KMPB School 2024 : Algebraic structures in Quantum Field Theory

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Operads and homotopy algebras

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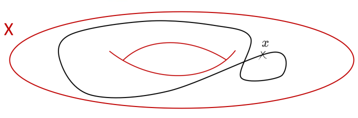
Introduction

X a topological space



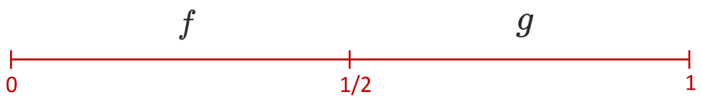
Space of loops :

$$\Omega X = \{ f : [0,1] \to X \mid f(0) = f(1) = x \}$$

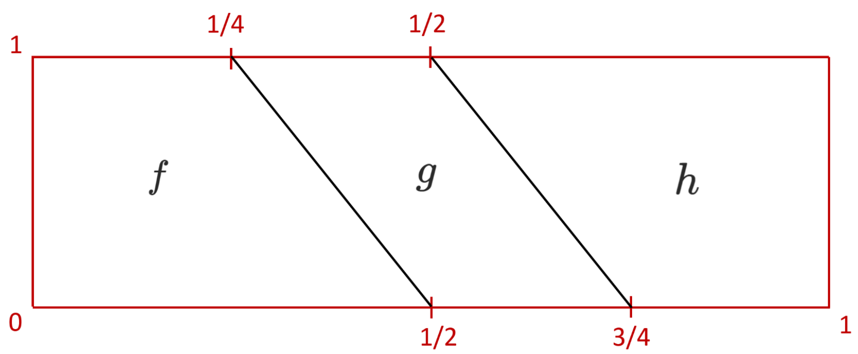
In X, we don't necessarily have a notion of sum, so we don't know how to add two loops. However, what we can do is to concatenate two such loops, that is we can first go through a first loop twice as fast, we come back to the starting point and then we go through a second loop g. We have the following product on the space of loops:

$$\Omega X \times \Omega X \longrightarrow \Omega X$$

$$(f, g) \longmapsto f * g = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}.$$



We can ask ourselves if we want to concatenate 3 loops, let us say f, g and k? We have at least two ways to proceed; to go through f, then g, then k, corresponding to two different parametrizations of [0,1]



$$(f * g) * k \neq f * (g * k)$$

concatenation is not associative

Actually, if we allow ourselves to change the travel time of each of the paths, we see that concatenation is not far from being associative. One can pass continuously from one way of composing to another.

There exists a continuous map $H : [0,1] \times [0,1] \longrightarrow [0,1]$

$$H(-, 0) = (f * g) * k$$

$$H(-, 1) = f * (g * k)$$

$$H(t, s) = \begin{cases} f\left(\frac{4t}{1+s}\right) & 0 \leq t \leq \frac{1+s}{4} \\ g(4t-s-1) & \frac{1+s}{4} \leq t \leq \frac{2+s}{4} \\ k\left(\frac{4t-s-2}{2-s}\right) & \frac{2+s}{4} \leq t \leq 1 \end{cases}$$

$\Rightarrow$ $*$ is associative up to homotopy.

For 4 loops, it is exactly the same : we many ways of composing f, g, k and h. (corresponding to embeddings of 4 copies of $[0,1]$ into $[0,1]$)

$$\Phi_0$$

$$((f * g) * k) * h$$

$$\downarrow H_1$$

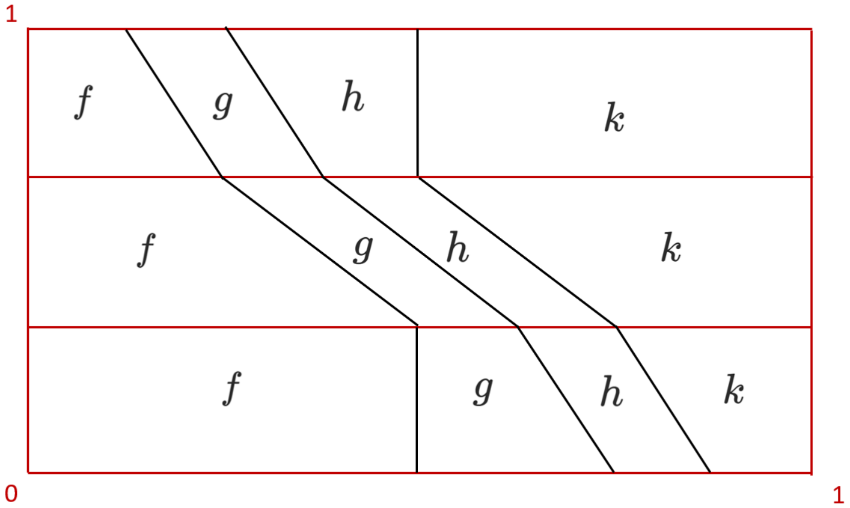
$$(f * (g * k)) * h$$

$$\downarrow H_2$$

$$f * ((g * k) * h)$$

$$\downarrow H_3$$

$$f * (g * (k * h))$$



we have a homotopy that comes from the composition of three successive homotopies. H_1, H_2 and H_3.

We have a second way to connect these two ways of composing by a homotopy given by :

$$\Phi_1 : ((f*g)*k)*h \xrightarrow{H_4} (f*g)*(k*h) \xrightarrow{H_5} f*(g*(k*h))$$

Φ_1 and Φ_2 are not the same homotopies but once again, we can find a homotopy between those :

$$\psi : [0,1]^2 \times [0,1] \longrightarrow [0,1]$$

$$\psi(-,-,0) = \Phi_0$$

$$\psi(-,-,1) = \Phi_1$$

Iterating this process, we see that concatenation is associative up to homotopy, with homotopies between these homotopies, and homotopies between those, and so on.

Remarque : we can encode each way of composing k loops as an embedding of k copies of $[0,1]$ into itself.

m-th iterated loop space of X : $\Omega^m X = \{ f : S^m \longrightarrow X \mid f(0,\dots,0,1) = x \}$

S^m is the unit sphere $= \{ (x_0, \dots, x_m) \in \mathbb{R}^{m+1} \mid \sum |x_i|^2 = 1 \}$

It is the space of all maps from the unit sphere S^m to X which sends the north pole to $*$. As before, we can define the composition of iterated loops and encode ways of composing k iterated loops in $\Omega^m X$ as embeddings of k copies of $[0,1]^m$ into itself. The concatenation is again associative up to homotopy, with homotopies between these homotopies, etc.

composing k iterated loop spaces in $\Omega^m X$ = embedding of k copies of $[0,1]^m$ into itself.

$\rightsquigarrow$ How to encode this entire structure ?

How to properly encode all theses structures and in particular the data of the different homotopies ?

This is to answer this question that May, Boardman and Vogt introduced operads in 1970.

Operades are tools which allow to encode structures such as the one on iterated loop spaces. We are going to study intensively their def but roughly an operad is :

Operad = collection of operations, equipped with a certain coherent manner to compose them.

Operads allow to encode algebraic structures through the notion of algebras over an operad.

Let $\mathcal{P}$ be an operad.

$\mathcal{P}$-algebra = a structure encoded by the operad $\mathcal{P}$

In other words, we can see a $\mathcal{P}$-algebra as a concrete realization of the operations of $\mathcal{P}$. Operads will allow to encode various different structures.

Exemple :

operad $\mathcal{P}$	$\mathcal{P}$-algebras
As	Associative algebras
Com	Commutative algebras
Lie	Lie algebras
Poisson	Poisson algebras

Theorem : (Recognition principle - Boardman - Vogt, 1968)
May, 1972

For all m, there exists an operad E_m, whose operations are embeddings of copies of $[0,1]^m$ into itself, such that every loop space $\Omega^m X$ is a E_m-algebra.

Conversely, if a connected topological space Y is a E_m-algebra, there exists X and a $Y \simeq \Omega^m X$

weak homotopy equivalence ($= \pi_m(g)$ is bijective $m \geq 0$.)

Connected = cannot be divided into two disjoint non-empty open sets

we can understand operads through the following analogy :

Before introducing the notion of groups, we were already working with automorphism groups that are concrete representations of a group structure

For operads, this is exactly the same thing, we were already working with particular cases of algebras in mathematical physics and elsewhere associative algebras, commutative algebras etc.
There is an abstraction of all this which is the operad and we can see these structures as concrete representations of the operad.

Goals of operads :
- provide a unified framework for dealing with all types of algebraic structures
- compare different types of algebraic structures
- apply classically known results in a certain type of algebra to other algebras, etc, ..
- Ease the treatment of more complicated types of algebras.

For simple types of algebras, involving only one binary product, we can deal without operads. we can encounter more sophisticated types of algebras with many compatible operations, multiple inputs and outputs
In those cases, it is going to be really convenient to have another language, another way of counting, to deal with all this.

There is an analogy that I like : It is like different numeral systems .
To deal with small numbers like the chapters of a book, roman numerals are sufficient . But to count that we are 8 billion people on earth and that in 20 years we will be 10 % more , its better to be with decimal numerals .

Here its exactly the same ; to deal with small types of algebras , we don't really need operads but to deal with more complicated types of algebras , it is convinient to have another language , another way of counting .

