

ON THE PICARD NUMBER OF $K3$ SURFACES OVER NUMBER FIELDS

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ABSTRACT. We discuss some aspects of the behavior of specialization at a finite place of Néron-Severi groups of $K3$ surfaces over number fields. We give optimal lower bounds for the Picard number of such specializations, thus answering a question of Elsenhans and Jahnel. As a consequence of these results, we show that it is possible to compute explicitly the Picard number of any given $K3$ surface over a number field.

1. INTRODUCTION

This paper deals with two questions concerning the arithmetic and the geometry of $K3$ surfaces. Let X be a polarized $K3$ surface over a number field k , and let \mathfrak{p} be a finite place of k where X has good reduction. Denote by $X_{\mathfrak{p}}$ the special fiber of a smooth model of X over the ring of integers of $k_{\mathfrak{p}}$. Denote by \overline{X} (resp. $\overline{X_{\mathfrak{p}}}$) the base change of X (resp. $X_{\mathfrak{p}}$) to an algebraic closure of k (resp. the residue field of \mathfrak{p}). Specialization of divisors induces a specialization map between the Néron-Severi groups of \overline{X} and $\overline{X_{\mathfrak{p}}}$.

Question 1. *What can be said about the specialization map*

$$(1) \quad sp : NS(\overline{X}) \rightarrow NS(\overline{X_{\mathfrak{p}}}) ?$$

A standard argument using the cycle class map and the smooth base change theorem shows that this specialization map is always injective up to torsion. We are here interested in the defect of surjectivity. This problem has been featured prominently in recent works of Bogomolov-Hassett-Tschinkel and Li-Liedtke [2, 13].

The second question is the following. Recall that the Picard number of a variety is by definition the rank of its Néron-Severi group.

Question 2. *Given a projective embedding of X , is it possible to compute the geometric Picard number of X ?*

This question is raised by Shioda in [22].

Using the Weil conjectures [5], it is possible to compute the Picard numbers of smooth projective varieties over finite fields, at least in the case the Tate conjecture holds. Indeed, counting points in sufficiently many extensions of the base field, one can compute the characteristic polynomial of the Frobenius acting on the second étale cohomology group, and determine the multiplicity of 1 as an eigenvalue. If the Tate conjecture holds, this multiplicity is equal to the Picard number.

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In characteristic zero, Question 2 is more difficult. In particular, the first explicit example of a K3 surface over a number field with Picard rank 1 has been recently given by van Luijk in [29]. Van Luijk's method provides a link between both questions. Indeed, it proceeds by computing Picard numbers at sufficiently many finite places in order to get information over the field of definition. In the past few years, the problem of computing Picard numbers of K3 surfaces has been featured for instance in the work of Elsenhans-Jahnel [7, 8], with recent geometric applications in the work of Hassett, Várilly-Alvarado and Várilly, [11, 12]. Recent work of Hassett-Kresch-Tschinkel [10] tackles this question in some cases.

With this approach, one of the main problems is finding finite places \mathfrak{p} such that the specialization map (1) is as close to being surjective as possible, that is., such that $\rho(\overline{X_{\mathfrak{p}}})$ is as small as possible.

Note that the situation in this mixed characteristic setting is in stark contrast with the case of equal characteristic zero. Indeed, for K3 surfaces defined over function fields over \mathbb{C} or $\overline{\mathbb{Q}}$, most specializations induce isomorphisms at the level of the Néron-Severi group. This is a consequence of Baire's theorem over \mathbb{C} , see for instance [31, Chapter 13], and of the Hilbert irreducibility theorem over $\overline{\mathbb{Q}}$, as was first noticed by Terasoma, see [27] and [1, Theorem 5.2(3)]. A different approach to this problem can be found in the paper [14] of Maulik-Poonen.

On the other hand, over finite fields, there are obstructions for the map (1) to be surjective, as was first noticed by Shioda in [22, 23]. Indeed, it is a consequence of the Tate conjecture that the geometric Picard number of a K3 surface over a number field is always even, see for instance [4]. This striking fact has been recently used in a surprising way by Bogomolov-Hassett-Tschinkel in [2] and Li-Liedtke in [13] to prove that any complex K3 surface with odd Picard rank contains infinitely many rational curves.

In this paper, we describe the Shioda-type obstructions that can prevent the map (1) from being surjective, and we give optimal lower bounds for the Picard number of the specialization. One of our results is that Hodge theory can force the existence of such obstructions even when the Picard number is even, see part (2) of Theorem 1 below.

Let X be a K3 surface over a number field k , and choose a complex embedding of k . Let ρ be the geometric Picard number of X and, for any finite place \mathfrak{p} of k where X has good reduction, let $\rho_{\mathfrak{p}}$ be the geometric Picard number of $X_{\mathfrak{p}}$. Note that we always have

$$\rho_{\mathfrak{p}} \geq \rho.$$

We need to control the Hodge theory of $X_{\mathbb{C}}$. Let T be the orthogonal complement of $NS(X_{\mathbb{C}})$ in the singular cohomology group $H^2(X_{\mathbb{C}}, \mathbb{Q})$ with respect to cup-product. The space T is a sub-Hodge structure of $H^2(X_{\mathbb{C}}, \mathbb{Q})$. Let E be the algebra of endomorphisms of T that respect the Hodge structure. In [32], Zarhin shows that E is either a totally real field or a CM field.

The following result can be considered as a number field analog of the specialization results over function fields mentioned above.

