



Kaledin classes & formality criteria

Coline Emprin

Mid-term workshop of the ANR HighAGT

Université de Strasbourg

May 31, 2023

Kaledin classes & formality criteria

1. The notion of formality

- The formality can be addressed as a deformation problem, using the operadic calculus.

2. Kaledin obstruction classes

- Construction of an obstruction theory to formality over any coefficient ring.

3. Formality criteria

- Formality descent with torsion coefficients
- Intrinsic formality criterium
- Degree twisting & automorphisms lifts



The notion of formality



Formal topological spaces

R : commutative ground ring

Definition

A topological space X is **formal** if there exists a zig-zag of quasi-isomorphisms of dga algebras,

$$C_{\text{sing}}^{\bullet}(X; R) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} H_{\text{sing}}^{\bullet}(X; R) .$$

→ Origins in rational homotopy theory (for $\mathbb{Q} \subset R$)

X formal $\implies$ The cohomology ring $H_{\text{sing}}^{\bullet}(X, \mathbb{Q})$ completely determines the rational homotopy type of X .

Examples

- Spheres, complex projective spaces, Lie groups
- Compact Kähler manifolds [DGMS, 1975]

Formality of an algebraic structure

A : chain complex over R

 $\mathcal{P}$: graded algebraic operad $\phi : \mathcal{P} \rightarrow \text{End}_A$ a dg $\mathcal{P}$ -algebra structure

Definition

The dg $\mathcal{P}$ -algebra (A, ϕ) is **formal** if

$$\exists (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \dots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (H(A), \varphi_0),$$

where φ_0 is the canonical $\mathcal{P}$ -algebra structure on $H(A)$.

Examples

- X is formal $:= (C_{\text{sing}}^\bullet(X; R), \cup)$ is formal as dga algebra
- $C(\mathcal{D}_k; \mathbb{R})$ is formal as an operad [Kontsevich, 1999]

Operadic homological algebra

A : chain complex over R

Assumptions:

- reduced connected weight-graded (co)operads
- R is a $\mathbb{Q}$ -algebra in the case of symmetric operads

$\mathcal{P}$: graded Koszul operad

$$\mathcal{P}_\infty \xrightarrow{\sim} \mathcal{P} \text{ with } \mathcal{P}_\infty := \Omega \mathcal{P}^i$$
$$\{\mathcal{P}_\infty\text{-algebra structures on } A\} := \mathrm{Hom}_{\mathrm{dgOp}}(\Omega^{\mathcal{P}^i}, \mathrm{End}_A)$$

Proposition

$$\mathrm{Hom}_{\mathrm{dgOp}}\left(\Omega^{\mathcal{P}^i}, \mathrm{End}_A\right) \cong \mathrm{Codiff}\left(\mathcal{P}^i(A)\right)$$

Definition

An ∞ -morphism $F : (A, \varphi) \rightsquigarrow (B, \psi)$ between $\mathcal{P}_\infty$ -algebra structures is a morphism of dg $\mathcal{P}^i$ -coalgebras:

$$(\mathcal{P}^i(A), \varphi) \rightarrow (\mathcal{P}^i(B), \psi) .$$

F is an ∞ -quasi-isomorphism if $F_0 : A \rightarrow B$ is a quasi-isomorphism.

Proposition (R is a characteristic zero field)

zig-zag of quasi-isos of $\mathcal{P}$ -algebras

∞ -quasi-iso

$$\exists (A, \phi) \xleftarrow{\sim} \cdot \xrightarrow{\sim} \cdots \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \phi') \iff \exists (A, \phi) \rightsquigarrow (B, \phi')$$

Corollary

A dg $\mathcal{P}$ -algebra (A, ϕ) is formal $\iff \exists (A, \phi) \rightsquigarrow (H(A), \varphi_0)$.

Homotopy transfer

Theorem (Homotopy transfer theorem)

Let (A, d) be a chain complex s.t. $H(A)$ is a homotopy retract:

$$h \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} (A, d) \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \end{array} (H(A), 0)$$

where $\text{id}_A - ip = d_A h + h d_A$ and i is a quasi-isomorphism.

