Kaledin classes & formality criteria

Coline Emprin

ANR Higher Algebra, Geometry, and Topology
May 6, 2024 | CIRM | Luminy

arXiv : 2404.17529

The notion of formality

Formal topological spaces

R : commutative ground ring

Definition

A topological space X is formal if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C^\bullet_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet_{\text{sing}}(X;R) \ .$$

$\rightarrow$ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal $\implies$ The cohomology ring $H^\bullet_{\text{sing}}(X,\mathbb{Q})$ completely determines the rational homotopy type of X.

Examples

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [Deligne, Griffiths, Morgan & Sullivan, 1975]

Formality of an algebraic structure

A : chain complex over R

$\mathscr{P}$: colored operad or properad

$\phi : \mathscr{P} \to \mathrm{End}_A$: a dg $\mathscr{P}$-algebra structure

Definition

The dg $\mathscr{P}$-algebra (A, ϕ) is formal if

$$\exists\, (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_0) \ ,$$

where φ_0 is the canonical $\mathscr{P}$-algebra structure on $H(A)$.

Examples

- X is formal $= (C^\bullet_{\mathrm{sing}}(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Purity implies formality

(A, ϕ) : dg $\mathscr{P}$-algebra encoded by an operad $\mathscr{P}$

α : unit of infinite order in R

σ_α : the degree twisting by α = automorphism of $(H(A), \varphi_0)$ which acts via $\alpha^k \times$ on $H_k(A)$.

Theorem

If σ_α admits a chain-level lift, i.e. $\exists\, f \in \mathrm{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$, then (A, ϕ) is formal.

→ Deligne, Griffiths, Morgan, Sullivan [1975]

→ Sullivan [1977]

→ Guillén Santos, Navarro, Pascual, Roig [2005]

→ Drummond-Cole and Horel [2021]

Examples

- Petersen [2014], Boavida de Brito and Horel [2021]
  The little disks operad $\mathcal{D}_k$ & Grothendieck-Teichmüller group

- Riche, Soergel, Williamson [2014]
  The extensions of parity sheaves on the flag variety.

- Drummond-Cole and Horel [2021]
  X : a complement of a hyperplane arrangement over $\mathbb{C}$
  defined over a finite extension K of $\mathbb{Q}_p$.

  ℓ : a prime number different from p

  $\rightarrow C^\bullet(X_{an}, \mathbb{Z}_\ell) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ [Artin].

  $\rightarrow$ A Frobenius action on $H^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is σ_q [Kim, 1994].

Questions

- Can we descend these results to other coefficient rings? (e.g. $\mathbb{Z}_{(\ell)}, \dots$)

- Does the degree twisting criteria hold for other types of algebras? (e.g. Hopf algebras, involutive Lie bialgebras,...)

- Is the degree twisting the only homology automorphism satisfying this property?

Kaledin classes & formality criteria

1. Gauge formality
   - Formality seen as a deformation problem

2. Kaledin classes
   - An obstruction theory to the formality over any ring

3. Formality criteria
   - Formality descent with torsion coefficient
   - Automorphism lifts

Gauge formality

Operadic homological algebra

A : chain complex over R

$\mathscr{P}$: reduced connected weight-graded Koszul operad

$\mathscr{P}_\infty \xrightarrow{\sim} \mathscr{P}$ with $\mathscr{P}_\infty := \Omega \mathscr{P}^{\text{i}}$

$\{\mathscr{P}_\infty - \text{algebra structures on } A\} := \mathrm{Hom}_{\mathrm{dgOp}}\left(\Omega \mathscr{P}^{\text{i}}, \mathrm{End}_A\right)$

Proposition

$$\mathrm{Hom}_{\mathrm{dgOp}}\left(\Omega \mathscr{P}^{i}, \mathrm{End}_A\right) \cong \mathrm{Codiff}\left(\mathscr{P}^{i}(A)\right)$$

Definition

An ∞-morphism $F : (A, \varphi) \rightsquigarrow (B, \psi)$ between $\mathscr{P}_\infty$-algebra structures is a morphism of dg $\mathscr{P}^{\text{i}}$-coalgebras:

$$(\mathscr{P}^{\text{i}}(A), \varphi) \to (\mathscr{P}^{\text{i}}(B), \psi) \ .$$

F is an ∞-quasi-isomorphism if $F_0 : A \to B$ is a quasi-isomorphism.

