Operadic calculus and formality criteria

Coline Emprin

Young Topologists Meeting 2022
Department of Mathematical Sciences - University of Copenhagen

July 18, 2022

The notion of formality

Formal topological spaces

R : commutative ground ring

Formal topological spaces

R : commutative ground ring

Definition

A topological space X is formal if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C^\bullet_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet(X;R) \ .$$

Formal topological spaces

R : commutative ground ring

Definition

A topological space X is formal if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C^\bullet_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet(X;R) \ .$$

$\rightarrow$ Origins in rational homotopy theory (for $R \subset \mathbb{Q}$)

X formal $\implies$ $H^\bullet(X,\mathbb{Q})$ completely determines the rational homotopy type of the space.

Examples

→ Formal spaces

- Spheres, complex projective spaces, Lie groups

Examples

→ Formal spaces

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [DGMS, 1975]

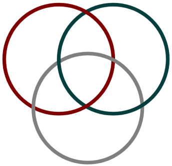
Examples

→ Formal spaces

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [DGMS, 1975]

→ Nonformal spaces

- The complement of the Borromean rings



Formality of an algebraic structure

A : cochain complex over R

Formality of an algebraic structure

A : cochain complex over R

(A, ν) : differential graded algebraic structure over A, e.g.

- $\rightarrow$ a dg associative algebra,
- $\rightarrow$ a dg Lie algebra,
- $\rightarrow$ a dg operad,
- $\rightarrow$ $\cdots$

Formality of an algebraic structure

A : cochain complex over R

(A, ν) : differential graded algebraic structure over A, e.g.

- $\rightarrow$ a dg associative algebra,
- $\rightarrow$ a dg Lie algebra,
- $\rightarrow$ a dg operad,
- $\rightarrow$ $\cdots$

Definition

The dg algebra (A, ν) is formal if

$$\exists\ (A, \nu) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \overline{\nu})\ .$$

Formality of an algebraic structure

A : cochain complex over R

(A, ν) : differential graded algebraic structure over A, e.g.

- $\rightarrow$ a dg associative algebra,
- $\rightarrow$ a dg Lie algebra,
- $\rightarrow$ a dg operad,
- $\rightarrow$ $\cdots$

Definition

The dg algebra (A, ν) is formal if

$$\exists\, (A, \nu) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \overline{\nu}) \ .$$

Examples

- X is formal $\iff$ $(C^{\bullet}_{\mathrm{sing}}(X; R), \cup)$ is formal as dga algebra

Formality of an algebraic structure

A : cochain complex over R

(A, ν) : differential graded algebraic structure over A, e.g.

→ a dg associative algebra,
→ a dg Lie algebra,
→ a dg operad,
→ $\cdots$

Definition

The dg algebra (A, ν) is formal if

$$\exists\, (A, \nu) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \overline{\nu})\ .$$

Examples

- X is formal $\iff (C^\bullet_{\mathrm{sing}}(X; R), \cup)$ is formal as dga algebra
- $C^\bullet_{\mathrm{sing}}(\mathcal{D}_k; \mathbb{R})$ is formal as an operad, where $\mathcal{D}_k$ is the little k-discs operad [Kontsevich, 1999]

Higher structures & operadic calculus

Homotopy retracts

Homotopy retracts

Definition

(W, d_W) is a homotopy retract of (V, d_V) if there are maps of cochain complexes

$$h \circlearrowright (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, d_W)$$

where $\mathrm{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism

Homotopy retracts

Definition

(W, d_W) is a homotopy retract of (V, d_V) if there are maps of cochain complexes

$$h \circlearrowleft (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, d_W)$$

where $\mathrm{id}_V - ip = d_V h + hd_V$ and i is a quasi-isomorphism

Proposition

If R is a field, any chain complex admits its cohomology as a homotopy retract

$$h \circlearrowleft (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0)$$

Transfer of algebraic structure

Transfer of algebraic structure

→ (A, d, ν) a dga algebra and a homotopy retract:

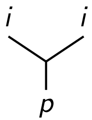
$$h \circlearrowright (A, d_A, \nu) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

Transfer of algebraic structure

→ (A, d, ν) a dga algebra and a homotopy retract:

$$h \circlearrowright (A, d_A, \nu) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

→ Transferred product: $\mu_2 := p \circ \nu \circ i^{\otimes 2} : H^{\otimes 2} \to H$

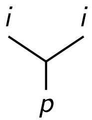


Transfer of algebraic structure

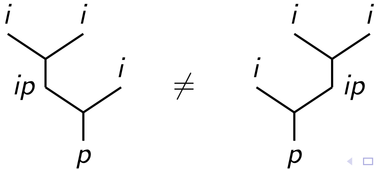
→ (A, d, ν) a dga algebra and a homotopy retract:

$$h \circlearrowright (A, d_A, \nu) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

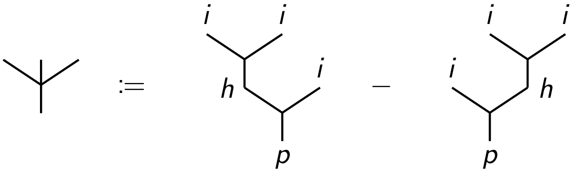
→ Transferred product: $\mu_2 := p \circ \nu \circ i^{\otimes 2} : H^{\otimes 2} \to H$



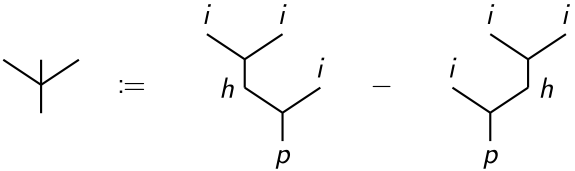
Not associative in general!



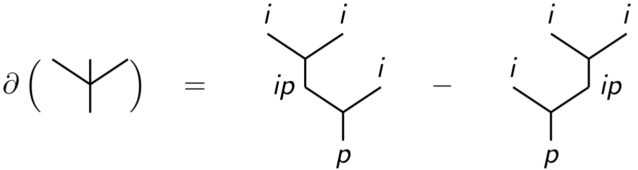
→ Consider $\mu_3 : H^{\otimes 3} \to H$



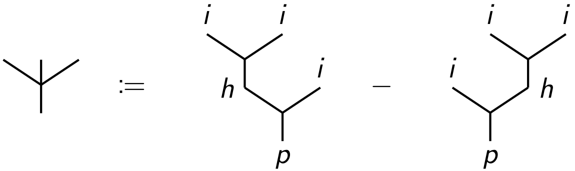
→ Consider $\mu_3 : H^{\otimes 3} \to H$



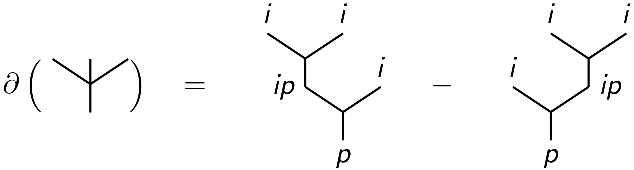
→ In $\mathrm{Hom}(H^{\otimes 3}, H)$:



→ Consider $\mu_3 : H^{\otimes 3} \to H$

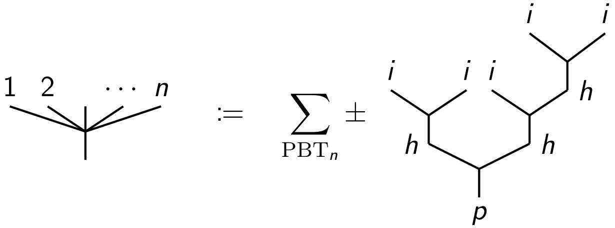


→ In $\mathrm{Hom}(H^{\otimes 3}, H)$:

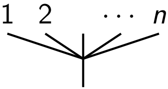


→ μ_2 is associative up to the homotopy μ_3.