Theorem 1. *Let X , T and E be as above.*

- (1) *If E is a CM field or the dimension of T as an E -vector space is even, then there exist infinitely many places \mathfrak{p} of good reduction such that $\rho_{\mathfrak{p}} = \rho$.*

Furthermore, after replacing k by a finite extension, this equality holds for a set of places of density 1.

- (2) Assume E is a totally real field and the dimension of T as an E -vector space is odd.

Let \mathfrak{p} be a finite place of k where X has good reduction. Assume that the characteristic of the residue field of \mathfrak{p} is at least 5. Then

$$\rho_{\mathfrak{p}} \geq \rho + [E : \mathbb{Q}].$$

There exist infinitely many places \mathfrak{p} of good reduction such that $\rho_{\mathfrak{p}} = \rho + [E : \mathbb{Q}]$. Furthermore, after replacing k by a finite extension, this equality holds for a set of places of density 1.

Remark 2. Note that if ρ is odd, the dimension of T over \mathbb{Q} is odd, hence X satisfies the assumptions of the second part of the theorem.

Remark 3. In [9], Elsenhans and Jahnel ask whether, with notations as in the theorem, there exists \mathfrak{p} such that $\rho_{\mathfrak{p}} - \rho \leq 1$. The result above shows that it is not the case if E is a totally real field of degree at least 2 over \mathbb{Q} , such that the dimension of T over E is odd. This is however true in all other cases.

This result shows that the Picard number can be forced to jump in specializations even when the Picard number of X is even. Using the method of Li and Liedtke in [13], we get the following corollary.

Corollary 4. *Let X be either a $K3$ surface of Picard rank 2 with E a totally real field of degree 4 or a $K3$ surface of Picard rank 4 with E a totally real field of even degree. Then X contains infinitely many rational curves.*

There exist such $K3$ surfaces by [28, Section 3], and they give new examples of $K3$ surfaces with infinitely many rational curves. Note that complex $K3$ surfaces of Picard rank different from 2 and 4 are known to contain infinitely many rational curves by [13].

The second main result of this paper is a solution to Question 2. Recall that van Luijk's method in [29] to prove that a $K3$ surface X over \mathbb{Q} has Picard number 1 was to first find two primes p and q of good reduction such that X specializes to a $K3$ surface of Picard number 2 modulo p and q . If the discriminant of the Néron-Severi lattices modulo p and q differ by a non-square factor, van Luijk shows that this implies that X has Picard number 1.

By Remark 3, there are cases where we cannot expect van Luijk's method to work directly for all $K3$ surfaces of rank 1. However, the second part of Theorem 1 can be used to show that reduction at finite places does indeed give enough information to compute Picard numbers over number fields.

This gives a theoretical explanation to the computations in [29, 7, 8, 12, 11].

Theorem 5. *There exists an algorithm which, given a projective $K3$ surface X over a number field, either returns its geometric Picard number or does not terminate.*

If $X \times X$ satisfies the Hodge conjecture for codimension 2 cycles, then the algorithm applied to X terminates.

Remark 6. In case the algorithm terminates, it is possible to compute the Picard number over the base field by searching for divisors spanning the geometric Néron-Severi group with rational coefficients and taking invariants under the action of the Galois group.

Remark 7. Let X be a $K3$ surface over \mathbb{C} . With the notations of Theorem 1, $X \times X$ satisfies the Hodge conjecture if and only if the field E acts by algebraic correspondences. By [1], this would be a consequence of the standard conjectures. In [16], Mukai has announced a proof in the case E is a CM field.

Remark 8. The proof of the theorem actually shows that the only case where the algorithm would not terminate is, with the notations of Theorem 1, if E is a totally real field that does not act on $H^2(X, \mathbb{Q})$ by algebraic correspondences and T is of odd dimension as a vector space over E .

In particular, the algorithm always terminates for surfaces with $E = \mathbb{Q}$.

While we only consider $K3$ surfaces in this paper, some of the methods we consider have a wider range of applications. Assuming general conjectures on algebraic cycles, it is a general fact that the Mumford-Tate group associated to the second cohomology group of a variety controls specialization of Néron-Severi groups, in a fashion that is similar to the way the monodromy representation appears in [1] or [14]. The multiplicity of the weight zero in the corresponding representation is what forces the Picard number to jump after specialization. This is related to algorithmic computations of Néron-Severi groups as in our paper.

For $K3$ surfaces, the work of Zarhin and Tankeev in [32] and [24, 25] allows us to give precise and unconditional results. The results of our paper conjecturally hold for varieties with $h^{2,0} = 1$. It seems likely that one can prove them unconditionally for holomorphic symplectic varieties by extending the work of Tankeev cited above.

In section 2, we recall results of Zarhin and Tankeev on the second cohomology group of a $K3$ surface. This allows us to prove Theorem 1 in section 3. Section 4 is devoted to discriminant computations which will allow us to prove Theorem 5 in the last section.

2. ALGEBRAIC MONODROMY GROUPS OF $K3$ SURFACES OVER NUMBER FIELDS

The results of this section are mostly contained in the work of Zarhin and Tankeev. After recalling some preliminary material, we describe the algebraic monodromy group of a $K3$ surface defined over a number field.

2.1. Mumford-Tate groups and the Mumford-Tate conjecture. Let \mathbb{S} be the Deligne torus, that is, the algebraic group over \mathbb{R} defined as

$$\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m.$$

Let H be a finite-dimensional vector space over \mathbb{Q} . Giving a Hodge structure on H is equivalent to giving an action of \mathbb{S} on $H_{\mathbb{R}} = H \otimes \mathbb{R}$.

