Invent math
DOI 10.1007/s00222-012-0443-y

The Tate conjecture for $K3$ surfaces over finite fields

François Charles

Received: 6 July 2012 / Accepted: 29 November 2012
© Springer-Verlag Berlin Heidelberg 2012

Abstract Artin's conjecture states that supersingular $K3$ surfaces over finite fields have Picard number 22. In this paper, we prove Artin's conjecture over fields of characteristic $p \geq 5$. This implies Tate's conjecture for $K3$ surfaces over finite fields of characteristic $p \geq 5$. Our results also yield the Tate conjecture for divisors on certain holomorphic symplectic varieties over finite fields, with some restrictions on the characteristic. As a consequence, we prove the Tate conjecture for cycles of codimension 2 on cubic fourfolds over finite fields of characteristic $p \geq 5$.

Mathematics Subject Classification (2010) 14C22 · 14C25 · 14G15 · 14J28

1 Introduction

The goal of this paper is to study the Tate conjecture for varieties with $h^{2,0} = 1$ over finite fields. The main result is the following. Recall that Artin conjectured in [3] that the rank of the Néron-Severi group of a supersingular $K3$ surface over a finite field—in the sense of Artin, that is, a $K3$ surface whose formal Brauer group has infinite height—has the maximal possible value, that is, 22.

Theorem 1 *Artin's conjecture holds for supersingular $K3$ surfaces over algebraically closed fields of characteristic $p \geq 5$.*

F. Charles (✉)
IRMAR–UMR 6625 du CNRS, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France
e-mail: francois.charles@univ-rennes1.fr

Published online: 29 December 2012

Let X be a smooth projective variety over a finite field k. Let ℓ be a prime number different from the characteristic of k. Tate conjectured in [35] that the Frobenius invariants of the space $H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i))$ are spanned by cohomology classes of algebraic cycles over k of codimension i. Using results of Nygaard-Ogus in [26], Theorem 1 implies the following.

Corollary 2 *The Tate conjecture holds for $K3$ surfaces over finite fields of characteristic $p \geq 5$.*

As a consequence of the main theorem of [23], this implies the following finiteness result.

Corollary 3 *Let k be a finite field of characteristic $p \geq 5$. There are only finitely many $K3$ surfaces over k up to isomorphism.*

With the extra assumption that p is large enough with respect to the degree of a polarization of the $K3$ surface, Theorem 1 is the main result of [24]. Our strategy uses that of [24] as a starting point. In particular, we also use, as a key geometric input, Borcherds' construction of automorphic forms for $O(2, n)$ [7, 8], which allows one to find ample divisors supported on the Noether-Lefschetz locus for $K3$ surfaces.

A key point of Maulik's argument is to show that $K3$ surfaces have semistable reduction in equal positive characteristic. This is where the restrictions on the characteristic of the base field appear. Maulik then proceeds to showing that supersingular $K3$ are elliptic, which is enough to conclude that they satisfy the Tate conjecture by a result of Artin in [3].

In this paper, we manage to circumvent the use of both semistable reduction for $K3$ surfaces and Artin's theorem on elliptic $K3$ surfaces, thus offering a direct proof of the Tate conjecture that gets rid of restrictions on the characteristic of the base field that appeared in [24]. These arguments allow us to prove the Tate conjecture for divisors on certain holomorphic symplectic varieties in any dimension, where showing semistable reduction seems out of reach at the moment, and where it is not clear what the analog of Artin's result might be.

Recall that a complex irreducible holomorphic symplectic variety is a complex smooth, simply-connected variety X such that $H^0(X, \Omega^2_X)$ is spanned by a unique holomorphic form that is everywhere non-degenerate. An important example is given by varieties of $K3^{[n]}$ type defined as the deformations of the Hilbert scheme of points on a $K3$ surface, see [4]. The second singular cohomology group of a complex irreducible holomorphic symplectic variety is endowed with a canonical form called the Beauville-Bogomolov form, see [4, 21].

We deduce Theorem 1 from the following result on the Tate conjecture for higher-dimensional varieties, with some restrictions on p.

Theorem 4 *Let Y be a complex projective irreducible holomorphic symplectic variety of dimension $2n$ with second Betti number $b_2 > 5$. Let h be the cohomology class of an ample line bundle on Y, let $d = h^{2n}$ and let q be the Beauville-Bogomolov form.*

Let $p \geq 5$ be a prime number. Assume that p is prime to d and that $p > 2n$. Suppose that Y can be defined over a finite unramified extension of $\mathbb{Q}_p$ and that Y has good reduction at p. Assume also that q induces a non-degenerate quadratic form on the reduction modulo p of the primitive lattice in the second cohomology group of Y. Then the reduction X of Y at p satisfies the Tate conjecture for divisors.

Remark At the moment, it is not known whether there exists a complex projective irreducible holomorphic symplectic variety with second Betti number different from 7, 10, 22 and 23.

In the case of varieties of $K3^{[n]}$ type, the assumptions of the theorem have the following explicit form.

Corollary 5 *Let Y be a complex polarized irreducible holomorphic symplectic variety of $K3^{[n]}$ type. Let h be the cohomology class of an ample line bundle on Y, and let $d = q(h)$, where q is the Beauville-Bogomolov form.*

Let $p \geq 5$ be a prime number. Assume that p is prime to d and that $p > 2n$. Suppose that Y can be defined over a finite unramified extension of $\mathbb{Q}_p$ and that Y has good reduction at p. Then the reduction X of Y at p satisfies the Tate conjecture.

Remark The assumption on p and the fact that Y is defined over an unramified extension of $\mathbb{Q}_p$ ensure that the Hodge to de Rham spectral sequence of X degenerates at E_1 by [17].

Remark For $K3$ surfaces, Theorem 4 is weaker than Theorem 1. However, proving Theorem 4 for fourfolds is a key step in removing the assumptions on the characteristic of the base field to get Theorem 1. We strongly expect that an extension of our method might relax the hypotheses on p even in the higher-dimensional case.

Using the correspondence between cubic fourfolds and certain holomorphic symplectic varieties, we get the following instance of the Tate conjecture.

Corollary 6 *Let k be a finite field of characteristic $p \geq 5$, and let X be a smooth cubic hypersurface in $\mathbb{P}^5_k$. Then X satisfies the Tate conjecture for cycles of codimension 2.*

Note that the Tate conjecture for cubic fourfolds and divisors on holomorphic symplectic varieties over number fields was proved by André in [1].

As in André's work, we make heavy use of the Kuga-Satake correspondence between Hodge structures of $K3$ type and certain abelian varieties. We use this correspondence as well as general ideas on the deformation of cycle classes to prove the algebraicity of some cohomology classes. This type of argument is quite close in spirit to well-known results in Hodge theory around questions of algebraicity of Hodge loci.

The main point, which appears in a slightly more involved form in Proposition 22, is that while the Kuga-Sataka correspondence is not known to be algebraic as predicted by the Hodge conjecture, its existence is enough to provide mixed characteristic analogs of the Noether-Lefschetz loci, namely, universal deformation spaces for pairs (X, α), where α is a suitable Galois-invariant cohomology class. This allows one to study the lifting of such pairs to characteristic zero.

This method has the advantage of replacing degeneracy issues for family of holomorphic symplectic varieties by similar problems for abelian varieties, which are much easier to deal with. As a consequence, we do not use any of the birational arguments of [24]. Of course, these results are deep and beautiful in their own right.

The plan of the paper is the following. We start by explaining how the results stated above can all be reduced to Theorem 4 for supersingular varieties.

In Sect. 3, we gather some generalities around the deformation problems we deal with and recall some results of [24]. We state and prove them in the generality we need.

In order to facilitate the exposition, we show in Sect. 4 that, with the notations of Theorem 4 when X is supersingular, the Picard number of X is at least 2. We achieve this result by introducing a partial compactification of the Kuga-Satake mapping in mixed characteristic and using arguments related to the geometry of Hodge loci. Proposition 22 contains the main geometric idea.

In the last section of the paper, we prove Theorem 4 using the ideas of Sect. 4 and an induction process. Some of the lifting results there might be of independent interest. A surprising phenomenon is that the induction process does not allow us to directly show the Tate conjecture. However, we are able to use known cases of the Hodge conjecture for low-dimensional abelian varieties to conclude the proof.

We learned that Keerthi Madapusi Pera has recently announced results on the Tate conjecture for $K3$ surfaces when p^2 does not divide the degree of a polarization. His proof seems to involve very different methods building on recent advances on the theory of canonical integral models of Shimura varieties.

2 Preliminary reductions

2.1 Reduction to Theorem 4

In this section, we show how Theorem 1, Corollaries 5 and 6 can be deduced from Theorem 4.

Proof of Corollary 5 Let X and Y be as in Corollary 5. We only need to show that X and Y satisfy the hypotheses of Theorem 4, that is, that p is prime to h^{2n} and that the Beauville-Bogomolov form q induces a non-degenerate quadratic form on $H^2(Y, \mathbb{Z})_{prim} \otimes \mathbb{Z}/p\mathbb{Z}$.

