Infinite-Dimensional Geometry of Numbers: Hermitian Quasi-coherent Sheaves and Theta Finiteness

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Introduction

0.1. The Analogy between Number Fields and Function Fields and Arakelov Geometry

0.1.1. Since the end of the nineteenth century, the analogy between number fields, namely the field extensions of $\mathbb{Q}$ of finite degree, and function fields in one variable, defined as the fields of rational functions $k(C)$ on an algebraic curve C over some base field k, has played a central role in algebraic geometry and arithmetics.

The "arithmetization" of the theory of algebraic curves in the paper [**DW82**] by Dedekind and Weber, and the work of Kronecker [**Kro82**], devoted in substance to schemes of finite type over $\mathbb{Z}$ and their fields of rational functions — both published in 1882 — mark the beginning of the "modern" developments of the analogy between number fields and function fields.[1]

During the first decades of the twentieth century, this analogy has been completed by the discovery that, to make it more satisfactory, the embeddings of some number field K in the Archimedean fields $\mathbb{R}$ and $\mathbb{C}$ — or more precisely, the associated "Archimedean places" — had to be put on the same footing as the non-zero prime ideals of the ring of integers $\mathcal{O}_K$ of K, or equivalently as the associated embeddings of K in p-adic fields.

Indeed, only by adjoining the Archimedean places of K to the closed points of $\operatorname{Spec} \mathcal{O}_K$ does one obtain an analogue of a smooth *projective* curve C over some base field k, and not of an affine curve. In this improved analogy, the number field K plays the role of the function field $k(C)$, the set of Archimedean places of K admits as counterpart a non-empty finite set Σ of closed points of C, and the "arithmetic curve" $\operatorname{Spec} \mathcal{O}_K$ corresponds to the affine curve $\overset{\circ}{C} := C \setminus \Sigma$.

This discovery crucially enters in the definition of the adeles of a number fields, and through this path became extremely influential.[2]

[1] Establishing a precise history of the early phases of its exploration is a delicate task. Indeed this analogy appears to have been for a long time a part of "mathematical folklore," rather than the subject of academic publications.

For instance, in [**Her79**], when discussing the work of Chebyshev on continued fractions, Hermite mentions as self-evident the analogy between (i) the ring $\mathbb{Z}$, its fraction field $\mathbb{Q}$, and the usual Archimedean absolute value, and (ii) the ring $\mathbb{C}[X]$ of regular functions on the affine line $\mathbb{A}^1_{\mathbb{C}}$, its fraction field $\mathbb{C}(X)$, and the absolute value on $\mathbb{C}(X)$ attached to the valuation at the point at infinity.

Similarly the analogy between number fields and complex algebraic curve is mentioned by Hilbert in the twelfth of his *Mathematical Problems* [**Hil01**] in an extremely suggestive (and intriguing) form, and to a large extent as well-known.

[2] Hensel and his then doctoral student Hasse seem to have been the first to be fully aware of the importance of considering simultaneously all the places — finite and Archimedean — of a number field. See [**Hen13**, Kap. XII] and the *Geleitwort* in [**Has75**, p. viii-ix], where Hasse discusses the origins of his work on the "local-global principle" for quadratic forms in [**Has23, Has24**], and reproduces the content of a postcard sent to him by Hensel in October 1920, where the role of the "prime at infinity" is emphasized.

To the best of our knowledge, the first published explicit mention of global results, involving all the places of a number field and similar to classical results on algebraic curves and Riemann surfaces, is Weil's article [**Wei39**]. Shortly thereafter, in a letter to his sister [**Wei80**], Weil attributes to Artin and Hasse the discovery of the significance of Archimedean places.

0.1.2. The transfer to function fields of Diophantine problems — for instance the study of rational points of algebraic varieties defined over number fields — has naturally led to investigate the properties of algebraic varieties defined over a function field $k(C)$ as above.[3]

To study such an algebraic variety V, say projective over $k(C)$, one usually introduces a model of V over C, namely a projective k-variety $\mathcal{V}$ equipped with a flat morphism $\mathcal{V} \to C$ whose generic fiber is isomorphic to V. If V has dimension n, then $\mathcal{V}$ has dimension $n+1$. For instance, the study of curves over $k(C)$ leads one to rely on the properties of surfaces over k.

This approach may be emulated when studying algebraic varieties over a number field K. A model of a projective variety V over K is a scheme $\mathcal{V}$ projective and flat over $\mathcal{O}_K$ such that the K-scheme

$$\mathcal{V}_K := \mathcal{V} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} K$$

is isomorphic to V. If for instance V is curve, then $\mathcal{V}$ is a scheme of Krull dimension 2, a so-called arithmetic surface.

Arakelov discovered how to bring into play the Archimedean places of K when pursuing this approach, at least when $\mathcal{V}$ is an arithmetic surface: by relying on the Hermitian geometry of the compact Riemann surface $\mathcal{V}(\mathbb{C})$ and of the analytic vector bundles over $\mathcal{V}(\mathbb{C})$, and on the theory of Green functions over $\mathcal{V}(\mathbb{C})$.

In this way, after endowing $\mathcal{V}(\mathbb{C})$ and the line bundles over $\mathcal{V}$ — more precisely the complex analytic line bundles over $\mathcal{V}(\mathbb{C})$ they define — with some Hermitian metric, Arakelov ([**Ara74**]), then Faltings ([**Fal84**]) and Deligne ([**Del87**]) could extend to projective arithmetic surfaces various classical results concerning projective surfaces over some base field.

In the "calculus on arithmetic surfaces" developed by Arakelov, Faltings, and Deligne, the classical intersection numbers, with values in $\mathbb{Z}$, between pairs of line bundles over a projective surface over some field k are replaced by *arithmetic intersection numbers,* with values in $\mathbb{R}$, attached to pairs of Hermitian line bundles over projective arithmetic surfaces $\mathcal{V}$ as above — a Hermitian line bundle over $\mathcal{V}$ being defined as a pair $(L, \|.\|)$ where L denotes a line bundle over the scheme $\mathcal{V}$ and $\|.\|$ a (regular enough) Hermitian metric on the $\mathbb{C}$-analytic line bundle $L_{\mathbb{C}}$ on the compact Riemann surface $\mathcal{V}(\mathbb{C})$ attached to the complex algebraic curve

$$\mathcal{V}_{\mathbb{C}} := \mathcal{V} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C}.$$

0.1.3. After this pioneering work concerning arithmetic surfaces, the Arakelov geometry of higher dimensional schemes has been developed in several directions, which we want to briefly recall.[4]

In Arakelov geometry, a central role is played by *Hermitian vector bundles* over schemes of finite type over $\operatorname{Spec} \mathbb{Z}$. Recall that if $\mathcal{V}$ is a regular separated scheme of finite type over $\operatorname{Spec} \mathbb{Z}$, a Hermitian vector bundle over $\mathcal{V}$ is a pair $(E, \|.\|)$ where E is vector bundle — that is, a locally free coherent sheaf — over the scheme $\mathcal{V}$, and where $\|.\|$ is a C^∞ Hermitian metric, invariant under complex conjugation, on the $\mathbb{C}$-analytic vector bundle $E^{\mathrm{an}}_{\mathbb{C}}$ on the complex manifold $\mathcal{V}(\mathbb{C})$, deduced from E by the base change $\mathbb{Z} \hookrightarrow \mathbb{C}$ and analytification.

For instance, when $\mathcal{V}$ is an arithmetic curve $X := \operatorname{Spec} \mathcal{O}_K$ defined by the ring of integers $\mathcal{O}_K$ of some number field K, the complex manifold $X(\mathbb{C})$ is a finite set of cardinal $[K : \mathbb{Q}]$, namely the set of field embeddings $x : K \longrightarrow \mathbb{C}$.

[3] A remarkable instance of this line of thought has been the proof of Mordell conjecture for curves over function fields; see [**Man63**], [**Gra65**], and [**Sam66**].

[4] We do not claim to give a complete view of Arakelov geometry — in particular of its developments during the last decades — but only to provide some points of reference concerning its now classical part, in order to put in perspective the content of this monograph.

Consequently a Hermitian vector bundle over X is nothing but a pair:

$$\overline{E} := \big(E, (\|.\|_x)_{x\in X(\mathbb{C})}\big),$$

where E is a finitely generated projective $\mathcal{O}_K$-module, and $(\|.\|_x)_{x\in X(\mathbb{C})}$ a family, invariant under complex conjugation, of Hermitian norms on the finite dimensional $\mathbb{C}$-vector spaces:

$$E_x := E \otimes_{\mathcal{O}_K, x} \mathbb{C}$$

deduced from E by the base changes $x : \mathcal{O}_K \to \mathbb{C}$.

Basic tensor operations on vector bundles — for instance the formation of direct sums, of tensor products, of exterior and symmetric powers, of the dual... — still make sense for Hermitian vector bundles. These operations are defined by their "classical version" applied to the underlying vector bundles, equipped with the Hermitian metric defined by these same operations in Hermitian geometry.

Similarly, if $f : \mathcal{V}' \to \mathcal{V}$ is a morphism of schemes of finite type over $\operatorname{Spec}\mathbb{Z}$ as above, we may define the pullback $f^*\overline{E}$ by f of some Hermitian vector $\overline{E} := (E, \|.\|)$ over $\mathcal{V}$ as the Hermitian vector bundle over $\mathcal{V}'$:

$$f^*\overline{E} := (E', \|.\|')$$

defined by the vector bundle over $\mathcal{V}'$:

$$E' := f^*E,$$

equipped with the Hermitian metric $\|.\|'$ defined as the pull-back of $\|.\|$ by the map of complex varieties $f_{\mathbb{C}} : \mathcal{V}'_{\mathbb{C}} \to \mathcal{V}_{\mathbb{C}}$.[5]

A basic invariant of Hermitian vector bundles over $\operatorname{Spec}\mathcal{O}_K$ is their *Arakelov degree*. This is a real number, defined as follows. When $\overline{E}$ is Hermitian line bundle – that is a Hermitian vector bundle of rank 1 — it is defined by the expression:

$$\widehat{\deg}\,\overline{E} := \log |E/\mathcal{O}_K s| - \sum_{x\in X(\mathbb{C})} \log \|s\|_x, \tag{0.1.1}$$

valid for any s in $E \setminus \{0\}$. When the rank of $\overline{E}$ is arbitrary, it is defined by the equality:

$$\widehat{\deg}\,\overline{E} := \widehat{\deg}\,\Lambda^{\operatorname{rk} E}\overline{E}.$$

When X is $\operatorname{Spec}\mathbb{Z}$, a Hermitian vector bundle $\overline{E} = (E, \|.\|)$ over X is nothing but a Euclidean lattice, defined by a free $\mathbb{Z}$-module E of finite rank and some Euclidean norm $\|.\|$ on the $\mathbb{R}$-vector space $E_{\mathbb{R}} := E \otimes_{\mathbb{Z}} \mathbb{R}$. Then the Arakelov degree of $\overline{E}$ may be expressed in terms of the covolume $\operatorname{covol}\overline{E}$ of this Euclidean lattice:

$$\widehat{\deg}\,\overline{E} = -\log \operatorname{covol}\overline{E}. \tag{0.1.2}$$

A first instance of arithmetic intersection numbers is provided by the *logarithmic height*:

$$h_{\overline{L}} := V(K) \longrightarrow \mathbb{R}. \tag{0.1.3}$$

associated to a Hermitian line bundle $\overline{L}$ over a projective $\mathcal{O}_K$-scheme $\mathcal{V}$ of generic fiber $V := \mathcal{V}_K$. It is defined by the relation:

$$h_{\overline{L}}(P) := \widehat{\deg}\, P^*\overline{L}, \tag{0.1.4}$$

for every point P in the set $V(K)$ of K-rational points of V, which according to the projectivity of $\mathcal{V}$ may be identified with the set $\mathcal{V}(\mathcal{O}_K)$ of sections of the structural morphism $\mathcal{V} \to \operatorname{Spec}\mathcal{O}_K$.

Indeed $P^*\overline{L}$ is a Hermitian line bundle over $\operatorname{Spec}\mathcal{O}_K$, and we may consider its Arakelov degree, the real number defined by (0.1.1) with $\overline{E} := P^*\overline{L}$. The heights defined by (0.1.4) constitute a

[5] In other words, for any $P' \in \mathcal{V}'(\mathbb{C})$, the Hermitian norm $\|.\|'_{P'}$ on the fiber $E'_{P'}$ of $E'_{\mathbb{C}} \simeq f^*_{\mathbb{C}}(E_{\mathbb{C}})$ coincides with the norm $\|.\|_{f(P')}$ on the fiber $E_{\mathbb{C},f(P')}$ of E_C when one takes into account the canonical isomorphism $E'_{\mathbb{C},P'} \simeq E_{\mathbb{C},f(P')}$.

refined version of the height formalism which plays a central role in classical proofs of Diophantine geometry and transcendence theory.

More generally, as established by Gillet and Soulé ([**GS90a, GS90b**]), to Hermitian vector bundles on a flat projective regular scheme $\mathcal{V}$ over an arithmetic curve $\operatorname{Spec}\mathcal{O}_K$, we may attach their arithmetic characteristic classes in the arithmetic Chow groups $\widehat{\mathrm{CH}}^{\bullet}(\mathcal{V})$, and consider the image of suitable products of these by the "arithmetic degree map":

$$\widehat{\deg}: \widehat{\mathrm{CH}}^{d}(\mathcal{V}) \longrightarrow \mathbb{R} \tag{0.1.5}$$

— where d denotes the Krull dimension of $\mathcal{V}$ – which is a generalization of the Arakelov degree of Hermitian line bundles over arithmetic curves defined by (0.1.1). The real numbers so defined by means of arithmetic intersection theory — the so-called *arithmetic intersection numbers* — may be seen as higher dimensional generalizations of the refined classical heights (0.1.3) defined by (0.1.4) (see for instance [**Fal91**] and [**BGS94**]).

0.1.4. Consider a regular flat projective scheme over an arithmetic curve:

$$\pi : \mathcal{V} \longrightarrow X := \operatorname{Spec}\mathcal{O}_K,$$

and assume that the complex manifold:

$$\mathcal{V}(\mathbb{C}) := \coprod_{x \in X(\mathbb{C})} \mathcal{V}_x(\mathbb{C})$$

is equipped with a C^∞ positive volume form μ, invariant under complex conjugation.

Then, to every Hermitian vector bundle $\overline{E} := (E, \|.\|)$ over $\mathcal{V}$, we may associate its direct image over X:

$$\pi_{\mu *}\overline{E} := \big(\pi_* E, (\|.\|_x)_{x \in X(\mathbb{C})}\big), \tag{0.1.6}$$

namely the Hermitian vector bundle over X defined as follows.

In the right hand side of (0.1.6), $\pi_* E$ denotes the direct image of the sheaf E over $\mathcal{V}$ by the morphism of schemes π, and is equivalently defined by the $\mathcal{O}_K$-module:

$$\pi_* E(X) := \Gamma(\mathcal{V}, E),$$

which is finitely generated and torsion free, hence projective.

Besides, for every field embedding $x \in X(\mathbb{C})$, the finite dimensional vector space:

$$(\pi_* E)_x = \Gamma(\mathcal{V}, E)_x \simeq \Gamma(\mathcal{V}_x, E_x)$$

is endowed with the L^2-norm $\|.\|_x$ defined by the Hermitian norm $\|.\|_{\mathcal{V}_x(\mathbb{C})}$ on the complex vector bundle E_x over $\mathcal{V}_x$ and the volume form $\mu_{|\mathcal{V}_x(\mathbb{C})}$. Namely, for every $s \in \Gamma(\mathcal{V}_x, E_x)$, we let:

$$\|s\|_x^2 := \int_{\mathcal{V}_x(\mathbb{C})} \|s(x)\|^2 \, d\mu(x). \tag{0.1.7}$$

The rank of the direct image $\pi_{\mu *}\overline{E}$ is the dimension $\dim_K \Gamma(\mathcal{V}_K, E_K)$ of the space of sections of E_K over the smooth projective K-scheme $\mathcal{V}_K$. Computing the dimension of spaces of sections of vector bundles over projective varieties is a basic problem in classical algebraic geometry. A solution to this problem — or more properly to a "derived variant" of it — is provided by the Grothendieck-Hirzebruch-Riemann-Roch theorem, which expresses the Euler-Poincaré characteristic:

$$\chi(\mathcal{V}_K, E_K) := \dim_K R^\bullet\Gamma(\mathcal{V}_K, E_K) = \sum_{i \geq 0} (-1)^i \dim_K H^i(\mathcal{V}_K, E_K)$$

in terms of intersections numbers attached to characteristic classes of E_K and $T_{\mathcal{V}_K}$.

Similarly the arithmetic Riemann-Roch theorem[6] provides an expression for a derived variant of the Arakelov degree $\widehat{\deg}\, \pi_{\mu *}\overline{E}$, namely for:

$$\widehat{\deg}\, R^\bullet \pi_{\omega *}\overline{E} = \widehat{\deg}\, \det R^\bullet \pi_{\omega *}\overline{E}, \tag{0.1.8}$$

the Arakelov degree of the determinant of the cohomology of E, $\det R^\bullet \pi_* E$, equipped with the Quillen metrics attached to the Hermitian metrics $\|.\|_{|\mathcal{V}_x(\mathbb{C})}$ and to some Kähler forms ω_x on the complex manifolds $\mathcal{V}_x(\mathbb{C})$ inducing the volume forms $\mu_{|\mathcal{V}_x(\mathbb{C})}$. This expression for the real number $\widehat{\deg}\, R^\bullet \pi_{\omega *}\overline{E}$ involves the arithmetic intersection numbers of arithmetic characteristic classes attached to $\overline{E}$ and to $\overline{T}_{\mathcal{V}/X} := (T_{\mathcal{V}/X}, \omega)$.

In a related vein, a comprehensive theory of arithmetically ample Hermitian line bundles has been developed, notably by Zhang [**Zha92, Zha95**]. It relates the positivity properties of the heights attached to a Hermitian line bundles $\overline{L}$ over a projective regular scheme $\mathcal{V}$ over $\operatorname{Spec} \mathcal{O}_K$ with the existence of elements of small norms in the Hermitian vector bundles $\pi_{\mu *}\overline{L}^{\otimes n}$ attached as above to the large positive powers $\overline{L}^{\otimes n}$ of $\overline{L}$, and admit striking applications to Diophantine geometry.[7]

0.1.5. The developments of Arakelov geometry discussed in the last paragraphs constitute beautiful counterparts of some of the deepest results of classical algebraic geometry, concerning algebraic schemes over some base field. It turns out that most of these results require some assumptions of regularity and projectivity, and are actually counterparts of classical results concerning smooth projective varieties over a field.

These assumptions of projectivity are directly related to the fact that, to define a non trivial arithmetic degree map (0.1.5), the scheme $\mathcal{V}$ has to be proper over $\operatorname{Spec} \mathbb{Z}$. Moreover the compactness of the complex manifold $\mathcal{V}(\mathbb{C})$ is crucial in the analytic construction behind the definition of the Arakelov degree (0.1.8) of the determinant of the cohomology: the definition of the Quillen metric uses the fact that the Laplacians of the $\overline{\partial}$-operators associated to the complex analytic vector bundle $E^{\mathrm{an}}_{\mathbb{C}}$ over $\mathcal{V}(\mathbb{C})$ have well-behaved zeta functions, and their construction in terms of the associated heat kernels relies on the compactness of $\mathcal{V}(\mathbb{C})$.

In the present state of technology, these analytic constructions make sense on complex analytic manifolds, and not on singular analytic spaces, and consequently require the smoothness of the complex scheme:

$$\mathcal{V}_{\mathbb{C}} := \coprod_{x \in X(\mathbb{C})} \mathcal{V}_x,$$

or equivalently of the generic fiber $\mathcal{V}_K$ of $\mathcal{V}$.

0.2. Hermitian Quasi-coherent Sheaves over Arithmetic Curves and the Theta Invariants h^0_θ and h^1_θ

0.2.1. It is rather intriguing that, while deep results of classical algebraic geometry — concerning algebraic schemes over some base field k — have been transferred to the arithmetic framework of Arakelov geometry, often by relying on sophisticated results of analysis on complex compact manifolds,[8] many basic constructions of algebraic geometry have no analogue in this framework yet, due to these requirements of regularity and projectivity.

[6] Established in [**GS92**] by Gillet and Soulé, who relied on the analytic work of Bismut and Lebeau [**BL91**]; see also [**Fal92**].

[7] See for instance [**Zha98**] for a survey and references concerning arithmetic ampleness and its applications. Since then, the theory has been expanded both to more general settings and weaker notions of positivity. See for instance [**YZ21**] for a general exposé in the adelic setting, with applications to general equidistribution results and the Mordell-Lang conjecture as in [**Küh21, DGH21**].

[8] See [**Bis98**] for a survey of advances concerning analytic torsion, motivated to a large extent by the proof of the arithmetic Riemann-Roch theorem and its generalizations.

Consider for instance the following problems, which will provide a rationale for the constructions we pursue in this monograph: what should play the role, in Arakelov geometry, of affine schemes of finite type over the base field k, or of vanishing criteria for higher cohomology groups over some possibly non-regular or non-projective algebraic k-schemes ?

The geometric model for these "arithmetic affine schemes" would be affine schemes V of finite type over k that are fibered over the smooth projective curve C, namely that are endowed with a morphism of k-schemes:

$$f : V \longrightarrow C.$$

Then f is clearly an affine morphism of finite type. Conversely, according to Serre's affineness criterion, for any affine morphism of finite type f as above, the schemes V is an affine k-scheme (necessarily of finite type) if and only if, for every coherent sheaf $\mathcal{C}$ over V, the following cohomology group vanishes:

$$H^1(V, \mathcal{C}) \simeq H^1(C, f_*\mathcal{C}),$$

or equivalently, if this holds for every coherent ideal subsheaf $\mathcal{C}$ of $\mathcal{O}_V$.

Observe that, in the framework of 0.1.1 above, where the curve C is endowed with a non-empty finite set Σ of closed points (which play the role of Archimedean places, and whose complement $\overset{\circ}{C} := C \setminus \Sigma$ plays the role of $\operatorname{Spec} \mathcal{O}_K$), an affine morphism $f : V \to C$ determines the following data:

(i) a morphism between affine k-schemes:

$$\overset{\circ}{f} := f_{|\overset{\circ}{V}} : \overset{\circ}{V} := f^{-1}(\overset{\circ}{C}) \longrightarrow \overset{\circ}{C};$$

(ii) for every $x \in \Sigma$, the base change:

$$f_x := f_{\widehat{\mathcal{O}}_{C,x}} : \widehat{V}_x := V_{\widehat{\mathcal{O}}_{C,x}} \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{C,x}$$

of f to the completion $\widehat{\mathcal{O}}_{C,x}$ of the discrete valuation ring $\mathcal{O}_{C,x}$.

Moreover these data satisfy the following compatibility condition:

(iii) if we denote by $k(C)_x$ the fraction field of $\widehat{\mathcal{O}}_{C,x}$ (or equivalently, the x-adic completion of $k(C)$), then the $k(C)_x$-schemes $\overset{\circ}{V} \otimes_{\mathcal{O}(\overset{\circ}{C})} k(C)_x$ and $\widehat{V}_x \otimes_{\widehat{\mathcal{O}}_{C,x}} k(C)_x$ may be identified and the morphisms $\overset{\circ}{f}_{k(C)_x}$ and $f_{x,k(C)_x}$ coincide.[9]

Descent theory shows that the data of an affine morphism of finite type $f : V \to C$ is actually equivalent to the data of an affine morphism of finite type $\overset{\circ}{f} : \overset{\circ}{V} \to \overset{\circ}{C}$ and, for every $x \in \Sigma$, of an affine morphism of finite type $f_x : \widehat{V}_x \to \operatorname{Spec} \widehat{\mathcal{O}}_{C,x}$, together with some "glueing data" as in (iii) above; see for instance [**BLR90**, Section 6.2, Example D].

Moreover the coherent sheaves $\mathcal{C}$ over V and their direct images $f_*\mathcal{C}$ over C admit a similar description in terms of coherent sheaves over the schemes $\overset{\circ}{V}$ and $\widehat{V}_x$ and of their direct images over $\overset{\circ}{C}$ and $\operatorname{Spec} \widehat{\mathcal{O}}_{C,x}$, for x in Σ.

0.2.2. When looking for arithmetic counterparts of affine varieties, in the definition of which Archimedean places would be taken into account, one is led by the previous discussion to introduce the following variations on the construction in 0.1.4 above.[10]

[9]These morphisms are the morphisms from $\overset{\circ}{V} \otimes_{\mathcal{O}(\overset{\circ}{C})} k(C)_x \simeq \widehat{V}_x \otimes_{\widehat{\mathcal{O}}_{C,x}} k(C)_x$ to $\operatorname{Spec} k(C)_x$ deduced from $\overset{\circ}{f}$ and f_x by base change to $k(C)_x$.

[10]The constructions in this subsection will be developed in full generality in [**BCa**]. The simplified framework of the next paragraphs should already provide a fair idea of the Diophantine setting where the formalism developed in this monograph applies.

Consider an affine scheme:

$$\pi : \mathcal{V} \longrightarrow X := \operatorname{Spec} \mathcal{O}_K$$

flat of finite type over $X := \operatorname{Spec} \mathcal{O}_K$, that for simplicity we will assume to be integral, and a Hermitian vector bundle $\overline{E} := (E, \|.\|)$ over $\mathcal{V}$, defined by a vector bundle E over $\mathcal{V}$ and by some continuous Hermitian metric $\|.\|$, invariant under complex conjugation, on the complex analytic vector bundle $E^{\mathrm{an}}_{\mathbb{C}}$ on the reduced analytic space:

$$\mathcal{V}^{\mathrm{an}}_{\mathbb{C}} = \mathcal{V}(\mathbb{C}) = \coprod_{x \in X(\mathbb{C})} \mathcal{V}_x(\mathbb{C}),$$

defined by the smooth complex scheme $\mathcal{V}_{\mathbb{C}} := \mathcal{V} \otimes_{\mathbb{C}} \mathbb{Z}$.

Assume furthermore that we are given a family $(K_x)_{x \in X(\mathbb{C})}$, invariant under complex conjugation, of holomorphically convex compact subsets of the Stein spaces $\mathcal{V}_x(\mathbb{C})$, and a family $(\mu_x)_{x \in X(\mathbb{C})}$, invariant under complex conjugations, of positive Radon measures on the compact sets the K_x's, the supports of which contain the Shilov boundaries of the K_x's.

The morphism $\pi : \mathcal{V} \to \operatorname{Spec} \mathcal{O}_K$ plays the role of the morphism $\mathring{f} : \mathring{V} \to \mathring{C}$ in 0.2.1 above, and the pairs $(\mathcal{V}_x, K_x)$ may be seen as complex avatars of the pairs defined by the $k(C)_x$-scheme

$$V_{k(C)_x} := \mathring{V} \otimes_{\mathcal{O}(\mathring{C})} k(C)_x \simeq \widehat{V}_x \otimes_{\widehat{\mathcal{O}}_{C,x}} k(C)_x$$

and by the affinoid in the associated rigid analytic space over the complete valued field $k(C)_x$ which is attached to the model $\widehat{V}_x$ of $V_{k(C)_x}$ over $\widehat{\mathcal{O}}_{C,x}$.

Then we may consider the pair:

$$\pi_{\mu *}\overline{E} := \big(\pi_* E, (\|.\|_x)_{x \in X(\mathbb{C})}\big)$$

defined by a construction similar as the one in 0.1.4 above. Namely, $\pi_* E$ is the quasi-coherent sheaf over $\operatorname{Spec} \mathcal{O}_K$ defined as the "classical" direct image by the morphism of schemes π. It is defined by the $\mathcal{O}_K$-module $\Gamma(\mathcal{V}, E)$, which, in general, is *not* a finitely generated $\mathcal{O}_K$-module.[11] However it is easily seen to be countably generated. Moreover, for every $x \in X(\mathbb{C})$, $\|.\|_x$ is the L^2- Hermitian seminorm on the complex vector space:

$$\Gamma(\mathcal{V}, E)_x \simeq \Gamma(\mathcal{V}_x, E_x)$$

defined by (0.1.7).

The "Arakelovian direct image" $\pi_{\mu *}\overline{E}$ plays the role of the direct image $f_*\mathcal{C}$ considered in the geometric situation of 0.2.1 above, in the special case when the coherent sheaf $\mathcal{C}$ is locally free.

This leads us to consider *Hermitian quasi-coherent sheaves* over the arithmetic curve X — which we define as pairs:

$$\overline{F} := \big(F, (\|.\|_x)_{x \in X(\mathbb{C})}\big),$$

where F a countably generated $\mathcal{O}_K$-module, and where $(\|.\|_x)_{x \in X(\mathbb{C})}$ is a family, invariant under complex conjugation, of Hermitian seminorms on the $\mathbb{C}$-vector spaces $F_x := F \otimes_{\mathcal{O}_K, x} \mathbb{C}$ — and to look for invariants attached to these pairs that would play the role of the invariant:

$$h^1(C, \mathcal{F}) := \dim_k H^1(C, \mathcal{F})$$

attached to a quasi-coherent sheaf $\mathcal{F}$ over C.

When the arithmetic curve X is $\operatorname{Spec} \mathbb{Z}$, we will talk of *Euclidean quasi-coherent sheaves* instead of Hermitian quasi-coherent sheaves. A Euclidean quasi-coherent sheaves is a pair $\overline{F} := (F, \|.\|)$, where F is a countably generated $\mathbb{Z}$-module and $\|.\|$ a Euclidean seminorm on the $\mathbb{R}$-vector space $E_{\mathbb{R}} \simeq E \otimes_{\mathbb{Z}} \mathbb{R}$.

[11] When $E \neq 0$, the $\mathcal{O}_K$-module $\Gamma(\mathcal{V}, E)$ is finitely generated if and only if $\pi : \mathcal{V} \to \operatorname{Spec} \mathcal{O}_K$ is a finite morphism.

0.2.3. Constructions similar to the ones in 0.2.2 above also arise when dealing with "classical questions" of Arakelov geometry, for instance in relation with arithmetically ample Hermitian line bundles on flat projective schemes over $\operatorname{Spec}\mathbb{Z}$.

0.2.3.1. Recall that Grauert has developed an approach to ampleness of line bundles over compact complex analytic spaces based on the geometry of the associated "tubes," namely of the germs of analytic spaces defined by the total spaces of the dual line bundles and their zero sections [**Gra62**].

This approach has been transposed in the framework of schemes in EGA-II [**Gro61**, Sections 8.8-10], where the following "Grauert's ampleness criterion" is established: if $p : X \to Y$ is a proper morphism of schemes, a line bundle $\mathcal{L}$ over X is ample relatively to p if and only if the zero section of the "vector bundle" $\mathbb{V}(\mathcal{L}) \to X$ may be contracted into the base scheme Y.[12]

0.2.3.2. When investigating the counterpart of these approaches in Arakelov geometry, one is led to to consider a flat projective (say regular) scheme $\mathcal{X}$ over an arithmetic curve $X := \operatorname{Spec}\mathcal{O}_K$, endowed with a Hermitian line bundle $\overline{L} := (L, \|.\|)$, and to investigate the Arakelov geometry of the "vector bundle":

$$p : \mathcal{V} := \mathbb{V}(L) \longrightarrow \mathcal{X}.$$

For $x \in X(\mathbb{C})$, the complex manifold $\mathcal{V}_x(\mathbb{C})$ is the total space of the $\mathbb{C}$-analytic line bundle $L_x^{\vee\mathrm{an}}$ over the projective smooth complex variety $\mathcal{X}_x(\mathbb{C})$. This line bundle is equipped with the Hermitian metric $\|.\|_x^\vee$ dual of $\|.\|_x$, and we may consider the associated unit disk bundle $K_x := D_x$ — which is a compact subset of $\mathcal{V}_x(\mathbb{C})$ — and its boundary ∂D_x,, the unit circle bundle, which is a principal $U(1)$-bundle over $\mathcal{X}(\mathbb{C})$.

We may finally consider a family $(\mu_x)_{x\in X(\mathbb{C})}$, invariant under conjugation, of positive Radon measures of supports the unit circle bundles $(\partial D_x)_{x\in X(\mathbb{C})}$, and perform the construction in 0.2.2, associated to the structural morphism:

$$\pi_{\mathcal{V}} : \mathcal{V} := \mathbb{V}(\mathcal{L}) \longrightarrow X := \operatorname{Spec}\mathcal{O}_K$$

— which is *not* proper when $\mathcal{X}$ is non-empty — to the family of measures $(\mu_x)_{x\in X(\mathbb{C})}$, and to the trivial Hermitian line bundle $\overline{\mathcal{O}}_{\mathcal{V}} := (\mathcal{O}_{\mathcal{V}}, |.|)$ over $\mathcal{V}$. In this way, we define a Hermitian quasi-coherent sheaf over $\operatorname{Spec}\mathcal{O}_K$:

$$\pi_{\mathcal{V},\mu*}\overline{\mathcal{O}}_{\mathcal{V}} := \big(\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}}), (\|.\|_x)_{x\in X(\mathbb{C})}\big).$$

0.2.3.3. By the very definition of $\mathcal{V}$ as $\operatorname{Spec}_{\mathcal{X}} \bigoplus_{n\in\mathbb{N}} L^{\otimes n}$, we have a canonical isomorphism of $\mathcal{O}_K$-modules:

$$\Gamma(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) \xrightarrow{\sim} \bigoplus_{n\in\mathbb{N}} \Gamma(\mathcal{X}, L^{\otimes n}). \tag{0.2.1}$$

It is natural to assume that each measure μ_x is invariant under the action of $U(1)$ over ∂D_x, and that its direct image:

$$\nu_x := p_{x*}\mu_x$$

under the map:

$$p_x : \mathcal{V}_x(\mathbb{C}) \longrightarrow \mathcal{X}_x(\mathbb{C})$$

is a C^∞ positive volume form on $\mathcal{X}_x(\mathbb{C})$. When this holds, the isomorphism (0.2.1) defines an isomorphism of Hermitian quasi-coherent sheaves over X:

$$\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}} \xrightarrow{\sim} \bigoplus_{n\in\mathbb{N}} \pi_{\mathcal{X},\nu*}\overline{L}^{\otimes n}, \tag{0.2.2}$$

where the right-hand side of (0.2.2) is the orthogonal direct sum[13] of the Hermitian vector bundles $\pi_{\mathcal{X},\nu*}\overline{L}^{\otimes n}$, which are defined by the "classical" construction recalled in 0.1.4 above, applied to the

[12]Recall that $\mathbb{V}(\mathcal{L}) := \operatorname{Spec}_X \bigoplus_{n\in\mathbb{N}} \mathcal{L}^{\otimes n}$ is, in naive terms, the total space of the dual line bundle $\mathcal{L}^\vee$.

[13]The isomorphism (0.2.2) shows that the Hermitian quasi-coherent sheaf $\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}}$ is actually a ind-Hermitian vector bundles, as defined in [**Bos20b**]; see 0.2.4 below.

structural morphism:

$$\pi_{\mathcal{X}} : \mathcal{X} \longrightarrow X := \operatorname{Spec} \mathcal{O}_K.$$

0.2.3.4. According to the classical versions of Grauert's ampleness criterion mentioned in 0.2.3.1 above, the relative ampleness of the Hermitian line bundle $\overline{L} := (L, \|.\|)$ with respect to the morphism $\pi_{\mathcal{X}}$ — namely, the ampleness of the line bundle L over $\mathcal{X}$, and the positivity of the Chern form $c_1(L^{\mathrm{an}}_{\mathbb{C}}, \|.\|)$ over $\mathcal{X}(\mathbb{C}) = \coprod_{x \in X(\mathbb{C})} \mathcal{X}_x(\mathbb{C})$ — is equivalent to the contractibility into $\operatorname{Spec} \mathcal{O}_K$ of the zero section of the morphism $p : \mathcal{V} := \mathcal{V}(L) \to \mathcal{X}$, and to the strict pseudoconvexity of the compact submanifold with boundary $K_x := D_x$ of $\mathcal{V}_x(\mathbb{C})$.

When this relative ampleness holds, it is natural to expect — on the grounds of the analogy with varieties fibered over a smooth projective curve C — that the Hermitian line bundle $\overline{L}$ on $\mathcal{X}$ is arithmetically ample if and only if a suitable analogue of the invariant $h^1(C, \mathcal{F})$ is finite when evaluated on the Hermitian quasi-coherent sheaf $\pi_{\mu *}\overline{\mathcal{O}}_{\mathcal{V}}$ over $\operatorname{Spec} \mathcal{O}_K$.

0.2.3.5. The simplest non-trivial instance of the previous construction arises when:

$$\mathcal{X} = X = \operatorname{Spec} \mathbb{Z}.$$

Then the Hermitian line bundle $\overline{L}$ over $\mathcal{X}$ is isomorphic the Hermitian line bundle:

$$\overline{\mathcal{O}}(\lambda) := (\mathbb{Z}, e^{-\lambda}|.|),$$

where the real number λ satisfies:

$$\lambda = \widehat{\deg}\, \overline{L}.$$

In this case, $\mathcal{V}$ is $\operatorname{Spec} \mathbb{Z}[X] =: \mathbb{A}^1_{\mathbb{Z}}$, the circle bundle ∂D_x is the circle:

$$C(e^{-\lambda}) := \left\{ z \in \mathbb{C} \mid |z| = e^{-\lambda} \right\} \subset \mathbb{C} = \mathbb{A}^1_{\mathbb{Z}}(\mathbb{C}),$$

and in the situation of 0.2.3.3, the measure μ is of the form $\alpha\, d\theta$, for some α in $\mathbb{R}^*_+$, where $d\theta$ denotes the rotation invariant measure of total mass 2π on $C(e^{-\lambda})$. Then the direct image $\pi_{\mu *}\overline{\mathcal{O}}_{\mathcal{V}}$ is easily seen to be the Euclidean quasi-coherent sheaf:

$$\pi_{\mu *}\overline{\mathcal{O}}_{\mathcal{V}} = (\mathbb{Z}[X], \|.\|_\lambda),$$

where, for every polynomial $\sum_n a_n X^n$ in $\mathbb{C}[X]$, the norm $\|.\|_\lambda$ is defined by:

$$\Big\| \sum_n a_n X^n \Big\|^2_\lambda := 2\pi\alpha \sum_n e^{-2\lambda n} |a_n|^2.$$

In other words, $\pi_{\mu *}\overline{\mathcal{O}}_{\mathcal{V}}$ is isomorphic to the orthogonal direct sum:

$$\bigoplus_{n \in \mathbb{N}} \overline{\mathcal{O}}\big(n\lambda - (1/2)\log(2\pi\alpha)\big).$$

0.2.4. In the monograph [**Bos20b**], invariants that play the role of the dimension $h^1(C, \mathcal{F})$ attached to some quasi-coherent sheaf $\mathcal{F}$ have been constructed for a special class of Hermitian quasi-coherent sheaves, the so-called ind-Hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$, namely the pairs $\big(F, (\|.\|_x)_{x \in X(\mathbb{C})}\big)$ as above where the $\mathcal{O}_K$-module F is projective and the Hermitian seminorms $(\|.\|_x)_{x \in X(\mathbb{C})}$ are norms.

