# Obstruction theory to formality and homotopy equivalences

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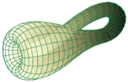
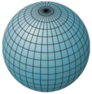
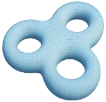
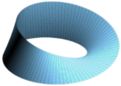
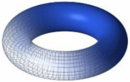
# Algebraic topology

$\rightarrow$ Henri Poincaré (1854–1912)

Goal: Classify topological spaces up to homotopy equivalences $\sim$

$=$ *up to continuous deformation*

Method: Use invariants $=$ algebraic objects ($n \in \mathbb{N}$, groups, $\ldots$) that remain unchanged under homotopy equivalence.

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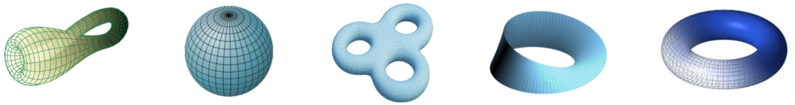
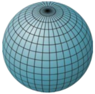
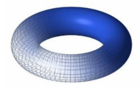


Method: Use algebraic invariants

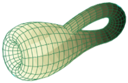
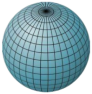
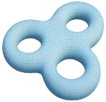
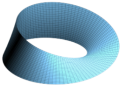
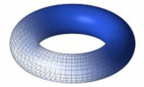
- Genus = number of holes

 $\not\sim$ 

$\text{genus} = 0$ $\qquad$ $\text{genus} = 1$

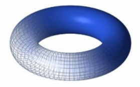
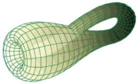
# Algebraic topology

Goal: Classify topological spaces up to homotopy equivalences $\sim$

Method: Use algebraic invariants

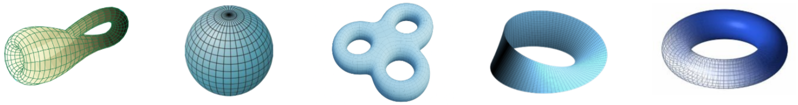
- Genus $=$ number of holes
- Cohomology groups $H^\bullet$

 $\not\sim$ 

$H^1 = \mathbb{Z} \oplus \mathbb{Z}$ $\qquad$ $H^1 = \mathbb{Z}$

# Algebraic topology

Goal: Classify topological spaces up to homotopy equivalences $\sim$



Method: Use algebraic invariants

- Genus = number of holes
- Cohomology groups $H^\bullet$
- Cohomology ring $H^\bullet$ with the cup product $\cup$

$$\mathbb{CP}^2 \qquad \not\sim \qquad \mathbb{S}^2 \vee \mathbb{S}^4$$

$$H^\bullet \cong \mathbb{Z}[x]/(x^3) \qquad H^\bullet \cong \mathbb{Z}[x,y]/(x^2,y^2,xy)$$

$$\deg(x) = 2 \qquad \deg(x) = 2,\ \deg(y) = 4$$

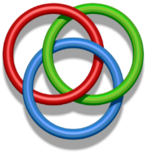
# Algebraic topology

Goal: Classify topological spaces up to homotopy equivalences $\sim$

Method: Use algebraic invariants

- Genus = number of holes
- Cohomology groups $H^\bullet$
- Cohomology ring $H^\bullet$ with the cup product $\cup$
- Massey products = $n$-ary operations generalizing $\cup$

$\not\sim$



# Algebraic topology

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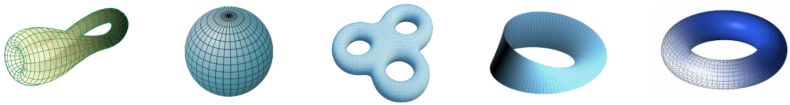

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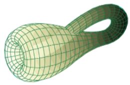
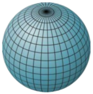
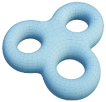
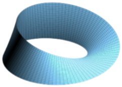
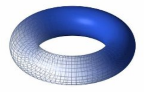
    

Method: Use algebraic invariants

- Genus = number of holes
- Cohomology groups $H^\bullet$
- Cohomology ring $H^\bullet$ with the cup product $\cup$
- Massey products = $n$-ary operations generalizing $\cup$

$\Longrightarrow$ These invariants are not faithful !

