Kaledin obstruction classes and formality criteria

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The notion of formality

Formal topological spaces

R : commutative ground ring

Definition

A topological space X is formal if there exists a zig-zag of quasi-isomorphisms of differential graded associative algebras,

$$C^\bullet_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet_{\text{sing}}(X;R) \ .$$

$\to$ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal $\implies$ $H^\bullet(X,\mathbb{Q})$ completely determines the rational homotopy type of X.

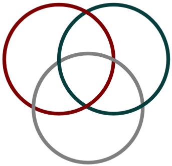
Examples

→ Formal spaces

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [DGMS, 1975]

→ Nonformal spaces

- The complement of the Borromean rings



Formality of an algebraic structure

A : cochain complex over R

(A, ϕ) : differential graded algebraic structure over A, e.g.

→ a dg associative algebra,
→ a dg Lie algebra,
→ $\cdots$

Definition

The dg algebra (A, ϕ) is formal if

$$\exists\, (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_*)\ ,$$

where φ_* denotes the induced structure on $H(A)$.

Examples

- X formal $= (C^\bullet_{\mathrm{sing}}(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Higher structures

Homotopy retracts

Definition

(W, d_W) is a homotopy retract of (V, d_V) if there are maps

$$h \circlearrowright (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, d_W)$$

where $\mathrm{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism .

Proposition

If R is a field, the cohomology of any cochain complex is a homotopy retract:

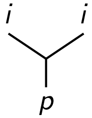
$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0) \ .$$

Transfer of algebraic structure

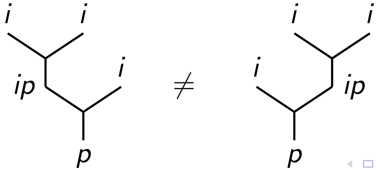
(A, d_A, ϕ) : a dga algebra and a homotopy retraction:

$$h \circlearrowright (A, d_A, \phi) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

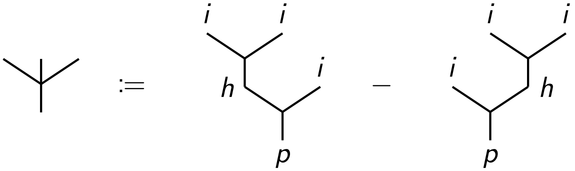
$\rightarrow$ Transferred product: $\varphi_2 := p \circ \phi \circ i^{\otimes 2} : H^{\otimes 2} \rightarrow H$



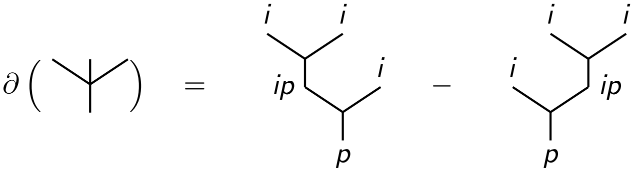
Not associative in general!



→ Consider $\varphi_3 : H^{\otimes 3} \to H$

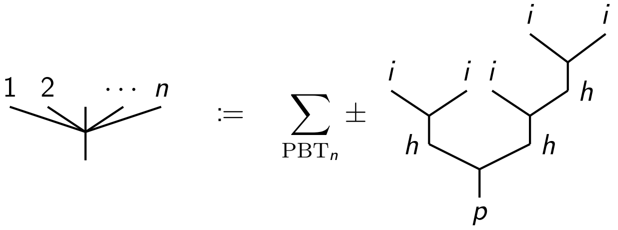


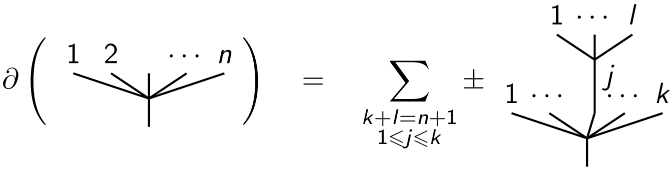
→ In $\mathrm{Hom}(H^{\otimes 3}, H)$:



→ φ_2 is associative up to the homotopy φ_3.