Definition 9. *Let H be a rational Hodge structure. The Mumford-Tate group of H is the smallest algebraic subgroup $MT(H)$ of $GL(H)$ such that $MT(H)_{\mathbb{R}}$ contains the image of \mathbb{S} in $GL(H_{\mathbb{R}})$.*

We refer to [6, Chapter I] for general properties of Mumford-Tate groups. Since \mathbb{S} is connected, this definition implies that Mumford-Tate groups are connected. Note that the Mumford-Tate group of a polarized Hodge structure is reductive.

Let i, j be nonnegative integers, and consider the Hodge structure

$$V = H^{\otimes i} \otimes (H^*)^{\otimes j}.$$

The Mumford-Tate group $MT(H)$ acts on V . If v is a Hodge class in V , then the line $\mathbb{Q}v$ is globally invariant under the action of $MT(H)$. Conversely, it follows from Chevalley's theorem on affine groups that $MT(H)$ is the largest algebraic subgroup of $GL(H_{\mathbb{C}})$ that leaves all such lines globally invariant, see [6, Chapter I, Proposition 3.4].

We now turn to the ℓ -adic theory. General results can be found in [20]. Let k be a number field and fix an algebraic closure \bar{k} . Let X be a smooth projective variety over k , and denote by \bar{X} the variety $X \times_{\text{Spec } k} \text{Spec } \bar{k}$. Fixing a prime number ℓ , we can consider the étale cohomology group $H^i(\bar{X}, \mathbb{Q}_{\ell})$ for some integer i . Let ρ_{ℓ} denote the continuous representation

$$\rho_{\ell} : G_k \rightarrow GL(H^i(\bar{X}, \mathbb{Q}_{\ell}))$$

of the absolute Galois group G_k of k . The image of ρ_{ℓ} is an ℓ -adic Lie group.

Definition 10. *With notations as above, let G_{ℓ} be the Zariski closure of the image of ρ_{ℓ} in the algebraic group $GL(H^i(\bar{X}, \mathbb{Q}_{\ell}))$. The algebraic group G_{ℓ} is called the algebraic monodromy group associated to the Galois representation ρ_{ℓ} .*

Note that replacing k by a finite extension replaces G_{ℓ} by an open subgroup of finite index. In particular, the identity component of the algebraic monodromy group does not depend on the choice of a field of definition for X .

General conjectures on algebraic cycles give important information on Mumford-Tate and algebraic monodromy groups. In particular, the latter are expected to be reductive. The expected relationship between those two groups is described by the Mumford-Tate conjecture as follows, see [20].

Conjecture 11. *Let k be a number field and fix a complex embedding of k . Let X be a smooth projective variety over k .*

Let G_{ℓ} be the algebraic monodromy group associated to the étale cohomology group $H^i(X_{\mathbb{C}}, \mathbb{Q}_{\ell})$ for some prime number ℓ , and let G_{ℓ}° be its identity component. Then, there is a canonical isomorphism

$$G_{\ell}^{\circ} \simeq MT(H^i(X_{\mathbb{C}}, \mathbb{Q}))_{\mathbb{Q}_{\ell}}.$$

The Mumford-Tate conjecture is implied by the conjunction of the Tate and Hodge conjectures. A lot of work has been done in its direction in the case of abelian varieties, see for instance [21, 19, 30].

In this paper, we will focus on the case of K3 surfaces, where the Mumford-Tate conjecture holds. However, an important part of our method concerning specialization of Néron-Severi groups holds in a general setting if one assumes the Mumford-Tate conjecture.

2.2. Mumford-Tate groups and algebraic monodromy groups of K3 surfaces. The following result is due to Tankeev and is crucial to this paper.

Theorem 12 (Tankeev, [24, 25]). *The Mumford-Tate conjecture holds for the second cohomology group of K3 surfaces over number fields.*

This result allows us to get a Hodge-theoretic description of the Galois action on the second cohomology group of a $K3$ surface.

Let us now recall the description due to Zarhin in [32] of the Mumford-Tate group of a $K3$ surface. Let X be a $K3$ surface over \mathbb{C} , and consider the singular cohomology $H = H^2(X, \mathbb{Q})$ endowed with its weight 2 Hodge structure. The Hodge structure H splits as a direct sum

$$H = NS(X) \oplus T,$$

where $NS(X)$ is the Néron-Severi group of X with rational coefficients, and T is the orthogonal of $NS(X)$ in H with respect to the cup-product. The Hodge structure T is called the transcendental part of $H^2(X, \mathbb{Q})$.

The Hodge structure T is simple. By Lefschetz's theorem on $(1, 1)$ classes, T is the smallest sub-Hodge structure of H such that $T \otimes \mathbb{C}$ contains $H^2(X, \mathcal{O}_X)$. By the Hodge index theorem, cup-product on $H^2(X, \mathbb{Q})$ restricts to a polarization $\psi : T \otimes T \rightarrow \mathbb{Q}$ on T .

Since $NS(X)$ is spanned by Hodge classes, the Mumford-Tate group of H acts by a character on $NS(X)$ and identifies with the Mumford-Tate group of T . Since T is polarized by ψ , $MT(T)$ is contained in the group of orthogonal similitudes $GO(T, \psi)$.