The second Betti number of Y is either 22 or 23, hence it is strictly larger than 5. Since $p > 2n$ and $\frac{(2n)!}{n!2^n} q(h)^n = h^{2n}$, see for instance [28, 4.1.4], p is prime to h^{2n}. Furthermore, the explicit description of the Beauville-Bogomolov form on the lattice $H^2(Y, \mathbb{Z})$ as in [4, Sect. 9] shows that the q induces a non-degenerate quadratic form on $H^2(Y, \mathbb{Z}) \otimes \mathbb{Z}/p\mathbb{Z}$. Since p is prime to $q(h)$, q induces a non-degenerate quadratic form on $H^2(Y, \mathbb{Z})_{prim} \otimes \mathbb{Z}/p\mathbb{Z}$. □

Proof of Corollary 6 Let k be a finite field of characteristic at least 5, and let X be a smooth cubic hypersurface in $\mathbb{P}^5_k$. Let F be the Fano variety of lines in X. It is a smooth projective variety of dimension 4 over k.

Let W be the ring of Witt vectors of k, and let K be the fraction field of W. The hypersurface X lifts to a cubic hypersurface $\mathcal{X}$ over W. Taking the Fano variety of lines gives a smooth lifting $\mathcal{F}$ of F over W.

By results of Beauville-Donagi in [5], given an embedding of K into $\mathbb{C}$, the variety $\mathcal{F}_{\mathbb{C}}$ is of $K3^{[2]}$ type. If q is the Beauville-Bogomolov form, and h is the ample class on $\mathcal{F}_{\mathbb{C}}$ is the ample class corresponding to the Plücker embedding, then $q(h) = 6$. As a consequence, Corollary 5 shows that F satisfies the Tate conjecture for divisors.

Let ℓ be a prime number invertible in k. The incidence correspondence between X and its variety of lines induces a morphism

$$\phi : H^4\big(X_{\overline{k}}, \mathbb{Q}_\ell(2)\big) \to H^2\big(F_{\overline{k}}, \mathbb{Q}_\ell(1)\big)$$

that is equivariant with respect to the Frobenius action on both sides.

In [5], Beauville and Donagi show that the corresponding morphism over $\mathbb{C}$ induces an isomorphism between the primitive parts of the cohomology groups. By the smooth base change theorem, ϕ induces an isomorphism between the primitive parts of $H^4(X_{\overline{k}}, \mathbb{Q}_\ell(2))$ and $H^2(F_{\overline{k}}, \mathbb{Q}_\ell(1))$ as well. Consider the Poincaré dual of ϕ

$$\psi : H^6\big(F_{\overline{k}}, \mathbb{Q}_\ell(3)\big)_{prim} \to H^4\big(X_{\overline{k}}, \mathbb{Q}_\ell(2)\big)_{prim}.$$

It is also induced by the incidence correspondence. In particular, it sends cohomology classes of cycles of dimension 1 in $F_{\overline{k}}$ to classes of cycles of codimension 2 in $X_{\overline{k}}$.

By the hard Lefschetz theorem, and since F satisfies the Tate conjecture for divisors, the group of Frobenius-invariant classes in $H^6(F_{\overline{k}}, \mathbb{Q}_\ell(3))$ is spanned by cohomology classes of cycles of dimension 1. This shows that the Frobenius-invariant part of $H^4(X_{\overline{k}}, \mathbb{Q}_\ell(2))$ is spanned by cohomology classes of codimension 2 cycles and shows that cubic fourfolds satisfy the Tate conjecture. □

We now show how Corollary 5 implies the Tate conjecture for $K3$ surfaces in any characteristic different from 2 and 3.

Proof of Theorem 1 Let S be a supersingular $K3$ surface over a finite field k of characteristic at least 5. Let $S^{[2]}$ be the Hilbert scheme that parametrizes length 2 zero-dimensional subschemes of S. By [19], $S^{[2]}$ is a smooth projective variety of dimension 4.

The variety $S^{[2]}$ is the quotient of the blow-up of $S \times S$ along the diagonal by the involution exchanging the two factors. By [4, Proposition 6], the second cohomology group of $S^{[2]}$ is generated by the second cohomology group of S and the class $[E]$ of the exceptional divisor. As a consequence, S satisfies the Tate conjecture if and only if $S^{[2]}$ satisfies the Tate conjecture for divisors.

By [14], S lifts to a projective $K3$ surface $\mathcal{S}$ over the ring W of Witt vectors of k. The variety $S^{[2]}$ lifts to the relative Hilbert scheme $\mathcal{S}^{[2]}$. Let K be the fraction field of W, and fix an embedding of K into $\mathbb{C}$. Let q be the Beauville-Bogomolov form on $H^2(\mathcal{S}^{[2]}_{\mathbb{C}}, \mathbb{Z})$, and let h be an ample cohomology class. Let $d = q(h)$.

For any large enough integer N, $Nh - [E]$ is an ample cohomology class. Since $q(E) = -2$, we have

$$q(Nh + E) = N^2 d - 2Nq(h, E) - 2.$$

If p divides N, then the variety $\mathcal{S}^{[2]}_{\mathbb{C}}$ with the polarization given by $Nh - [E]$ satisfies the hypothesis of Theorem 4. This shows that S satisfies the Tate conjecture. □

Remark The idea of finding prime-to-p polarizations on $S^{[2]}$ in order to study the Tate conjecture on S is somewhat reminiscent of Zarhin's trick of finding a principal polarization on $(A \times \hat{A})^4$, where A is an abelian variety, see [38].

Remark Note that the proof of Theorem 1 only requires the special case of Theorem 4 in the supersingular case, that is, when the Galois action on the second cohomology group of the reduction of X at p is trivial.

2.2 The universal deformation space and reduction of Theorem 4 to the supersingular case

Let us keep the notations of Theorem 4. By the theorem of Deligne and Illusie in [17], the Hodge to de Rham spectral sequence of X degenerates. By upper semicontinuity of cohomology groups, this implies that the Hodge numbers of X and Y are the same. Using the universal coefficients theorem, it is easy to check that the crystalline cohomology groups of X are torsion free.

The versal formal deformation space of X is smooth over the ring of Witt vectors W. Indeed, by the Bogomolov-Tian-Todorov theorem [6, 36, 37], the versal deformation space of Y, that is, in characteristic zero, is smooth of dimension the dimension of $H^1(Y, T_Y)$. It follows that the versal deformation space of X over W has relative dimension at least the dimension of $H^1(Y, T_Y) \simeq H^1(Y, \Omega^1_Y)$, which is equal to the dimension of $H^1(X, \Omega^1_X) \simeq H^1(X, T_X)$ since the Hodge to de Rham spectral sequence degenerates at E_1. This implies that the versal formal deformation space of X is smooth over W

As a consequence of these results, the deformation theory of X is very similar to the deformation theory of $K3$ surfaces. In a more precise way, the second crystalline cohomology group of X is a $K3$ crystal as in [30]. The results of [30], Sects. 1 and 2 on the versal deformation space of polarized $K3$ surfaces, as well as the results of [29] hold without any change for the deformation of X. We will freely refer to these results.

Definition 7 Let X be as in Theorem 4. We say that X is supersingular if the Frobenius morphism acts on the second étale cohomology group of X through a finite group. Otherwise, we say that X has finite height.

As in the case of $K3$ surfaces, the general results of [3] apply and show that X is supersingular if and only if the formal Brauer group of X has finite height. These remarks show that the proof of [26] gives without any change the following theorem.

Theorem 8 (Nygaard-Ogus, [26]) *Let X be as in Theorem 4. If X has finite height, then X satisfies the Tate conjecture.*

The supersingular case is thus the only remaining case of Theorem 4. The proof of this case will be logically independent of the work of Nygaard-Ogus.

3 Deformation theory and the Kuga-Satake morphism

From now on, and through the remainder of this article, we will fix the following notations. Let k be the algebraic closure of a finite field of characteristic

$p \geq 5$. Let W be the ring of Witt vectors of k, and let K be the fraction field of W. By abuse of notation, we will again denote by X the base change over k of a variety satisfying the hypotheses of Theorem 4.

We assume that X is supersingular, that is, that the Frobenius morphism acts on the second étale cohomology group of X through a finite group. Let $b = b_2(X) > 5$ be the second Betti number of X (which is equal to the second Betti number of Y by the smooth base change theorem). We will show that the Néron-Severi group of X has rank b.

In this section, we gather results and notations around the deformation space of X, the Kuga-Satake mapping in that setting, and the Noether-Lefschetz locus. While some of these results are quite similar to those in [24], and some are taken directly from there, we state them in our context and sometimes give different proofs and constructions.

3.1 Deformation spaces

Let $\widehat{\mathcal{X}} \to \widehat{S}$ be the formal versal deformation space of X over W. We showed in Sect. 2.2 that the assumptions on X ensure that $\widehat{S}$ is formally smooth of relative dimension $b-2$ and that the deformation is universal.

Let L be the ample line bundle on X induced by the ample line bundle on Y with cohomology class h. Since p is prime to h^{2n}, the class $c_1(L)^n$ is nonzero in $H^{2n}_{dR}(X/k)$, which in turn implies that $c_1(L)$ doesn't lie in $F^2H^2_{dR}(X/k)$, where $F^\bullet$ is the Hodge filtration on de Rham cohomology.