For every $\mathcal{P}$ -algebra structure (A, ϕ) , there exists a $\mathcal{P}_\infty$ -algebra structure φ on $H(A)$ s.t. p extends to an ∞ -quasi-isomorphism:

$$\begin{array}{ccc} (A, \phi) & \xrightarrow{p_\infty} & (H(A), \varphi) \\ & \searrow \text{Formality} & \downarrow \exists ? \\ & & (H(A), \varphi_0) \end{array}$$

$\mathcal{P}_\infty$ -algebra structures on $H(A)$

The **convolution dg Lie algebra** associated to $H(A)$:

$$\mathfrak{g} := \left(\text{Hom}(\overline{\mathcal{P}}^i, \text{End}_{H(A)}), [-, -], d \right)$$

$$\rightarrow \text{Hom}(\overline{\mathcal{P}}^i, \text{End}_{H(A)}) := \prod_{n>0} \text{Hom}(\overline{\mathcal{P}}^i(n), \text{End}_{H(A)}(n))$$

$$\rightarrow d(\varphi) := -(-1)^{|\varphi|} \varphi \circ d_{\overline{\varphi}^i}$$

$$\rightarrow \varphi \star \psi := \overline{\mathcal{P}}^i \xrightarrow{\Delta(\mathbf{1})} \overline{\mathcal{P}}^i \circ_{(\mathbf{1})} \overline{\mathcal{P}}^i \xrightarrow{\varphi \circ (\mathbf{1}) \psi} \text{End}_{H(A)} \circ_{(\mathbf{1})} \text{End}_{H(A)} \xrightarrow{\gamma(\mathbf{1})} \text{End}_{H(A)}$$

$$\rightarrow [\varphi, \psi] := \varphi \star \psi - (-1)^{|\varphi||\psi|} \psi \star \varphi$$

Every $\varphi \in \text{Hom}(\overline{\mathcal{P}}^i, \text{End}_{H(A)})$ decomposes as

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots)$$

where φ_k is the restriction $\varphi_k : \overline{\mathcal{P}}^i(k+2) \implies \text{End}_{H(A)}(k+2)$.

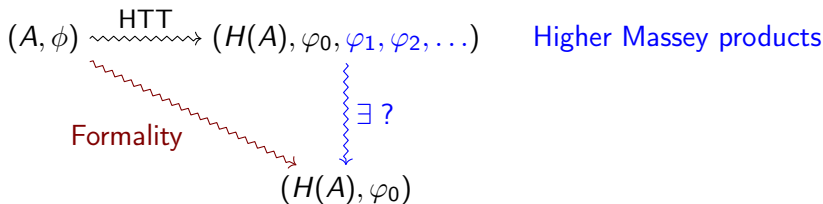
Its set of **Maurer–Cartan elements**:

$$\mathrm{MC}(\mathfrak{g}) = \{\varphi \in \mathfrak{g}_{-1}, d(\varphi) + \frac{1}{2}[\varphi, \varphi] = 0\}$$

Proposition

$$\{\mathcal{P}_\infty - \text{algebra structures on } H(A)\} \cong \mathrm{MC}(\mathfrak{g})$$

Remark



$\implies$ If the higher Massey products vanish, then (A, ϕ) is formal.

The gauge group

The **convolution dg Lie algebra**:

$$\mathfrak{g} := \left(\text{Hom}(\overline{\mathcal{P}}^i, \text{End}_{H(A)}), [-, -], d \right)$$

Its set of **degree zero elements**:

$$\mathfrak{g}_0 := \text{Hom}(\overline{\mathcal{P}}^i, \text{End}_{H(A)})_0$$

The **Baker–Campbell–Hausdorff** formula, with $\text{ad}_\lambda := [\lambda, -]$:

$$\lambda, \mu \in \mathfrak{g}_0, \quad e^{\text{ad}_{\text{BCH}(\lambda, \mu)}} = e^{\text{ad}_\lambda} \circ e^{\text{ad}_\mu}.$$

$$\text{BCH}(\lambda, \mu) = \lambda + \mu + \frac{1}{2}[\lambda, \mu] + \frac{1}{12}([\lambda, [\lambda, \mu]] + [\mu, [\mu, \lambda]]) + \cdots$$

$$\Gamma := (\mathfrak{g}_0, \text{BCH}, 0)$$

The gauge action

$$\Gamma := (\mathfrak{g}_0, \text{BCH}, 0)$$

$$\{\mathcal{P}_\infty - \text{algebra structures on } H(A)\} := \text{MC}(\mathfrak{g})$$