0.2.4.1. The starting point of the constructions in [**Bos20b**] is the fact that the invariants $h^0(C, \mathcal{F})$ and $h^1(C, \mathcal{F})$ attached to a *vector bundle* $\mathcal{F}$ over the projective curve C admit as arithmetic analogues the theta invariants $h^0_\theta(\overline{E})$ and $h^1_\theta(\overline{E})$ attached to a Hermitian vector bundle $\overline{E} := \big(E, (\|.\|_x)_{x \in X(\mathbb{C})}\big)$ over the arithmetic curve $X = \operatorname{Spec} \mathcal{O}_K$.

When X is $\operatorname{Spec} \mathbb{Z}$, and therefore $\overline{E} = (E, \|.\|)$ is a Euclidean lattice, these invariants are the non-negative real numbers defined as follows in terms of theta series associated to $\overline{E}$ and to the dual[14] Euclidean lattice $\overline{E}^\vee := (E^\vee, \|.\|^\vee)$:

$$h^0_\theta(\overline{E}) := \log \sum_{v\in E} e^{-\pi\|v\|^2}, \tag{0.2.3}$$

and:

$$h^1_\theta(\overline{E}) := h^0_\theta(\overline{E}^\vee) = \log \sum_{\xi\in E^\vee} e^{-\pi\|\xi\|^{\vee 2}}. \tag{0.2.4}$$

When X is an arbitrary arithmetic curve, we may consider the finite flat morphism:

$$\pi : X = \operatorname{Spec} \mathcal{O}_K \longrightarrow \operatorname{Spec} \mathbb{Z}, \tag{0.2.5}$$

and the direct image $\pi_*\overline{E}$ of $\overline{E}$ over $\operatorname{Spec}\mathbb{Z}$. It is the Euclidean lattice[15] $(\pi_*E, \|.\|)$ defined by the $\mathbb{Z}$-module π_*E — namely E seen as a $\mathbb{Z}$-module — equipped with the Euclidean norm $\|.\|$ defined by the relation:

$$\|v\|^2 := \sum_{x\in X(\mathbb{C})} \|v\|^2_x \quad \text{for every } v \in E.$$

Then the theta invariants of $\overline{E}$ are defined by “reduction to $\operatorname{Spec}\mathbb{Z}$”:

$$h^i_\theta(\overline{E}) := h^i_\theta(\pi_*\overline{E}) \quad \text{for } i = 0, 1, \tag{0.2.6}$$

and the classical Poisson formula for theta series implies the following equality:

$$h^0_\theta(\overline{E}) - h^1_\theta(\overline{E}) = \widehat{\deg}\,\overline{E} - (\mathrm{rk} E/2).\log|\Delta_K|, \tag{0.2.7}$$

where Δ_K denotes the discriminant of the number field K. The relation (0.2.7) is formally similar to the Riemann-Roch formula for vector bundles over curves.

Recall that, when $\overline{E}$ is a Hermitian line bundle over $\operatorname{Spec}\mathcal{O}_K$, the “Poisson-Riemann-Roch formula” (0.2.7) is a key point in Hecke’s proof [**Hec17**] of the analytic continuation and of the functional equation of the Dedekind zeta function attached to an arbitrary number field.

Moreover, a few years after Hecke’s work, F. K. Schmidt [**Sch31**] established the analogous properties for the zeta functions attached to function fields in one variable over a finite field. To achieve this, Schmidt had to establish the Riemann-Roch formula in this setting. Moreover, in his study of these zeta functions, the dimensions $h^0(C, L)$ and $h^1(C, L) = h^0(C, L^\vee \otimes \omega_C)$ of the cohomology groups of a line bundle L over C are the analogues of the non-negative real numbers $h^0_\theta(\overline{L})$ and $h^1_\theta(\overline{L})$ attached to a Hermitian line bundle $\overline{L}$ over $\operatorname{Spec}\mathcal{O}_K$ — which appear in substance in Hecke’s proof — and the Riemann-Roch formula over C play the role of the Poisson-Riemann-Roch formula (0.2.7) for Hermitian line bundles.

Although never completely forgotten, the analogy between the dimensions of the cohomology groups of vector bundles over a projective curve and the theta invariants of Hermitian vector bundles over an arithmetic curve receded into the background during the second half of the twentieth century. However it recently experienced a revival thanks to the work of van der Geer and Schoof [**vdGS00**], which notably emphasizes its relations with Arakelov geometry.[16]

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[14] Recall that $\overline{E}^\vee$ is defined by the dual $\mathbb{Z}$-module $E^\vee := \operatorname{Hom}_\mathbb{Z}(E, \mathbb{Z})$ and by the Euclidean norm $\|.\|^\vee$ on $E^\vee \otimes \mathbb{R} \simeq \operatorname{Hom}_\mathbb{R}(E_\mathbb{R}, \mathbb{R})$ dual of the Euclidean norm $\|.\|$

[15] This construction is actually the special case of the construction of $\pi_{\mu *}$ in 0.1.4 applied to the morphism (0.2.5) and to the counting measure μ on $X(\mathbb{C})$.

[16] See [**Bos20b**, 0.1.2] for a more complete discussion and references.

0.2.4.2. Besides the Poisson-Riemann-Roch formula, the theta invariants h^0_θ and h^1_θ satisfy further properties similar to the ones of the dimension of cohomology groups $h^0(C,E)$ and $h^1(C,E)$ attached to a vector bundle E over a projective curve C.

Consider for instance two Hermitian vector bundles over $X = \operatorname{Spec} \mathcal{O}_K$:

$$\overline{E} := \big(E, (\|.\|_x)_{x\in X(\mathbb{C})}\big) \quad \text{and} \quad \overline{E}' := \big(E', (\|.\|'_x)_{x\in X(\mathbb{C})}\big),$$

and a morphism:

$$f : \overline{E} \longrightarrow \overline{E}',$$

namely an $\mathcal{O}_K$-linear map $f : E \to E'$ such that, for any $x \in X(\mathbb{C})$, the $\mathbb{C}$-linear map:

$$f_x : E_x \longrightarrow E'_x$$

is norm decreasing:

$$\|f_x(v)\|'_x \leq \|v\|_x \quad \text{for every } v \in E_x.$$

Then, if f is injective (resp. if $f_K : E_K \to E'_K$ is surjective), the following inequality holds:

$$h^0_\theta(\overline{E}) \leq h^0_\theta(\overline{E}') \quad (\text{resp. } h^1_\theta(\overline{E}') \leq h^1_\theta(\overline{E})). \tag{0.2.8}$$

Moreover, for every admissible short exact sequence of Hermitian vector bundles over X:

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0,$$

the following estimates holds, for $i = 0, 1$:

$$h^i_\theta(\overline{F}) \leq h^i_\theta(\overline{E}) + h^i_\theta(\overline{G}). \tag{0.2.9}$$

Let us emphasize a simple but crucial feature of the theta invariants h^0_θ and h^1_θ: contrary to most classical estimates in geometry of numbers, which relates elementary invariants of Euclidean lattices like their successive minima or their covering radius, the dimension of the Hermitian vector bundles $\overline{E}$, $\overline{E}'$, $\overline{F}$, or $\overline{G}$ do not appear in the estimates (0.2.8) and (0.2.9). This already hints at the possibility to define theta invariants in some infinite dimensional setting.

Indeed these estimates play a central role in the construction in [**Bos20b**], by means of limit procedures, of extensions of the invariants h^0_θ and h^1_θ attached to certain infinite dimensional analogues of Hermitian vector bundles over X, the ind-Hermitian vector bundles and their "duals", the pro-Hermitian vector bundles.

The construction of the invariant $h^0_\theta(\overline{E})$ attached to some ind-Hermitian vector bundle $\overline{E}$ is straightforward. Actually $h^0_\theta(\overline{E})$ may also be directly defined by formulas (0.2.3) and (0.2.6), which still make sense in the infinite rank setting.

The construction of the invariant $h^1_\theta(\overline{E})$ is more elaborate, and reminiscent of the construction of a measure starting from a suitable additive set function, in abstract measure theory. Indeed, the use of limit procedures leads to introduce *two* natural possible extensions $\underline{h}^1_\theta(\overline{E})$ and $\overline{h}^1_\theta(\overline{E})$ for the theta invariant h^1_θ, defined for a general ind-Hermitian $\overline{E}$ over $\operatorname{Spec} \mathcal{O}_K$. These extensions satisfy the estimates:

$$0 \leq \underline{h}^1_\theta(\overline{E}) \leq \overline{h}^1_\theta(\overline{E}) \leq +\infty,$$

but $\underline{h}^1_\theta(\overline{E})$ and $\overline{h}^1_\theta(\overline{E})$ may differ in general. A central result in [**Bos20b**] is a flexible criterion on a ind-Hermitian vector bundle $\overline{E}$ for these invariant $\underline{h}^1_\theta(\overline{E})$ and $\overline{h}^1_\theta(\overline{E})$ to be finite and coincide.[17]

Let us finally stress that the Arakelov degree of Hermitian vector bundles over arithmetic curves, which plays a central role in the diverse developments of Arakelov geometry recalled in Section 0.1,

[17]The presentation in [**Bos20b**] focuses on the extensions of the invariant h^0_θ to pro-Hermitian vector bundles. The construction of extensions of h^1_θ to ind-Hermitian vector bundles is a straightforward reformulation of these constructions using duality. The emphasis in [**Bos20b**] on the invariants h^0_θ attached to pro-Hermitian vector bundles is justified by their applications in Diophantine situations which involve both formal geometry over arithmetic curves and complex analytic geometry; see for instance [**Bos20b**, Chapter9] and [**BC22**, Chapter 8].

makes no sense for their infinite dimensional generalizations[18]. The introduction of invariants of a different nature appears necessary for the investigation of the latter.

0.2.4.3. The theta invariants of a ind-Hermitian vector bundle over $\operatorname{Spec}\mathbb{Z}$ of the form:

$$\bigoplus_{n\in\mathbb{N}} \overline{\mathcal{O}}(\lambda_n),$$

where $(\lambda_n)_{n\in\mathbb{N}}$ denotes an arbitrary sequence of real numbers are easily determined, and provide a simple but suggestive illustration of the general formalism.

Namely we have:

$$\underline{h}^1_\theta\big(\bigoplus_{n\in\mathbb{N}} \overline{\mathcal{O}}(\lambda_n)\big) = \overline{h}^1_\theta\big(\bigoplus_{n\in\mathbb{N}} \overline{\mathcal{O}}(\lambda_n)\big) = \sum_{n\in\mathbb{N}} h^1_\theta\big(\overline{\mathcal{O}}(\lambda_n)\big) \quad (\in [0,+\infty]),$$

where, for every $\lambda \in \mathbb{R}$:

$$h^1_\theta(\overline{\mathcal{O}}(\lambda)) = \log \sum_{k\in\mathbb{Z}} \exp(-\pi k^2 e^{2\lambda});$$

see for instance [**Bos20b**, Example 6.4.1].

Actually, as observed in [**Bos20b**, Section 2.4], we have:

$$h^1_\theta(\overline{\mathcal{O}}(\lambda)) = \lambda^- + \eta(\lambda),$$

where $\lambda^- := \max(0, -\lambda)$, and:

$$0 < \eta(\lambda) = 2\exp(-\pi e^{2|\lambda|}) + O\big(\exp(-2\pi e^{2|\lambda|})\big) \quad \text{when } |\lambda| \longrightarrow +\infty.$$

Consequently, the following equivalence holds:

$$h^1_\theta\big(\bigoplus_{n\in\mathbb{N}} \overline{\mathcal{O}}(\lambda_n)\big) := \sum_{n\in\mathbb{N}} h^1_\theta\big(\overline{\mathcal{O}}(\lambda_n)\big) < +\infty \Longleftrightarrow \sum_{n\in\mathbb{N}} \exp(-\pi e^{2\lambda_n}) < +\infty.$$

This applies to the Euclidean quasi-coherent sheaf $\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}}$ associated to a Hermitian line bundle $\overline{L}$ over $\mathcal{X} = X = \operatorname{Spec}\mathbb{Z}$ in 0.2.3.5 above, and establishes the following implications:

$$\lambda := \widehat{\deg}\,\overline{L} > 0 \Longrightarrow \underline{h}^1_\theta(\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}}) = \overline{h}^1_\theta(\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}}) < +\infty$$

and:

$$\lambda := \widehat{\deg}\,\overline{L} \leq 0 \Longrightarrow \underline{h}^1_\theta(\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}}) = \overline{h}^1_\theta(\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}}) = +\infty.$$

This shows that, as hinted in 0.2.3.5, the Hermitian line bundle $\overline{L}$ is arithmetically ample if and only if the theta invariant:

$$h^1_\theta(\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}}) := \underline{h}^1_\theta(\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}}) = \overline{h}^1_\theta(\pi_{\mu*}\overline{\mathcal{O}}_{\mathcal{V}})$$

is finite.

0.2.5. This monograph pursues the development of an "infinite dimensional geometry of numbers" in which theta invariants play a central role, initiated in [**Bos20b**]. It includes a number of new features, which we discuss in the next paragraphs.

[18] With the exception of a few very special situations.

0.2.5.1. We are interested in realizing in full generality the program concerning "arithmetic affine schemes" sketched in 0.2.1 above. The construction of Hermitian quasi-coherent sheaves over $\operatorname{Spec} \mathcal{O}_K$ by direct images in the relatively affine situation considered in 0.2.2 produces Hermitian quasi-coherent sheaves $\overline{F} = \big(F, (\|.\|_x)_{x \in X(\mathbb{C})}\big)$ where the $\mathcal{O}_K$-module F is definitely not projective, and some of the seminorms $\|.\|_x$ are not norms. Consequently, to achieve this program, it is crucial to extend the construction and the properties of the theta invariant h^1_θ to Hermitian quasi-coherent sheaves more general than ind-Hermitian vector bundles.

Working with general countably generated $\mathcal{O}_K$-modules F — and not only projective ones — and with general Hermitian seminorms $\|.\|_x$ introduces a number of technical difficulties that did not occur in [**Bos20b**]. Ultimately we are able to develop a satisfactory formalism of h^1_θ invariants in this general setting which, compared to the ones in [**Bos20b**], satisfies some simple additional properties — the "compatibility with canonical dévissage" and the "downward continuity" as a function of the defining Hermitian seminorms.[19]

0.2.5.2. The theta series in the right hand sides of (0.2.3) and (0.2.4) defining the theta invariants $h^0_\theta(\overline{E})$ and $h^1_\theta(\overline{E})$ of some Euclidean lattice $\overline{E}$ also occur in the seminal work of Banaszczyk [**Ban93**], who used them to establish remarkable new estimates concerning classical invariants of geometry of numbers — independently of their role in the analogy between function fields and number fields.

More precisely, to investigate various invariants attached to a Euclidean lattice $\overline{E} := (E, \|.\|)$, Banaszczyk introduces the following discrete measure on the real vector space $E_\mathbb{R}$:

$$\gamma_{\overline{E}} := \sum_{v \in E} e^{-\pi \|v\|^2}\, \delta_v$$

— whose total mass is $\exp(h^0_\theta(\overline{E}))$ — and its Fourier transform, which is a $E^\vee$-periodic function on the dual space $E^\vee_\mathbb{R}$, and may also be expressed as a suitable theta series according to the Poisson formula.

During the last decades, the techniques introduced by Banaszczyk in in [**Ban93**] to establish optimal transference estimates have been used to investigate various properties of Euclidean lattices and their invariants which play a central role in "lattice based cryptography," the domain of computer science devoted to the construction of cryptosystems based on Euclidean lattices.

In particular the probability measure on $E_\mathbb{R}$:

$$\beta_{\overline{E}} := e^{-h^0_\theta(\overline{E})} \sum_{v \in E} e^{-\pi \|v\|^2}\, \delta_v$$

and its Fourier transform $B_{\overline{E}^\vee}$ on $E^\vee_\mathbb{R}$ — which we will call the *Banaszczyk measure* and the *Banaszczyk function* attached to $\overline{E}$ and to $\overline{E}^\vee$ — have been shown to satisfy remarkable estimates, with striking consequences for the theta invariants of Euclidean lattices, by Banaszczyk [**Ban92, Ban22**] and by experts of lattice based cryptography, notably by Dadush, Regev, and Stephens-Davidowitz in [**DR16, DRSD14, RSD17b**].

In this monograph, we establish some of these estimates in a suitably generalized version, and we use them to derive some of the most delicate properties of the invariant h^1_θ and its extensions to Hermitian quasi-coherent sheaves.[20]

[19] These properties are introduced in Chapter 4 as the axioms **NSAp** and **Cont**$^+$.

[20] In the monograph [**Bos20b**], some aspects of Banaszczyk seminal paper [**Ban93**] are discussed, and the techniques of [**Ban93**] are used to compare the invariant $h^0_\theta(\overline{E})$ attached to a Euclidean lattice $\overline{E}$ and its "naive" variant $h^0_{\mathrm{Ar}}(\overline{E})$, defined as $\log |\{v \in E \mid \|v\| \leq 1\}|$. The measures $\gamma_{\overline{E}}$ and $\beta_{\overline{E}}$ also play a key role in the construction in [**Bos20b**, Chapter 7] of the invariants h^0_θ associated to pro-Hermitian vector bundles, and dually of the invariant h^1_θ attached to ind-Hermitian vector bundles. However the more delicate results in [**Ban22**] and [**RSD17a**] do not enter in this construction.

0.2.5.3. Theta invariants, in the infinite dimensional setting of Hermitian quasi-coherent sheaves, constitute our main objet of study in this monograph. However we also study invariants attached to Hermitian quasi-coherent sheaves which admit a more elementary definition and have an obvious geometric meaning. Among them, a special role will be played by the *covering radius* of Euclidean quasi-coherent sheaves, defined by the equality:

$$\rho(E, \|.\|) := \inf \left\{ r \in \mathbb{R}_+ \mid E_{/\mathrm{tor}} + B_{\|.\|}(0, r) = E_{\mathbb{R}} \right\} \quad (\in [0, +\infty]),$$

where $(E, \|.\|)$ denotes a Euclidean quasi-coherent sheaf, $E_{/\mathrm{tor}}$ the image of E in $E_{\mathbb{R}}$, and $B_{\|.\|}(0, r)$ the open ball of center 0 and radius r in $E_{\mathbb{R}}$ equipped with the semi-norm $\|.\|$.

Diverse results established in this monograph are illustrations of the following principle: *the invariant $h^1_\theta(E, \|.\|)$ attached to a Euclidean lattice $(E, \|.\|)$, or more generally to a suitable Euclidean quasi-coherent sheaf, is "small" when every element of the seminormed $\mathbb{R}$-vector space $(E_{\mathbb{R}}, \|.\|)$ is "close enough" to some element of $E_{/\mathrm{tor}}$.*

This principle provides an intuitively appealing interpretation of the "cohomological condition" on some Euclidean quasi-coherent sheaf $(E, \|.\|)$ to have a vanishing or a finite theta invariant $h^1_\theta(E, \|.\|)$, and is formally instantiated by comparison estimates relating the theta invariants h^1_θ and the covering radii attached to Euclidean quasi-coherent sheaves.

We may illustrate this principle by the Euclidean quasi-coherent sheaf $\pi_{\mu *}\overline{\mathcal{O}}_{\mathcal{V}}$ considered in 0.2.3.5 and 0.2.4.3 above. Its covering radius is easily seen to satisfy:

$$\rho(\pi_{\mu *}\overline{\mathcal{O}}_{\mathcal{V}})^2 = \pi^2 \alpha^2 \sum_{n \in \mathbb{N}} e^{-2\lambda n}.$$

It is finite if and only if $\lambda := \widehat{\deg}\, \overline{L}$ is positive, and we have seen in 0.2.4.3 that this holds if and only if $h^1_\theta\big(\pi_{\mu *}\overline{\mathcal{O}}_{\mathcal{V}}\big)$ is finite.

0.2.5.4. Applications of our infinite dimensional geometry of numbers to Arakelov geometry and Diophantine problems led us to include two significant innovations in our formalism, which with hindsight also shed some light on various questions of classical geometry of numbers.

Firstly, several of our main results concern not only objects, but also *morphisms,* in suitable categories of Euclidean quasi-coherent sheaves.

For instance, a central result about Banaszczyk functions is their monotonicity property with respect to arbitrary norm decreasing maps between Euclidean quasi-coherent sheaves. Moreover the comparison estimates investigated in the final chapter of this monograph, which provide a control on covering radii in terms of theta invariants, are established in a relative setting, concerning relative covering radii and theta ranks attached to morphisms of Euclidean quasi-coherent sheaves.

Having at one's disposal such relative results will be not only conceptually more satisfactory, but also crucial in applications.

Secondly, several of our results about invariants attached to Euclidean quasi-coherent sheaves and to their morphisms are concerned with a countably generated $\mathbb{Z}$-module E equipped with *two* Euclidean seminorms $\|.\|$ and $\|.\|'$ on the real vector space $E_{\mathbb{R}}$. Indeed we shall establish estimates involving suitable invariants of the Euclidean quasi-coherent sheaves $\overline{E} := (E, \|.\|)$ and $\overline{E}' := (E, \|.\|')$ and the relative traces $\mathrm{Tr}(\|.\|'/\|.\|)$ or $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$.

When $\overline{E}$ and $\overline{E}'$ are one and the same Euclidean lattice, of rank n, these relative traces equal n, and our estimates becomes estimates in the usual style of geometry of numbers, where the rank of Euclidean lattices appear.

It is quite remarkable that various classical estimates in geometry of numbers involving the rank of Euclidean lattices admit an infinite dimensional generalization involving a pair of Euclidean quasi-coherent sheaves $\overline{E}$ and $\overline{E}'$ as above, and where the rank is replaced by the relative trace $\mathrm{Tr}(\|.\|'/\|.\|)$ or $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$.

Moreover the nuclearity properties of the spaces of analytic sections of coherent analytic sheaves on complex analytic spaces show that pairs of Euclidean quasi-coherent sheaves $\overline{E}$ and $\overline{E}'$ as above, for which these relative traces are finite, naturally arise when one considers Hermitian quasi-coherent sheaves constructed as direct images, as in 0.2.1.

For this reason, our formalism involving pairs of Euclidean norms satisfying suitable trace conditions, which may appear rather *ad hoc* at first sight, fits quite nicely the applications to Diophantine geometry.

0.2.5.5. Although motivated by questions of arithmetic geometry, the construction in [**Bos20b**] of extensions of theta invariants to suitable categories of infinite rank Hermitian vector bundles has a fundamentally analytic character. This is reflected by the role played in this construction by measure theory on Polish, but non-locally compact, spaces; see [**Bos20b**, Sections 7.3-6]

The "measure theoretic" character of the constructions in this monograph is arguably still stronger. For instance, the analogy between our constructions of invariants attached to general Hermitian quasi-coherent sheaves and various classical constructions of measures and capacities becomes very precise when we investigate the "rank invariants" associated to morphisms in Chapter 5. Moreover the fine properties of theta invariants are established in Chapter 8 by using some classical theorems concerning Borel probability measures on infinite dimensional locally convex spaces, due notably to Bochner, Prokhorov, Sazonov, and Minlos.

0.2.6. In a sequel [**BCa**] to this monograph, we shall develop a theory of "arithmetic affine schemes" and of their modifications in Arakelov geometry. It will realizes the program hinted at in 0.2.2 above, and also have applications to the arithmetical theory of ampleness — notably to some arithmetic analogues of Grauert's ampleness criteria, as hinted at in 0.2.3 — and to integral points.

Besides the main results of this monograph — concerning Hermitian quasi-coherent sheaves, their theta invariants h^1_θ, and their covering radii — these applications to Diophantine geometry will also rely on the tools developed in two more technical works, of independent interest.

Firstly, in [**BCb**] we shall investigate infinite dimensional geometry of numbers in a slightly different perspective, by studying countably generated $\mathbb{Z}$-module endowed with the topologies defined by the embedding of the associated $\mathbb{R}$-vector spaces into duals of nuclear Fréchet spaces. A thorough use of the results of this monograph concerning theta invariants and their relationship to covering radii makes it possible to develop a somewhat qualitative version of geometry of numbers for such "nuclear quasi-coherent sheaves over arithmetic curves", focusing on the study of their bounded subsets.

We will also show that nuclear quasi-coherent sheaves occur naturally as spaces of sections of coherent sheaves on a quasi-projective scheme $\mathcal{V}$ over $\mathbb{Z}$ endowed with a conjugation invariant compact subset K of its space $\mathcal{V}(\mathbb{C})$ of complex points. Such pairs $(\mathcal{V}, K)$ will be studied systematically in [**BCa**] under the name of *A-schemes*.

Secondly, the monograph [**Cha**] will be devoted to diverse results of complex analytic geometry concerning "analytic pairs" — namely pairs (X, K) consisting in a complex analytic space X and some compact subset K of $X_{\rm red}$ — and the topological and bornological properties of the spaces $\Gamma(K, \mathcal{C})$ of (germs of) sections on K of a coherent analytic sheaf $\mathcal{C}$ on X. It will cover the foundational work in complex analytic geometry necessary to the investigation of the A-schemes mentioned above.

0.3. Contents

We now proceed to a brief synopsis of the main results of this monograph. More detailed presentations of these results are given in the introductory paragraphs of each chapter.

Chapter 1 collects various results concerning countably generated modules over Dedekind rings. The main technical result in this chapter is the existence, for every countably generated module M

over some Dedekind ring R, of a largest projective quotient of M:

$$\delta_M : M \longrightarrow M^{\vee\vee}.$$

This chapter presents further results, either of motivational nature[21] or of a technical nature, that may be skipped at first reading.

The main definitions concerning Hermitian quasi-coherent sheaves that will be used in this monograph are introduced in Chapter 2. We notably introduce the categories $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$, and their subcategories $\overline{\mathbf{Coh}}_X$ and $\overline{\mathbf{Coh}}_X^{\leq 1}$, defined by Hermitian quasi-coherent sheaves whose underlying $\mathcal{O}_K$-modules are finitely generated. These subcategories contain the categories $\overline{\mathbf{Vect}}_X$ and $\overline{\mathbf{Vect}}_X^{\leq 1}$ of Hermitian vector bundles over X, classically considered in Arakelov geometry.

We also introduce the admissible short exact sequences of Hermitian quasi-coherent sheaves, and their canonical dévissages, which will play a central role in the study of theta invariants.

The remaining parts of Chapter 2 are devoted to more technical results concerning Hermitian quasi-coherent sheaves. Firstly concerning the vectorization functor from $\overline{\mathbf{Coh}}_X^{\leq 1}$ to $\overline{\mathbf{Vect}}_X^{\leq 1}$, which constitutes a left adjoint to the inclusion functor from $\overline{\mathbf{Vect}}_X^{\leq 1}$ to $\overline{\mathbf{Coh}}_X^{\leq 1}$, and secondly concerning the duality functor from the category $\overline{\mathbf{Coh}}_X$ to a suitably defined category $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$ of pro-Hermitian vector bundles over X. These developments are intended for later reference, and their study may be postponed until they are referred to in later chapters.

Before investigating invariants of Hermitian quasi-coherent sheaves, motivated by the analogy between number fields and function fields, in Chapter 3 we study various properties of the invariant of a quasi-coherent sheaf $\mathcal{F}$ over a smooth projective curve C over some base field k defined by the dimension $h^1(C, \mathcal{F})$ of its first cohomology group $H^1(C, \mathcal{F})$. This invariant $h^1(C, \mathcal{F})$ plays the role of a "geometric model" for the invariants of Hermitian quasi-coherent sheaves investigated in this monograph.

The results of Chapter 3 are not used in the following chapters. However they isolate the main properties of the invariant $\mathcal{F} \mapsto h^1(C, \mathcal{F})$ that will play a central role in our axiomatic approach to invariants of Hermitian quasi-coherent sheaves in Chapters 4 and 5.

Chapter 4 is devoted to our main constructions of invariants with values in $[0, +\infty]$ attached to Hermitian quasi-coherent sheaves on X, starting from some invariant attached to Hermitian vector bundles with values in $\mathbb{R}^+$. These constructions requires the validity of some basic properties of this invariant on $\overline{\mathbf{Vect}}_X$, namely its *monotonicity, subadditivity,* and *downward continuity.*

These properties are easily shown to hold for the theta invariant h^1_θ which constitutes the main subject of this monograph, and are also satisfied by other significant invariants attached to some Euclidean lattice, for instance by the square $\rho(\overline{E})^2$ of its covering radius, or its Gauss-Voronoi invariant $\mathrm{gv}(\overline{E})$ introduced in Section 9.1. Specialized to the invariant h^1_θ, the results of Chapter 4 will establish an extension to the larger category $\mathbf{q}\overline{\mathbf{Coh}}_X$ of the constructions in [**Bos20b**, Chapters 7 and 9] of theta invariants on the category $\mathbf{ind}\overline{\mathbf{Vect}}_X$ of ind-Hermitian vector bundles over X.[22]

In Chapter 5, we pursue our axiomatic approach of invariants attached to Hermitian quasi-coherent sheaves, starting from invariants on $\overline{\mathbf{Vect}}_X$ or $\overline{\mathbf{Coh}}_X$, by studying the consequence of some additional axiom, which we name *strong monotonicity.*

When strong monotonicity holds, it becomes possible to attach well behaved invariants, not only to objects of these categories, but also to morphisms in these categories. The invariants associated to objects of $\overline{\mathbf{Vect}}_X$ or $\overline{\mathbf{Coh}}_X$, or $\mathbf{q}\overline{\mathbf{Coh}}_X$ are supposed to play the role of the dimension of some

[21] They "explain" how Archimedean places are taken into account in our definition of Hermitian quasi-coherent sheaves over an arithmetic curve.

[22] The category $\mathbf{ind}\overline{\mathbf{Vect}}_X$ is the full subcategory of $\mathbf{q}\overline{\mathbf{Coh}}_X$ whose objects are the Hermitian quasi-coherent sheaves $(E, (\|.\|_x)_{x \in X(\mathbb{C})})$ such that the underlying $\mathcal{O}_K$-module E is projective and the Hermitian seminorms $\|.\|_x$ are actually norms.

elusive cohomology group associated to these objects, while their "relative versions," associated to morphisms in these categories, are analogues of the rank of the morphisms they induce between cohomology groups.[23]

We also show in Chapter 5 that, under the assumption of strong monotonicity, the constructions of invariants on $\mathbf{q}\overline{\mathbf{Coh}}_X$ starting from invariants on $\overline{\mathbf{Vect}}_X$ developed in Chapter 4 satisfies remarkable additional properties. These turn out to be quite useful in applications.

At this point, we should emphasize that the existence of significant invariants which, besides the monotonocity, subadditivity, and downward continuity which entered as basic axioms in Chapter 4 also satisfy strong monotonicity, is by no means trivial. The only instances of such invariants which we are aware of are the theta invariant h^1_θ and its variants. The proof of their strong monotonicity, which we shall establish in Chapter 7 relies on some non-trivial estimates on theta functions attached to Euclidean lattices established in the work of Banaszczyk [**Ban92, Ban22**] and Regev and Stephens-Davidowitz [**RSD17a**].

Chapter 6 is devoted to the study of various invariants attached to Euclidean quasi-coherent sheaves that are defined in elementary terms, in the spirit of the classical geometry of numbers. Notably we investigate the properties of the covering radius $\rho(\overline{M})$ attached to some object $\overline{M}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and its relations with other elementary invariants, and we also study a relative version of the covering radius, attached to morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

In Chapter 7, we investigate the theta invariants in the finite rank setting. Notably we establish that the invariant h^1_θ on the categories $\overline{\mathbf{Vect}}_X$ and $\overline{\mathbf{Coh}}_X$ is strongly monotonic. Our proof relies on the properties of the so-called Banaszczyk function $B_{\overline{E}}$ associated to an object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and of its Fourier transform $\beta_{\overline{E}^\vee}$, the Banaszczyk measure of $\overline{E}$, which will also play a key role in the next chapters.

Chapter 8 is devoted to the construction and to the properties of the theta invariant in the infinite dimensional situation. We apply the general formalism developed in Chapters 4 and 5 to the invariant h^1_θ, and in this way we define the invariants $\underline{h}^1_\theta$ and $\overline{h}^1_\theta$ on $\mathbf{q}\overline{\mathbf{Coh}}_X$, and the subcategories of θ^1-summable and θ^1-finite objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$. We also extend to objects $\overline{E}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ the definitions and the properties of the Banaszczyk function $B_{\overline{E}}$ and of the Banaszczyk measure $\beta_{\overline{E}^\vee}$, and we use them to investigate the objects $\overline{E}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that the theta invariants $\underline{h}^1_\theta(\overline{E})$ and $\overline{h}^1_\theta(\overline{E})$ are finite and coincide.

Finally, in Chapter 9, we explore the relations between the theta invariants attached to some Euclidean quasi-coherent sheaf $\overline{E}$, notably $\overline{h}^1_\theta(\overline{E})$, and the more naive invariants introduced in Chapter 6, especially its covering radius $\rho(\overline{E})$.

In the Diophantine applications of theta invariants developed in the sequel of this monograph, the formalism developed in the previous chapters will be used to transpose in Diophantine geometry various cohomological techniques familiar in classical algebraic geometry. It will lead to results concerning the smallness or the vanishing of the invariants $\overline{h}^1_\theta(\overline{E})$ associated to Euclidean quasi-coherent sheaves $\overline{E}$ naturally associated to the Diophantine problems under study.

To derive "concrete" consequences — for instance density results — from these results about the theta invariants $\overline{h}^1_\theta(\overline{E})$, it will be crucial to know that they imply similar smallness properties of the covering radii $\rho(\overline{E})$ of these Euclidean quasi-coherent sheaves. The main result of Chapter 9 will provide the needed control of covering radii in terms of theta invariants.

[23] A similar philosophy appears in the work of McMurray Price [**MP17**]. This work focuses on invariants of Hermitian vector bundles that play the role of the dimension $h^0(C, E)$ of the space of sections of vector bundle E over a smooth projective curve over some field k, while we consider invariants which are arithmetic counterparts of the dimension $h^1(C, \mathcal{F})$ of the cohomology group $H^1(C, \mathcal{F})$ associated to an arbitrary quasi-coherent sheaf $\mathcal{F}$ over C.

Our main theorem will actually concern not single objects $\overline{E}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, but pairs of objects $\overline{E} := (E, \|.\|)$ and $\overline{E}' := (E, \|.\|')$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ admitting the same underlying $\mathbb{Z}$-module E, and whose defining seminorms $\|.\|$ and $\|.\|'$ satisfy a Hilbert-Schmidt condition of the form $\mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty$. We will actually work in a relative setting — crucial for Diophantine applications — and establish bound on the relative covering radius attached to a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ in terms of its theta rank.

These chapters are followed by three Appendices.

Appendices A and B gather various properties associated to a pair of Hermitian seminorms on a complex vector space, concerning notably their singular values and the associated relative traces.

Appendix C presents various classical results concerning positive Borel measures on some topological real vector space in the specific form needed for their application in Chapter 8, in a form that should be accessible with only some familiarity with basic notions of measure theory.

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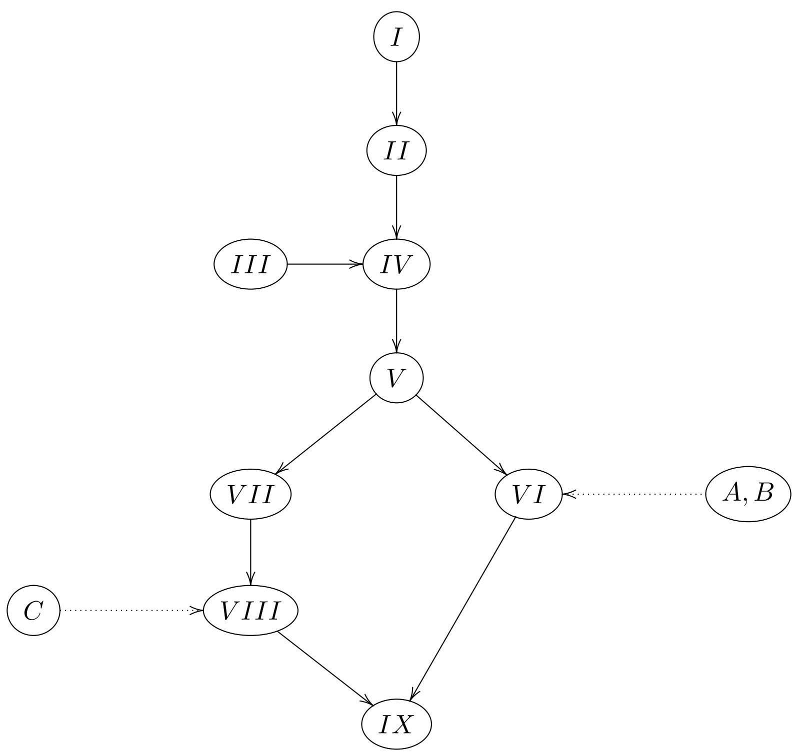

0.4. Acknowledgements

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0.5. Notation and Conventions

0.5.1. Basic Notations.

0.5.1.1. We denote by $\mathbb{N}$ the set of nonnegative integers, and for $k \in \mathbb{N}$, we denote by $\mathbb{N}_{>k}$ (resp., by $\mathbb{N}_{\geq k}$) the set of nonnegative integers greater than k (resp., greater than or equal to k).