# Faithful invariants

$X$ : a topological space (simply connected and of finite type)

**Theorem (Mandell, 2005)**

*The $E_\infty$-algebra structure on $C^\bullet_{\mathrm{sing}}(X;\mathbb{Z})$ is a faithful invariant.*

# Faithful invariants

$X$ : a topological space (simply connected and of finite type)

**Theorem (Mandell, 2005)**

*The $E_\infty$-algebra structure on $C^\bullet_{\mathrm{sing}}(X;\mathbb{Z})$ is a faithful invariant.*

Rational homotopy type : class of $X$ up to maps inducing isomorphisms in rational cohomology.

**Theorem (Sullivan, 1977)**

*The commutative algebra of polynomial forms $\mathcal{A}^\bullet_{\mathrm{PL}}(X)$ is a faithful invariant of the rational homotopy type.*

# The notion of formality

# Formal topological spaces

### Definition

A topological space $X$ is formal if there exists a zig-zag

$$\mathcal{A}^{\bullet}_{\mathrm{PL}}(X) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^{\bullet}_{\mathrm{sing}}(X;\mathbb{Q})$$

of quasi-isomorphisms of differential graded (dg) commutative algebras, i.e. morphisms inducing isomorphisms in cohomology.

### Remark

$X$ formal $\implies$ The cohomology ring $H^{\bullet}_{\mathrm{sing}}(X,\mathbb{Q})$ completely determines the rational homotopy type of $X$.

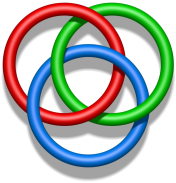
## Examples

→ Formal spaces

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [Deligne–Griffiths–Morgan–Sullivan, '75]

→ Nonformal spaces

- The complement of the Borromean rings



# Formality over any coefficient ring

### Theorem (Saleh, 2017)

*A space $X$ is formal if and only if if there exists a zig-zag of quasi-isomorphisms of dg associative algebras*

$$C^\bullet_{\text{sing}}(X;\mathbb{Q}) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet_{\text{sing}}(X;\mathbb{Q}) .$$

$R$ : commutative ground ring

### Definition

A topological space $X$ is formal if there exists a zig-zag of quasi-isomorphisms of dg associative algebras

$$C^\bullet_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet_{\text{sing}}(X;R) .$$

# Formality of an algebraic structure

$A$ : cochain complex over $R$

$\mathscr{P}$ : colored operad or properad

$\phi : \mathscr{P} \to \mathrm{End}_A$ : a dg $\mathscr{P}$-algebra structure

**Definition**

The dg $\mathscr{P}$-algebra $(A, \phi)$ is formal if there exists a zig-zag

$$(A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_1) \ ,$$

where $\varphi_1$ is the canonical $\mathscr{P}$-algebra structure on $H(A)$.

**Remark**

$X$ is formal if $(C^\bullet_{\mathrm{sing}}(X; R), \cup)$ is formal as dg associative algebra

# Examples

1. Differential geometry [Deligne–Griffiths–Morgan–Sullivan, 1975]
   Compact Kähler manifolds

2. Mathematical physics [Kontsevich's quantization theorem, 1997]
   Hochschild complex of a polynomial algebra

3. Algebraic topology [Kontsevich, 1999]
   The chains over the little $k$-disks operad

4. Representation theory [Riche–Soergel–Williamson, 2014]
   The extensions of parity sheaves on the flag variety

5. Algebraic geometry [Cirici–Horel, 2018, Drummond-Cole–Horel, 2021]
   Étale cohomology of complements of hyperplane arrangements with coefficients in $\mathbb{Z}_\ell$

# Purity implies formality

$(A, \phi)$ : dg $\mathscr{P}$-algebra encoded by an operad $\mathscr{P}$

$\alpha$ : unit of infinite order in $R$

$\sigma_\alpha$ : the degree twisting by $\alpha$ = automorphism of $(H(A), \varphi_1)$ which acts via $\alpha^k \times$ on $H^k(A)$.