Let E be the algebra of endomorphisms of the Hodge structure T . In [32, Theorem 1.5.1], Zarhin proves that E is either a totally real field or a CM field. The field E is equipped with an involution induced by the polarization on T , which is either the identity if E is totally real or complex conjugation in case E is CM.

Since E consists of endomorphisms of Hodge structures, the Mumford-Tate group of T commutes with E . By the discussion above, the Mumford-Tate group of T is a subgroup of the centralizer of E in the group $GO(T, \psi)$.

Theorem 13 (Zarhin, [32, Theorem 2.2.1]). *The Mumford-Tate group of T is the centralizer of E in the group of orthogonal similitudes $GO(T, \psi)$.*

Now keep the same notation, and assume X can be defined over a number field k . Fix a prime number ℓ . The action of the absolute Galois group G_k on $H^2(X, \mathbb{Q}_\ell)$ leaves the \mathbb{Q}_ℓ -span of the Néron-Severi group of X globally invariant, as well as its orthogonal complement $T_\ell = T \otimes \mathbb{Q}_\ell$. As above, the identity component of the algebraic monodromy group G_ℓ of $H^2(X, \mathbb{Q}_\ell)$ identifies with the algebraic monodromy group of T_ℓ .

The polarization ψ on T extends to a symmetric bilinear form ψ_ℓ . The representation of G_k in the automorphism group of T_ℓ factors through the group $GO(T_\ell, \psi_\ell)$.

Since Hodge cycles on products of $K3$ surfaces are absolute Hodge, see [6, Chapter I, 6.26], the field E corresponding to endomorphisms of the Hodge structure T acts on T_ℓ and commutes with a finite-index subgroup of G_k . As a consequence, the identity component of G_ℓ commutes with the action of $E \otimes \mathbb{Q}_\ell$.

By Theorem 12, the Mumford-Tate conjecture holds for X . As an immediate corollary of Theorem 13, we get the following description of the identity component of the algebraic monodromy group of X .

Corollary 14. *With notations as above, the identity component of the algebraic monodromy group associated to T_ℓ is the centralizer of $E \otimes \mathbb{Q}_\ell$ in the group of orthogonal similitudes $GO(T_\ell, \psi_\ell)$.*

3. PICARD NUMBERS OF SPECIALIZATIONS

This section is devoted to the proof of Theorem 1. We start by the following result which encompasses the elementary linear algebra needed in Theorem 1.

Let T be a finite dimensional vector space endowed with a non-degenerate symmetric bilinear form ψ . If f is any linear endomorphism of T , let f' be the adjoint of f with respect to ψ .

Let E be a number field acting on T . Assume that E is stable under $e \mapsto e'$, and that E is either a totally real field with $e = e'$ for all $e \in E$, or a CM field such that $e \mapsto e'$ acts as complex conjugation on E .

Let H be the centralizer of E in the special orthogonal group $SO(T, \psi)$. Let ℓ be a prime number, and let $H_\ell = H \otimes \mathbb{Q}_\ell$.

Proposition 15. *The following holds.*

- (1) *If E is a CM field or the dimension of T as an E -vector space is even, then there exists $h \in H_\ell$ such that h does not have any root of unity as an eigenvalue.*
- (2) *If E is a totally real field and the dimension of T as an E -vector space is odd, then the eigenspace of any $h \in H_\ell$ associated to the eigenvalue 1 is of dimension at least $[E : \mathbb{Q}]$. Furthermore, there exists $h \in H_\ell$ for which this dimension is exactly $[E : \mathbb{Q}]$ and such that no root of unity different from 1 appears as an eigenvalue of h .*

Proof. Let us first assume that E is a totally real field. By [32, 2.1], there exists a unique E -bilinear form $\phi : T \times T \rightarrow E$ such that $\psi = \text{Tr}_{E/\mathbb{Q}}(\phi)$. With this notation, the centralizer of E in $SO(T, \psi)$ is equal, as a subgroup of $GL(T)$, to the Weil restriction $\text{Res}_{E/\mathbb{Q}}(SO_E(T, \phi))$, where $SO_E(T, \phi)$ denotes the group of orthogonal similitudes of the E -vector space T with respect to ϕ .

Assume furthermore that the dimension of T as a vector space over E is even, and let us show that there is an element $h \in H_\ell$ such that h does not have any root of unity as an eigenvalue.

Considering an orthogonal decomposition of T as an E -vector space endowed with the bilinear form ϕ , we can assume T is of dimension 2 over E . Let h be an orthogonal automorphism of the E -vector space T of determinant 1 that is not of finite order. Then h corresponds to an element of H_ℓ with the desired property.

Now if the dimension of T as a vector space over E is odd, recall that any element of $SO_E(T, \psi)$ admits 1 as an eigenvalue. It follows from the description of H_ℓ as a Weil restriction that any $h \in H_\ell$ has 1 as an eigenvalue, and that the corresponding eigenspace is invariant under the action of E . As a consequence, its dimension is at least $[E : \mathbb{Q}]$. One can then argue as in the previous paragraph to conclude the proof of the proposition in this case.

Let us now assume that E is a CM field. Let e be an element of E such that $ee' = 1$ and e is not a root of unity. Then multiplication by e on T corresponds to an element of H_ℓ as in the theorem. \square

We now turn to the proof of Theorem 1. From now on, we use the notations there. Let us start with a straightforward lemma.

