Kaledin classes & formality criteria

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The notion of formality

Formal topological spaces

R : commutative ground ring

Definition

A topological space X is formal if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C^\bullet_{\text{sing}}(X;R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H^\bullet_{\text{sing}}(X;R)\ .$$

$\rightarrow$ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal $\implies$ The cohomology ring $H^\bullet_{\text{sing}}(X,\mathbb{Q})$ completely determines the rational homotopy type of X.

Examples

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [Deligne, Griffiths, Morgan & Sullivan, 1975]

Formality of an algebraic structure

A : chain complex over R

$\mathscr{P}$: colored operad or properad

$\phi : \mathscr{P} \to \mathrm{End}_A$: a dg $\mathscr{P}$-algebra structure

Definition

The dg $\mathscr{P}$-algebra (A, ϕ) is formal if

$$\exists\ (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_*)\ ,$$

where φ_* is the canonical $\mathscr{P}$-algebra structure on $H(A)$.

Examples

- X is formal $=$ $(C^\bullet_{\mathrm{sing}}(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Purity implies formality

(A, ϕ) : dg $\mathscr{P}$-algebra encoded by an operad $\mathscr{P}$

α : unit of infinite order in R

σ_α : the degree twisting by α = automorphism of $(H(A), \varphi_*)$ which acts via $\alpha^k \times$ on $H_k(A)$.

Theorem

If σ_α admits a chain-level lift, i.e. $\exists\, f \in \mathrm{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$, then (A, ϕ) is formal.

→ Deligne, Griffiths, Morgan, Sullivan [1975]

→ Sullivan [1977]

→ Guillén Santos, Navarro, Pascual, Roig [2005]

→ Drummond-Cole and Horel [2021]

Example (Drummond-Cole and Horel, 2021)

X : a complement of a hyperplane arrangement over $\mathbb{C}$ defined over a finite extension K of $\mathbb{Q}_p$.

ℓ : a prime number different from p

$\rightarrow C^\bullet(X_{an}, \mathbb{Z}_\ell) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ [Artin].

$\rightarrow$ A Frobenius action on $H^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Z}_\ell)$ is σ_q [Kim, 1994].

Questions

- Can we descend these results to other coefficient rings? (e.g. $\mathbb{Z}_{(\ell)}, \ldots$)
- Does the degree twisting criteria hold for other types of algebras? (e.g. Hopf algebras, involutive Lie bialgebras,...)
- Is the degree twisting the only homology automorphism satisfying this property?

Higher structures

Homotopy retracts

Definition

(W, d_W) is a homotopy retract of (V, d_V) if there are maps

$$h \circlearrowright (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, d_W)$$

where $\mathrm{id}_V - ip = d_V h + h d_V$ and i is a quasi-isomorphism .

Proposition

If R is a field, the cohomology of any cochain complex is a homotopy retract:

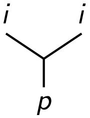
$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H(A), 0) \ .$$

Transfer of algebraic structure

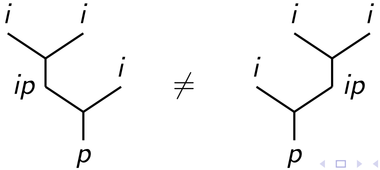
(A, d_A, ϕ) : a dga algebra and a homotopy retraction:

$$h \circlearrowright (A, d_A, \phi) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

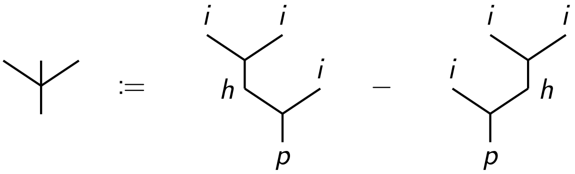
$\rightarrow$ Transferred product: $\varphi_2 := p \circ \phi \circ i^{\otimes 2} : H^{\otimes 2} \to H$



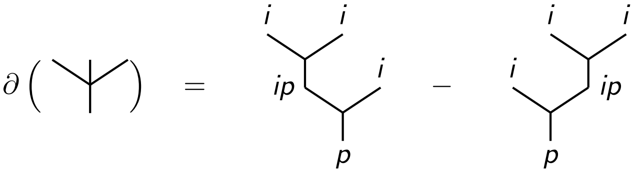
Not associative in general!