**Theorem**

*If $\sigma_\alpha$ admits a chain-level lift, i.e. $\exists\, f \in \mathrm{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$, then $(A, \phi)$ is formal.*

→ Deligne–Griffiths–Morgan–Sullivan [1975]

→ Sullivan [1977]

→ Guillén Santos–Navarro–Pascual–Roig [2005]

→ Cirici–Horel [2022]

# Example

5. Algebraic geometry [Cirici-Horel, 2018, Drummond-Cole–Horel, 2021]

   $X$ : a complement of a hyperplane arrangement over $\mathbb{C}$
   defined over a finite extension $K$ of $\mathbb{Q}_p$.

   $\ell$ : a prime number different from $p$.

   $\rightarrow C^\bullet_{\mathrm{sing}}(X_{an}, \mathbb{Z}_\ell) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ [Artin].

   $\rightarrow$ A Frobenius action on $H^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is $\sigma_q$ [Kim, 1994].

# Questions

- Can we descend these results to other coefficient rings? (e.g. $\mathbb{Z}_{(\ell)}$)
- Does the degree twisting criteria hold for other types of algebras? (e.g. Hopf algebras, Lie bialgebras,...)
- Is the degree twisting the only homology automorphism satisfying this property?
- Can we incorporate all the aformentionned examples into a single theory?

# Outline

## Kaledin classes

- A faithful obstruction theory to the formality over any ring

## Formality criteria

- Formality descent
- Intrinsic formality
- Automorphism lifts

## Beyond formality

- Generalizing Kaledin classes to detect homotopy equivalences
- Models for highly connected manifolds

# Kaledin classes

# Gauge formality

$A$ : cochain complex over $R$

$\mathscr{C}$ : reduced weight-graded differential graded coproperad

$(A, \phi)$ : dg $\Omega\mathscr{C}$-algebra

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$A$ : cochain complex over $R$

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$(A, \phi)$ : dg $\Omega\mathscr{C}$-algebra

**Theorem (Hoffbeck–Leray–Vallette, 2025)**

*$R$ is a characteristic zero field and $(B, \phi')$ a dg $\Omega\mathscr{C}$-algebra*

$$\underbrace{\exists\, (A,\phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B,\phi')}_{\textit{zig-zag of quasi-isos of } \Omega\mathscr{C}\textit{-algebras}} \quad \Longleftrightarrow \quad \underbrace{\exists\, (A,\phi) \overset{\sim}{\rightsquigarrow} (B,\phi')}_{\infty\textit{-quasi-iso}}$$

**Definition**

The $\Omega\mathscr{C}$-algebra $(A, \phi)$ is *gauge formal* if $\exists\, (A, \phi) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_1)$.

# Homotopy transfer theorem

## Theorem

*Suppose that $H(A)$ is a contraction of $A$ if $\mathscr{C}$ is*

1. *a symmetric cooperad then $R$ is a $\mathbb{Q}$-algebra* [Berglund, 14];
2. *a coproperad then $R$ is a characteristic zero field* [HLV, 20];
3. *a symmetric quasi-planar cooperad then $R$ is a field* [GRiL, 23].