Gauge action

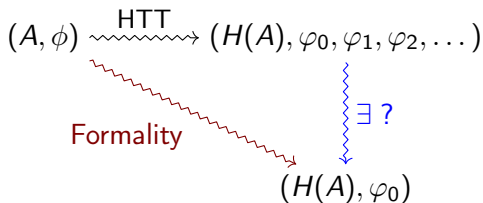
$$\begin{aligned} \Gamma \times \mathrm{MC}(\mathfrak{g}) &\longrightarrow \mathrm{MC}(\mathfrak{g}) \\ (\lambda, \varphi) &\longmapsto \lambda \cdot \varphi := e^{\mathrm{ad}_\lambda}(\varphi) - \frac{e^{\mathrm{ad}_\lambda} - \mathrm{id}}{\mathrm{ad}_\lambda}(d\lambda) \end{aligned}$$

Proposition (Dotsenko – Shadrin – Vallette, 2016)

$$\begin{array}{c} \exists \text{ } \infty\text{-quasi-isomorphism } (H(A), \varphi) \xrightarrow{\sim} (H(A), \varphi_0) \\ \iff \\ \exists \lambda \in \Gamma \text{ such that } \lambda \cdot \varphi = \varphi_0 \end{array}$$

An equivalent characterization of formality

(A, ϕ) : a $\mathcal{P}$ -algebra s.t. $H(A)$ is a homotopy retract



Definition

- (A, ϕ) is **gauge formal** if $\exists \lambda \in \Gamma$ such that $\lambda \cdot \varphi = \varphi_0$
- (A, ϕ) is **gauge n -formal** if $\exists \lambda \in \Gamma$ such that

$$\lambda \cdot \varphi = (\varphi_0, 0, \dots, 0, \psi_{n+1}, \dots) .$$



Kaledin obstruction classes



Formal deformation

$$\varphi = (\varphi_0, \varphi_1, \varphi_2, \dots) \in \text{MC}(\mathfrak{g})$$

A formal deformation of φ_0 :

$$\Phi := \varphi_0 + \varphi_1 h + \varphi_2 h^2 + \cdots + \varphi_k h^k + \cdots$$

in the dg Lie algebra $\mathfrak{g}[[h]] := \mathfrak{g} \hat{\otimes} R[[h]]$.

Remark

$$\Phi \in \mathrm{MC}(\mathfrak{g}[[h]]), \text{ i.e. } d(\Phi) + \frac{1}{2}[\Phi, \Phi] = 0.$$

Proposition

$d^\Phi := d + [\Phi, -]$ is a differential on $\mathfrak{g}[[h]]$

Twisted dg Lie algebra:

$$g[[h]]^\Phi := (g[[h]], [-, -], d^\Phi)$$

The Kaledin classes

$$\partial_h \Phi := \varphi_1 + 2\varphi_2 h + \cdots + k\varphi_k h^{k-1} + \cdots \in \mathfrak{g}[[h]]$$

Lemma

$\partial_h \Phi$ is a cycle in $\mathfrak{g}[[h]]^\Phi$, i.e. $d^\Phi(\partial_h \Phi) = 0$.