Lemma 16. *The identity component of the algebraic monodromy group associated to $T_\ell(1)$ is the centralizer of $E \otimes \mathbb{Q}_\ell$ in the special orthogonal group $SO(T_\ell, \psi_\ell)$.*

Proof. The representation of G_k on $T_\ell(1)$ is equal to the representation of G_k on T_ℓ twisted by the cyclotomic character. On the other hand, since the map

$$T_\ell(1) \otimes T_\ell(1) \rightarrow \mathbb{Q}_\ell$$

given by cup-product is G_k -equivariant, G_k acts on $T_\ell(1)$ through the orthogonal group $O(T_\ell, \psi_\ell)$.

The lemma then follows from Corollary 14 and the fact that the special orthogonal group is the identity component of the orthogonal group. \square

Proof of Theorem 1. We use the notations of the theorem. First note that since specialization of Néron-Séveri groups is injective, the inequality $\rho_{\mathfrak{p}} \geq \rho$ always holds.

Let $F_{\mathfrak{p}}$ be the geometric Frobenius at \mathfrak{p} acting on the étale cohomology group $H^2(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell(1))$, where ℓ is a prime number prime to \mathfrak{p} . By the smooth base change theorem, the group $H^2(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_\ell(1))$ identifies with $H^2(\overline{X}, \mathbb{Q}_\ell(1))$, and $F_{\mathfrak{p}}$ leaves both the Néron-Severi group and $T_\ell(1)$ globally invariant.

Let H be the centralizer of $E \otimes \mathbb{Q}_\ell$ in the special orthogonal group $SO(T_\ell, \psi_\ell)$. Let n be the dimension of T as a vector space over \mathbb{Q} , and let S be the finite set of complex roots of unity of degree at most n over \mathbb{Q} .

Assume first that E is a CM field or E is a totally real field and the dimension of T as a vector space over E is even. By Proposition 15, the set of $h \in H_\ell$ such that h does not have any eigenvalue in S is a dense, Zariski-open subset V_ℓ of H_ℓ .

By Lemma 16 and Chebotarev's density theorem, we can find a finite extension k' of k with the property that if U is the set of finite places \mathfrak{p} of k' such that for any $\mathfrak{p} \in U$, X has good reduction at \mathfrak{p} and the geometric Frobenius $F_{\mathfrak{p}}$ acting on $T_\ell(1)$ does not have any eigenvalue in S , the U has density 1.

Indeed, let k' be a finite extension of k such that the image of absolute Galois group $G_{k'}$ in the automorphism group of $T_\ell(1)$ lands in H_ℓ . By Lemma 16, the image of $G_{k'}$ is Zariski-dense in H_ℓ . By [21, I.8, Théorème 2], this implies that the set U of finite places \mathfrak{p} of k' such that for any $\mathfrak{p} \in U$, X has good reduction at \mathfrak{p} and the geometric Frobenius $F_{\mathfrak{p}}$ acting on $T_\ell(1)$ belongs to V_ℓ , has density 1.

Choose U as above, and let \mathfrak{p} be in U . By the Weil conjectures, the characteristic polynomial of the geometric Frobenius $F_{\mathfrak{p}}$ has rational coefficients. By definition of S , this implies that it does not have any eigenvalue that is a root of unity.

As a consequence, $F_{\mathfrak{p}}$ acting on the whole cohomology group $H^2(\overline{X}, \mathbb{Q}_\ell(1))$ admits 1 as an eigenvalue of multiplicity ρ and does not have any other eigenvalue that is a root of unity. It follows that $\rho_{\mathfrak{p}} \leq \rho$, and finally that $\rho_{\mathfrak{p}} = \rho$. This proves the first part of Theorem 1.

Now assume that E is a totally real field and that the dimension of T as a vector space over E is odd. By Proposition 15, every element of H_ℓ has 1 as an eigenvalue with multiplicity at least $[E : \mathbb{Q}]$. By definition of the algebraic monodromy group, if \mathfrak{p} is a finite place of k , then some power of the geometric Frobenius belongs to H_ℓ .

By work of Nygaard and Ogus [18, 17] in the finite height case and the author in the supersingular case [3], K3 surfaces over finite fields of characteristic at least

5 satisfy the Tate conjecture, namely, their geometric Picard number is equal to the dimension of the largest subspace of the second cohomology group where the Frobenius morphism acts by roots of unity. It follows that $\rho_{\mathfrak{p}} \geq \rho + [E : \mathbb{Q}]$.

By Proposition 15 again, the set of $h \in H_{\ell}$ such that h admits 1 as an eigenvalue of multiplicity $[E : \mathbb{Q}]$ and does not have any other eigenvalue in S is a dense, Zariski-open subset of H_{ℓ} .

By Lemma 16 and Chebotarev's density theorem, we can find a finite extension k' of k and a set U of finite places \mathfrak{p} of k' that has density 1 such that for any $\mathfrak{p} \in U$, X has good reduction at \mathfrak{p} and the geometric Frobenius $F_{\mathfrak{p}}$ acting on $T_{\ell}(1)$ admits 1 as an eigenvalue of multiplicity $[E : \mathbb{Q}]$ and does not have any other eigenvalue in S .

Choose U as above, and let \mathfrak{p} be in U . Since $X_{\mathfrak{p}}$ satisfies the Tate conjecture, we can argue as above to finish the proof of Theorem 1. \square

Remark 17. Using Frobenius tori as in [20] and the fact that Frobenius tori are maximal tori of the Mumford-Tate groups for infinitely many primes, one can work directly in the group of orthogonal similitudes instead of reducing to the special orthogonal group as in Lemma 16.