Let $\widehat{T}$ be the universal deformation space of the pair (X, L). By [30, Proposition 2.3], $\widehat{T}$ is formally smooth of relative dimension $b-3$ over W. We also denote by $\widehat{\mathcal{X}} \to \widehat{T}$ the universal formal deformation of the polarized variety (X, L).

By Artin's algebraization theorem, we can find a smooth scheme T of finite type over W, and a smooth projective morphism $\pi : \mathcal{X} \to T$, together with a relatively ample line bundle $\mathcal{L}$ on $\mathcal{X}$ that extends the universal formal deformation of the pair (X, L) over $\widehat{T}$. After shrinking T, we can assume that π is a universal deformation at every point.

The Beauville-Bogomolov form on Y is a rational multiple of the usual intersection form on the primitive cohomology lattice by [4, p. 775, Remarque 2]. As a consequence, it descends to a quadratic form on the relative primitive cohomology over T, for the étale as well as for the crystalline theories. We will denote this extension by q as well.

3.2 Spin level structures

We briefly recall basic definitions about spin level structures, see [1, 24, 33] to which we refer for further details. Let $n \geq 3$ be an integer, and assume n is prime to p.

Let L be an abstract lattice that is isomorphic to the primitive cohomology lattice of Y endowed with the Beauville-Bogomolov quadratic form. Let

$$CSpin(L) = CSpin(L)(\mathbb{A}_f) \cap Cl_+(L \otimes \widehat{\mathbb{Z}}),$$

where $Cl_+(L)$ is the even part of the Clifford algebra of L.

Let $\mathbb{K}_n^{sp}$ be the subgroup of $CSpin(L)$ consisting of elements congruent to 1 modulo n, and let $\mathbb{K}_n^{ad}$ be its image in $SO(L \otimes \widehat{\mathbb{Z}})$. If ℓ is a prime number, let $\mathbb{K}_{n,\ell}^{ad}$ be the ℓ-adic part of $\mathbb{K}_n^{ad}$. By assumption, q is a non-degenerate quadratic form on $L \otimes \mathbb{Z}/p\mathbb{Z}$ and by [1, 4.4], $\mathbb{K}_{n,p}^{ad}$ is the whole special orthogonal group.

We say that $\pi : \mathcal{X} \to T$ admits a spin level n structure if the algebraic fundamental group of T acts on primitive cohomology with ℓ-adic coefficients through the group $\mathbb{K}_n^{ad}$ for any ℓ such that $\mathbb{K}_{n,\ell}^{ad}$ is a proper subgroup of the special orthogonal group, choosing a base point corresponding to the complex lift Y of X.

After replacing T by an étale cover, we can and will assume that $\pi : \mathcal{X} \to T$ admits a spin level n structure.

3.3 The Kuga-Satake construction

The Kuga-Satake mapping plays a major role in this paper. We refer to [24], as well as to the papers of André and Rizov [1, 34], for most definitions and results. However, for reasons that will become clear below, we need to work with a slightly different definition as follows.

Let $V = H^2(Y, \mathbb{Q})_{prim}$ be the primitive part of the second Betti cohomology group of the holomorphic symplectic variety Y, and let C be the Clifford algebra of V. The classical Kuga-Satake construction, see [15] or [12] for details, endows C with a polarized Hodge structure of weight 1. As a consequence, there exists a polarized abelian variety A with $H^1(A, \mathbb{Q}) \simeq V$ as polarized Hodge structures. The integer lattice in V determines A uniquely.

Let g be the dimension of A and d'^2 be the degree of the polarization. An explicit computation shows that d' is prime to p, see [24, 5.1]. The polarized abelian variety A is the *Kuga-Satake variety* of Y.

Elements of V act on C by multiplication on the left. This induces a canonical primitive embedding of polarized Hodge structures

$$H^2(Y, \mathbb{Z})_{prim} \hookrightarrow End\big(H^1(A, \mathbb{Z})\big). \qquad (1)$$

Note that this canonical embedding only exists if we define the Kuga-Satake variety using the full Clifford algebra C and not only its even part C^+ as in the references above. This is the reason we make this slight change in definition. The Kuga-Satake variety associated to C is isogenous to the square of the Kuga-Satake variety associated to C^+.

The following is an easy consequence of the construction of the Kuga-Satake mapping.

Proposition 9 *Let v be an element of $H^2(Y,\mathbb{Z})_{prim}$ mapping to the endomorphism ϕ of $H^1(A,\mathbb{Z})$. Then $\phi\circ\phi = q(v)\mathrm{Id}_{H^1(A,\mathbb{Z})}$.*

Proof By [1, 15], there exists a canonical morphism of algebras

$$C \hookrightarrow End\big(H^1(A,\mathbb{Z})\big)$$

where C is the Clifford algebra of $H^2(Y,\mathbb{Z})_{prim}$, compatible with (1). This proves the result. □

3.4 The Kuga-Satake mapping

We now proceed to the construction of a Kuga-Satake mapping over the deformation space T. First, let $\pi_K : \mathcal{X}_K \to T_K$ be the generic fiber of π. Let $\mathcal{A}_{g,d',n}$ be the moduli space of abelian varieties of dimension g with a polarization of degree d'^2 and a level n structure over W, and let $\mathcal{A}_{g,d',n,K}$ be its generic fiber.

The following result is proved in [1, Theorem 8.4.3], see also [34] for the case of $K3$ surfaces. It follows from the fact that $\pi : \mathcal{X} \to T$ admits a spin level n structure.

Proposition 10 *There exists a morphism*

$$\kappa_K : T_K \to \mathcal{A}_{g,d',n,K}$$

which for a complex point t sends the variety $\mathcal{X}_t$ to its Kuga-Satake variety.

Given any prime number ℓ, there is a canonical primitive embedding of ℓ-adic sheaves on T_K

$$R^2_{et}\pi_*\mathbb{Z}_\ell(1)_{prim} \hookrightarrow \mathrm{End}\big(R^1_{et}\psi_*\mathbb{Z}_\ell\big), \tag{2}$$

where $R^2_{et}\pi_*\mathbb{Z}_\ell(1)_{prim}$ is the relative primitive cohomology of π and $\psi : \mathcal{A}_K \to T_K$ is the abelian scheme over T induced by κ_K.

Remark The result of André is actually stated for the usual intersection pairing on primitive cohomology. Since the Beauville-Bogomolov form q is proportional to this intersection pairing, the same result holds using q.

We can now use Proposition 6.1.2 of [34] to conclude that the Kuga-Satake mapping extends to T and get the following.

Proposition 11 *The Kuga-Satake mapping κ_K extends uniquely to a morphism*

$$\kappa : T \to \mathcal{A}_{g,d',n}.$$

3.5 Quasi-finiteness of the Kuga-Satake mapping

The following result is due to Maulik in the case X is a $K3$ surface.

Proposition 12 ([24], Proposition 5.10) *The Kuga-Satake map $\kappa : T \to \mathcal{A}_{g,d',n}$ is quasifinite.*

The proof of Maulik can easily be adapted to our setting. For the sake of completeness, let us however sketch a slightly more direct proof. We start with the following analog of (2). We use the language of filtered Frobenius crystals as in [24, Definition 6.3].

Proposition 13 *Let b be a k-point of T, and let $\widehat{B}$ be the formal neighborhood of b in T. Denote by $\psi : \mathcal{A} \to T$ the Kuga-Satake abelian scheme associated to $\mathcal{X} \to T$. There is a canonical primitive strict embedding of filtered Frobenius crystals on $\widehat{B}$*

$$R^2\pi_*\Omega^\bullet_{\widehat{\mathcal{X}}/\widehat{B}}(1)_{prim} \hookrightarrow \mathrm{End}\big(R^1\psi_*\Omega^\bullet_{\widehat{\mathcal{A}}/\widehat{B}}\big). \tag{3}$$

This morphism is compatible with (2) via the comparison theorems.

Remark Saying that the morphism above is strict means that the filtration on the left side is induced by the filtration on the right side.

Proof Aside from the strictness property, this is proven in [24, Sect. 6]. First, one argues that the morphism exists at the level of isocrystals by the comparison theorems of Andreatta-Iovita in [2]. To check that the morphism is integral, one uses the theory of Fontaine-Messing in [20]. Note that these arguments are general and do not use any property of the Beauville-Bogomolov form contrary to the subtler Morita arguments of [24].

The morphism (3) is primitive because (2) is. Strictness of (3) can be checked at the fibers, and is a general property of the theory of Fontaine-Messing, see for instance [10, Proposition 3.1.1.1]. □

Proof of Proposition 12 This is an easy consequence of Proposition 13 as in [24, 6.4]. □

3.6 Period maps

One of the main points of this paper is that a large part of [24] can be carried out at the level of period spaces. We briefly gather some results on period maps for families of holomorphic symplectic varieties. André's paper [1] contains related results.