I. Definition of an operad

To understand the definition of an operad, on can look at the fundamental example given by the endomorphism operad.

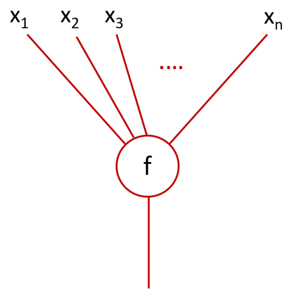
1.1. The endomorphism operad

X be a set

$\mathrm{End}_X(m) = \mathrm{Hom}(X^m, X)$ the set of all maps from the product of m copies of X to X

An element of $\mathrm{End}_X(m)$ can be thought as a tree where the number of leaves corresponds to the inputs of f and the root to the output.

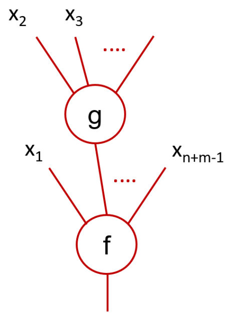
$f : X^m \longrightarrow X$



$f(x_1, x_2, \dots, x_m)$

Given $g : X^m \longrightarrow X$, there is an obvious manner to compose them by replacing one of the inputs of f by the output of g.

By choosing the second input of f, we get



$f(x_1, g(x_2, \dots, x_{m+1}), \dots, x_m$

we denote it $f \circ_2 g$.

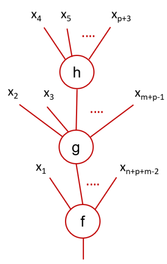
we replaced the second input of f but we could have chosen the i-th input for each $i \in \{1, \dots, m\}$. This leads to other compositions that we denote $f \circ_i g$.

$\forall i \in \{1, \dots, m\}$ $f \circ_i g$.

If one composes several maps this way, the order of the compositions does not matter. In other words, the compositions are associative

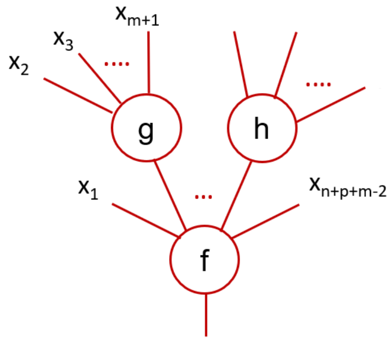
Compositions are associative : $h : X^p \to X$

$(f \circ_2 g) \circ_4 h = f \circ_2 (g \circ_3 h)$



Sequential compositi

$$(f \circ_2 g) \circ_{n+m-2} h = (f \circ_{n-1} h) \circ_2 g$$



parallel composition

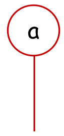
The map $id_X \in End_X(1)$ is a neutral element, that is for any $f : X^n \to X$ and $i \in \{1, \cdots, n\}$

$$f \circ_i id_X = f \quad \text{and} \quad id_X \circ_1 f = f.$$

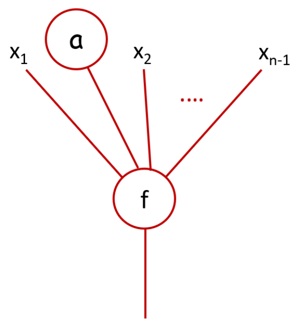
We can also allow maps $X^0 \to X$ of arity 0.

X^0 is a singleton and f is fully determined by the data of one element $a \in X$.

$End_X(0) \simeq X$. We can depict these maps of arity 0 as corks:



The composition $f \circ_i a$ for $i \in \{1, \dots, n\}$ amounts to blocking the input i of f by forcing it to be equal to a.



1.2. Definition of an operad.

Definition: A (nonsymmetric) operad in sets is the data of

- a family of sets $\mathcal{P}(0), \mathcal{P}(1), \mathcal{P}(2), \dots$ indexed by natural numbers.
- composition map $\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n+m-1)$ for all n, m and $i \in \{1, \dots, n\}$.
- a unit element $id \in \mathcal{P}(1)$ such that for all $f \in \mathcal{P}(n)$, we have $id \circ_1 f = f$ and $f \circ_i id = f$.

which satisfy for $f \in \mathcal{P}(n)$, $g \in \mathcal{P}(m)$ and $h \in \mathcal{P}(p)$

- sequential composition: if $1 \leq i \leq n$ and $1 \leq j \leq m$ then
$$(f \circ_i g) \circ_{i+j-1} h = f \circ_i (g \circ_j h)$$
- parallel composition: if $1 \leq i < k \leq n$, then
$$(f \circ_i g) \circ h = (f \circ_k h) \circ_i g$$

Example :

The endomorphism operad End_X. (The mother of set theoretic operad)

Definition : A morphism $\phi : \mathcal{P} \to \mathcal{Q}$ between two ns operads is a family of maps $\{\phi_m : \mathcal{P}(m) \to \mathcal{Q}(m)\}_{m \geq 0}$ which preserve the units $\phi_1(id_{\mathcal{P}}) = id_{\mathcal{Q}}$ and which commute with the composition maps

$$\phi_{m+m'-1}(f \circ_i^{\mathcal{P}} g) = \phi_m(f) \circ_i^{\mathcal{Q}} \phi_{m'}(g).$$

Remark : One can define operads not only in sets, but also in (set-theoretic operad)

- vector spaces (algebraic operads)
- topological spaces (topological operad)
- chain complexes (differential graded operad)
- simplicial sets (simplicial operad)

$\rightsquigarrow$ in any symmetric monoidal category $(\mathcal{C}, \otimes)$.

One of the main interest of working with operads is that they allow to encode algebraic structures through the notion of algebra over an operad.

Definition : Let $\mathcal{P}$ be an operad in $\mathcal{C}$ (e.g. in sets, vector spaces, etc)
Let X be an objet of $\mathcal{C}$.
A $\mathcal{P}$-algebra structure on X is a morphism of ns operad

$$\mathcal{P} \longrightarrow \mathrm{End}_X.$$

For each element $f \in \mathcal{P}$, we associate a morphism $A \otimes \cdots \otimes A \to A$ which is a concrete representation of these operations.

Example : The operad As encoding associative algebras

K be a field

For all $m \geq 1$, $\mu_m : K \to K$ $\quad (x_1, \dots, x_m) \mapsto x_1 \cdots x_m$

- $As(m) := K\mu_m$ ← one dim vector space generated by μ_m
  $As(0) := 0$
- As forms an algebraic operad

Proposition : As - algebras $\Longleftrightarrow$ Associative algebras

proof : $\Rightarrow$ Let $\varphi : As \to \mathrm{End}_X$ be an As-algebra structure.

$\forall m$ we have $\mu_m := \varphi(\mu_m)$. It is an operation $X \otimes \cdots \otimes X \to X$.

In particular, there is a linear map $\mu_2 : X \otimes X \to X$. 
The compositions $\mu_2 \circ_1 \mu_2$ and $\mu_2 \circ_2 \mu_2$ both corresponds to μ_3 so they coincide : μ_2 is associative.

$\Leftarrow$ Exercise.

Remark: For unital associative algebras, we define uAs in the same way except for $uAs_0 := \mathbb{K}1$.

1.3. Symmetric operads

Let us now extend the definition of a nonsymmetric operad in order to take care of the possible symmetries of the operations that we aim to encode.

Remark: $\mathrm{End}_X(n)$ carries a natural right action of the symmetric group $\mathbb{S}_n$

$$f \cdot \sigma (x_1, \dots, x_n) = f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

The composition of multilinear functions satisfies natural equivariance properties w.r.t. this action.

$\mathbb{S}_n$: group of bijections of the set $\{1, \dots, n\}$

Definition: A symmetric operad in C is the data of

- an $\mathbb{S}$-module in C, i.e. a family $\mathcal{P}(0), \mathcal{P}(1), \dots$ of objects in C equipped with an $\mathbb{S}_n$-right action on $\mathcal{P}(n)$ for all n.
- composition maps $\circ_i : \mathcal{P}(n) \times \mathcal{P}(m) \to \mathcal{P}(n+m-1)$ $\forall\, n, m \geq 0$ and $i \in \{1, \dots, n\}$
- a unit element $id \in \mathcal{P}(1)$ such that for all $f \in \mathcal{P}(n)$
  $id \circ_1 f = f$ and $f \circ_i id = f$

which satisfy sequential and parallel composition and equivariance with respect to the symmetric groups :

For any $\sigma \in \mathbb{S}_m$, we have : $\mu \circ_i \nu^{\sigma} = (\mu \circ_i \nu)^{\sigma'}$
where $\sigma' \in \mathbb{S}_{n-1+m}$ acts by id except on $\{i, \dots, i-1+m\}$ on which it acts via $\sigma(- - i)$.

Second, for any $\sigma \in \mathbb{S}_m$, we have:

$$\mu^\sigma \circ_i \nu = (\mu \circ_{\sigma(i)} \nu)^{\sigma''}$$

where $\sigma'' \in \mathbb{S}_{m-1+n}$ is acting like σ on $\{1, \dots, m-1+n\} \setminus \{i, \dots, i-1+n\}$ with values in $\{1, \dots, m-1+n\} \setminus \{\sigma(i), \dots, \sigma(i)-1+n\}$ and identically on the block $\{i, \dots, i-1+n\}$ with values in $\{\sigma(i), \dots, \sigma(i)-1+n\}$.