→ Consider $\varphi_3 : H^{\otimes 3} \to H$

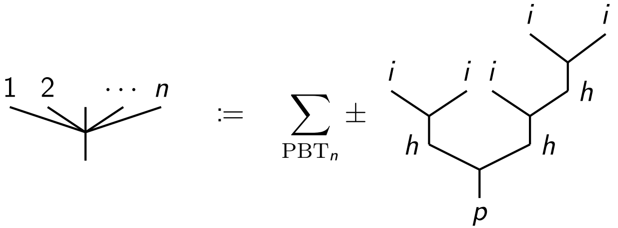


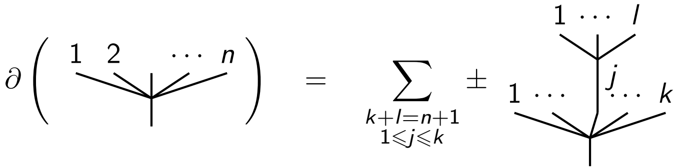
→ In $\mathrm{Hom}(H^{\otimes 3}, H)$:



→ φ_2 is associative up to the homotopy φ_3.

→ $\varphi_n : H^{\otimes n} \to H$, for all $n \geqslant 2$





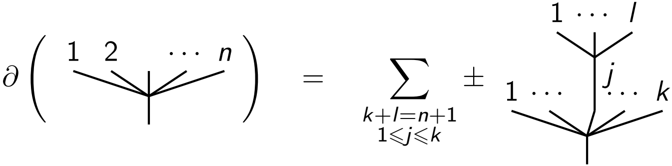
Homotopy associative algebras

Definition (Stasheff, 1963)

A_∞-algebra: a cochain complex H with a collection of maps

$$\varphi_n : H^{\otimes n} \to H$$

of degree $2-n$, for all $n \geqslant 2$, which satisfy the relations



Examples

- Every dga algebra (A, ϕ) is an A_∞-algebra with $\varphi_n = 0$ for all $n \geqslant 3$.
- $(H, d_H, \varphi_2, \varphi_3, \dots)$

Homotopy transfer theorem

Theorem (Kadeishvili, 1982)

Given a dga algebra (A, d_A, ϕ) *and a homotopy retract*

$$h \circlearrowright (A, d_A, \phi) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

there exists an A_∞*-algebra structure on* H *such that* p *(and* i*) extend to* A_∞*-quasi-isomorphisms:*

$$(A, d_A, \phi) \rightsquigarrow (H, d_H, \varphi_2, \varphi_3, \varphi_4, \dots)$$

Homotopy morphisms

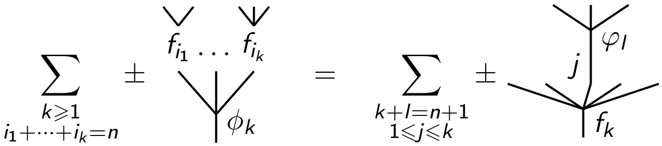
$(A, d_A, \phi_2, \dots), (H, d_H, \varphi_2, \dots)$: A_∞-algebras

Definition

A_∞-morphism $f : A \rightsquigarrow H$ is a collection of linear maps

$$f_n : A^{\otimes n} \longrightarrow H, \quad n \geqslant 1 ,$$

of degree $1 - n$, which satisfy the relations



where $\varphi_1 = d_H$ and $\phi_1 = d_A$.

Homotopy quasi-isomorphisms

Definition

A_∞-quasi-isomorphism $f : A \xrightarrow{\sim} H$ is an A_∞-morphism where $f_1 : A \to H$ is a quasi-isomorphism .