We first describe the period domain, and refer to [16, Sect. 1] for details. Let V be a vector space over $\mathbb{Q}$ of dimension $b-1$, and let ψ be a non-degenerate bilinear form on V of signature $(2, b-3)$ on V. Let G be the algebraic group $SO(V, \psi)$, and let Ω be the period domain, that is,

$$\Omega = \left\{\omega \in \mathbb{P}(V_{\mathbb{C}}), \psi(\omega, \omega) = 0 \text{ and } \psi(\omega, \overline{\omega}) > 0\right\}.$$

To any $\omega \in \Omega$, we can associate a polarized Hodge structure of weight 2 on V such that $F^2 V_{\mathbb{C}} = \mathbb{C}\omega$. The period domain can be naturally identified with a conjugacy class of morphisms $h : \mathbb{S} \to G_{\mathbb{R}}$, where $\mathbb{S}$ is the Deligne torus. The pair (G, Ω) is a Shimura datum with reflex field $\mathbb{Q}$.

Now let (L, ψ) be the lattice of Sect. 3.2, that is, a lattice isomorphic to the primitive lattice of Y. We consider the Shimura datum above associated to $V = L \otimes \mathbb{Q}$.

Let $n \geq 3$ be as before, and let S_n be the Shimura variety defined over $\mathbb{Q}$ such that

$$\mathcal{S}_{n,\mathbb{C}} = G(\mathbb{Q})\backslash\Omega \times G(\mathbb{A}_f)/\mathbb{K}_n^{ad}.$$

Fix an embedding of K into $\mathbb{C}$. Since $\mathcal{X} \to T_K$ admits a level n spin structure, the classical period map takes the form of an étale morphism of quasi-projective varieties

$$j : T_{\mathbb{C}} \to \mathcal{S}_{n,\mathbb{C}}.$$

The local Torelli theorem for holomorphic symplectic varieties in [4, Théorème 5] implies the following result.

Proposition 14 *The map*

$$j : T_{\mathbb{C}} \to \mathcal{S}_{n,\mathbb{C}}$$

is étale.

The Kuga-Satake construction actually defines a morphism of Shimura varieties

$$KS_{\mathbb{C}} : \mathcal{S}_{n,\mathbb{C}} \to \mathcal{A}_{g,d',n,\mathbb{C}}.$$

It is a finite morphism, defined over K by [1]. We get the following decomposition of the Kuga-Satake mapping.

Proposition 15 *The Kuga-Satake mapping*

$$\kappa_{\mathbb{C}} : T_{\mathbb{C}} \to \mathcal{A}_{g,d',n,\mathbb{C}}$$

factorizes as $KS_{\mathbb{C}} \circ j$, *where*

$$j : T_{\mathbb{C}} \to \mathcal{S}_{n,\mathbb{C}}$$

is étale and

$$KS_{\mathbb{C}} : \mathcal{S}_{n,\mathbb{C}} \to \mathcal{A}_{g,d',n,\mathbb{C}}$$

is finite.

Using the Kuga-Satake mapping, one can show the following.

Proposition 16 *The period map* $j : T_{\mathbb{C}} \to \mathcal{S}_{n,\mathbb{C}}$ *is defined over* K.

Proof This is almost contained in [1, Appendix 1]. In this paper, André studies the Kuga-Satake mapping $KS_{\mathbb{C}} : \mathcal{S}_{n,\mathbb{C}} \to \mathcal{A}_{g,d',n,\mathbb{C}}$ by proving it admits a factorization $KS_{\mathbb{C}} = p \circ s$, where $\widetilde{\mathcal{S}}_{\mathbb{C}}$ is a Shimura variety of spinorial type that parametrizes Kuga-Satake abelian varieties, $s : \mathcal{S}_{n,\mathbb{C}}$ is a section of a canonical map of Shimura varieties $\widetilde{\mathcal{S}}_{\mathbb{C}} \to \mathcal{S}_{n,\mathbb{C}}$—in particular, s is a closed immersion—and $p : \widetilde{\mathcal{S}}_{\mathbb{C}} \to \mathcal{A}_{g,d',n,\mathbb{C}}$ is defined by the aforementioned moduli interpretation of $\widetilde{\mathcal{S}}_{\mathbb{C}}$.

Furthermore, André shows that Proposition 10 can be refined by showing that $\kappa_{\mathbb{C}} : T_{\mathbb{C}} \to \mathcal{A}_{g,d',n,\mathbb{C}}$ admits as well a factorization $\kappa_{\mathbb{C}} = p \circ \widetilde{\kappa}_{\mathbb{C}}$ for some $\widetilde{\kappa}_{\mathbb{C}} : T_{\mathbb{C}} \to \widetilde{\mathcal{S}}_{\mathbb{C}}$.

By [1, Theorem 8.4.3 and Appendix 1], $\widetilde{\mathcal{S}}_{\mathbb{C}}$ is defined over K, and so are $\widetilde{\kappa}_{\mathbb{C}}$, p and s. We have

$$\widetilde{\kappa}_{\mathbb{C}} = s \circ j.$$

Since s is a closed immersion, j is defined over K. □

3.7 Divisors on the period space

In this section, we recall a slightly adapted version of Theorem 3.1 of [24] that proves the ampleness of some components of the Noether-Lefschetz locus in the moduli space of polarized $K3$ surfaces.

Recall that $d = q(h)$, where h is the class of the polarization of $\mathcal{X} \to T$. Let L_0 be a lattice containing L such that the embedding $L \subset L_0$ is isomorphic to the embedding of the primitive cohomology of Y into its full second cohomology group. We also denote by $h \in L$ the image of the ample class of Y.

Let Λ be a rank 2 lattice of the form

$$\Lambda = \begin{pmatrix} d & a \\ a & b \end{pmatrix}.$$

If its discriminant $bd - a^2$ is negative, let H_Λ be the locus in the period domain Ω of points ω such that there exists $v \in L$ with $\psi(v, v) = b$, $\psi(v, h) = a$ and $\psi(\omega, v) = 0$.

By definition of $\mathbb{K}_n^{ad}$, the divisor H_Λ descends to a divisor D_Λ in $S_{n,\mathbb{C}}$. We will also denote by D_Λ the divisor on the generic fiber of T obtained via the period map.

Let λ be the Hodge bundle on $\mathcal{S}_{n,\mathbb{C}}$. By definition, it is induced by the tautological line bundle over the period space. The Hodge bundle pulls back by the period map to the Hodge bundle on $T_{\mathbb{C}}$. Recall that the Hodge bundle on T, which we denote by λ as well, is defined as

$$\lambda = \pi_* \Omega^2_{\mathcal{X}/T}.$$

A stronger version of Theorem 17 is stated in [24] for period spaces of $K3$ surfaces. The proof extends to give the following statement in the higher-dimensional case. It relies in an essential way on Borcherds' results in [7, 8].

Theorem 17 ([24], Theorem 3.1) *There exist lattices $\Lambda_1, \ldots, \Lambda_r$ and a Cartier divisor D supported on the D_{Λ_i} such that*

$$\mathcal{O}(D) = \lambda^{\otimes a}$$

for some positive integer a.

4 Partial compactifications of the moduli space and existence of a line bundle

4.1 Making the Kuga-Satake mapping finite

One of the main results of [24], and one that we are wishing to avoid, is the fact that families of supersingular $K3$ surfaces with semi-stable reduction do not degenerate. The analogous result for supersingular abelian varieties is well-known, see [31, Proof of Theorem 1.1.a], essentially because of the criterion of Néron-Ogg-Shafarevich. As a consequence, the result for $K3$ surfaces, or more generally for varieties as in Theorem 4, would follow if the Kuga-Satake mapping were finite.

In this section, we give a very simple construction of a canonical partial compactification of T over which the Kuga-Satake mapping extends to a finite morphism to $\mathcal{A}_{g,d',n}$.

As in Proposition 15, the Kuga-Satake map $\kappa : T \to \mathcal{A}_{g,d',n}$ admits a factorization over $\mathbb{C}$ through

$$j : T_{\mathbb{C}} \to \mathcal{S}_{n,\mathbb{C}}$$

and

$$KS_{\mathbb{C}} : \mathcal{S}_{n,\mathbb{C}} \to \mathcal{A}_{g,d',n,\mathbb{C}}.$$

By Zariski's main theorem, there exists a normal variety $\widetilde{T}_{\mathbb{C}}$ over $\mathbb{C}$ containing $T_{\mathbb{C}}$ as an open subvariety such that j extends to a finite morphism $\widetilde{T}_{\mathbb{C}} \to \mathcal{S}_{n,\mathbb{C}}$. Since j is defined over K, we can assume that $\widetilde{T}_{\mathbb{C}}$ and the map

$$\widetilde{T}_{\mathbb{C}} \to \mathcal{S}_{n,\mathbb{C}}$$

are defined over K. Let $\widetilde{T}_K$ be a model of $\widetilde{T}_{\mathbb{C}}$ over K.

Let T' be the normal scheme over W defined by gluing the W-schemes T and $\widetilde{T}_K$ along their common open subscheme T_K. By definition, the Kuga-Satake map extends to a morphism

$$\kappa' : T' \to \mathcal{A}_{g,d',n}.$$

Since κ is quasifinite by Proposition 12, κ' is quasifinite as well. The map $KS_{\mathbb{C}}$ above is finite by Proposition 15. This proves that κ' is a finite morphism when restricted to the generic fiber of T'.