Exercise: $\mathrm{Com}(m) = \mathbb{K}\,\upsilon_m$ equipped with the trivial action of $\mathbb{S}_m$.

Check that Com-algebras $\longleftrightarrow$ commutative algebras.

Remark: One can also encode associative algebras with a symmetric operad that we denote Ass.

$\mathrm{Ass}(m) = \mathbb{K}[\mathbb{S}_m]$ regular representation of the symmetric group.

$$\left\{ \sum_{g \in \mathbb{S}_m} a_g\, g \;\middle|\; a_g \in \mathbb{K} \right\}$$

Example: The operad Lie encoding Lie-algebras

Let $V_m = \mathbb{K}x_1 \oplus \dots \oplus \mathbb{K}x_m$

$\mathrm{Lie}(V_m)$ the free Lie algebra generated by V_m

$\mathrm{Lie}(m) \subset \mathrm{Lie}(V_m)$ subspace linear in each x_i

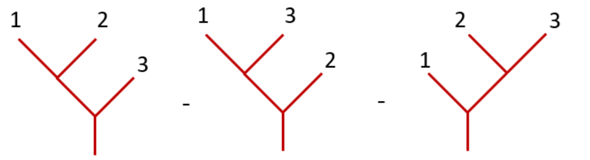
This defines a symmetric operad Lie whose algebras are Lie algebras.

A Lie algebra is a vector space X with a binary operation

$$X \otimes X \to X$$
$$x \otimes y \mapsto [x, y]$$

which is antisymmetric: $[x, y] = -[y, x]$

Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$



$= 0$

Remark: The above definition is the partial definition of an operad. There are several equivalent definitions of the data of an operad.

$\mathrm{Lie}(m)$ is the induced representation $\mathrm{Ind}_{\mathbb{Z}/m\mathbb{Z}}^{\mathbb{S}_m}(\rho)$

where (ρ) is the one dim rep of cyclic group $\mathbb{Z}/m\mathbb{Z}$ given by an irreducible mth root of unity

• The classical definition is the data of an $\mathbb{S}$-module, unit element endowed with associative, unital and equivariant composite maps

$$\gamma_{i_1,\dots,i_k} : \mathcal{P}(k)\otimes\mathcal{P}(i_1)\otimes\cdots\otimes\mathcal{P}(i_k) \longrightarrow \mathcal{P}(i_1+\cdots+i_k)$$

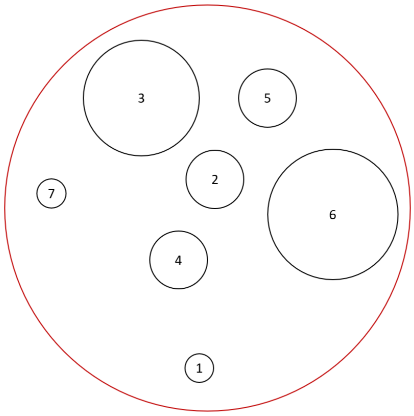
1.4. The exemple of the little disks (cubes) operad.

↳ mother of topological operads

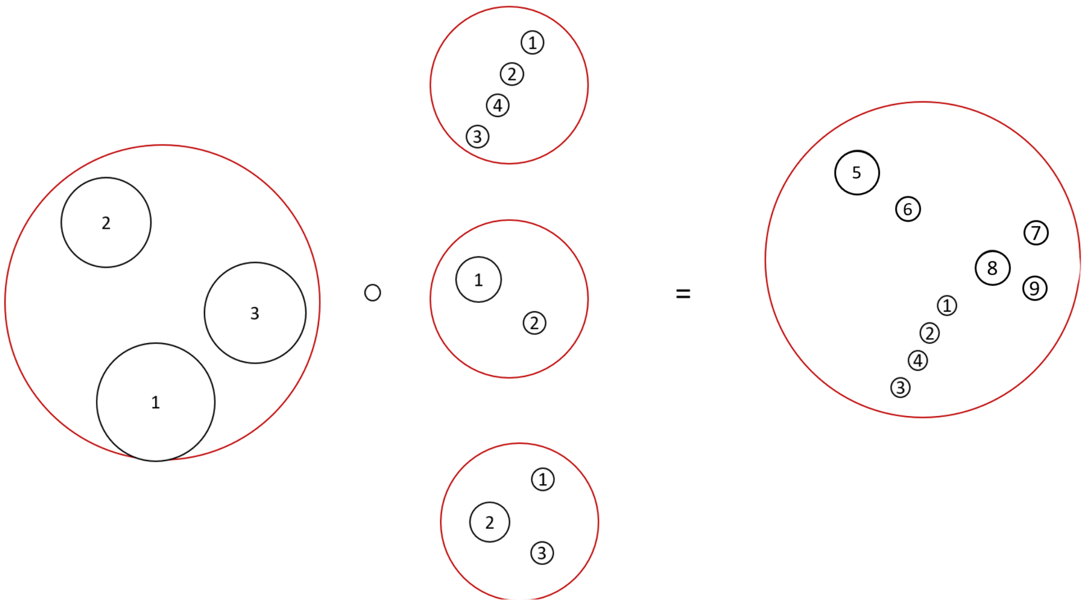
$$E_m(k) = \Big\{ (s_1,\dots,s_k) \;\Big|\; \underbrace{s_i : [0,1]^m \hookrightarrow [0,1]^m}_{\text{embedding}} \Big\}$$

+ non intersection condition

The symmetric group action is given by permuting the labels.



The composition is given by insertion of a cube inside an interior cube:



Examples of E_m-algebras:

1. $\Omega^m X$ the m-th iterated loop space.

2 (Heuristic) $\|x\|_\infty = \max(|x_1|, \dots, |x_m|)$

- Suppose that we have a certain field theory.
- Consider observables with compact support on the unit cube / disk.
- Suppose that the given theory is topological
  ↳ the theory is not sensitive to geometric quantities (metric, ...) but only to topology / homotopy.
- Thus, extend by zero the observables with compact support of a smaller cube to the unit cube is an iso (or an equivalence if we are in an homotopical framework).
- Playing the same game than for iterated loop spaces, we have an E_m-algebra structure on observables with compact support on $[0,1]^m$.

3. The Hochshild cochain complex of an associative algebra $C(A)$ has an E_2-algebra structure. [Tamarkin '03 - Deligne conjecture]

4. For $m=1$, E_1-algebras are homotopy associative algebras

Remark: Any E_m-algebra is a factorization algebra on $\mathbb{R}^m$. To any $U \subset \mathbb{R}^m$, we associate A.

1.5. Gerstenhaber and BV algebras

Theorem: [Cohen 76] differential graded operads

The singular homology $H_\bullet(E_2)$ of the little discs operad is isomorphic to the operad Gerst, encoding Gerstenhaber algebras.

Definition: A Gerstenhaber algebra is a graded vector space A endowed with
- a commutative binary product $*$ of degree 0,
- a commutative bracket $\langle \cdot, \cdot \rangle$ of degree $+1$, i.e.
  $|\langle a,b\rangle| = |a| + |b| + 1$

Such that
- the product $*$ is associative
- the bracket satisfies Jacobi identity
- the product $\bullet$ and the bracket $\langle \cdot,\cdot \rangle$ satisfy the Leibniz relation.

$$\langle a, b \bullet c \rangle = \langle a,b\rangle \bullet c + b \langle a, c \rangle$$

Remark : Any s.m functor between two s.m categories induces a functor between the associated notions of operads.
Since we are working over a field, the homology functor
$$H^\bullet : (Top, \times) \to (gr\,Mod, \otimes)$$
defines such a s.m functor.

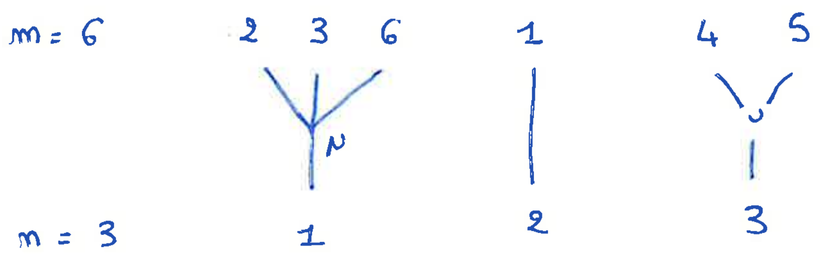
Proposition : [Gerstenhaber 63]
For any associative algebra A, the Hochschild cohomology of A with coefficients in itself inherits a Gerstenhaber algebra structure.

1.5. Category associated to an operad.

Definition : Let $\mathcal{P}$ be an operad in $\mathcal{C}$.
We associate to it a symmetric monoidal category $\mathrm{cat}\,\mathcal{P}$ as follows.