Proposition (R is a field)

quasi-isos of associative algebras $\qquad$ *A_∞-quasi-iso*

$$\exists\, (A,\phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B,\phi') \iff \exists\, (A,\phi) \overset{\sim}{\rightsquigarrow} (B,\phi')$$

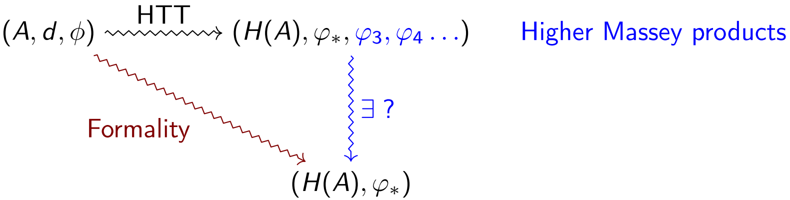
Corollary

A dga algebra (A,ϕ) is formal if and only if

$$\exists\, (A,\phi) \overset{\sim}{\rightsquigarrow} (H(A),\varphi_*)\ .$$

An equivalent characterization of formality

(A, d, ϕ) a dga algebra such that $H(A)$ is a homotopy retract



$\Longrightarrow$ If the higher Massey products vanish, then (A, d, ϕ) is formal.

Definition

- (A, d, ϕ) is gauge formal if $\exists\, (H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*)$.
- (A, d, ϕ) is gauge *n*-formal if

$$\exists\, (H(A), \varphi_*, \varphi_3, \varphi_4 \ldots) \overset{\sim}{\rightsquigarrow} (H(A), \varphi_*, 0, \ldots, 0, \varphi'_{n+1}, \ldots)\ .$$

Kaledin classes

Hochschild complex

Transfered structure: $(H(A), \varphi_*, \varphi_3, \varphi_4, \dots)$

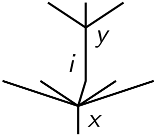
$$\varphi_n \in Hom(H(A)^{\otimes n}, H(A)), \quad |\varphi_n| = 2 - n$$

Hochschild cochain complex:

$$\mathfrak{g} := \prod_{n \geqslant 1} s^{-n+1} Hom(H(A)^{\otimes n}, H(A))$$

Lie bracket : $[x, y] := x \star y - (-1)^{|x||y|} y \star x$

$$x \star y := \sum_{i=1}^{n} (-1)^{(i-1)(m-1)}$$



for $x \in Hom(H(A)^{\otimes n}, H(A))$ and $y \in Hom(H(A)^{\otimes m}, H(A))$.

A formal deformation

Transfered structure:

$$(\varphi_*, \varphi_3, \varphi_4, \ldots) \in \mathfrak{g} := \prod_{n \geqslant 1} s^{-n+1} Hom(H(A)^{\otimes n}, H(A))$$

A formal deformation:

$$\Phi := \varphi_* + \varphi_3 \hbar + \varphi_4 \hbar^2 + \cdots \in \mathfrak{g}[\![\hbar]\!] := \mathfrak{g} \widehat{\otimes} R[\![\hbar]\!]$$

Proposition : $\mathrm{ad}_\Phi := [\Phi, -]$ defines a differential on $\mathfrak{g}[\![\hbar]\!]$

Twisted dg Lie algebra:

$$\mathfrak{g}[\![\hbar]\!]^\Phi := (\mathfrak{g}[\![\hbar]\!], [-,-], \mathrm{ad}_\Phi)$$

Kaledin classes

$$\partial_\hbar \Phi := \varphi_3 + 2\varphi_4 \hbar + 3\varphi_5 \hbar^2 + \cdots \in \mathfrak{g}[\![\hbar]\!]$$

Lemma : $\partial_\hbar \Phi$ is a cycle in $\mathfrak{g}[\![\hbar]\!]^\Phi := (\mathfrak{g}[\![\hbar]\!], [-,-], \mathrm{ad}_\Phi)$,

$$\mathrm{ad}_\Phi(\partial_\hbar \Phi) := [\Phi, \partial_\hbar \Phi] = 0 \ .$$

Kaledin class:

$$K_\Phi := [\partial_\hbar \Phi] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right) \ .$$

n^{th}-truncated Kaledin class :

$$K_\Phi^n := \left[\varphi_3 + 2\varphi_4 \hbar + \cdots + (n-2)\varphi_n \hbar^{n-3}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^{n-2}\right)^{\widetilde{\Phi}}\right) \ .$$

Kaledin classes

Kaledin class:

$$K_\Phi := \left[\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots\right] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