Now applying Zariski's main theorem, we can find a normal W-scheme $\overline{T}$ and a dominant open immersion $T' \hookrightarrow \overline{T}$ such that

$$\kappa' : T' \to \mathcal{A}_{g,d',n}$$

extends to a finite morphism

$$\overline{\kappa} : \overline{T} \to \mathcal{A}_{g,d',n}.$$

We can summarize the preceding construction in the following statement.

Proposition 18 *There exists a normal, separated W-scheme $\overline{T}$, and a dominant open immersion*

$$i : T \hookrightarrow \overline{T}$$

such that

1. *The Kuga-Satake map κ extends to a finite morphism*

$$\overline{\kappa} : \overline{T} \to \mathcal{A}_{g,d',n}.$$

2. *The generic fiber of* $\overline{\kappa} \circ i$ *factorizes through the period map*

$$j : T_{\mathbb{C}} \to \mathcal{S}_{n,\mathbb{C}}.$$

The second condition above shows that the complex points of $\overline{T}$ parametrize Hodge structures of weight 2. Again, we denote by D_Λ the divisor on the generic fiber of $\overline{T}$ that is the inverse image of the divisor D_Λ in $\mathcal{S}_{n,\mathbb{C}}$.

Remark Through the Kuga-Satake map, it should actually possible to give a modular interpretation of the space $\overline{T}$ and of its special fiber. It is for instance likely that the p-divisible group associated to the formal Brauer group of the universal variety over T_k actually extends to a p-divisible group over $\overline{T}_k$. However, one of the points of this paper is that this modular interpretation is not needed.

4.2 The supersingular locus in $\overline{T}$

We start by defining the supersingular locus in $\overline{T}$.

Definition 19 The supersingular locus in $\overline{T}$ is the inverse image by $\overline{\kappa}$ of the locus of supersingular abelian varieties in $\mathcal{A}_{g,d',n}$.

The following is one of the main points of our proof. It is a straightforward consequence of a result of Oort.

Proposition 20 *The supersingular locus in* $\overline{T}$ *is projective.*

Proof In the course of the proof of Theorem 1.1.a in [31], Oort proves that the locus of supersingular abelian varieties in $\mathcal{A}_{g,d',n}$ is a projective subscheme of $\mathcal{A}_{g,d',n}$. Since $\overline{\kappa}$ is finite, the supersingular locus in $\overline{T}$ is projective. □

Over T, the supersingular locus above coincides with the locus of points t such that the fiber X_t of $\mathcal{X}$ at t is supersingular, as the next proposition shows. The analogous result in the ordinary case was proved by Nygaard in [27, Proposition 2.5].

Proposition 21 *Let* X_t *be a fiber of* π *over a* k*-point of* T. *Then* X_t *is supersingular if and only if the Kuga-Satake abelian variety of* X_t *is supersingular.*

Proof Let ℓ be a prime number different from the characteristic of k. Let A be the Kuga-Satake variety of X_t. The varieties X_t and A are defined over a finite field k_0. We denote by G_{k_0} the absolute Galois group of k_0. Let $P = H^2(X_t, \mathbb{Q}_\ell(1))_{prim}$ and let $C(P)$ be the Clifford algebra associated to P. By standard properties of the Kuga-Satake construction, there is an isomorphism of G_{k_0}-modules

$$C(P) \simeq \mathrm{End}_{C(P)}\big(H^1(A, \mathbb{Q}_\ell)\big). \tag{4}$$

We have to show that X_t is supersingular if and only if A is.

Let us first assume that the dimension of P is even. By [9, Paragraphe 9, n. 4, Corollaire after Théorème 2], $C(P)$ is a central simple algebra. Assume that A is supersingular. Up to replacing k_0 by a finite extension, we can assume that the Frobenius acts on $H^1(A, \mathbb{Q}_\ell)$ by a scalar. Equation (4) then shows that the Frobenius action on $C(P)$ is trivial. Since P is a Frobenius invariant subspace of $C(P)$, this shows that the Frobenius action on P is trivial and implies that X_t is supersingular.

Conversely, if X_t is supersingular, we can assume that the Frobenius morphism acts trivially on P, hence on $C(P) \simeq \mathrm{End}_{C(P)}(H^1(A, \mathbb{Q}_\ell))$. By the bicommutant theorem, it acts on $H^1(A, \mathbb{Q}_\ell)$ by an element of $C(P)$. Since it commutes with the action of $C(P)$, it acts through the center of $C(P)$. Since this center is trivial, Frobenius acts on $H^1(A, \mathbb{Q}_\ell)$ by a homothety. This implies that A is supersingular.

In case the dimension of P is odd, the even part $C^+(P)$ of the Clifford algebra $C(P)$ is a central simple algebra by [9, Paragraphe 9, n. 4, Théorème 3]. By standard properties of the Kuga-Satake construction, and up to replacing k_0 by a finite extension, A is isogenous to the square of an abelian variety B over k_0, such that there is an isomorphism of G_{k_0}-modules

$$C^+(P) \simeq \mathrm{End}_{C^+(P)}\big(H^1(B, \mathbb{Q}_\ell)\big).$$

Here B is the Kuga-Satake variety used in [24]. We have to show that X_t is supersingular if and only if B is. The same argument as above shows this equivalence. □

4.3 The closure of some Hodge loci

The main result of this section is the key to avoiding the degeneration results of [24]. We investigate the geometrical properties of the Zariski closure of the Hodge locus D_Λ of Sect. 3.7 in $\overline{T}$.

Before stating the result, we introduce the following notation. If

$$\Lambda = \begin{pmatrix} d & a \\ a & b \end{pmatrix}$$

is a lattice and N is a nonzero integer, let

$$\Lambda_N = \begin{pmatrix} d & Na \\ Na & N^2 b \end{pmatrix}.$$

Note that there is an embedding of lattices $\Lambda_N \hookrightarrow \Lambda$ that sends the second base vector v to Nv. As a consequence, with the notations of Sect. 3.7, the divisor D_{Λ_N} contains the divisor D_Λ.

Recall that by construction the family $\pi : \mathcal{X} \to T$ contains X as a special fiber.

Proposition 22 *Let Λ be a lattice as in Sect. 3.7, and let $\overline{D}_\Lambda$ be the Zariski closure of D_Λ in $\overline{T}$. Let C be the connected component of the supersingular locus of $\overline{T}$ passing through the point of T corresponding to X.*

If the intersection of C and $\overline{D}_\Lambda$ is nonempty, then there exists a nonzero integer N such that $\overline{D}_{\Lambda_N}$ contains the support of C.

Remark By the second part of Proposition 18 and the comment that follows, we can view D_Λ as a divisor on the generic fiber of $\overline{T}$, which is why the statement above makes sense.

The proposition above is close in spirit to the statement of [3, Theorem 1.1.a], and our point of view is somewhat similar to that of [13]. The main idea of our argument is very simple and can be summarized as follows.

Assume that C actually parametrizes a family of supersingular polarized varieties—this is not true as C does not need to lie in T. Then the assumption of the proposition means that there exists a polarized supersingular deformation X_0 of X, together with a line bundle $\mathcal{L}_0$ on X_0 such that the lattice generated by $\mathcal{L}_0$ and the polarization in the Néron-Severi group of X_0 is isomorphic to Λ.

Now consider the versal deformation space of $(X_0, \mathcal{L}_0)$ inside T. The arguments from [14, Theorem 1.6] show that it is a divisor D in T which is flat over W. By the argument of Artin above, see also [13, Theorem 1], and up to replacing $\mathcal{L}_0$ by $\mathcal{L}_0^{\otimes p^r}$ for some positive r, this divisor contains C, and the intersection properties of $\mathcal{L}_0$ imply that its generic fiber is contained in $D_{\Lambda_{p^r}}$. This implies the result.

In our situation, we do not know that C parametrizes a family of $K3$ surfaces. However, the Kuga-Satake map provides us with a family of supersingular abelian varieties over C, which will be enough for our purposes. We now proceed with the proof.

Proof By assumption, there exists a finite extension R of W with residue field k and fraction field F, and an R-point P of $\overline{T}$, such that the generic fiber of P is an F-point t of $\overline{T}$ lying on D_Λ and such that the special fiber of P is a k-point t_0 of C. Up to replacing Λ by Λ_d and changing basis for Λ, we can assume that

$$\Lambda = \begin{pmatrix} d & 0 \\ 0 & b \end{pmatrix}.$$

This means that D_Λ parametrizes varieties X with an element $v \in H^2(X, \mathbb{Z})_{prim}$ such that $q(v) = b$.

Let $\pi : \mathcal{A} \to \overline{T}_R$ be the family of polarized abelian varieties obtained by pulling back the universal family over $\mathcal{A}_{g,d',n}$ by the Kuga-Satake map $\overline{\kappa}$ of Proposition 18. Using (1) of Sect. 3.3 and Proposition 11, the second base element of Λ corresponds to an endomorphism ϕ of the polarized abelian variety $\mathcal{A}_t$ such that $\phi \circ \phi$ is multiplication by b on $\mathcal{A}_t$.

