

→ Transferred product: $\varphi_2 := p \circ \phi \circ i^{\otimes 2} : H^{\otimes 2} \rightarrow H$

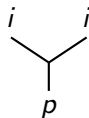


Transfer of algebraic structure

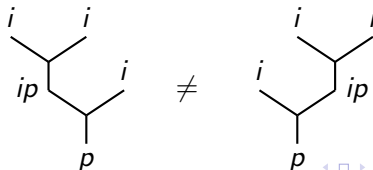
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$$h \circlearrowleft (A, d_A, \phi) \xrightleftharpoons[i]{p} (H, d_H)$$

→ Transferred product: $\varphi_2 := p \circ \phi \circ i^{\otimes 2} : H^{\otimes 2} \rightarrow H$



Not associative in general!



→ Consider $\varphi_3 : H^{\otimes 3} \rightarrow H$

$$\text{Y-junction} := \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ h \quad \quad i \\ \diagup \quad \diagdown \\ p \end{array} - \begin{array}{c} i \quad i \\ \diagup \quad \diagdown \\ i \quad \quad h \\ \diagdown \quad \diagup \\ p \end{array}$$

→ In $\text{Hom}(H^{\otimes 3}, H)$:

$$\partial \left(\text{Y-junction} \right) = \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ ip \quad \quad i \\ \diagup \quad \diagdown \\ p \end{array} - \begin{array}{c} i \quad i \\ \diagup \quad \diagdown \\ i \quad \quad ip \\ \diagdown \quad \diagup \\ p \end{array}$$

→ φ_2 is associative up to the homotopy φ_3 .

→ $\varphi_n : H^{\otimes n} \rightarrow H$, for all $n \geq 2$

$$\begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array} := \sum_{\text{PBT}_n} \pm \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ i \quad i \\ \diagdown \quad \diagup \\ h \quad h \\ | \\ p \end{array}$$

$$\partial \left(\begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \end{array} \right) = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm \begin{array}{c} 1 \quad \dots \quad l \\ \diagdown \quad \diagup \\ j \\ \diagdown \quad \diagup \\ 1 \quad \dots \quad k \end{array}$$

Homotopy associative algebras

Definition (Stasheff, 1963)

A_∞ -algebra: a cochain complex H with a collection of maps

$$\varphi_n : H^{\otimes n} \rightarrow H$$

of degree $2 - n$, for all $n \geq 2$, which satisfy the relations

$$\partial \left(\begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \diagdown \quad | \quad \diagup \\ \text{ } \end{array} \right) = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm \begin{array}{c} 1 \quad \dots \quad l \\ \diagdown \quad | \quad \diagup \\ \text{ } \\ \diagdown \quad | \quad \diagup \\ 1 \quad \dots \quad j \quad \dots \quad k \\ \diagdown \quad | \quad \diagup \\ \text{ } \end{array}$$

Examples

- Every dga algebra (A, ϕ) is an A_∞ -algebra with $\varphi_n = 0$ for all $n \geq 3$.
- $(H, d_H, \varphi_2, \varphi_3, \dots)$

Homotopy morphisms

$(A, d_A, \phi_2, \dots), (H, d_H, \varphi_2, \dots) : A_\infty$ -algebras

Definition

A_∞ -morphism $f : A \rightsquigarrow H$ is a collection of linear maps

$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geq 1,$$

of degree $1 - n$, which satisfy the relations

$$\sum_{\substack{k \geq 1 \\ i_1 + \dots + i_k = n}} \pm \begin{array}{c} \vee \quad \vee \\ f_{i_1} \dots f_{i_k} \\ | \\ \phi_k \end{array} = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm \begin{array}{c} \vee \quad \vee \\ \quad \quad \varphi_l \\ | \quad j \\ f_k \end{array}$$

where $\varphi_1 = d_H$ and $\phi_1 = d_A$.

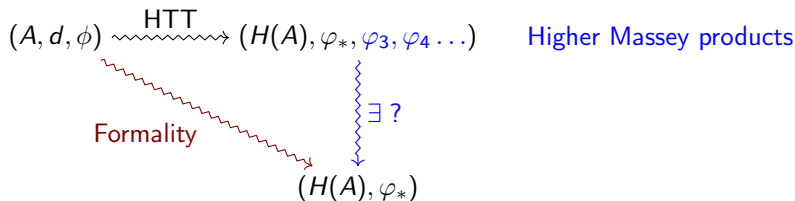
Homotopy quasi-isomorphisms

Definition

A_∞ -quasi-isomorphism $f : A \xrightarrow{\sim} H$ is an A_∞ -morphism where $f_1 : A \rightarrow H$ is a quasi-isomorphism .