Definition (Kaledin class)

The **Kaledin class** of $\varphi \in \text{MC}(\mathfrak{g})$ is the homology class

$$K_\Phi := [\partial_h \Phi] \in H_{-1} \left(\mathfrak{g}[[h]]^\Phi \right) .$$

Its n^{th} -truncated Kaledin class is

$$K_\Phi^n := [\varphi_1 + 2\varphi_2 h + \cdots + n\varphi_n h^{n-1}] \in H_{-1} \left((\mathfrak{g}[[h]]/h^n)^{\tilde{\Phi}} \right).$$

Kaledin class:

$$K_\Phi := [\varphi_1 + 2\varphi_2 h + 3\varphi_3 h^2 + \cdots] \in H_{-1}(\mathfrak{g}[[h]]^\Phi)$$

n^{th} -truncated Kaledin class :

$$K_\Phi^n := [\varphi_1 + 2\varphi_2 h + \cdots + n\varphi_n h^{n-1}] \in H_{-1} \left((\mathfrak{g}[[h]]/h^n)^{\tilde{\Phi}} \right)$$

Theorem ([Kaledin, 2007], [Lunts, 2007], [Melani–Rubió, 2019])

 $R : \mathbb{Q}\text{-algebra}$

$\mathcal{P}$: Koszul operad

$(A, \phi) : dg \mathcal{P}\text{-algebra s.t. } H(A) \text{ is a homotopy retract}$

- (A, ϕ) is gauge formal $\iff K_\Phi = 0$.
- (A, ϕ) is gauge n -formal $\iff K_\Phi^n = 0$.

Kaledin class:

$$K_\Phi := [\varphi_1 + 2\varphi_2 h + 3\varphi_3 h^2 + \cdots] \in H_{-1}(\mathfrak{g}[[h]]^\Phi)$$

n^{th} -truncated Kaledin class :

$$K_\Phi^n := [\varphi_1 + 2\varphi_2 h + \cdots + n\varphi_n h^{n-1}] \in H_{-1} \left((\mathfrak{g}[[h]]/h^n)^{\tilde{\Phi}} \right)$$

Theorem (E., 2023)

R : commutative ground ring

 $\mathcal{P}$: Koszul properad, colored operad
$$(A, \phi) : dg \mathcal{P}\text{-}(al)\text{gebra s.t. } H(A) \text{ is a homotopy retract}$$

- (A, ϕ) is gauge formal $\iff K_\Phi = 0$.
- (A, ϕ) is gauge n -formal $\iff K_\Phi^n = 0$.

Heuristic behind these obstruction classes

Gauge action: $\lambda \cdot \varphi := e^{\text{ad}_\lambda}(\varphi) - \frac{e^{\text{ad}_\lambda} - \text{id}}{\text{ad}_\lambda}(d\lambda)$

Vector field Υ_λ on $\text{MC}(\mathfrak{g})$: $\forall \lambda \in \Gamma, \quad \Upsilon_\lambda(\varphi) := -d\lambda - [\varphi, \lambda]$

Associated flow:

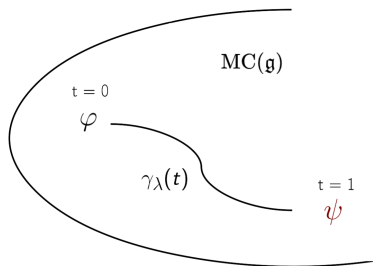
$$\frac{d}{dt}\gamma_\lambda(t) = \Upsilon_\lambda(\gamma_\lambda(t))$$

Integration of the flow starting at $\gamma_\lambda(0) = \varphi$:

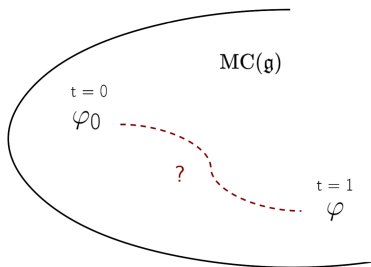
$$\gamma_\lambda(t) = e^{t\text{ad}_\lambda}(\varphi) - \frac{e^{t\text{ad}_\lambda} - \text{id}}{t\text{ad}_\lambda}(td\lambda)$$

$$\lambda \cdot \varphi = \psi \iff \gamma_\lambda(1) = \psi$$

$$\lambda \cdot \varphi = \psi \iff \gamma_\lambda(1) = \psi$$

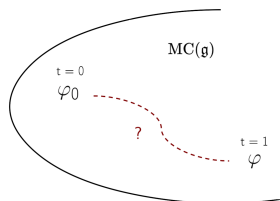


The case of formality



Does there exists $\lambda \in \Gamma$, such that $\gamma_\lambda(0) = \varphi_0$ and $\gamma_\lambda(1) = \varphi$?