Let D be the component of the intersection of D_Λ with T_F such that, locally, D is the image in T_F of the versal deformation space of the pair $(\mathcal{A}_t, \phi)$. In other words, D is the locus in T_F where ϕ deforms with $\mathcal{A}_t$.

The point t specializes to t_0. Let us write $\mathcal{A}_0$ for $\mathcal{A}_{t_0}$. The endomorphism ϕ specializes to an endomorphism ϕ_0 of $\mathcal{A}_0$. There exists a positive integer r such that the pair $(\mathcal{A}_0, p^r\phi_0)$ deforms over C. Indeed, the generic fiber $\mathcal{A}_{k(C)}$ of the universal polarized abelian scheme $\mathcal{A}$ over C is supersingular by Proposition 21, which implies that the cokernel of the specialization map

$$\mathrm{End}(\mathcal{A}_{k(C)}) \to \mathrm{End}(\mathcal{A}_0)$$

is killed by a power of p. In particular, the pair $(\mathcal{A}_0, p^r\phi_0)$ deforms to a pair $(\mathcal{A}_1, \psi_1)$ such that $\mathcal{A}_1$ is the Kuga-Satake abelian variety associated to the variety X itself over k. Note that $\psi_1 \circ \psi_1$ is multiplication by $p^{2r}b$ on $\mathcal{A}_1$. Let t_1 be the corresponding point of T.

We want to study the versal deformation space of the pair $(\mathcal{A}_1, \psi_1)$ by making use of X. This can be done using the following canonical embedding

$$H^2_{crys}(X/W)_{prim} \hookrightarrow \mathrm{End}\big(H^1_{crys}(\mathcal{A}_1/W)\big) \tag{5}$$

induced by (3). Given a lifting of the polarized variety X to characteristic zero, the groups above are identified to relative de Rham cohomology groups, and (5) is flat and respects the Hodge filtration.

Lemma 23 *Let* $[\psi_1] \in \mathrm{End}(H^1_{crys}(\mathcal{A}_1/W))$ *be the crystalline cohomology class of* ψ_1. *Then* $[\psi_1]$ *lies in the image of* $H^2_{crys}(X_1/W)_{prim}$ *by the morphism* (5).

Proof Since (5) is primitive by Proposition 13, its cokernel is torsion-free. We can work with crystalline cohomology groups tensored with K and only show that $[\psi_1]$ lies in the image of $H^2_{crys}(X_1/K)$.

Let $\widehat{B}$ be the formal neighborhood of t_0 in $\overline{T}$. The formal scheme $\widehat{B}$ is flat over $\mathrm{Spf}(R)$ since $\overline{T}$ is flat over $\mathrm{Spec}(W)$. Recall the morphism of (3)

$$R^2\pi_*\Omega^\bullet_{\widehat{\mathcal{X}}/\widehat{B}}(1)_{prim} \hookrightarrow \mathrm{End}\big(R^1\psi_*\Omega^\bullet_{\widehat{\mathcal{A}}/\widehat{B}}\big).$$

Let α be the cohomology class of ϕ in $\mathrm{End}(H^1_{dR}(\mathcal{A}_t/F))$. By [22, Proposition 8.9], there exists a unique flat section $\widetilde{\alpha}$ of $\mathrm{End}(R^1\pi_*\Omega^\bullet_{\mathcal{A}/\widehat{B}[1/p]})$ over $\widehat{B}[1/p]$ passing through α.[1]

Since $\widetilde{\alpha}$ is flat and α belongs to $H^2_{dR}(\mathcal{A}_t/F)_{prim}$ by assumption, $\widetilde{\alpha}$ comes from a section of $R^2\pi_*\Omega^\bullet_{\widehat{\mathcal{X}}/\widehat{B}[1/p]}(1)_{prim}[1/p]$ over $\widehat{B}[1/p]$.

On the other hand, the endomorphism ψ_1 deforms by assumption over C to an endomorphism $\widetilde{\psi}$ of the abelian scheme $\mathcal{A}/C$. The crystalline cohomology class $[\widetilde{\psi}]$ of $\widetilde{\psi}$ induces a section of the convergent isocrystal $R^2\pi_*\Omega^\bullet_{\widehat{\mathcal{X}}/\widehat{B}}(1)_{prim}[1/p]_{|C}$.

By definition of α, we have $\widetilde{\alpha}_t = [\widetilde{\psi}]_{t_0}$ under the various identifications. By flatness, if t' is a point of $\widehat{B}[1/p]$ that specializes to the generic point η_C of $\widehat{C}$, we get $\widetilde{\alpha}_{t'} = [\widetilde{\psi}]_{\eta_C}$. By the remarks above, this implies the result. □

The preceding lemma allows us to control the deformation theory of the pair $(\mathcal{A}_1, \psi_1)$ as in the sketch of proof above. Indeed, standard deformation theory arguments and, for instance, Grothendieck-Messing theory as in [25]—see also [11, Theorem 2.4] for a summary—shows that the obstruction to deforming ψ_1 with $\mathcal{A}_1$ is controlled by the group $H^2(X, \mathcal{O}_X)$. As this k-vector space is one-dimensional, the versal deformation space of $(\mathcal{A}_1, \psi_1)$ is a divisor Σ in the formal neighborhood $\widehat{T}$ of t_1 in T.

By the argument of [30, Theorem 2.9], no component of Σ can dominate the special fiber T_k. Since the result is not stated as such in [30] as we do not know a priori that ψ_1 comes from a line bundle on X_1, let us sketch how Ogus' proof works in our setting.

Ogus shows by a dimension count that T_k contains an ordinary point. As a consequence, if some component of Σ dominates T_k, then one can deform $(\mathcal{A}_1, \psi_1)$ to a pair $(\mathcal{A}_2, \psi_2)$ where $\mathcal{A}_2$ is the Kuga-Satake variety of an ordinary fiber of π. There we can assume that ψ_2 is not divisible by p, and the deformation arguments of [30, Proposition 2.2] give the result.

This shows that Σ is a flat divisor in $\widehat{T}$. Furthermore, one of its components contains the supersingular component C by assumption. As a consequence, the Zariski closure of the generic fiber Σ_K of Σ contains C.

We claim that Σ_K is included in some D_{Λ_N}. First, let s be any complex point of Σ_K. By definition, s corresponds to a polarized variety X_s and an endomorphism ψ of its Kuga-Satake abelian variety $\mathcal{A}_t$. Since ψ is a deformation of ψ_1, $\psi \circ \psi$ is multiplication by $p^{2r}b$ on $\mathcal{A}_s$. Furthermore, again by assumption, the cohomology class of ψ lies in the subspace $H^2(X, \mathbb{Z})_{prim}$ of $\mathrm{End}(H^1(A_t, \mathbb{Z}))$.

[1]The result of [22] above only works over a smooth base, and $\widehat{B}[1/p]$ might not be smooth. However, the abelian scheme $\mathcal{A}$ over $\widehat{B}$ comes from the universal abelian scheme over $\mathcal{A}_{g,d',n}$, which is smooth and where Katz' result applies.

Since the Kuga-Satake correspondence is induced by a Hodge class, the cohomology class $[\psi]$ of ψ in Betti cohomology is a Hodge class. As a consequence, it comes from a Hodge class in $H^2(X, \mathbb{Z})_{prim}$, which shows by the Lefschetz (1, 1) theorem that it is the cohomology class of a line bundle $\mathcal{L}$ on X. By Proposition 11, we have $q([\psi]) = q(c_1(\mathcal{L})) = p^{2r}b$. As a consequence, the lattice generated by $c_1(\mathcal{L})$ and the polarization in the Néron-Severi group of X is isomorphic to Λ_{p^r}. This concludes the proof. □

In the course of the proof, we actually obtained the following result.

Proposition 24 *Let t_0 be a k-point t_0 of T that lies on a component C of the supersingular locus. Let ψ be an endomorphism of $\mathcal{A}_{t_0}$ such that the cohomology class of ψ acting on $H^1_{crys}(\mathcal{A}_{t_0}/W)$ belongs to the image of $H^2_{crys}(X_{t_0}/W)_{prim}$ by the Kuga-Satake correspondence.*

There exists a lattice Λ as in Sect. 3.7 and a positive integer r such that the versal deformation space of $(\mathcal{A}_{t_0}, p^r\psi)$ in $\overline{T}$ contains C and is contained with the Zariski closure $\overline{D}_\Lambda$ of the Noether-Lefschetz divisor D_Λ in $\overline{T}$.

4.4 Finding one line bundle

Choose lattices Λ_i and a divisor D supported on the D_{Λ_i} as in the conclusion of Theorem 17. There exists a positive integer a such that $\mathcal{O}(D) = \lambda^{\otimes a}$.

Proposition 25 *Let $\overline{D}$ be the Zariski closure of D in $\overline{T}$. Then the special fiber $\overline{D}_k$ of $\overline{D}$ is an ample $\mathbb{Q}$-Cartier divisor.*

Proof Consider the Kuga-Satake mapping

$$KS_{\mathbb{C}} : \mathcal{S}_{n,\mathbb{C}} \to \mathcal{A}_{g,d',n,\mathbb{C}}$$

of Sect. 3.6.