4. DISCRIMINANTS OF NÉRON-SEVERI GROUPS

In this section, we discuss properties of the Néron-Severi lattices of specializations of $K3$ surfaces. Once again, we use the notations of Theorem 1.

Proposition 18. *Assume that E is a totally real field and that the dimension of T over E is odd. If \mathfrak{p} is a finite place of k such that X has good reduction at \mathfrak{p} , denote by $\delta(\mathfrak{p}) \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$ the discriminant of the lattice $NS(\overline{X}_{\mathfrak{p}})$ with respect to the intersection product.*

There exist infinitely many finite places \mathfrak{p}_i , $i \in \mathbb{N}$ and infinitely many finite places \mathfrak{q}_j , $j \in \mathbb{N}$ of k such that for any integers i, j

- (1) X has good, ordinary reduction at both \mathfrak{p}_i and \mathfrak{q}_j .
- (2) $\rho_{\mathfrak{p}_i} = \rho_{\mathfrak{q}_j} = \rho + [E : \mathbb{Q}]$.
- (3) $\delta(\mathfrak{p}_i) \neq \delta(\mathfrak{q}_j)$.

Remark 19. A special case of this result is that the method developed in [29] to prove that a given $K3$ surface over a number field has Picard number 1 always works in the case $E = \mathbb{Q}$. We noted in Remark 3 that it cannot work directly otherwise.

In the next section, we will adapt the method so as to make it work in every case.

We start with some easy linear algebra.

Lemma 20. *Let ℓ be a prime number, and let V be a free module of finite rank over \mathbb{Z}_{ℓ} . Let g be an endomorphism of V such that $g \otimes \mathbb{Q}_{\ell}$ is a semisimple automorphism of $V \otimes \mathbb{Q}_{\ell}$, and denote by r the multiplicity of 1 as an eigenvalue of g . Let W be the eigenspace associated to the eigenvalue 1 of g . Let d be a positive integer.*

Then there exists an integer N with the following property. Let h be an endomorphism of V such that $h \otimes \mathbb{Q}_{\ell}$ is a semisimple automorphism of $V \otimes \mathbb{Q}_{\ell}$. Assume that r is the multiplicity of 1 as an eigenvalue of h , and let W' be the eigenspace associated to the eigenvalue 1 of h . If h is congruent to g modulo ℓ^N , then $W \otimes \mathbb{Z}/\ell^d\mathbb{Z} = W' \otimes \mathbb{Z}/\ell^d\mathbb{Z}$.

Remark 21. In particular, if V is endowed with a symmetric bilinear form ϕ such that the restriction of ϕ to W is not degenerate, and N is sufficiently large, then the discriminants of W and W' are equal in $\mathbb{Q}_\ell^*/(\mathbb{Q}_\ell^*)^2$.

Proof. Write $V = W \oplus \widetilde{W}$, where \widetilde{W} is a g -invariant submodule of V . Since $g \otimes \mathbb{Q}_\ell$ does not fix any nonzero element of $\widetilde{W} \otimes \mathbb{Q}_\ell$, a compactness argument shows that there exists an integer N such that if $g(v) - v \in l^N V$ for some $v \in \widetilde{W}$, then $v \in \ell^k \widetilde{W}$.

Let h be as in the statement of the lemma. By definition of N , if $v \in V$ is fixed by h , then $v \otimes \mathbb{Z}/\ell^k \mathbb{Z} \in W \otimes \mathbb{Z}/\ell^k \mathbb{Z}$. With the notation of the lemma, it follows that $W' \otimes \mathbb{Z}/\ell^k \mathbb{Z} \subset W \otimes \mathbb{Z}/\ell^k \mathbb{Z}$. Since both W and W' are saturated submodules of V of the same rank r , equality follows. \square

Proof of Proposition 18. First note that the dimension of T as a vector space over E is at least 3. Indeed, let ω be a generator of $T^{2,0} \subset T \otimes \mathbb{C}$, and let $\sigma : E \rightarrow \mathbb{C}$ be the complex embedding of E satisfying

$$\forall e \in E, e.\omega = \sigma(e)\omega.$$

The complex lines $\mathbb{C}\omega$ and $\mathbb{C}\bar{\omega}$ are two distinct one-dimensional subspaces of $T_E \otimes \mathbb{C}$, where T_E denotes T endowed with the structure of a vector space over E . As a consequence, the dimension of T as a vector space over E is at least 2, and at least 3 since we assumed it to be odd.

Recall that ψ is the bilinear form on T induced by cup-product. As in Proposition 15, there exists a unique E -bilinear form $\phi : T \times T \rightarrow E$ such that $\psi = Tr_{E/\mathbb{Q}}(\phi)$. Any orthogonal basis of T_E with respect to ϕ induces an orthogonal decomposition of T with respect to ψ .

$$T = T_1 \oplus \dots \oplus T_r$$

where the T_i are stable under the action of E and of dimension 1 as E -vector spaces.

By the same reasoning as above, since the T_i are one-dimensional over E , there is no integer i such that $T_i \otimes \mathbb{C}$ contains the two-dimensional space $T^{2,0} \oplus T^{0,2}$.