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Theorem ([Kaledin, 2007], [Lunts, 2007])

R : $\mathbb{Q}$-algebra

(A, ϕ) : dg associative algebra, $H(A)$ is a homotopy retract

- *(A, ϕ) is gauge formal $\iff K_\Phi = 0$.*
- *(A, ϕ) is gauge n-formal $\iff K_\Phi^n = 0$.*

Kaledin class:

$$K_\Phi := \left[\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots\right] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

n^{th}-truncated Kaledin class :

$$K_\Phi^n := \left[\varphi_3 + 2\varphi_4\hbar + \cdots + (n-2)\varphi_n\hbar^{n-3}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^{n-2}\right)^{\widetilde{\Phi}}\right)$$

Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

R : *$\mathbb{Q}$-algebra*

$\mathcal{P}$: *Koszul operad*

(A, ϕ) : *dg $\mathcal{P}$-algebra such that $H(A)$ is a homotopy retract*

- *(A, ϕ) is gauge formal $\Longleftrightarrow K_\Phi = 0$.*
- *(A, ϕ) is gauge n-formal $\Longleftrightarrow K_\Phi^n = 0$.*

Kaledin class:

$$K_\Phi := \left[\varphi_3 + 2\varphi_4\hbar + 3\varphi_5\hbar^2 + \cdots\right] \in H^1\left(\mathfrak{g}[\![\hbar]\!]^\Phi\right)$$

n^{th}-truncated Kaledin class :

$$K_\Phi^n := \left[\varphi_3 + 2\varphi_4\hbar + \cdots + (n-2)\varphi_n\hbar^{n-3}\right] \in H^1\left(\left(\mathfrak{g}[\![\hbar]\!]/\hbar^{n-2}\right)^{\widetilde{\Phi}}\right)$$

Theorem (E., 2023)

R : *commutative ring*

$\mathcal{P}$: *(Pr)operad, possibly coloured in groupoids*

(A, ϕ) : *dg* $\mathcal{P}$*-algebra such that* $H(A)$ *is a homotopy retract*

- (A, ϕ) *is gauge formal* $\iff K_\Phi = 0$.
- (A, ϕ) *is gauge n-formal* $\iff K_\Phi^n = 0$.

Formality criteria

Formality descent

(A, ϕ) : a dg $\mathscr{P}$-algebra that admits a transferred structure

$H_i(A)$: projective, finitely generated for all i.

S : faithfully flat commutative R-algebra.

Proposition (E., 2024)

(A, ϕ) is gauge n-formal $\iff$ $(A \otimes_R S, \phi \otimes 1)$ is gauge n-formal.

Proof.

$$H_{-1}\left(\mathfrak{g}_{H(A)}[\![\hbar]\!]^{\Phi}\right) \otimes_{R[\![\hbar]\!]} S[\![\hbar]\!] \quad \cong \quad H_{-1}\left(\mathfrak{g}_{H(A\otimes_R S)}[\![\hbar]\!]^{\Phi\otimes 1}\right) \qquad \square$$

Examples

- $C(\mathcal{D}_k;\mathbb{R})$ is formal $\iff$ $C(\mathcal{D}_k;\mathbb{Q})$ is formal [GSNPR, 2005]
- $\mathbb{Z}_{(\ell)} \subset \mathbb{Z}_\ell$

Complement of hyperplane arrangements

X : a complement of a hyperplane arrangement over $\mathbb{C}$
$\rightarrow$ complement of a finite collection of affine hyperplanes in $\mathbb{A}^n_{\mathbb{C}}$.

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

s : order of q in $\mathbb{F}_\ell^\times$

Proposition (Dummond-Cole – Horel, 2021)

If X is defined over K, i.e. $\exists\ K \hookrightarrow \mathbb{C}$ and $\exists\ \mathcal{X}$ a complement of a hyperplane arrangement over K s.t. $\mathcal{X} \times_K \mathbb{C} \cong X$, then $C^\bullet(X_{an}, \mathbb{Z}_\ell)$ is gauge $(s-1)$-formal.

Formality descent $\implies$ $C^\bullet(X_{an}, \mathbb{Z}_{(\ell)})$ is gauge $(s-1)$-formal.