By the argument of [24, Proposition 5.8], if $\lambda_{\mathcal{A}}$ denotes the Hodge bundle on $\mathcal{A}_{g,d',n,\mathbb{C}}$, there exists a positive integer r such that

$$KS_{\mathbb{C}}^*\left(\lambda_{\mathcal{A}}^{\otimes r}\right) = \lambda^{\otimes(2^{b-1}r)},$$

where b is the second Betti number of X.

As a consequence, after pulling back to T_K, we can write

$$\mathcal{O}(MD) = \kappa_K^*\left(\lambda_{\mathcal{A}}^N\right)$$

for some positive integers M and N—taking powers of line bundles to descend the equality from $T_{\mathbb{C}}$ to T_K.

Let U be the smooth locus of $\overline{T} \to \operatorname{Spec} W$, and let D' be the closure of D in U. By Lemma 5.12 in [24], $\mathcal{O}(MD') = \overline{\kappa}^*(\lambda_{\mathcal{A}}^N)_{|U}$. Let s be a section of $\overline{\kappa}^*(\lambda_{\mathcal{A}}^N)$ over U with divisor MD'.

Since $\overline{T}$ is normal, the complement of U in $\overline{T}$ has codimension at most 2 in $\overline{T}$ and s extends to a section of $\overline{\kappa}^*(\lambda_{\mathcal{A}}^N)$ over $\overline{T}$. The divisor of s is the closure of MD' in $\overline{T}$, which precisely means that the divisor of s in $\overline{T}$ is $M\overline{D}$ and shows that

$$\mathcal{O}(M\overline{D}) = \overline{\kappa}^*\big(\lambda_{\mathcal{A}}^N\big).$$

By [18, V.2.3], the Hodge line bundle $\lambda_{\mathcal{A}}$ is ample on $\mathcal{A}_{g,d',n,k}$. Since $\overline{\kappa}$ is finite, this proves that $M\overline{D}_k$ is an ample Cartier divisor and concludes the proof. □

We now prove the following first step in the direction of Theorem 4. We keep the notations as above.

Proposition 26 *There exists a complex point t of T such that $\mathcal{X}_t$ specializes to X and has Picard number at least* 2.

Proof Let x be the k-point of T corresponding to X. By [29, Theorem 15], the supersingular locus in T is closed of dimension $s = b - 3 - E((b-1)/2)$, where E is the integer part function. Indeed, the height of varieties parametrized by $\mathcal{X} \to T$ varies between 1 and $E((b-1)/2)$ if it is finite, and the locus of points in T with height at least h has codimension $h-1$. Together with Proposition 20, this shows that there is a nontrivial proper s-dimensional component C in the supersingular locus of $\overline{T}$ containing x. Note that $s > 0$ because $b > 5$.

Choose Λ_i and D as above, and let $\overline{D}$ be as in the previous proposition. By Proposition 25, some multiple of $\overline{D}_k$ is an ample Cartier divisor. As a consequence, the intersection of $\overline{D}$ and C is not empty. By Proposition 22, there exists a lattice Λ such that the Zariski closure of D_Λ in $\overline{T}$ contains C. As a consequence, X is the specialization of a variety with Picard number at least 2. □

5 Proof of Theorem 4

We now adapt the techniques of the preceding section to prove Theorem 4. We keep the notations as above.

5.1 Lifting many line bundles to characteristic zero

In this section, we prove the following result.

Proposition 27 *Let x be the point of $\overline{T}$ corresponding to X, and let C be the connected component of the supersingular locus of $\overline{T}$ containing x. There exist a k-point y of C and a complex point z of $\overline{T}$ with the following properties.*

1. *Under the identifications of Proposition 18, the point z specializes to y.*
2. *The weight 2 Hodge structure parametrized by z has Picard rank $b-3$.*

We start with a generalization of Proposition 22 to higher Picard numbers which might be of independent interest. Before stating it, let us introduce some notations.

Let Λ be a nondegenerate lattice containing a primitive element h of square d. We denote by Z_Λ the locus in $\mathcal{S}_{n,\mathbb{C}}$ of points s such that if H_s is the weight 2 Hodge structure on L corresponding to s, there exists an embedding of Λ in the Néron-Severi group of H_s mapping h to the class of the polarization. In case the rank of Λ is 2, we recover the divisor D_Λ we used above. As before, Z_Λ is defined over $\mathbb{Q}$.

Proposition 28 *Let x be the point of $\overline{T}$ corresponding to X and let C be a component of the supersingular locus in $\overline{T}$ passing through x. Then there exists a lattice Λ of rank $E((b-1)/2)$ such that the Zariski-closure of Z_Λ in $\overline{T}$ contains C.*

Corollary 29 *The variety X admits a lift to a polarized variety of Picard rank at least $E((b-1)/2)$ in characteristic* 0.

Proof of Proposition 28 We prove by induction on $n \leq E((b-1)/2)$ that there exists a lattice Λ of rank n such that the Zariski closure of Z_Λ in $\overline{T}$ contains C. We will argue as in Proposition 22, which deals with the rank 2 case.

Let $n < E((b-1)/2)$ be a positive integer, and assume that there exists a lattice Λ of rank n such that the Zariski-closure of Z_Λ in $\overline{T}$ contains C. Let us first remark that Z_Λ is itself a union of Shimura subvarieties of $\mathcal{S}_n$, associated to the orthogonal of copies of Λ in the lattice L. As such, its components are Shimura varieties of orthogonal type corresponding to a lattice of signature $(2, b-2-n)$.

The Noether-Lefschetz locus on Z_Λ is a countable union of divisors satisfying the following analog of Theorem 17. Let Λ' be a nondegenerate rank $n+1$ lattice containing Λ. The variety $Z_{\Lambda'}$ is naturally a divisor in Z_Λ. The proof of [24, Theorem 3.1] translates immediately to show that the analog of Theorem 17 holds for the divisors $Z_{\Lambda'}$ in Z_Λ.

We can now use the ampleness arguments of Proposition 25, working this time with the Zariski closure $\overline{Z}_\Lambda$ of Z_Λ in $\overline{T}$, to show that there exists a lattice

Λ' of rank $n+1$ containing Λ such that the intersection of $\overline{Z}_{\Lambda'}$ and C is not empty.[2]

At this point, we can repeat the proof of Proposition 22 to show that some $\overline{Z}_{\Lambda'}$ actually contains the support of C. The only step that does not go through is the following. Let $\mathcal{X}_t$ be a fiber of π over a k-point t of T with an embedding of Λ in $\mathrm{Pic}(X_t)$, and let $(\mathcal{X}_t, S)$ be an irreducible component of a versal k-deformation of the pair (X_t, Λ) in T. We need to show that the geometric generic fiber $\mathcal{X}_{\overline{\eta}}$ is ordinary with Picard group of rank n. This is a generalization of [30, Theorem 2.9].

We first remark that by standard deformation theory, the dimension of $\overline{Z}_{\Lambda}$ is at least $b-2-n$. On the other hand, Ogus shows in [30, Theorem 2.9] that the dimension of the non-ordinary locus in S is at most $\max(n, b-3-n)$. Since $n < E((b-1)/2)$, this shows that $\mathcal{X}_{\overline{\eta}}$ is ordinary. Usual deformation theory of ordinary $K3$ crystals allows us to conclude that S is of dimension $b-2-n$ and that the conclusion holds. □

Proof of Proposition 27 Let us show by induction on $n \le b-3-E((b-1)/2)$ that there exists a nondegenerate lattice Λ of rank $E((b-1)/2)+n$ such that the intersection of $\overline{Z}_{\Lambda}$ with C is a non-empty subscheme C_{Λ} of C of dimension at least $b-3-E((b-1)/2)-n$. For $n = b-3-E((b-1)/2)$, this gives the conclusion of Proposition 27.

For $n=0$, this is the statement of Proposition 28, since C is of dimension $b-3-E((b-1)/2)$. Assume that the result we just stated holds for some $n < b-3-E((b-1)/2)$. As in the proof of Proposition 28 above, since the dimension of C_{Λ} is positive, we can find a nondegenerate lattice Λ' of rank $n+1$ containing Λ such that $\overline{Z}_{\Lambda'}$ has non-empty intersection $C_{\Lambda'}$ with C_{Λ}. Since the dimension of C_{Λ} is at least $b-3-E((b-1)/2)-n$, the dimension of $C_{\Lambda'}$ is at least $b-3-E((b-1)/2)-(n+1)$. This concludes the proof of Proposition 27. □

5.2 From Picard rank $b-3$ to Picard rank b

In this section, we show how to derive Theorem 4 from Proposition 27.

Proof of Theorem 4 We start with a Hodge-theoretic lemma.

Lemma 30 *Let H be a weight 2 polarized Hodge structure with $h^{2,0}=1$. Assume that the codimension of the space of Hodge classes in H is at most 3. Let A be the Kuga-Satake variety of H together with a polarization, and let*

$$p : \mathrm{End}\big(H^1(A,\mathbb{Q})\big) \to \mathrm{End}\big(H^1(A,\mathbb{Q})\big)$$

[2] The only difference with Proposition 25 is that $\overline{Z}_{\Lambda}$ is not normal a priori. However, one can work on the normalization of $\overline{Z}_{\Lambda}$ and carry on with the proof.

be the orthogonal projector onto H. Then p is induced by an algebraic correspondence on $(A \times A)^2$.