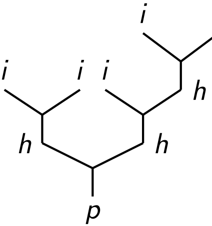
$\rightarrow \mu_n : H^{\otimes n} \rightarrow H$, for all $n \geq 2$



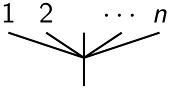
$\rightarrow \mu_n : H^{\otimes n} \rightarrow H$, for all $n \geq 2$



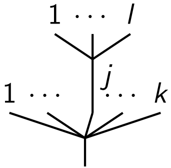
$$:= \sum_{\mathrm{PBT}_n} \pm$$



$$\partial \Bigg($$



$$\Bigg) = \sum_{\substack{k+l=n+1 \\ 1 \leq j \leq k}} \pm$$



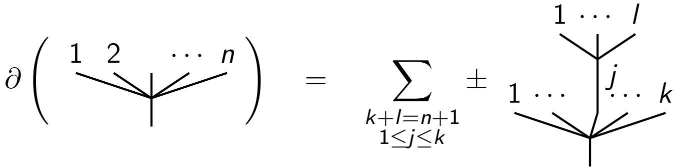
Homotopy associative algebras

Definition

A_∞-algebra: a chain complex H with a collection of maps

$$\mu_n : H^{\otimes n} \to H$$

of degree $n-2$, for all $n \geq 2$, which satisfy the relations



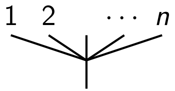
Homotopy associative algebras

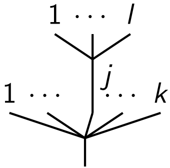
Definition

A_∞-algebra: a chain complex H with a collection of maps

$$\mu_n : H^{\otimes n} \to H$$

of degree $n-2$, for all $n \geq 2$, which satisfy the relations

$\partial\Big($ 

$\Big) = \sum_{\substack{k+l=n+1 \\ 1\leq j\leq k}} \pm$ 

Examples

- Every dga algebra (A, ν) is an A_∞-algebra with $\mu_n = 0$ for all $n \geq 3$.

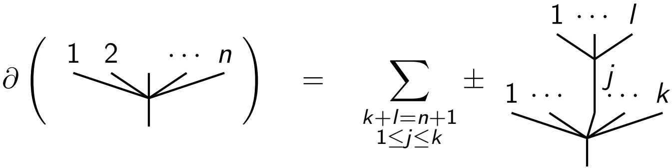
Homotopy associative algebras

Definition

A_∞-algebra: a chain complex H with a collection of maps

$$\mu_n : H^{\otimes n} \to H$$

of degree $n-2$, for all $n \geq 2$, which satisfy the relations



Examples

- Every dga algebra (A,ν) is an A_∞-algebra with $\mu_n = 0$ for all $n \geq 3$.
- $(H, d_H, \mu_2, \mu_3, \dots)$

Homotopy transfer theorem

Theorem (Kadeishvili, 1982)

Given a dga algebra (A, d_A, ν) and a homotopy retract

$$h \circlearrowright (A, d_A, \nu) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

there exists an A_∞-algebra structure on H such that p (and i) extend to A_∞-quasi-isomorphisms:

$$p_\infty : (A, d_A, \nu) \overset{\sim}{\rightsquigarrow} (H, d_H, \mu_2, \mu_3, \dots)$$

