Obstruction theory to formality and homotopy equivalences

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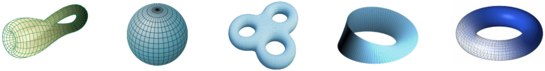
Rational homotopy theory

Goal: Study the rational homotopy type of spaces, i.e. their class up to maps inducing isomorphisms in rational cohomology.



Method: Use algebraic invariants

- Genus = number of holes
- Cohomology ring $H^\bullet(X;\mathbb{Q})$ with the cup product $\cup$
- Massey products = n-ary operations generalizing $\cup$

Rational homotopy theory

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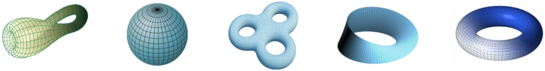

Method: Use algebraic invariants

- Genus = number of holes
- Cohomology ring $H^\bullet(X;\mathbb{Q})$ with the cup product $\cup$
- Massey products = n-ary operations generalizing $\cup$

$\Longrightarrow$ These invariants are not faithful !

Faithful invariants

X : a topological space (simply connected and of finite type)

Theorem (Sullivan, 1977)

The commutative algebra of polynomial forms $\mathcal{A}^\bullet_{\mathrm{PL}}(X)$ is a faithful invariant of the rational homotopy type.

Formal topological spaces

Definition

A space X is formal if there exists a zig-zag of quasi-isomorphisms

$$\mathcal{A}^\bullet_{\mathrm{PL}}(X) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet_{\mathrm{sing}}(X;\mathbb{Q})$$

i.e. morphisms inducing isomorphisms in cohomology.

Remark

X formal $\implies$ The cohomology ring $H^\bullet_{\mathrm{sing}}(X,\mathbb{Q})$ completely determines the rational homotopy type of X.

Examples

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [Deligne–Griffiths–Morgan–Sullivan, '75]

Formality of an algebraic structure

A : cochain complex over R

$\mathscr{P}$: colored operad or properad

$\phi : \mathscr{P} \to \mathrm{End}_A$: a dg $\mathscr{P}$-algebra structure

Definition

The dg $\mathscr{P}$-algebra (A, ϕ) is formal if there exists a zig-zag

$$(A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_*) ,$$

where φ_1 is the canonical $\mathscr{P}$-algebra structure on $H(A)$.

Examples

- X formal if $\mathcal{A}^\bullet_{\mathrm{PL}}(X)$ is formal as dg commutative algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Higher structures

Homotopy retracts

Definition

(W, d_W) is a homotopy retract of (V, d_V) if there are maps

$$\begin{tikzcd} (V, d_V) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (W, d_W) \arrow[l, shift left, "i"] \end{tikzcd}$$

where $\mathrm{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism .

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Proposition

If R is a field, the cohomology of any cochain complex is a homotopy retract:

$$\begin{tikzcd} (A, d_A) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H(A), 0) \arrow[l, shift left, "i"] \end{tikzcd} .$$

Transfer of algebraic structure

(A, d_A, ϕ) : a dga algebra and a homotopy retraction:

$$\begin{tikzcd} (A, d_A, \phi) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H, d_H) \arrow[l, shift left, "i"] \end{tikzcd}$$

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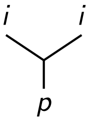
$\rightarrow$ Transferred product: $\varphi_2 := p \circ \phi \circ i^{\otimes 2} : H^{\otimes 2} \to H$



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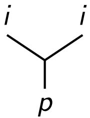
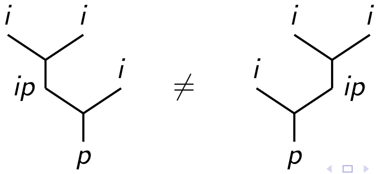
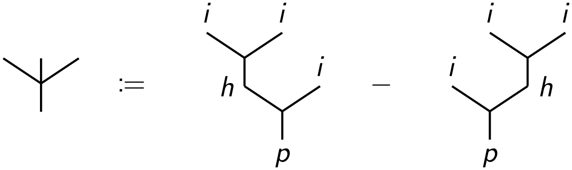


Not associative in general!