$\to \varphi_n : H^{\otimes n} \to H$, for all $n \geqslant 2$





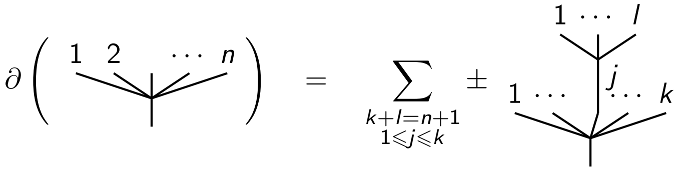
Homotopy associative algebras

Definition

A_∞-algebra: a cochain complex H with a collection of maps

$$\varphi_n : H^{\otimes n} \to H$$

of degree $2-n$, for all $n \geqslant 2$, which satisfy the relations



Examples

- Every dga algebra (A, ϕ) is an A_∞-algebra with $\varphi_n = 0$ for all $n \geqslant 3$.
- $(H, d_H, \varphi_2, \varphi_3, \dots)$

Homotopy transfer theorem

Theorem (Kadeishvili, 1982)

Given a dga algebra (A, d_A, ϕ) and a homotopy retract

$$h \circlearrowright (A, d_A, \phi) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

there exists an A_∞-algebra structure on H such that p (and i) extend to A_∞-quasi-isomorphisms:

$$(A, d_A, \phi) \rightsquigarrow (H, d_H, \varphi_2, \varphi_3, \varphi_4, \dots)$$

Homotopy morphisms

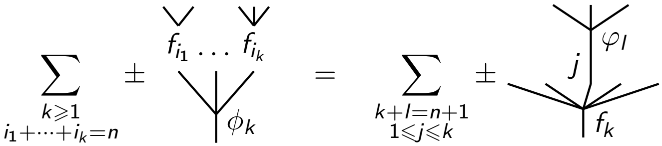
$(A, d_A, \phi_2, \dots), (H, d_H, \varphi_2, \dots)$: A_∞-algebras

Definition

A_∞-morphism $f : A \rightsquigarrow H$ is a collection of linear maps

$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geqslant 1 ,$$

of degree $1 - n$, which satisfy the relations



where $\varphi_1 = d_H$ and $\phi_1 = d_A$.

Homotopy quasi-isomorphisms

Definition

A_∞-quasi-isomorphism $f : A \stackrel{\sim}{\rightsquigarrow} H$ is an A_∞-morphism where $f_1 : A \to H$ is a quasi-isomorphism .

Proposition (R is a characteristic zero field)

quasi-isos of associative algebras		*A_∞-quasi-iso*
$\exists\,(A,\phi) \stackrel{\sim}{\longleftarrow} \cdot \stackrel{\sim}{\longrightarrow} \cdots \stackrel{\sim}{\longleftarrow} \cdot \stackrel{\sim}{\longrightarrow} (B,\phi')$	$\Longleftrightarrow$	$\exists\,(A,\phi) \stackrel{\sim}{\rightsquigarrow} (B,\phi')$

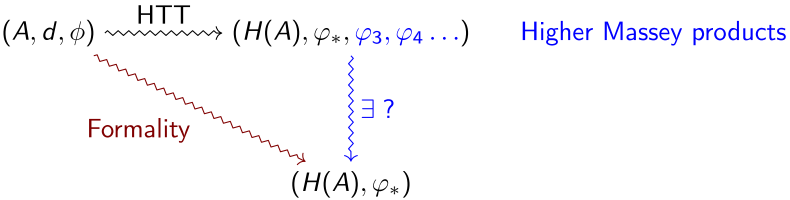
Corollary

A dga algebra (A,ϕ) is formal if and only if

$$\exists\,(A,\phi) \stackrel{\sim}{\rightsquigarrow} (H(A),\varphi_*)\ .$$

An equivalent characterization of formality

(A, d, ϕ) a dga algebra such that $H(A)$ is a homotopy retract



$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

Definition

- (A, d, ϕ) is gauge formal if $\exists\, (H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*)$.
- (A, d, ϕ) is gauge *n*-formal if

$$\exists\, (H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*, 0, \ldots, 0, \varphi'_{n+1}, \ldots)\ .$$

Kaledin classes

Hochschild complex

Transfered structure: $(H(A), \varphi_*, \varphi_3, \varphi_4, \ldots)$

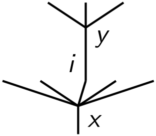
$$\varphi_n \in Hom(H(A)^{\otimes n}, H(A)), \quad |\varphi_n| = 2 - n$$

Hochschild cochain complex:

$$\mathfrak{g} := \prod_{n \geqslant 1} s^{-n+1} Hom(H(A)^{\otimes n}, H(A))$$

Lie bracket : $[x, y] := x \star y - (-1)^{|x||y|} y \star x$

$$x \star y := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)}$$



for $x \in Hom(H(A)^{\otimes n}, H(A))$ and $y \in Hom(H(A)^{\otimes m}, H(A))$.