By *countable*, we mean "of cardinality at mots the cardinality of $\mathbb{N}$."

Concerning maps between partially ordered sets, we follow the French usage, and by *increasing*, we mean *non-decreasing.*

0.5.1.2. If M is a module over a ring A, and if B is a commutative A-algebra, we denote by M_B the "base changed" module $M \otimes_A B$. Similarly, if $\varphi : M \to N$ is a morphism of A-modules, we let:

$$\varphi_B := \varphi \otimes_A \mathrm{Id}_B : M_B \longrightarrow N_B,$$

and if $\mathcal{X}$ is a scheme over $\operatorname{Spec} A$, we denote by:

$$\mathcal{X}_B := \mathcal{X} \times_{\operatorname{Spec} A} \operatorname{Spec} B$$

is base change to $\operatorname{Spec} B$.

0.5.1.3. We will often denote by K a number field and by $\mathcal{O}_K$ its ring of integer. The associated *arithmetic curve* is the scheme:

$$X := \operatorname{Spec} \mathcal{O}_K.$$

The set $X(\mathbb{C})$ of morphisms of schemes from $\operatorname{Spec} \mathbb{C}$ to X may be identified with the set of field embeddings:

$$x : K \longrightarrow \mathbb{C},$$

and for every $\mathcal{O}_K$-module M (resp. any morphism of $\mathcal{O}_K$-modules $\varphi : M \to N$, resp. any $\mathcal{O}_K$-scheme $\mathcal{X}$), its base change by $x : \mathcal{O}_K \to \mathbb{C}$ will be denoted by:

$$M_x := M \otimes_{\mathcal{O}_K, x} \mathbb{C}. \quad \text{(resp. by } \varphi_x : M_x \longrightarrow N_x\text{), resp. by } \mathcal{X}_x\text{).}$$

We denote by π the morphism of schemes from X to $\operatorname{Spec} \mathbb{Z}$, and by $\overline{\omega}_\pi$ its relative dualizing sheaf equipped with its natural Hermitian metrics (which give norm 1 to the trace map $\mathrm{Tr}_{K/\mathbb{Q}}$; see for instance [**Bos20b**, 1.2.1]).

0.5.2. Lebesgue measures, Fourier transforms, and Poisson formula.

0.5.2.1. A *Lebesgue measure* on a finite dimensional $\mathbb{R}$-vector space V is a translation invariant positive Radon measure on V.

To any Lebesgue measure λ on V is canonically attached a Lebesgue measure $\lambda^\vee$ on the dual vector space $V^\vee := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$, characterized by the following property. If $(e_i)_{1 \leq i \leq n}$ denotes a $\mathbb{R}$-basis of V and $(\xi_i)_{1 \leq i \leq n}$ the dual basis of $V^\vee$, then:

$$\lambda\Big(\sum_{i=1}^{n} [0,1) e_i\Big)\; \lambda^\vee\Big(\sum_{i=1}^{n} [0,1) \xi_i\Big) = 1.$$

If $\|.\|$ is a Euclidean norm on the finite dimensional $\mathbb{R}$-vector space V, the Lebesgue measure $\lambda_{\overline{V}}$ associated to the Euclidean vector space $\overline{V} := (V, \|.\|)$ is the Lebesgue measure on V that satisfies the following equivalent normalization conditions, where $(v_1, \ldots, v_n)$ denotes an orthonormal basis of the Euclidean space $\overline{V}$:

$$\lambda_{\overline{V}}\Big(\sum_{i=1}^{n} [0,1) v_i\Big) = 1,$$

and

$$\int_V e^{-\pi \|x\|^2}\, d\lambda_{\overline{V}}(x) = 1.$$

If we denote by $\|.\|^\vee$ the dual Euclidean norm on $V^\vee$, defined by the relation:

$$\|\xi\|^\vee := \max_{v \in V, \|v\| \leq 1} \langle \xi, v \rangle,$$

and by $\overline{V}^\vee := (V^\vee, \|.\|^\vee)$ the Euclidean vector space dual to $\overline{V}$, then we have:

$$\lambda_{\overline{V}^\vee} = \lambda_{\overline{V}}^\vee.$$

0.5.2.2. Let V a Hausdorff locally convex topological vector space, and let us denote by:

$$V^\vee := \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{R}}(V, \mathbb{R})$$

its topological dual. To any complex Borel measure with finite total mass ν on V is associated its *Fourier transform*:

$$\mathcal{F}\nu : V^\vee \longrightarrow \mathbb{C}, \quad \xi \longmapsto \int_V e^{-2\pi i \langle \xi, x \rangle}\, d\nu(x).$$

It is a continuous function on $V^\vee$ equipped with the weak topology (namely, the $\sigma(V^\vee, V)$-topology), and is a function of positive type when ν is a positive finite Borel measure.

We will use the notation $\mathcal{F}^\vee \nu$ for the function on $V^\vee$ defined by:

$$\mathcal{F}^\vee \nu(\xi) := \mathcal{F}\nu(-\xi) = \int_V e^{2\pi i \langle \xi, x \rangle}\, d\nu(x).$$

0.5.2.3. Let V be a finite dimensional $\mathbb{R}$-vector, $V^\vee := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ its dual, and λ a Lebesgue measure on V.

We shall define the Fourier transform:

$$\mathcal{F}_{V,\lambda} : L^1(V) \longrightarrow \mathcal{C}_0(V^\vee)$$

by the relation, for every $f \in L^1(V)$ and any $\xi \in V^\vee$:

$$\mathcal{F}_{V,\lambda} f(\xi) := \mathcal{F}(f \cdot \lambda)(\xi) = \int_V f(x)\, e^{-2\pi i \langle \xi, x \rangle} d\lambda(x).$$

We shall also define:

$$\mathcal{F}^\vee_{V^\vee, \lambda^\vee} : L^1(V^\vee) \longrightarrow \mathcal{C}_0(V)$$

by the relation, for every $g \in L^1(V^\vee)$ and any $x \in V \simeq V^{\vee\vee}$:

$$\mathcal{F}^\vee_{V^\vee, \lambda^\vee} g(x) := \mathcal{F}^\vee(g \cdot \lambda^\vee)(\xi) = \int_V f(\xi)\, e^{2\pi i \langle \xi, x \rangle} d\lambda^\vee(\xi).$$

Then, if we denote by $\mathcal{S}(V)$ and $\mathcal{S}(V^\vee)$ the Schwartz spaces of complex valued $\mathcal{C}^\infty$–functions with rapid decay over V and $V^\vee$, the Fourier transforms $\mathcal{F}_{V,\lambda}$ and $\mathcal{F}_{V^\vee, \lambda^\vee}$ establish isomorphisms of Fréchet spaces, inverse of each other:

$$\mathcal{F}_{V,\lambda} : \mathcal{S}(V) \xrightarrow{\sim} \mathcal{S}(V^\vee) \quad \text{and} \quad \mathcal{F}^\vee_{V^\vee, \lambda^\vee} : \mathcal{S}(V^\vee) \xrightarrow{\sim} \mathcal{S}(V). \tag{0.5.1}$$

0.5.2.4. If $\overline{V} := (V, \|.\|)$ is a finite dimensional Euclidean vector space, of dual $\overline{V}^\vee := (V^\vee, \|.\|^\vee)$, we shall denote by $\mathcal{F}_{\overline{V}}$ and $\mathcal{F}^\vee_{\overline{V}^\vee}$ the Fourier transform $\mathcal{F}_{V, \lambda_{\overline{V}}}$ and its inverse $\mathcal{F}^\vee_{V^\vee, \lambda_{\overline{V}^\vee}}$.

Using this notation, the Fourier transform of Gaussian functions is given by the following expression:

$$\mathcal{F}_{\overline{V}}\big(e^{-\pi \|.\|^2}\big) = e^{-\pi \|.\|^{\vee 2}}, \tag{0.5.2}$$

and more generally, for every $t \in \mathbb{R}_+^*$:

$$\mathcal{F}_{\overline{V}}\big(e^{-\pi t \|.\|^2}\big) = t^{-\dim_{\mathbb{R}} V/2} e^{-\pi t^{-1} \|.\|^{\vee 2}}. \tag{0.5.3}$$

0.5.2.5. Let us keep the notation introduced in 0.5.2.3 above, and consider Λ a lattice in V, namely a (free, discrete) subgroup of V generated by a $\mathbb{R}$-basis of V.

The $\mathbb{R}$-vector space $\Lambda_{\mathbb{R}}$ may be canonically identified to V, and the dual $\mathbb{Z}$-module $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ to the dual lattice in $V^{\vee}$:

$$\Lambda^{\vee} := \{\xi \in V^{\vee} \mid \xi(\Lambda) \subseteq \mathbb{Z}\}.$$

By definition, the *covolume* $\mathrm{covol}_{\lambda}(\Lambda)$ of Λ with respect to the Lebesgue measure λ is defined as:

$$\mathrm{covol}_{\lambda}(\Lambda) := \lambda(\Delta),$$

where Δ is any Borel fundamental domain for the action by translation of Λ on V.

Then the following equality holds:

$$\mathrm{covol}_{\lambda}(\Lambda) \cdot \mathrm{covol}_{\lambda^{\vee}}(\Lambda^{\vee}) = 1. \tag{0.5.4}$$

Moreover the *Poisson formula* asserts that, for any function $f \in \mathcal{S}$ and any $x \in V$, the following equality holds:[24]

$$\mathrm{covol}_{\lambda}(\Lambda). \sum_{v \in \Lambda} f(x+v) = \sum_{\xi \in \Lambda^{\vee}} \mathcal{F}_{V,\lambda} f(\xi)\, e^{2\pi i \langle \xi, x \rangle}. \tag{0.5.5}$$

In particular, we have:

$$\mathrm{covol}_{\lambda}(\Lambda). \sum_{v \in \Lambda} f(v) = \sum_{\xi \in \Lambda^{\vee}} \mathcal{F}_{V,\lambda} f(\xi). \tag{0.5.6}$$

[24]The reader may refer to [**Hör90**, 7.1-2] for a concise and elegant exposition of the basic properties of the Fourier transform in the spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)'$ of Schwartz functions and tempered distributions which emphasizes the role of the Poisson formula.

Part 1

Hermitian Quasi-coherent Sheaves over Arithmetic Curves

CHAPTER 1

Countably Generated Modules over Dedekind Rings

An *arithmetic curve* is a scheme:

$$X := \operatorname{Spec} \mathcal{O}_K$$

defined by the ring of integers $\mathcal{O}_K$ of some number field K. Its set of complex points $X(\mathbb{C})$ is the set, of cardinality $[K : \mathbb{Q}]$, of field embeddings:

$$x : K \longrightarrow \mathbb{C}.$$

The main object of study of this monograph are the *Hermitian quasi-coherent sheaves* over such an arithmetic curve X. These are defined as pairs:

$$\overline{F} := \big(F, (\|.\|_x)_{x \in X(\mathbb{C})}\big),$$

where F is a countably generated $\mathcal{O}_K$-module, and where $(\|.\|_x)_{x \in X(\mathbb{C})}$ is a family, invariant under complex conjugation, of Hermitian seminorms on the complex vector spaces:

$$F_x := F \otimes_x \mathbb{C},$$

deduced from F by the base changes $x : \mathcal{O}_K \to \mathbb{C}$.

In this chapter, we gather diverse preliminary results concerning countably generated modules over the ring of integer $\mathcal{O}_K$ of some number field K, and more generally, over some Dedeking ring R.

In Section 1.1, we discuss various properties of countably generated modules over a Noetherian ring, and more generally of quasi-coherent sheaves of countable type over a Noetherian scheme.

These properties notably imply that the quasi-coherent sheaves over an arithmetic curve $X = \operatorname{Spec} \mathcal{O}_K$ that arise in practice when investigating the algebraic geometry of schemes of finite type over K or over $\mathcal{O}_K$ are defined by countably generated $\mathcal{O}_K$-modules. This demonstrates that considering only countably generated $\mathcal{O}_K$-modules in our definition of Hermitian quasi-coherent sheaves is an innocuous restriction.

Section 1.2 is devoted to the main technical result in this chapter. We show that, for every countably generated module M over some Dedekind ring R, there is a largest projective quotient of M:

$$\delta_M : M \longrightarrow M^{\vee\vee},$$

constructed as its double dual, suitably defined.

In Section 1.3, we discuss the compatibility of various constructions and properties of modules over Dedekind rings with finite flat morphisms $R \hookrightarrow S$ of the base rings.

In Section 1.4, we present a topological interpretation of the construction of the largest projective quotient $M^{\vee\vee}$ in Section 1.2 when the base ring R is $\mathbb{Z}$, and more generally when R is the ring of integers $\mathcal{O}_K$ of a number field K.

In Sections 1.5 and 1.6, we discuss the role of ultrametric seminorms when studying modules over Dedekind rings. The content of these two sections will not be explicitly used in our investigation of Hermitian quasi-coherent sheaves in the remaining of this monograph. However they will provide a motivation for the use of Hermitian seminorms when developing the theory of Hermitian quasi-coherent sheaves over an arithmetic curve X, which play the role of quasi-coherent sheaves over X "compactified" by its Archimedean places.

More specifically, in Section 1.5, we review various (more or less) classical facts concerning the description of torsion free modules over a discrete valuation ring R in terms of vector spaces on the fraction field K of R and of ultrametric seminorms on these vector spaces. We also extend this description to the situation where R is a valuation ring defining a non-discrete valuation of rank 1 on its fraction field K — which arguably is a better analogue of the Archimedean places of number fields than discrete valuations.

Finally in Section 1.6, we discuss the compatibility of the constructions of the dual module $M^\vee$ and of the largest projective quotient $M^{\vee\vee}$ with extensions of Dedekind ring $R \hookrightarrow \mathring{R}$ defining an open immersion $\operatorname{Spec} \mathring{R} \hookrightarrow \operatorname{Spec} R$ between their associated schemes.

The results in Sections 1.1, 1.5 and 1.6 are either folklore or motivational, the ones in Sections 1.3 and 1.4 are of a rather technical nature, and the detail of their proofs could be skipped without inconvenience.

In contrast, the construction of the largest projective quotient in Section 1.2 — which does not seem to appear in the literature in spite of its basic character — will play an important role in this monograph. Indeed the theta invariants $\underline{h}^1_\theta$, $\overline{h}^1_\theta$, $h^1_\theta, \ldots$ that we shall attach to some Hermitian quasi-coherent sheaf $\overline{F} := \big(F, (\|.\|_x)_{x\in X(\mathbb{C})}\big)$ over an arithmetic curve $X := \operatorname{Spec} \mathcal{O}_K$ will turn out to be unaltered when $\overline{F}$ is replaced by the Hermitian quasi-coherent sheaf:

$$\overline{F}^{\vee\vee} := \big(F^{\vee\vee}, (\|.\|^\sim_x)_{x\in X(\mathbb{C})}\big)$$

defined by the largest projective quotient $F^{\vee\vee}$ of the $\mathcal{O}_K$-module F and by seminorms $(\|.\|^\sim_x)_{x\in X(\mathbb{C})}$ quotients of the seminorms $(\|.\|_x)_{x\in X(\mathbb{C})})$ by the surjective $\mathbb{C}$-linear maps:

$$\delta_{F,x} : F_x \longrightarrow F_x^{\vee\vee}.$$

It should be emphasized that the study of countably generated $\mathbb{Z}$-modules, and more generally of countably generated $\mathcal{O}_K$-modules, displays considerably delicate phenomena.[1] The invariance of the theta invariants under the replacement of $\overline{F}$ by $\overline{F}^{\vee\vee}$ show that, ultimately, we will not need to take them into account.

In this chapter, by R, we always denote a Dedeking ring, in the sense of Bourbaki [**Bou65**, VII.2.1]; in other words, R is either a field, or a Noetherian integrally closed domain of dimension 1.

Moreover, for every R-module M, we denote by M_{tor} its torsion submodule:

$$M_{\mathrm{tor}} := \{m \in M \mid \exists a \in R \setminus \{0\}, am = 0\},$$

and we let:

$$M_{/\mathrm{tor}} := M/M_{\mathrm{tor}}.$$

1.1. Countably Generated Modules and Quasi-coherent Sheaves of Countable Type

1.1.1. Countably generated A-modules. Let A be a ring (commutative, with unit). The following proposition gathers various properties of countably generated A-modules which are straightforward consequences of definitions.

[1]When R is $\mathbb{Z}$, countably generated modules are nothing but countable abelian groups. The study of not finitely generated countable abelian groups was initiated in the Habilitationschrift of F. W. Levi [**Lev17**] — where are constructed examples of countable abelian groups such that the short exact sequence $0 \to M_{\mathrm{tor}} \to M \to M/M_{\mathrm{tor}} \to 0$ is not split — and pursued in the pionneering works of Prüfer [**Prü23, Prü24, Prü25**], Ulm [**Ulm33**], and Baer [**Bae36, Bae37**]. We refer the reader to the lecture notes by Kaplansky [**Kap69**] for a gentle presentation of various classical results, and to the monograph of Fuchs [**Fuc15**] for a synthesis of one century of investigations of abelian groups.

All these results demonstrate the difficulty of establishing general classification results concerning countable abelian groups. During the last decades, various advances concerning the level in the hierarchy of Borel equivalence relations of the classification of various classes of countable abelian groups, due notably to Hjorth, Kechris, and Thomas, have established the "hardness" of these classification problems in a precise sense; see for instance [**Tho06**].

PROPOSITION 1.1.1. *(1) Let M be a A-module, equipped with a filtration by submodules*

$$M_0 = \{0\} \subseteq M_1 \subseteq \ldots \subset M_n \subseteq M_{n+1} \subseteq \ldots$$

that is exhausting, namely that satisfies:

$$\bigcup_{n \geq 0} M_n = M.$$

If the successive quotients M_{n+1}/M_n, $n \geq 0$, are countably generated, then M is countably generated.

Conversely, if a A-module M is countably generated, then it admits an exhausting filtration $(M_n)_{n\in\mathbb{N}}$ as above such that the quotients M_{n+1}/M_n, $n \geq 0$, are finitely generated.

(2) Any quotient of countably generated A-module is countably generated. If the ring A is Noetherian, any submodule of a countably generated A-module is countably generated.

(3) For any A-algebra B and any countably generated A-module M, the B-module $M_B := M \otimes_A B$ is countably generated.

(4) If B is an A-algebra of finite type, and if M is a countably generated B-module, then M is also countably generated as a A-module.

1.1.2. Quasi-coherent sheaves of countable type over Noetherian schemes.

PROPOSITION AND DEFINITION 1.1.2. *Let X be a Noetherian scheme, and let $(U_\alpha)_{\alpha\in I}$ be a covering of X by affine open subschemes. For every quasi-coherent sheaf $\mathcal{F}$ over X, the following conditions are equivalent:*

(i) There exists an increasing sequence:

$$\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_n \subseteq \mathcal{C}_{n+1} \subseteq \cdots$$

of coherent subsheaves of $\mathcal{F}$ such that $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$.

(ii) For every affine open subscheme U of X, the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is countably generated.

(iii) For every $\alpha \in I$, the $\mathcal{O}_X(U_\alpha)$-module $\mathcal{F}(U_\alpha)$ is countably generated.

When these conditions are satisfied, we shall say that $\mathcal{F}$ is a quasi-coherent sheaf of countable type.

The proof of Proposition 1.1.2 relies on the following classical theorem concerning extensions of coherent susbsheaves of a quasi-coherent sheaf:

THEOREM 1.1.3 (see [**GD71**, I.6.9.7]). *Let $\mathcal{F}$ be a quasi-coherent sheaf over a Noetherian scheme X. For every open subscheme U of X and every coherent subsheaf $\mathcal{C}$ of the restriction $\mathcal{F}_{|U}$ of $\mathcal{F}$ to U, there is a coherent subsheaf $\widetilde{\mathcal{C}}$ of $\mathcal{F}$ such that $\widetilde{\mathcal{C}}_{|U} = \mathcal{C}$.*

PROOF OF PROPOSITION 1.1.3. Assume that Condition (i) is satisfied. Then, for every affine open subscheme U of X, the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is the increasing union of its finitely generated submodules $\mathcal{C}_n(U)$ and therefore is countably generated. Therefore (ii) is satisfied.

The implication $(ii) \Rightarrow (iii)$ is clear.

Finally assume that (iii) is satisfied.

Since X is Noetherian and therefore quasi-compact, there exists a finite subset J of I such that $(U_\alpha)_{\alpha\in J}$ is an open covering of X. For every $\alpha \in J$, we may choose a sequence $(s_{\alpha,n})_{n\in\mathbb{N}}$ of generators of the $\mathcal{O}(U_\alpha)$-module $\mathcal{F}(U_\alpha)$. For every (α, n) in $J \times \mathbb{N}$, we may consider the coherent subsheaf $\mathcal{O}_{U_\alpha} s_{\alpha,n}$ of $\mathcal{F}_{|U_\alpha}$. According to Theorem 1.1.3, this coherent subsheaf is the restriction to U_α of some coherent subsheaf $(\mathcal{O}_{U_\alpha} s_{\alpha,n})^\sim$ of $\mathcal{F}$.

We may define inductively an increasing sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of coherent subsheaves of $\mathcal{F}$ as follows:

- $\mathcal{C}_0 = 0$;

- For every $n \in \mathbb{N}$, $\mathcal{C}_{n+1} = \mathcal{C}_n + \sum_{\alpha \in J} (\mathcal{O}_{U_\alpha} s_{\alpha,n})^\sim$.

By construction, for every $\alpha \in J$, the $\mathcal{O}_X(U_\alpha)$-submodule $\bigcup_{n\in\mathbb{N}} \mathcal{C}_n(U_\alpha)$ of $\mathcal{F}(U_\alpha)$ contains all the terms of the sequence $(s_{\alpha,n})_{n\in\mathbb{N}}$, and therefore coincides with $\mathcal{F}(U_\alpha)$. This establishes the equality $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ and completes the proof of (i). □

PROPOSITION 1.1.4. *Let X be a Noetherian scheme, and let:*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short sequence of quasi-coherent sheaf over X. The sheaf $\mathcal{G}$ is of countable type if and only if both $\mathcal{F}$ and $\mathcal{H}$ are of countable type.

This directly follows from Proposition 1.1.1 (1) and (2), and from the characterization of quasi-coherent sheaves of countable type by Condition (ii) in Proposition 1.1.2.

COROLLARY 1.1.5. *Let X be a Noetherian scheme. For any morphism $u : \mathcal{F}_1 \to \mathcal{F}_2$ of quasi-coherent sheaves over X, if $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) is countably generated, then* $\ker u$ *and* $\operatorname{im} u$ *(resp.* $\operatorname{im} u$ *and* $\operatorname{coker} u$*) are countably generated.*

PROPOSITION 1.1.6. *Let $f : X \to Y$ be a morphism of Noetherian schemes.*

(1) For every quasi-coherent sheaf of countable type $\mathcal{F}$ over Y, the quasi-coherent sheaf $f^*\mathcal{F}$ over X is of countable type.

(2) When X and Y are separated and when f is a morphism of finite type, then for every quasi-coherent sheaf of countable type $\mathcal{F}$ over X, the (higher) direct images $R^i f_* \mathcal{F}$ are quasi-coherent sheaves of countable type over Y.

PROOF. Assertion (1) is a direct consequence of the definition of quasi-coherent sheaves of countable type — for instance from its characterization by Condition (i) in Proposition 1.1.2.

With the notation of (2), the direct image $R^* f_* \mathcal{F}$ in the bounded derived category of quasi-coherent sheaves over Y is actually represented by a bounded complex of quasi-coherent sheaves of countable type on X. This follows from the description of $R^* f_* \mathcal{F}$ in terms of the Cech complex associated to finite affine open covering of X and from Proposition 1.1.1. We leave the details to the interested reader. □

For later reference, we spell out the following simple consequence of the previous propositions:

COROLLARY 1.1.7. *Let X be a Noetherian scheme, Σ a closed subset of X, and $U := X \setminus \Sigma$ the open subscheme of X defined as its complement.*

If $\mathcal{F}$ is a quasi-coherent sheaf over X such that its subsheaf $\Gamma_\Sigma \mathcal{F}$ of sections supported by Σ vanishes, then $\mathcal{F}$ is of countable type if and only if its restriction $\mathcal{F}_{|U}$ to U is of countable type.

PROOF. Let us denote by $j : U \to X$ the inclusion morphism.

If $\mathcal{F}$ is of countable type, then $\mathcal{F}_{|U} = j^*\mathcal{F}$ also is, as a special case of Proposition 1.1.6, (1). Conversely, when $\mathcal{F}_U$ is of countable type, then the quasi-coherent sheaf $j_*\mathcal{F}_{|U}$ over X also is, according to Proposition 1.1.6, (2). When moreover the sheaf $\Gamma_\Sigma \mathcal{F}$ vanishes, or equivalently when the canonical morphism of sheaves $\mathcal{F} \to j_* j^* \mathcal{F} = j_* \mathcal{F}_{|U}$ is injective, this implies that $\mathcal{F}$ is of countable type by Proposition 1.1.4. □

Propositions 1.1.4 and 1.1.6 and their corollaries show that a large part of algebraic geometry, which concerns schemes of finite type over a Noetherian base, may be developed by considering only quasi-coherent sheaves that are of countable type.

A notable exception to this principle would be the sheaves of meromorphic functions. Indeed, if X is an integral scheme of finite type over some base field k, the sheaf of meromorphic functions on X

— namely the constant sheaf defined by field $k(X)$ of rational functions over X — is quasi-coherent, and this sheaf is of countable type if and only if X is zero dimensional or k is countable.

The formalism of this monograph is developed with a view toward applications to Diophantine geometry, where such countability assumptions on the base fields are satisfied, since Diophantine geometry involves (mostly) base fields that are finitely generated extensions of the prime field.

For these reasons, considering only countably generated $\mathcal{O}_K$-modules in our definition of Hermitian quasi-coherent sheaves will never appear as an actual restriction in practice.

1.2. The Largest Projective Quotient of a Countably Generated R-Module

1.2.1. Topological modules and duality. Let us begin by preliminaries concerning the dual module associated to some (topological) module over an arbitrary base ring.

In this subsection, we denote by A an arbitrary (commutative, unital) ring A.

1.2.1.1. *The duality functors.* Let $\mathbf{Mod}_A$ denote the category of A-modules, and $\mathbf{Mod}_A^{\mathrm{top}}$ the category of topological modules and continuous A-linear maps over the ring A equipped with the discrete topology.

We may define as follows two adjoint A-linear duality functors:

$$.^\vee : \mathbf{Mod}_A \longrightarrow (\mathbf{Mod}_A^{\mathrm{top}})^{\mathrm{opp}} \quad \text{and} \quad .^{\vee^t} : \mathbf{Mod}_A^{\mathrm{top}} \longrightarrow (\mathbf{Mod}_A)^{\mathrm{opp}}.$$

Firstly, to any A-module M, we may attach the A-module:

$$M^\vee := \mathrm{Hom}_A(M, A),$$

equipped with the topology of simple convergence — namely with the topology induced by the product topology on the set A^M of set-theoretic maps from M to A, where each factor A is endowed with the discrete topology. And to any A-linear morphism $\varphi : M \to M'$, we may attach its transpose:

$$\varphi^\vee : M'^\vee \longrightarrow M^\vee, \quad f \longmapsto f \circ \varphi,$$

which indeed is A-linear and continuous when $M'^\vee$ and $M^\vee$ are equipped with the topology of simple convergence.

Secondly, to any topological A-module N, we may attach the A-module

$$N^{\vee^t} := \mathrm{Hom}_A^{\mathrm{top}}(N, A)$$

of continuous A-linear maps from M to A, where A is endowed with the discrete topology, and to any continuous A-linear map $\psi : N \to N'$ between topological A-modules, we may attach its transpose:

$$\varphi^{\vee^t} : N'^{\vee^t} \longrightarrow N^{\vee^t}, \quad f \longmapsto f \circ \varphi.$$

If $(I, \preccurlyeq)$ is a directed set and $(M_i)_{i \in I}$ is an increasing family of finitely generated A-submodules of some A-module M that satisfies:

$$\bigcup_{i \in I} M_i = M,$$

then M may be identified with the inductive limit $\varinjlim_{i \in I} M_i$, and the topological A-module $M^\vee$ with the projective limit $\varprojlim_{i \in I} M_i^\vee$ of the discrete A-modules $M_i^\vee := \mathrm{Hom}_A(M_i, A)$, which are finitely generated when A is Noetherian.

Using this description of the topological module $M^\vee$, one easily establishes that the functors $.^\vee$ and $.^{\vee^t}$ are adjoint of each other, in the form of functorial A-isomorphisms:

$$\mathrm{Hom}_A(M, N^{\vee^t}) \xrightarrow{\sim} \mathrm{Hom}_A^{\mathrm{top}}(N, M^\vee),$$

where M (resp. N) denotes an object of $\mathbf{Mod}_A$ (resp. of $\mathbf{Mod}_A^{\mathrm{top}}$). Actually both $\mathrm{Hom}_A(M, N^{\vee^t})$ and $\mathrm{Hom}_A^{\mathrm{top}}(N, M^\vee)$ may be identified with the A-module of A-bilinear maps from $M \times N$ to N that are continuous in the second variable.

1.2.1.2. *Biduality.* If M (resp. N) is an A-module (resp. a topological A-module), we denote by δ_M (resp. by δ_N) the biduality morphism

$$\delta_M : M \longrightarrow M^{\vee\vee^t} \quad (\text{resp.} \quad \delta_N : N \longrightarrow N^{\vee^t\vee})$$

defined by:

$$\delta_M(x)(\xi) := \xi(x) \quad (\text{resp.} \quad \delta_N(x)(\xi) := \xi(x))$$

for any (x, ξ) in $M \times M^\vee$ (resp. in $N \times N^{\vee^t}$).These morphisms define a natural transformation from the identity functor to the functor $.^{\vee\vee^t}$ (resp. $.^{\vee^t\vee}$) from $\mathbf{Mod}_A$ (resp. $\mathbf{Mod}_A^{\mathrm{top}}$) to itself.

PROPOSITION 1.2.1. *Let M (resp. N) be an A-module (resp. a topological A-module). Then the composition*

$$(\delta_M)^\vee \circ \delta_{M^\vee} : M^\vee \xrightarrow{\delta_{M^\vee}} M^{\vee\vee^t\vee} \xrightarrow{(\delta_M)^\vee} M^\vee \quad (\textit{resp.} \quad (\delta_N)^{\vee^t} \circ \delta_{N^{\vee^t}} : N^{\vee^t} \xrightarrow{\delta_{N^{\vee^t}}} N^{\vee^t\vee\vee^t} \xrightarrow{(\delta_N)^{\vee^t}} N^{\vee^t})$$

is the identity morphism $\mathrm{Id}_{M^\vee}$ *(resp.* $\mathrm{Id}_{N^{\vee^t}}$*).*

Proposition 1.2.1 shows that the dual topological module $M^\vee$ (resp. the dual module $N^{\vee^t}$) may be canonically identified with a direct summand of the "tridual" $M^{\vee\vee^t\vee}$ (resp. $N^{\vee^t\vee\vee^t}$).

PROOF. For every $(x, \xi) \in M \times M^\vee$, we have:

$$\big((\delta_M)^\vee \circ \delta_{M^\vee}(\xi)\big)(x) = (\delta_M)^\vee\big(\delta_{M^\vee}(\xi)\big)(x) = \delta_{M^\vee}(\xi)\big(\delta_M(x)\big) = \delta_M(x)(\xi) = \xi(x).$$

This establishes the equality: $(\delta_M)^\vee \circ \delta_{M^\vee} = \mathrm{Id}_{M^\vee}$. The proof of the equality $(\delta_N)^{\vee^t} \circ \delta_{N^{\vee^t}} = \mathrm{Id}_{N^{\vee^t}}$ is similar. □

Observe that the duality functor $.^\vee$ (resp. $.^{\vee^t}$) transform direct sums in direct products (resp. direct products in direct sums). In particular, if M is a free A-module $A^{(I)}$, then $M^\vee$ may be identified with A^I (equipped with the product topology of the discrete topology on each factor A) and the double dual $M^{\vee\vee^t}$ with $A^{(I)}$, so that the biduality morphism $\delta_M : M \to M^{\vee\vee}$ is an isomorphism of A-modules.

This immediately implies that δ_M is an isomorphism if M is a projective A-module.

1.2.2. The categories $\mathbf{CP}_R$ and $\mathbf{CTC}_R$. In the remaining of this section, we denote by R a Dedekind ring.

As in [**Bos20b**, Chapter 4], we will denote by $\mathbf{CP}_R$ the category of countably generated, projective R-modules, and by $\mathbf{CTC}_R$ the category of linearly compact Tate spaces over R with countable basis – that is, the full subcategory of $\mathbf{Mod}_R^{\mathrm{top}}$ consisting of those topological R-modules M such that the following equivalent conditions hold:

(i) the topology of M is Hausdorff and complete, and there exists a countable basis of neighborhoods U of 0 in M consisting in R-submodules of N such that N/U is finitely generated and projective;
(ii) there exists a countably generated projective R-module N – that is, an object of $\mathbf{CP}_R$ – and an isomorphism of topological R-modules between M and $N^\vee$ equipped as above with the topology of simple convergence.

Equivalently, the objects of $\mathbf{CTC}_R$ are the topological R-modules that are isomorphic to the "prodiscrete" R-modules defined as the projective limit $\varprojlim E_\bullet$ of a projective system:

$$E_\bullet : E_0 \longleftarrow E_1 \longleftarrow \cdots \longleftarrow E_i \longleftarrow E_{i+1} \longleftarrow \cdots$$

of surjective morphisms of finitely generated projective R-modules.

Recall from [**Bos20b**, 4.3.2] that, restricted to $\mathbf{CP}_R$ and $\mathbf{CTC}_R$, the duality functors $.^\vee$ introduced above define an adjoint equivalence:

$$.^\vee : \mathbf{CP}_R \leftrightarrows \mathbf{CTC}_R^{\mathrm{op}} : .^{\vee^t}$$

Actually, for any object M in $\mathbf{CP}_R$ (resp. for any object N in $\mathbf{CTC}_R$), the biduality morphism δ_M is an isomorphism

$$\delta_M : M \xrightarrow{\sim} M^{\vee\vee^t} \quad (\text{resp.} \quad \delta_N : N \xrightarrow{\sim} N^{\vee^t\vee})$$

in $\mathbf{CP}_R$ (resp. in $\mathbf{CTC}_R$).

From now on, for simplicity, as in [**Bos20b**], we shall denote by $N^\vee$, instead of $N^{\vee^t}$, the topological dual of an object N in $\mathbf{CTC}_R$. Actually when R is neither a field nor a complete discrete valuation ring, the R-module $N^\vee$ and its submodule $N^{\vee^t}$ coincide; see [**Bos20b**, 4.2.3 and Appendix B].

1.2.3. The biduality morphism $\delta_M : M \to M^{\vee\vee}$ associated to a countably generated R-module M. The following theorem will play a key role in this article by allowing us to handle arbitrary countably generated R-modules by reducing to *projective* countably generated R-modules.

THEOREM 1.2.2. *Let M be a countably generated R-module. Then the topological R-module $M^\vee$ is an object of $\mathbf{CTC}_R$. The bidual R-module $M^{\vee\vee}$ is an object of $\mathbf{CP}_R$ and the canonical biduality morphism*

$$\delta_M : M \longrightarrow M^{\vee\vee}$$

is surjective.

If P is a projective R-module, any morphism of R-modules $M \to P$ factors through δ_M.

In particular, the bidual module $M^{\vee\vee}$ is the "largest projective quotient" of M.

PROOF. Write M as an increasing union of finitely generated submodules M_i, $i \geq 0$. Then, as a topological R-module, the topological R-module $M^\vee$ may be identified with a limit of topological R-modules

$$M^\vee = \lim_i M_i^\vee,$$

defined from the projective system

$$M_0^\vee \xleftarrow{p_0} M_1^\vee \longleftarrow \cdots \longleftarrow M_i^\vee \xleftarrow{p_i} M_{i+1}^\vee \longleftarrow \cdots,$$

where p_i is the transpose of the inclusion morphism $M_i \hookrightarrow M_{i+1}$.

Let N_i be the image of $M^\vee$ in $M_i^\vee$. The R-modules $M_i^\vee$ and N_i are finitely generated and torsion free, hence they are projective. The maps

$$p_{i|N_{i+1}} : N_{i+1} \longrightarrow N_i$$

are surjective by construction, and therefore the topological R-module $\lim_i N_i$ is an object of $\mathbf{CTC}_R$ – see [**Bos20b**, 4.2.1]. Since the inclusions $N_i \hookrightarrow M_i^\vee$ define an isomorphism of topological R-modules

$$\lim_i N_i \xrightarrow{\sim} \lim_i M_i^\vee,$$

this shows that $M^\vee$ is an object of $\mathbf{CTC}_R$.

Since $M^\vee$ is an object of $\mathbf{CTC}_R$, its dual $M^{\vee\vee} = \mathrm{Hom}_R^{\mathrm{top}}(M^\vee, R)$ is an object of $\mathbf{CP}_R$.

To prove that the biduality morphism δ_M is surjective, we consider its image $T = \delta_M(M)$. It is a submodule of the countably generated projective R-module $M^{\vee\vee}$, and therefore, by [**Bos20b**, Proposition 4.1.1, (3)], T is countably generated and projective as well. The biduality morphism δ_M factors as

$$\delta_M = i \circ p : M \xrightarrow{p} T \xhookrightarrow{i} M^{\vee\vee}$$

We are reduced to showing that i is an isomorphism in $\mathbf{CP}_R$. By duality, it is enough to show that its transpose:

$$i^\vee : M^{\vee\vee\vee} \longrightarrow T^\vee$$

is an isomorphism in $\mathbf{CTC}_R$.