Automorphism lifts

(A, ϕ) : a dg $\mathscr{P}$-algebra that admits a transferred structure

Theorem (E., 2024)

Suppose that $u \in \mathrm{Aut}(H(A), \varphi_0)$ *admits a chain lift. Let*

$$\mathrm{Ad}_u : \mathrm{End}_{H(A)} \to \mathrm{End}_{H(A)} \quad \psi \longmapsto u^{\otimes q} \circ \psi \circ (u^{-1})^{\otimes p} ,$$

for $\psi \in \mathsf{End}_{H(A)}(p, q) = \mathsf{Hom}\left(H(A)^{\otimes p}, H(A)^{\otimes q}\right)$

1. *If* $\mathrm{Ad}_u - \mathrm{id}$ *is invertible, then* (A, ϕ) *is gauge formal and every homology automorphism admits a chain level lift.*
2. *If* $\mathrm{Ad}_u - \mathrm{id}$ *is invertible on the elements of degree* k *for all* $k < n$*, then* (A, ϕ) *is gauge* n*-formal.*

Automorphism lifts

R : a characteristic zero field

(A, ϕ) : a dg $\mathscr{P}$-algebra that admits a transferred structure s.t. $H(A)$ is finite dimensional.

Corollary (E., 2024)

Suppose that there exists $u \in \mathrm{Aut}(H(A), \varphi_0)$ such that for all $k < n$, and all p-tuples $(k_1, \ldots, k_p)$,

$$\mathrm{Spec}(u_{k_1+\cdots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p}) = \varnothing \ ,$$

where $u_i := u_{|H_i(A)}$. If u admits a lift at the level of chains then (A, ϕ) is gauge n-formal.

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers $\mathcal{O}_K$

ℓ : a prime number different from p

X : a smooth proper K-scheme

Definition

$\alpha \in \overline{\mathbb{Q}}_\ell$ is a Weil number of weight n if

$$\forall\, \iota : \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2} \ .$$

Theorem (Deligne, 1974)

For all n, the eigenvalues of a Frobenius action on $H^n_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight n.

Theorem

Let X be a smooth and proper scheme over $\mathbb{C}$. The algebra $C^\bullet(X_{\mathrm{an}}, \mathbb{Q})$ is formal.

Proof.

- There exists a smooth and proper model $\mathcal{X}$ over $\mathcal{O}_K$.

$$C^\bullet(X_{an}, \mathbb{Q}_\ell) \cong C^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$$

- Let u be the Frobenius action on $H^\bullet_{et}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell)$.
- For all $k \geq 1$, $(k_1, \ldots, k_p)$ and $s := k_1 + \cdots + k_p$,

$$\underset{\substack{\Uplus \\ \alpha \\ |\iota(\alpha)| = q^{\frac{s+k}{2}}}}{\mathrm{Spec}(u_{s+k})} \quad \cap \quad \underset{\substack{\Uplus \\ \beta \\ |\iota(\beta)| = q^{\frac{s}{2}}}}{\mathrm{Spec}(u_{k_1} \otimes \cdots \otimes u_{k_p})} = \varnothing \, .$$

$$|\iota(\alpha)| = q^{\frac{s+k}{2}} \quad > \quad |\iota(\beta)| = q^{\frac{s}{2}}$$

□

Previous work: [Deligne, 1980]

Let Sch_K be the category of smooth and proper schemes over K of *good reduction*, i.e. for which there exists a smooth and proper model over $\mathcal{O}_K$.

Theorem (E., 2024)

Let $\mathbb{V}$ be a groupoid and let $\mathscr{P}$ be a $\mathbb{V}$-colored operad in sets. Let X be a $\mathscr{P}$-algebra in Sch_K. The dg $\mathscr{P}$-algebra $C_\bullet(X_{\mathrm{an}}, \mathbb{Q})$ is formal.

Example (Guillén Santos, Navarro, Pascual, & Roig, 2005)

$\overline{\mathcal{M}}$ the cyclic operad of moduli spaces of stable algebraic curves $C_\bullet(\overline{\mathcal{M}}_{an}; \mathbb{Q})$ is formal

Thank you for your attention!