Proposition (R is a characteristic zero field)

zig-zag of quasi-isos of $\mathscr{P}$-algebras — *∞-quasi-iso*

$$\exists\, (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \phi') \quad \Longleftrightarrow \quad \exists\, (A, \phi) \overset{\sim}{\rightsquigarrow} (B, \phi')$$

Corollary

A dg $\mathscr{P}$-algebra (A, ϕ) is formal $\Longleftrightarrow \ \exists\, (A, \phi) \ \overset{\sim}{\rightsquigarrow} \ (H(A), \varphi_0)$.

Homotopy transfer

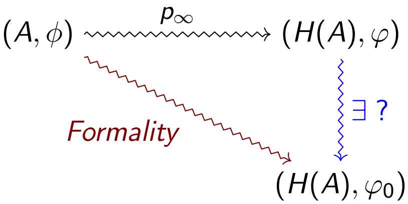
Theorem (Homotopy transfer theorem)

Let (A, d) be a chain complex s.t. $H(A)$ is a contraction :

$$h \circlearrowleft (A, d) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0)$$

$$\mathrm{id}_A - ip = d_A h + h d_A, \quad pi = \mathrm{id}_{H(A)}, \quad h^2 = 0, \quad ph = 0, \quad hi = 0\ .$$

For every $\mathscr{P}$-algebra structure (A, ϕ), there exists a $\mathscr{P}_\infty$-algebra structure φ s.t. p extends to an ∞-quasi-isomorphism:



$\mathscr{P}_\infty$-algebra structures on $H(A)$

The convolution dg Lie algebra associated to $H(A)$:

$$\mathfrak{g} := \left(\mathrm{Hom}(\overline{\mathscr{P}}^{\mathsf{i}}, \mathrm{End}_{H(A)}), [-,-], d\right)$$

→ $\mathrm{Hom}\left(\overline{\mathscr{P}}^{\mathsf{i}}, \mathrm{End}_{H(A)}\right) := \prod_{n\geq 0} \mathrm{Hom}\left(\overline{\mathscr{P}}^{\mathsf{i}}(n), \mathrm{End}_{H(A)}(n)\right)$

→ $d(\varphi) := (-1)^{|\varphi|+1}\varphi \circ d_{\overline{\mathscr{P}}^{\mathsf{i}}}$

→ $\varphi \star \psi := \overline{\mathscr{P}}^{\mathsf{i}} \xrightarrow{\Delta_{(1)}} \overline{\mathscr{P}}^{\mathsf{i}} \circ_{(1)} \overline{\mathscr{P}}^{\mathsf{i}} \xrightarrow{\varphi \circ_{(1)} \psi} \mathrm{End}_{H(A)} \circ_{(1)} \mathrm{End}_{H(A)} \xrightarrow{\gamma_{(1)}} \mathrm{End}_{H(A)}$

→ $[\varphi, \psi] := \varphi \star \psi - (-1)^{|\varphi||\psi|}\psi \star \varphi$

Every $\varphi \in \mathrm{Hom}\left(\overline{\mathscr{P}}^{\mathsf{i}}, \mathrm{End}_{H(A)}\right)$ decomposes as

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots)$$

where φ_k is the restriction $\varphi_k : \overline{\mathscr{P}}^{\mathsf{i}(k)} \longrightarrow \mathrm{End}_{H(A)}$.

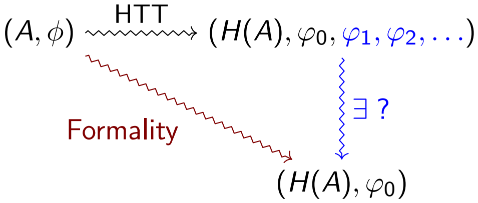
Its set of Maurer–Cartan elements:

$$\mathrm{MC}(\mathfrak{g}) = \{\varphi \in \mathfrak{g}_{-1},\ d(\varphi) + \tfrac{1}{2}[\varphi, \varphi] = 0\}$$

Proposition

$$\{\mathscr{P}_\infty - \textit{algebra structures on } H(A)\} \cong \mathrm{MC}(\mathfrak{g})$$

Remark



Higher Massey products

$\Longrightarrow$ If the higher Massey products vanish, then (A, ϕ) is formal.