The signature of ψ on T is $(2, \dim(T) - 2)$. By the Hodge index theorem and the remark above, the signature of the restriction of ψ to T_i is either $(0, [E : \mathbb{Q}])$ or $(1, [E : \mathbb{Q}] - 1)$. Since the dimension of T over E is at least 3, both these signatures appear, and this implies that, up to reordering, we can assume that the discriminant of T_1 is negative and the discriminant of T_2 is positive. Let δ and δ' be these two discriminants in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$.

Since $\delta \neq \delta'$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, there exists a prime number ℓ such that the images of δ and δ' in $\mathbb{Q}_\ell^*/(\mathbb{Q}_\ell^*)^2$ are different. If W is any subspace of T_ℓ such that the restriction of ψ_ℓ to W is non-degenerate, let $\delta(W)$ denote the discriminant of W in $\mathbb{Q}_\ell^*/(\mathbb{Q}_\ell^*)^2$.

By Lemma 20, Proposition 15 and Chebotarev's density theorem, we can find, for any positive integer d , infinitely many finite places \mathfrak{p}_i , $i \in \mathbb{N}$ and infinitely many finite places \mathfrak{q}_j , $j \in \mathbb{N}$ of k such that for any integers i, j

- (1) X has good, ordinary reduction at both \mathfrak{p}_i and \mathfrak{q}_j .
- (2) $\rho_{\mathfrak{p}_i} = \rho_{\mathfrak{q}_j} = \rho + [E : \mathbb{Q}]$.
- (3) If $F_{\mathfrak{p}_i}$ (resp. $F_{\mathfrak{q}_j}$) denotes the geometric Frobenius at \mathfrak{p}_i (resp. \mathfrak{q}_j) acting on $T_\ell(1)$, and $W_{\mathfrak{p}_i}$ (resp. $W_{\mathfrak{q}_j}$) denotes the eigenspace associated to the eigenvalue 1 of $F_{\mathfrak{p}_i}$ (resp. $F_{\mathfrak{q}_j}$), then $W_{\mathfrak{p}_i} \otimes \mathbb{Z}/\ell^d \mathbb{Z} = T_1 \otimes \mathbb{Z}/\ell^d \mathbb{Z}$ (resp. $W_{\mathfrak{q}_j} \otimes \mathbb{Z}/\ell^d \mathbb{Z} = T_2 \otimes \mathbb{Z}/\ell^d \mathbb{Z}$).

- (4) The geometric Frobenius $F_{\mathfrak{p}_i}$ (resp. $F_{\mathfrak{q}_j}$) acting on $T_\ell(1)$ does not have any eigenvalue different from 1 that is a root of unity.

If d is big enough, this implies that $\delta(W_{\mathfrak{p}_i}) = \delta$ and $\delta(W_{\mathfrak{q}_j}) = \delta'$ in $\mathbb{Q}_\ell^*/(\mathbb{Q}_\ell^*)^2$. Proposition 18 immediately follows by the Tate conjecture for ordinary $K3$ surfaces. \square

Remark 22. The proof above shows that the density of pairs $(\mathfrak{p}, \mathfrak{q})$ as in the proposition is positive.

5. COMPUTING THE PICARD NUMBER OVER NUMBER FIELDS

This section is devoted to a proof of Theorem 5. Given a projective $K3$ surface over a number field k , we want to compute the Picard number of X using the equations of X in a projective embedding.

There are two steps in our approach. The first one is finding sufficiently many divisors on X , and the second is proving that the rational span of these divisors is the whole Néron-Severi group of X .

In case we want to prove that the $K3$ surface has Picard number 1, the first step is vacuous, as we already have a divisor given by a hyperplane section. In general, the first step is done by going through the Hilbert schemes of curves in the projective space we are working in and doing elimination theory to find curves on X . After a finite number of computations, this will allow us to find divisors on X that span the Néron-Severi group.

The second step will be done by reducing to finite characteristic and using our results above.

However, this is not sufficient. Indeed, the field E of endomorphisms of the transcendental part of the Hodge structure of X plays a role in the behavior of the Picard number after specialization, and in case E is a totally real field strictly containing \mathbb{Q} such that T is of odd dimension over E , this leads to some loss of accuracy in the estimates reduction at finite places can provide.

This problem will be solved by studying codimension 2 varieties in $X \times X$. Assuming the Hodge conjecture for $X \times X$, these determine the field E , which will allow us to conclude.

We start by the following result.

Proposition 23. *Let X be a $K3$ surface over a number field k . Assume we are given the equations of X in some projective embedding.*

Let T be the transcendental part of $H^2(X, \mathbb{Q})$, and let E be the field of endomorphisms of the Hodge structure T .

Assume that we know that the Picard number of X is greater or equal than some integer ρ , and that the degree of E over \mathbb{Q} is greater or equal than some integer d . Then there exists an algorithm with the following properties:

- (1) *If the algorithm terminates, it proves that Picard number of X is ρ .*
- (2) *Suppose that the Picard number of X is actually ρ . Then the algorithm terminates unless E is totally real, the dimension of T as a vector space over E is odd and $d < [E : \mathbb{Q}]$.*

Proof. Let ρ' be the actual Picard number of X . We know that $\rho' \geq \rho$. Using the Weil conjectures [5], we can compute the characteristic polynomial of Frobenius at any finite place \mathfrak{p} of k of good reduction, see [29, 9]. This allows in particular to

compute the numbers $\rho_{\mathfrak{p}}$. Using the Artin-Tate formula [26, Conjecture (C)] which holds for surfaces satisfying the Tate conjecture by [15], one can also compute the discriminants $\delta(\mathfrak{p})$ as in Proposition 18.