*For any $\Omega\mathscr{C}$-algebra structure $(A, \phi)$, there exists an $\Omega\mathscr{C}$-algebra $(H(A), \varphi)$ and an $\infty$-quasi-isomorphism:*

$$(A, \phi) \underset{\sim}{\overset{p_\infty}{\rightsquigarrow}} (H(A), \varphi)$$

# Homotopy transfer theorem

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$$\begin{tikzcd}
(A, \phi) \arrow[rr, squiggly, "p_\infty", "\sim"'] \arrow[drr, squiggly, "\sim", "\text{gauge formality}"'] & & (H(A), \varphi) \\
& & (H(A), \varphi_1)
\end{tikzcd}$$

# Homotopy transfer theorem

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*For any $\Omega\mathscr{C}$-algebra structure $(A, \phi)$, there exists an $\Omega\mathscr{C}$-algebra $(H(A), \varphi)$ and an $\infty$-quasi-isomorphism:*

$$
\begin{tikzcd}
(A,\phi) \arrow[r, rightsquigarrow, "p_\infty", "\sim"'] \arrow[dr, rightsquigarrow, "\sim", "\text{gauge formality}"'] & (H(A),\varphi) \arrow[d, rightsquigarrow, "\wr\ \exists\ ?"] \\
& (H(A),\varphi_1)
\end{tikzcd}
$$

$$\{\Omega\mathscr{C}\text{-algebra structures on } \mathrm{H(A)}\} := \mathrm{MC}(\mathfrak{g})$$

The convolution dg Lie algebra associated to $H(A)$:

$$\mathfrak{g} := \left(\mathrm{Hom}\left(\overline{\mathscr{C}}, \mathrm{End}_{H(A)}\right), [-,-], d\right)$$

→ $\mathrm{Hom}(\overline{\mathscr{C}}, \mathsf{End}_{H(A)}) := \prod_{n\geq 0} \mathrm{Hom}\left(\overline{\mathscr{C}}(n), \mathsf{End}_{H(A)}(n)\right)$

→ $d(\varphi) := -(-1)^{|\varphi|}\varphi \circ d_{\overline{\mathscr{C}}}$

→ $\varphi \star \psi := \overline{\mathscr{C}} \xrightarrow{\Delta_{(1,1)}} \overline{\mathscr{C}} \underset{(1,1)}{\boxtimes} \overline{\mathscr{C}} \xrightarrow{\varphi \underset{(1,1)}{\boxtimes} \psi} \mathsf{End}_{H(A)} \underset{(1,1)}{\boxtimes} \mathsf{End}_{H(A)} \xrightarrow{\gamma_{(1,1)}} \mathsf{End}_{H(A)}$

→ $[\varphi, \psi] := \varphi \star \psi - (-1)^{|\varphi||\psi|}\psi \star \varphi$

Every $\varphi \in \mathrm{Hom}\left(\overline{\mathscr{C}}, \mathrm{End}_{H(A)}\right)$ decomposes as

$$\varphi = (\varphi_1, \varphi_2, \varphi_3, \dots)$$

where $\varphi_k$ is the restriction $\varphi_k : \mathscr{C}^{(k)} \Longrightarrow \mathsf{End}_{H(A)}$.

# An equivalent characterization of formality

$(A,\phi)$: an $\Omega\mathscr{C}$-algebra that admits a transferred structure

$$\begin{tikzcd}
(A,\phi) \arrow[r, rightsquigarrow, "\mathrm{HTT}", "\sim"'] \arrow[dr, rightsquigarrow, "\sim", "\text{gauge formality}"'] & (H(A),\varphi_1,\varphi_2,\varphi_3,\dots) \arrow[d, rightsquigarrow, "\exists\,?", "\wr"'] \\
& (H(A),\varphi_1)
\end{tikzcd}$$

Higher Massey products

$\Longrightarrow$ If the higher Massey products vanish, then $(A,d,\phi)$ is formal.

## Definition

- $(A,\phi)$ is gauge formal if $\exists\,(H(A),\varphi_1,\varphi_2,\dots) \overset{\sim}{\rightsquigarrow} (H(A),\varphi_*)$.
- $(A,\phi)$ is gauge $n$-formal if

$$\exists\,(H(A),\varphi_*,\varphi_3,\varphi_4\dots) \overset{\sim}{\rightsquigarrow} (H(A),\varphi_*,0,\dots,0,\varphi'_{n+1},\dots)\,.$$

# Formal deformation

$$\varphi = (\varphi_1, \varphi_2, \varphi_3, \dots) \in \mathrm{MC}(\mathfrak{g})$$

A formal deformation of $\varphi_1$:

$$\Phi := \varphi_1 + \varphi_2 \hbar + \varphi_3 \hbar^2 + \cdots + \varphi_{k+1} h^k + \cdots$$

in the dg Lie algebra $\mathfrak{g}[\![\hbar]\!] := \mathfrak{g} \widehat{\otimes} R[\![\hbar]\!]$.