An equivalent characterization of formality

(A, d, ϕ) a dga algebra such that $H(A)$ is a homotopy retract



$\implies$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

Definition

- (A, d, ϕ) is **gauge formal** if $\exists (H(A), \varphi_*, \varphi_3, \varphi_4 \dots) \rightsquigarrow (H(A), \varphi_*)$.
- (A, d, ϕ) is **gauge n -formal** if

$$\exists (H(A), \varphi_*, \varphi_3, \varphi_4 \dots) \rightsquigarrow (H(A), \varphi_*, 0, \dots, 0, \varphi'_{n+1}, \dots) .$$



Kaledin classes



Hochschild complex

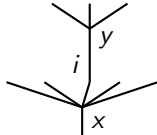
Transferred structure: $(H(A), \varphi_*, \varphi_3, \varphi_4, \dots)$

$$\varphi_n \in \text{Hom}(H(A)^{\otimes n}, H(A)), \quad |\varphi_n| = 2 - n$$

Hochschild cochain complex:

$$\mathfrak{g} := \prod_{n \geq 1} s^{-n+1} \text{Hom}(H(A)^{\otimes n}, H(A))$$

Lie bracket : $[x, y] := x \star y - (-1)^{|x||y|} y \star x$

$$x \star y := \sum_{i=1}^n (-1)^{(i-1)(m-1)}$$


for $x \in \text{Hom}(H(A)^{\otimes n}, H(A))$ and $y \in \text{Hom}(H(A)^{\otimes m}, H(A))$.

A formal deformation

Transferred structure:

$$(\varphi_*, \varphi_3, \varphi_4, \dots) \in \mathfrak{g} := \prod_{n \geq 1} s^{-n+1} \operatorname{Hom}(H(A)^{\otimes n}, H(A))$$

A formal deformation:

$$\Phi := \varphi_* + \varphi_3 \hbar + \varphi_4 \hbar^2 + \dots \in \mathfrak{g}[[\hbar]] := \mathfrak{g} \hat{\otimes} R[[\hbar]]$$

Proposition : $\operatorname{ad}_\Phi := [\Phi, -]$ defines a differential on $\mathfrak{g}[[\hbar]]$

Twisted dg Lie algebra:

$$\mathfrak{g}[[\hbar]]^\Phi := (\mathfrak{g}[[\hbar]], [-, -], \operatorname{ad}_\Phi)$$

Kaledin classes

$$\partial_{\hbar}\Phi := \varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots \in \mathfrak{g}[[\hbar]]$$

Lemma : $\partial_{\hbar}\Phi$ is a cycle in $\mathfrak{g}[[\hbar]]^{\Phi} := (\mathfrak{g}[[\hbar]], [-, -], \text{ad}_{\Phi})$,

$$\text{ad}_{\Phi}(\partial_{\hbar}\Phi) := [\Phi, \partial_{\hbar}\Phi] = 0 .$$

Kaledin class:

$$K_{\Phi} := [\partial_{\hbar}\Phi] \in H^1\left(\mathfrak{g}[[\hbar]]^{\Phi}\right) .$$

n^{th} -truncated Kaledin class :

$$K_{\Phi}^n := [\varphi_3 + 2\varphi_4\hbar + \cdots + (n-2)\varphi_n\hbar^{n-3}] \in H^1\left((\mathfrak{g}[[\hbar]]/\hbar^{n-2})^{\tilde{\Phi}}\right) .$$

Kaledin classes

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Theorem ([Kaledin, 2007], [Lunts, 2007])

$R : \mathbb{Q}$ -algebra

$(A, \phi) : dg \text{ associative algebra, } H(A) \text{ is a homotopy retract}$

- (A, ϕ) is gauge formal $\iff K_{\Phi} = 0$.
- (A, ϕ) is gauge n -formal $\iff K_{\Phi}^n = 0$.

