Kaledin classes & formality criteria

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   - Construction of an obstruction theory to formality over any coefficient ring.

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   - Construction of an obstruction theory to formality over any coefficient ring.

4. Formality criteria
   - Formality descent with torsion coefficients
   - Intrinsic formality criterium
   - Degree twisting & automorphisms lifts

The notion of formality

Formal topological spaces

R : commutative ground ring

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Definition

A topological space X is formal if there exists a zig-zag of quasi-isomorphisms of differential graded associative algebras,

$$C^{\bullet}_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^{\bullet}_{\text{sing}}(X;R) .$$

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$\rightarrow$ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal $\implies$ $H^\bullet(X,\mathbb{Q})$ completely determines the rational homotopy type of X.

Examples

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- Spheres, complex projective spaces, Lie groups

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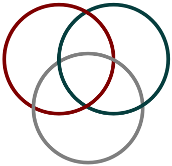
→ Nonformal spaces

- The complement of the Borromean rings



Formality of an algebraic structure

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The dg algebra (A, ϕ) is formal if

$$\exists\ (A, \phi) \overset{\sim}{\longleftarrow} \cdot \overset{\sim}{\longrightarrow} \cdots \overset{\sim}{\longleftarrow} \cdot \overset{\sim}{\longrightarrow} (H(A), \varphi_*)\ ,$$

where φ_* denotes the induced structure on $H(A)$.

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Examples

- X formal $= (C^\bullet_{\mathrm{sing}}(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Higher structures

Homotopy retracts

Homotopy retracts

Definition

(W, d_W) is a homotopy retract of (V, d_V) if there are maps

$$h \circlearrowright (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, d_W)$$

where $\mathrm{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism .

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Proposition

If R is a field, the cohomology of any cochain complex is a homotopy retract:

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0) \ .$$

Transfer of algebraic structure

Transfer of algebraic structure

(A, d_A, ϕ) : a dga algebra and a homotopy retraction:

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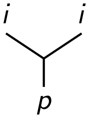
$$h \circlearrowright (A, d_A, \phi) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

$\rightarrow$ Transferred product: $\varphi_2 := p \circ \phi \circ i^{\otimes 2} : H^{\otimes 2} \to H$



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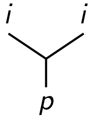
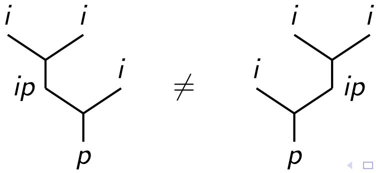
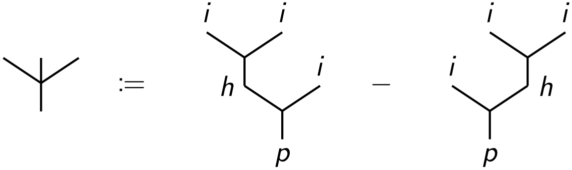


Not associative in general!



→ Consider $\varphi_3 : H^{\otimes 3} \to H$



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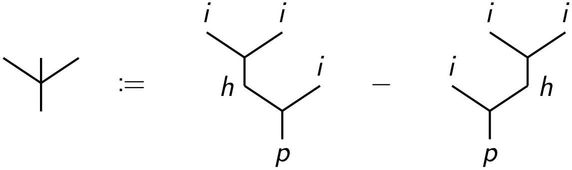
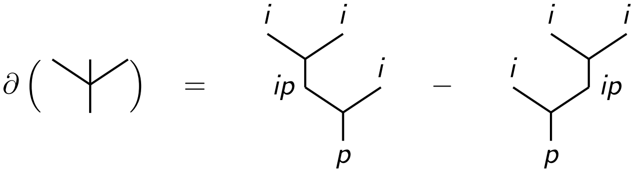


$\rightarrow$ In $\mathrm{Hom}(H^{\otimes 3}, H)$:



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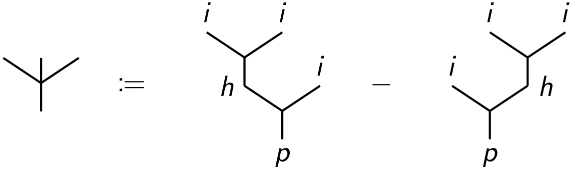