Does there exist $\lambda \in \Gamma$, such that
 $\gamma_\lambda(0) = \varphi_0$ and $\gamma_\lambda(1) = \varphi$?



Attempt: “ $\Phi(h) = \varphi_0 + \varphi_1 h + \varphi_2 h^2 + \dots$ ”

Does there exist $\lambda \in \Gamma$, such that $\Phi = \gamma_\lambda$?

$$\Phi = \gamma_\lambda \iff \partial_h \Phi = \Upsilon_\lambda(\Phi) = -d\lambda - [\Phi, \lambda]$$

$$\iff \partial_h \Phi = d^\Phi(-\lambda)$$

$$\iff \partial_h \Phi \text{ is a boundary in } \mathfrak{g}[[h]]^\Phi$$

$$\iff K_\Phi = [\partial_h \Phi] = 0$$



Formality criteria



Formality descent

(A, ϕ) : a dg $\mathcal{P}$ -algebra s.t. $H(A)$ is a homotopy retract

$H_i(A)$: projective, finitely generated for all i .

S : faithfully flat commutative R -algebra.

Proposition (E., 2023)

(A, ϕ) is gauge n -formal $\iff (A \otimes_R S, \phi \otimes 1)$ is gauge n -formal.

Proof.

$$\begin{array}{ccc} H_{-1} \left(\underset{\Psi}{\mathfrak{g}_{H(A)}[[h]]^\Phi} \right) \otimes_{R[[h]]} S[[h]] & \cong & H_{-1} \left(\underset{\Psi}{\mathfrak{g}_{H(A \otimes_R S)}[[h]]^{\Phi \otimes 1}} \right) \\ K_\Phi \otimes 1 = 0 & \iff & K_{\Phi \otimes 1} = 0 \end{array}$$

□

Examples

- $C(\mathcal{D}_k; \mathbb{R})$ is formal $\iff C(\mathcal{D}_k; \mathbb{Q})$ is formal [GSNPR, 2005]
- $\mathbb{Z}_{(\ell)} \subset \mathbb{Z}_\ell$

Intrinsic formality

A graded $\mathcal{P}$ -algebra (H, φ_0) is **intrinsically formal** if every $\mathcal{P}$ -algebra (A, ϕ) such that $(H(A), \varphi_0) = (H, \varphi_0)$ is itself gauge formal.

$$\mathfrak{g}^{\varphi_0} : (\mathfrak{g}, [-, -], d + [\varphi_0, -])$$

Proposition (E., 2023)

$$H_{-1}(\mathfrak{g}^{\varphi_0}) = 0 \implies (H, \varphi_0) \text{ intrinsically formal.}$$

Proof.

For all (A, ϕ) such that $(H(A), \varphi_0) = (H, \varphi_0)$, then

$$K_\Phi = 0 \in H_{-1}(\mathfrak{g}[[h]]^\Phi) .$$



Previous works: [Hinich, 2003]

Tamarkin's proof of Kontsevich formality

k : a characteristic zero field

A : a polynomial algebra over k

Theorem (Hinich, 2003)

The shifted cohomological Hochschild complex $C(A; A)[1]$ is formal as a dg Lie algebra.

Proof.

- $\text{Lie}[1] \subset \text{Gerst}$;
- $(HH^\bullet(A), \varphi_0)$ has a *Gerst*-algebra structure;
- $C(A; A)$ has a Gerst_∞ -algebra structure inducing φ_0 ;
- $(HH^\bullet(A), \varphi_0)$ is intrinsically formal as a *Gerst*-algebra;
 $\rightarrow H_{-1}(\mathfrak{g}^{\varphi_0}) = 0$, where $\mathfrak{g} = \text{Hom}(\overline{\text{Gerst}}^i, \text{End}_{HH^\bullet(A)})$.