Homotopy morphisms

$(A, d_A, \nu_2, \dots)$, $(H, d_H, \mu_2, \dots)$: A_∞-algebras

Homotopy morphisms

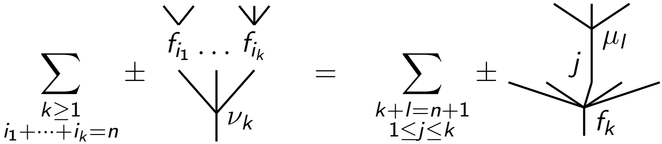
$(A, d_A, \nu_2, \dots), (H, d_H, \mu_2, \dots)$: A_∞-algebras

Definition

A_∞-morphism $f : A \rightsquigarrow H$ is a collection of linear maps

$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geq 1 ,$$

of degree $n - 1$, which satisfy the relations



where $\mu_1 = d_H$ and $\nu_1 = d_A$.

Homotopy quasi-isomorphisms

Definition

A_∞-quasi-isomorphism $f : A \overset{\sim}{\rightsquigarrow} H$ is an A_∞-morphism where $f_1 : A \to H$ is a quasi-isomorphism.

Homotopy quasi-isomorphisms

Definition

A_∞-quasi-isomorphism $f : A \overset{\sim}{\rightsquigarrow} H$ is an A_∞-morphism where $f_1 : A \to H$ is a quasi-isomorphism.

Proposition

A, H dga algebras

$$\exists\, A \overset{\sim}{\longleftarrow} \cdot \overset{\sim}{\longrightarrow} \cdots \overset{\sim}{\longleftarrow} \cdot \overset{\sim}{\longrightarrow} H \quad \Longleftrightarrow \quad \exists\, A \overset{\sim}{\rightsquigarrow} H$$

Homotopy quasi-isomorphisms

Definition

A_∞-quasi-isomorphism $f : A \overset{\sim}{\rightsquigarrow} H$ is an A_∞-morphism where $f_1 : A \to H$ is a quasi-isomorphism.

Proposition

A, H dga algebras

$$\exists\, A \overset{\sim}{\longleftarrow} \cdot \overset{\sim}{\longrightarrow} \cdots \overset{\sim}{\longleftarrow} \cdot \overset{\sim}{\longrightarrow} H \quad \Longleftrightarrow \quad \exists\, A \overset{\sim}{\rightsquigarrow} H$$

Corollary

A dga algebra (A, d, ν) *is formal* $\Longleftrightarrow \exists\, (A, d, \nu) \overset{\sim}{\rightsquigarrow} (H(A), 0, \overline{\nu})$.

An equivalent characterization of formality

(A, d, ν) a dga algebra such that $H(A)$ is a homotopy retract

$(A, d, \nu) \overset{\text{HTT}}{\rightsquigarrow} (H(A), \overline{\nu}, \mu_3, \mu_4 \ldots)$ Higher Massey products

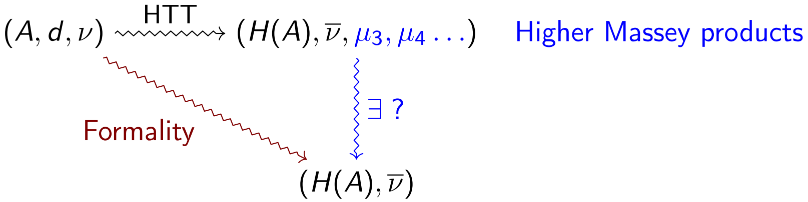
Formality: $(A, d, \nu) \rightsquigarrow (H(A), \overline{\nu})$

$\exists$?: $(H(A), \overline{\nu}, \mu_3, \mu_4 \ldots) \rightsquigarrow (H(A), \overline{\nu})$

$(H(A), \overline{\nu})$

An equivalent characterization of formality

(A, d, ν) a dga algebra such that $H(A)$ is a homotopy retract



$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ν) is formal.