The gauge group

The convolution dg Lie algebra:

$$\mathfrak{g} := \left(\mathrm{Hom}(\overline{\mathscr{P}}^{\text{i}}, \mathrm{End}_{H(A)}), [-,-], d\right)$$

Its set of degree zero elements:

$$\mathfrak{g}_0 := \mathrm{Hom}\left(\overline{\mathscr{P}}^{\text{i}}, \mathrm{End}_{H(A)}\right)_0$$

The Baker–Campbell–Hausdorff formula, with $\mathrm{ad}_\lambda := [\lambda, -]$:

$$\lambda, \mu \in \mathfrak{g}_0, \qquad e^{\mathrm{ad}_{\mathrm{BCH}(\lambda,\mu)}} = e^{\mathrm{ad}_\lambda} \circ e^{\mathrm{ad}_\mu} \ .$$

$$\mathrm{BCH}(\lambda,\mu) = \lambda + \mu + \frac{1}{2}[\lambda,\mu] + \frac{1}{12}([\lambda,[\lambda,\mu]] + [\mu,[\mu,\lambda]]) + \cdots$$

$$\Gamma := (\mathfrak{g}_0, \mathrm{BCH}, 0)$$

The gauge action

$\Gamma := (\mathfrak{g}_0, \mathrm{BCH}, 0)$

$\{\mathscr{P}_\infty - \text{algebra structures on } \mathsf{H}(A)\} \cong \mathrm{MC}(\mathfrak{g})$

Gauge action

$$\begin{array}{ccc} \Gamma \times \mathrm{MC}(\mathfrak{g}) & \longrightarrow & \mathrm{MC}(\mathfrak{g}) \\ (\lambda, \varphi) & \longmapsto & \lambda \cdot \varphi := e^{\mathrm{ad}_\lambda}(\varphi) - \frac{e^{\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda) \end{array}$$

Proposition (Dotsenko – Shadrin – Vallette, 2016)

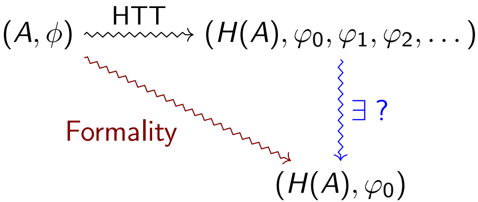
$$\exists\ \infty\text{-quasi-isomorphism } (H(A), \varphi) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_0)$$

$$\Longleftrightarrow$$

$$\exists\ \lambda \in \Gamma \text{ such that } \lambda \cdot \varphi = \varphi_0$$

An equivalent characterization of formality

(A, ϕ): a $\mathscr{P}$-algebra that admits a transferred structure



Definition

- (A, ϕ) is gauge formal if $\exists\ \lambda \in \Gamma$ such that $\lambda \cdot \varphi = \varphi_0$
- (A, ϕ) is gauge n-formal if $\exists\ \lambda \in \Gamma$ such that

$$\lambda \cdot \varphi = (\varphi_0, 0, \ldots, 0, \psi_{n+1}, \ldots)\ .$$

Kaledin classes

Formal deformation

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots) \in \mathrm{MC}(\mathfrak{g})$$

A formal deformation of φ_0:

$$\Phi := \varphi_0 + \varphi_1 \hbar + \varphi_2 \hbar^2 + \cdots + \varphi_k \hbar^k + \cdots$$

in the dg Lie algebra $\mathfrak{g}[\![\hbar]\!] := \mathfrak{g} \widehat{\otimes} R[\![\hbar]\!]$.

Remark

$\Phi \in \mathrm{MC}(\mathfrak{g}[\![\hbar]\!])$, i.e. $d(\Phi) + \frac{1}{2}[\Phi, \Phi] = 0$.

Proposition

$d^\Phi := d + [\Phi, -]$ *is a differential on* $\mathfrak{g}[\![\hbar]\!]$

Twisted dg Lie algebra:

$$\mathfrak{g}[\![\hbar]\!]^\Phi := (\mathfrak{g}[\![\hbar]\!], [-,-], d^\Phi)$$

The Kaledin classes

$$\partial_\hbar \Phi := \varphi_1 + 2\varphi_2 \hbar + \cdots + k\varphi_k \hbar^{k-1} + \cdots \in \mathfrak{g}[\![\hbar]\!]$$

Lemma

$\partial_\hbar \Phi$ is a cycle in $\mathfrak{g}[\![\hbar]\!]^\Phi$, i.e. $d^\Phi(\partial_\hbar \Phi) = 0$.