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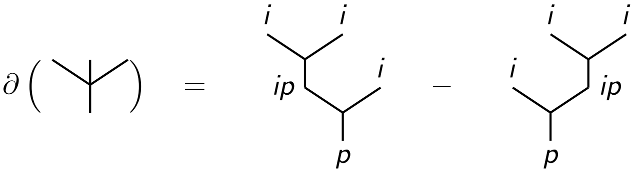
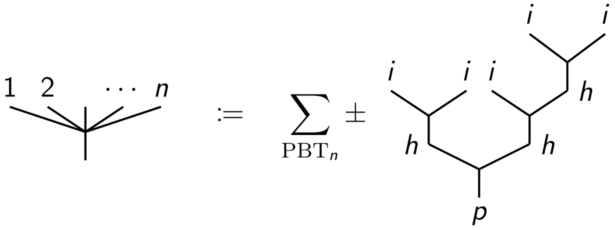


→ φ_2 is associative up to the homotopy φ_3.

$\rightarrow \varphi_n : H^{\otimes n} \rightarrow H$, for all $n \geqslant 2$



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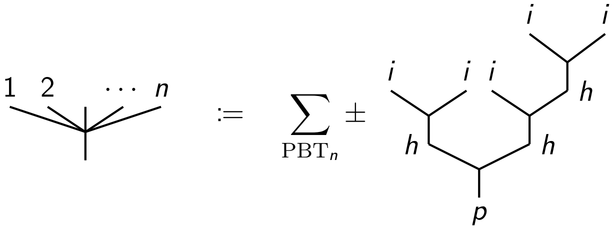
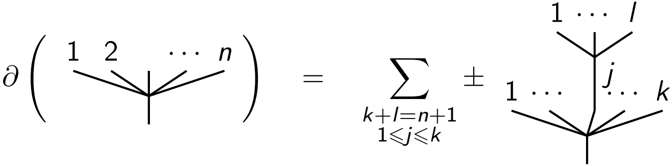




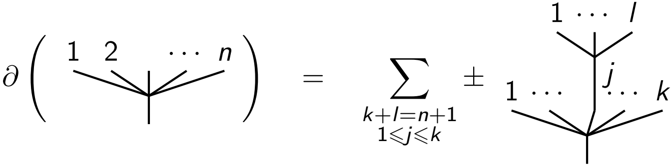
Homotopy associative algebras

Definition (Stasheff, 1963)

A_∞-algebra: a cochain complex H with a collection of maps

$$\varphi_n : H^{\otimes n} \to H$$

of degree $2 - n$, for all $n \geqslant 2$, which satisfy the relations



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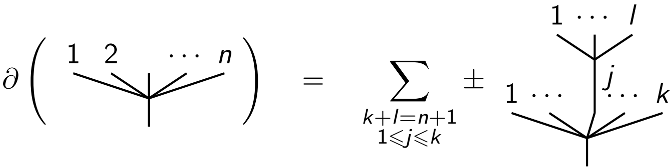

Examples

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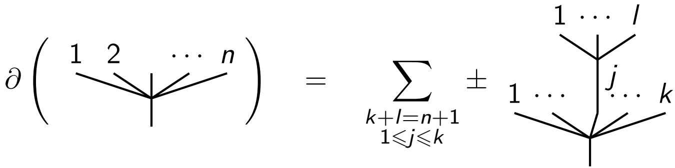

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- $(H, d_H, \varphi_2, \varphi_3, \dots)$

Homotopy transfer theorem

Theorem (Kadeishvili, 1982)

Given a dga algebra (A, d_A, ϕ) and a homotopy retract

$$h \circlearrowright (A, d_A, \phi) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

there exists an A_∞-algebra structure on H such that p (and i) extend to A_∞-quasi-isomorphisms:

$$(A, d_A, \phi) \rightsquigarrow (H, d_H, \varphi_2, \varphi_3, \varphi_4, \dots)$$

Homotopy morphisms

$(A, d_A, \phi_2, \dots),\ (H, d_H, \varphi_2, \dots)\ :\ A_\infty$-algebras

Homotopy morphisms

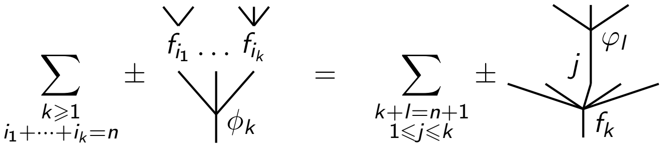
$(A, d_A, \phi_2, \dots), (H, d_H, \varphi_2, \dots)$: A_∞-algebras

Definition

A_∞-morphism $f : A \rightsquigarrow H$ is a collection of linear maps

$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geqslant 1 ,$$

of degree $1 - n$, which satisfy the relations



where $\varphi_1 = d_H$ and $\phi_1 = d_A$.