Lemma 1.2.1 shows that the composition:

$$(\delta_M)^\vee \circ \delta_{M^\vee} : \; M^\vee \xrightarrow{\delta_{M^\vee}} M^{\vee\vee\vee} \xrightarrow{(\delta_M)^\vee} M^\vee$$

is the identity map. Since $M^\vee$ is an object of $\mathbf{CTC}_R$, the map $\delta_{M^\vee}$ is an isomorphism of topological R-modules. Therefore $(\delta_M)^\vee$ also is an isomorphism of topological R-modules.

Consider the factorization of $(\delta_M)^\vee$ as

$$(\delta_M)^\vee = p^\vee \circ i^\vee : M^{\vee\vee\vee} \xrightarrow{i^\vee} T^\vee \xhookrightarrow{p^\vee} M^\vee.$$

As p is onto, the morphism of topological R-modules $p^\vee$ is injective and strict; in other words, $p^\vee$ induces an R-linear homeomorphism between $T^\vee$ and its image in $M^\vee$. Since $(\delta_M)^\vee$ is an isomorphism, this implies that $p^\vee$ is an isomorphism, hence that $i^\vee : M^{\vee\vee\vee} \to T^\vee$ is an isomorphism. This completes the proof of the surjectivity of δ_M.

Finally, let $\varphi : M \to P$ be a morphism from M to a projective R-module P. Consider the commutative diagram:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\delta_M"] & P \arrow[d, "\delta_P"] \\ M^{\vee\vee} \arrow[r, "\varphi^{\vee\vee}"] & P^{\vee\vee}. \end{tikzcd}$$

Since P is projective, the biduality morphism δ_P is an isomorphism, and therefore φ factors as

$$\varphi = \delta_P^{-1} \circ \varphi^{\vee\vee} \circ \delta_M. \qquad \square$$

We note two consequences of independent interest of Theorem 1.2.2.

Corollary 1.2.3. *Let M be a countably generated R-module. Then the transpose*

$$(\delta_M)^\vee : M^{\vee\vee\vee} \longrightarrow M^\vee$$

of the biduality map $\delta_M : M \to M^{\vee\vee}$ is an isomorphism.

Proof. Lemma 1.2.1 shows that $(\delta_M)^\vee$ is surjective. By Theorem 1.2.2, it is injective. $\square$

Corollary 1.2.4. *A countably generated R-module M is projective if and only if, for any nonzero $m \in M$, there exists $\xi \in \operatorname{Hom}_R(M, R)$ such that $\xi(m) \neq 0$.*

Proof. The R-module M is projective if and only if the surjective morphism $\delta_M : M \longrightarrow M^{\vee\vee}$ is an isomorphism; this holds precisely when it is injective. $\square$

When $R = \mathbb{Z}$, Corollary 1.2.4 above has been established by Ramspott and Stein [**RS62**].

1.2.4. Antiprojective R-modules and canonical dévissage. We shall say that an R-module M is *antiprojective* when, for any projective R-module P, we have

$$\operatorname{Hom}_R(M, P) = 0,$$

or, equivalently, when $M^\vee = \operatorname{Hom}_R(M, R) = 0$. For instance, any torsion R-module is antiprojective. When the Dedekind ring R is *not* a field, then its filed of fractions K, and more generally any K-vector space, is antiprojective when considered as an R-module.

Let M be a countably generated R-module. We may consider its submodule

$$M_{\mathrm{ap}} := \ker \delta_M.$$

According to Theorem 1.2.2, it fits into a short exact sequence

$$0 \longrightarrow M_{\mathrm{ap}} \longrightarrow M \xrightarrow{\delta_M} M^{\vee\vee} \longrightarrow 0, \tag{1.2.1}$$

which splits, since $M^{\vee\vee}$ is projective. As a consequence, we obtain a (non-canonical) isomorphism of R-modules:

$$M \xrightarrow{\sim} M_{\mathrm{ap}} \oplus M^{\vee\vee}.$$

Lemma 1.2.1 implies the equality

$$M_{\mathrm{ap}}^{\vee} = 0.$$

In other words, $M_{\mathrm{ap}}^{\vee}$ is antiprojective. The short exact sequence (1.2.1) actually shows that M_{ap} is the largest antiprojective submodule of M. In particular, it contains the torsion submodule of M.

The short exact sequence (1.2.1) associated to a countably generated R-module M will be called the *canonical dévissage of* M. Observe that its construction is functorial, namely, to any two countably generated R-modules M_1 and M_2 and to any morphism of R-modules

$$\varphi : M_1 \longrightarrow M_2$$

is attached a commutative diagram of R-modules with admissible exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M_{1,\mathrm{ap}} \arrow[r] \arrow[d, "\varphi_{|M_{1,\mathrm{ap}}}"] & M_1 \arrow[r, "\delta_{M_1}"] \arrow[d, "\varphi"] & M_1^{\vee\vee} \arrow[r] \arrow[d, "\varphi^{\vee\vee}"] & 0 \\
0 \arrow[r] & M_{2,\mathrm{ap}} \arrow[r] & M_2 \arrow[r, "\delta_{M_2}"] & M_2^{\vee\vee} \arrow[r] & 0.
\end{tikzcd}$$

1.2.5. The sets $\mathrm{coft}(M)$, $\mathrm{scoft}(M)$, $\mathrm{coh}(M)$, and $\mathcal{U}(M^{\vee})$.

DEFINITION 1.2.5. For every R-module M, we denote by $\mathrm{coft}(M)$ (resp. by $\mathrm{scoft}(M)$) the set of R-submodules N of M such that the quotient $\mathbb{R}$-module M/N is finitely generated (resp. is finitely generated and torsion free), and we denote by $\mathrm{coh}(M)$ the set of finitely generated R-submodules of M.

The acronym coft (resp. scoft) stands for "of **co**-**f**inite **t**ype" (resp. "**s**aturated of **co**-**f**inite **t**ype"), and coh for "**coh**erent."

1.2.5.1. The following properties of $\mathrm{coft}(M)$ and $\mathrm{scoft}(M)$ are easy consequences of their definition, and the details of their proof will be left to the reader.

PROPOSITION 1.2.6. *(1) For every R-module M, $\mathrm{coft}(M)$ and $\mathrm{scoft}(M)$ are stable under finite intersection. In particular $(\mathrm{coft}(M), \supseteq)$ and $(\mathrm{scoft}(M), \supseteq)$ are directed sets, with smallest element M. Moreover, for every $N \in \mathrm{coft}(M)$, the saturation of N in M — namely the R-submodule N^{sat} of M such that $N \subseteq N^{\mathrm{sat}}$ and $N^{\mathrm{sat}}/N = (M/N)_{\mathrm{tor}}$ — belongs to $\mathrm{scoft}(M)$.*

(2) Every morphism of R-modules $f : M_1 \to M_2$ induces order preserving maps:

$$f^{-1} : \mathrm{coft}(M_2) \longrightarrow \mathrm{coft}(M_1) \quad \textit{and} \quad f^{-1} : \mathrm{scoft}(M_2) \longrightarrow \mathrm{scoft}(M_1).$$

(3) A R-submodule N of some R-module M belongs to $\mathrm{scoft}(M)$ if and only if there exists a finite family $(\xi_i)_{i \in F}$ of elements of $M^{\vee}$ such that:

$$M = \bigcap_{i \in F} \ker \xi_i.$$

Actually the proof of (1) and (2) uses only the noetherianity of the base ring R. Assertion (3) follows from the fact that a torsion free finitely generated R-module is projective, and therefore a R-submodule of some finitely generated free R-module.

1.2.5.2. For any object N of $\mathbf{CTC}_R$, we shall denote the set of open saturated submodules of N by $\mathcal{U}(N)$. For any U in $\mathcal{U}(N)$, the quotient N/U is finitely generated and projective; see [**Bos20b**, 4.2.1].

For any object M in $\mathbf{CP}_R$ and any $C \in \mathrm{coh}(M)$, the submodule of $M^\vee$:

$$C^\perp := \{\xi \in M^\vee \mid \xi_{|C} = 0\}$$

is an element of $\mathcal{U}(M^\vee)$. Moreover the saturation C^{sat} of C in M is also an element of $\mathrm{coh}(M)$, and satisfies:

$$C^{\mathrm{sat}\perp} = C^\perp.$$

If $\mathrm{coh}^{\mathrm{sat}}(M)$ denotes the set of saturated and finitely generated R-submodules of M, one defines in this way an order reversing bijection:

$$.^\perp : \mathrm{coh}^{\mathrm{sat}}(M) \xrightarrow{\sim} \mathcal{U}(M^\vee), \quad C \longmapsto C^\perp.$$

The inverse bijection maps $U \in \mathcal{U}(M^\vee)$ to

$$U^\perp := \{m \in M,\, \delta_M(m)_{|U} = 0\} = \bigcap_{\xi \in U} \ker \xi.$$

Moreover, for any $C \in \mathrm{coh}^{\mathrm{sat}}(M)$ with image $C^\perp$ in $\mathcal{U}(M^\vee)$, the restriction map $M^\vee \to C^\vee$, namely the transpose of the inclusion morphism $C \hookrightarrow M$, induces an isomorphism of finitely generated projective R-modules:

$$M^\vee / C^\perp \xrightarrow{\sim} C^\vee;$$

see [**Bos20b**, Corollary 4.4.9].

1.2.5.3. The construction of the sets $\mathrm{scoft}(M)$ and $\mathrm{coh}(M)$ associated to a countably generated R-module M and the duality properties in 1.2.5.2 above are compatible with the canonical dévissage, as demonstrated by the following proposition, which we leave as an exercise for the reader.

Proposition 1.2.7. *Let M be a countably generated R-module.*

Every $N \in \mathrm{scoft}(M)$ contains M_{ap} and the map δ_M induces a bijection:

$$\delta_M^{-1} : \mathrm{scoft}(M^{\vee\vee}) \xrightarrow{\sim} \mathrm{scoft}(M).$$

The map:

$$\mathrm{coh}(M) \longrightarrow \mathrm{coh}(M^{\vee\vee}), \quad C \longmapsto \delta_M(C) \tag{1.2.2}$$

is onto, and the map:

$$\mathrm{coh}(M^{\vee\vee}) \longrightarrow \mathcal{U}(M^{\vee\vee\vee}) \simeq \mathcal{U}(M^\vee), \quad \widetilde{C} \longmapsto \widetilde{C}^\perp := \big\{\xi \in M^{\vee\vee\vee} \mid \xi_{|\widetilde{C}} = 0\big\} \simeq \delta_M^\vee(\widetilde{C}^\perp) \tag{1.2.3}$$

restricts to a bijection:

$$\mathrm{coh}^{\mathrm{sat}}(M^{\vee\vee}) \xrightarrow{\sim} \mathcal{U}(M^\vee). \tag{1.2.4}$$

The surjective map:

$$\mathrm{coh}(M) \longrightarrow \mathcal{U}(M^\vee)$$

defined as the composition of (1.2.2) *and* (1.2.3) *maps $C \in \mathrm{coh}(M)$ to*

$$C^\perp := \big\{\xi \in M^\vee \mid \xi_{|C} = 0\big\}.$$

Moreover, for any $U \in \mathcal{U}(M^\vee)$, the submodule of M:

$$U^\perp := \bigcap_{\xi \in U} \ker \xi$$

contains M_{ap}, the quotient $U^{\perp}/M_{\mathrm{ap}}$ is a finitely generated projective R-module, and the transposes of the inclusion morphism $U^{\perp} \rightarrow M$ and of the quotient morphism $U^{\perp} \rightarrow U^{\perp}/M_{\mathrm{ap}}$ define isomorphisms:

$$M^{\vee}/U \xrightarrow{\sim} (U^{\perp})^{\vee} \xrightarrow{\sim} (U^{\perp}/M_{\mathrm{ap}})^{\vee}.$$

1.3. Modules over Dedekind Rings and Finite Morphisms

Let L be a finite field extension of the field of fractions K of R, assume that the integral closure S of R in K is a finitely generated R-module,[2]. and denote by X (resp. Y) the spectrum of the Dedekind ring R (resp. S), and by

$$\pi : Y = \operatorname{Spec} S \longrightarrow X = \operatorname{Spec} R$$

the finite flat morphism of schemes defined by the inclusion of rings $R \hookrightarrow S$.

Any S-module M defines a quasi-coherent sheaf over Y that we will still denote by M. Its direct image $\pi_* M$ is (the quasi-coherent sheaf over X defined by) M considered as an R-module by means of the inclusion $R \hookrightarrow S$.

Similarly, if N is an object of of $\mathbf{Mod}_S^{\mathrm{top}}$, we define $\pi_* N$ as the object of $\mathbf{Mod}_R^{\mathrm{top}}$ defined by N seen as a R-module, equipped with its given topology.

1.3.1. Permanence properties of the functor $\pi_* : \mathbf{Mod}_S \rightarrow \mathbf{Mod}_R$.

LEMMA 1.3.1. *Let M be an S-module. Then M is torsion-free if and only if the R-module $\pi_* M$ is torsion-free. An S-submodule N of M is saturated in M if and only if the R-module $\pi_* N$ is saturated in $\pi_* M$.*

PROOF. The norm map $N_{L/K}$ sends a (nonzero) element a of S to a (nonzero) element $N_{L/K}(a)$ of R in the ideal aS. As a consequence, if the S-module M is not torsion-free, we may find a nonzero $m \in M$ and a nonzero $a \in S$ such that $am = 0$. Then $N_{L/K}(a) m = 0$, so that $\pi_* M$ is not torsion-free. The converse is obvious, and a similar argument proves the second equivalence. □

PROPOSITION 1.3.2. *Let M be an S-module. The following two conditions are equivalent:*

(i) M is an object of $\mathbf{CP}_S$;
(ii) $\pi_* M$ is an object of $\mathbf{CP}_R$.

PROOF. The implication $(i) \Rightarrow (ii)$ follows from the fact that, as an R-module, the ring S is finitely generated and torsion-free, hence projective.

To prove the converse implication $(ii) \Rightarrow (i)$, we shall use the fact that a module M over a Dedekind ring is countably generated and projective if and only if it admits an exhaustive filtration

$$M_0 \subseteq M_1 \subseteq \ldots \subseteq M_i \subseteq M_{i+1} \subseteq \ldots$$

by finitely generated, torsion-free submodules M_i that are saturated in M ; see for instance [**Bos20b**, Proposition 4.1.1].

When (ii) is satisfied, we may consider such a filtration $(M_i)_{i \geq 0}$ of M considered as an R-module. For any $i \geq 0$, we may consider the S-submodule M_i' of M generated by M_i, that is, $M_i' = S M_i$, and the saturation M_i'' of M_i' considered as an R-submodule of M. Clearly, M_i' is finitely generated as an S-module, hence as an R-module.This implies that its saturation M_i'' in the projective R-module M is a finitely generated R-module. The R-submodule M_i'' is easily seen to be an S-submodule of M. As such, it is saturated in M by Lemma 1.3.1. Finally,

$$M_0'' \subseteq M_1'' \subseteq \ldots \subseteq M_i'' \subseteq M_{i+1}'' \subseteq \ldots$$

[2]This holds when for instance L is a separable extension of K, or when R is a finitely generated algebra over a field.

is an exhaustive filtration of S-module M by saturated, finitely generated and torsion-free S-submodules. This proves that M is an object of $\mathbf{CP}_S$. $\square$

1.3.2. Relative duality and biduality. To any S-module M, we may associate the dual topological S-module

$$M^\vee := \mathrm{Hom}_S(M, S),$$

which is an object in $\mathbf{CTC}_S$, and the bidual module

$$M^{\vee\vee} := \mathrm{Hom}_S^{\mathrm{cont}}(M^\vee, S),$$

which is an object in $\mathbf{CP}_S$.

We may also consider the R-module $\pi_* M$ and its dual and bidual objects in $\mathbf{CTC}_R$ and $\mathbf{CP}_R$ respectively:

$$(\pi_* M)^\vee := \mathrm{Hom}_R(M, R)$$

and

$$(\pi_* M)^{\vee\vee} := \mathrm{Hom}_R^{\mathrm{cont}}(M^\vee, R).$$

1.3.2.1. To describe the compatibility between these constructions, we need to introduce some – very classical – definitions.

Observe that for any topological S-module N, and any finitely generated projective S-module E, the tensor product $N \otimes_S E$ has a natural topology as an S-module: it is defined by by identifying E with a direct summand of some free S-module of finite rank $S^{\oplus n}$, and $N \otimes_S E$ with a direct summand of $N^{\oplus n}$.

Moreover, the natural morphisms of S-modules[3]

$$N^\vee \otimes_S E = \mathrm{Hom}_S^{\mathrm{cont}}(N, S) \otimes_S E \longrightarrow \mathrm{Hom}_S^{\mathrm{cont}}(N, E) \tag{1.3.1}$$

and

$$N^\vee \otimes_S E^\vee = \mathrm{Hom}_S^{\mathrm{cont}}(N, S) \otimes_S E^\vee \longrightarrow \mathrm{Hom}_S^{\mathrm{cont}}(N \otimes_S E, S) = (N \otimes_S E)^\vee \tag{1.3.2}$$

are isomorphisms. Indeed, this is clear when E is a free S-module of finite rank; by considering direct summands of such free modules, the general case follows.

Recall that the morphism π admits a relative dualizing sheaf $\omega_{Y/X}$, namely, the invertible sheaf over Y attached to the invertible S-module[4]

$$\omega_{S/R} := \mathrm{Hom}_R(S, R).$$

Moreover, for any S-module M, we may define a morphism of "relative duality":

$$I_M : \pi_*(M^\vee \otimes_S \omega_{S/R}) \longrightarrow (\pi_* M)^\vee,$$

by mapping a simple tensor $\xi \otimes \lambda$ in

$$M^\vee \otimes_S \omega_{S/R} = \mathrm{Hom}_S(M, S) \otimes_S \mathrm{Hom}_R(S, R)$$

to the composition $\lambda \circ \xi$ in $\mathrm{Hom}_R(M, R)$.

Similarly, for any topological S-module N, we may consider the morphism of R-modules

$$I_N : \pi_*(N^\vee \otimes_S \omega_{S/R}) \longrightarrow (\pi_* N)^\vee,$$

which maps an element $\xi \otimes \lambda$ in $\mathrm{Hom}_S^{\mathrm{cont}}(N, S) \otimes_S \mathrm{Hom}(S, R)$ to the element $\lambda \circ \xi$ in $\mathrm{Hom}_R^{\mathrm{cont}}(N, R)$. Using the isomorphism (1.3.1), which takes the form

$$N^\vee \otimes_S \omega_{S/R} \simeq \mathrm{Hom}_S^{\mathrm{cont}}(N, \omega_{S/R}),$$

[3]For simplicity, we denote by $N^\vee$ instead of $N^{\vee^t}$ the topological dual in $\mathbf{Mod}_R$ (resp. in $\mathbf{Mod}_S$) of an object N of $\mathbf{Mod}_R^{\mathrm{top}}$ (resp. of $\mathbf{Mod}_S^{\mathrm{top}}$).

[4]The structure of S-module on the additive group $\mathrm{Hom}_R(S, R)$ is defined by setting $s\xi = \xi(s.)$ for any $s \in S$ and any $\xi \in \mathrm{Hom}_R(S, R)$.

we see that, for any $\alpha \in \mathrm{Hom}_S^{\mathrm{cont}}(N, \omega_{S/R})$, we have:

$$I_N(\alpha) = \mathrm{Tr} \circ \alpha \in \mathrm{Hom}_R(N, R) = (\pi_* N)^\vee,$$

where Tr denotes the R-linear map:

$$\mathrm{Tr} : \omega_{S/R} \longrightarrow R$$

defined by $\mathrm{Tr}(\lambda) = \lambda(1)$ for any $\lambda \in \omega_{S/R} = \mathrm{Hom}_S(S, R)$.

Since the S-module $\omega_{S/R}$ is invertible, the isomorphism (1.3.2) with $E = \omega_{S/R}$ defines an isomorphism of S-modules

$$L_N : N^\vee \xrightarrow{\sim} (N \otimes_S \omega_{S/R})^\vee \otimes_S \omega_{S/R}$$

for every topological S-module N.

1.3.2.2. The following lemma is a straightforward consequence of the definitions of the relative duality morphisms. We leave its proof to the reader.

Lemma 1.3.3. *For any S-module M, the following diagram of R-modules commutes:*

$$\begin{tikzcd}
\pi_* M \arrow[rr, "\delta_{\pi_* M}"] \arrow[d, "\delta_M"] & & (\pi_* M)^{\vee\vee} \arrow[d, "(I_M)^\vee"] \\
\pi_*(M^{\vee\vee}) \arrow[r, "L_{M^\vee}", "\sim"'] & \pi_*((M^\vee \otimes \omega_{S/R})^\vee \otimes_S \omega_{S/R} \arrow[r, "I_{M^\vee \otimes_S \omega_{S/R}}"] & (\pi_*(M^\vee \otimes_S \omega_{S/R}))^\vee.
\end{tikzcd} \tag{1.3.3}$$

Proposition 1.3.4. *The relative duality morphisms I_M and I_N defined above satisfy the following properties:*

(i) For every S-module M, the map I_M is an isomorphism of topological R-modules.
(ii) For every topological S-module N in $\mathbf{CTC}_S$, the map I_N is an isomorphism of R-modules.

Proof. To prove (i), note that, clearly, the morphism I_M is an isomorphism when M is a torsion S-module; in this case, its range and its source are both zero. It is also an isomorphism when M is a free S-module of finite rank, and therefore when M is a direct summand of a free S-module of finite rank, that is, when M is a projective S-module of finite type; in this case, the range and the source of I_M are discrete projective R-modules. This implies that I_M is an isomorphism of topological R-modules when M is finitely generated. This proves (i), by considering an arbitrary S-module M as the colimit of its finitely generated submodules.

Let us prove (ii). When M is an object of $\mathbf{CP}_S$, the R-module $\pi_* M$ is an object of $\mathbf{CP}_R$, and both δ_M and $\delta_{\pi_* M}$ are isomorphisms. Moreover, I_M, and therefore its transpose $(I_M)^\vee$, are isomorphisms by (i). The commutative diagram (1.3.3) then shows that $I_{M^\vee \otimes_S \omega_{S/R}}$ is an isomorphism. Moreover, as the S-module $\omega_{S/R}$ is invertible, any object N of $\mathbf{CTC}_S$ is isomorphic to $M^\vee \otimes_S \omega_{S/R}$ for some object M of $\mathbf{CP}_S$. □

1.3.2.3. From Lemma 1.3.3 and Proposition 1.3.4, we deduce the following compatibility between canonical dévissage and direct images:

Corollary 1.3.5. *For every S-module M, the R-submodule $(\pi_* M)_{\mathrm{ap}}$ of $\pi_* M$ coincides with $\pi_*(M_{\mathrm{ap}})$, and the R-modules $(\pi_* M)^{\vee\vee}$ and $\pi_*(M^{\vee\vee})$ are canonically isomorphic.*

Proof. Proposition 1.3.4 shows that, in the diagram (1.3.3), the morphisms $I_{M^\vee \otimes_S \omega_{S/R}}$ and $(I_M)^\vee$ are isomorphisms. Therefore, the kernels $(\pi_* M)_{\mathrm{ap}}$ and $\pi_*(M_{\mathrm{ap}})$ of $\delta_{\pi_* M}$ and δ_M coincide, and $(\pi_* M)^{\vee\vee}$ and $\pi_*(M^{\vee\vee})$ – which may be identified with the coimages of these morphisms – are canonically isomorphic. □

We may finally establish the following compatibility between direct images and (saturated) submodules of co-finite type:

Corollary 1.3.6. *For every S-module M, a S-submodule N of M belongs to* $\mathrm{coft}(M)$ *(resp. to* $\mathrm{scoft}(M)$*)) if and only if $\pi_* N$ — that is, N seen as a R-submodule of M — belongs to* $\mathrm{coft}(\pi_* M)$ *(resp. to* $\mathrm{scoft}(\pi_* M)$*). Moreover for every $N' \in \mathrm{scoft}(\pi_* M)$, there exists N in* $\mathrm{scoft}(M)$ *such that $\pi_* N' \subseteq N'$.*

In other words the image of the "inclusion map":

$$\pi_* : \mathrm{scoft}(M) \longrightarrow \mathrm{scoft}(\pi_* M)$$

is cofinal in the directed set $(\mathrm{scoft}(\pi_* M), \supseteq)$.

Proof. Observe that a S-module M is finitely generated (resp. is torsion free) if and only if the R-module $\pi_* M$ is. Applied to M/N, this establishes the second assertion.

Consider an element N' of $\mathrm{scoft}(\pi_* M)$. According to Proposition 1.2.6 (3) applied to $\pi_* M$, there exists a finite family $(\xi'_i)_{i \in F}$ of elements of $(\pi_* M)^\vee$ such that:

$$N' := \cap_{i \in I} \ker \xi'_i.$$

Proposition 1.3.4 (i) shows that, for every $i \in F$, there exists a finite family $(\xi_j, \lambda_j)_{j \in F_i}$ of elements of $M^\vee \times \omega_{S/R}$ such that:

$$\xi'_i = \sum_{j \in F_i} \lambda_j \circ \xi_j.$$

Then by Proposition 1.2.6 (3) again, the S-submodule

$$N := \bigcap_{i \in F} \bigcap_{j \in F_i} \ker \xi_j$$

of M belongs to $\mathrm{scoft}(M)$. It is clearly contained in N'. □

1.4. The Largest Projective Quotient of a Countably Generated $\mathbb{Z}$-module: Topological Interpretation

In this section, we present a topological interpretation of the canonical dévissage (1.2.1) when the base ring is $\mathbb{Z}$, or more generally the ring of integers $\mathcal{O}_K$ of a number field K.

1.4.1. A theorem of Brown, Higgins, and Morris. Any real vector space V may be endowed with a canonical topology of Hausdorff topological vector space, by considering V as the colimit of its finite-dimensional vector subspaces endowed with their (unique) topology of Hausdorff vector spaces. We will call this topology the *inductive topology* of V. By definition, a subset X of V is open (resp. closed) for the inductive topology if and only if the intersection $X \cap F$ of X with any finite-dimensional vector subspace F of V is open (resp. closed) in F.

The inductive topology is the finest topology for which V is a topological vector space. In other words, any morphism from V to a real topological vector space is continuous.

When the real vector space V admits a countable basis $(v_i)_{i \in I}$, the inductive topology on V may be described as the vector space topology such that, when $\underline{\varepsilon} = (\varepsilon_i)_{i \in I}$ runs over $(\mathbb{R}^*_+)^I$, the "rectangles"

$$\mathcal{R}(\underline{\varepsilon}) := \left\{ \sum_{i \in I} x_i v_i \mid (x_i)_{i \in I} \in \mathbb{R}^{(I)} \cap \prod_{i \in I}]-\varepsilon_i, \varepsilon_i[\right\}$$

form a basis of (open) neighborhoods of 0 in V; see for instance [**Bou81**, Section III.1.4, Lemme 1, p. III.6] and [**BHM75**, Proposition 1]. In particular, V equipped with the inductive topology is a locally convex topological vector space, and may be identified with the inductive limit of its finite-dimensional vector subspaces in the category of locally convex topological real vector spaces.

In [**BHM75**], Brown, Higgins and Morris have established the following infinite dimensional generalization of the classical description of closed subgroups of finite dimension real vector spaces.

THEOREM 1.4.1 ([**BHM75**, Theorem 1]). *Let V be a countably generated real vector space. For any additive subgroup G of V that is closed in the inductive topology, there exists a basis $(v_i)_{i\in I}$ of the real vector space V and two disjoint subsets K and L of I such that the following equality holds:*

$$G = \bigoplus_{k\in K} \mathbb{R}v_k \oplus \bigoplus_{l\in L} \mathbb{Z}v_l.$$

In particular, the connected component G^0 of 0 in G is the largest $\mathbb{R}$-vector subspace of V contained in G, it is open and closed in G, and the quotient G/G^0 is a countably generated free $\mathbb{Z}$-module.

1.4.2. Application to countably generated $\mathbb{Z}$-modules. Let us consider a countably generated $\mathbb{Z}$-module M. To M, we may attach the countably generated real vector space

$$M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R},$$

and the map

$$\iota : M \longrightarrow M_{\mathbb{R}},\ m \mapsto m \otimes 1.$$

This map defines an isomorphism from M/M_{tor} onto its image $M_{/\mathrm{tor}} = \iota(M)$. Lastly, we may consider the closure $\overline{M}_{/\mathrm{tor}}$ of $M_{/\mathrm{tor}}$ in $M_{\mathbb{R}}$ endowed with its inductive topology.

Applied to the subgroup $\overline{M}_{/\mathrm{tor}}$ of $M_{\mathbb{R}}$, Theorem 1.4.1 admits the following consequence.

PROPOSITION 1.4.2. *With the previous notation, the following equality holds:*

$$M_{\mathrm{ap}} = \iota^{-1}(\overline{M_{/\mathrm{tor}}}^{0}). \tag{1.4.1}$$

Moreover the morphism $\iota : M \to \overline{M_{/\mathrm{tor}}}$ defines an isomorphism of countably generated free $\mathbb{Z}$-modules:

$$M/M_{\mathrm{ap}} \xrightarrow{\sim} \overline{M_{/\mathrm{tor}}}/\overline{M_{/\mathrm{tor}}}^{0}.$$

Proposition 1.4.2 may be summarized by the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M_{\mathrm{ap}} \arrow[r] \arrow[d, "\iota_{|M_{\mathrm{ap}}}"] & M \arrow[r, "\delta_M"] \arrow[d, "\iota"] & M^{\vee\vee} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \overline{M_{/\mathrm{tor}}}^{0} \arrow[r] & \overline{M_{/\mathrm{tor}}} \arrow[r] & \overline{M_{/\mathrm{tor}}}/\overline{M_{/\mathrm{tor}}}^{0} \arrow[r] & 0.
\end{tikzcd}$$

PROOF OF PROPOSITION 1.4.2. Consider the composition

$$M \overset{\iota}{\twoheadrightarrow} M_{/\mathrm{tor}} \hookrightarrow \overline{M_{/\mathrm{tor}}} \twoheadrightarrow \overline{M_{/\mathrm{tor}}}/\overline{M_{/\mathrm{tor}}}^{0}.$$

Since its range is a free $\mathbb{Z}$-module, it vanishes on M_{ap}. This proves the inclusion:

$$M_{\mathrm{ap}} \subseteq \iota^{-1}(\overline{M_{/\mathrm{tor}}}^{0}).$$

Any ξ in $M^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ defines an $\mathbb{R}$-linear form $\xi_{\mathbb{R}}$ in $\mathrm{Hom}_{\mathbb{R}}(M_{\mathbb{R}},\mathbb{R})$ which satisfies

$$\xi_{\mathbb{R}} \circ \iota = \xi. \tag{1.4.2}$$

We have:

$$\xi_{\mathbb{R}}(M_{/\mathrm{tor}}) = \xi(M) \subset \mathbb{Z}.$$

Therefore, by continuity of $\xi_{\mathbb{R}}$ for the inductive topology, we have:

$$\xi_{\mathbb{R}}(\overline{M_{/\mathrm{tor}}}) \subseteq \mathbb{Z}.$$

This implies that $\xi_{\mathbb{R}}$ vanishes on $M^0_{/\mathrm{tor}}$, which is an $\mathbb{R}$-vector space. According to (1.4.2), this implies that ξ vanishes on $\iota^{-1}(\overline{M_{/\mathrm{tor}}}^{0})$ and establishes the inclusion:

$$\iota^{-1}(\overline{M_{/\mathrm{tor}}}^{0}) \subseteq M_{\mathrm{ap}}.$$

This completes the proof of (1.4.1) and shows that ι defines an injective map

$$M/M_{\mathrm{ap}} \hookrightarrow \overline{M_{/\mathrm{tor}}}/\overline{M_{/\mathrm{tor}}}^0, \ [m] \mapsto [\iota(m)]. \tag{1.4.3}$$

As $\overline{M_{/\mathrm{tor}}}^0$ is open in $\overline{M_{/\mathrm{tor}}}$, the quotient topology on $\overline{M_{/\mathrm{tor}}}/\overline{M_{/\mathrm{tor}}}^0$ is discrete. Since ι has dense image in $\overline{M_{/\mathrm{tor}}}$, the map (1.4.3) has dense image as well. This shows that it is surjective. $\square$

COROLLARY 1.4.3. *Let M be a torsion free countably generated $\mathbb{Z}$-module. If its image $\iota(M)$ in $M_{\mathbb{R}}$ equipped with the inductive topology is discrete, then M is projective, hence free.*

1.4.3. Application to countably generated $\mathcal{O}_K$-modules. Combined with the compatibility of the formation of the largest projective quotient with finite morphisms established in Section 1.3, Proposition 1.4.2 allows us to give a topological description of the canonical dévissage of a countably generated R-module when the base ring R is the ring of integers of an arbitrary number field.

Let K be a number field, let $\mathcal{O}_K$ be its ring of integers, and let X denote the arithmetic curve $\operatorname{Spec} \mathcal{O}_K$. The set $X(\mathbb{C})$ of complex points of the $\mathbb{Z}$-scheme X may be identified with the set of field embeddings of K into $\mathbb{C}$. Consequently, we have a canonical isomorphism of $\mathbb{C}$-algebras:

$$K_{\mathbb{C}} := K \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}^{X(\mathbb{C})}, \quad a \otimes \lambda \mapsto (x(a)\lambda)_{x \in X(\mathbb{C})}. \tag{1.4.4}$$

The $\mathbb{C}$-algebra $K_{\mathbb{C}}$ is equipped with a distinguished $\mathbb{C}$-antilinear automorphism, namely the "complex conjugation" which maps $a \otimes \lambda \in K \otimes_{\mathbb{Q}} \mathbb{C}$ to $a \otimes \overline{\lambda}$. Under the isomorphism (1.4.4), it becomes the $\mathbb{C}$-antilinear involution

$$\mathbb{C}^{X(\mathbb{C})} \xrightarrow{\sim} \mathbb{C}^{X(\mathbb{C})}, \quad (a_x)_{x \in X(\mathbb{C})} \mapsto (\overline{a_{\overline{x}}})_{x \in X(\mathbb{C})}.$$

Its set of fixed points is the $\mathbb{R}$-subalgebra of $K_{\mathbb{C}}$:

$$K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2},$$

where, as usual, r_1 (resp. r_2) denotes the number of real (resp. complex) places of K.

Observe that the fields K and $\mathbb{R}$ are canonically embedded in $K_{\mathbb{R}}$ by the morphisms $(a \mapsto a \otimes 1_{\mathbb{Q}})$ and $(\lambda \mapsto 1_K \otimes \lambda)$ respectively, and that the subring $\mathcal{O}_K$ of $K_{\mathbb{R}}$ generates $K_{\mathbb{R}}$ as a real vector space. Moreover, for any $\mathcal{O}_K$-module, we have canonical isomorphisms:

$$M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} M \otimes_{\mathcal{O}_K} (\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{R}) = M \otimes_{\mathcal{O}_K} K_{\mathbb{R}}.$$

In particular $M_{\mathbb{R}}$ is naturally endowed with the structure of a $K_{\mathbb{R}}$-vector space.

Together with Proposition 1.3.4, Corollary 1.3.4 and Proposition 1.4.2, these observations easily imply the following amplification of Proposition 1.4.2, which we spell out for future reference.

THEOREM 1.4.4. *Let M be a countably generated $\mathcal{O}_K$-module, let*

$$\iota : M \longrightarrow M_{\mathbb{R}}, \quad m \longmapsto m \otimes 1_{\mathbb{R}}$$

be the canonical morphism from M to $M_{\mathbb{R}}$. Identify $M_{/\mathrm{tor}}$ with the image of ι.

The closure $\overline{M_{/\mathrm{tor}}}$ of $M_{/\mathrm{tor}}$ in $M_{\mathbb{R}}$ endowed with its inductive topology is an $\mathcal{O}_K$-submodule of $M_{\mathbb{R}}$, and its connected component $M^0_{/\mathrm{tor}}$ is a $K_{\mathbb{R}}$-submodule of $M_{\mathbb{R}}$. Moreover, the map ι defines a commutative diagram of $\mathcal{O}_K$-modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M_{\mathrm{ap}} \arrow[r] \arrow[d, "\iota_{|M_{\mathrm{ap}}}"] & M \arrow[r, "\delta_M"] \arrow[d, "\iota"] & M^{\vee\vee} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \overline{M_{/\mathrm{tor}}}^0 \arrow[r] & \overline{M_{/\mathrm{tor}}} \arrow[r] & \overline{M_{/\mathrm{tor}}}/\overline{M_{/\mathrm{tor}}}^0 \arrow[r] & 0.
\end{tikzcd}$$

The arguments leading to Theorem 1.4.4 may actually be used to give an alternate proof of Theorem 1.2.2 for countably generated modules over $\mathcal{O}_K$.

1.5. Torsion-free Modules over Valued Rings and Non-archimedean Seminorms

1.5.1. Torsion-free modules over discrete valuation rings and nonarchimedean seminorms. In this subsection and the next one, we assume that R is a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K.

We shall denote by

$$v : K \longrightarrow \mathbb{Z} \cup \{+\infty\}$$

the associated valuation on K, by ϖ a uniformizer of R – namely, an element of $v^{-1}(1) = \mathfrak{m} \setminus \mathfrak{m}^2$ – by q a real number with $q > 1$, and by $|.|$ the absolute value on K defined by setting

$$|x| = q^{-v(x)} \in \mathbb{R}_+$$

for any $x \in K$. The value set of K is defined as

$$|K| = q^{\mathbb{Z}} \cup \{0\}.$$

1.5.1.1. Let V be a K-vector space. To V, we may associate the set $\mathcal{M}(V)$ of R-submodules M of V that satisfy the following equivalent conditions:

(i) M generates the K-vector space V ;
(ii) $V = \bigcup_{n \geq 0} \varpi^{-n} M$;
(iii) the K-linear map

$$M \otimes_R K \longrightarrow V, \quad m \otimes \lambda \longmapsto \lambda m$$

is an isomorphism.

