Kaledin classes & formality criteria

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Kaledin classes & formality criteria

1. The notion of formality
   - The formality can be addressed as a deformation problem, using the operadic calculus.
2. Kaledin obstruction classes
   - Construction of an obstruction theory to formality over any coefficient ring.
3. Formality criteria
   - Formality descent with torsion coefficients
   - Intrinsic formality criterium
   - Degree twisting & automorphisms lifts

The notion of formality

Formal topological spaces

R : commutative ground ring

Definition

A topological space X is formal if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C^\bullet_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet_{\text{sing}}(X;R)\ .$$

$\rightarrow$ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal $\implies$ The cohomology ring $H^\bullet_{\text{sing}}(X,\mathbb{Q})$ completely determines the rational homotopy type of X.

Examples

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [DGMS, 1975]

Formality of an algebraic structure

A : chain complex over R

$\mathscr{P}$: graded algebraic operad

$\phi : \mathscr{P} \to \mathrm{End}_A$ a dg $\mathscr{P}$-algebra structure

Definition

The dg $\mathscr{P}$-algebra (A, ϕ) is formal if

$$\exists\,(A,\phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_0)\ ,$$

where φ_0 is the canonical $\mathscr{P}$-algebra structure on $H(A)$.

Examples

- X is formal $:= (C^\bullet_{\mathrm{sing}}(X;R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k;\mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Operadic homological algebra

A : chain complex over R

Assumptions:
- reduced connected weight-graded (co)operads
- R is a $\mathbb{Q}$-algebra in the case of symmetric operads

$\mathscr{P}$: graded Koszul operad

$\mathscr{P}_\infty \xrightarrow{\sim} \mathscr{P}$ with $\mathscr{P}_\infty := \Omega\mathscr{P}^{\text{i}}$

$\{\mathscr{P}_\infty - \text{algebra structures on } A\} := \mathsf{Hom}_{\mathrm{dgOp}}\left(\Omega\mathscr{P}^{\text{i}}, \mathsf{End}_A\right)$

Proposition

$$\mathsf{Hom}_{\mathrm{dgOp}}\left(\Omega\mathscr{P}^{i}, \mathsf{End}_A\right) \cong \mathrm{Codiff}\left(\mathscr{P}^{i}(A)\right)$$

Definition

An ∞-morphism $F : (A, \varphi) \rightsquigarrow (B, \psi)$ between $\mathscr{P}_\infty$-algebra structures is a morphism of dg $\mathscr{P}^{\text{i}}$-coalgebras:

$$(\mathscr{P}^{\text{i}}(A), \varphi) \to (\mathscr{P}^{\text{i}}(B), \psi) \ .$$

F is an ∞-quasi-isomorphism if $F_0 : A \to B$ is a quasi-isomorphism.

Proposition (R is a characteristic zero field)

zig-zag of quasi-isos of $\mathscr{P}$-algebras ⟺ *∞-quasi-iso*

$$\exists\, (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \phi') \iff \exists\, (A, \phi) \overset{\sim}{\rightsquigarrow} (B, \phi')$$

Corollary

A dg $\mathscr{P}$-algebra (A, ϕ) is formal $\iff \exists\, (A, \phi) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_0)$.

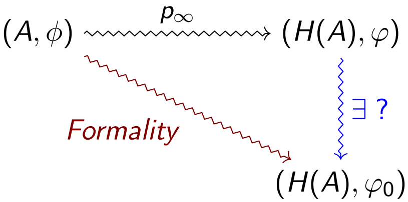
Homotopy transfer

Theorem (Homotopy transfer theorem)

Let (A, d) be a chain complex s.t. $H(A)$ is a homotopy retract:

$$h \circlearrowright (A, d) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0)$$

where $\mathrm{id}_A - ip = d_A\hbar + hd_A$ and i is a quasi-isomorphism.
For every $\mathscr{P}$-algebra structure (A, ϕ), there exists a $\mathscr{P}_\infty$-algebra structure φ on $H(A)$ s.t. p extends to an ∞-quasi-isomorphism:



$\mathscr{P}_\infty$-algebra structures on $H(A)$

The convolution dg Lie algebra associated to $H(A)$:

$$\mathfrak{g} := \left(\mathrm{Hom}(\overline{\mathscr{P}}^{\text{!`}}, \mathrm{End}_{H(A)}), [-,-], d\right)$$

- $\to$ $\mathrm{Hom}(\overline{\mathscr{P}}^{\text{!`}}, \mathrm{End}_{H(A)}) := \prod_{n\geq 0} \mathrm{Hom}(\overline{\mathscr{P}}^{\text{!`}}(n), \mathrm{End}_{H(A)}(n))$
- $\to$ $d(\varphi) := -(-1)^{|\varphi|}\varphi \circ d_{\overline{\mathscr{P}}^{\text{!`}}}$
- $\to$ $\varphi \star \psi := \overline{\mathscr{P}}^{\text{!`}} \xrightarrow{\Delta_{(1)}} \overline{\mathscr{P}}^{\text{!`}} \circ_{(1)} \overline{\mathscr{P}}^{\text{!`}} \xrightarrow{\varphi \circ_{(1)} \psi} \mathrm{End}_{H(A)} \circ_{(1)} \mathrm{End}_{H(A)} \xrightarrow{\gamma_{(1)}} \mathrm{End}_{H(A)}$
- $\to$ $[\varphi, \psi] := \varphi \star \psi - (-1)^{|\varphi||\psi|}\psi \star \varphi$

Every $\varphi \in \mathrm{Hom}(\overline{\mathscr{P}}^{\text{!`}}, \mathrm{End}_{H(A)})$ decomposes as

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots)$$

where φ_k is the restriction $\varphi_k : \overline{\mathscr{P}}^{\text{!`}}(k+2) \Longrightarrow \mathrm{End}_{H(A)}(k+2)$.

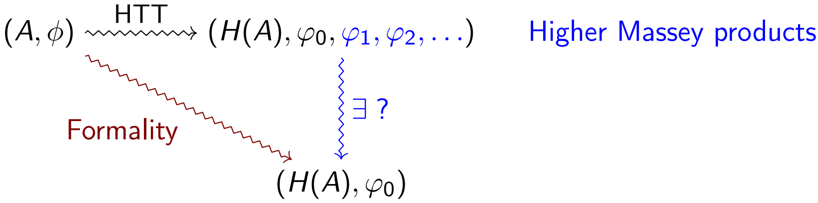
Its set of Maurer–Cartan elements:

$$\mathrm{MC}(\mathfrak{g}) = \{\varphi \in \mathfrak{g}_{-1},\ d(\varphi) + \tfrac{1}{2}[\varphi, \varphi] = 0\}$$

Proposition

$$\{\mathscr{P}_\infty - \textit{algebra structures on } H(A)\} \cong \mathrm{MC}(\mathfrak{g})$$

Remark



$\Longrightarrow$ If the higher Massey products vanish, then (A, ϕ) is formal.

The gauge group

The convolution dg Lie algebra:

$$\mathfrak{g} := \left(\mathrm{Hom}(\overline{\mathscr{P}}^{\mathrm{i}}, \mathrm{End}_{H(A)}), [-,-], d\right)$$

Its set of degree zero elements:

$$\mathfrak{g}_0 := \mathrm{Hom}(\overline{\mathscr{P}}^{\mathrm{i}}, \mathrm{End}_{H(A)})_0$$