An equivalent characterization of formality

(A, d, ν) a dga algebra such that $H(A)$ is a homotopy retract

$$(A, d, \nu) \overset{\text{HTT}}{\rightsquigarrow} (H(A), \overline{\nu}, \mu_3, \mu_4 \ldots)$$ Higher Massey products

Formality: $(A, d, \nu) \rightsquigarrow (H(A), \overline{\nu})$

$\exists$?: $(H(A), \overline{\nu}, \mu_3, \mu_4 \ldots) \rightsquigarrow (H(A), \overline{\nu})$

$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ν) is formal.

Definition

- (A, d, ν) is formal if $\exists\ (H(A), \overline{\nu}, \mu_3, \mu_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \overline{\nu})$.
- (A, d, ν) is n-formal if $\exists\ (H(A), \overline{\nu}, \mu_3, \mu_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \overline{\nu}, 0, \ldots, 0, \mu'_{n+1}, \ldots)$.

There is nothing special with the dga algebra structure!

There is nothing special with the dga algebra structure!

Examples

- dg commutative algebras and C_∞-algebras
- dg Lie algebras and L_∞-algebras
- dg Frobenius algebras and $Frobenius_\infty$-algebras
- dg Lie bialgebras and $LieBi_\infty$-algebras
- ...
- dg $\mathcal{P}$-algebra and $\mathcal{P}_\infty$-algebras, for any Koszul (pr)operad $\mathcal{P}$.

There is nothing special with the dga algebra structure!

Examples

- dg commutative algebras and C_∞-algebras
- dg Lie algebras and L_∞-algebras
- dg Frobenius algebras and $Frobenius_\infty$-algebras
- dg Lie bialgebras and $LieBi_\infty$-algebras
- ...
- dg $\mathcal{P}$-algebra and $\mathcal{P}_\infty$-algebras, for any Koszul (pr)operad $\mathcal{P}$.

$\Longrightarrow$ Operadic calculus provides a unified framework to deal with all types of algebraic structures

Formality criteria

The degree twisting

(A, d, ν) : a dg $\mathcal{P}$-algebra s.t. $H(A)$ is a homotopy retract

The degree twisting

(A, d, ν) : a dg $\mathcal{P}$-algebra s.t. $H(A)$ is a homotopy retract

α : a unit in R.

σ_α : the degree twisting by α

$\rightarrow$ linear automorphism of $H(A)$ which acts via $\alpha^k \times$ on $H^k(A)$.

The degree twisting

(A, d, ν) : a dg $\mathcal{P}$-algebra s.t. $H(A)$ is a homotopy retract

α : a unit in R.

σ_α : the degree twisting by α
$\rightarrow$ linear automorphism of $H(A)$ which acts via $\alpha^k \times$ on $H^k(A)$.

Theorem (Drummond-Cole – Horel, 2021)

Suppose that σ_α admits a lift, i.e. $\exists\, f \in \mathrm{End}(A, \nu)$ s.t. $H(f) = \sigma_\alpha$.

- $\forall k,\ \alpha^k - 1 \in R^\times \implies (A, d, \nu)$ *is formal.*
- $\forall k \leq n,\ \alpha^k - 1 \in R^\times \implies (A, d, \nu)$ *is n-formal.*

Theorem (Drummond-Cole – Horel, 2021)

Suppose that σ_α admits a lift, i.e. $\exists\, f \in \mathrm{End}(A,\nu)$ s.t. $H(f) = \sigma_\alpha$.

- $\forall k,\ \alpha^k - 1 \in R^\times \implies (A,d,\nu)$ *is formal.*
- $\forall k \leq n,\ \alpha^k - 1 \in R^\times \implies (A,d,\nu)$ *is n-formal.*

Heuristic :

→ Higher Massey products have to be compatible with the lift.

→ They intertwine multiplication by α^l with multiplication by α^k with $l \neq k$.

→ They have to vanish

Complement of subspace arrangements

X : a complement of a hyperplane arrangement over $\mathbb{C}$
$\to$ complement of a finite collection of affine hyperplanes in $\mathbb{A}^n_{\mathbb{C}}$.