## Remark

$\Phi \in \mathrm{MC}(\mathfrak{g}[\![\hbar]\!])$, i.e. $d(\Phi) + \frac{1}{2}[\Phi, \Phi] = 0$.

## Proposition

$d^{\Phi} := d + [\Phi, -]$ *is a differential on* $\mathfrak{g}[\![\hbar]\!]$

Twisted dg Lie algebra:

$$\mathfrak{g}[\![\hbar]\!]^{\Phi} := (\mathfrak{g}[\![\hbar]\!], [-,-], d^{\Phi})$$

# The Kaledin classes

$$\partial_\hbar \Phi := \varphi_2 + 2\varphi_3\hbar + \cdots + k\varphi_{k+1}\hbar^{k-1} + \cdots \in \mathfrak{g}[\![\hbar]\!]$$

## Lemma

*$\partial_\hbar \Phi$ is a cycle in $\mathfrak{g}[\![\hbar]\!]^\Phi$, i.e. $d^\Phi(\partial_\hbar \Phi) = 0$.*

# The Kaledin classes

$$\partial_\hbar \Phi := \varphi_2 + 2\varphi_3 \hbar + \cdots + k\varphi_{k+1}\hbar^{k-1} + \cdots \in \mathfrak{g}[\![\hbar]\!]$$

## Lemma

*$\partial_\hbar \Phi$ is a cycle in $\mathfrak{g}[\![\hbar]\!]^\Phi$, i.e. $d^\Phi(\partial_\hbar \Phi) = 0$.*

## Definition

Kaledin class:

$$K_\Phi := [\partial_\hbar \Phi] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right) .$$

$n^{th}$-truncated Kaledin class:

$$K_\Phi^n := \left[\varphi_2 + 2\varphi_3\hbar + \cdots + n\varphi_{n+1}\hbar^{n-1}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^n\right)^{\widetilde{\Phi}}\right) .$$

Kaledin class:

$$K_\Phi := \left[\varphi_2 + 2\varphi_3\hbar + 3\varphi_4\hbar^2 + \cdots\right] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

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## Theorem (E., 2024)

*Let $(A, \phi)$ be an $\Omega\mathscr{C}$-algebra that admits a transferred structure.*

- *If $R$ is a $\mathbb{Q}$-algebra, $(A, \phi)$ is gauge formal $\iff K_\Phi = 0$.*
- *If $n!$ is invertible in $R$, $(A, \phi)$ is gauge $n$-formal $\iff K_\Phi^n = 0$.*

Previous works: [Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019]

# Formality criteria

# Formality descent

$(A, \phi)$ : an dg $\Omega\mathscr{C}$-algebra that admits a transferred structure

$H^i(A)$ : projective, finitely generated for all $i$.

$S$ : faithfully flat commutative $R$-algebra.

**Theorem (E., 2024)**

*$(A, \phi)$ is gauge $n$-formal $\iff$ $(A \otimes_R S, \phi \otimes 1)$ is gauge $n$-formal.*

**Examples**

- $D_k$ : little $k$-disks operads [Guillén Santos–Navarro–Pascual–Roig]
  $C(\mathcal{D}_k; \mathbb{R})$ is formal $\iff$ $C(\mathcal{D}_k; \mathbb{Q})$ is formal
- $\mathbb{Z}_{(\ell)} \subset \mathbb{Z}_\ell$

# Complement of hyperplane arrangements

$X$ : a complement of an hyperplane arrangement over $\mathbb{C}$
$\to$ complement of a finite collection of affine hyperplanes in $\mathbb{A}^n_{\mathbb{C}}$.