Definition 1.5.1. A *ultrametric* $|.|$-*seminorm*, or, for short, a *seminorm* on V is a map

$$||.|| : V \longrightarrow \mathbb{R}_+$$

such that the following conditions hold:

- **NA**: for any $(v, w) \in V^2$, $||v + w|| \leq \max(||v||, ||w||)$;
- **H**: for any $(\lambda, v) \in K \times V$, $||\lambda v|| = |\lambda| \, ||v||$.

We will denote by $\mathcal{N}(V)$ the set of ultrametric seminorms

$$||.|| : V \longrightarrow |K|$$

with values in $|K|$ on the K-vector space V.

Proposition 1.5.2. *The map*

$$\mu_V : \mathcal{N}(V) \longrightarrow \mathcal{M}(V), \quad ||.|| \longmapsto M_{||.||} := \big\{ v \in V | \, ||v|| \leq 1 \big\}$$

is a bijection. Its inverse sends an R-submodule M in $\mathcal{M}(V)$ to the seminorm $||.||_M$ defined by

$$||v||_M = \inf \big\{ |\lambda|, \, \lambda \in K, \, v \in \lambda M \big\}.$$

The proof follows easily from the definitions of $\mathcal{M}(V)$ and $\mathcal{N}(V)$, and is left to the reader.

Observe that, for any $v \in V$, we have $||v||_M = q^{-s}$, where

$$s = \sup \big\{ k \in \mathbb{Z}, \, v \in \varpi^k M \big\} \in \mathbb{Z} \cup \big\{ +\infty \big\}.$$

In particular,

$$||v||_M = 0 \Longleftrightarrow Kv \subseteq M.$$

When $||v||_M \neq 0$, there exists a unique integer k such that v belongs to $\varpi^k M \setminus \varpi^{k+1} M$, and

$$||v||_M = |\varpi^k| = q^{-k}.$$

For any two submodules M_1 and M_2 in $\mathcal{M}(V)$, we have:

$$M_1 \subseteq M_2 \Longleftrightarrow ||.||_{M_2} \leq ||.||_{M_1}.$$

Moreover, $M_1 \cap M_2$ and $M_1 + M_2$ also belong to $\mathcal{M}(V)$,. The seminorm

$$||.||_{M_1 \cap M_2} = \max\left(||.||_{M_1}, ||.||_{M_2}\right)$$

(resp. $||.||_{M_1+M_2}$) is the smallest (resp. largest) seminorm $||.||$ on V such that $||.||_{M_1} \leq ||.||$ and $||.||_{M_2} \leq ||.||$ (resp. $||.||_{M_1} \geq ||.||$ and $||.||_{M_2} \geq ||.||$).

The bijection between $\mathcal{N}(V)$ and $\mathcal{M}(V)$ described in Proposition 1.5.2 is easily seen to compatible with restriction to and quotient by vector spaces, in the following sense.

Consider a K-vector subspace V' of V.

If M is an element of $\mathcal{M}(V)$ and $\|.\| := \mu_V(M)$ denotes the associated seminorm in $\mathcal{N}(V)$, then $M \cap V'$ is an element of $\mathcal{M}(V')$ and its image $\mu_{V'}(M \cap V')$ in $\mathcal{M}(V')$ is the seminorm $\|.\|_{|V'}$ restriction of $\|.\|$ to V'.

Moreover the image $p(M)$ of M by the quotient K-linear map:

$$p : V \longrightarrow V/V'$$

belongs to $\mathcal{M}(V/V')$, and the associated seminorm $\mu_{V/V'}(p(M))$ in $\mathcal{N}(V/V')$ is the quotient seminorm $\|.\|^\sim$ on V/V' defined by:

$$\|w\|^\sim := \inf_{v \in p^{-1}(w)} \|v\|.$$

1.5.1.2. Every seminorm $||.||$ on V endows V with the structure of a topological K-vector space. If V' is another K-vector space endowed with a seminorm $||.||'$, and if $\varphi : V \to V'$ is a K-linear map, then φ is continuous, when V and V' are equipped with the topology defined by $||.||$ and $||.||'$ respectively, if and only if the *operator norm*[5] of φ, defined as

$$|||\varphi||| = \sup_{v \in V, ||v|| \leq 1} ||\varphi(v)||' \in \mathbb{R}_+ \cup \{+\infty\}$$

is finite.

Now assume that $||.||$ and $||.||'$ both take value in $|K|$, and let

$$M = \mu_V(||.||), \ M' = \mu_{V'}(||.||').$$

Observe that there is a canonical embedding

$$\mathrm{Hom}_R(M, M') \hookrightarrow \mathrm{Hom}_K(V, V')$$

that maps $\psi \in \mathrm{Hom}_R(M, M')$ to

$$\psi \otimes_R \mathrm{Id}_K : V \simeq M \otimes_R K \longrightarrow M' \otimes_R K = V'.$$

The following proposition is a straightforward consequence of the definitions.

PROPOSITION 1.5.3. *With the notation above, the R-submodule* $\mathrm{Hom}_R(M, M')$ *of* $\mathrm{Hom}_K(V, V')$ *coincides with*

$$\left\{\varphi \in \mathrm{Hom}_K(V, V') \mid |||\varphi||| \leq 1\right\}.$$

Moreover the K-vector subspace of $\mathrm{Hom}_K(V, V')$ *generated by* $\mathrm{Hom}_R(M, M')$*, i.e., the K-vector space*

$$\mathrm{Hom}_R(M, M') \otimes_R K = \bigcup_{n \geq 0} \varpi^{-n} \mathrm{Hom}_R(M, M'),$$

is the space

$$\mathrm{Hom}^{\mathrm{cont}}_K(V, V') = \left\{\varphi \in \mathrm{Hom}_K(V, V') \mid |||\varphi||| < +\infty\right\}$$

of continuous K-linear maps $\varphi : V \to V'$.

Finally observe that a R-submodule of K is either $K = \bigcup_{k \geq 0} \varpi^{-k} R$ or of the form αR for some $\alpha \in K$, and therefore is countably generated. This easily implies:

[5]Since $|||\varphi|||$ may take the value $+\infty$, the terminology *operator quasinorm* would be more appropriate.

PROPOSITION 1.5.4. *A torsion-free R module M is countably generated if and only if the K-vector space $V := M \otimes_R K$ admits a countable basis.*

This proposition is actually a special case of Corollary 1.1.7.

1.5.2. Compatibility with completion. Let us keep the notation of the previous subsection, and let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. It is a discrete valuation ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$. Its fraction field may be identified with the completion $\widehat{K}$ of K with respect to the absolute value $|.|$.

We shall still denote by v the $\widehat{\mathfrak{m}}$-adic valuation on $\widehat{K}$, and by $|.| = q^{-v}$ the absolute value on $\widehat{K}$. Note that the absolute value $|.|$ has the same value set $q^{\mathbb{Z}} \cup \{0\}$ on K and $\widehat{K}$.

1.5.2.1. Extending scalars from K to $\widehat{K}$, the K-vector space V defines a $\widehat{K}$-vector space:

$$V_{\widehat{K}} := V \otimes_K \widehat{K},$$

in which V is naturally embedded via the map:

$$V \hookrightarrow V_{\widehat{K}}, \quad v \longmapsto v \otimes 1.$$

Consequently we may form the following commutative diagram:

$$\begin{tikzcd} \mathcal{N}(V_{\widehat{K}}) \arrow[r, ".|_V"] \arrow[d, "\mu_{V_{\widehat{K}}}", "\wr"'] & \mathcal{N}(V) \arrow[d, "\mu_V", "\wr"'] \\ \mathcal{M}(V_{\widehat{K}}) \arrow[r, ".\cap V"] & \mathcal{M}(V), \end{tikzcd} \tag{1.5.1}$$

where the first horizontal map sends a seminorm $\|.\|$ on $V_{\widehat{K}}$ to its restriction $||.||_{|V}$ to V, and the second one maps a $\widehat{R}$-submodule $\widehat{M}$ of $V_{\widehat{K}}$ to $\widehat{M} \cap V$.

Let M be an element of $\mathcal{M}(V)$. Making use of flatness of $\widehat{R}$ over R and the canonical isomorphism $K \otimes_R \widehat{R} \xrightarrow{\sim} \widehat{K}$, we get an injective map:

$$M_{\widehat{R}} := M \otimes_R \widehat{R} \hookrightarrow V \otimes_R \widehat{R} \simeq V \otimes_K \widehat{K} = V_{\widehat{K}}.$$

Moreover it is clear that the $\widehat{R}$-submodule $M_{\widehat{R}}$ of $V_{\widehat{K}}$ defined above generates the $\widehat{K}$-vector space $V_{\widehat{K}}$. As a consequence, the construction above defines a map:

$$. \otimes_R \widehat{R} : \mathcal{M}(V) \longrightarrow \mathcal{M}(V_{\widehat{K}}), \quad M \longmapsto M_{\widehat{R}}.$$

PROPOSITION 1.5.5. *The maps $._{|V}$ and $. \cap V$ in* (1.5.1) *are bijective. Moreover, the maps $. \cap V$ and $. \otimes_R \widehat{R}$ are inverse to one another.*

PROOF. Since $(\widehat{K}, |.|)$ is complete, any separated topological vector space over $(\widehat{K}, |.|)$ with finite dimension n is isomorphic to $\widehat{K}^n$; see for instance [**Bou81**, Section I.2.3, Théorème 1, p. I.14]. In other words, any finite-dimensional $\widehat{K}$-vector space E admits a unique structure of separated topological vector space over $(\widehat{K}, |.|)$. Moreover any $|.|$-seminorm on E equipped with this canonical topology is continuous.

Let F be a finite-dimensional K-subvector space of V. Then F is dense in $F_{\widehat{K}}$ endowed with its topology of separated topological vector space. As a consequence, for any $||.||$ in $\mathcal{N}(V_{\widehat{K}})$, the restriction $||.||_{F_{\widehat{K}}}$ of $||.||$ to $F_{\widehat{K}}$ is uniquely determined by its restriction $||.||_{|F}$ to F. As a consequence, $||.||$ is determined by its restriction to V. This proves that the map $._{|V}$ in (1.5.1) is injective.

To complete the proof of the proposition, we are left to showing that, for any M in $\mathcal{M}(V)$, the following equality holds:

$$M_{\widehat{R}} \cap V = M. \tag{1.5.2}$$

Indeed, since $_{|V}$ is injective, (1.5.1) shows that $. \cap V$ is injective as well. If (1.5.2) holds, then in particular $. \cap V$ is surjective, so that it is a bijection. The remaining statements then follow directly from (1.5.1).

To prove (1.5.2), observe that the diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, hook] & K \arrow[d, hook] \\ \widehat{R} \arrow[r, hook] & \widehat{K} \end{tikzcd} \tag{1.5.3}$$

is both a cartesian and a cocartesian diagram of injective morphisms of R-modules. Since any torsion-free R-module is flat, the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & V \arrow[d] \\ M_{\widehat{R}} \arrow[r] & V_{\widehat{K}} \end{tikzcd}$$

deduced from (1.5.3) by applying the functor $.\otimes_R M$ is cartesian and cocartesian as well, with all the maps in the diagram being injective. In particular, M is the intersection of $M_{\widehat{R}}$ and V in $V_{\widehat{K}}$. □

1.5.2.2. The fact that the maps $.\cap V$ and $.\otimes_R \widehat{R}$ are inverse to one another is essentially equivalent to the following classical descent results, in the special case of torsion-free R-modules:

Proposition 1.5.6 ([**BLR90**, Section 6.2, Proposition D.4 (a)]). *The functor which associates to each R-module M the triple $(M_K, M_{\widehat{R}}, \tau)$ where M_K and $M_{\widehat{R}}$ are defined by extension of scalar:*

$$M_K := M \otimes_R K, \quad M_{\widehat{R}} := M \otimes_R \widehat{R}$$

and where

$$\tau : M_K \otimes_K \widehat{K} \xrightarrow{\sim} M_{\widehat{R}} \otimes_{\widehat{R}} \widehat{K}$$

is the canonical isomorphism, is an equivalence of categories from the category of R-modules to the category of pairs of K- and $\widehat{R}$-modules endowed with some glueing datum over $\widehat{K}$.

For later reference, it is convenient to observe that the results in this subsection and in the previous one establish *an equivalence between the category of torsion-free R-modules and the category whose objects are the pairs $(V, \|.\|)$ consisting in a K-vector space V and an ultrametric $|.|$-seminorm on the $\widehat{K}$-vector space $V_{\widehat{K}} := V \otimes_K \widehat{K}$:*

$$\|.\| : V_{\widehat{K}} \longrightarrow |K|,$$

and whose morphisms are K-linear maps that, after base change to $\widehat{K}$, have operator norm ≤ 1.

In this equivalence, to the pair $(V, \|.\|)$ is associated the R-module:

$$M := \big\{v \in V \mid \|v\| \leq 1\big\}. \tag{1.5.4}$$

Moreover M is a countably generated R-module if and only if the K vector space V has a countable basis.

We leave it as an exercise to the reader to establish the following additional property of this equivalence of category:

Proposition 1.5.7. *Assume that the K-vector space V has a countable basis. Then the countably generated R-module M defined by* (1.5.4) *is a free if and only if $\|.\|$ is a norm on $V_{\widehat{K}}$, namely if and only if:*

$$\big\{v \in V_{\widehat{K}} \mid \|v\| = 0\big\} = \{0\}.$$

Observe that, when $K \neq \widehat{K}$, the fact that the restriction $\|.\|_{|V}$ is a norm on V does not imply in general that $\|.\|$ is a norm on $V_{\widehat{K}}$.[6]

1.5.3. The case of a non-discrete valuation of rank 1. In this subsection, we discuss how the correspondence described in Proposition 1.5.2 may be extended to the case of a non-discrete valuation. This framework is arguably a better "model" for Archimedean places of number fields than the one of discrete valuations previously considered.

Let K be a field endowed with a non-archimedean absolute value:

$$|.| : K \longrightarrow \mathbb{R}_+.$$

Let R be the valuation ring of $(K, |.|)$: it is the subring of K consisting of those $x \in K$ with $|x| \leq 1$. Let I be the maximal ideal of R, that is,

$$I = \{x \in K | \, |x| < 1\}.$$

We assume that the value group $|K^*|$ is a dense subgroup of $\mathbb{R}_+^*$. This is equivalent to R not being a discrete valuation ring.

Let V be a K-vector space. As in 1.5.1, let $\mathcal{M}(V)$ be the set of R-submodules M of V such that M generates the K-vector space V. Let $\mathcal{N}(V)$ be the set of non-archimedean $|.|$-seminorms

$$||.|| : V \longrightarrow \mathbb{R}_+$$

on the K-vector space V. We do not require those elements of $\mathcal{N}(V)$ to take values in $|K|$.

As in 1.5.1 again, there are natural maps

$$\mu_V : \mathcal{N}(V) \longrightarrow \mathcal{M}(V), \quad ||.|| \longmapsto M_{||.||} := \big\{v \in V | \, ||v|| \leq 1\big\}$$

and

$$\nu_V : \mathcal{M}(V) \longrightarrow \mathcal{N}(V), \; M \mapsto ||.||_M$$

where the seminorm $||.||_M$ is defined by

$$||v||_M = \inf \big\{|\lambda|, \, \lambda \in K, \, v \in \lambda M\big\}.$$

PROPOSITION 1.5.8. *The composition*

$$\nu_V \circ \mu_V : \mathcal{N}(V) \longrightarrow \mathcal{N}(V)$$

is $\mathrm{Id}_{\mathcal{N}(V)}$.

PROOF. Let $||.||$ be an element of $\mathcal{N}(V)$, and let M be the R-submodule of V:

$$M = \mu_V(||.||).$$

Then by definition:

$$M = \big\{v \in V | \, ||v|| \leq 1\big\}$$

so that, for any nonzero $\lambda \in K$, we have:

$$\lambda M = \big\{v \in V | \, ||v|| \leq \lambda\big\}.$$

Let $||.||'$ be the archimedean seminorm:

$$||.||' = \nu_V(M) = (\nu_V \circ \mu_V)(||.||).$$

Then by definition, for any $v \in V$, we have:

$$||v||' = \inf \big\{|\lambda|, \, \lambda \in K, \, v \in \lambda M\big\} = \inf \big\{|\lambda|, \, ||v|| \leq \lambda\big\} = ||v||,$$

the last equality holding because $|K|$ is dense in $\mathbb{R}_+$. □

[6] Consider for instance $V := K^2$ and the seminorm $\|.\|$ on $V_{\widehat{K}} = \widehat{K}^2$ defined by $\|(x, y)\| := |y - \alpha x|$ for some $\alpha \in \widehat{K} \setminus K$.

In general, it is not true that $\mu_V \circ \nu_V = \mathrm{Id}_{\mathcal{M}(V)}$. It follows from the definitions that if M is an element of $\mathcal{M}(V)$, we have:

$$(\mu_V \circ \nu_V)(M) = \{v \in V | \forall \lambda \in K, |\lambda| > 1 \implies v \in \lambda M\}. \tag{1.5.5}$$

In particular, we have an inclusion:

$$M \subset (\mu_V \circ \nu_V)(M).$$

Define a set $\mathcal{M}^a(V)$ as the quotient of $\mathcal{M}(V)$ by the equivalence relation $\sim$ defined by the following: if M and N are elements of $\mathcal{M}(V)$, $M \sim N$ if and only if

$$I(M+N)/(M \cap N) = 0,$$

i.e.

$$I(M+N) \subset M \cap N.$$

The set $\mathcal{M}^a(V)$ may be understood more conceptually as follows: since $|.|$ is not discrete, the maximal ideal I of R satisfies $I^2 = I$. It is clearly flat over R. As a consequence, we may consider the category of *almost R-modules* as in [**GR03**]. Then $\mathcal{M}^a(V)$ is the set of isomorphism classes of almost R-submodules of V that generate V as a K-vector space.

Let $p : \mathcal{M}(V) \to \mathcal{M}^a(V)$ be the quotient map, and let

$$\mu_V^a : \mathcal{N}(V) \longrightarrow \mathcal{M}^a(V)$$

be the composition $\mu_V^a = p \circ \mu_V$.

Lemma 1.5.9. *The map*

$$\nu_V : \mathcal{M}(V) \longrightarrow \mathcal{N}(V)$$

factors through the quotient map

$$p : \mathcal{M}(V) \longrightarrow \mathcal{M}^a(V).$$

Proof. Let M and N be two elements of $\mathcal{M}(V)$ with $M \sim N$. We need to show the equality

$$\nu_V(M) = \nu_V(N).$$

Note that $M \sim M \cap N \sim M + N$. In particular, we may assume that there is an inclusion $M \subset N$. Since $M \sim N$, we have $IN \subset M$. Write $||.||_M = \nu_V(M)$, $||.||_N = \nu_V(N)$.

Let v be an element of V. For any element λ of K, if v lies in λM, then v lies in λN. In particular, we have the inequality:

$$||v||_M = \inf\{|\lambda|, \lambda \in K, v \in \lambda M\} \leq \inf\{|\lambda|, \lambda \in K, v \in \lambda N\} = ||v||_N.$$

To prove the converse inequality, choose $\varepsilon > 0$. Let λ be an element of K with

$$|\lambda| \leq ||v||_N + \varepsilon$$

and $v \in \lambda N$. Let μ be an element of I. Since $\mu N \subset N$, we have:

$$\mu v \in \lambda \mu N \subset \lambda M,$$

so that:

$$|\mu|\, ||v||_M = ||\mu v||_M \leq |\lambda| \leq \varepsilon + ||v||_N.$$

Since $|\mu|$ may be chosen as close to 1 as needed, this proves:

$$||v||_M \leq ||v||_N.$$

□

Using Lemma 1.5.9, let
$$\nu_V^a : \mathcal{M}^a(V) \longrightarrow \mathcal{N}(V)$$
be the unique map through which ν_V factors.

The following statement is the variant of Proposition 1.5.2 that holds in the non-discrete case.

PROPOSITION 1.5.10. *The maps*
$$\mu_V^a : \mathcal{N}(V) \longrightarrow \mathcal{M}^a(V)$$
and
$$\nu_V^a : \mathcal{M}^a(V) \longrightarrow \mathcal{N}(V)$$
defined above are bijections. They are the inverse of one another.

PROOF. By Proposition 1.5.8, we have $\nu_V \circ \mu_V = \mathrm{Id}_{\mathcal{N}(V)}$. This readily implies the equality:
$$\nu_V^a \circ \mu_V^a = \mathrm{Id}_{\mathcal{N}(V)}.$$

We prove the equality:
$$\mu_V^a \circ \nu_V^a = \mathrm{Id}_{\mathcal{M}^a(V)}.$$
This amounts to proving that if M is an element of $\mathcal{M}(V)$, then:
$$(\mu_V \circ \nu_V)(M) \sim M.$$
Let $N = (\mu_V \circ \nu_V)(M)$. Recall from (1.5.5) that we have:
$$N = \big\{v \in V \,|\, \forall \lambda \in K,\, |\lambda| > 1 \implies v \in \lambda M\big\}.$$
We have $M \subset N$. Let μ be an element of I, and let v be an element of N. If $\mu = 0$, then $\mu v \in M$. If $\mu \neq 0$, then $|\mu^{-1}| > 1$, so that v belongs to $\mu^{-1}M$ and μv belongs to M. This proves that $IN \subset M$, and finally that $M \sim N$. □

1.6. Localization and Ultrametric Seminorms

In this section, we consider a Dedekind ring R that is not a field, the associated affine scheme:
$$X := \operatorname{Spec} R,$$
and a nonempty open subscheme $\mathring{X}$ of X. This subscheme is the complement:
$$\mathring{X} = X \setminus \Sigma$$
of a finite set Σ of closed points of X, and is actually affine, defined by a Dedekind ring $\mathring{R} := \mathcal{O}_X(\mathring{X})$ which satisfies:
$$R \subseteq \mathring{R} \subseteq K,$$
where K denotes the faction field of R.

We want to discuss the propertis of the base change[7] of modules from R to $\mathring{R}$ — which geometrically corresponds to restricting to $\mathring{X}$ quasi-coherent sheaves over X — and particular its compatibility with duality and with the construction of the largest projective quotient. We will notably emphasize the role of the ultrametric $|.|_{\mathfrak{p}}$-seminorms associated to the $\mathfrak{p}$-adic absolute values $|.|_{\mathfrak{p}}$ on K associated to the elements $\mathfrak{p}$ of Σ.

Throughout this section, we shall use the following notation.

We shall denote by $\eta := \operatorname{Spec} K$ the generic point of X and $\mathring{X}$. The elements of
$$X_0 := |X| \setminus \{\eta\}$$
are the closed points of X, or equivalently the nonzero prime ideals of R.

[7]We will refer to this base change as "localization", although strcitly speaking the ring $\mathring{R}$ is not always a localization of R. However, when K is a number field and R its ring of integers $\mathcal{O}_K$, $\mathring{R}$ is always a localization of $\mathcal{O}_K$, as a consequence of the finiteness of the ideal class group of K.

For every $\mathfrak{p} \in X_0$, we denote by $R_{(\mathfrak{p})}$ the discrete valuation ring $\mathcal{O}_{X,\mathfrak{p}}$, that is, the localization of R at $\mathfrak{p}$, by $R_{\mathfrak{p}}$ (resp. $K_{\mathfrak{p}}$) the $\mathfrak{p}$-adic completion of $R_{(\mathfrak{p})}$ (resp. K), and by $v_{\mathfrak{p}}$ the $\mathfrak{p}$-adic valuation on $K_{\mathfrak{p}}$. We also choose a real number $q_{\mathfrak{p}} > 1$, and we define the $\mathfrak{p}$-adic absolute value on $K_{\mathfrak{p}}$ associated to $v_{\mathfrak{p}}$ as:

$$|.|_{\mathfrak{p}} := q_{\mathfrak{p}}^{-v_{\mathfrak{p}}}.$$

We have:

$$R \subseteq R_{(\mathfrak{p})} \subseteq K, \quad \text{for every } \mathfrak{p} \in X_0,$$

and:

$$R = \bigcap_{\mathfrak{p} \in X_0} R_{(\mathfrak{p})} = \big\{x \in K \mid \forall\, \mathfrak{p} \in X_0, |x|_{\mathfrak{p}} \leq 1\big\}.$$

Similarly we have:

$$\mathring{R} \subseteq R_{(\mathfrak{p})} \subset K, \quad \text{for every } \mathfrak{p} \in X_0 \setminus \Sigma,$$

and:

$$\mathring{R} = \bigcap_{\mathfrak{p} \in X_0 \setminus \Sigma} R_{(\mathfrak{p})} = \big\{x \in K \mid \forall\, \mathfrak{p} \in X_0 \setminus \Sigma, |x|_{\mathfrak{p}} \leq 1\big\},$$

and consequently:

$$R = \big\{x \in \mathring{R} \mid \forall\, \mathfrak{p} \in \Sigma, |x|_{\mathfrak{p}} \leq 1\big\}.$$

If M be a R-module, and if $\widetilde{M}$ denotes the associated quasi-coherent sheaf over X. We shall say that *the R-module has no Σ-torsion* when the following equivalent properties are satisfied:

(i) the R-module $\Gamma_{\Sigma}(X, \widetilde{M})$ of sections of $\widetilde{M}$ over X that are supported by Σ vanishes;
(ii) for every $\mathfrak{p} \in \Sigma$, the $R_{(\mathfrak{p})}$-module $M_{R_{(\mathfrak{p})}}$ is torsion free;
(iii) for every $\mathfrak{p} \in \Sigma$, the $R_{\mathfrak{p}}$-module $M_{R_{\mathfrak{p}}}$ is torsion free;
(iv) the canonical map

$$M \longrightarrow M_{\mathring{R}}, \quad m \longmapsto m \otimes_R 1_{\mathring{R}}$$

is injective.

This section has been included mainly[8] to provide a motivation for the construction of suitable categories of Hermitian quasi-coherent sheaves over arithmetic curves in the next chapter. From this perspective, the finite set of closed points Σ has to be thought as the analogue of Archimedean places in the arithmetic framework.

The results in this section will not be used in the remaining of this monograph, and some details of their proofs are left to the interested reader.

1.6.1. Seminormed $\mathring{R}$-modules and R-modules. We may define the category $\overline{\mathbf{Mod}}^{\leq 1}_{\mathring{R},\Sigma}$ of Σ-*seminormed $\mathring{R}$-modules* as follows.

The objects of $\overline{\mathbf{Mod}}^{\leq 1}_{\mathring{R},\Sigma}$ are pairs:

$$\overline{\mathring{M}} := (\mathring{M}, (\|.\|_{\mathfrak{p}})_{\mathfrak{p} \in \Sigma})$$

consisting in a $\mathring{R}$-module $\mathring{M}$ and in a family $(\|.\|_{\mathfrak{p}})_{\mathfrak{p} \in \Sigma}$ of ultrametric $|.|_{\mathfrak{p}}$-seminorms:

$$\|.\|_{\mathfrak{p}} : M_{K_{\mathfrak{p}}} \longrightarrow |K_{\mathfrak{p}}|_{\mathfrak{p}} := q_{\mathfrak{p}}^{\mathbb{Z}} \cup \{0\}$$

on the $K_{\mathfrak{p}}$-vector spaces $M_{K_{\mathfrak{p}}}$.

A morphism in $\overline{\mathbf{Mod}}^{\leq 1}_{\mathring{R},\Sigma}$ from $\overline{\mathring{M}}$ to another object

$$\overline{\mathring{M}'} := (\mathring{M}', (\|.\|'_{\mathfrak{p}})_{\mathfrak{p} \in \Sigma})$$

---

[8] and secondarily to discuss the compatibility with localization of the construction of the largest projective quotient in Section 1.4, which is a slightly delicate issue.

of $\overline{\mathbf{Mod}}^{\leq 1}_{\mathring{R},\Sigma}$ is a morphism of $\mathring{R}$-modules:

$$\varphi : \mathring{M} \longrightarrow \mathring{M}'$$

such that, for every $\mathfrak{p} \in \Sigma$, the operator norm $|||\varphi_{K_\mathfrak{p}}|||_\mathfrak{p}$ of the $K_\mathfrak{p}$-linear map $\varphi_{K_\mathfrak{p}}$ between the seminormed $K_\mathfrak{p}$-vector spaces $(\mathring{M}_{K_\mathfrak{p}}, \|.\|_p)$ and $(\mathring{M}'_{K_\mathfrak{p}}, \|.\|'_p)$ satisfies:

$$|||\varphi_{K_\mathfrak{p}}|||_\mathfrak{p} \leq 1. \tag{1.6.1}$$

We may also introduce a category $\overline{\mathbf{Mod}}_{\mathring{R},\Sigma}$ with the same objects as $\overline{\mathbf{Mod}}^{\leq 1}_{\mathring{R},\Sigma}$, where, in the definition of morphisms, condition (1.6.1) is replaced by the following weaker one:

$$|||\varphi_{K_\mathfrak{p}}|||_\mathfrak{p} < +\infty. \tag{1.6.2}$$

To any R-module M, we may attach the object $\overline{\mathring{M}}$ of $\overline{\mathbf{Mod}}^{\leq 1}_{\mathring{R},\Sigma}$ defined by the $\mathring{R}$-module:

$$\mathring{M} := M_{\mathring{R}}$$

and by the seminorms $(\|.\|_\mathfrak{p})_{\mathfrak{p}\in\Sigma}$ on the $K_\mathfrak{p}$-vector spaces:

$$\mathring{M}_{K_\mathfrak{p}} \simeq M_{K_\mathfrak{p}}$$

attached to the image in $M_{K_\mathfrak{p}}$ of the $R_\mathfrak{p}$-module $M_{R_\mathfrak{p}}$ by the inverse of the bijection:

$$\mu_{M_{K_\mathfrak{p}}} : \mathcal{N}(M_{K_\mathfrak{p}}) \xrightarrow{\sim} \mathcal{M}(M_{K_\mathfrak{p}})$$

investigated in Subsections 1.5.1 and 1.5.2.

This constructions actually defines a functor from $\mathbf{Mod}_R$ to $\overline{\mathbf{Mod}}^{\leq 1}_{\mathring{R},\Sigma}$, by sending a morphism

$$\psi : M \longrightarrow M'$$

of R-module to its base change:

$$\varphi := \psi_{\mathring{R}} : M_{\mathring{R}} \longrightarrow M'_{\mathring{R}},$$

which indeed satisfies conditions (1.6.1), since $\varphi_{K_\mathfrak{p}}$ sends the image in $M_{K_\mathfrak{p}}$ of $M_{R_\mathfrak{p}}$ to the image in $M'_{K_\mathfrak{p}}$ of $M'_{R_\mathfrak{p}}$.

Recall that basic descent theory shows that the category of quasi-coherent sheaves on X is equivalent to the category of quasi-coherent sheaves $(\mathring{\mathcal{F}}, (\mathcal{F}_\mathfrak{p})_{\mathfrak{p}\in\Sigma})$ on the scheme

$$\mathring{X} \amalg \coprod_{\mathfrak{p}\in\Sigma} \operatorname{Spec} R_{(\mathfrak{p})},$$

equipped with "glueing data," namely with isomorphisms of $K_\mathfrak{p}$-vector spaces:

$$\mathring{\mathcal{F}}_{K_\mathfrak{p}} \xrightarrow{\sim} \mathcal{F}_{\mathfrak{p},K_\mathfrak{p}}, \quad \text{for } \mathfrak{p} \in \Sigma.$$

By combining this equivalence of category with the descriptions of torsion free-modules over discrete valuation rings and of their morphisms in terms of seminorms in Subsections 1.5.1 and 1.5.2, we obtain:

Proposition 1.6.1. *Restricted to the full subcategory of* $\mathbf{Mod}_R$ *defined by modules with no* Σ*-torsion, the functor from* $\mathbf{Mod}_R$ *to* $\overline{\mathbf{Mod}}^{\leq 1}_{\mathring{R},\Sigma}$ *constructed above is an equivalence of category.*

1.6.2. Seminorms and Duality. Let M be an R-module. We may consider its dual,

$$M^\vee := \mathrm{Hom}_R(M, R),$$

which defines an object of $\mathbf{Mod}_R^{\mathrm{top}}$, and the dual of the K-vector space M_K,

$$M_K^\vee := \mathrm{Hom}_K(M_K, K),$$

which defines an object of $\mathbf{Mod}_K^{\mathrm{top}}$.

The topology of $M^\vee$ and of $M_K^\vee$ is the topology of pointwise convergence, defined by the discrete topology on R and on K.[9] The natural map, defined by extension of scalars:

$$M^\vee \longrightarrow M_K^\vee, \quad \xi \longmapsto \xi_K := \xi \otimes_R \mathrm{Id}_K,$$

is clearly injective, and a homeomorphism onto its image, which is the closed R-submodule of $M_K^\vee$ consisting in the K-linear forms η in $M_K^\vee$ such that the image $\eta(M_{/\mathrm{tor}})$ of $M_{/\mathrm{tor}}$ is contained in R.[10]

For every $\mathfrak{p} \in X_0$, we may consider the ultrametric $|.|_\mathfrak{p}$-seminorm

$$\|.\|_\mathfrak{p} : M_{K_\mathfrak{p}} \longrightarrow |K_\mathfrak{p}|_\mathfrak{p} = q_\mathfrak{p}^\mathbb{Z} \cup \{0\}$$

deduced from the image of $M_{R_\mathfrak{p}}$ in $M_{K_\mathfrak{p}}$ by the bijection $\mu_{M_{K_\mathfrak{p}}}$. Every element $\eta \in M_{K_\mathfrak{p}}^\vee := \mathrm{Hom}_{K_\mathfrak{p}}(M_{K_\mathfrak{p}}, K_\mathfrak{p})$ admits an operator norm, as defined in 1.5.1.2, namely:

$$\|\eta\|_\mathfrak{p}^\vee := \sup_{v \in M_{R_\mathfrak{p}/\mathrm{tor}}} |\eta(v)|_\mathfrak{p} = \sup_{v \in M_{R_{(\mathfrak{p})}/\mathrm{tor}}} |\eta(v)|_\mathfrak{p}.$$

Observe that $M_K^\vee$ naturally embeds in $M_{K_\mathfrak{p}}^\vee$, and that, for every $\xi \in M_K^\vee$, the following equivalence holds:

$$\xi(M_{/\mathrm{tor}}) \subseteq R_{(\mathfrak{p})} \Longleftrightarrow \xi_{K_\mathfrak{p}}(M_{R_\mathfrak{p}/\mathrm{tor}}) \subseteq R_\mathfrak{p} \Longleftrightarrow \|\xi\|_\mathfrak{p}^\vee \leq 1.$$

Consequently, the dual R-module $M^\vee$ of M, identified with its image in $M_K^\vee$, satisfies:

$$M^\vee = \big\{\xi \in M_K^\vee \mid \|\xi\|_\mathfrak{p} \leq 1\big\}.$$

1.6.3. Duality and localization.

1.6.3.1. The discussion of the previous subsection applies with the Dedekind ring R replaced by the ring $\mathring{R}$, which admits the same fraction field K.

This shows that, for every R-module M, if we let:

$$\mathring{M} := M_{\mathring{R}},$$

we have injective maps of topological modules:

$$M^\vee = \mathrm{Hom}_R(M, R) \hookrightarrow \mathring{M}^\vee := \mathrm{Hom}_{\mathring{R}}(\mathring{M}, \mathring{R}) \hookrightarrow M_K^\vee, \tag{1.6.3}$$

that are homeomorphisms onto their images, which themselves are closed in $M_K^\vee$.

The canonical morphism of $\mathring{R}$-module:

$$(M^\vee)_{\mathring{R}} := \mathrm{Hom}_R(M, R) \otimes_R \mathring{R} \longrightarrow \mathring{M}^\vee := \mathrm{Hom}_{\mathring{R}}(\mathring{M}, \mathring{R}), \tag{1.6.4}$$

that maps $\xi \otimes_R \lambda$ to $\lambda(\xi \otimes_R \mathrm{Id}_{\mathring{R}})$ for any $\xi \in M^\vee$ and any $\lambda \in \mathring{R}$, is also easily seen to be injective.

Thanks to the injective maps in (1.6.3) and in (1.6.4), we shall identify $M^\vee$, $\mathring{M}^\vee$, and $(M^\vee)_{\mathring{R}}$ to submodules of $M_K^\vee$, which therefore satisfy:

$$M^\vee \subseteq (M^\vee)_{\mathring{R}} \subseteq \mathring{M}^\vee \subseteq M_K^\vee.$$

[9] For instance, the topology of $M_K^\vee$ is the topology induced by the topology on K^{M_K} defined as the product topology of the discrete topology on each factor K.

[10] We identify $M_{/\mathrm{tor}} := M/M_{\mathrm{tor}}$ with the image of M in M_K.

PROPOSITION 1.6.2. *With the notation above, for every $\xi \in \mathring{M}^\vee$, the following equivalence hold:*

$$\xi \in M^\vee \Longleftrightarrow \forall\, \mathfrak{p} \in \Sigma,\ ||\xi||^\vee_\mathfrak{p} \leq 1, \tag{1.6.5}$$

and:

$$\xi \in (M^\vee)_{\mathring{R}} \Longleftrightarrow \forall\, \mathfrak{p} \in \Sigma,\ ||\xi||^\vee_\mathfrak{p} < +\infty. \tag{1.6.6}$$

The proof of (1.6.5) is straightforward, and also the proof of (1.6.6) when $\mathring{R}$ is of form $R[1/f]$ for some $f \in R \setminus \{0\}$ (for instance, when R is the ring of integers of some number field). We leave the details of the proof of (1.6.6) in the general situation to the interested reader.

If we introduce the "trivial rank one object":

$$\overline{\mathring{R}} := (\mathring{R}, (|.|_\mathfrak{p})_{\mathfrak{p}\in\Sigma})$$

in the category $\overline{\mathbf{Mod}}^{\leq 1}_{\mathring{R},\Sigma}$, the equivalence (1.6.5) states that $M^\vee$ may be identified with the set of morphisms from $\overline{\mathring{M}}$ to $\overline{\mathring{R}}$ in this category. This is also a consequence of the equivalence of category in Proposition 1.6.1. The equivalence 1.6.6 shows that $(M^\vee)_{\mathring{R}}$ may be identified with the set of morphisms from $\overline{\mathring{M}}$ to $\overline{\mathring{R}}$ in the category $\overline{\mathbf{Mod}}_{\mathring{R},\Sigma}$.