Definition (Kaledin class)

The Kaledin class of $\varphi \in \mathrm{MC}(\mathfrak{g})$ is the homology class

$$K_\Phi := [\partial_\hbar \Phi] \in H_{-1}\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right) .$$

Its n^{th}-truncated Kaledin class is

$$K_\Phi^n := \left[\varphi_1 + 2\varphi_2 \hbar + \cdots + n\varphi_n \hbar^{n-1}\right] \in H_{-1}\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^n\right)^{\widetilde{\Phi}}\right) .$$

Kaledin class:

$$K_\Phi := \left[\varphi_1 + 2\varphi_2\hbar + 3\varphi_3\hbar^2 + \cdots\right] \in H_{-1}\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

n^{th}-truncated Kaledin class :

$$K_\Phi^n := \left[\varphi_1 + 2\varphi_2\hbar + \cdots + n\varphi_n\hbar^{n-1}\right] \in H_{-1}\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^n\right)^{\widetilde{\Phi}}\right)$$

Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

R : *$\mathbb{Q}$-algebra*

$\mathcal{P}$: *Koszul operad*

(A, ϕ) : *dg $\mathcal{P}$-algebra that admits a transferred structure*

- (A, ϕ) *is gauge formal* $\iff K_\Phi = 0$.
- (A, ϕ) *is gauge n-formal* $\iff K_\Phi^n = 0$.

Kaledin class:

$$K_\Phi := \left[\varphi_1 + 2\varphi_2\hbar + 3\varphi_3\hbar^2 + \cdots\right] \in H_{-1}\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

n^{th}-truncated Kaledin class :

$$K_\Phi^n := \left[\varphi_1 + 2\varphi_2\hbar + \cdots + n\varphi_n\hbar^{n-1}\right] \in H_{-1}\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^n\right)^{\widetilde{\Phi}}\right)$$

Theorem (E., 2024)

R : *commutative ground ring*

$\mathcal{P}$: *(pr)operad colored in groupoids*

n : *integer such that $n!$ is invertible in R*

(A, ϕ) : *dg $\mathcal{P}$-algebra that admits a transferred structure*

- (A, ϕ) *is gauge formal* $\iff K_\Phi = 0$.
- (A, ϕ) *is gauge n-formal* $\iff K_\Phi^n = 0$.

Formality criteria

Formality descent

(A, ϕ) : a dg $\mathscr{P}$-algebra that admits a transferred structure

$H_i(A)$: projective, finitely generated for all i.

S : faithfully flat commutative R-algebra.

Proposition (E., 2024)

(A, ϕ) is gauge n-formal $\iff$ $(A \otimes_R S, \phi \otimes 1)$ is gauge n-formal.

Proof.

$$H_{-1}\left(\mathfrak{g}_{H(A)}[\![\hbar]\!]^{\Phi}\right) \otimes_{R[\![\hbar]\!]} S[\![\hbar]\!] \quad \cong \quad H_{-1}\left(\mathfrak{g}_{H(A \otimes_R S)}[\![\hbar]\!]^{\Phi \otimes 1}\right)$$

□

Examples

- $C(\mathcal{D}_k; \mathbb{R})$ is formal $\iff$ $C(\mathcal{D}_k; \mathbb{Q})$ is formal [GSNPR, 2005]
- $\mathbb{Z}_{(\ell)} \subset \mathbb{Z}_\ell$

Complement of hyperplane arrangements

X : a complement of a hyperplane arrangement over $\mathbb{C}$
$\rightarrow$ complement of a finite collection of affine hyperplanes in $\mathbb{A}^n_{\mathbb{C}}$.

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

s : order of q in $\mathbb{F}_\ell^\times$

Proposition (Dummond-Cole – Horel, 2021)

If X is defined over K, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_\ell)$ is gauge $(s-1)$-formal.

Formality descent $\implies$ $C^\bullet(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s-1)$-formal.

Triviality of fibrations

Theorem (E., 2024)

X : a simply connected topological space

F : a nilpotent space of finite $\mathbb{Q}$-type.

A fibration $\xi : E \to X$ with fiber $F_{\mathbb{Q}}$ is trivial up to homotopy iff $\xi \otimes \mathbb{R}$ is trivial up to homotopy.