- The objects of $\mathrm{cat}\,\mathcal{P}$ are the natural numbers :
$$\underline{0}, \underline{1}, \underline{2}, \dots, \underline{n}, \dots$$
We also consider $\underline{n}$ as the set $\{1, \dots, n\}$, so $\underline{0} = \emptyset$.
- The morphisms of $\mathrm{cat}\,\mathcal{P}$ are defined as
$$\mathrm{cat}\,\mathcal{P}(\underline{m}, \underline{n}) = \bigoplus_{f : \underline{m} \to \underline{n}} \bigotimes_{i=1}^{n} \mathcal{P}(f^{-1}(i))$$
where f is a set map from $\{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$

Example : Here is an example of a morphism in $\mathrm{Cat}\,\mathcal{P}$:



$\mu \in \mathcal{P}(3)$
$\nu \in \mathcal{P}(2)$

Proposition : The operad structure of $\mathcal{P}$ induces on $\mathrm{Cat}\,\mathcal{P}$ a structure of symmetric monoidal category which is the addition of integers on objects.

proof :
- The composition of morphisms in $\mathrm{Cat}\,\mathcal{P}$ is obtained through the composition in $\mathcal{P}$.
- Associativity follows readily from associativity of the composition in $\mathcal{P}$.
- The $\mathbb{S}_n$-module structure of $\mathcal{P}(n)$ accounts for the action of the automorphism group of $\underline{n}$.

Definition: A Batalin-Vilkovisky algebra is a Gerstenhaber algebra equipped with an extra square-zero unary operator Δ of degree $+1$ satisfying the following quadratic-linear relation:

$$\langle a,b\rangle = \Delta(a\cdot b) - \Delta(a)\cdot b - a\cdot\Delta(b).$$

↳ the bracket is the obstruction to Δ being a derivation wrt $\cdot$.

Example: Let M be a smooth oriented m-dim manifold equipped with a volume form ω.
Let $\Gamma(M,TM)$ denote the space of vector fields; i.e the sections of the tangent bundle, endowed with the classical Lie bracket.

$$\Gamma(M,TM)\cong \mathrm{Der}(\mathcal{C}^\infty(M))$$

The space of polyvector fields $\Gamma(M,\Lambda(TM))$ is equipped with a Gerstenhaber algebra structure, whose Lie bracket is obtained from the previous one under the Leibniz rule.
The contraction of ω along polyvector fields defines the following iso with the differential forms

$$\Gamma(M,\Lambda^\bullet(TM)) \longrightarrow \Omega^{m-\bullet}(M)$$
$$\pi \longmapsto i_\pi(\omega) := \omega(\pi,-)$$

The transfer of the de Rham differential map d_{DR}, defines the divergence operator $\Delta := \mathrm{div}_\omega$, which endows $\Gamma(M,\Lambda(TM))$ with a BV-algebra structure [Tamarkin - Tsygan 2000]

Let A be a cyclic unital associative algebra, the Gerstenhaber algebra structure on the Hochschild cohomology of an associative algebra lifts to a BV-algebra structure.

Proposition: For any cyclic unital associative algebra A, the Hochschild cohomology of A inherits a BV-algebra structure.

$$\mathfrak{g} = \prod_{m\geq 1} s^{-m+1}\mathrm{Hom}(A^{\otimes m}, A)$$

$$[x,y] = x * y - (-1)^{|x||y|}\, y * x$$

$$x * y = \sum_{i=1}^{m} (-1)^{(i-1)(m-1)}\, x \circ_i y$$

$$HH(A) = H(\mathfrak{g}, d^\varphi = [\varphi,-])$$

Remark : • The framed little disks operad $\mathcal{D}_2^{fr}$ is made of the configurations of n framed disks, that is with a point on the boundary, inside the unit disks $\mathbb{D}_2$.

Proposition : [Getzler 94]
The singular Homology $H_\cdot(\mathcal{D}_2^{fr})$ of the framed little disks operad is an algebraic operad isomorphic to the operad BV.

proof : The framing induces the square-zero degree 1 operator Δ on homology.

Example. $\Omega^2 X$ endowed with an action of S^1 is an algebra over $\mathcal{D}_2^{fr}$.
$H(\Omega^2 X)$ has a BV-algebra structure.

II. Other types of operads

Depending on the type of algebraic structure that one wants to encode (several spaces, scalar product, trace, operations with multiple inputs and multiple outputs, etc.), it is often possible to extend the previous definition of an operad.

2.1 Colored operad

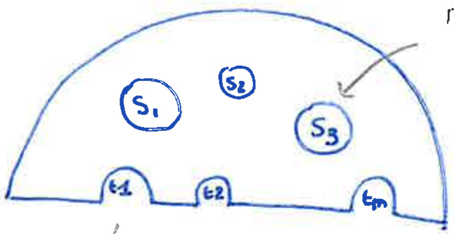
A colored operad is an operad in which each input or output comes with a color chosen in a given set.

A composition is possible whenever the colors of the corresponding input and output involved match.

Example : - $End_{V_1 \oplus \cdots \oplus V_n}$ paradigm : algebraic structures acting on sevel underlying spaces

- The Swiss-cheese operad

Standard unit upper semi-disk.



non-overlapping disks labeled 1 though n

$\in S(3, m)$

↳ upper semidisks labeled 1 though m

Definition : Let I be a set of colors.
An I-colored operad in C is a set $\{ \mathcal{P}(n, i) \}_i$ of objects of C indexed by all maps $i : \{0, \dots, n\} \to I$ with

- composition maps

$$\circ_\ell : \mathcal{P}(n, i) \otimes \mathcal{P}(m, j) \longrightarrow \mathcal{P}(n+m-1, i \circ_\ell j)$$

where $i(\ell) = j(0)$ and $i \circ_\ell j : \{0, \dots, m+n-1\} \to I$

$$k \mapsto \begin{cases} i(k) & k \leq \ell \\ j(k-\ell+1) & \ell \leq k < \ell + m \\ i(k-m) & \ell + m \leq k \end{cases}$$

- A right $\mathbb{S}_n$-action on

$$\bigoplus_{i : \{0, \dots, n\} \to I} \mathcal{P}(n, i)$$

- An identity $id_\alpha \in \mathcal{P}(1, c_\alpha)$ for each $\alpha \in I$, where $c_\alpha : \langle 1 \rangle \to I$ is the constant map with value α.

+ compatibility relations.

2.2. Cyclic operad

↳ to encode structure on spaces endowed with a non-degenerate symmetric pairing $\langle -,- \rangle : V \oplus V \to \mathbb{K}$.

In this case $\mathrm{End}_V(n) = \mathrm{Hom}(V^{\otimes n}, V) \cong \mathrm{Hom}(V^{\otimes n+1}, \mathbb{K})$ is equipped with a natural action of $\mathbb{S}_{n+1}$.

⇝ we are given an action of the cycle $(1\,2 \cdots n+1)$ which exchanges input and output :

Examples :

1. The symmetric operad Ass can be seen as a cyclic operad as follows. we endow $\mathrm{Ass}(n) \cong \mathbb{K}[\mathbb{S}_n]$ with the following right action of $\mathbb{S}_{n+1}$:

   → $\mathbb{S}_{n+1} \circlearrowleft \mathbb{S}_{n+1}$ by conjugation $\omega^g = g^{-1} \omega g$.

   → $U_{n+1} \subset \mathbb{S}_{n+1}$ permutation which have only on cycle = the orbit of $(0\,1 \ldots n)$

   → The bijection $\mathbb{S}_n \to U_{n+1}$ $\sigma \mapsto (0\ \sigma(1) \cdots \sigma(n))$ allow to wiew $\mathbb{S}_n$ as a $\mathbb{S}_{n+1}$-set.

   An algebra over the cyclic operad Ass is a finite dim associative algebra equipped with a non-degenerate pairing $\langle -,- \rangle$ which is invariant :

$$\langle ab, c \rangle = \langle a, bc \rangle.$$

2. A Frobenius algebra is a cyclic uAs-algebra.

   ↳ the nonsymmetric operad uAs can be endowed with a "cyclic nonsymmetric operad" structure, that is with an action of the cyclic groups $\mathbb{Z}/(n+1)\mathbb{Z}$

3. A symmetric Frobenius algebra is a cyclic uAss-algebra. (in bijection with 2-dimensional oriented Topological field theory)

   ↳ this is a unital associative algebra, equipped with a nondegenerate and symmetric bilinear form, such that

$$\langle ab, c \rangle = \langle a, bc \rangle.$$

→ A Topological Conformal field theory is an algebra over the singular chains $C_\bullet(\mathcal{R})$ of $\mathcal{R}_{g,n,m}$

Definition: The Riemann spheres $\mathcal{R}$, ie Riemann surfaces of genus 0, with n input discs and 1 output disc form an operad.

Proposition: There is an isomorphism $H_\bullet(\mathcal{R}) = BV$.

Proposition: [Getzler] The homology of a TCFT carries a natural BV-algebra structure, whose product is given by the deg 0 homology class of $\mathcal{R}(2)$ and whose degree 1 operator Δ is given by the fundamental class of the circle $\mathcal{R}(1) \simeq S^1$.

Proposition: [Huang 97] introduced by Richard Borcherds.
A vertex operator algebra is a "partial" algebra over the Riemann sphere operad $\mathcal{R}$.