Proof We can write $H = H' \oplus V$ as polarized Hodge structures, where H' is of dimension at most 3 and V is contained in the space of Hodge classes of H. Standard computations show that the Kuga-Satake variety of H is isogenous—in a functorial way—to a power of the Kuga-Satake variety associated to H'. As a consequence, we can assume that the dimension of H is at most 3.

First assume that the dimension of H is exactly 3. In that case, A is an abelian variety of dimension $2^{3-1} = 4$. However, we know that A is isogenous to the square of the Kuga-Satake variety obtained by considering the even Clifford algebra of H. It follows that A is isogenous to the square of an abelian surface. In particular, $(A \times A)^2$ is isomorphic to a product of abelian surfaces and satisfies the Hodge conjecture by the main result of [32]. This proves the theorem, as the projector p is indeed given by a Hodge class.

If the dimension of H is 2, A is an abelian surface—that is actually isogenous to the square of an elliptic curve—and the same proof applies. □

We now use the notations of Proposition 27, and we want to prove that the Picard rank of X is b. Let A_z be the Kuga-Satake variety of the weight 2 Hodge structure H_z parametrized by z. By assumption, A_z has good, supersingular reduction at p. Let A_y be this smooth reduction. By the preceding lemma, we have an algebraic correspondence of codimension 2 on $(A_z \times A_z)^2$ which acts as the orthogonal projector

$$p : \mathrm{End}\big(H^1(A_z, \mathbb{Q})\big) \to \mathrm{End}\big(H^1(A_z, \mathbb{Q})\big)$$

onto H_z.

The correspondence p specializes to a correspondence p_y on $(A_y \times A_y)^2$. By the smooth base-change theorem, the image of p_y acting on crystalline cohomology is $(b-1)$-dimensional. Furthermore, since A_y is supersingular, its crystalline cohomology is spanned by algebraic cycles. In particular, we can find endomorphisms $\psi_1, \dots, \psi_{b-1}$ of A_y that lie in the image of p and span the image of p.

By Proposition 24, the ψ_i deform to classes of line bundles on X spanning the primitive cohomology of X. This concludes the proof of Theorem 4. □

Acknowledgements The work on this paper started when I had the chance of discussing the paper [24] with Daniel Huybrechts during the workshop “Advances in hyperkähler and holomorphic symplectic geometry” at the BIRS in Banff.

I am very grateful to D. Huybrechts for the many discussions we had on this subject and his many remarks on early versions of this paper. I would also like to thank the organizers of the workshop for creating a stimulating atmosphere. I thank Davesh Maulik for helpful email correspondence, Olivier Benoist for interesting discussions and Hélène Esnault for helpful remarks on a first draft of this paper. I thank the referee for his careful reading and helpful remarks.

References

1. André, Y.: On the Shafarevich and Tate conjectures for hyper-Kähler varieties. Math. Ann. **305**(2), 205–248 (1996)
2. Andreatta, F., Iovita, A.: Comparison isomorphisms for formal schemes. To appear in J. Inst. Math. Jussieu, available at http://www.mathstat.concordia.ca/faculty/iovita/
3. Artin, M.: Supersingular $K3$ surfaces. Ann. Sci. École Norm. Super. (4) **7**, 543–567 (1974)
4. Beauville, A.: Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differ. Geom. **18**(4), 755–782 (1983)
5. Beauville, A., Donagi, R.: La variété des droites d'une hypersurface cubique de dimension 4. C. R. Acad. Sci. Paris Sér. I Math. **301**(14), 703–706 (1985)
6. Bogomolov, F.A.: Hamiltonian Kählerian manifolds. Dokl. Akad. Nauk SSSR **243**(5), 1101–1104 (1978)
7. Borcherds, R.E.: Automorphic forms with singularities on Grassmannians. Invent. Math. **132**(3), 491–562 (1998)
8. Borcherds, R.E.: The Gross-Kohnen-Zagier theorem in higher dimensions. Duke Math. J. **97**(2), 219–233 (1999)
9. Bourbaki, N.: Éléments de Mathématique. Algèbre. Chapitre 9. Springer, Berlin (2007). Reprint of the 1959 original
10. Breuil, C., Messing, W.: Torsion étale and crystalline cohomologies. Astérisque **279**, 81–124 (2002). Cohomologies p-adiques et applications arithmétiques, II
11. Chai, C.-L., Oort, F.: Moduli of Abelian varieties and p-divisible groups. In: Arithmetic Geometry. Clay Math. Proc., vol. 8, pp. 441–536. Am. Math. Soc., Providence (2009)
12. Charles, F., Schnell, C.: Notes on absolute Hodge classes. Preprint
13. de Jong, J.: On a result of Artin. Preprint
14. Deligne, P.: Relèvement des surfaces $K3$ en caractéristique nulle. In: Algebraic Surfaces, Orsay, 1976–1978. Lecture Notes in Math., vol. 868, pp. 58–79. Springer, Berlin (1981). Prepared for publication by Luc Illusie
15. Deligne, P.: La conjecture de Weil pour les surfaces $K3$. Invent. Math. **15**, 206–226 (1972)
16. Deligne, P.: Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. In: Automorphic Forms, Representations and L-Functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2. Proc. Sympos. Pure Math., vol. XXXIII, pp. 247–289. Am. Math. Soc., Providence (1979)
17. Deligne, P., Illusie, L.: Relèvements modulo p^2 et décomposition du complexe de de Rham. Invent. Math. **89**(2), 247–270 (1987)
18. Faltings, G., Chai, C.-L.: Degeneration of Abelian Varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22. Springer, Berlin (1990). With an appendix by David Mumford
19. Fogarty, J.: Algebraic families on an algebraic surface. Am. J. Math. **90**, 511–521 (1968)
20. Fontaine, J.-M., Messing, W.: p-adic periods and p-adic étale cohomology. In: Current Trends in Arithmetical Algebraic Geometry, Arcata, Calif., 1985. Contemp. Math., vol. 67, pp. 179–207. Am. Math. Soc., Providence (1987)
21. Huybrechts, D.: Compact hyper-Kähler manifolds: basic results. Invent. Math. **135**(1), 63–113 (1999)
22. Katz, N.M.: Nilpotent connections and the monodromy theorem: applications of a result of Turrittin. Inst. Hautes Études Sci. Publ. Math. **39**, 175–232 (1970)
23. Maulik, D., Lieblich, M., Snowden, A.: Finiteness of K3 surfaces and the Tate conjecture. arXiv:1107.1221

24. Maulik, D.: Supersingular $K3$ surfaces for large primes (2012). arXiv:1203.2889
25. Messing, W.: The Crystals Associated to Barsotti-Tate Groups: with Applications to Abelian Schemes. Lecture Notes in Mathematics, vol. 264. Springer, Berlin (1972)
26. Nygaard, N., Ogus, A.: Tate's conjecture for $K3$ surfaces of finite height. Ann. Math. (2) **122**(3), 461–507 (1985)
27. Nygaard, N.O.: The Tate conjecture for ordinary $K3$ surfaces over finite fields. Invent. Math. **74**(2), 213–237 (1983)
28. O'Grady, K.G.: Involutions and linear systems on holomorphic symplectic manifolds. Geom. Funct. Anal. **15**(6), 1223–1274 (2005)
29. Ogus, A.: Singularities in the height strata in the moduli of $K3$ surfaces. Preprint
30. Ogus, A.: Supersingular $K3$ crystals. In: Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. II. Astérisque, vol. 64, pp. 3–86. Soc. Math. France, Paris (1979)
31. Oort, F.: Subvarieties of moduli spaces. Invent. Math. **24**, 95–119 (1974)
32. Ramón Marí, J.J.: On the Hodge conjecture for products of certain surfaces. Collect. Math. **59**(1), 1–26 (2008)
33. Rizov, J.: Moduli stacks of polarized $K3$ surfaces in mixed characteristic. Serdica Math. J. **32**(2–3), 131–178 (2006)
34. Rizov, J.: Kuga-Satake abelian varieties of K3 surfaces in mixed characteristic. J. Reine Angew. Math. **648**, 13–67 (2010)
35. Tate, J.T.: Algebraic cycles and poles of zeta functions. In: Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), pp. 93–110. Harper & Row, New York (1965)
36. Tian, G.: Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. In: Mathematical Aspects of String Theory, San Diego, Calif., 1986. Adv. Ser. Math. Phys., vol. 1, pp. 629–646. World Scientific Publishing, Singapore (1987)
37. Todorov, A.N.: The Weil-Petersson geometry of the moduli space of SU($n \geq 3$) (Calabi-Yau) manifolds. I. Commun. Math. Phys. **126**(2), 325–346 (1989)
38. Zarhin, Ju.G.: A remark on endomorphisms of abelian varieties over function fields of finite characteristic. Izv. Akad. Nauk SSSR, Ser. Mat. **38**, 471–474 (1974)