Example

The Fadell–Neuwirth fibration :

$$\xi : \mathrm{Conf}_{n-1}\left(\mathbb{R}^d\right) \longrightarrow \mathrm{Conf}_n\left(\mathbb{S}^d\right) \longrightarrow \mathbb{S}^d \ .$$

If $d \geqslant 5$ odd, $\xi \otimes \mathbb{R}$ is trivial up to homotopy [Haya Enriquez, 2022]

$\implies$ ξ is trivial up to homotopy.

Automorphism lifts

(A, ϕ) : a dg $\mathscr{P}$-algebra that admits a transferred structure

Theorem (E., 2024)

Suppose that $u \in \mathrm{Aut}(H(A), \varphi_0)$ admits a chain lift. Let

$$\mathrm{Ad}_u : \mathrm{End}_{H(A)} \to \mathrm{End}_{H(A)} \quad \psi \longmapsto u^{\otimes q} \circ \psi \circ (u^{-1})^{\otimes p} \ ,$$

for $\psi \in \mathsf{End}_{H(A)}(p, q) = \mathsf{Hom}\left(H(A)^{\otimes p}, H(A)^{\otimes q}\right)$

1. *If $\mathrm{Ad}_u - \mathrm{id}$ is invertible, then (A, ϕ) is gauge formal and every homology automorphism admits a chain level lift.*
2. *If $\mathrm{Ad}_u - \mathrm{id}$ is invertible on the elements of degree k for all $k < n$, then (A, ϕ) is gauge n-formal.*

Automorphism lifts

R : a characteristic zero field

(A, ϕ) : a dg $\mathscr{P}$-algebra that admits a transferred structure s.t. $H(A)$ is finite dimensional.

Corollary (E., 2024)

Suppose that there exists $u \in \mathrm{Aut}(H(A), \varphi_0)$ such that for all $k < n$, and all p-tuples $(k_1, \ldots, k_p)$,

$$\mathrm{Spec}(u_{k_1+\cdots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing \ ,$$

where $u_i := u_{|H_i(A)}$. If u admits a lift at the level of chains then (A, ϕ) is gauge n-formal.

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers $\mathcal{O}_K$

ℓ : a prime number different from p

X : a smooth proper K-scheme

Definition

$\alpha \in \overline{\mathbb{Q}}_\ell$ is a Weil number of weight n if

$$\forall\, \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2}\ .$$

Theorem (Deligne, 1974)

For all n, the eigenvalues of a Frobenius action on $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight n.

Theorem

Let X be a smooth and proper scheme over $\mathbb{C}$. The algebra $C^\bullet(X_{\mathrm{an}}, \mathbb{Q})$ is formal.

Proof.

- There exists a smooth and proper model $\mathcal{X}$ over $\mathcal{O}_K$.

$$C^\bullet(X_{an}, \mathbb{Q}_\ell) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$$

- Let u be the Frobenius action on $H^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$.
- For all $k \geq 1$, $(k_1, \ldots, k_p)$ and $s := k_1 + \cdots + k_p$,

$$\underset{\substack{\Uparrow \\ \alpha \\ |\iota(\alpha)| = q^{\frac{s+k}{2}}}}{\mathrm{Spec}(u_{s+k})} \quad \cap \quad \underset{\substack{\Uparrow \\ \beta \\ |\iota(\beta)| = q^{\frac{s}{2}}}}{\mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p})} \quad = \varnothing \, .$$

$$|\iota(\alpha)| = q^{\frac{s+k}{2}} \quad > \quad |\iota(\beta)| = q^{\frac{s}{2}}$$

□

Previous work: [Deligne, 1980]

Let Sch_K be the category of smooth and proper schemes over K of *good reduction*, i.e. for which there exists a smooth and proper model over $\mathcal{O}_K$.

Theorem (E., 2024)

Let $\mathbb{V}$ be a groupoid and let $\mathscr{P}$ be a $\mathbb{V}$-colored operad in sets. Let X be a $\mathscr{P}$-algebra in Sch_K. The dg $\mathscr{P}$-algebra $C_\bullet(X_{\mathrm{an}}, \mathbb{Q})$ is formal.

Example (Guillén Santos, Navarro, Pascual, & Roig, 2005)

$\overline{\mathcal{M}}$ the cyclic operad of moduli spaces of stable algebraic curves $C_\bullet(\overline{\mathcal{M}}_{an}; \mathbb{Q})$ is formal

Thank you for your attention!