Definition: A $\mathbb{Z}_{\geq 0}$ graded vertex superalgebra is a $\mathbb{Z}_{\geq 0}$-graded super space $V = \bigoplus_{i \geq 0} V^i$ where the component in each conformal degree i has a parity decomposition $V^i = V^i_{ev} \oplus V^i_{odd}$ equipped with a distinguished vacuum vector $\mathbb{1} \in V^0_{ev}$ and a family of bilinear operations

$$(n) : V \times V \longrightarrow V$$

for $n \in \mathbb{Z}$, such that

$$p(a_{(n)} b) = p(a) + p(b) \qquad V^i_{(n)} V^j \subset V^{i+j-n-1}.$$

A topological vertex algebra is a vertex superalgebra V with certain extra structure: there are distinguished elements $L \subset V^2_{ev}$, $G \subset V^2_{odd}$, $Q \in V^1_{odd}$, $J \in V^1_{ev}$ satisfying (Virasoro element, superpartners and the current)

$$Q_{(0)}^2 = 0 = G_{(1)}^2 \qquad [Q_{(0)}, G_{(1)}] = L_{(1)}.$$

Theorem: On any topological vertex algebra V, the operations

$$x \bullet y = x_{(-1)} y \qquad \{x, y\} = (-1)^{p(x)} (G_{(0)} x)_{(0)} y$$

are cochain maps with respect to the diff $d = Q_{(0)}$ and induce a Gerst structure on the cohomology

Batalin - Vilkovisky formalism

Let W be a finite dim chain complex.
We consider it as a manifold with the structure sheaf of formal functions which vanish at 0:

$$\hat{S}(W^*) := \prod_{m\geq 1} S^m(W^*)$$

where $S^m(W^*) := ((W^*)^{\otimes m})_{\mathfrak{S}_m}$.

To any finite dim chain complex V, we associate its "cotangent bundle" $W := V \oplus sV^*$. It has a canonical non degenerate form.
Its commutative algebra of functions $\hat{S}(V^* \oplus sV)$ carries the following deg-1 operator Δ.

$$\alpha \in \hat{S}(V^* \oplus sV) \qquad \Delta(\alpha) = \sum_{i=1}^{m} \frac{\partial \alpha}{\partial v_i} \frac{\partial \alpha}{\partial \nu_i} \qquad \begin{array}{l} \{v_i\} \text{ basis } V \\ \{\nu_i\} \text{ basis } V^* . \end{array}$$

Proposition: This forms a BV-algebra.

Quantized version: $\hat{S}(V^* \oplus sV)[[\hbar]]$ (again a BV-algebra)

The BV formalism relies on the functions which are solution to the master equation:

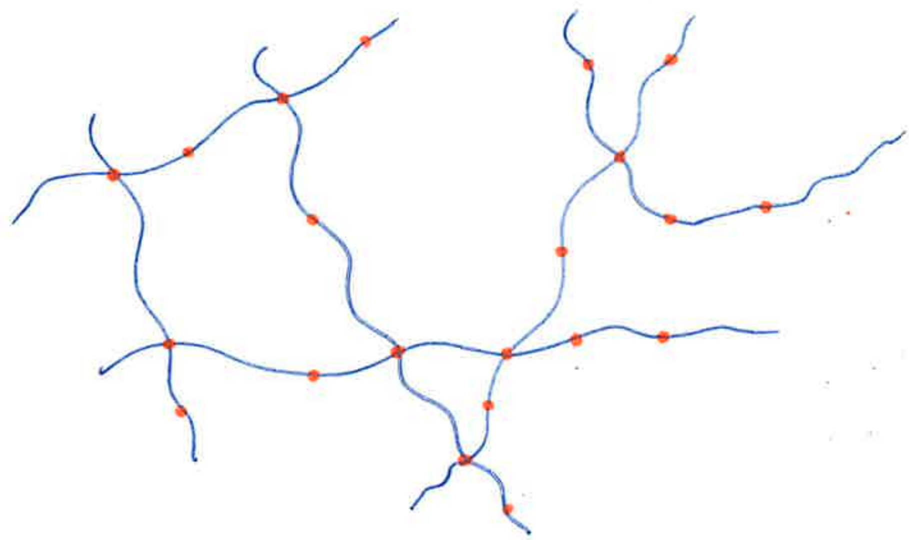
$$d(\alpha) + \hbar \Delta(\alpha) + \frac{1}{2}\langle \alpha, \alpha \rangle = 0$$

2.3. Modular operad

↳ In a cyclic operad, one makes no distinction between inputs and outputs. So one can compose operations along genus 0 graphs.

The idea of modular operads is to allow to compose operations along any graph of any genus.

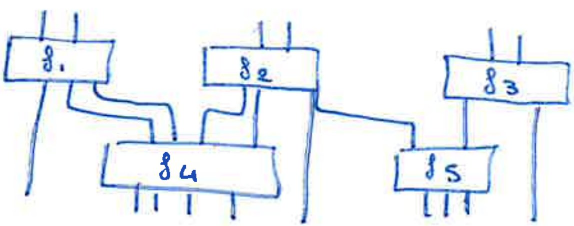
Example: The Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{g,m+1}$ of stable curves of genus g with $m+1$ marked points is the mother of modular operads. The operadic composite maps are defined by intersecting curves along their marked points:



- The Gromov-Witten invariants endow the cohomology groups $H^\cdot(X)$ of any projective or symplectic variety with a $H_\cdot(\overline{\mathcal{M}}_{g,m+1})$-algebra structure. Such a structure is also called cohomological field theory (CoF
- A Frobenius manifold is an algebra over the cyclic operad $H_\cdot(\overline{\mathcal{M}}_{0,m+1})$
- The Quantum cohomology ring is the $H_\cdot(\overline{\mathcal{M}}_{0,m+1})$-algebra structure on $H^\cdot(X)$.

2.4. Properad

A properad $\{\mathcal{P}(n,m)\}_{n,m\in\mathbb{N}}$ is meant to encode operations with several inputs and several outputs. But, in contrast to modular operads, where inputs and outputs are confused, one keeps track of the inputs and the outputs in a properad. Moreover, we consider only compositions along connected graphs:



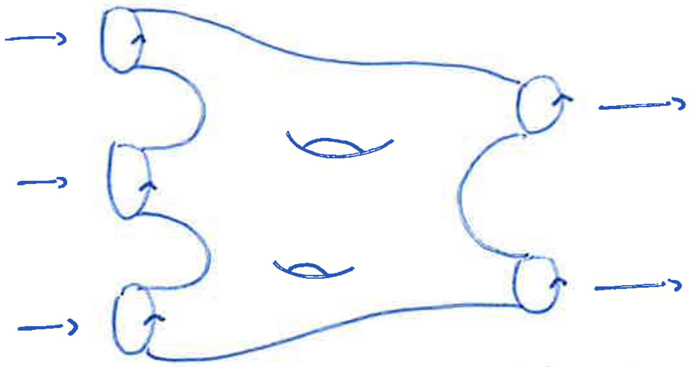
Ex: Endomorphism properad

$\mathrm{End}_X(n,m) = \mathrm{Hom}(X^m, X^n)$

Example : Riemann surfaces, i.e. smooth compact complex curves, with parametrized holomorphic holes form a properad.

$\mathcal{R}_{g,n,m}$: component of genus g with n input holes and m output holes.

The properadic composite maps are defined by sewing the Riemann surfaces along the holes.



A conformal Field theory (as defined by Graeme Segal) is an algebra over the properad of Riemann surfaces.

↳ is a QFT which is invariant under conformal transformations,

A conformal map is a function that locally preserves angles, but not necessarily lengths

2.5. Prop

↳ like a properad, but where one can also compose along non-necessarily connected graphs.

↳ introduced by Saunders Mac-Lane

Definition : sm category whose objects are the natural numbers and whose monoidal product is their sum

↳ elements of $\mathrm{Hom}_{\mathcal{C}}(n, m)$ can be seen as the operations with n inputs and m outputs.

↳ Such a PROP $\mathcal{C}$ is an $\mathbb{S}$-bimodule $\{\mathcal{C}(m,n)\}_{m,n \in \mathbb{N}}$ together with two types of compositions :

- horizontal composition :

$$\mathcal{C}(m_1, n_1) \otimes \mathcal{C}(m_2, n_2) \longrightarrow \mathcal{C}(m_1 + m_2, n_1 + n_2)$$

induced by the monoidal product, and the vertical composition $C(m, \ell) \otimes C(n, m) \longrightarrow C(n, \ell)$ given by the categorical composition.

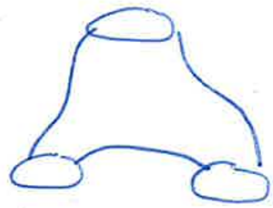
Remark: Every prop $\mathcal{C}$ has an underlying operad $\mathcal{P}_{\mathcal{C}}$ given by $\mathcal{P}_{\mathcal{C}}(n) = \mathrm{Hom}_{\mathcal{C}}(n, 1)$.

Example: To any object X in $\mathcal{C}$, we associate the PROP denoted $\mathcal{E}nd_X$ defined by $\mathcal{E}nd_X(n,m) = \mathrm{Hom}(X^{\otimes n}, X^{\otimes m})$.

Definition: An algebra over a prop, i.e. a s.m functor $F: \mathcal{P} \to \mathcal{C}$, is what F.W. Lawvere called an algebraic theory.

Examples: Let M, N be closed $(m-1)$-dim manifolds

A cobordism Σ from M to N is a compact m-manifold with boundary given by $M \sqcup N$.