→ Consider $\varphi_3 : H^{\otimes 3} \to H$



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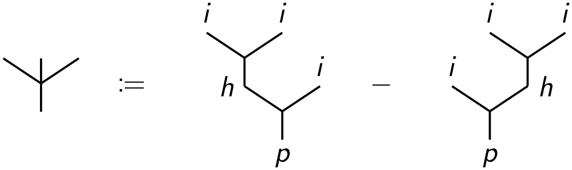
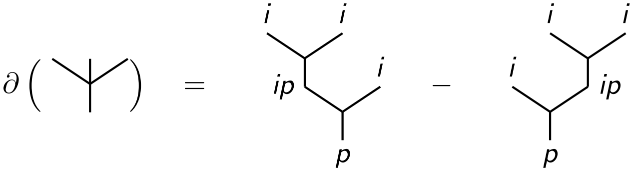


→ In $(H^{\otimes 3}, H)$:



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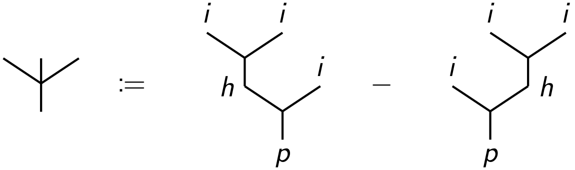

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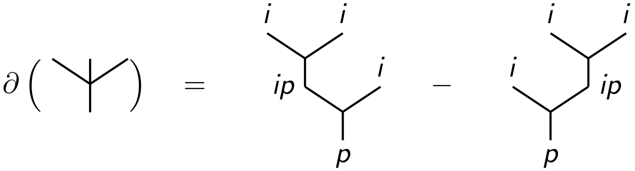
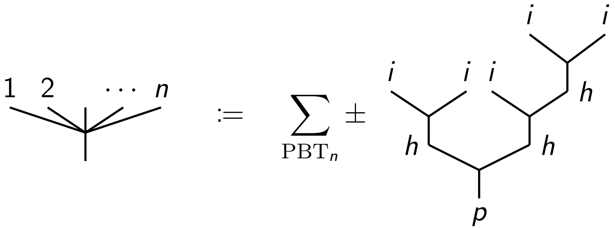


→ φ_2 is associative up to the homotopy φ_3.

$\rightarrow \varphi_n : H^{\otimes n} \rightarrow H$, for all $n \geqslant 2$



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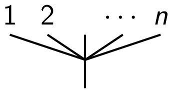
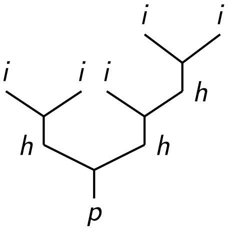
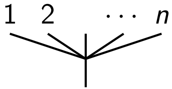


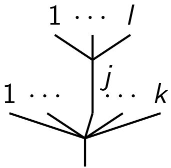
$$:= \sum_{\mathrm{PBT}_n} \pm$$



$$\partial \Bigg($$



$$\Bigg) = \sum_{\substack{k+l=n+1 \\ 1 \leqslant j \leqslant k}} \pm$$



Homotopy associative algebras

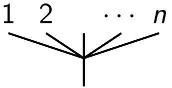
Definition

A_∞-algebra: a cochain complex H with a collection of maps

$$\varphi_n : H^{\otimes n} \to H$$

of degree $2-n$, for all $n \geqslant 2$, which satisfy the relations

$\partial \Bigg($



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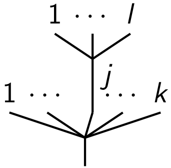


Examples

- Every dga algebra (A, ϕ) is an A_∞-algebra with $\varphi_n = 0$ for all $n \geqslant 3$.
- $(H, d_H, \varphi_2, \varphi_3, \dots)$

Homotopy transfer theorem

Theorem (Kadeishvili, 1982)