The Baker–Campbell–Hausdorff formula, with $\mathrm{ad}_\lambda := [\lambda, -]$:

$$\lambda, \mu \in \mathfrak{g}_0, \qquad e^{\mathrm{ad}_{\mathrm{BCH}(\lambda,\mu)}} = e^{\mathrm{ad}_\lambda} \circ e^{\mathrm{ad}_\mu} \ .$$

$$\mathrm{BCH}(\lambda,\mu) = \lambda + \mu + \frac{1}{2}[\lambda,\mu] + \frac{1}{12}([\lambda,[\lambda,\mu]] + [\mu,[\mu,\lambda]]) + \cdots$$

$$\Gamma := (\mathfrak{g}_0, \mathrm{BCH}, 0)$$

The gauge action

$\Gamma := (\mathfrak{g}_0, \mathrm{BCH}, 0)$

$\{\mathscr{P}_\infty - \text{algebra structures on } \mathsf{H}(\mathsf{A})\} := \mathrm{MC}(\mathfrak{g})$

Gauge action

$$\begin{array}{ccc} \Gamma \times \mathrm{MC}(\mathfrak{g}) & \longrightarrow & \mathrm{MC}(\mathfrak{g}) \\ (\lambda, \varphi) & \longmapsto & \lambda \cdot \varphi := e^{\mathrm{ad}_\lambda}(\varphi) - \frac{e^{\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda) \end{array}$$

Proposition (Dotsenko – Shadrin – Vallette, 2016)

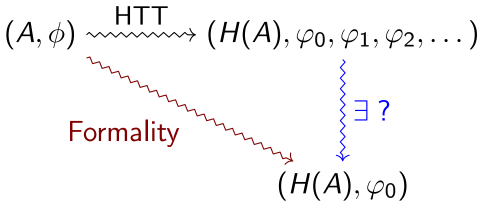
$$\exists\ \infty\text{-}\textit{quasi-isomorphism}\ \ (H(A), \varphi)\ \overset{\sim}{\rightsquigarrow}\ (H(A), \varphi_0)$$

$$\Longleftrightarrow$$

$$\exists\ \lambda \in \Gamma \textit{ such that } \lambda \cdot \varphi = \varphi_0$$

An equivalent characterization of formality

(A, ϕ): a $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract



Definition

- (A, ϕ) is gauge formal if $\exists\ \lambda \in \Gamma$ such that $\lambda \cdot \varphi = \varphi_0$
- (A, ϕ) is gauge n-formal if $\exists\ \lambda \in \Gamma$ such that

$$\lambda \cdot \varphi = (\varphi_0, 0, \dots, 0, \psi_{n+1}, \dots)\ .$$

Kaledin obstruction classes

Formal deformation

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots) \in \mathrm{MC}(\mathfrak{g})$$

A formal deformation of φ_0:

$$\Phi := \varphi_0 + \varphi_1 \hbar + \varphi_2 \hbar^2 + \cdots + \varphi_k h^k + \cdots$$

in the dg Lie algebra $\mathfrak{g}[\![\hbar]\!] := \mathfrak{g} \widehat{\otimes} R[\![\hbar]\!]$.

Remark

$\Phi \in \mathrm{MC}(\mathfrak{g}[\![\hbar]\!])$, i.e. $d(\Phi) + \frac{1}{2}[\Phi, \Phi] = 0$.

Proposition

$d^\Phi := d + [\Phi, -]$ *is a differential on* $\mathfrak{g}[\![\hbar]\!]$

Twisted dg Lie algebra:

$$\mathfrak{g}[\![\hbar]\!]^\Phi := (\mathfrak{g}[\![\hbar]\!], [-,-], d^\Phi)$$

The Kaledin classes

$$\partial_\hbar \Phi := \varphi_1 + 2\varphi_2\hbar + \cdots + k\varphi_k\hbar^{k-1} + \cdots \in \mathfrak{g}[\![\hbar]\!]$$

Lemma

$\partial_\hbar \Phi$ is a cycle in $\mathfrak{g}[\![\hbar]\!]^{\Phi}$, i.e. $d^{\Phi}(\partial_\hbar \Phi) = 0$.