The cobordism category Cobm is the category with objects $(m-1)$-dim manifolds and morphisms m-dim cobordisms between them.

composition is given by gluing along the boundary $(m-1)$-manifold.

The cobordism category is a sm category with monoidal structure coming from disjoint union of manifolds.

Definition: Sir Michael Atiyah proposed to define a Topological Quantum Field theory as an algebra over the category of cobordism. [1988]

↳ correlation function do not depend on the metric of space time

↳ not very interesting on flat Minkowski spacetime.

↳ applied to curved spacetime such as for example Riemann surfaces

III. Homotopy algebras

3.1. Homotopy associative algebras

(or E_1-algebras, or $\mathcal{A}_\infty$-algebras)

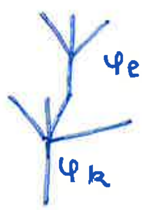
(Jim Stasheff in his PhD '63)

Definition: An $\mathcal{A}_\infty$-algebra is a chain complex (A, d) endowed with a family of operations

$$\{\varphi_m : A^m \longrightarrow A\}_{m \geq 2} \qquad |\varphi_m| = m-2$$

satisfying the following relations:

$$\partial(\varphi_m) = \sum_{\substack{k+l=m+1 \\ 1 \leq j \leq k}} (-1)^{j+lk}$$



Remark: • The differential d of A induces a differential ∂ on $\mathrm{Hom}(A^m, A)$

$$\partial(f) = d \circ f - (-1)^{|f|} f \circ d_{A^{\otimes m}} \quad \text{where } d_{A^{\otimes m}} := (d_A, id, \ldots, id) + \cdots + (id, \ldots, id, d_A)$$

• $\mathcal{A}_\infty$-algebras are algebras over the operad $\mathcal{A}_\infty$:

$$\mathcal{A}_\infty = \mathcal{T}\Big(\bigoplus_{m \geq 2} \mathbb{K}\mu_m\Big) + \text{relations}.$$

Example :

1. ΩX

2. Every differential graded associative algebra (A, Φ) is an A_∞-algebra structure with $\varphi_m = 0 \quad \forall\, m \geq 3$.

3. Costello's papers on the "B-model" and Gromov-Witten theory.

- The A_∞-operad and the moduli space of curves. [2004]

   Give a new proof of the result :

   Thm : [Harer - Mumford - Thurston - Penner - Kontsevich]
   A cell complex built from ribbon graphs is homotopy equivalent to the moduli space of curves.

   ⇝ by studying the modular enveloppe of the A_∞ (cyclic) operad.

   The modular enveloppe of a cyclic operad is the smallest modular operad containing it.

   The constructed operad correspond to a modular operad constructed from moduli spaces of Riemann surfaces.

- Topological conformal field theory and Calabi-Yau categories

- If X is a CY-manifold, Witten describes two different topological twistings of the non linear sigma model of maps from a Riemann Surface to X, which he calls the A and B models.
- If $X, X^\vee$ are a mirror pair of CY varieties, then the A model on X is equivalent to the B model on $X^\vee$ and vice-versa.
- The A model has been mathematically constructed as the theory of Gromov-Witten invariants.
- The genus 0 part of the B model has been constructed by Barannikov - Kontsevich (they constructed a Frobenius manifold from the variations of Hodge structure of a CY.
- The higher genus B model is more mysterious.
  In the physics litterature, it is constructed as a kind of quantization of the Kodaira - Spencer deformation theory of complex structure on a CY

The aim of the paper is to construct a purely algebraic counterpart to the theory of GW invariants (at all genera).

- He uses an appropriate A_∞-version of the derived category of coherent sheaves on a CY variety to construct the B model at all genera.

This story has had many ramifications, including his recent collaboration with Caldararu and Tu:

- <u>Categorical Enumerative invariants, I: String vertices</u> [2020]

↳ define combinatorial counterparts to the geometric string vertices of Sen-Zwiebach and Costello-Zwiebach, which are certain closed subsets of the moduli spaces of curves.

↳ This act on the Hochschild chains of a cyclic A_∞-algebra

↳ First of two papers where they define enumerative invariants associated to a pair consisting of a cyclic A_∞-algebra and a splitting of the Hodge filtration on its cyclic homology.

⇝ these invariants conjecturally generalize the GW and Fan-Jarvis-Ruan-Witten invariants.

Good notion of morphisms:

Definition: (∞-morphism / A_∞-morphism)
An ∞-morphism between two A_∞-algebra structures

$$f : (A, d_A, \phi_2, \phi_3, \dots) \rightsquigarrow (B, d_B, \psi_2, \psi_3, \dots)$$

is a collection of linear maps

$$f_m : A^{\otimes m} \longrightarrow H, \quad m \geq 1,$$

of degree $1-m$, which satisfy the following relations

$$\partial(f_m) + \sum_{\substack{k \geq 1 \\ i_1 + \dots + i_k = m}} (-1)^s \; \phi_k(f_{i_1}, \dots, f_{i_k}) = \sum_{\substack{k+\ell = m+1 \\ 1 \leq j \leq k}} (-1)^{j+\ell k} \; f_k(\dots, \psi_\ell, \dots)$$

where $\psi_1 = d_B$ and $\phi_1 = d_A$

$$s = (k-1)(i_1 - 1) + (k-2)(i_2 - 1) + \dots + 2(i_{k-2} - 1) + (i_{k-1} - 1)$$
$$= \sum_{t=1}^{k-1} (k-t)(i_t - 1)$$

An ∞-morphism is an ∞-quasi-isomorphism if f_1 is a quasi-isomorphism
∞-isomorphism isomorphism

Proposition: 1. A_∞-algebras with A_∞-morphisms form a category.
2. A_∞-isomorphisms are invertible morphisms in this category.

↳ this proof is similar to the proof the power series $a_1 x + a_2 x^2 + \dots$ with first term invertible are invertible.

3.2. Homotopy commutative algebras (or C_∞-algebras)

$\mathcal{C}_\infty = \Omega\, Com^{i}$

$Com^{i} = (s^{-1})^c \otimes Lie^\vee$

↳ this is a commutative algebra, where the associativity relation has been relaxed up to homotopy.

Definition: A C_∞-algebra structure on a chain complex (A, d) is an A_∞-algebra $(A, d, \varphi_2, \varphi_3, \dots)$ such that each map $\varphi_m : A^{\otimes m} \to A$ is a Harrison cochain, i.e. vanishes on the sum of all (p,q)-shuffles for $p+q=m$, $p \geq 1$:

$$\sum_{\sigma \in Ш(p,q)} (-1)^{\overset{\text{Kozul sign}}{\varepsilon}} a_{\sigma(1)} \otimes \dots \otimes a_{\sigma(p+q)} \qquad a_i \in A$$

By definition, a (p,q)-shuffle is a sequence of integers

$$[i_1 \dots i_p \mid j_1 \dots j_q]$$

which is a permutation of $\{1, \dots, p+q\}$ such that $i_1 < \dots < i_p$ and $j_1 < \dots < j_q$.
Let us denote by $Ш(p,q)$ all these permutations.

An ∞-morphism $f : A \rightsquigarrow B$ of $\mathcal{C}_\infty$-algebras is an A_∞-morphism such that each of its components $f_m : A^{\otimes m} \to B$ vanishes over all non-trivial (p,q)-shuffle of elements of A for $p+q=m$.

Example: The cohomology of the Lie algebra L_1 of polynomial vector fields over the line $\mathbb{K}^1$ is a C_∞-algebra generated by $H^1_{CE}(L_1)$.
[Millionshchikov, 2010 – Massey products in Lie algebra cohomology]

On $L_1 = \bigoplus_{m \geq 1} \mathbb{K} x_m$ the operation $\{x_p, x_q\} := (p+1)\, x_{p+q}$ is a pre-Lie product. The associated Lie bracket is given by

$$[x_p, x_q] = (p-q)\, x_{p+q}$$

Remark: The universal enveloppe $\mathcal{U}(L_1)$ is the dual of the celebrated Faà di Bruno Hopf algebra.

$$\mathcal{U}(L) = T(L) \big/ \left(x \otimes y - y \otimes x - [x,y] \quad \forall\, x, y \in L \right)$$

Remark : (E_∞ - algebras)

- One also needs to relax the symmetry of the commutative product "up to homotopy".
- By definition, an E_∞ - operad is a dg operad $\mathcal{E}$ which is a model for Com.
- Over a field $\mathbb{K}$ of characteristic zero, $\mathcal{C}_\infty$ is an E_∞ - operad.

Example : [Man06] The singular cochain complex with $\mathbb{Q}$ - coeff (resp $\mathbb{Z}_{(p)}$) of a simply connected CW complex, satisfying finitness and completeness conditions, carries an E_∞ - algebra structure, which faithfully determines its homotopy type (resp. its p - adic homotopy type).

- An example of an E_∞ - operad in given by the Barratt - Eccles operad $\mathcal{E}$. For any $k \geq 1$, one defines $\mathcal{E}(k)$ to be the normalized bar construction of the symmetric group $\mathbb{S}_k$.