1.6.3.2. We now assume that the R-module M, and therefore the $\mathring{R}$-module $\mathring{M}$, is countably generated.

Then the topology of $M^\vee_K$ is metrizable, hence also the one of $M^\vee$ and of $\mathring{M}^\vee$. Actually, according to Theorem 1.2.2, the topological R-module $M^\vee$ is an object of $\mathbf{CTC}_R$, and $\mathring{M}^\vee$ is an object of $\mathbf{CTC}_{\mathring{R}}$.

We may also consider the image $M^\vee \widehat{\otimes}_R \mathring{R}$ of $M^\vee$ by the functor "completed tensor product":

$$.\widehat{\otimes}_\mathbb{R} \mathring{R} : \mathbf{CTC}_R \longrightarrow \mathbf{CTC}_{\mathring{R}};$$

it contains the "algebraic tensor product":

$$(M^\vee)_{\mathring{R}} := M^\vee \otimes_R \mathring{R}$$

as a dense $\mathring{R}$-submodule; see [**Bos20b**, Section 4.2.1].

The following proposition is established by unwinding the definitions, and by using the uniqueness of the completion of a metric space; see also [**Bos20b**, Section 4.2.1-2].

PROPOSITION 1.6.3. *As a topological $\mathring{R}$-module, the completed tensor product $M^\vee \widehat{\otimes}_R \mathring{R}$ may be identified with the closure of $(M^\vee)_{\mathring{R}}$ in $M^\vee_K$, and is contained in $\mathring{M}^\vee$.*

1.6.4. Canonical dévissage and localization. Let us still consider a countably generated R-module M. We may wish to compare the largest projective quotients of the R-module M and of the $\mathring{R}$-module $\mathring{M} := M_{\mathring{R}}$, which are defined by the biduality morphisms:

$$\delta_M : M \longrightarrow M^{\vee\vee} \quad \text{and} \quad \delta_{\mathring{M}} : \mathring{M} \longrightarrow \mathring{M}^{\vee\vee}.$$

To achieve this, observe that the base change from R to $\mathring{R}$ of δ_M defines a morphism of $\mathring{R}$-modules:

$$\delta_{M,\mathring{R}} : \mathring{M} := M_{\mathring{R}} \longrightarrow (M^{\vee\vee})_{\mathring{R}},$$

which, like δ_M, is surjective with projective range.

By applying Theorem 1.2.2 to the $\mathring{R}$-module $\mathring{M}$, we obtain the first assertion in the following proposition.

Proposition 1.6.4. *There exists a unique morphism of $\mathring{R}$-modules:*

$$p : \mathring{M}^{\vee\vee} \longrightarrow (M^{\vee\vee})_{\mathring{R}}$$

such that $\delta_{M,\mathring{R}} = p \circ \delta_{\mathring{M}}$. The morphism p is split surjective, and its kernel $\ker p$ is a countably generated projective $\mathring{R}$-module.

Moreover the topological $\mathring{R}$-module $(\ker p)^\vee$ is isomorphic to the quotient topological $\mathring{R}$-module $\mathring{M}^\vee/(M^\vee \widehat{\otimes}_R \mathring{R})$, which consequently is an object of $\mathbf{CTC}_{\mathring{R}}$.

The second assertion in Proposition 1.6.4 is straightforward, and the last one follows from Proposition 1.6.3. Combined with Propositions 1.6.2 and 1.6.3, it provides a description of $\ker p$ in terms of the seminormed $\mathring{R}$-module $(\mathring{M}, (\|.\|_\mathfrak{p})_{\mathfrak{p}\in\Sigma})$.

The existence of the surjective morphism p implies that the rank of the projective R-module $M^{\vee\vee}$ is at most the one of the projective $\mathring{R}$-module $\mathring{M}$. The following examples that p may actually vanish while $\mathring{M}$ does not, or be an isomorphism.

Example 1.6.5. Assume that Σ is nonempty and that M is $\mathring{R}$, seen as a R-module.

Then on the one hand, we have:

$$M^\vee = 0.$$

Indeed, any element of $\mathrm{Hom}_R(\mathring{R}, R)$ extends to a linear form

$$\varphi_K : \mathring{R} \otimes_R K = K \longrightarrow R \otimes_K K = K,$$

and accordingly may be written as $(x \mapsto \varphi(1)x)$. If $\mathfrak{p}$ is a prime in Σ, then, for every $x \in \mathring{R}$, the following inequality holds in $\mathbb{Z} \cup \{+\infty\}$:

$$v_\mathfrak{p}(\varphi(1)) + v_\mathfrak{p}(x) = v_p(\varphi(x)) \geq 0.$$

Since $v_\mathfrak{p}(S) = \mathbb{Z}$, this implies the equality $v_\mathfrak{p}(\varphi(1)) = +\infty$, that is the vanishing of $\varphi(1)$.

On the other hand, the $\mathring{R}$-module $\mathring{M} \simeq \mathring{R} \otimes_R \mathring{R}$, and therefore its dual $\mathring{M}^\vee$, may be identified with $\mathring{R}$.

Observe also that $\mathfrak{p}$ is an element of Σ, the $R_\mathfrak{p}$-module $M_{R_\mathfrak{p}}$ coincides with $K_\mathfrak{p}$, so that the seminorm $||.||_\mathfrak{p}$ on $M_{K_\mathfrak{p}}$ vanishes identically. As a consequence, for any $\eta \in \mathrm{Hom}_{K_\mathfrak{p}}(M_{K_\mathfrak{p}}, K_\mathfrak{p})$, the operator norm of η satisfies:

$$\|\eta\|_\mathfrak{p}^\vee = \begin{cases} 0 & \text{if } \xi = 0; \\ +\infty & \text{if } \xi \neq 0. \end{cases}$$

Example 1.6.6. Assume that $M = R^{(I)}$ for some index set I.

The the following identifications hold:

$$M^\vee \simeq R^I, \quad \mathring{M} \simeq \mathring{R}^{(I)}, \quad \text{and} \quad \mathring{M}^\vee \simeq \mathring{R}^I,$$

and therefore:

$$M^{\vee\vee} \simeq R^{(I)}, \quad (M^{\vee\vee})_{\mathring{R}} \simeq \mathring{R}^{(I)}, \quad \text{and} \quad \mathring{M}^{\vee\vee} \simeq \mathring{R}^{(I)}.$$

In terms of these identifications, we also have:

$$\delta_M = \mathrm{Id}_{R^{(I)}}, \quad \delta_{\mathring{M}} = \mathrm{Id}_{\mathring{R}^{(I)}}, \quad \text{and} \quad p = \mathrm{Id}_{\mathring{R}^{(I)}}.$$

Morover, for every $\mathfrak{p}$ in Σ, and any $\xi = (\xi_i)_{i\in I}$ in $\mathring{M}^\vee \simeq \mathring{R}^I$, we have:

$$||\xi||_\mathfrak{p}^\vee = \sup_{i\in I} |\xi_i|_\mathfrak{p}.$$

CHAPTER 2

Hermitian Quasi-coherent Sheaves over Arithmetic Curves

This chapter is devoted to the definition and to the basic properties of *Hermitian quasi-coherent sheaves* over an arithmetic curve $X := \operatorname{Spec} \mathcal{O}_K$ attached to a number field K.

These are defined as pairs $\overline{F} := \big(F, (\|.\|_x)_{x \in X(\mathbb{C})}\big)$, where F is a countably generated $\mathcal{O}_K$-module, and where $(\|.\|_x)_{x \in X(\mathbb{C})}$ is a family, invariant under complex conjugation, of Hermitian seminorms on the complex vector spaces: $F_x := F \otimes_x \mathbb{C}$.

Hermitian quasi-coherent sheaves constitute the main object of study in this monograph. and are natural from the perspective of the classical analogy between number fields and function fields. Indeed they constitute the analogues of quasi-coherent sheaves of countable type on a smooth projective curve C over some field k.

In the preliminary section 2.1, we consider a smooth projective curve C, equipped with a finite non-empty subset Σ of closed points — which play the role of the Archimedean places in the arithmetic setting — and we briefly describe various categories of quasi-coherent sheaves over C that are the geometric counterparts of the categories of Hermitian quasi-coherent sheaves whose definitions constitute the main topic of this chapter. The notation for these categories of quasi-coherent sheaves over C has been chosen to make clear the correspondence between these geometric categories and their arithmetic counterparts.

Section 2.2 introduces the main definitions concerning Hermitian quasi-coherent sheaves that will be used in this monograph. In particular, we introduce the *categories* $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ *and* $\mathbf{q}\overline{\mathbf{Coh}}_X$, whose objects are the Hermitian quasi-coherent sheaves over the arithmetic curve X, the notion of *admissible short exact sequence* of Hermitian quasi-coherent sheaves, and the *canonical dévissage* of a Hermitian quasi-coherent sheaf $\overline{F} := \big(F, (\|.\|_x)_{x \in X(\mathbb{C})}\big)$, which is defined in terms of the maximal projective quotient $F^{\vee\vee}$ of the $\mathcal{O}_K$-module F.

The remaining sections are devoted to diverse complements to the basic definitions introduced in Section 2.2. Strictly speaking, the main results of this monograph are independent of these complements. However the constructions and results of these sections shed some light on various developments in the next chapters. The reader could skip the contents of Sections 2.3 to 2.5 at first reading, and postpone the study of the results and the proofs in these sections — some of which are somewhat technical — until they are needed in later chapters.

Section 2.3 is devoted to the *vectorization functor*. Namely we shall consider the full subcategories $\overline{\mathbf{Coh}}_X^{\leq 1}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$, the objects of which are the Hermitian quasi-coherent sheaves whose underlying $\mathcal{O}_K$-module is finitely generated. The category $\overline{\mathbf{Coh}}_X^{\leq 1}$ contains as full subcategory the category $\overline{\mathbf{Vect}}_X^{\leq 1}$ of Hermitian vector bundles over X, classically considered in Arakelov geometry, and the vectorization functor:

$$.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_X^{\leq 1} \longrightarrow \overline{\mathbf{Vect}}_X^{\leq 1}$$

is a left adjoint to the inclusion functor:

$$\overline{\mathbf{Vect}}_X^{\leq 1} \hookrightarrow \overline{\mathbf{Coh}}_X^{\leq 1}.$$

The vectorization functor will naturally occur when we shall establish that the study of certain invariants of objects in $\overline{\mathbf{Coh}}_X^{\leq 1}$ may actually be reduced to the one of invariants of the more classical Hermitian vector bundles in $\overline{\mathbf{Vect}}_X^{\leq 1}$.

Sections 2.4 and 2.5 are devoted to the constructions of two contravariant duality functors:

$$.^{\vee} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]} \tag{2.0.1}$$

and:

$$.^{\vee} : \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]} \longrightarrow \mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]} \tag{2.0.2}$$

that extend the duality functors between the categories of categories of ind- and pro-Hermitian vector bundles over X studied in [**Bos20b**]:

$$.^{\vee} : \mathbf{ind}\overline{\mathbf{Vect}}_X \longrightarrow \mathbf{pro}\overline{\mathbf{Vect}}_X \quad \text{and} \quad .^{\vee} : \mathbf{pro}\overline{\mathbf{Vect}}_X \longrightarrow \mathbf{ind}\overline{\mathbf{Vect}}_X, \tag{2.0.3}$$

While the functors (2.0.3) establish adjoint equivalences of categories,[1] the new duality functor (2.0.1) is not an equivalence. Actually the composite biduality functor:

$$.^{\vee\vee} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow \mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$$

provides an alternative construction of the canonical dévissage of an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

The construction of the category $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$, which constitutes the range and the source of the functors (2.0.1) and (2.0.2), requires to work with pro-vector bundles over X equipped with suitable generalizations of Euclidean or Hermitian seminorms — namely with *quasinorms*, that are allowed to take the value $+\infty$. Section 2.4 is devoted to various results concerning Euclidean and Hermitian quasi-norms, that are preliminary to the construction of the category $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$ and of the duality functors (2.0.1) and (2.0.2) in Section 2.5.

In this chapter, we use the following notation.

We denote by K a number field and by $\mathcal{O}_K$ its ring of integers. The set of complex points $X(\mathbb{C})$ of the associated arithmetic curve

$$X := \operatorname{Spec} \mathcal{O}_K$$

is the set, of cardinality $[K : \mathbb{Q}]$, of field embeddings of K in $\mathbb{C}$.

We also denote by:

$$\pi : X \longrightarrow \operatorname{Spec} \mathbb{Z}$$

the (unique) morphism of schemes, which is defined by the inclusion of $\mathbb{Z}$ into $\mathcal{O}_K$, and by ω_π its dualizing sheaf, namely the line bundle over X defined by $\mathcal{O}_K$-module $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$.

2.1. Preliminary: Categories of Quasi-coherent Sheaves over Marked Smooth Projective Curves

In this section, we denote by C a smooth, projective, geometrically connected curve over some base field k, and by Σ a non-empty finite set of closed points of C.

The open subscheme of C:

$$\mathring{C} := C \setminus \Sigma$$

is a smooth affine geometrically connected curve over K, and for every $x \in \Sigma$, the local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring of field of fractions K. We shall denote by $\widehat{\mathcal{O}}_{X,x}$ the completion of the local ring $\mathcal{O}_{X,x}$ and by K_x its fraction field, and by v_x the x-adic valuation on K and its extension to K_x.

[1] See [**Bos20b**, Section 5.5].

We shall also choose a real number[2] $q_x > 1$, and we shall define the x-adic absolute value on K_x associated to v_x as :

$$|.|_x := q_x^{-v_x}.$$

Moreover we shall say that a quasi-coherent sheaf $\mathcal{F}$ on C *has no* Σ*-torsion* when, for every $x \in \Sigma$, the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x is a torsion free $\mathcal{O}_{X,x}$-module. If we denote by:

$$i : \mathring{C} \longrightarrow C$$

the inclusion morphism, this is equivalent to the injectivity of the tautological morphism of quasi-coherent sheaves over C:

$$\mathcal{F} \longrightarrow i_* i^* \mathcal{F}.$$

Observe that the k-scheme $\mathring{C}$ uniquely determines C and Σ. This will legitimate diverse notations below, where $\mathring{C}$ explicitly appears, but neither C, nor Σ.

The data (C, Σ) of a marked smooth projective curve as above constitute a geometric analogue of the arithmetic curve $X = \operatorname{Spec} \mathcal{O}_K$. In this analogy, the function field $k(C) = k(\mathring{C})$ of C plays the role of the number field K and the k-algebra

$$R := \mathcal{O}_C(\mathring{C})$$

the role of the ring of integers $\mathcal{O}_K$. The affine curve $\mathring{C} = \operatorname{Spec} R$ plays the role of $X = \operatorname{Spec} \mathcal{O}_K$, and the finite set Σ the role of the set of Archimedean places of K, or equivalently of the set $X(\mathbb{C})$ of field embeddings of K into $\mathbb{C}$, modulo complex conjugation.

In this section, we describe various categories of quasi-coherent sheaves over C in a way which should make clear their analogy with the various categories of Hermitian quasi-coherent sheaves over the arithmetic curve X that we shall introduce in the next section. This description will be in terms of modules over the algebra $\mathcal{O}_C(\mathring{C})$ and of ultrametric $|.|_x$-seminorms associated to the point x in Σ.

Hopefully this will convince the reader that the objects we shall introduce in the arithmetic setting of the next section are the counterparts of various natural classes of quasi-coherent sheaves in the geometric setting.

2.1.1. The categories $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{(\leq 1)}$, $\overline{\mathbf{Coh}}_{\mathring{C}}^{(\leq 1)}$, $\overline{\mathbf{Coh}}_{C}^{(\leq 1)}$ and $\overline{\mathbf{Vect}}_{\mathring{C}}^{(\leq 1)}$.

2.1.1.1. The categories $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$, which play a central role in this monograph, will be the arithmetic counterparts of the categories $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ and $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$ defined as follows.

The objects of both categories are the *quasi-coherent sheaves of countable type, with no* Σ*-torsion* over C.

The morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ are the morphisms of $\mathcal{O}_C$ modules.

If $\mathcal{F}$ and $\mathcal{G}$ are two quasi-coherent sheaves over C that define objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$, the set of morphisms from $\mathcal{F}$ to $\mathcal{G}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$ is the inductive limit:

$$\varinjlim_i \operatorname{Hom}_{\mathcal{O}_C}\big(\mathcal{F}, \mathcal{G} \otimes \mathcal{O}_C(i\Sigma)\big),$$

where i describes the directed set $(\mathbb{N}, \leq)$. If $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ are two morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$, defined by some morphisms of $\mathcal{O}_C$-modules:

$$\varphi \in \operatorname{Hom}_{\mathcal{O}_C}\big(\mathcal{F}, \mathcal{G} \otimes \mathcal{O}_C(i\Sigma)\big) \quad \text{and} \quad \psi \in \operatorname{Hom}_{\mathcal{O}_C}\big(\mathcal{G}, \mathcal{H}(j\Sigma)\big),$$

for some non-negative integers i and j, there composition in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$ is the morphism:

$$\psi \circ \varphi \in \operatorname{Hom}_{\mathcal{O}_C}\big(\mathcal{F}, \mathcal{H}((i+j)\Sigma)\big),$$

[2]When k is a finite field, it is natural to choose for q_x the cardinality $|\kappa(x)|$ of the residue field $\kappa(x)$ of x, which is a finite extension of k.

defined by means of the identification:

$$\mathrm{Hom}_{\mathcal{O}_C}\big(\mathcal{G}, \mathcal{H}(j\Sigma)\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_C}\big(\mathcal{G}(i\Sigma), \mathcal{H}((i+j)\Sigma)\big).$$

2.1.1.2. To a quasi-coherent sheaf $\mathcal{F}$ over C, we may attach the R-module:

$$F := \mathcal{F}(\mathring{C}).$$

The K-vector space $F_K := F \otimes_R K$ is the stalk of $\mathcal{F}$ at the generic point of C.

For every $x \in \Sigma$, we may also consider the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x. It is a $\mathcal{O}_{X,x}$-module. There is a canonical identification:

$$\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} K \xrightarrow{\sim} F_K,$$

and therefore an isomorphism of K_x-vector spaces:

$$\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} K_x \xrightarrow{\sim} F_{K_x}.$$

Let us now assume that $\mathcal{F}$ has no Σ-torsion. Then for every $x \in \Sigma$, we may consider the canonical injective morphism of $\mathcal{O}_{X,x}$-modules:

$$\mathcal{F}_x \hookrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} K \simeq F_K.$$

It induces an injective morphism of $\widehat{\mathcal{O}}_{X,x}$-modules:

$$\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \widehat{\mathcal{O}}_{X,x} \hookrightarrow \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} K_x \simeq F_{K_x},$$

and $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \widehat{\mathcal{O}}_{X,x}$ generates the K_x-vector space F_{K_x}. By the correspondance between submodules and ultrametric seminorms discussed in Section 1.5, it is associated to an ultrametric $|.|_x$-seminorm $\|.\|_x$ on F_{K_x} with values in $|K|_x = q_x^{\mathbb{Z}} \cup \{0\}$. The seminorm $\|.\|_x$ is characterized by the equivalence, for every $v \in F_{K_x}$:

$$v \in \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \widehat{\mathcal{O}}_{X,x} \Longleftrightarrow \|v\|_x \leq 1.$$

In this way, to any quasi-coherent sheaf $\mathcal{F}$ over C without Σ-torsion we associate a *metrized R-module*:

$$\overline{F} := (F, (\|.\|_x)_{x\in\Sigma}),$$

consisting in some R-module F and in a family $(\|.\|_x)_{x\in\Sigma}$ of ultrametric $|.|_x$-seminorms with value in $|K|_x$ on the K_x-vector spaces F_{K_x}.

Moreover this construction is functorial. Indeed, if $\mathcal{F}$ and $\mathcal{F}'$ are two quasi-coherent sheaf $\mathcal{F}$ over C without Σ-torsion, and if $\overline{F} := (F, (\|.\|_x)_{x\in\Sigma})$ and $\overline{F}' := (F', (\|.\|'_x)_{x\in\Sigma})$ denote the associated metrized R-modules, then any morphism of $\mathcal{O}_C$-modules:

$$\Phi : \mathcal{F} \longrightarrow \mathcal{F}'$$

defines a morphism of R-modules:

$$\varphi := \Phi_{|\mathring{C}} : F := \mathcal{F}(\mathring{C}) \longrightarrow F' := \mathcal{F}'(\mathring{C}), \tag{2.1.1}$$

and, for every $x \in \Sigma$, the operator norm of the K_x-linear map:

$$\varphi_x := \varphi \otimes_R \mathrm{Id}_{K_x} : F_{K_x} \longrightarrow F'_{K_x},$$

computed with respect to the seminorms $\|.\|_x$ and $\|.\|'_x$, satisfy:

$$|||\varphi_x||| \leq 1.$$

More generally, for every $i \in \mathbb{N}$, a morphism of $\mathcal{O}_C$-module:

$$\Phi : \mathcal{F} \longrightarrow \mathcal{F}' \otimes \mathcal{O}_C(i\Sigma)$$

still defines a morphism of R-modules $\varphi := \Phi_{|\mathring{C}} : F \to F'$, which now satisfies:

$$|||\varphi_x||| \leq q_x^i \tag{2.1.2}$$

for every $x \in \Sigma$.

Using the results concerning modules over Dedekind rings established in Sections 1.1, 1.5, and 1.6 (see notably Corollary 1.1.7 and Proposition 1.6.1), one easily checks that the above constructions[3] define *an equivalence of categories from the category* $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ *(resp.* $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$*) to the category whose objects are the metrized R-modules* $\overline{F} := (F, (\|.\|_x)_{x\in\Sigma})$ *where F is a countably generated R-module, and whose morphisms from* $\overline{F} := (F, (\|.\|_x)_{x\in\Sigma})$ *to* $\overline{F}' := (F', (\|.\|'_x)_{x\in\Sigma})$ *are the morphisms of R-modules:*

$$\varphi : F \longrightarrow F'$$

such that, for every $x \in \Sigma$*, the operator norm of the* K_x*-linear map:*

$$\varphi_x := \varphi \otimes_R \mathrm{Id}_{K_x} : F_{K_x} \longrightarrow F'_{K_x}$$

satisfies:

$$|||\varphi_x||| \leq 1 \quad (\textit{resp. } |||\varphi_x||| < +\infty).$$

2.1.1.3. The subcategories $\overline{\mathbf{Coh}}_X^{\leq 1}$ and $\overline{\mathbf{Coh}}_X$, $\overline{\mathbf{Coh}}_{\overline{X}}^{\leq 1}$ and $\overline{\mathbf{Coh}}_{\overline{X}}$, and $\overline{\mathbf{Vect}}_X^{\leq 1}$, of $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$, introduced in the next section of this chapter, will be the counterparts of the full subcategories of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ and $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$ defined as follows.

The category $\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ (resp. $\overline{\mathbf{Coh}}_{\mathring{C}}$) is the full subcategory of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ (resp. $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$) whose objects are the quasi-coherent sheaves $\mathcal{F}$ of countable type over C, with no Σ-torsion, such that the restriction $\mathcal{F}_{|\mathring{C}}$ to $\mathring{C}$ is coherent — or equivalently such that the R-module $F := \mathcal{F}(\mathring{C})$ is finitely generated.

The category $\overline{\mathbf{Coh}}_C^{\leq 1}$ (resp. $\overline{\mathbf{Coh}}_C$) is the full subcategory of $\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ (resp. $\overline{\mathbf{Coh}}_{\mathring{C}}$) whose objects are the coherent sheaves $\mathcal{F}$ over C with no Σ-torsion. Equivalently the R-module $F := \mathcal{F}(\mathring{C})$ is finitely generated and, for every $x \in \Sigma$, the associated seminorm $\|.\|_x$ on F_{K_x} is a norm.

The categories $\overline{\mathbf{Coh}}_{\mathring{C}}$ and $\overline{\mathbf{Coh}}_C$ are actually equivalent to the category of coherent sheaves over $\mathring{C}$, or to the category of finitely generated modules over R.

The category $\overline{\mathbf{Vect}}_{\mathring{C}}^{\leq 1}$ is the full subcategory of $\overline{\mathbf{Coh}}_C^{\leq 1}$ whose objects are the vector bundles (or equivalently the locally free coherent sheaves) over C. An object $\mathcal{F}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ belongs to $\overline{\mathbf{Vect}}_C^{\leq 1}$ if and only if the R-module $F := \mathcal{F}(\mathring{C})$ is finitely generated and projective, and the associated seminorms $(\|.\|_x)_{x\in\Sigma}$ are norms.

2.1.2. The categories $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ and $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$ as exact categories.

2.1.2.1. The additive (actually k-linear) category $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ is *not* an abelian category, due to the requirement on its objects to Σ-torsion free quasi-coherent sheaves. However it becomes an exact category[4] if we endow it with the class $\mathcal{E}$ of kernel-cokernel pairs consisting in short exact sequences:

$$0 \longrightarrow \mathcal{F}' \xrightarrow{j} \mathcal{F} \xrightarrow{p} \mathcal{F}''' \longrightarrow 0 \tag{2.1.3}$$

of $\mathcal{O}_C$-modules, where $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ denote objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$.

It is indeed straightforward that this class $\mathcal{E}$ satisfies the axioms defining an exact category (see for instance [**Bos20b**, D.1.1]). Moreover the allowable epimorphisms are the surjective morphisms $p : \mathcal{F} \to \mathcal{F}''$ of $\mathcal{O}_C$-modules between objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$, the allowable monomorphisms are the injective morphisms $i : \mathcal{F}' \to \mathcal{F}$ of $\mathcal{O}_C$-modules between objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ whose cokernel $\mathcal{F}/i(\mathcal{F}')$ has no Σ-torsion.

[3] of the metrized R-module $\overline{F} := (\mathcal{F}(\mathring{C}), (\|.\|_x)_{x\in\Sigma})$ associated to a quasi-coherent sheaf $\mathcal{F}$ and of the morphism φ of R-modules associated to a morphism Φ of $\mathcal{O}_C$-modules.

[4] In the sense of [**Qui73**, §2]. We refer the reader to [**Kel96**] and [**Büh10**] for details and references concerning exact categories; see also the short summary in [**Bos20b**, Appendix D], whose terminology we follow — namely we say "allowable" instead of "admissible" or "strict" morphisms.

This implies that, more generally, the allowable morphisms are the morphisms $f : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_C$-modules between objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ whose cokernel $\mathcal{G}/f(\mathcal{F})$ has no Σ-torsion.

2.1.2.2. The short exact sequences (2.1.3) that define the structure of an exact category on $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$ admit a simple interpretation in terms of the description of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$ as a category of metrized R-modules.

Consider an object $\mathcal{F}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$, and $\overline{F} := (\mathcal{F}(\mathring{C}), (\|.\|_x)_{x\in\Sigma})$ the associated metrized R-module.

To any R-submodule F' of $F := \mathcal{F}(\mathring{C})$, we may attach the quasi-coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$ defined as follows, in terms of the quasi-coherent subsheaf $\widetilde{F}'$ of $i^*\mathcal{F} := \mathcal{F}_{|\mathring{C}}$ attached to F'. Since $\mathcal{F}$ has no Σ-torsion, it may be identified to a subsheaf of the quasi-coherent sheaf $i_*i^*\mathcal{F}$ over C. The direct image $i_*\widetilde{F}'$ also is a quasi-coherent subsheaf of $i_*i^*\mathcal{F}$, and therefore we may form the intersection:

$$\mathcal{F}' := i_*\widetilde{F}' \cap \mathcal{F},$$

which is a quasi-coherent subsheaf of $\mathcal{F}$.

In more concrete terms, for every open subset V of C, we have:

$$\mathcal{F}'(V) := \left\{ s \in \widetilde{\mathcal{F}}(V) \mid s_{|\mathring{C}\cap V} \in \widetilde{F}'(\mathring{C} \cap V) \right\}.$$

The sheaf $\mathcal{F}'$ is the largest subsheaf of $\mathcal{F}$ whose restriction to $\mathring{C}$ is $\widetilde{F}'$. It is also the unique quasi-coherent subsheaf of $\mathcal{F}$ whose restriction to $\mathring{C}$ is $\widetilde{F}'$, and such that the quasi-coherent sheaf:

$$\mathcal{F}'' := \mathcal{F}/\mathcal{F}'$$

over C has no Σ-torsion.

The metrized R-modules associated to $\mathcal{F}'$ and to $\mathcal{F}/\mathcal{F}'$ are easily seen to be:

$$\overline{F'} := (F', (\|.\|_{x|F'_{K_x}})_{x\in\Sigma} \tag{2.1.4}$$

and

$$\overline{F/F'} := (F/F', (\|.\|_x^{\sim})_{x\in\Sigma}), \tag{2.1.5}$$

defined by means of the restrictions $\|.\|_{x|F'_{K_x}}$ to F'_{K_x} and to the quotient seminorms $\|.\|_x^{\sim}$ on $(F/F')_{K_x} \simeq F_{K_x}/F'_{K_x}$ of the seminorms $\|.\|_x$ on the K_x-vector spaces F_{K_x}.

These remarks show that, written in terms of metrized R-modules, the short exact sequences (2.1.3) coincide, up to isomorphism, with the diagrams of the form:

$$0 \longrightarrow \overline{F'} \xrightarrow{j} \overline{F} \xrightarrow{p} \overline{F/F'} \longrightarrow 0,$$

associated by means of the constructions (2.1.4) and (2.1.5) to a metrized R-module:

$$\overline{F} := (F, (\|.\|_x)_{x\in\Sigma})$$

— with F a countably generated R-module — and to some R-submodule F' of F, and to the inclusion morphism $j : F' \to F$ and the quotient map $p : F \to F/F'$.

2.1.2.3. Similar considerations apply to the k-linear category $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$. Again, it is not an abelian category, but it becomes an exact category if we endow it with the class $\dot{\mathcal{E}}$ of kernel-cokernel pairs consisting in the diagrams deduced by "saturation" under isomorphisms in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}$ from the short exact sequences of the form (2.1.3) that define $\mathcal{E}$.

These diagrams may also be described as follows in terms of metrized R-modules. Up to isometric isomorphisms,[5] these are the diagrams:

$$0 \longrightarrow \dot{\overline{F'}} \xrightarrow{j} \overline{F} \xrightarrow{p} \ddot{\overline{F/F'}} \longrightarrow 0,$$

[5]in other words, up to isomorphisms in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathring{C}}^{\leq 1}$.

associated to a metrized R-module $\overline{F}$ with F countably generated and to some R-submodule F' of F as above, where the seminorms $\|.\|^{\cdot}_x$ and $\|.\|^{\cdot\cdot}_x$ defining the metrized R-modules:

$$\dot{\overline{F'}} := (F', (\|.\|^{\cdot}_x)_{x\in\Sigma})$$

and:

$$\ddot{\overline{F/F'}} := (F/F', (\|.\|^{\cdot\cdot}_x)_{x\in\Sigma}),$$

are now allowed to be arbitrary $|K|_v$-valued ultrametric seminorms that are equivalent to the seminorms $\|.\|_{x|F'_{K_x}}$ and $\|.\|^{\sim}_x$ on F'_{K_x} and F_{K_x}/F'_{K_x}.

In terms of metrized R-modules, the morphisms and the allowable morphisms in the exact category $\mathbf{q\overline{Coh}}_{\hat{C}}$ admit the following description.

The morphisms from $\overline{F}_1 := (F_1, (\|.\|_{1,x})_{x\in\Sigma})$ to $\overline{F}_2 := (F_2, (\|.\|_{2,x})_{x\in\Sigma})$ in $\mathbf{q\overline{Coh}}_{\hat{C}}$ are the morphisms of R-modules:

$$\varphi : F_1 \longrightarrow F_2$$

such that, for every $x \in \Sigma$, the K_x-linear map:

$$\varphi_x : F_{1,K_x} \longrightarrow F_{2,K_x}$$

is continuous, when F_{1,K_x} and F_{2,K_x} are endowed with the seminorms $\|.\|_{1,x}$ and $\|.\|_{2,x}$. Among these morphisms, the allowable morphisms are those such that the two seminorms on $\varphi_x(F_{1,K_x})$ defined as the quotient of $\|.\|_{1,x}$ and as the restriction of $\|.\|_{2,x}$ are equivalent.

2.2. Main Definitions: the Categories $\mathbf{q\overline{Coh}}_X^{(\leq 1)}$, $\overline{\mathbf{Coh}}_X^{(\leq 1)}$, $\overline{\mathbf{Coh}}_{\overline{X}}^{(\leq 1)}$ and $\overline{\mathbf{Vect}}_X^{(\leq 1)}$

2.2.1. The categories $\mathbf{q\overline{Coh}}_X$ and $\mathbf{q\overline{Coh}}_X^{\leq 1}$.

2.2.1.1. Recall that we denote by K a number field and by $\mathcal{O}_K$ its ring of integers. The category of quasi-coherent sheaves on the associated arithmetic curve:

$$X := \operatorname{Spec} \mathcal{O}_K$$

is equivalent to the category of $\mathcal{O}_K$-modules, and we shall use somewhat interchangeably the terminology "quasi-coherent sheaf over X," "quasi-coherent $\mathcal{O}_X$-module," and "$\mathcal{O}_K$-module."

If $\mathcal{F}$ is a $\mathcal{O}_K$-module, and $x \in X(\mathbb{C})$ is a field embedding of K in $\mathbb{C}$, we shall denote by:

$$\mathcal{F}_x := \mathcal{F} \otimes_{\mathcal{O}_K,x} \mathbb{C}$$

the complex vector space deduced from $\mathcal{F}$ by the base change $x : \operatorname{Spec}\mathbb{C} \to X$.

Observe that, if x is an element of $X(\mathbb{C})$ and $\overline{x}$ its complex conjugate, then there is a canonical $\mathbb{C}$-antilinear isomorphism defined by complex conjugation:

$$\overline{\cdot} : \mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_{\overline{x}}, \quad f \otimes_x \lambda \longmapsto f \otimes_{\overline{x}} \overline{\lambda}. \tag{2.2.1}$$

DEFINITION 2.2.1. A *Hermitian quasi-coherent sheaf* over X is a pair:

$$\overline{\mathcal{F}} := \big(\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})}\big),$$

where F is a countably generated $\mathcal{O}_K$-module, and where $(\|.\|_x)_{x\in X(\mathbb{C})}$ is a family, invariant under complex conjugation[6] of Hermitian seminorms on the complex vector spaces $(\mathcal{F}_x)_{x\in X(\mathbb{C})}$.

If $\overline{\mathcal{F}}_1 := \big(\mathcal{F}_1, (\|.\|_{1,x})_{x\in X(\mathbb{C})}\big)$ and $\overline{\mathcal{F}}_2 := \big(\mathcal{F}_1, (\|.\|_{2,x})_{x\in X(\mathbb{C})}\big)$ are two Hermitian quasi-coherent sheaves over X, a *morphism of Hermitian quasi-coherent sheaves*:

$$\varphi : \overline{\mathcal{F}}_1 \longrightarrow \overline{\mathcal{F}}_2$$

is a morphism of $\mathcal{O}_K$-modules $\varphi \in \operatorname{Hom}_{\mathcal{O}_K}(\mathcal{F}_1, \mathcal{F}_2)$ such that, for every $x \in X(\mathbb{C})$, the $\mathbb{C}$-linear map:

$$\varphi_x := \varphi \otimes_{\mathcal{O}_K,x} \operatorname{Id}_{\mathbb{C}} : \mathcal{F}_{1,x} \longrightarrow \mathcal{F}_{2,x}$$

[6] In other words, with the notation (2.2.1), the equality $\|\overline{v}\|_{\overline{x}} = \|v\|_x$ holds for every $x \in X(\mathbb{C})$ and every $v \in \mathcal{F}_x$.

is a continuous map between the seminormed vector spaces $(\mathcal{F}_{1,x}, \|.\|_{1,x})$ and $(\mathcal{F}_{2,x}, \|.\|_{2,x})$.

The continuity of φ_x is equivalent to the finiteness of its operator norm:

$$|||\varphi_x|||_x := \sup_{v \in \mathcal{F}_{1,x}, \|v\|_{1,x} \leq 1} \|\varphi_x(v)\|_{2,x}. \tag{2.2.2}$$

The Hermitian quasi-coherent sheaves over X and their morphisms, as defined above, constitute an $\mathcal{O}_K$-linear category, that we will denote by $\mathbf{q}\overline{\mathbf{Coh}}_X$.

The subcategory of $\mathbf{q}\overline{\mathbf{Coh}}_X$ whose objects are the Hermitian quasi-coherent sheaves over X and whose morphisms are the morphisms $\varphi : \overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_2$ as defined above of operator norm at most one — namely those morphisms such that:

$$|||\varphi_x|||_x \leq 1 \quad \text{for every } x \in X(\mathbb{C})$$

— will be denoted by $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$.

2.2.1.2. If $\overline{\mathcal{F}} := \big(\mathcal{F}, (\|.\|_x)_{x \in X(\mathbb{C})}\big)$ is a Hermitian quasi-coherent sheaf over X, we will denote by:

$$\overline{\mathcal{F}}_x := (\mathcal{F}_x, \|.\|_x)$$

the underlying seminormed $\mathbb{C}$-vector space attached to the field embedding $x \in X(\mathbb{C})$.

Let $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ be two Hermitian quasi-coherent sheaves over X. An isomorphism φ between $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ is precisely an *isometric isomorphism* between $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$, namely an isomorphism:

$$\varphi : \mathcal{F} \xrightarrow{\sim} \mathcal{G}$$

between the underlying $\mathcal{O}_K$-modules such that, for every x in $X(\mathbb{C})$, the invertible $\mathbb{C}$-linear map:

$$\varphi_x : \mathcal{F}_x \xrightarrow{\sim} \mathcal{G}_x$$

is an isometry between the seminormed spaces $\overline{\mathcal{F}}_x$ and $\overline{\mathcal{G}}_x$.