Definition (Kaledin class)

The Kaledin class of $\varphi \in \mathrm{MC}(\mathfrak{g})$ is the homology class

$$K_\Phi := [\partial_\hbar \Phi] \in H_{-1}\left(\mathfrak{g}[\![\hbar]\!]^{\Phi}\right) .$$

Its n^{th}-truncated Kaledin class is

$$K_\Phi^n := \left[\varphi_1 + 2\varphi_2\hbar + \cdots + n\varphi_n\hbar^{n-1}\right] \in H_{-1}\left((\mathfrak{g}[\![\hbar]\!]/\hbar^n)^{\widetilde{\Phi}}\right) .$$

Kaledin class:

$$K_\Phi := \left[\varphi_1 + 2\varphi_2\hbar + 3\varphi_3\hbar^2 + \cdots\right] \in H_{-1}\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

n^{th}-truncated Kaledin class :

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Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

R : *$\mathbb{Q}$-algebra*

$\mathcal{P}$: *Koszul operad*

(A,ϕ) : *dg $\mathcal{P}$-algebra s.t. $H(A)$ is a homotopy retract*

- (A,ϕ) *is gauge formal* $\iff K_\Phi = 0$.
- (A,ϕ) *is gauge n-formal* $\iff K_\Phi^n = 0$.

Kaledin class:

$$K_\Phi := \left[\varphi_1 + 2\varphi_2\hbar + 3\varphi_3\hbar^2 + \cdots\right] \in H_{-1}\left(\mathfrak{g}[\![\hbar]\!]^{\Phi}\right)$$

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Theorem (E., 2023)

R : *commutative ground ring*

$\mathcal{P}$: *Koszul (pr)operad colored in groupoids*

(A, ϕ) : *dg $\mathcal{P}$-(al)gebra s.t. $H(A)$ is a homotopy retract*

- (A, ϕ) *is gauge formal* $\iff K_\Phi = 0$.
- (A, ϕ) *is gauge n-formal* $\iff K_\Phi^n = 0$.

Heuristic behind these obstruction classes

Gauge action: $\lambda \cdot \varphi := e^{\mathrm{ad}_\lambda}(\varphi) - \dfrac{e^{\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda)$

Vector field Υ_λ on $\mathrm{MC}(\mathfrak{g})$: $\forall\, \lambda \in \Gamma, \quad \Upsilon_\lambda(\varphi) := -d\lambda - [\varphi, \lambda]$

Associated flow:

$$\frac{d}{dt}\gamma_\lambda(t) = \Upsilon_\lambda\left(\gamma_\lambda(t)\right)$$

Integration of the flow starting at $\gamma_\lambda(0) = \varphi$:

$$\gamma_\lambda(t) = e^{t\mathrm{ad}_\lambda}(\varphi) - \frac{e^{t\mathrm{ad}_\lambda} - \mathrm{id}}{t\mathrm{ad}_\lambda}(td\lambda)$$

$$\boxed{\lambda \cdot \varphi = \psi \Longleftrightarrow \gamma_\lambda(1) = \psi}$$

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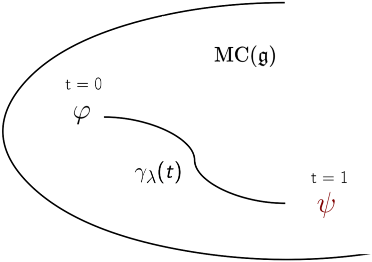
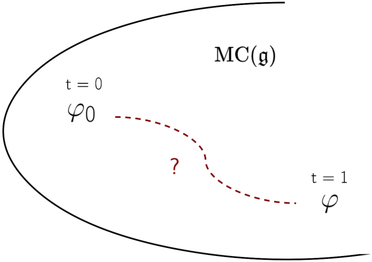


The case of formality



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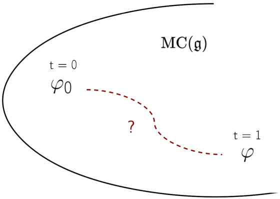

Attempt: "$\Phi(\hbar) = \varphi_0 + \varphi_1\hbar + \varphi_2\hbar^2 + \cdots$"

Does there exists $\lambda \in \Gamma$, such that $\Phi = \gamma_\lambda$?