  $\mathcal{E}_d(\mathbb{K})$ is the quotient of the free $\mathbb{K}$ - module on the set of $d+1$ - tuples $(w_0, \cdots, w_d)$ of elements of $\mathbb{S}_k$ by the degenerate tuples such that $w_i = w_{i+1}$ $\quad 0 \leq i \leq d-1$.

  The suboperad of $\mathcal{E}$ made of of degree 0 elements $\mathcal{E}_0(k) = \mathbb{K}[\mathbb{S}_k]$ is the symmetric operad Ass.

- The Barratt - Eccles operad comes equipped with a filtration, defined by the number of descents of permutations, such that

$$\mathrm{Ass} \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_m \subset \cdots \subset \operatorname{Colim}_m \mathcal{E}_m = \mathcal{E}$$

  This gives intermediate ways to relax the notion of commutative algebra up to homotopy.

- For each $m \geq 1$, the operad $\mathcal{E}_m$ is isomorphic over $\mathbb{Z}$ to the chains of the little m - discs operad. Such a dg operad is called an E_m - operad.

3.3. Homotopy Lie algebras

($\mathcal{L}_\infty$-algebras)

Definition: A homotopy Lie algebra is a chain complex (A, d) equipped with a family of skew-symmetric maps* $\ell_m : A^{\otimes m} \to A$ of degree $|\ell_m| = m-2$ for all $m \geq 2$, which satisfy the relation

$$\partial(\ell_m) = \sum_{\substack{p+q=m+1 \\ p,q>1}} \sum_{\sigma \in Ш(p,q)} \operatorname{sgn}(\sigma)(-1)^{(p-1)q} (\ell_p \circ_1 \ell_q)^\sigma$$

* $f^\sigma(x_1, \dots, x_m) = f(x_{\sigma(1)}, \dots$

* $f = \operatorname{sgn}(\sigma) f^\sigma \quad \forall \sigma \in S$

An ∞-morphism of $\mathcal{L}_\infty$-algebras $f : (A, d_A, \ell_2, \dots) \rightsquigarrow (B, d_A, \ell_2, \ell_3, \dots)$ is a collection of linear maps $f_m : A^{\otimes m} \longrightarrow B$ of degree $|f_m| = m-1$ that are antisymmetric and satisfy $d_B \circ f_1 = f_1 \circ d_A$

$$\sum_{\substack{p+q=m+1 \\ p,q>1}} \sum_{\sigma \in Ш(p,q)} \operatorname{sgn}(\sigma)(-1)^{(p-1)|q|} (f_p \circ_1 \ell_q^A)^\sigma$$

$$= \sum_{\substack{k \geq 2 \\ i_1 + \dots + i_k = m}} \sum_{\sigma \in Ш(i_1, \dots, i_k)} \operatorname{sgn}(\sigma)(-1)^\varepsilon \ell_k^B \circ (f_{i_1}, \dots, f_{i_k})^\sigma + \partial(f_m)$$

Examples: Pioneers of the use of homotopy structures:

- String theory - Barton Swiebach
  - ↳ A paper with Witten which goes back to '92.
    Algebraic Structures and differential geometry in 2D String theory
- Andrey Losev - Pavel Mnev

3.4. Koszul duality theory

↳ This process of considering homotopy versions of algebras is not specific to associative, commutative and Lie algebras. We can do the same thing for every type of algebra whose corresponding operad is called Koszul.

Let $\mathcal{P}$ be an algebraic operad which is Koszul; i.e there exists a minimal model :

$$\mathcal{P}_\infty \xrightarrow{\sim} \mathcal{P}$$
↑ quasi-isomorphism

We can consider the set of $\mathcal{P}_\infty$-algebras as algebras encoded by $\mathcal{P}_\infty$ + we have a notion of $\mathcal{P}_\infty$-morphisms.

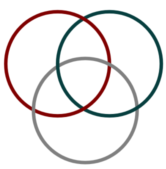
Proposition : 1. $\mathcal{P}_\infty$-algebras with $\mathcal{P}_\infty$-morphisms form a category
2. $\mathcal{P}_\infty$-isomorphisms are invertible morphisms in this category.

Remark : $\mathcal{P}$-algebras are particular cases of $\mathcal{P}_\infty$-algebras.

Proposition : Let (A, ϕ) and (B, ψ) be two $\mathcal{P}$-algebras.
The following propositions are equivalent
1. There exists a zig-zag of quasi-iso of $\mathcal{P}$-alg :
$$(A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \psi).$$
2. There exists an ∞-quasi-isomorphism
$$(A, \phi) \rightsquigarrow (B, \psi).$$

Corollary : The following propositions are equivalent.
1. A dg $\mathcal{P}$-algebra (A, d_A, ϕ) is formal, i.e
$$\exists \quad (A, d_A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \phi_*)$$
2. There exists an ∞-quasi-isomorphism.
$$(A, d_A, \phi) \rightsquigarrow (H(A), \phi_*).$$

Examples : • A topological space X is formal if $(C_\cdot(X; \mathbb{Q}), \cup)$ is formal as an associative algebra.



ex : • sphere, projective spaces, Lie groups
• compact Kähler manifolds [DGMS'75]

Counter-example : complement of Borromean ring.

• The little disks operad $\mathcal{D}_2$ [Kontsevich 99]
• $\overline{\mathcal{M}}_{g,n}$ is formal [GSNPR'05].

3.5. Other examples of Koszul operad

- Pois$_\infty$ - algebras

↳ A Gerstenhaber algebra is a poisson algebra with the Lie bracket in degree 1.

↳ Poisson algebras mix a commutative operation and a Lie bracket

- Gerst$_\infty$ - algebras [Getzler and Jones]

- BV$_\infty$-algebras

Proposition : [Galvez - Carrillo, Tonks, Vallette, 2009]

1. The singular chains $C_\bullet(\Omega^2 X)$ where X is a top space endowed with an action of the circle S^1 carries a homotopy BV-algebra structure.

2. Any TCFT carries a BV$_\infty$ - algebra structure, which is homotopy equivalent to the action of the operadic part $C_\bullet(R)$.

Theorem : [Lian, Zuckerman, 1993]
The BRST cohomology of a TCFT has a Gerst - algebra structure.

Lian - Zuckerman conjecture : The product and bracket operations on a topological vertex operator algebra can be extended into a Gerst$_\infty$ - algebra structure

↳ [Gálvez - Gorbounov - Tonks 2010]

Extension : Any topological vertex operator algebra A, with nonnegatively graded conformal weight, carries an explicit homotopy BV - algebra structure, which extends the Lian - Zuckerman operations in conformal weight zero and which induces the Lian - Zuckerman BV - algebra structure on $H_\bullet(V)$.

IV. Kontsevich's deformation quantization and QFT.

↳ One famous application of homotopy algebras in mathematical physics.

⇝ any finite dimensional Poisson manifold can be canonically quantized

⇝ Kontsevich's proof boils down to the construction of an L_∞-qi.

⇝ Tamarkin's proof uses operads.

4.1. Deformations and Poisson structures

K a commutative ring

A a K-algebra ; i.e. a K-module endowed with a k-bilinear map from $A \times A \to A$.

$A[[t]]$ the $K[[t]]$-module of formal power series.

Definition: • A formal deformation of the multiplication of A $*$ is a $k[[t]]$-bilinear map $A[[t]] \times A[[t]] \to A[[t]]$ such that we have

$$u * v = uv \mod tA[[t]] \quad \forall\, u, v \in A[[t]].$$

It is of the form

$$a * b = ab + B_1(a,b)t + \dots + \underset{\uparrow\ k\text{-bilinear map}}{B_m(a,b)}t^m + \dots$$

- Let $\mathcal{J}$ be the group of $K[[t]]$-module automorphisms g of $A[[t]]$ such that $g(u) \equiv u \mod tA[[t]]$ for all $u \in A[[t]]$.
- We define two formal deformations $*$ and $*'$ to be equivalent if there is an element $g \in \mathcal{J}$ such that

$$g(u * v) = g(u) *' g(v) \quad \forall\, u, v \in A[[t]].$$

Lemma: Suppose that A is commutative and associative.
Let $*$ be an associative formal def of the multiplication of A.
For all $a, b \in A$, put $\{a,b\} = B_1(a,b) - B_1(b,a)$.

a. The map $\{\, ,\}$ is a poisson bracket on A.

b. It only depends on the eq. class of $*$.

⇝ Given a Poisson bracket on A, when does it lift to an associative formal deformation?

It is going to be the case when A is the algebra of smooth functions on a differentiable manifold M

4.2. Kontsevich's theorem

M a differentiable manifold

A the algebra of smooth functions on M.

Definition: Let $m \geq 1$.

- A multidifferential operator on M is a map $P : A^m \to A$ compatible with restrictions to open subsets and such that, in each system $x_1, \ldots, x_m$ of local coordinates on M, we have
$$P(f_1, \ldots, f_m) = \sum a_{\nu_1, \ldots, \nu_m} \left(\frac{\partial^{|\nu_1|}}{\partial x_i^{\nu_1}} f_1 \right) \cdots \left(\frac{\partial^{|\nu_m|}}{\partial x_i^{\nu_m}} f_m \right)$$
where the ν_i are multi-indices and the $a_{\nu_1, \ldots, \nu_m}$ are smooth functions which vanish for allmost all $(\nu_1, \ldots, \nu_m)$.