Kaledin class:

$$K_{\Phi} := [\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots] \in H^1(\mathfrak{g}[[\hbar]]^{\Phi})$$

n^{th} -truncated Kaledin class :

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Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

R : $\mathbb{Q}$ -algebra

$\mathcal{P}$: Binary Koszul operad

(A, ϕ) : dg $\mathcal{P}$ -algebra that admits a transferred structure

- (A, ϕ) is gauge formal $\iff K_{\Phi} = 0$.
- (A, ϕ) is gauge n -formal $\iff K_{\Phi}^n = 0$.



Formality criteria



Formality descent

(A, ϕ) : a dg $\mathcal{P}$ -algebra that admits a transferred structure

$H_i(A)$: projective, finitely generated for all i .

S : faithfully flat commutative R -algebra.

Proposition (E., 2024)

(A, ϕ) is gauge n -formal $\iff (A \otimes_R S, \phi \otimes 1)$ is gauge n -formal.

Proof.

$$H_{-1}(\mathfrak{g}_{H(A)}[[\hbar]]^\Phi) \otimes_{R[[\hbar]]} S[[\hbar]] \cong H_{-1}(\mathfrak{g}_{H(A \otimes_R S)}[[\hbar]]^{\Phi \otimes 1}) \quad \square$$

Examples

- $C(\mathcal{D}_k; \mathbb{R})$ is formal $\iff C(\mathcal{D}_k; \mathbb{Q})$ is formal [GSNPR, 2005]
- $\mathbb{Z}_{(\ell)} \subset \mathbb{Z}_\ell$

Complement of hyperplane arrangements

X : a **complement of a hyperplane arrangement** over $\mathbb{C}$
 $\rightarrow$ complement of a finite collection of affine hyperplanes in $\mathbb{A}_{\mathbb{C}}^n$.

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

s : order of q in $\mathbb{F}_{\ell}^{\times}$

Proposition (Dummond-Cole – Horel, 2021)

If X is defined over K , i.e. $\exists K \hookrightarrow \mathbb{C}$ and $\exists \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^{\bullet}(X_{an}, \mathbb{Z}_{\ell})$ is gauge $(s - 1)$ -formal.

Formality descent $\implies C^{\bullet}(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s - 1)$ -formal.

Triviality of fibrations

Theorem (E., 2024)

X : a simply connected topological space

F : a nilpotent space of finite $\mathbb{Q}$ -type.

A fibration $\xi : E \rightarrow X$ with fiber $F_{\mathbb{Q}}$ is trivial up to homotopy iff $\xi \otimes \mathbb{R}$ is trivial up to homotopy.

Example

The Fadell–Neuwirth fibration :

$$\xi : \operatorname{Conf}_{n-1}(\mathbb{R}^d) \longrightarrow \operatorname{Conf}_n(\mathbb{S}^d) \longrightarrow \mathbb{S}^d .$$

If d is odd, $\xi \otimes \mathbb{R}$ is trivial up to homotopy [Haya Enriquez, 2022]

$\implies \xi$ is trivial up to homotopy.

Automorphism lifts

(A, ϕ) : a dg $\mathcal{P}$ -algebra that admits a transferred structure

Theorem (E., 2024)

Suppose that $u \in \text{Aut}(H(A), \varphi_0)$ admits a chain lift. Let

$$\text{Ad}_u : \text{End}_{H(A)} \rightarrow \text{End}_{H(A)} \quad \psi \longmapsto u^{\otimes q} \circ \psi \circ (u^{-1})^{\otimes p},$$

for $\psi \in \text{End}_{H(A)}(p, q) = \text{Hom}(H(A)^{\otimes p}, H(A)^{\otimes q})$

- 1. If $\text{Ad}_u - \text{id}$ is invertible, then (A, ϕ) is gauge formal and every homology automorphism admits a chain level lift.*
- 2. If $\text{Ad}_u - \text{id}$ is invertible on the elements of degree k for all $k < n$, then (A, ϕ) is gauge n -formal.*

Automorphism lifts

R : a field

(A, ϕ) : a dg $\mathcal{P}$ -algebra that admits a transferred structure s.t.
 $H(A)$ is finite dimensional.

Corollary (E., 2024)

Suppose that there exists $u \in \text{Aut}(H(A), \varphi_0)$ such that for all $k < n$, and all p -tuples $(k_1, \dots, k_p)$,

$$\text{Spec}(u_{k_1+\dots+k_p+k}) \cap \text{Spec}(u_{k_1} \otimes \dots \otimes u_{k_p}) = \emptyset ,$$

where $u_i := u|_{H_i(A)}$. If u admits a lift at the level of chains then (A, ϕ) is gauge n -formal.