$$\Phi = \gamma_\lambda \iff \partial_\hbar \Phi = \Upsilon_\lambda(\Phi) = -d\lambda - [\Phi, \lambda]$$

$$\iff \partial_\hbar \Phi = d^\Phi(-\lambda)$$

$$\iff \partial_\hbar \Phi \text{ is a boundary in } \mathfrak{g}[\![\hbar]\!]^\Phi$$

$$\iff K_\Phi = [\partial_\hbar \Phi] = 0$$

Formality criteria

Formality descent

(A, ϕ) : a dg $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract

$H_i(A)$: projective, finitely generated for all i.

S : faithfully flat commutative R-algebra.

Proposition (E., 2023)

(A, ϕ) is gauge n-formal $\iff$ $(A \otimes_R S, \phi \otimes 1)$ is gauge n-formal.

Proof.

$$\begin{array}{ccc} H_{-1}\left(\mathfrak{g}_{H(A)}[\![\hbar]\!]^{\Phi}\right) \otimes_{R[\![\hbar]\!]} S[\![\hbar]\!] & \cong & H_{-1}\left(\mathfrak{g}_{H(A\otimes_R S)}[\![\hbar]\!]^{\Phi\otimes 1}\right) \\ \Cup & & \Cup \\ K_{\Phi} \otimes 1 = 0 & \iff & K_{\Phi\otimes 1} = 0 \end{array}$$

□

Examples

- $C(\mathcal{D}_k; \mathbb{R})$ is formal $\iff$ $C(\mathcal{D}_k; \mathbb{Q})$ is formal [GSNPR, 2005]
- $\mathbb{Z}_{(\ell)} \subset \mathbb{Z}_\ell$

Intrinsic formality

A graded $\mathscr{P}$-algebra (H, φ_0) is intrinsically formal if every $\mathscr{P}$-algebra (A, ϕ) such that $(H(A), \varphi_0) = (H, \varphi_0)$ is itself gauge formal.

$\mathfrak{g}^{\varphi_0}$: $(\mathfrak{g}, [-,-], d + [\varphi_0, -])$

Proposition (E., 2023)

$$H_{-1}(\mathfrak{g}^{\varphi_0}) = 0 \implies (H, \varphi_0) \textit{ intrinsically formal.}$$

Proof.

For all (A, ϕ) such that $(H(A), \varphi_0) = (H, \varphi_0)$, then

$$K_\Phi = 0 \in H_{-1}\left(\mathfrak{g}[\![\hbar]\!]^{\Phi}\right) .$$

□

Previous works: [Hinich, 2003]

Tamarkin's proof of Kontsevich formality

k : a characteristic zero field

A : a polynomial algebra over k

Theorem (Hinich, 2003)

The shifted cohomological Hochschild complex $C(A;A)[1]$ is formal as a dg Lie algebra.

Proof.

- $\mathrm{Lie}[1] \subset \mathit{Gerst}$;
- $(HH^\bullet(A), \varphi_0)$ has a *Gerst*-algebra structure;
- $C(A;A)$ has a Gerst_∞-algebra structure inducing φ_0;
- $(HH^\bullet(A), \varphi_0)$ is intrinsically formal as a *Gerst*-algebra;

$\rightarrow H_{-1}(\mathfrak{g}^{\varphi_0}) = 0$, where $\mathfrak{g} = \mathrm{Hom}(\overline{\mathit{Gerst}}^{\text{i}}, \mathrm{End}_{HH^\bullet(A)})$.

The degree twisting

(A, ϕ) : a dg $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract

α : a unit in R.

σ_α : the degree twisting by α

$\rightarrow$ linear automorphism of $H(A)$ which acts via $\alpha^k \times$ on $H_k(A)$.

Theorem (Drummond-Cole – Horel, 2021)

Suppose that σ_α admits a lift, i.e. $\exists\, f \in \mathrm{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$.