Given a dga algebra (A, d_A, ϕ) and a homotopy retract

$$\begin{tikzcd} (A, d_A, \phi) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H, d_H) \arrow[l, shift left, "i"] \end{tikzcd}$$

there exists an A_∞-algebra structure on H such that p (and i) extend to A_∞-quasi-isomorphisms:

$$(A, d_A, \phi) \rightsquigarrow (H, d_H, \varphi_2, \varphi_3, \varphi_4, \dots)$$

Homotopy morphisms

$(A, d_A, \phi_2, \dots)$, $(H, d_H, \varphi_2, \dots)$: A_∞-algebras

Homotopy morphisms

$(A, d_A, \phi_2, \ldots), \ (H, d_H, \varphi_2, \ldots) \ : \ A_\infty$-algebras

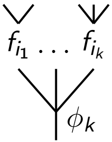
Definition

A_∞-morphism $f : A \rightsquigarrow H$ is a collection of linear maps

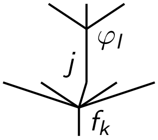
$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geqslant 1 \ ,$$

of degree $1 - n$, which satisfy the relations

$$\sum_{\substack{k \geqslant 1 \\ i_1 + \cdots + i_k = n}} \pm$$



$$= \sum_{\substack{k+l=n+1 \\ 1 \leqslant j \leqslant k}} \pm$$



where $\varphi_1 = d_H$ and $\phi_1 = d_A$.

Homotopy quasi-isomorphisms

Definition

A_∞-quasi-isomorphism $f : A \overset{\sim}{\rightsquigarrow} H$ is an A_∞-morphism where $f_1 : A \to H$ is a quasi-isomorphism .

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Proposition (R is a characteristic zero field)

quasi-isos of associative algebras $\qquad\qquad$ *A_∞-quasi-iso*

$$\exists\ (A,\phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B,\phi') \quad \Longleftrightarrow \quad \exists\ (A,\phi) \overset{\sim}{\rightsquigarrow} (B,\phi')$$

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Corollary

A dga algebra (A,ϕ) is formal if and only if

$$\exists\ (A,\phi) \overset{\sim}{\rightsquigarrow} (H(A),\varphi_*)\ .$$

An equivalent characterization of formality

(A, d, ϕ) a dga algebra such that $H(A)$ is a homotopy retract

$$
\begin{tikzcd}
(A, d, \phi) \arrow[r, rightsquigarrow, "\text{HTT}"] \arrow[dr, rightsquigarrow, "\text{Formality}"'] & (H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \arrow[d, rightsquigarrow, "\exists\ ?"] \\
 & (H(A), \varphi_*)
\end{tikzcd}
$$

Higher Massey products

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Higher Massey products

$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

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Higher Massey products

$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

Definition

- (A, d, ϕ) is gauge formal if $\exists\ (H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*)$.

An equivalent characterization of formality

(A, d, ϕ) a dga algebra such that $H(A)$ is a homotopy retract

$$\begin{tikzcd}
(A,d,\phi) \arrow[r, squiggly, "\mathrm{HTT}"] \arrow[dr, squiggly, "\text{Formality}"'] & (H(A),\varphi_*,\varphi_3,\varphi_4\ldots) \arrow[d, squiggly, "\exists\ ?"] \\
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Definition

- (A, d, ϕ) is gauge formal if $\exists\, (H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*)$.
- (A, d, ϕ) is gauge n-formal if

$$\exists\, (H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*, 0, \ldots, 0, \varphi'_{n+1}, \ldots)\ .$$

Kaledin classes

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Kaledin class:

$$K_\Phi := \left[\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots\right] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

n^{th}-truncated Kaledin class :

$$K_\Phi^n := \left[\varphi_3 + 2\varphi_4\hbar + \cdots + (n-2)\varphi_n\hbar^{n-3}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^{n-2}\right)^{\widetilde{\Phi}}\right)$$

Theorem ([Kaledin, 2007], [Lunts, 2007])

R : $\mathbb{Q}$-algebra

(A, ϕ) : dg associative algebra, $H(A)$ is a homotopy retract

- *(A, ϕ) is gauge formal $\iff K_\Phi = 0$.*
- *(A, ϕ) is gauge n-formal $\iff K_\Phi^n = 0$.*

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Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