- A star product on M is an associative formal deformation $* = \sum B_m t^m$ such that the B_m are bidifferential operators.

- Let J_d denotes the group of $\mathbb{R}[[t]]$-module automorphisms $g = \sum g_m t^m$ of $A[[t]]$ such that g_0 is the identity and all g_m are differential operators.

- Two star products $*$ and $*'$ are equivalent if there is a $g \in J_d$ such that $g(u * v) = g(u) *' g(v)$ for all $u, v \in A[[t]]$.

Remark: As in the lemma above, each star product $*$ on M gives rise to a Poisson bracket $\{,\}$. We call $*$ a formal deformation of $\{,\}$.

Theorem: [Kontsevich, 97]

a. Each poisson bracket on A admits a formal quantization, canonical up to equivalence.

b. There is an equivalence

$$\left\{ \begin{array}{l} \text{equivalence classes of} \\ \quad \text{Poisson brackets} \\ \pi = 0 + \pi_1 t + \cdots + \pi_m t^m + \cdots \\ \text{on } A[[t]] \end{array} \right\} \xrightarrow{\qquad} \left\{ \begin{array}{l} \text{equivalence classes} \\ \text{of star products} \\ \text{on } M. \end{array} \right\}$$
$$[\pi] \mapsto [*_\pi]$$

Moreover, the poisson bracket on A associated to $*_\pi$ equals the coefficient π_1.

Remark: The canonical quantization of a given poisson bracket $\{\cdot,\cdot\}$ in a. is obtained by applying b. to $\pi = \{\cdot,\cdot\} t$.

Example: The Moyal - Weyl product

$M = \mathbb{R}^2$ Consider the Poisson bracket given by

$$\{f, g\} = \mu \circ \left(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \right) (f \otimes g)$$ where μ is the multiplication of function on M.

Kontsevich's construction yields the associative formal deformation given by

$$f * g = \sum_{m=0}^{\infty} \frac{\partial^m f}{\partial x_1^m} \frac{\partial^m g}{\partial x_2^m} \frac{t^m}{m!} .$$

4.3. On the proof

↳ The above thm asserts that two deformation problems are equivalent :
- deforming the zero Poisson bracket on A
- deforming the multiplication μ on A.

↳ every deformation problem can be encoded by a dg Lie algebra.

We denote the corresponding dg Lie algebras by $L_{Pois}(M)$ and $L_{Star}(M)$.

⇝ $L_{Star}(M)$ is the Hochschild cochain complex associated to A up to a shift : $CC_\cdot(A)[1]$.

⇝ [Hochschild - Kostant - Rosenberg theorem]
For a differentiable manifold M, $L_{Pois}(M)$ is isomorphic to the homology of $L_{Star}(M)$.

To prove the main theorem above, it is enough to show that there is a zig-zag of Lie quasi-isomorphisms

$$L_{Star}(M) \leftarrow \cdot \rightarrow \cdots \leftarrow \cdot \rightarrow L_{Pois}(M)$$

In other words, $L_{Star}(M)$ is formal as a Lie algebra.

Theorem : [Kontsevich '97]
For every differentiable manifold M, there is an $\mathcal{L}_\infty$ - quasi-iso

$$\mathcal{U}^M : L_{Star}(M) \rightsquigarrow L_{Pois}(M) .$$

Remark : Cattaneo and Felder have shown that Kontsevich's formula can actually be formulated in terms of a perturbative expression of the functional integral of a certain 2-dim TFT : the Poisson sigma model discovered independently by Ikeda and Schaller - Strobel.

4. 4. Tamarkin's proof

1. → $(L_{Pois}(M), \phi_*)$ has a Gerst-algebra structure.

2. → $(L_{star}(M), \phi)$ has a $Gerst_\infty$-algebra structure inducing the Gerst structure in homology.

We have to prove that ϕ is formal.

3. → $(L_{Pois}(M), \phi_*)$ is intrinsically formal as a Gerst-algebra structure, i.e. any algebra admitting this homology is formal.

Remark: With 2., Tamarkin gave the solution to Deligne conjecture.
The operad E_2 is formal $C.(E_2) \leftarrow \cdot \rightarrow \cdots \leftarrow \cdot \rightarrow Gerst \leftarrow Gerst_\infty$
ϕ extends to a $C.(E_2)$-algebra structure on $L_{star}(M)$.

Remark: Deformation quantization originated in the field of theoretical physics, mainly from the idea of Dirac and Weyl, in order to understand the mathematical structure when passing from a commutative classical algebra of observables.
In order to describe such a quantization procedure, one considers the theory of infinitesimally deforming algebras by using formal power series and methods of non-commutative geometry.

V. Homotopy transfer theorem

↳ really useful to compute amplitudes / integrals / effective actions
↳ perturbation lemma (HTT applied to the algebra of dual numbers)

Definition: (W, d_W) is a homotopy retract of (V, d_V) if there are maps

$$h \circlearrowright (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, d_W)$$

where $id_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism

Exercise: If R is a field, the cohomology of any cochain complex is a homotopy retract:

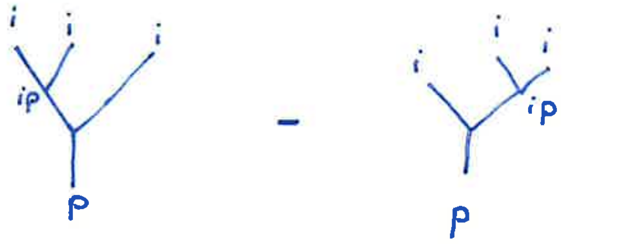
$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0).$$

Let (A, d_A, ϕ) be a dg algebra structure and a homotopy retraction

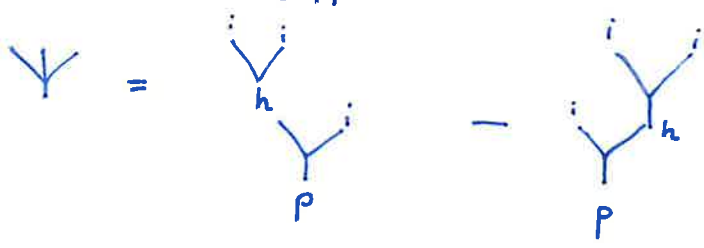
$$h \circlearrowright (A, d_A, \phi) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

→ Transferred product : $\varphi_2 := p \circ \phi \circ i^{\otimes 2} : H^{\otimes 2} \longrightarrow H$

Not associative in general!

$D =$  $\neq 0$

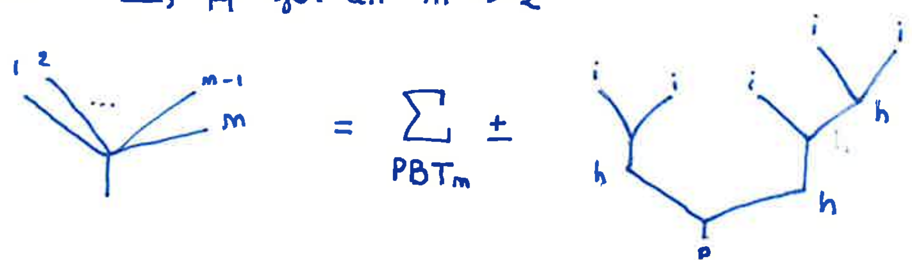
Consider $\varphi_3 : H^{\otimes 3} \longrightarrow H$

 $\in \mathrm{Hom}(H^{\otimes 3}, H)$

$$\partial(\varphi_3) = D.$$

→ φ_2 is associative up to the homotopy φ_3.

→ $\varphi_n : H^{\otimes n} \longrightarrow H$ for all $n \geq 2$



Theorem (Kadeishvili, 1982) Let $\mathcal{P}$ be a Koszul (pr) operad

Given a dg$\mathcal{P}_\infty$a algebra (A, d_A, ϕ) and a homotopy retract

$$h \circlearrowright (A, d_A, \phi) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

there exists an $\mathcal{P}_\infty$ A_∞- algebra structure on H such that p (and i) extend to A_∞ $\mathcal{P}_\infty$ - quasi-isomorphisms :

$$(A, d_A, \phi) \rightsquigarrow (H, d_H, \varphi_2, \varphi_3, \varphi_4).$$

Examples :

- TQFT, homological algebra and elements of K. Saito's theory of primitive form, Andrey Losev. (L_∞)
- Two field-theoretic viewpoints on the Fukaya-Morse A_∞-category. O. Chekeres, A. Losev, P. Mnev, D. R. Youmans. (A_∞)
- BV quantization, A. S. Cattaneo, P. Mnev, M. Schiavina.
- Quantum L_∞- algebras and the Homological perturbation Lemma. M. Doubek, B. Jurčo, Ján Pulmann.
  - ↳ construct a minimal model of a given quantum L_∞-alg via the HTT and show that its given by a Feynman diagram expansion, computing the effective action in the finite dim BV formalism.
- Wheeled pro(p) file of BV - formalism, S.A. Merkulov
  - ↳ BV formalism is equivalent to HTT for unimodular Lie bialgebras. Classical Feynman diagrams are exactly the graphs appearing in the HTT formula for wheeled properad encoding unimodular Lie bialgebras.