By contrast, an isomorphism φ between $\overline{\mathcal{F}}$ and $\overline{G}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ is an isomorphism between the underlying $\mathcal{O}_K$-modules $\mathcal{F}$ and $\mathcal{G}$ such that, for any $x \in X(\mathbb{C})$, the invertible map φ_x is a homeomorphism when $\mathcal{F}_x$ and $\mathcal{G}_x$ are equipped with the topology defined by the seminormed spaces $\overline{\mathcal{F}}_x$ and $\overline{\mathcal{G}}_x$.

In other words, an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is determined, up to a (unique) isomorphism in $\mathbf{q}\overline{\mathbf{Coh}}_X$, by the underlying $\mathcal{O}_K$-module $\mathcal{F}$ and the structure of topological vector space on the complex vector spaces $(\mathcal{F}_x)_{x \in X(\mathbb{C})}$ defined by the seminormed vector spaces $(\overline{\mathcal{F}}_x)_{x \in X(\mathbb{C})}$.

2.2.1.3. We will systematically denote Hermitian quasi-coherent sheaves over the arithmetic curve $X = \operatorname{Spec} \mathcal{O}_K$ by means of "overlined" characters, such as $\overline{E}$, $\overline{\mathcal{E}}$,... and the underlying $\mathcal{O}_K$-modules (or equivalently, the quasi-coherent sheaf over X) by same characters without overline, such as E, $\mathcal{E}$,...

The Hermitian seminorms on the complex vectors spaces E_x, $\mathcal{E}_x$,... deduced from E, $\mathcal{E}$,... by the complex embeddings $x \in X(\mathbb{C})$ that define the Hermitian structure of $\overline{E}$, $\overline{\mathcal{E}}$,... will be denoted by $\|.\|_{\overline{E},x}$, $\|.\|_{\overline{\mathcal{E}},x}$,..., and the Hermitian scalar products that define these seminorms by $\langle .,. \rangle_{\overline{E},x}$, $\langle .,. \rangle_{\overline{\mathcal{E}},x}$,...,

When X is $\operatorname{Spec} \mathbb{Z}$, we shall denote by $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ and $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ the categories $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ and $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$. An object of these categories is a pair $\overline{\mathcal{F}} := (\mathcal{F}, \|.\|)$ where $\mathcal{F}$ is a countably generated $\mathbb{Z}$-module — that is, a countable abelian group — and $\|.\|$ a Hermitian seminorm on the $\mathbb{C}$-vector space $\mathcal{F}_{\mathbb{C}} := \mathcal{F} \otimes \mathbb{C}$ that is invariant under complex conjugation. The data of the Hermitian seminorm $\|.\|$ on $\mathcal{F}_{\mathbb{C}}$ is equivalent to the data of its restriction $\|.\|_{|\mathcal{F}_{\mathbb{R}}}$, which is a Euclidean seminorm on the $\mathbb{R}$-vector space $\mathcal{F}_{\mathbb{R}}$.

Accordingly, the Hermitian quasi-coherent sheaves over $\operatorname{Spec}\mathbb{Z}$ will often be defined as pairs $(\mathcal{F}, \|.\|)$ with $\mathcal{F}$ is countably generated $\mathbb{Z}$-module and $\|.\|$ a Euclidean seminorm on $\mathcal{F}_{\mathbb{R}}$, and will be called *Euclidean quasi-coherent sheaves.*

If $\mathcal{F}$ is a countably generated torsion $\mathcal{O}_K$-module, then the $\mathbb{C}$-vector spaces $(\mathcal{F}_x)_{x\in\mathbb{C}}$ all vanish. These vector spaces admit a unique Hermitian seminorm, namely the zero norm (!), and equipped with these norms, the $\mathcal{O}_K$-module $\mathcal{F}$ defines a Hermitian quasi-coherent sheaf over X, which we will still denote by $\mathcal{F}$.

2.2.1.4. The category $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ will play a central role in this monograph, and the analogy between $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ and the category $\mathbf{q}\overline{\mathbf{Coh}}_{\tilde{C}}^{\leq 1}$ of quasi-coherent sheaves on a smooth projective curve C introduced in Section 2.1 will be a guiding theme in our work.

However we should emphasize that the category $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ does *not* share some of the basic properties of the category $\mathbf{q}\overline{\mathbf{Coh}}_{\tilde{C}}^{\leq 1}$. In particular, $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ is very far from being an additive category — due to the Archimedean character of Archimedean places. For instance, in general the product of two objects does not exist in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$. A related issue is that the set of morphisms between two objects in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$, in general, is not closed under addition.[7]

In spite of its lack of simple categorical interpretation, it is natural to introduce the following definition of direct sums of Hermitian quasi-coherent sheaves.

DEFINITION 2.2.2. If $(\overline{\mathcal{E}}_i)_{i\in I} := ((\mathcal{E}_i, (\|.\|_{i,x})_{x\in X(\mathbb{C})}))_{i\in I}$ is a countable family of Hermitian quasi-coherent sheaves over X, their direct sum is the Hermitian quasi-coherent sheaf over X:

$$\bigoplus_{i\in I}\overline{\mathcal{E}}_i := \Big(\bigoplus_{i\in I}\mathcal{E}_i, (\|.\|_x)_{x\in X(\mathbb{C})}\Big),$$

where, for every $x\in X(\mathbb{C})$ and every $(v_i)_{i\in I}$ in $\Big(\bigoplus_{i\in I}\mathcal{E}_i\Big)_x \simeq \bigoplus_{i\in I}\mathcal{E}_{i,x}$, we let:

$$\|(v_i)_{i\in I}\|_x^2 := \sum_{i\in I}\|v_i\|_{i,x}^2.$$

For any $k\in I$, the inclusion map $\iota_k : \mathcal{E}_k \to \bigoplus_{i\in I}\mathcal{E}_i$ and the projection map $p_k : \bigoplus_{i\in I}\mathcal{E}_i \to \mathcal{E}_k$ define morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$:

$$\iota_k : \overline{\mathcal{E}}_k \to \bigoplus_{i\in I}\overline{\mathcal{E}}_i \quad \text{and} \quad p_k : \bigoplus_{i\in I}\overline{\mathcal{E}}_i \to \overline{\mathcal{E}}_k.$$

However, if $\overline{\mathcal{E}}$ is a Hermitian quasi-coherent sheaf over X, the diagonal map:

$$\Delta_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{E}\oplus\mathcal{E}, \quad e \longmapsto (e,e)$$

defines a morphism $\Delta_{\mathcal{E}} : \overline{\mathcal{E}} \to \overline{\mathcal{E}}\oplus\overline{\mathcal{E}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, but not in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ unless all the Hermitian seminorms defining $\overline{\mathcal{E}}$ vanish.

2.2.2. Admissible short exact sequences of Hermitian quasi-coherent sheaves.

[7] This basic failure of the analogy between the categories $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ and $\mathbf{q}\overline{\mathbf{Coh}}_{\tilde{C}}^{\leq 1}$ makes all the more remarkable that, as will be shown in Chapters 7 and 8, the theta invariants attached to Hermitian quasi-coherent sheaves satisfy formal properties astonishingly similar to the ones of cohomological invariants attached to quasi-coherent sheaves over a projective curve C.

2.2.2.1. *Admissible short exact sequences and admissible morphisms.* Let us consider a Hermitian quasi-coherent sheaf over X:

$$\overline{\mathcal{E}} := \big(\mathcal{E}, (\|.\|_x)_{x\in X(\mathbb{C})}\big).$$

For any $\mathcal{O}_K$-submodule $\mathcal{F}$ of $\mathcal{E}$, we may perform the following construction.

We may introduce the short exact sequence of $\mathcal{O}_K$-modules defined by the inclusion of $\mathcal{F}$ into $\mathcal{E}$ and by the quotient map from $\mathcal{E}$ onto $\mathcal{E}/\mathcal{F}$:

$$0 \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{E}/\mathcal{F} \longrightarrow 0.$$

We may equip each complex vector space $\mathcal{F}_x$, $x \in X(\mathbb{C})$, with the restriction of the Hermitian seminorm $\|.\|_x$ over $\mathcal{E}_x$, and thus define a Hermitian quasi-coherent sheaf over X:

$$\overline{\mathcal{F}} := \big(\mathcal{F}, (\|.\|_{x|\mathcal{F}_x})_{x\in X(\mathbb{C})}\big).$$

We may also equip the complex vector spaces $(\mathcal{E}/\mathcal{F})_x \simeq \mathcal{E}_x/\mathcal{F}_x$ with the quotient seminorm $\|.\|^{\sim}_x$ deduced from $\|.\|_x$ by means of the surjective $\mathbb{C}$-linear map $p_x : \mathcal{E}_x \to \mathcal{E}_x/\mathcal{F}_x$, namely with the Hermitian seminorm defined by the equality:

$$\|v\|^{\sim}_x := \inf_{\tilde{v}\in p_x^{-1}(v)} \|\tilde{v}\|_x$$

for every $v \in \mathcal{E}_x/\mathcal{F}_x$. We may thus define a Hermitian quasi-coherent sheaf over X:

$$\overline{\mathcal{E}/\mathcal{F}} := \big(\mathcal{E}/\mathcal{F}, (\|.\|^{\sim}_x)_{x\in X(\mathbb{C})}\big).$$

Finally we may associate to $\overline{\mathcal{E}}$ and $\mathcal{F}$ the following diagram in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$:

$$0 \longrightarrow \overline{\mathcal{F}} \xrightarrow{i} \overline{\mathcal{E}} \xrightarrow{p} \overline{\mathcal{E}/\mathcal{F}} \longrightarrow 0. \tag{2.2.3}$$

An admissible short exact of Hermitian quasi-coherent sheaves over X is defined as a diagram in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ which is, up to isometric isomorphism, of the form (2.2.3).

This definition may be equivalently reformulated as follows.

Consider three Hermitian quasi-coherent sheaves over X:

$$\overline{\mathcal{E}}_i := \big(\mathcal{E}_i, (\|.\|_{i,x})_{x\in X(\mathbb{C})}\big), \quad i \in \{1, 2, 3\}.$$

DEFINITION 2.2.3. An *admissible injective morphism* $i : \overline{\mathcal{E}}_1 \to \overline{\mathcal{E}}_2$ is an injective morphism of $\mathcal{O}_K$-modules $i : \mathcal{E}_1 \to \mathcal{E}_2$ such, for every $x \in X(\mathbb{C})$, the injective $\mathbb{C}$-linear map $i_x : \mathcal{E}_{1,x} \to \mathcal{E}_{2,x}$ is an isometry from the seminormed space $\overline{\mathcal{E}}_{1,x}$ into $\overline{\mathcal{E}}_{2,x}$.

An *admissible surjective morphism* $p : \overline{\mathcal{E}}_2 \to \overline{\mathcal{E}}_3$ is a surjective morphism of $\mathcal{O}_K$-modules $p : \mathcal{E}_2 \to \mathcal{E}_3$ such that, for every $x \in X(\mathbb{C})$, the seminorm $\|.\|_{3,x}$ on $\mathcal{E}_{3,x}$ is the quotient seminorm deduced from $\|.\|_{2,x}$ by means of the surjective $\mathbb{C}$-linear map $p_x : \mathcal{E}_{2,x} \to \mathcal{E}_{3,x}$.

An *admissible short exact sequence* of Hermitian quasi-coherent sheaves over X — or shortly, an admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$ — is a diagram of the form:

$$0 \longrightarrow \overline{\mathcal{E}}_1 \xrightarrow{i} \overline{\mathcal{E}}_2 \xrightarrow{p} \overline{\mathcal{E}}_3 \longrightarrow 0,$$

where i (resp. p) is an admissible injective (resp. surjective) morphism, and where the diagram:

$$0 \longrightarrow \mathcal{E}_1 \xrightarrow{i} \mathcal{E}_2 \xrightarrow{p} \mathcal{E}_3 \longrightarrow 0$$

is a short exact sequence of $\mathcal{O}_K$-modules.

On the model of allowable morphisms in an exact category,[8] we could define more generally an *admissible morphism* $f : \overline{\mathcal{E}} \to \overline{\mathcal{E}}'$ between two Hermitian quasi-coherent sheaves $\overline{\mathcal{E}} := (\mathcal{E}, (\|.\|_x)_{x\in X(\mathbb{C})})$ and $\overline{\mathcal{E}}' := (\mathcal{E}', (\|.\|'_x)_{x\in X(\mathbb{C})})$ over X to be a morphism which admits a factorization:

$$f = i \circ p : \overline{\mathcal{E}} \xrightarrow{p} \overline{\mathcal{F}} \xrightarrow{i} \overline{\mathcal{E}}',$$

where p is an admissible surjective morphism and i an admissible injective morphism.

This holds precisely when, for every $x \in X(\mathbb{C})$, the restriction $\|.\|'_{x|f_x(\mathcal{E}_x)}$ of the seminorm $\|.\|'_x$ to the image of the $\mathbb{C}$-linear map $f_x : \mathcal{E}_x \to \mathcal{E}'_x$ coincides with the quotient seminorm of $\|.\|_x$, defined by means of the surjection $f_x : \mathcal{E}_x \to f_x(\mathcal{E}_x)$.

EXAMPLE 2.2.4. A basic instance of the above construction of the admissible short exact sequence (2.2.3) associated to a submodule arises from its application to the torsion submodule $\mathcal{E}_{\mathrm{tor}}$ of the $\mathcal{O}_K$-module $\mathcal{E}$.

In this way, we associate to any Hermitian quasi-coherent sheaf $\overline{\mathcal{E}}$ over X the following admissible short exact sequence:

$$0 \longrightarrow \mathcal{E}_{\mathrm{tor}} \longrightarrow \overline{\mathcal{E}} \longrightarrow \overline{\mathcal{E}}_{/\mathrm{tor}} := \overline{\mathcal{E}/\mathcal{E}_{\mathrm{tor}}} \longrightarrow 0. \tag{2.2.4}$$

The quotient map $\mathcal{E} \to \mathcal{E}_{/\mathrm{tor}} := \mathcal{E}/\mathcal{E}_{\mathrm{tor}}$ induces isomorphisms of $\mathbb{C}$-vector spaces $\mathcal{E}_x \simeq \mathcal{E}_{/\mathrm{tor},x}$, and using this identification, we may write:

$$\overline{\mathcal{E}}_{/\mathrm{tor}} := \big(\mathcal{E}_{/\mathrm{tor}}, (\|.\|_{\overline{\mathcal{E}},x})_{x\in X(\mathbb{C})}\big).$$

EXAMPLE 2.2.5. Consider a Hermitian quasi-coherent sheaf $\overline{\mathcal{E}} := \big(\mathcal{E}, (\|.\|_x)_{x\in X(\mathbb{C})}\big)$ over X, and a surjective morphism of $\mathcal{O}_K$-modules:

$$q : \mathcal{E} \longrightarrow \mathcal{G}.$$

To these data is canonically associated an admissible short exact sequence of Hermitian quasi-coherent sheaves:

$$0 \longrightarrow \overline{\ker q} \xrightarrow{i} \overline{\mathcal{E}} \xrightarrow{q} \overline{\mathcal{G}} \longrightarrow 0. \tag{2.2.5}$$

In (2.2.5), we denote by $\overline{\ker q}$ the Hermitian quasi-coherent sheaves over X defined by $\ker q$ and the restrictions to $(\ker q)_x \simeq \ker q_x$ of the seminorms $\|.\|_x$, and by i the inclusion morphism. Moreover $\overline{\mathcal{G}}$ is the Hermitian quasi-coherent sheaves over X defined by $\mathcal{G}$ equipped with the quotient seminorms of the seminorms $\|.\|_x$ defined by means of the surjections $q_x : \mathcal{E}_x \to \mathcal{G}_x$.

In the following paragraphs, we describe two constructions involving admissible short exact sequences which will appear recurrently in this monograph.

2.2.2.2. *Short exact sequences of admissible short exact sequences.* Consider an admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\mathcal{E}} \xrightarrow{i} \overline{\mathcal{F}} \xrightarrow{p} \overline{\mathcal{G}} \longrightarrow 0,$$

and a submodule $\mathcal{F}'$ of $\mathcal{F}$.

We may introduce the following $\mathcal{O}_K$-submodules of $\mathcal{E}$ and $\mathcal{G}$:

$$\mathcal{E}' := i^{-1}(\mathcal{F}') \quad \text{and} \quad \mathcal{G}' := p(\mathcal{F}'),$$

[8] See for instance [**Bos20b**, D.1.3].

and construct a commutative diagram in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$, whose columns are admissible short exact sequences:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \overline{\mathcal{E}}' \arrow[r, "i'"] \arrow[d] & \overline{\mathcal{F}}' \arrow[r, "p'"] \arrow[d] & \overline{\mathcal{G}}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\mathcal{E}} \arrow[r, "i"] \arrow[d] & \overline{\mathcal{F}} \arrow[r, "p"] \arrow[d] & \overline{\mathcal{G}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{\mathcal{E}/\mathcal{E}'} \arrow[r, "\tilde{i}"] \arrow[d] & \overline{\mathcal{F}/\mathcal{F}'} \arrow[r, "\tilde{p}"] \arrow[d] & \overline{\mathcal{G}/\mathcal{G}'} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

In this diagram, i' and $\tilde{p}$ are admissible (respectively injective and surjective) morphisms, and p' and $\tilde{i}$ are morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$, but may be not admissible.

In other words, if we denote by $\overline{\mathcal{E}/\mathcal{E}'}^{\sim}$ the $\mathcal{O}_K$-module $\mathcal{E}/\mathcal{E}'$ equipped with the Hermitian seminorms that make the injection $\tilde{i}$ isometric, then the diagram:

$$0 \longrightarrow \overline{\mathcal{E}/\mathcal{E}'}^{\sim} \xrightarrow{\tilde{i}} \overline{\mathcal{F}/\mathcal{F}'} \xrightarrow{\tilde{p}} \overline{\mathcal{G}/\mathcal{G}'} \longrightarrow 0$$

is an admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$, and the identity map

$$\mathrm{Id}_{\mathcal{E}/\mathcal{E}'} : \overline{\mathcal{E}/\mathcal{E}'} \longrightarrow \overline{\mathcal{E}/\mathcal{E}'}^{\sim}$$

is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$.

Similarly, if we denote $\overline{\mathcal{G}}'^{\sim}$ the $\mathcal{O}_K$-module $\mathcal{G}'$ equipped with the quotient Hermitian seminorms deduced from the ones on $\overline{\mathcal{F}}'$ by means of the surjections $p'_x : \mathcal{F}'_x \to \mathcal{G}'_x$, then the diagram:

$$0 \longrightarrow \overline{\mathcal{E}}' \xrightarrow{i'} \overline{\mathcal{F}}' \xrightarrow{p'} \overline{\mathcal{G}}'^{\sim} \longrightarrow 0$$

is an admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$, and the identity map

$$\mathrm{Id}_{\mathcal{G}'} : \overline{\mathcal{G}}'^{\sim} \longrightarrow \overline{\mathcal{G}}'$$

is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$.

2.2.2.3. *Admissible short exact sequences and exhaustive filtrations.* Consider an admissible short exact sequences in $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\mathcal{E}} \xrightarrow{i} \overline{\mathcal{F}} \xrightarrow{p} \overline{\mathcal{G}} \longrightarrow 0,$$

an exhaustive filtration[9] $(\mathcal{E}_n)_{n\geq 0}$ of $\mathcal{E}$ by $\mathcal{O}_K$-submodules, and a $\mathcal{O}_K$-submodule $\mathcal{F}'$ of $\mathcal{F}$.

For any $n \geq 0$, we have a short exact sequence

$$0 \longrightarrow \mathcal{E}_n + i^{-1}(\mathcal{F}') \xrightarrow{i} i(\mathcal{E}_n) + \mathcal{F}' \xrightarrow{p} p(\mathcal{F}') \longrightarrow 0$$

[9] In other words, $(\mathcal{E}_n)_{n\geq 0}$ is a sequence of $\mathcal{O}_K$-submodules of $\mathcal{E}$ such that $\mathcal{E}_n \subseteq \mathcal{E}_{n+1}$ for every $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} \mathcal{E}_n = \mathcal{E}$.

of $\mathcal{O}_K$-modules. We may define objects $\overline{\mathcal{E}_n + i^{-1}(\mathcal{F}')}$, $\overline{i(\mathcal{E}_n) + \mathcal{F}'}$ and $\overline{p(\mathcal{F}')}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ by endowing $\mathcal{E}_n + i^{-1}(\mathcal{F}')$, $i(\mathcal{E}_n) + \mathcal{F}'$ and $p(\mathcal{F}')$ with the restrictions of the Hermitian seminorms on $\overline{\mathcal{E}}$, $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ respectively.

Moreover, for any $n \geq 0$, we may define an object $\overline{p(\mathcal{F}')_n}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ by requiring the diagram:

$$0 \longrightarrow \overline{\mathcal{E}_n + i^{-1}(\mathcal{F}')} \xrightarrow{i} \overline{i(\mathcal{E}_n) + \mathcal{F}'} \xrightarrow{p} \overline{p(\mathcal{F}')}_n \longrightarrow 0$$

to be an admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$, by the construction in Example 2.2.5 applied to the Hermitian quasi-coherent sheaf $\overline{i(\mathcal{E}_n) + \mathcal{F}'}$ and to the surjective morphism:

$$p : i(\mathcal{E}_n) + \mathcal{F}' \longrightarrow p(\mathcal{F}').$$

In other words, for any $x \in X(\mathbb{C})$, the Hermitian seminorm $||.||_{\overline{p(\mathcal{F}')}_n, x}$ on $p(\mathcal{F}')_x$ is the quotient seminorm deduced from the restriction of the seminorm $||.||_{\mathcal{F},x}$ to $i(\mathcal{E}_n)_x + \mathcal{F}'_x$ by means of the surjective $\mathbb{C}$-linear map:

$$p_x : i(\mathcal{E}_n)_x + \mathcal{F}'_x \longrightarrow p(\mathcal{F}')_x.$$

PROPOSITION 2.2.6. *For any $x \in X(\mathbb{C})$ and any $v \in p(\mathcal{F}')_x$, the sequence $\big(||v||_{\overline{p(\mathcal{F}')}_n, x}\big)_{n\geq 0}$ is decreasing, and satisfies:*

$$\lim_{n\to+\infty} ||v||_{\overline{p(\mathcal{F}')}_n, x} = ||v||_{\overline{p(\mathcal{F}')}, x}.$$

PROOF. The sequence of $\mathcal{O}_K$-modules $(i(\mathcal{E}_n)+\mathcal{F}')_{n\geq 0}$ is an exhaustive filtration of $p^{-1}(p(\mathcal{F}')) = i(\mathcal{E}) + \mathcal{F}'$, and therefore, for any $x \in X(\mathbb{C})$, the sequence of $\mathbb{C}$-vector spaces $(i(\mathcal{E}_n)_x + \mathcal{F}'_x)_{n\geq 0}$ is an exhaustive filtration of $p_x^{-1}(p_x(\mathcal{F}'_x))$. The fact that the sequence of seminorms $\big(||.||_{\overline{p(\mathcal{F}')}_n, x}\big)_{n\geq 0}$ is decreasing and converges pointwise to $||.||_{\overline{p(\mathcal{F}')}, x}$ now follows from the definition of these seminorms as quotient seminorms. □

2.2.3. Inverse and direct images.

2.2.3.1. Let L be a finite extension of the number field K, and let:

$$f : Y := \operatorname{Spec} \mathcal{O}_L \longrightarrow X := \operatorname{Spec} \mathcal{O}_K$$

be the morphism of schemes defined by the inclusion of rings $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$.

To f are attached the *inverse image functors*:

$$f^* : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow \mathbf{q}\overline{\mathbf{Coh}}_Y \quad \text{and} \quad f^* : \mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1} \longrightarrow \mathbf{q}\overline{\mathbf{Coh}}_Y^{\leq 1} \tag{2.2.6}$$

defined as follows.

To a Hermitian quasi-coherent sheaf $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})})$ over X, they associate the Hermitian quasi-coherent sheaf over Y:

$$f^*\overline{\mathcal{F}} := \big(f^*\mathcal{F}, (\|.\|_{f(y)})_{y\in Y(\mathbb{C})}\big), \tag{2.2.7}$$

where:

$$f^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_K} \mathcal{O}_L.$$

To clarify the meaning of the right-hand side of (2.2.7), observe that an element $y \in Y(\mathbb{C})$ is a field embedding $y : L \to \mathbb{C}$, that its image $f(y)$ in $X(\mathbb{C})$ is the composite field embedding:

$$f(y) : K \hookrightarrow L \xrightarrow{y} \mathbb{C},$$

and that the $\mathbb{C}$-vector space:

$$(f^*\mathcal{F})_y := (F \otimes_{\mathcal{O}_K} \mathcal{O}_L) \otimes_{\mathcal{O}_L, y} \mathbb{C}$$

may be identified with:

$$\mathcal{F}_{f(y)} := \mathcal{F} \otimes_{\mathcal{O}_K, f(y)} \mathbb{C}.$$

Therefore $\|.\|_{f(y)}$ defines a Hermitian seminorm on $(f^*\mathcal{F})_y$.

The morphism:
$$f^*\varphi : f^*\overline{\mathcal{F}} \longrightarrow f^*\overline{\mathcal{G}}$$
associated by the functors (2.2.6) to a morphism:
$$\varphi : \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}}$$
is the map:
$$\varphi_{\mathcal{O}_L} := \varphi \otimes_{\mathcal{O}_K} \mathrm{Id}_{\mathcal{O}_L} : f^*\mathcal{F} \longrightarrow f^*\mathcal{G}.$$
Since, for every $y \in Y(\mathbb{C})$, the $\mathbb{C}$-linear map:
$$(f^*\varphi)_y : (f^*\mathcal{F})_y \longrightarrow (f^*\mathcal{G})_y$$
may be identified to:
$$\varphi_{f(y)} : \mathcal{F}_{f(y)} \longrightarrow \mathcal{G}_{f(y)},$$
it indeed defines a morphism from $f^*\overline{\mathcal{F}}$ to $f^*\overline{\mathcal{G}}$.

2.2.3.2. The functor:
$$f^* : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow \mathbf{q}\overline{\mathbf{Coh}}_Y$$
admits a right adjoint:
$$f_* : \mathbf{q}\overline{\mathbf{Coh}}_Y \longrightarrow \mathbf{q}\overline{\mathbf{Coh}}_X.$$
This functor f_* may be defined in such a way that it restricts to a functor:
$$f_* : \mathbf{q}\overline{\mathbf{Coh}}_Y^{\leq 1} \longrightarrow \mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}. \tag{2.2.8}$$

In this monograph, the *direct image functor* (2.2.8) will be used only when f is the morphism from an arithmetic curve to $\operatorname{Spec}\mathbb{Z}$:
$$\pi : X := \operatorname{Spec}\mathcal{O}_K \longrightarrow \operatorname{Spec}\mathbb{Z},$$
and we shall spell out its definition in this special case only, and leave its general construction to the reader.

Let $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})})$ be a Hermitian quasi-coherent sheaf over X.

We may consider the $\mathbb{Z}$-module $\pi_*\mathcal{F}$, defined by $\mathcal{F}$ seen as a $\mathbb{Z}$-module. The isomorphism of $\mathbb{C}$-algebras:
$$\mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}^{X(\mathbb{C})}, \quad \alpha\otimes\lambda \longmapsto (x(\alpha)\lambda)_{x\in X(\mathbb{C})}$$
induces an isomorphism of $\mathbb{C}$-vector spaces:
$$i = (i_x)_{x\in X(\mathbb{C})} : \pi_*\mathcal{F}_{\mathbb{C}} := \mathcal{F}\otimes_{\mathbb{Z}}\mathbb{C} \simeq \mathcal{F}\otimes_{\mathcal{O}_K}(\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{C}) \xrightarrow{\sim} \bigoplus_{x\in X(\mathbb{C})} \mathcal{F}_x.$$
We define a Hermitian seminorm $\|.\|$ on $\pi_*\mathcal{F}_{\mathbb{C}}$, invariant under complex conjugation, by the relation:
$$\|f\|^2 := \sum_{x\in X(\mathbb{C})} \|i_x(f)\|_x^2,$$
and we let:
$$\pi_*\overline{\mathcal{F}} := (\pi_*\mathcal{F}, \|.\|).$$

This construction is easily seen to define a functor:
$$\pi_* : \mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1} \longrightarrow \mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$$
when, to a morphism $\varphi : \overline{\mathcal{E}} \to \overline{\mathcal{F}}$ between two Hermitian quasi-coherent sheaves on X, we attach the map:
$$\pi_*\varphi := \varphi : \pi_*\mathcal{E} := \mathcal{E} \longrightarrow \pi_*\mathcal{F} := \mathcal{F}.$$
It is indeed $\mathbb{Z}$-linear (!), and the induced $\mathbb{C}$-linear map:
$$(\pi_*\varphi)_{\mathbb{C}} : (\pi_*\mathcal{E})_{\mathbb{C}} \longrightarrow (\pi_*\mathcal{F})_{\mathbb{C}}$$

may be identified with the map:

$$\operatorname{diag}(\varphi_x)_{x\in X(\mathbb{C})} : \bigoplus_{x\in X(\mathbb{C})} \mathcal{E}_x \longrightarrow \bigoplus_{x\in X(\mathbb{C})} \mathcal{F}_x, \quad (e_x)_{x\in X(\mathbb{C})} \longmapsto (\varphi_x(e_x))_{x\in X(\mathbb{C})}.$$

If $\overline{\mathcal{F}}$ is an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ as above, we shall often use the following notation:

$$\mathcal{F}_{\mathbb{C}} := (\pi_*\mathcal{F})_{\mathbb{C}} = \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{C},$$

and:

$$\mathcal{F}_{\mathbb{R}} :=:= (\pi_*\mathcal{F})_{\mathbb{R}} = \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{R}.$$

The $\mathbb{R}$-vector space $\mathcal{F}_{\mathbb{R}}$ may be identified with the fixed points under complex conjugation in

$$\mathcal{F}_{\mathbb{C}} \simeq \bigoplus_{x\in X(\mathbb{C})} \mathcal{F}_x,$$

and the $\mathcal{O}_K$-module $\mathcal{F}_{/\mathrm{tor}}$ with the image of the canonical map:

$$\mathcal{F} \longrightarrow \mathcal{F}_{\mathbb{R}}, \quad f \longmapsto f \otimes 1.$$

2.2.3.3. Admissible short exact sequences, and consequently admissible morphisms, are compatible with inverse and direct images, as shown by the following proposition:

PROPOSITION 2.2.7. *For every diagram in* $\mathbf{q}\overline{\mathbf{Coh}}_X$*:*

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0, \tag{2.2.9}$$

the following conditions are equivalent:

(i) the diagram (2.2.9) *is an admissible short exact sequence in* $\mathbf{q}\overline{\mathbf{Coh}}_X$*;*

(ii) the diagram:

$$0 \longrightarrow \pi_*\overline{E} \xrightarrow{\pi_* i} \pi_*\overline{F} \xrightarrow{\pi_* p} \pi_*\overline{G} \longrightarrow 0$$

is an admissible short sequence in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$*;*

(iii) the diagram:

$$0 \longrightarrow f^*\overline{E} \xrightarrow{f^* i} f^*\overline{F} \xrightarrow{f^* p} f^*\overline{G} \longrightarrow 0$$

is an admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_Y$*.*

This is a simple consequence of the definitions and of the faithful flatness of $\mathcal{O}_L$ over $\mathcal{O}_K$, and we leave the details of the proof to the reader.

2.2.4. The canonical dévissage of an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

2.2.4.1. Let $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})})$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

In Subsection 1.2.4, we have attached to the countably generated $\mathcal{O}_K$-module $\mathcal{F}$ its canonical dévissage:

$$0 \longrightarrow \mathcal{F}_{\mathrm{ap}} \hookrightarrow \mathcal{F} \xrightarrow{\delta_{\mathcal{F}}} \mathcal{F}^{\vee\vee} \longrightarrow 0. \tag{2.2.10}$$

The $\mathcal{O}_K$-module is projective, and $\mathcal{F}_{\mathrm{ap}}$ is the largest antiprojective $\mathcal{O}_K$-module contained in $\mathcal{F}$, and contains $\mathcal{F}_{\mathrm{tor}}$.

DEFINITION 2.2.8. The *canonical dévissage of the Hermitian quasi-coherent sheaf* $\overline{\mathcal{F}}$ is the admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\mathcal{F}}_{\mathrm{ap}} \longrightarrow \overline{\mathcal{F}} \xrightarrow{\delta_{\mathcal{F}}} \overline{\mathcal{F}}^{\vee\vee} \longrightarrow 0 \tag{2.2.11}$$

deduced from the short exact sequence (2.2.10) and the Hermitian structure on $\overline{\mathcal{F}}$ by the construction in 2.2.2.1 above.

By construction, $\overline{\mathcal{F}}^{\vee\vee}$ is the Hermitian quasi-coherent sheaf defined by the $\mathcal{O}_K$-module $\mathcal{F}^{\vee\vee}$ equiped with the Hermitian seminorms quotient of the seminorms $\|.\|_x$, defined by means of the surjective $\mathbb{C}$-linear maps: $\delta_{\mathcal{F},x} : \mathcal{F}_x \to (\mathcal{F}^{\vee\vee})_x$. Since the $\mathcal{O}_K$-module $\mathcal{F}^{\vee\vee}$ is projective, $\overline{\mathcal{F}}$ is an object of $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$.

In Subsection 2.5.4, we shall show that $\overline{\mathcal{F}}^{\vee\vee}$ may also be identified to the bidual of $\overline{\mathcal{F}}$, suitably defined.

2.2.4.2. The image of the antiprojective $\mathcal{O}_K$-module $\mathcal{F}_{\mathrm{ap}}$ in the real vector subspace $\mathcal{F}_{\mathrm{ap},\mathbb{R}}$ of $\mathcal{F}_{\mathbb{R}}$ satisfies strong density properties, as shown in the next proposition, which relies on the topological interpretation of the canonical dévissage of countably generated $\mathbb{Z}$- and $\mathcal{O}_K$-modules presented in Section 1.4.

This proposition will allow us to prove that various significant invariants attached to objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ are unaltered when an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is replaced by $\overline{\mathcal{F}}^{\vee\vee}$; see for instance Propositions 4.3.12, 6.3.2, and 8.2.2.

PROPOSITION 2.2.9. *Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that the $\mathcal{O}_K$-module $\mathcal{F}$ is torsion-free — and therefore may be identified with a submodule of $\mathcal{F}_{\mathbb{R}} := \mathcal{F} \otimes_{\mathbb{Z}} \mathbb{R}$ — and antiprojective*

Let k be a positive integer. For any $(f_1, \ldots, f_k)$ in $\mathcal{F}^k$ and any ε in $\mathbb{R}_+^*$, there exists an $\mathcal{O}_K$-submodule $\mathcal{C}$ in $\mathcal{F}$ such that:

(i) $\mathcal{C}$ contains $f_1, \ldots, f_k$;
(ii) there exists $(\widetilde{f}_1, \ldots, \widetilde{f}_{2k})$ in $\mathcal{F}^{2k}$ such that:

$$\mathcal{C} = \sum_{1\leq i\leq 2k} \mathcal{O}_K \widetilde{f}_i \quad \text{and} \quad \max_{1\leq i\leq 2k} ||\widetilde{f}_i||_{\pi_*\overline{\mathcal{F}}} < \varepsilon.$$

PROOF. According to Theorem 1.4.4, $\mathcal{F}$ is dense in $\mathcal{F}_{\mathbb{R}}$ endowed with its inductive topology. It is *a fortiori* dense in $\mathcal{F}_{\mathbb{R}}$ endowed with the seminorm $||.||_{\pi_*\overline{\mathcal{F}}}$.

Let N be an integer such that:

$$N \geq 2 \quad \text{and} \quad \frac{1}{N} \max_{1\leq i\leq k} ||f_i||_{\pi_*\overline{\mathcal{F}}} < \frac{\varepsilon}{2}.$$

For any $i \in \{1, \ldots, k\}$, we may choose $\widetilde{f}_i \in \mathcal{F}$ such that:

$$||\widetilde{f}_i - \frac{1}{N} f_i||_{\pi_*\overline{\mathcal{F}}} < \frac{\varepsilon}{N}.$$

Then we have:

$$||\widetilde{f}_i||_{\pi_*\overline{\mathcal{F}}} < \frac{\varepsilon}{N} + \frac{1}{N} \max_{1\leq j\leq k} ||f_j||_{\pi_*\overline{\mathcal{F}}} < \varepsilon,$$

and $\widetilde{f}_{i+k} := N\widetilde{f}_i - f_i$ satisfies the inequality:

$$||\widetilde{f}_{i+k}||_{\pi_*\overline{\mathcal{F}}} < \varepsilon.$$

Consequently the $\mathcal{O}_K$-submodule $\mathcal{C}$ of $\mathcal{F}$ generated by $(\widetilde{f}_1, \ldots, \widetilde{f}_{2k})$ satisfies (i) and (ii). □

2.2.4.3. The construction of the canonical dévissage (2.2.11) associated to an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is actually functorial.