$K$ : a finite extension of $\mathbb{Q}_p$

$q$ : order of the residue field of the ring of integers of $K$

$\ell$ : a prime number different from $p$

$s$ : order of $q$ in $\mathbb{F}_\ell^\times$

**Theorem (Cirici–Horel, 2018, Dummond-Cole–Horel, 2021)**
*If $X$ is defined over $K$ then $C^\bullet(X_{an}, \mathbb{Z}_\ell)$ is gauge $(s-1)$-formal.*

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**Theorem (E., 2024)**

*If $X$ is defined over $K$ then $C^\bullet(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s-1)$-formal.*

**Proof.**

Formality descent $\square$

# Intrinsic formality

A graded $\Omega\mathscr{C}$-algebra $(H, \varphi_*)$ is intrinsically formal if every dg $\Omega\mathscr{C}$-algebra $(A, \phi)$ such that $(H(A), \varphi_1) \cong (H, \varphi_*)$ is gauge formal.

$\mathfrak{g}^{\varphi_*}$ : $(\mathfrak{g}, [-,-], d + [\varphi_*, -])$

**Proposition (E., 2024)**

$$H^1(\mathfrak{g}^{\varphi_*}) = 0 \implies (H, \varphi_*) \text{ intrinsically formal.}$$

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## Proposition (E., 2024)

$$H^1(\mathfrak{g}^{\varphi_*}) = 0 \implies (H, \varphi_*) \text{ intrinsically formal.}$$

## Proof.

For all $(A, \phi)$ such that $(H(A), \varphi_1) = (H, \varphi_*)$, then

$$K_\Phi = 0 \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right) . \qquad \square$$

Previous works: [Hinich, 2003] → Tamarkin's proof of Kontsevich formality

# Properadic coformality of spheres

**Example (Kontsevich–Takeda–Vlassopoulos, 2021)**

$A = C_*(\Omega S^n; R)$ has an pre-Calabi-Yau (or $V_\infty$-algebra) structure

$$\phi = \underbrace{m_{(1)}}_{A_\infty\text{-alg}} + \underbrace{m_{(2)}}_{\substack{\text{Poisson bivector}\\ \text{up to homotopy}}} + m_{(3)} + \cdots$$

where $m_{(\ell)}$ is a cyclically anti-symmetric collection of maps

$$m_{(\ell)}^{k_1,\ldots,k_\ell} : sA^{k_1} \otimes \cdots \otimes sA^{k_\ell} \to A^\ell \ .$$

$\Longrightarrow$ encodes Poincaré duality.

# Properadic coformality of spheres

The pre-CY algebra on $C_*(\Omega S^n; R)$ has vanishing copairing:

$$m^{0,0}_{(2)} = 0$$

$\rightarrow$ always the case where $A$ is connective and $n \geqslant 1$.

**Theorem (E.-Takeda, 2025)**

*If $R$ is a $\mathbb{Q}$-algebra, $(H(A), \varphi_1)$ is intrinsically formal as an $n$-pre-CY algebra structure with vanishing copairing.*

# Automorphism lifts

$R$ : a field

$(A, \phi)$ : a dg $\Omega\mathscr{C}$-algebra that admits a transferred structure

$H^i(A)$ : projective, finitely generated for all $i$.
$H(A)$ is finite dimensional.

## Corollary (E., 2024)

*Suppose that there exists $u \in \mathrm{Aut}(H(A), \varphi_1)$ such that for all $k < n$, and all $p$-tuples $(k_1, \ldots, k_p)$,*

$$\mathrm{Spec}(u_{k_1+\cdots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing \ ,$$

*where $u_i := u_{|H^i(A)}$. If $u$ admits a lift at the level of chains then $(A, \phi)$ is gauge $n$-formal.*

# Frobenius & Weil numbers

$K$ : a finite extension of $\mathbb{Q}_p$

$q$ : order of the residue field of the ring of integers $\mathcal{O}_K$

$\ell$ : a prime number different from $p$

$X$ : a smooth proper $K$-scheme

**Definition**

$\alpha \in \overline{\mathbb{Q}}_\ell$ is a Weil number of weight $n$ if

$$\forall\ \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2}\ .$$

**Theorem (Deligne, 1974)**

*For all $n$, the eigenvalues of a Frobenius action on $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight $n$.*

Let $\mathrm{Sch}_K$ be the category of smooth and proper schemes over $K$ of *good reduction*, i.e. for which there exists a smooth and proper model over $\mathcal{O}_K$.