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

l : a prime number different from p

h : order of q in $\mathbb{F}_l^\times$

Proposition

If X is defined over K, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_l)$ is $(h-1)$-formal.

Proposition

If X is defined over K, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_l)$ is $(h-1)$-formal.

Heuristic :

- $\rightarrow$ $C^\bullet(X_{an}, \mathbb{Z}_l) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_l)$.
- $\rightarrow$ The action of a Frobenius on $H_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_l)$ is σ_q, [Kim, 1994].

Automorphism lifts

R : a field

(A, d, ν) : a dg $\mathcal{P}$-algebra s.t. $H(A)$ is a homotopy retract and finite dimensional.

Automorphism lifts

R : a field

(A, d, ν) : a dg $\mathcal{P}$-algebra s.t. $H(A)$ is a homotopy retract and finite dimensional.

Theorem (E., 2022)

Suppose that there exists $u \in \mathrm{Aut}(H(A), \overline{\nu})$ such that for all $k < n$, and all p-tuples $(k_1, \ldots, k_p)$,

$$\mathrm{Spec}(u_{k_1+\cdots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing \ ,$$

where $u_i := u_{|H^i(A)}$. If u admits a lift at the level of chains then (A, d, ν) is n-formal.

Automorphism lifts

R : a field

(A, d, ν) : a dg $\mathcal{P}$-algebra s.t. $H(A)$ is a homotopy retract and finite dimensional.

Theorem (E., 2022)

Suppose that there exists $u \in \mathrm{Aut}(H(A), \overline{\nu})$ such that for all $k < n$, and all p-tuples $(k_1, \ldots, k_p)$,

$$\mathrm{Spec}(u_{k_1+\cdots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing \ ,$$

where $u_i := u_{|H^i(A)}$. If u admits a lift at the level of chains then (A, d, ν) is n-formal.

Corollary

For every smooth projective K-scheme X, $C^\bullet(X_{an}, \mathbb{Q}_l)$ is formal.

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

l : a prime number different from p

X : a smooth projective K-scheme

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

l : a prime number different from p

X : a smooth projective K-scheme

Definition

$\alpha \in \overline{\mathbb{Q}}_l$ is a Weil number of weight n if

$$\forall\, \iota : \overline{\mathbb{Q}}_l \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2}\ .$$

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

l : a prime number different from p

X : a smooth projective K-scheme

Definition

$\alpha \in \overline{\mathbb{Q}}_l$ is a Weil number of weight n if

$$\forall\, \iota : \overline{\mathbb{Q}}_l \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2}\ .$$

Theorem (Deligne, 1974)

For all n, the eigenvalues of a Frobenius action on $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_l)$ are Weil numbers of weight n.

Corollary

For every smooth projective K-scheme X, $C^\bullet(X_{an}, \mathbb{Q}_l)$ is formal.

Corollary

For every smooth projective K-scheme X, $C^\bullet(X_{an}, \mathbb{Q}_l)$ is formal.

Proof.

- $C^\bullet(X_{an}, \mathbb{Q}_l) \xrightarrow{\sim} C^\bullet_{et}(X_{\overline{K}}, \mathbb{Q}_l)$
- Let u be the Frobenius action on $H^\bullet_{et}(X_{\overline{K}}, \mathbb{Q}_l)$ and fix $\iota : \overline{\mathbb{Q}}_l \hookrightarrow \mathbb{C}$.
- For all $k \geq 1$, $(k_1, \dots, k_p)$ and $s := k_1 + \cdots + k_p$,

$$\mathrm{Spec}(u_{s+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing .$$

$$\alpha \in \mathrm{Spec}(u_{s+k}), \quad \beta \in \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p})$$

$$|\iota(\alpha)| = q^{\frac{s+k}{2}} \quad > \quad |\iota(\beta)| = q^{\frac{s}{2}}$$

$\square$

Thank you for your attention!