R : $\mathbb{Q}$-algebra

$\mathscr{P}$: binary Koszul operad

(A, ϕ) : dg -algebra such that $H(A)$ is a homotopy retract

- *(A, ϕ) is gauge formal $\Longleftrightarrow K_\Phi = 0$.*
- *(A, ϕ) is gauge n-formal $\Longleftrightarrow K_\Phi^n = 0$.*

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Theorem (E., 2024)

R : *commutative ring*

$\mathscr{P}$: *(pr)operad, possibly coloured in groupoids*

(A, ϕ) : *dg -algebra such that $H(A)$ is a homotopy retract*

- *If R is a $\mathbb{Q}$-algebra, (A, ϕ) is gauge formal $\iff K_\Phi = 0$.*
- *If $n!$ is invertible in R, (A, ϕ) is gauge n-formal $\iff K_\Phi^n = 0$.*

Properadic coformality of spheres

Example (Kontsevich–Takeda–Vlassopoulos, 2021)

$A = C_*(\Omega S^n; R)$ has an pre-Calabi-Yau structure

$$\phi = \underbrace{m_{(1)}}_{A_\infty\text{-alg}} + \underbrace{m_{(2)}}_{\substack{\text{Poisson bivector} \\ \text{up to homotopy}}} + m_{(3)} + \cdots$$

where $m_{(\ell)}$ is a cyclically anti-symmetric collection of maps

$$m_{(\ell)}^{k_1,\dots,k_\ell} : sA^{k_1} \otimes \cdots \otimes sA^{k_\ell} \to A^\ell \ .$$

$\implies$ encodes Poincaré duality.

Theorem (E.-Takeda, 2025)

If R is a $\mathbb{Q}$-algebra, $(C_*(\Omega S^n; R), \phi)$ is gauge formal.

Complement of hyperplane arrangements

K : a finite extension of $\mathbb{Q}_p$

ℓ : a prime number different from p

q : order of the residue field of the ring of integers of K

s : order of q in $\mathbb{F}_\ell^\times$

X : a complement of a hyperplane arrangement over $\mathbb{C}$

Theorem (E., 2024)

If X is defined over K, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a complement of hyperplane arrangement over K such that $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet\left(X_{an}, \mathbb{Z}_{(\ell)}\right)$ is gauge $(s-1)$-formal.

Triviality of fibrations

Theorem (E., 2024)

X : a simply connected topological space

F : a nilpotent space of finite $\mathbb{Q}$-type.

A fibration $\xi : E \to X$ with fiber $F_{\mathbb{Q}}$ is trivial up to homotopy if and only if $\xi \otimes \mathbb{R}$ is trivial up to homotopy.

Example

The Fadell–Neuwirth fibration :

$$\xi : \mathrm{Conf}_{n-1}\left(\mathbb{R}^d\right) \longrightarrow \mathrm{Conf}_n\left(\mathbb{S}^d\right) \longrightarrow \mathbb{S}^d \ .$$

If d is odd, $\xi \otimes \mathbb{R}$ is trivial up to homotopy [Haya Enriquez, 2022]

$\implies$ ξ is trivial up to homotopy.

Beyond formality

Homotopy equivalences between algebraic structures

Definition

The dg $\mathscr{P}$-algebras (A, ϕ) and (B, ψ) are

- homotopy equivalent

$$\exists\, (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \psi)$$

- gauge homotopy equivalent if $\exists\, (A, \phi) \overset{\sim}{\rightsquigarrow} (B, \psi)$.

Example

(A, ϕ) is formal $\iff$ it is homotopy equivalent to $(H(A), \varphi_*)$

Minimal model on highly connected variety

Theorem (E., 2025)

Let $\mathbb{K}$ be a field. Let M^d be a compact k-connected oriented C^∞-manifold where d is smaller than $(\ell+1)k+2$. The algebra

$$C^\bullet_{\mathrm{sing}}(M, \mathbb{K})$$

is homotopy equivalent to an A_∞-algebra $(H^\bullet_{\mathrm{sing}}(M, \mathbb{K}), \varphi)$, with $\varphi_n = 0$ for $n \geqslant \ell$.

Thank you for your attention!