A formal deformation

Transfered structure:

$$(\varphi_*, \varphi_3, \varphi_4, \dots) \in \mathfrak{g} := \prod_{n \geqslant 1} s^{-n+1} Hom(H(A)^{\otimes n}, H(A))$$

A formal deformation:

$$\Phi := \varphi_* + \varphi_3 \hbar + \varphi_4 \hbar^2 + \cdots \in \mathfrak{g}[\![\hbar]\!] := \mathfrak{g} \widehat{\otimes} R[\![\hbar]\!]$$

Proposition : $\mathrm{ad}_\Phi := [\Phi, -]$ defines a differential on $\mathfrak{g}[\![\hbar]\!]$

Twisted dg Lie algebra:

$$\mathfrak{g}[\![\hbar]\!]^\Phi := (\mathfrak{g}[\![\hbar]\!], [-,-], \mathrm{ad}_\Phi)$$

Kaledin classes

$$\partial_\hbar \Phi := \varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots \in \mathfrak{g}[\![\hbar]\!]$$

Lemma : $\partial_\hbar \Phi$ is a cycle in $\mathfrak{g}[\![\hbar]\!]^\Phi := (\mathfrak{g}[\![\hbar]\!], [-,-], \mathrm{ad}_\Phi)$,

$$\mathrm{ad}_\Phi(\partial_\hbar \Phi) := [\Phi, \partial_\hbar \Phi] = 0 \ .$$

Kaledin class:

$$K_\Phi := [\partial_\hbar \Phi] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right) \ .$$

n^{th}-truncated Kaledin class :

$$K_\Phi^n := \left[\varphi_3 + 2\varphi_4\hbar + \cdots + (n-2)\varphi_n\hbar^{n-3}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^{n-2}\right)^{\widetilde{\Phi}}\right) \ .$$

Kaledin classes

Kaledin class:

$$K_\Phi := \left[\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots\right] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

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Theorem ([Kaledin, 2007], [Lunts, 2007])

R : $\mathbb{Q}$-algebra

(A,ϕ) : dg associative algebra, $H(A)$ is a homotopy retract

- *(A,ϕ) is gauge formal $\Longleftrightarrow K_\Phi = 0$.*
- *(A,ϕ) is gauge n-formal $\Longleftrightarrow K_\Phi^n = 0$.*

Kaledin class:

$$K_\Phi := \left[\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots\right] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

n^{th}-truncated Kaledin class :

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Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

R : $\mathbb{Q}$-algebra

$\mathcal{P}$: Koszul operad

(A, ϕ) : dg $\mathcal{P}$-algebra such that $H(A)$ is a homotopy retract

- *(A, ϕ) is gauge formal $\iff K_\Phi = 0$.*
- *(A, ϕ) is gauge n-formal $\iff K_\Phi^n = 0$.*

Kaledin class:

$$K_\Phi := \left[\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots\right] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

n^{th}-truncated Kaledin class :

$$K_\Phi^n := \left[\varphi_3 + 2\varphi_4\hbar + \cdots + (n-2)\varphi_n\hbar^{n-3}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^{n-2}\right)^{\widetilde{\Phi}}\right)$$

Theorem (E., 2023)

R : *commutative ring*

$\mathcal{P}$: *Koszul operad or properad, possibly coloured in groupoids*

(A, ϕ) : *dg $\mathcal{P}$-algebra such that $H(A)$ is a homotopy retract*

- (A, ϕ) *is gauge formal* $\Longleftrightarrow K_\Phi = 0$.
- (A, ϕ) *is gauge n-formal* $\Longleftrightarrow K_\Phi^n = 0$.

Examples

K : a finite extension of $\mathbb{Q}_p$

ℓ : a prime number different from p

Proposition

For every smooth proper K-scheme X, $C^\bullet(X_{an}, \mathbb{Q}_\ell)$ is formal.

q : order of the residue field of the ring of integers of K

s : order of q in $\mathbb{F}_\ell^\times$

Proposition

Let X be a complement of a hyperplane arrangement over $\mathbb{C}$. If X is defined over K, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a c.h.a. over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s-1)$-formal.

Thank you for your attention!