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers $\mathcal{O}_K$

ℓ : a prime number different from p

X : a smooth proper K -scheme

Definition

$\alpha \in \overline{\mathbb{Q}}_\ell$ is a **Weil number of weight n** if

$$\forall \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2}.$$

Theorem (Deligne, 1974)

For all n , the eigenvalues of a Frobenius action on $H_{\text{et}}^n(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight n .

Theorem

Let X be a smooth and proper scheme over $\mathbb{C}$. The algebra $C^\bullet(X_{\text{an}}, \mathbb{Q})$ is formal.

Proof.

- There exists a smooth and proper model $\mathcal{X}$ over $\mathcal{O}_K$.

$$C^\bullet(X_{\text{an}}, \mathbb{Q}_\ell) \cong C_{\text{et}}^\bullet(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$$

- Let u be the Frobenius action on $H_{\text{et}}^\bullet(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$.
- For all $k \geq 1$, $(k_1, \dots, k_p)$ and $s := k_1 + \dots + k_p$,

$$\begin{array}{ccc} \text{Spec}(u_{s+k}) & \cap & \text{Spec}(u_{k_1} \otimes \dots \otimes u_{k_p}) \\ \downarrow \Psi & & \downarrow \Psi \\ \alpha & & \beta \\ |\iota(\alpha)| = q^{\frac{s+k}{2}} & > & |\iota(\beta)| = q^{\frac{s}{2}} \end{array}$$



Previous work: [Deligne, 1980]

Let Sch_K be the category of smooth and proper schemes over K of *good reduction*, i.e. for which there exists a smooth and proper model over $\mathcal{O}_K$.

Theorem (E., 2024)

Let $\mathbb{V}$ be a groupoid and let $\mathcal{P}$ be a $\mathbb{V}$ -colored operad in sets. Let X be a $\mathcal{P}$ -algebra in Sch_K . The dg $\mathcal{P}$ -algebra $C_\bullet(X_{\mathrm{an}}, \mathbb{Q})$ is formal.

Example (Guillén Santos, Navarro, Pascual, & Roig, 2005)

$\overline{\mathcal{M}}$ the cyclic operad of moduli spaces of stable algebraic curves
 $C_\bullet(\overline{\mathcal{M}}_{\mathrm{an}}; \mathbb{Q})$ is formal



Beyond formality



Homotopy equivalences between algebraic structure

Definition

The dg $\mathcal{P}$ -algebras (A, ϕ) and (B, ψ) are

- **homotopy equivalent**

$$\exists (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \dots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \psi)$$

- **gauge homotopy equivalent** if $\exists (A, \phi) \xrightarrow{\sim} (B, \psi)$.

Example

(A, ϕ) is formal if it is homotopy equivalent to $(H(A), \varphi_*)$

Question

- Can we generalize Kaledin classes to study homotopy equivalences ?

Obstruction sequences to homotopy equivalences

Let (A, φ) and (B, ψ) be two $\mathcal{P}$ -algebras with $H(A) \cong H(B)$.

→ **obstruction sequence** $(\vartheta_k)_{1 \leq k \leq n}$ which is either

- an infinite sequence of vanishing classes, when $n = \infty$;
- a finite sequence of trivial classes that ends on $\vartheta_n \neq 0$.

The index $n \in \llbracket 1, \infty \rrbracket$ of the last class only depends on ϕ and ψ .

Theorem (E., 2024)

The algebras (A, φ) and (B, ψ) are gauge homotopy equivalent modulo $\mathcal{F}^k \mathfrak{g}$ for all k if and only if $n = \infty$.

Minimal model on highly connected variety

Theorem (E., 2024)

Let M^d be a compact k -connected oriented C^∞ -manifold where d is smaller than $(\ell + 1)k + 2$. For every prime number p ,

$$C_{\text{sing}}^*(M, \mathbb{F}_p)$$

is homotopy equivalent to an A_∞ -algebra $(H_{\text{sing}}^(M, \mathbb{F}_p), \varphi)$, with $\varphi_n = 0$ for $n \geq \ell$.*



Thank you for your attention!