Theorem (Drummond-Cole – Horel, 2021)

Suppose that σ_α admits a lift, i.e. $\exists f \in \text{End}(A, \phi)$ s.t. $H(f) = \sigma_\alpha$.

- $\forall k, \alpha^k - 1 \in R^\times \implies (A, \phi)$ is gauge formal.
- $\forall k \leq n, \alpha^k - 1 \in R^\times \implies (A, \phi)$ is gauge n -formal.

Heuristic :

- Higher Massey products have to be compatible with the lift.
- They intertwine multiplication by α^l with multiplication by α^k with $l \neq k$.
- They have to vanish

Previous works: [DGMS, 1975], [Sullivan, 1977], [GSNPR, 2005]

Automorphism lifts

(A, ϕ) : a dg $\mathcal{P}$ -algebra s.t. $H(A)$ is a homotopy retract

Theorem (E., 2023)

Suppose that $\exists u \in \text{Aut}(H(A), \varphi_0)$ admitting a chain lift. Let $\text{Ad}_u : \text{End}_{H(A)} \rightarrow \text{End}_{H(A)}$ s.t. $\forall p \in \mathbb{N}$, and $\forall \psi \in \text{End}_{H(A)}(p)$

$$\mathrm{Ad}_u(\psi) = u \circ \psi \circ (u^{-1})^{\otimes p}.$$

1. If $\text{Ad}_u - \text{id}$ is invertible, then (A, ϕ) is gauge formal.
2. If $\text{Ad}_u - \text{id}$ is invertible on the elements of degree k for all $k < n$, then (A, ϕ) is gauge n -formal.

Automorphism lifts

R : a characteristic zero field

(A, ϕ) : a dg $\mathcal{P}$ -algebra s.t. $H(A)$ is a homotopy retract and finite dimensional.

Corollary

Suppose that there exists $u \in \text{Aut}(H(A), \varphi_0)$ such that for all $k < n$, and all p -tuples $(k_1, \dots, k_p)$,

$$\mathrm{Spec}(u_{k_1+\dots+k_p+k}) \cap \mathrm{Spec}(u_{k_1} \otimes \dots \otimes u_{k_p}) = \emptyset,$$

where $u_i := u|_{H_i(A)}$. If u admits a lift at the level of chains then (A, ϕ) is gauge n -formal.

Frobenius & Weil numbers

K : a finite extension of $\mathbb{Q}_p$

q : order of the residue field of the ring of integers of K

ℓ : a prime number different from p

X : a smooth proper K -scheme

Definition

$\alpha \in \overline{\mathbb{Q}_\ell}$ is a **Weil number of weight n** if

$$\forall \iota: \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}, \quad |\iota(\alpha)| = q^{n/2}.$$

Theorem (Deligne, 1974)

For all n , the eigenvalues of a Frobenius action on $H_{\text{et}}^n(X_{\overline{K}}, \mathbb{Q}_\ell)$ are Weil numbers of weight n .

Corollary

For every smooth proper K -scheme X , $C^\bullet(X_{an}, \mathbb{Q}_\ell)$ is formal.

Proof.

- $C^\bullet(X_{an}, \mathbb{Q}_\ell) \xrightarrow{\sim} C_{et}^\bullet(X_{\overline{K}}, \mathbb{Q}_\ell)$
- Let u be the Frobenius action on $H_{et}^\bullet(X_{\overline{K}}, \mathbb{Q}_\ell)$ and fix $\iota : \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$.
- For all $k \geq 1$, $(k_1, \dots, k_p)$ and $s := k_1 + \dots + k_p$,

$$\begin{array}{ccc} \text{Spec}(u_{s+k}) & \cap & \text{Spec}(u_{k_1} \otimes \dots \otimes u_{k_p}) = \emptyset . \\ \Psi & & \Psi \\ \alpha & & \beta \\ |\iota(\alpha)| = q^{\frac{s+k}{2}} & > & |\iota(\beta)| = q^{\frac{s}{2}} \end{array}$$



Previous works: [Deligne, 1980], [GSNPR, 2005]



Thank you for your attention!