We start computing $\rho_{\mathfrak{p}}$ and $\delta(\mathfrak{p})$ for all places \mathfrak{p} of good reduction.

Let us distinguish three cases. First assume that E is a CM field. By Theorem 1, we can find \mathfrak{p} with $\rho_{\mathfrak{p}} = \rho'$. If it happens that ρ , the lower bound for the Picard number of X that we were given, is equal to the actual Picard number ρ' (that we do not know yet), the computation at \mathfrak{p} together with this lower bound allows us to prove that X has Picard number $\rho = \rho'$.

Now assume that E is totally real and the dimension of T as a vector space over E is even. In that case, Theorem 1 allows us to make the same conclusion.

The last case happens when E is totally real and the dimension of T as a vector space over E is odd. By Proposition 18, the finite field computations give us two finite places \mathfrak{p} and \mathfrak{q} of k where X has good, ordinary reduction, with $\rho_{\mathfrak{p}} = \rho_{\mathfrak{q}} = \rho' + [E : \mathbb{Q}]$, and $\delta(\mathfrak{p}) \neq \delta(\mathfrak{q})$.

Since $\delta(\mathfrak{p}) \neq \delta(\mathfrak{q})$, we know that the specialization maps $NS(\overline{X}) \rightarrow NS(\overline{X}_{\mathfrak{p}})$ and $NS(\overline{X}) \rightarrow NS(\overline{X}_{\mathfrak{q}})$ are not surjective. This means that $NS(X_{\mathfrak{p}}) \cap T_{\ell}(1)$ is nonzero in $H^2(\overline{X}_{\mathfrak{p}}, \mathbb{Q}_{\ell}(1))$.

Now we know by the analysis in the proof of Theorem 1 that this intersection is stable under the action of E . As a consequence, its dimension is at least $[E : \mathbb{Q}] \geq d$. This gives us the estimation

$$\rho' \leq \rho_{\mathfrak{p}} - d.$$

In case d happens to be equal to the actual degree $[E : \mathbb{Q}]$ and $\rho = \rho'$, these estimates allow us to prove that X has Picard number $\rho = \rho'$. \square

Remark 24. In case $\rho = \rho' = 1$ and $E = \mathbb{Q}$, this proves that the method of [29] always works.

Proof of Theorem 5. Let X , E and T be as above. Let ρ' be the Picard number of X and d' be the degree of E over \mathbb{Q} . By Proposition 23, we only need to be able to prove that the Picard number of X is at least ρ' and the degree of E over \mathbb{Q} is at least d' .

The assertion on the Picard number is theoretically – although not computationally – easy. One can go through Hilbert schemes of curves in the projective space where X is given and check, using elimination theory, for curves that happen to lie on X . Computing intersection matrices with this divisors on X , one can find divisors that span a ρ' -dimensional subset of the Néron-Severi group of X .

Running these Hilbert schemes computations alongside the computations of Proposition 23 allows for a computation of the Picard number of X unless E is a totally real field strictly containing \mathbb{Q} such that T is of odd dimension over E .

To deal with the latter case, one has to work on $X \times X$. If one assumes the Hodge conjecture for $X \times X$, then elements of E are induced by codimension 2 cycles in $X \times X$. As above, one can use Hilbert schemes to find codimension 2 subschemes in $X \times X$.

Given such a subscheme Z , the action of Z on T can be determined by first computing the characteristic polynomial of the correspondence $H^2(X, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q})$

by computing intersection numbers between T and the various subschemes obtained by composing the correspondence induced by Z with itself.

Factoring the characteristic polynomial, this gives candidates for the algebraic number λ such that $[Z]_*\eta = \lambda\eta$, where η is a nonzero algebraic 2-form on X . An approximate computation can then determine λ . The degree of λ over \mathbb{Q} is a lower bound for $[E : \mathbb{Q}]$.

By the primitive element theorem, it is easy to see that one can find Z such that this computation gives an optimal estimate for the degree of E . Using Proposition 23, this concludes the proof.

In conclusion, an algorithm to compute Picard number of $K3$ surfaces works as follows. Let X be a $K3$ surface. Run the three following algorithms alongside each other.

- (1) Going through Hilbert schemes of a suitable projective space, find divisors on X and compute the dimension of their span in the Néron-Severi group via intersection theory. This gives a lower bound for the Picard number.
- (2) Going through Hilbert schemes of a suitable projective space, find codimension 2 cycles in $X \times X$. Using intersection theory again, use these to get a lower bound on the field E of endomorphisms of the transcendental part of $H^2(X, \mathbb{Q})$.
- (3) Going through finite places \mathfrak{p} of k , compute the Picard number and the discriminant of the Néron-Severi group of $\overline{X}_{\mathfrak{p}}$ by counting points over finite fields. Using the preceding step, get an upper bound on the Picard number of X .

We showed that the estimates provided by the method solve the problem unconditionally unless E is a totally real field strictly containing \mathbb{Q} and the transcendental part of $H^2(X, \mathbb{Q})$ is of odd dimension over E . In the latter case, the estimates above are sufficiently precise to compute the Picard number if we assume the Hodge conjecture for $X \times X$.

□

Remark 25. It seems that the computations of the second step above would be very lengthy to do in practice. We however wanted to point out that they can be done.

Note that the computations terminate much faster in most cases, since $E = \mathbb{Q}$ for the majority of $K3$ complex surfaces, in the sense of Baire category.

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