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Proposition (R is a field)

quasi-isos of associative algebras $\qquad$ A_∞*-quasi-iso*

$$\exists\,(A,\phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B,\phi') \iff \exists\,(A,\phi) \overset{\sim}{\rightsquigarrow} (B,\phi')$$

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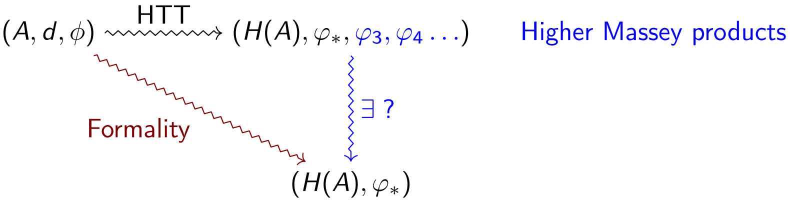
Corollary

A dga algebra (A,ϕ) is formal if and only if

$$\exists\, (A,\phi) \overset{\sim}{\rightsquigarrow} (H(A),\varphi_*)\ .$$

An equivalent characterization of formality

(A, d, ϕ) a dga algebra such that $H(A)$ is a homotopy retract



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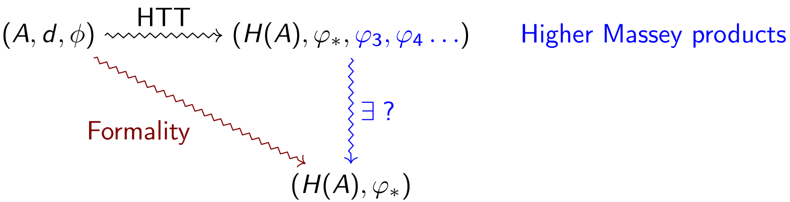

$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

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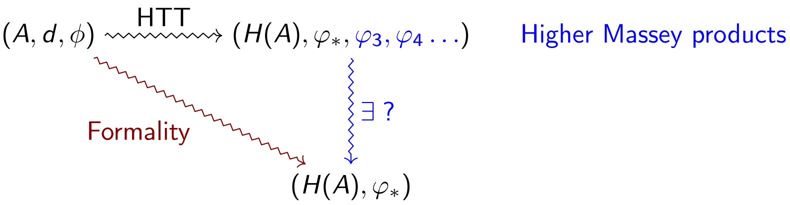

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Definition

- (A, d, ϕ) is gauge formal if $\exists$ $(H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*)$.

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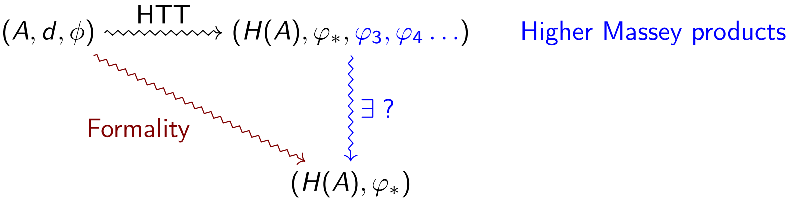

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- (A, d, ϕ) is gauge n-formal if

$$\exists\ (H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*, 0, \ldots, 0, \varphi'_{n+1}, \ldots) .$$

Kaledin classes

Hochschild complex

Transfered structure: $(H(A), \varphi_*, \varphi_3, \varphi_4, \dots)$

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Hochschild cochain complex:

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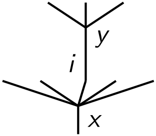
Lie bracket : $[x, y] := x \star y - (-1)^{|x||y|} y \star x$

$$x \star y := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)}$$



for $x \in Hom(H(A)^{\otimes n}, H(A))$ and $y \in Hom(H(A)^{\otimes m}, H(A))$.

