

**THE TATE CONJECTURE FOR $K3$ SURFACES OVER FINITE FIELDS
(ERRATUM)**

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In the statement of Theorem 4, it should be assumed that the second and third Betti cohomology groups of Y have no p -torsion. This is implicitly used in 2.2 to compute the second crystalline cohomology group of X as well as its Hodge number $h^{2,0}$. Furthermore, the Beauville-Bogomolov form should be assumed to be even so as to apply Borcherds' results – this is satisfied in all known cases.

Corollary 5 and 6 still hold: the second and third cohomology group of a variety of $K3^{[n]}$ type are torsion-free as can be easily checked by a direct computation, see e.g. [Cha14, Proposition 2.5].

As pointed out to us by Olivier Benoist, there is an error in the proof of [Cha12, Proposition 25]. This omission has consequences in [Cha12, Corollary 29]. We indicate how to fix accordingly the contents of Section 5 of the paper, which proves the main theorem. The beautiful survey [Ben14] briefly discusses how to work around the mistake and still recover part of the results.

We keep the notation of Proposition 25. Looking at the beginning of page 140, Lemma 5.12 in [Mau12] only shows the equality $\mathcal{O}(MD')_{U_k} = \bar{\kappa}^*(\lambda_{\mathcal{A}}^N)_{U_k}$ between restrictions of the line bundles to the special fiber. As a consequence, the ampleness of $M\bar{D}_k$ is proved only when $U_k = \bar{T}_k$, i.e., when \bar{T} is smooth.

In the setting of Proposition 25, the main theorem of [Kis10] ensures that \bar{T} is indeed smooth. Indeed, the normalization of the closure of the image of the Kuga-Satake map in the moduli space of abelian varieties is smooth, since we are considering varieties with polarization of degree prime to p . However, the argument of Proposition 25 is applied in Proposition 28 to families Z_Λ of lattice-polarized Hodge structures, where $\Lambda \otimes \mathbb{Z}/p\mathbb{Z}$ might be degenerate – since the induction step of the proof involves multiplication by powers of p of some elements of the lattices. We now explain the modifications one needs to apply in Section 5 of the paper.

We keep the notations of section 5. In particular, C is a connected component of the supersingular locus in \bar{T} . We show by induction on r that for all $r \leq b - 5$, there exists a lattice Λ of rank r such that $\Lambda \otimes \mathbb{Z}/p\mathbb{Z}$ is non-degenerate and \bar{Z}_Λ intersects C along a subscheme of positive dimension.

We first claim that for any lattice Λ of rank $r \leq b - 5$, any component of the supersingular locus of $(\bar{Z}_\Lambda)_k$ is positive-dimensional. Indeed, if h is any integer, any component of the locus in $(\bar{Z}_\Lambda)_k$ of points of height at least $h + 1$ has codimension at most h by a standard argument [Ogu, Proposition 11]¹. Furthermore, by [Art74, Theorem 0.1], at any point of finite height h of $(\bar{Z}_\Lambda)_k$, we have

$$2h \leq b - r.$$

This shows that the codimension of any component of the supersingular locus of $(\bar{Z}_\Lambda)_k$ is at most $E(\frac{b-r}{2})$, where E is the integer part function. Since the dimension of $(\bar{Z}_\Lambda)_k$ is $b - 2 - r$ and $r \leq b - 5$, and since $b - 2 - r > E(\frac{b-r}{2})$ if $r \leq b - 5$, the claim is proved.

¹By construction of \bar{T} , any point of \bar{T} defines a crystal – at a point of T , corresponding to a variety X , this is the second crystalline cohomology group of X – which shows that the height is well-defined.

The existence of Λ such that $\Lambda \otimes \mathbb{Z}/p\mathbb{Z}$ is non-degenerate and \overline{Z}_Λ intersects C along a subscheme of positive dimension now follows by induction as in the paper, noting that the non-degeneracy assumption allows us to apply the aforementioned result of Kisin once again so as to apply Proposition 25 as before.

Let Λ be a lattice of rank $b - 5$ as above. By section 5.2, the proof of Theorem 4 will be complete if we can find a lattice Λ' of rank $b - 3$ such that $\overline{Z}_{\Lambda'}$ has non-empty intersection with C . By assumption \overline{Z}_Λ is smooth and intersects C along a subscheme of dimension at least 1. As in the proof of Proposition 28, we can find a lattice Λ_1 – such that $\Lambda_1 \otimes \mathbb{Z}/p\mathbb{Z}$ is non-degenerate – of rank $b - 4$, containing Λ , such that \overline{Z}_{Λ_1} has nonempty intersection with C . We can assume that Λ_1 is generated by Λ and an element $v \in \Lambda_1$. By Proposition 22, we can find an integer N such that, writing $\Lambda_{1,N}$ for the lattice generated by Λ and Nv , $\overline{Z}_{\Lambda_{1,N}}$ contains C_Λ .

Once again, the proof of [Mau12, Theorem 3.1] shows that we can find a lattice Λ_2 of rank $b - 4$ containing Λ such that the intersection of \overline{Z}_{Λ_2} and C is not empty, and such that there is no lattice of rank $b - 4$ containing Λ into which both $\Lambda_{1,N}$ and Λ_2 map injectively by a morphism respecting Λ .

The intersection of $\overline{Z}_{\Lambda_{1,N}}$ and \overline{Z}_{Λ_2} contains a point c of C by construction, and the assumptions on the lattices implies that $\overline{Z}_{\Lambda_{1,N}}$ is a divisor in \overline{Z}_{Λ_2} , flat over the base. This readily shows that c lies in some $\overline{Z}_{\Lambda'}$, where Λ' has rank $b - 3$. This concludes the proof.

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