- $\forall k,\ \alpha^k - 1 \in R^\times \implies (A, \phi)$ *is gauge formal.*
- $\forall k \leq n,\ \alpha^k - 1 \in R^\times \implies (A, \phi)$ *is gauge n-formal.*

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Heuristic :

→ Higher Massey products have to be compatible with the lift.

→ They intertwine multiplication by α^l with multiplication by α^k with $l \neq k$.

→ They have to vanish

Previous works: [DGMS, 1975], [Sullivan, 1977], [GSNPR, 2005]

Complement of subspace arrangements

X : a complement of a hyperplane arrangement over $\mathbb{C}$
$\rightarrow$ complement of a finite collection of affine hyperplanes in $\mathbb{A}^n_{\mathbb{C}}$.

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

s : order of q in $\mathbb{F}_\ell^\times$

Proposition (Cirici – Horel, 2022)

If X is defined over K, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_\ell)$ is gauge $(s-1)$-formal.

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Heuristic :

- $\rightarrow$ $C^\bullet(X_{an}, \mathbb{Z}_\ell) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ [Artin]
- $\rightarrow$ The action of a Frobenius on $H_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is σ_q, [Kim, 1994].

Formality descent $\implies$ $C^\bullet(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s-1)$-formal.

Automorphism lifts

(A, ϕ) : a dg $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract

Theorem (E., 2023)

Suppose that $\exists\, u \in \mathrm{Aut}(H(A), \varphi_0)$ *admitting a chain lift. Let* $\mathrm{Ad}_u : \mathrm{End}_{H(A)} \to \mathrm{End}_{H(A)}$ *s.t.* $\forall\, p \in \mathbb{N}$, *and* $\forall\, \psi \in \mathrm{End}_{H(A)}(p)$

$$\mathrm{Ad}_u(\psi) = u \circ \psi \circ (u^{-1})^{\otimes p} \ .$$

1. *If* $\mathrm{Ad}_u - \mathrm{id}$ *is invertible, then* (A, ϕ) *is gauge formal.*
2. *If* $\mathrm{Ad}_u - \mathrm{id}$ *is invertible on the elements of degree* k *for all* $k < n$, *then* (A, ϕ) *is gauge* n*-formal.*

Automorphism lifts

R : a characteristic zero field

(A, ϕ) : a dg $\mathscr{P}$-algebra s.t. $H(A)$ is a homotopy retract and finite dimensional.

Corollary

Suppose that there exists $u \in \mathrm{Aut}(H(A), \varphi_0)$ such that for all $k < n$, and all p-tuples $(k_1, \dots, k_p)$,

$$\mathrm{Spec}(u_{k_1+\dots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing \ ,$$

where $u_i := u_{|H_i(A)}$. If u admits a lift at the level of chains then (A, ϕ) is gauge n-formal.

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

X : a smooth proper K-scheme

Definition

$\alpha \in \overline{\mathbb{Q}}_\ell$ is a Weil number of weight n if

$$\forall\ \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2}\ .$$

Theorem (Deligne, 1974)

For all n, the eigenvalues of a Frobenius action on $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight n.

Corollary

For every smooth proper K-scheme X, $C^\bullet(X_{an}, \mathbb{Q}_\ell)$ is formal.

Proof.

- $C^\bullet(X_{an}, \mathbb{Q}_\ell) \xrightarrow{\sim} C^\bullet_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$
- Let u be the Frobenius action on $H^\bullet_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ and fix $\iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.
- For all $k \geq 1$, $(k_1, \dots, k_p)$ and $s := k_1 + \cdots + k_p$,

$$\begin{array}{ccc} \operatorname{Spec}(u_{s+k}) & \cap & \operatorname{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) \quad = \varnothing \,. \\ \Cup & & \Cup \\ \alpha & & \beta \\ |\iota(\alpha)| = q^{\frac{s+k}{2}} & > & |\iota(\beta)| = q^{\frac{s}{2}} \end{array}$$

□

Previous works: [Deligne, 1980], [GSNPR, 2005]

Thank you for your attention!