A formal deformation

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Twisted dg Lie algebra:

$$\mathfrak{g}[\![\hbar]\!]^\Phi := (\mathfrak{g}[\![\hbar]\!], [-,-], \mathrm{ad}_\Phi)$$

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n^{th}-truncated Kaledin class :

$$K_\Phi^n := \left[\varphi_3 + 2\varphi_4 \hbar + \cdots + (n-2)\varphi_n \hbar^{n-3}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^{n-2}\right)^{\widetilde{\Phi}}\right) \ .$$

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Theorem ([Kaledin, 2007], [Lunts, 2007])

R : $\mathbb{Q}$-algebra

(A, ϕ) : dg associative algebra, $H(A)$ is a homotopy retract

- *(A, ϕ) is gauge formal $\Longleftrightarrow K_{\Phi} = 0$.*
- *(A, ϕ) is gauge n-formal $\Longleftrightarrow K_{\Phi}^{n} = 0$.*

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n^{th}-truncated Kaledin class :

$$K^n_\Phi := \left[\varphi_3 + 2\varphi_4\hbar + \cdots + (n-2)\varphi_n\hbar^{n-3}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^{n-2}\right)^{\widetilde{\Phi}}\right)$$

Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

R : *$\mathbb{Q}$-algebra*

$\mathcal{P}$: *Koszul operad*

(A, ϕ) : *dg $\mathcal{P}$-algebra such that $H(A)$ is a homotopy retract*

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Theorem (E., 2023)

R : *commutative ring*

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- $(HH^\bullet(A), \varphi_*)$ is intrinsically formal as a *Gerst*-algebra;
  $\rightarrow H^1(\mathfrak{g}^{\varphi_*}) = 0$, where $\mathfrak{g} = \mathrm{Hom}(\overline{\mathit{Gerst}}, \mathrm{End}_{HH^\bullet(A)})$.

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Theorem (Drummond-Cole – Horel, 2021)

Suppose that σ_α admits a lift, i.e. $\exists\ f \in \mathrm{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$.

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Previous works: [DGMS, 1975], [Sullivan, 1977], [GSNPR, 2005]

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If X is defined over K, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_\ell)$ is gauge $(s-1)$-formal.

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- $\rightarrow$ $C^\bullet(X_{an}, \mathbb{Z}_\ell) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ [Artin]
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Formality descent $\implies$ $C^\bullet(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s-1)$-formal.

Automorphism lifts

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Theorem (E., 2023)

Suppose that $\exists\, u \in \mathrm{Aut}(H(A), \varphi_*)$ *admitting a cochain lift. Let*

$$\mathrm{Ad}_u : \mathrm{End}_{H(A)} \to \mathrm{End}_{H(A)}$$

s.t. $\forall\, p \in \mathbb{N}$, *and* $\forall\, \psi \in \mathrm{End}_{H(A)}(p) = \mathrm{Hom}(H(A)^{\otimes p}, H(A))$

$$\mathrm{Ad}_u(p)(\psi) = u \circ \psi \circ (u^{-1})^{\otimes p} \ .$$

1. *If* $\mathrm{Ad}_u - \mathrm{id}$ *is invertible, then* (A, ϕ) *is gauge formal.*
2. *If* $\mathrm{Ad}_u - \mathrm{id}$ *is invertible on the elements of degree* k *for all* $k < n$, *then* (A, ϕ) *is gauge* n*-formal.*

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Corollary

Suppose that there exists $u \in \mathrm{Aut}(H(A), \varphi_*)$ such that for all $k < n$, and all p-tuples $(k_1, \ldots, k_p)$,

$$\mathrm{Spec}(u_{k_1+\cdots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing \ ,$$

where $u_i := u_{|H_i(A)}$. If u admits a lift at the level of chains then (A, ϕ) is gauge n-formal.

Frobenius & Weil numbers

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Theorem (Deligne, 1974)

For all n, the eigenvalues of a Frobenius action on $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight n.

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For every smooth proper K-scheme X, $C^\bullet(X_{an}, \mathbb{Q}_\ell)$ is formal.

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$$\begin{array}{ccccc} \operatorname{Spec}(u_{s+k}) & \cap & \operatorname{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) & = \varnothing \, . \\ \Cup & & \Cup & \\ \alpha & & \beta & \\ |\iota(\alpha)| = q^{\frac{s+k}{2}} & > & |\iota(\beta)| = q^{\frac{s}{2}} & \end{array}$$

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Thank you for your attention!