### Theorem (E., 2024)

*Let $\mathbb{V}$ be a groupoid and let $\mathscr{P}$ be a $\mathbb{V}$-colored operad in sets. Let $X$ be a $\mathscr{P}$-algebra in $\mathrm{Sch}_K$. The dg $\mathscr{P}$-algebra $C_\bullet(X_{\mathrm{an}}, \mathbb{Q})$ is formal.*

### Example (Guillén Santos–Navarro–Pascual–Roig, 2005)

Let $\overline{\mathcal{M}}$ the cyclic operad of moduli spaces of stable algebraic curves of genus zero. The cyclic operad $C_\bullet(\overline{\mathcal{M}}_{an}; \mathbb{Q})$ is formal.

# Beyond formality

# Homotopy equivalences between algebraic structure

## Definition

The dg $\Omega\mathscr{C}$-algebras $(A,\phi)$ and $(B,\psi)$ are

- homotopy equivalent

$$\exists\,(A,\phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B,\psi)$$

- gauge homotopy equivalent if $\exists\,(A,\phi) \overset{\sim}{\rightsquigarrow} (B,\psi)$.
- gauge $n$-homotopy equivalent if $(A,\phi)$ is gauge homotopy equivalent to an $\Omega\mathscr{C}$-algebra $(B,\varphi)$ such that

$$\varphi - \psi \in \mathcal{F}^{n+1}\mathfrak{g}\ .$$

## Example

$(A,\phi)$ is formal $\iff$ it is homotopy equivalent to $(H(A),\varphi_1)$

# Obstruction sequences to homotopy equivalences

Let $(A, \phi)$ and $(B, \psi)$ be two $\Omega\mathscr{C}$-algebras admitting transferred structures and such that $H(A) \cong H(B)$.

$\rightarrow$ obstruction sequence $(\vartheta_k)_{1 \leqslant k \leqslant m}$ which is either

- an infinite sequence of vanishing classes, when $m = \infty$ ;
- a finite sequence of trivial classes that ends on $\vartheta_m \neq 0$ .

## Proposition

*The index $m \in [\![1, \infty]\!]$ only depends on $\varphi$ and $\psi$: this is their homotopy equivalence degree.*

## Theorem (E., 2025)

*Let $(A, \phi)$ and $(B, \psi)$ be two $\Omega\mathscr{C}$-algebras admitting transferred structures and such that $H(A) \cong H(B)$.*

1. *If $R$ is a $\mathbb{Q}$-algebra, the algebras are gauge $k$-homotopy equivalent for all $k$ if and only if $m = \infty$ .*
2. *If $m!$ is invertible in $R$, the algebras are gauge $(m-1)$-homotopy equivalent but not gauge $m$-homotopy equivalent if and only if $m \in \mathbb{N}$ .*

# Minimal model on highly connected variety

## Theorem (E., 2025)

*Let $\mathbb{K}$ be a field. Let $M^d$ be a compact $k$-connected oriented $C^\infty$-manifold where $d$ is smaller than $(\ell+1)k+2$. The algebra*

$$C^\bullet_{\mathrm{sing}}(M,\mathbb{K})$$

*is homotopy equivalent to an $A_\infty$-algebra $(H^\bullet_{\mathrm{sing}}(M,\mathbb{K}),\varphi)$, with $\varphi_n = 0$ for $n \geqslant \ell$.*

Thank you for your attention!