Indeed, for any morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$\varphi : \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}},$$

we may consider the underlying morphism of $\mathcal{O}_K$-modules:

$$\varphi : \mathcal{F} \longrightarrow \mathcal{G}$$

and, as observed in 1.2.4 above, the attached commutative diagram with exact rows of $\mathcal{O}_K$-modules:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_{\mathrm{ap}} \arrow[r] \arrow[d, "\varphi_{|\mathcal{F}_{\mathrm{ap}}}"] & \mathcal{F} \arrow[r, "\delta_{\mathcal{F}}"] \arrow[d, "\varphi"] & \mathcal{F}^{\vee\vee} \arrow[r] \arrow[d, "\varphi^{\vee\vee}"] & 0 \\
0 \arrow[r] & \mathcal{G}_{\mathrm{ap}} \arrow[r] & \mathcal{G} \arrow[r, "\delta_{\mathcal{G}}"] & \mathcal{G}^{\vee\vee} \arrow[r] & 0.
\end{tikzcd}$$

Moreover it is straightforward that, for any $x \in X(\mathbb{C})$, the maps $\varphi_{|\mathcal{F}_{\mathrm{ap}},x}$ and $\varphi_x^{\vee\vee}$ define continuous maps of seminormed $\mathbb{C}$-vector spaces:

$$\varphi_{|\mathcal{F}_{\mathrm{ap}},x} : \overline{\mathcal{F}}_{\mathrm{ap},x} \longrightarrow \overline{\mathcal{G}}_{\mathrm{ap},x}$$

and:

$$\varphi_x^{\vee\vee} : \overline{\mathcal{F}}_x^{\vee\vee} \longrightarrow \overline{\mathcal{G}}_x^{\vee\vee}.$$

Consequently $\varphi_{|\mathcal{F}_{\mathrm{ap}}}$ and $\varphi^{\vee\vee}$ define morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_X$, which fit into a commutative diagram in $\mathbf{q}\overline{\mathbf{Coh}}_X$ whose rows are admissible short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \overline{\mathcal{F}}_{\mathrm{ap}} \arrow[r] \arrow[d, "\varphi_{|\mathcal{F}_{\mathrm{ap}}}"] & \overline{\mathcal{F}} \arrow[r, "\delta_{\mathcal{F}}"] \arrow[d, "\varphi"] & \overline{\mathcal{F}}^{\vee\vee} \arrow[r] \arrow[d, "\varphi^{\vee\vee}"] & 0 \\
0 \arrow[r] & \overline{\mathcal{G}}_{\mathrm{ap}} \arrow[r] & \overline{\mathcal{G}} \arrow[r, "\delta_{\mathcal{G}}"] & \overline{\mathcal{G}}^{\vee\vee} \arrow[r] & 0.
\end{tikzcd}$$

This construction is clearly compatible with composition of morphisms. Moreover, if $\varphi : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$, it is also the case of the morphisms:

$$\varphi_{|\mathcal{F}_{\mathrm{ap}}} : \overline{\mathcal{F}}_{\mathrm{ap}} \to \overline{\mathcal{G}}_{\mathrm{ap}} \quad \text{and} \quad \varphi^{\vee\vee} : \overline{\mathcal{F}}^{\vee\vee} \to \overline{\mathcal{G}}^{\vee\vee}.$$

2.2.5. The categories $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0](\leq 1)}$, $\mathbf{ind}\overline{\mathbf{Vect}}_X^{(\leq 1)}$, $\overline{\mathbf{Coh}}_X^{(\leq 1)}$, $\overline{\mathbf{Coh}}_{\overline{X}}^{(\leq 1)}$, and $\overline{\mathbf{Vect}}_X^{(\leq 1)}$.

2.2.5.1. The category $\mathbf{q}\overline{\mathbf{Coh}}_X$ (resp. $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$) admit various full subcategories defined as follows:

(i) the category $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$ (resp. $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]\leq 1}$) the objects of which are the Hermitian quasi-coherent sheaves $\overline{\mathcal{F}}$ whose underlying $\mathcal{O}_K$-module $\mathcal{F}$ is projective;

(ii) the category $\mathbf{ind}\overline{\mathbf{Vect}}_X$ (resp. $\mathbf{ind}\overline{\mathbf{Vect}}_X^{\leq 1}$), already introduced in [**Bos20b**], the objects of which are the Hermitian quasi-coherent sheaves $\overline{\mathcal{F}}$ whose underlying $\mathcal{O}_K$-module $\mathcal{F}$ is projective, and whose Hermitian seminorms $\|.\|_{\overline{\mathcal{F}},x}$ are norms;

(iii) the category $\overline{\mathbf{Coh}}_X$ (resp. $\overline{\mathbf{Coh}}_X^{\leq 1}$), the objects of which are the Hermitian quasi-coherent sheaves whose underlying $\mathcal{O}_K$-module $\mathcal{F}$ is finitely generated;

(iv) the category $\overline{\mathbf{Coh}}_{\overline{X}}$ (resp. $\overline{\mathbf{Coh}}_{\overline{X}}^{\leq 1}$), the objects of which are the Hermitian quasi-coherent sheaves $\overline{\mathcal{F}}$ whose underlying $\mathcal{O}_K$-module $\mathcal{F}$ is finitely generated, and whose Hermitian seminorms $\|.\|_{\overline{\mathcal{F}},x}$ are norms;

(v) the category $\overline{\mathbf{Vect}}_X$ (resp. $\overline{\mathbf{Vect}}_X^{\leq 1}$), the objects of which are the Hermitian vector bundles over X, classically considered in Arakelov geometry; these are the Hermitian quasi-coherent sheaves $\overline{\mathcal{F}}$ whose underlying $\mathcal{O}_K$-module $\mathcal{F}$ is finitely generated and projective (or equivalently, finitely generated and torsion free), and whose Hermitian seminorms $\|.\|_{\overline{\mathcal{F}},x}$ are norms.

When $X = \operatorname{Spec}\mathbb{Z}$, the categories $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$, $\mathbf{ind}\overline{\mathbf{Vect}}_X$, $\overline{\mathbf{Coh}}_X$,... will be denoted $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[0]}$, $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{Z}}$, $\overline{\mathbf{Coh}}_{\mathbb{Z}}$,...

The objects of $\overline{\mathbf{Vect}}_X$ whose underlying $\mathcal{O}_K$-module has rank one are the *Hermitian line bundles* over X.

The objects of $\overline{\mathbf{Vect}}_{\mathbb{Z}}$ are the *Euclidean lattices*, defined as the pairs $\overline{E} := (E, \|.\|)$ where E is a free $\mathbb{Z}$-module of finite rank and $\|.\|$ is a Euclidean norm on the $\mathbb{R}$-vector space $E_{\mathbb{R}} := E \otimes \mathbb{R}$.

The inverse and direct images functor constructed in Subsection 2.2.3 "preserve" these subcategories: with the notation of this subsection, the functor f^* (resp. π_*) maps objects of $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$, $\mathbf{ind}\overline{\mathbf{Vect}}_X$, $\overline{\mathbf{Coh}}_X$,... to objects of $\mathbf{ind}\overline{\mathbf{Vect}}_Y^{[0]}$, $\mathbf{ind}\overline{\mathbf{Vect}}_Y$, $\overline{\mathbf{Coh}}_Y$,... (resp. to objects of $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[0]}$, $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{Z}}$, $\overline{\mathbf{Coh}}_{\mathbb{Z}}$,...).

2.2.5.2. The inclusion functor from $\overline{\mathbf{Vect}}_X$ to $\overline{\mathbf{Coh}}_{\overline{X}}$ (resp. from $\overline{\mathbf{Vect}}_X^{\leq 1}$ to $\overline{\mathbf{Coh}}_{\overline{X}}^{\leq 1}$) admits as left adjoint the functor "killing the torsion":

$$\overline{\mathbf{Coh}}_{\overline{X}} \longrightarrow \overline{\mathbf{Vect}}_X \quad (\text{resp. } \overline{\mathbf{Coh}}_{\overline{X}}^{\leq 1} \longrightarrow \overline{\mathbf{Vect}}_X^{\leq 1}), \quad \overline{E} \longmapsto \overline{E}_{/\mathrm{tor}}. \tag{2.2.12}$$

.

We may also define a duality functor:

$$\overline{\mathbf{Coh}}_{\overline{X}} \longrightarrow \overline{\mathbf{Vect}}_X \quad (\text{resp. } \overline{\mathbf{Coh}}_{\overline{X}}^{\leq 1} \longrightarrow \overline{\mathbf{Vect}}_X^{\leq 1}), \quad \overline{E} \longmapsto \overline{E}^{\vee}. \tag{2.2.13}$$

It maps an object $\overline{E} := (E, (\|.\|_x)_{x \in X(\mathbb{C})})$ of $\overline{\mathbf{Coh}}_{\overline{X}}$ to the Hermitian vector bundle:

$$\overline{E}^{\vee} := (E^{\vee}, (\|.\|_x^{\vee})_{x \in X(\mathbb{C})})$$

— defined by the dual $\mathcal{O}_K$-module $E^{\vee} := \mathrm{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K)$ and the Hermitian norms on the complex vector spaces $(E^{\vee})_x \simeq \mathrm{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$ dual to the Hermitian norms $\|.\|_x$ — and a morphism $\varphi : \overline{E} \to \overline{F}$ between two objects of $\overline{\mathbf{Coh}}_{\overline{X}}$ to the transpose map $\varphi^{\vee} := . \circ \varphi : F^{\vee} \to E^{\vee}$, which indeed defines a morphism $\varphi^{\vee} : \overline{F}^{\vee} \to \overline{E}^{\vee}$.

The biduality functor:

$$.^{\vee\vee} : \overline{\mathbf{Coh}}_{\overline{X}}^{\leq 1} \longrightarrow \overline{\mathbf{Vect}}_X^{\leq 1} \quad (\text{resp. } ^{\vee\vee} : \overline{\mathbf{Coh}}_{\overline{X}}^{\leq 1} \longrightarrow \overline{\mathbf{Vect}}_X^{\leq 1}) \tag{2.2.14}$$

may be identified with the functor (2.2.12).

2.2.5.3. If $\overline{L} := (L, (\|.\|_x)_{x \in X(\mathbb{C})})$ is a Hermitian line bundle over X, its *Arakelov degree* is the real number:

$$\widehat{\deg}\, \overline{L} := \log |L/\mathcal{O}_K s| + \sum_{x \in X(\mathbb{C})} \log \|s\|_x^{-1},$$

where s denotes a non-zero element of the $\mathcal{O}_K$-module L.

More generally, to every object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\overline{X}}$, we may attach the Hermitian line bundle over X:

$$\det \overline{E} := (\det E, (\|.\|_{\wedge^{\mathrm{rk} E} \overline{E}_x})_{x \in X(\mathbb{C})}),$$

where the Hermitian norms $\|.\|_{\wedge^{\mathrm{rk} E} \overline{E}_x}$ on the one-dimensional $\mathbb{C}$-vector spaces $(\det E_x) \simeq \wedge^{\mathrm{rk} E} E_x$ are defined by the following equality:

$$\|v_1 \wedge \cdots \wedge v_{\mathrm{rk} E}\|^2_{\wedge^{\mathrm{rk} E} \overline{E}_x} := \det \left(\langle v_i, v_j \rangle_{\overline{E}, x} \right)_{1 \leq i,j \leq \mathrm{rk} E},$$

valid for every $x \in \mathbb{C}$, and every $v_1, \dots, v_{\mathrm{rk} E}$ in E_x.

Then the Arakelov degree of $\overline{E}$ is defined as the real number:

$$\widehat{\deg}\, \overline{E} := \widehat{\deg}\, \det \overline{E}.$$

It is straightforward that, for any two Hermitian line bundles over X, the following relation holds[10]:

$$\widehat{\deg}\, \overline{L}_1 \otimes \overline{L}_2 = \widehat{\deg}\, \overline{L}_1 + \widehat{\deg}\, \overline{L}_2.$$

[10] The definition of the tensor product of Hermitian vector bundles — and notably of Hermitian line bundles — is a special instance of Definition 2.2.10 below, valid in the more general framework of Hermitian quasi-coherent sheaves.

For every admissible short exact sequence in $\overline{\mathbf{Coh}}_{\overline{X}}$:

$$0 \longrightarrow \overline{E}_1 \longrightarrow \overline{E}_2 \longrightarrow \overline{E}_3 \longrightarrow 0,$$

there exists an isometric isomorphism:

$$\det \overline{E}_2 \xrightarrow{\sim} \det \overline{E}_1 \otimes \det \overline{E}_3,$$

and consequently the following additivity relation is satisfied by the Arakelov degree:

$$\widehat{\deg}\, \overline{E}_2 = \widehat{\deg}\, \overline{E}_1 + \widehat{\deg}\, \overline{E}_3. \tag{2.2.15}$$

For every object $\overline{E}$ in $\overline{\mathbf{Coh}}_{\overline{X}}$, the Arakelov degree also satisfies the following relations,

$$\widehat{\deg}\, \overline{E} = \widehat{\deg}\, \overline{E}_{/\mathrm{tor}} + \log |E_{\mathrm{tor}}|, \tag{2.2.16}$$

where $|E_{\mathrm{tor}}|$ denote the cardinality of the torsion module E_{tor}, which is finite, and:

$$\widehat{\deg}\, \pi_* \overline{E} = \widehat{\deg}\, \overline{E} - \mathrm{rk} E.(\log |\Delta_K|)/2, \tag{2.2.17}$$

where Δ_K denotes the discriminant of the number field K.

Using (2.2.16) and (2.2.17), we see that the computation of the Arakelov degree of an object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\overline{X}}$ reduces to the one of the Arakelov degree of the Euclidean lattice $\pi_* \overline{E}_{/\mathrm{tor}}$. In turn, the Arakelov degree of a Euclidean lattice $\overline{E}$ may be expressed in terms of its covolume $\mathrm{covol}(\overline{E})$:

$$\widehat{\deg}\, \overline{E} = -\log \mathrm{covol}(\overline{E}). \tag{2.2.18}$$

2.2.6. Tensor products.

2.2.6.1. We may define the tensor products of Hermitian quasi-coherent sheaves as follows.

DEFINITION 2.2.10. If $\overline{\mathcal{E}}$ and $\overline{\mathcal{E}}'$ are two objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$, we let:

$$\overline{\mathcal{E}} \otimes \overline{\mathcal{E}}' := (\mathcal{E} \otimes_{\mathcal{O}_K} \mathcal{E}', (\|.\|_x^{\otimes})_{x \in X(\mathbb{C})}),$$

where, for $x \in X(\mathbb{C})$, we denote by $\|.\|_x^{\otimes}$ the Hermitian seminorm on:

$$(\mathcal{E} \otimes_{\mathcal{O}_K} \mathcal{E}')_x \simeq \mathcal{E}_x \otimes_{\mathbb{C}} \mathcal{E}'_x$$

deduced by tensor product from the Hermitian seminorms $\|.\|_x$ and $\|.\|'_x$ on $\mathcal{E}_x$ and $\mathcal{E}'_x$.

Recall that, if we denote by $\langle .,. \rangle$ and $\langle .,. \rangle'$ the Hermitian scalar product on $\mathcal{E}_x$ and $\mathcal{E}'_x$ defining $\|.\|_x$ and $\|.\|'_x$, then the Hermitian scalar product $\langle .,. \rangle^{\otimes}$ on $\mathcal{E}_x \otimes_{\mathbb{C}} \mathcal{E}'_x$ defining $\|.\|^{\otimes}$ satisfies the relation:

$$\langle e_1 \otimes e'_1, e_2 \otimes e'_2 \rangle^{\otimes} := \langle e_1, e_2 \rangle \, \langle e'_1, e'_2 \rangle.$$

This construction is easily seen to be functorial. Namely, if

$$\varphi : \overline{\mathcal{E}} \longrightarrow \overline{\mathcal{F}} \quad \text{and} \quad \varphi' : \overline{\mathcal{E}}' \longrightarrow \overline{\mathcal{F}}'$$

are two morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_X$ (resp. in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$), then the tensor product:

$$\varphi \otimes \varphi' : \mathcal{E} \otimes_{\mathcal{O}_K} \mathcal{E}' \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_K} \mathcal{F}'$$

defines a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X$ (resp. in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$):

$$\varphi \otimes \varphi' : \overline{\mathcal{E}} \otimes \overline{\mathcal{E}}' \longrightarrow \overline{\mathcal{F}} \otimes \overline{\mathcal{F}}'.$$

2.2.6.2. In this monograph, with a few exceptions, tensor products of Hermitian quasi-coherent sheaves over X will occur when one of them is a Hermitian line bundle over X.

In particular, tensor products with the Hermitian line bundles $\overline{\mathcal{O}}_X(\delta)$ defined below will recurrently appear in the next chapters.

For every $\delta \in \mathbb{R}$, we define the Hermitian line bundle $\overline{\mathcal{O}}(\delta)$ over $\operatorname{Spec}\mathbb{Z}$ by:

$$\overline{\mathcal{O}}(\delta) := (\mathbb{Z}, e^{-\delta}\,|.|),$$

and we denote its inverse image over X by:

$$\overline{\mathcal{O}}_X(\delta) := \pi^*\overline{\mathcal{O}}(\delta) = (\mathcal{O}_K, (e^{-\delta}|.|)_{x\in X(\mathbb{C})}).$$

Observe that $\overline{\mathcal{O}}(\delta)$ satisfies:

$$\widehat{\deg}\,\overline{\mathcal{O}}(\delta) = \delta,$$

and that this properties characterizes $\overline{\mathcal{O}}(\delta)$ up to isometric isomorphism.

The tensor product of an object $\overline{\mathcal{E}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ by $\overline{\mathcal{O}}_X(\delta)$ may be identified with the Hermitian quasi-coherent sheaf over X deduced from $\overline{\mathcal{E}}$ by scaling its Hermitian norms by a factor $e^{-\delta}$:

$$\overline{\mathcal{E}} \otimes \overline{\mathcal{O}}_X(\delta) \simeq \big(\mathcal{E}, (e^{-\delta}\,\|.\|_{\overline{\mathcal{E}},x})_{x\in X(\mathbb{C})}\big),$$

and we shall often write $\overline{\mathcal{E}} \otimes \overline{\mathcal{O}}(\delta)$ instead of $\overline{\mathcal{E}} \otimes \overline{\mathcal{O}}_X(\delta)$.

Tensoring by a Hermitian line bundle — for instance by $\overline{\mathcal{O}}_X(\delta)$ — defines a functor from $\mathbf{q}\overline{\mathbf{Coh}}_X$ (resp. $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$) to itself, and is compatible with inverse images and admissible short exact sequence. We also have canonical isometric isomorphisms:

$$\pi_*\big(\overline{\mathcal{E}} \otimes \overline{\mathcal{O}}(\delta)\big) \simeq \pi_*\overline{\mathcal{E}} \otimes \overline{\mathcal{O}}(\delta).$$

2.2.6.3. The compatibility between duality and direct images by π will involve tensoring by the "Hermitian relative dualizing sheaf" $\overline{\omega}_\pi$, namely the Hermitian line bundle over X whose underlying $\mathcal{O}_K$-module[11] is:

$$\omega_\pi := \operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z}),$$

and whose Hermitian norms are defined by:

$$\|\operatorname{Tr}_{K/\mathbb{Q}}\|_{\overline{\omega}_\pi,x} = 1 \quad \text{for every } x \in X(\mathbb{C}),$$

where $\operatorname{Tr}_{K/\mathbb{Q}}$ denotes the trace map from K to $\mathbb{Q}$, which is an element of ω_π.

For instance, for every object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\overline{X}}$, the $\mathbb{Z}$-linear map:

$$\operatorname{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K) \otimes_{\mathcal{O}_K} \operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(E, \mathbb{Z}), \quad \eta \otimes \xi \longmapsto \xi \circ \eta$$

is an isomorphism of $\mathbb{Z}$-modules, and defines an isometric isomorphism of Euclidean lattices:

$$\pi_*(\overline{E}^\vee \otimes \overline{\omega}_\pi) \xrightarrow{\sim} (\pi_*\overline{E})^\vee; \tag{2.2.19}$$

see [**BK10**, Proposition 3.2.2].

If we denote by Δ_K the discriminant of the number field K, we also have:

$$\widehat{\deg}\,\overline{\omega}_\pi = \log|\Delta_K|. \tag{2.2.20}$$

2.3. The Vectorization Functor from $\overline{\mathbf{Coh}}_X$ to $\overline{\mathbf{Vect}}_X$

2.3.1. The vectorization functor $.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow \overline{\mathbf{Vect}}_{\mathbb{Z}}$.

[11] The $\mathcal{O}_K$-module structure on $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$ is defined by $\alpha\xi := (x \mapsto \xi(\alpha x))$ for every α in $\mathcal{O}_K$ and every ξ in $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$.

2.3.1.1. *The Euclidean lattice $\overline{E}^{\mathrm{vect}}$ associated to $\overline{E}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$.* In this paragraph, we fix an object $\overline{E} := (E, \|.\|)$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

To $\overline{E}$ is attached the free $\mathbb{Z}$-module of rank $\mathrm{rk} E$:

$$E_{/\mathrm{tor}} := E/E_{\mathrm{tor}},$$

which will be identified with its image in $E_{\mathbb{R}}$, and the vector subspace of $E_{\mathbb{R}}$:

$$K = \ker \|.\| := \{v \in E_{\mathbb{R}} \mid \|v\| = 0\}.$$

We may introduce the closure $\overline{E_{/\mathrm{tor}} + K}$ of the subgroup $E_{/\mathrm{tor}} + K$ in $E_{\mathbb{R}}$ equipped with its usual Hausdorff locally convex topology.[12] It is a closed subgroup of $E_{\mathbb{R}}$, and its connected component $\overline{E_{/\mathrm{tor}} + K}^{\circ}$ is a $\mathbb{R}$-vector subspace of $E_{\mathbb{R}}$. We will denote by

$$p : E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}/\overline{E_{/\mathrm{tor}} + K}^{\circ}$$

the quotient map.

PROPOSITION 2.3.1. *(i) The image $p(E_{/\mathrm{tor}})$ coincides with $p(\overline{E_{/\mathrm{tor}} + K})$ and is a discrete co-compact subgroup of $E_{\mathbb{R}}/\overline{E_{/\mathrm{tor}} + K}^{\circ}$.*

(ii) The quotient seminorm on $E_{\mathbb{R}}/\overline{E_{/\mathrm{tor}} + K}^{\circ}$,

$$\|.\|^{\mathrm{vect}} : E_{\mathbb{R}}/\overline{E_{/\mathrm{tor}} + K}^{\circ} \longrightarrow \mathbb{R}_+, \quad x \longmapsto \inf_{x' \in p^{-1}(x)} \|x\| \tag{2.3.1}$$

is a Euclidean norm.

PROOF. (i) According to the structure of closed subgroups of finite dimensional vector spaces, any subgroup of $\overline{E_{/\mathrm{tor}} + K}$ containing $\overline{E_{/\mathrm{tor}} + K}^{\circ}$ is closed in $\overline{E_{/\mathrm{tor}} + K}$. This implies that $E_{/\mathrm{tor}} + \overline{E_{/\mathrm{tor}} + K}^{\circ}$ is a closed subgroup of $\overline{E_{/\mathrm{tor}} + K}$. Since it contains $E_{/\mathrm{tor}}$ and K, it coincides with $\overline{E_{/\mathrm{tor}} + K}$.

This proves that the subgroups $p(E_{/\mathrm{tor}})$ and $p(\overline{E_{/\mathrm{tor}} + K})$ of the $\mathbb{R}$-vector space $E_{\mathbb{R}}/\overline{E_{/\mathrm{tor}} + K}^{\circ}$ coincide and are discrete. Since $E_{/\mathrm{tor}}$ generates the $\mathbb{R}$-vector space $E_{\mathbb{R}}$, its image $p(E_{/\mathrm{tor}})$ generates $E_{\mathbb{R}}/\overline{E_{/\mathrm{tor}} + K}^{\circ}$, hence is cocompact in $E_{\mathbb{R}}/\overline{E_{/\mathrm{tor}} + K}^{\circ}$.

(ii) This directly follows from the fact that $\ker p = \overline{E_{/\mathrm{tor}} + K}^{\circ}$ contains $K = \ker \|.\|$. □

Consequently the $\mathbb{Z}$-module:

$$E^{\mathrm{vect}} := p(E_{/\mathrm{tor}})$$

is a free of finite rank, the real vector space $E^{\mathrm{vect}}_{\mathbb{R}}$ may be identified with $E_{\mathbb{R}}/\overline{E_{/\mathrm{tor}} + K}^{\circ}$, and the pair

$$\overline{E}^{\mathrm{vect}} := (E^{\mathrm{vect}}, \|.\|^{\mathrm{vect}}) \tag{2.3.2}$$

defines a Euclidean lattice, the *vectorization* of $\overline{E}$.

Moreover the morphism of $\mathbb{Z}$-modules obtained as the composition:

$$E \longrightarrow E/E_{\mathrm{tor}} =: E_{/\mathrm{tor}} \overset{p_{|E_{/\mathrm{tor}}}}{\longrightarrow} p(E_{/\mathrm{tor}})$$

defines a surjective admissible morphism:

$$\nu_{\overline{E}} : \overline{E} \longrightarrow \overline{E}^{\mathrm{vect}}$$

in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, with $\nu_{\overline{E},\mathbb{R}} = p$.

This construction satisfies the following universal property:

[12] The "overline" symbol is used in $\overline{E}$ and in $\overline{E_{/\mathrm{tor}} + K}$ with different meanings. We hope this will not become too confusing.

PROPOSITION 2.3.2. *Every morphism $f : \overline{E} \longrightarrow \overline{F}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ (resp. in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$), where $\overline{F}$ is an object of $\overline{\mathbf{Vect}}_{\mathbb{Z}}$, uniquely factorizes as*

$$f = \tilde{f} \circ \nu_{\overline{E}}, \tag{2.3.3}$$

with $\tilde{f} : \overline{E}^{\mathrm{vect}} \longrightarrow \overline{F}$ a morphism in $\overline{\mathbf{Vect}}_{\mathbb{Z}}$ (resp. in $\overline{\mathbf{Vect}}_{\mathbb{Z}}^{\leq 1}$).

Moreover the operator norms $\|f_{\mathbb{R}}\|$ and $\|\tilde{f}_{\mathbb{R}}\|$ of $f_{\mathbb{R}} : \overline{E}_{\mathbb{R}} \to \overline{F}_{\mathbb{R}}$ and $\tilde{f}_{\mathbb{R}} : \overline{E}_{\mathbb{R}}^{\mathrm{vect}} \to \overline{F}_{\mathbb{R}}$ satisfy:

$$\|\tilde{f}_{\mathbb{R}}\| = \|f_{\mathbb{R}}\|. \tag{2.3.4}$$

PROOF. Consider a morphism $f : \overline{E} \to \overline{F}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, with $\overline{F} := (F, \|.\|_{\overline{F}})$ a Euclidean lattice. There exists $\lambda \in \mathbb{R}_+^*$ such that, for every $v \in E_{\mathbb{R}}$, we have:

$$\|f_{\mathbb{R}}(v)\|_{\overline{F}} \leq \lambda \|v\|_{\overline{F}}. \tag{2.3.5}$$

In particular $f_{\mathbb{R}}$ vanishes on K and therefore map $E_{/\mathrm{tor}} + K$, hence $\overline{E_{/\mathrm{tor}} + K}$, to the discrete subgroup F of $F_{\mathbb{R}}$. This implies that $f_{\mathbb{R}}$ vanishes on $\overline{E_{/\mathrm{tor}} + K}^{\circ}$ and therefore factorizes as $f_{\mathbb{R}} = \tilde{f}_{\mathbb{R}} \circ p$ for a uniquely determined $\mathbb{R}$-linear map

$$\tilde{f}_{\mathbb{R}} : E_{\mathbb{R}}^{\mathrm{vect}} := E_{\mathbb{R}} / \overline{E_{/\mathrm{tor}} + K}^{\circ} \longrightarrow F_{\mathbb{R}}.$$

Moreover $\tilde{f}_{\mathbb{R}}$ maps $E^{\mathrm{vect}} := p(E_{/\mathrm{tor}})$ to $f_{\mathbb{R}}(E_{/\mathrm{tor}}) = f(E)$, which is contained in F, and according to (2.3.5) and to the definition (2.3.1) of Vect.Vect$^{\mathrm{vect}}$ as a quotient norm, satisfies the inequality:

$$\|\tilde{f}_{\mathbb{R}}(w)\|_{\overline{F}} \leq \lambda \|w\|^{\mathrm{vect}} \tag{2.3.6}$$

for every $w \in E_{\mathbb{R}}^{\mathrm{vect}}$.

This establishes the inequality:

$$\|\tilde{f}_{\mathbb{R}}\| \leq \|f_{\mathbb{R}}\|. \tag{2.3.7}$$

Since $\nu_{\overline{E},\mathbb{R}}$ is norm decreasing, this equality is actually an equality, and (2.3.4) holds.

In particular $\tilde{f}_{\mathbb{R}}$ is the $\mathbb{R}$-linear map attached to a morphism $\tilde{f}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ from $\overline{E}^{\mathrm{vect}}$ to $\overline{F}$, which satisfies (2.3.3) by construction. Moreover the unicity of the factorization is clear, and (2.3.7) shows that, when f is a morphism in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$, then $\tilde{f}$ also is a morphism in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$. □

Since the morphism $\nu_{\overline{E}}$ is surjective admissible, the transpose map

$$\nu_{\overline{E}}^{\vee} : E^{\mathrm{vect}\vee} \longrightarrow E^{\vee}$$

is injective with a saturated image. The latter admits the following description, which is a straightforward corollary of Proposition 2.3.2 applied with $\overline{F} = \overline{\mathcal{O}}_{\mathbb{Z}} := (\mathbb{Z}, |.|)$.

COROLLARY 2.3.3. *For every $\xi \in E^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Z})$, the following two conditions are equivalent:*

(i) $\xi \in \nu_{\overline{E}}^{\vee}(E^{\mathrm{vect}\vee})$;
(ii) the $\mathbb{R}$-linear map $\xi_{\mathbb{R}} : E_{\mathbb{R}} \longrightarrow \mathbb{R}$ is continuous on $(E_{\mathbb{R}}, \|.\|)$.

Indeed condition (ii) precisely asserts that ξ defines a morphism in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$:

$$\xi : \overline{E} \longrightarrow \overline{\mathcal{O}}_{\mathbb{Z}}.$$

Corollary 2.3.3 provides an alternative construction of E^{vect} — as the dual of the saturated submodule of $E^{\vee}$ defined by condition (ii) — and therefore of $\overline{E}^{\mathrm{vect}}$ and of the surjective admissible morphism $\nu_{\overline{E}} : \overline{E} \to \overline{E}^{\mathrm{vect}}$.

2.3.1.2. *The functor* $.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow \overline{\mathbf{Vect}}_{\mathbb{Z}}$. Proposition 2.3.2 admits the following straightforward consequence:

COROLLARY 2.3.4. *For every morphism $f : \overline{E} \to \overline{F}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, there exists a unique morphism $f^{\mathrm{vect}} : \overline{E}^{\mathrm{vect}} \to \overline{F}^{\mathrm{vect}}$ in $\overline{\mathbf{Vect}}_{\mathbb{Z}}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \overline{E} & \xrightarrow{f} & \overline{F} \\ {\scriptstyle \nu_{\overline{E}}}\big\downarrow & & \big\downarrow{\scriptstyle \nu_{\overline{F}}} \\ \overline{E}^{\mathrm{vect}} & \xrightarrow{f^{\mathrm{vect}}} & \overline{F}^{\mathrm{vect}}. \end{array}$$

Moreover:

$$\|f_{\mathbb{R}}^{\mathrm{vect}}\| = \|\nu_{\overline{F},\mathbb{R}} \circ f_{\mathbb{R}}\| \leq \|f_{\mathbb{R}}\|.$$

Corollary 2.3.4 implies that assigning to an object $\overline{E}$ (resp. a morphism f) in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ its "vectorization" $\overline{E}^{\mathrm{vect}}$ (resp. the morphism f^{vect}) defines a functor:

$$.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow \overline{\mathbf{Vect}}_{\mathbb{Z}},$$

which restricts to a functor:

$$.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1} \longrightarrow \overline{\mathbf{Vect}}_{\mathbb{Z}}^{\leq 1}.$$

Proposition 2.3.2 establishes also that these functors are left adjoints to the inclusion functors $\overline{\mathbf{Vect}}_{\mathbb{Z}} \hookrightarrow \overline{\mathbf{Coh}}_{\mathbb{Z}}$ and $\overline{\mathbf{Vect}}_{\mathbb{Z}}^{\leq 1} \hookrightarrow \overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$.

2.3.1.3. *The objects $\overline{E}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ such that $\overline{E}^{\mathrm{vect}} = 0$.* In our constructions of invariants on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following characterization of its objects with trivial vectorization will be useful.

PROPOSITION 2.3.5. *For every object $\overline{E} := (E, \|.\|)$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following conditions are equivalent:*

(i) $\overline{E}^{\mathrm{vect}} = 0$;
(ii) $E_{/\mathrm{tor}} + K$ *is dense in* $E_{\mathbb{R}}$, *where* $K = \ker \|.\| := \{v \in E_{\mathbb{R}} \mid \|v\| = 0\}$;
(iii) for every $\varepsilon \in \mathbb{R}_+^*$, $E_{\mathbb{R}} = E_{/\mathrm{tor}} + B_{\overline{E}_{\mathbb{R}}}(0, \varepsilon)$;
(iv) for every $\varepsilon \in \mathbb{R}_+^*$, *there exists a generating family* $(e_i)_{i \in I}$ *of the* $\mathbb{Z}$*-module* E *such that* $\|e_i\| < \varepsilon$ *for every* $i \in I$;
(v) there exists $N \in \mathbb{N}$ *and, for every* $\varepsilon \in \mathbb{R}_+^*$, *a generating family* $(e_i)_{1 \leq i \leq N}$ *of the* $\mathbb{Z}$*-module* E *such that* $\|e_i\| < \varepsilon$ *for every* $i \in \{1, \ldots, N\}$;
(vi) for every $\varepsilon \in \mathbb{R}_+^*$, *there exists a family* $(e_i)_{i \in I}$ *of elements of* $E_{/\mathrm{tor}}$ *which generate the* $\mathbb{R}$*-vector space* $E_{\mathbb{R}}$ *such that* $\|e_i\| < \varepsilon$ *for every* $i \in I$.

When these conditions are satisfied and E is torsion-free[13]*, condition (v) is satisfied with $N = \mathrm{rk}_{\mathbb{Z}} E$.*

Among the above conditions, (iii), (iv), and (vi) may be reformulated in terms of the invariants ρ, γ, and $\lambda^{[0]}$ introduced in Chapter 6, Sections 6.1 and 6.4. Namely they are respectively equivalent to:

(iii') $\rho(\overline{E}) = 0$;
(iv') $\gamma(\overline{E}) = 0$;
(vi') $\lambda^{[0]}(\overline{E}) = 0$.

PROOF. We shall use, in the special case of Euclidean coherent sheaves, some of the relations between the invariant ρ, γ and $\lambda^{[0]}$ established in Chapter 6.[14]

[13] or equivalently, free.

[14] The proofs of these relations in Chapter 6 do not rely on Proposition 2.3.5.

The equivalence $(i) \Leftrightarrow (ii)$ follows from the definition of $\overline{E}^{\mathrm{vect}}$, and the implication $(ii) \Rightarrow (iii)$ from the inclusion:

$$K \subseteq B_{\overline{E}_R}(0, \varepsilon),$$

valid for every $\varepsilon \in \mathbb{R}_+^*$.

The implication $(iii') \Rightarrow (iv')$ is a special case of Proposition 6.4.10, and the implication $(vi') \Rightarrow (v)$ follows from Proposition 6.4.8, which also shows that, when E is torsion-free, we may take $N = \mathrm{rk}_{\mathbb{Z}} E$ in (v).

The implications $(iv) \Rightarrow (vi)$ and $(v) \Rightarrow (vi)$ are clear.

Finally consider an element ξ of $E^\vee = \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Z})$ such that the linear form $\xi_{\mathbb{R}} : E_{\mathbb{R}} \to \mathbb{R}$ is continuous on $(E_{\mathbb{R}}, \|.\|)$; there exists $C \in \mathbb{R}_+$ such that:

$$|\xi(v)| \leq C \,\|v\|$$

for every $v \in E_{\mathbb{R}}$. When (vi) holds, we may choose ε in $(O, 1/C)$ and find a family $(e_i)_{i \in I}$ in E_{tor} as in the statement of (vi). Then we have:

$$\xi_{\mathbb{R}}(e_i) \in \mathbb{Z} \quad \text{and} \quad |\xi_{\mathbb{R}}(e_i)| \leq C\|e_i\| \leq C\varepsilon < 1,$$

for every $i \in I$. This implies that the $\xi_{\mathbb{R}}(e_i)$ all vanish, and therefore that ξ vanishes. According to Corollary 2.3.3, this proves that $E^{\mathrm{vect}\vee}$ is zero. This establishes the implication $(vi) \Rightarrow (i)$. □

2.3.1.4. *The kernel of the admissible surjective morphism $\nu_{\overline{E}}$.* Let us return to the notation of 2.3.1.1 above, and consider the admissible short exact sequence in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ attached to $\nu_{\overline{E}}$:

$$0 \longrightarrow \overline{V} \hookrightarrow \overline{E} \xrightarrow{\nu_{\overline{E}}} \overline{E}^{\mathrm{vect}} \longrightarrow 0. \tag{2.3.8}$$

The object $\overline{V}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ admits the following description:

PROPOSITION 2.3.6. *With the above notation, we have:*

$$V_{\mathrm{tor}} = E_{\mathrm{tor}}, \quad V_{/\mathrm{tor}} = E_{/\mathrm{tor}} \cap \overline{E_{/\mathrm{tor}} + K}^{\circ}, \quad \textit{and} \quad V_{\mathbb{R}} = \overline{E_{/\mathrm{tor}} + K}^{\circ}. \tag{2.3.9}$$

Moreover $V_{/\mathrm{tor}} + K$ is dense in $V_{\mathbb{R}}$.

PROOF. The relations (2.3.9) are straightforward. Moreover any element v of $V_{\mathbb{R}} := V_{\mathbb{R}} = \overline{E_{/\mathrm{tor}} + K}^{\circ}$ may be written as a limit:

$$v = \lim_{i \to +\infty} (e_i + k_i)$$

with $e_i \in E_{/\mathrm{tor}}$ and $k_i \in K$. Then we have:

$$0 = \nu_{\overline{E}}(v) = \lim_{i \to +\infty} \nu_{\overline{E}}(e_i + k_i) = \lim_{i \to +\infty} \nu_{\overline{E}}(e_i).$$

Since $\nu_{\overline{E}}(e_i)$ belongs to the discrete subgroup E^{vect} of $E^{\mathrm{vect}}_{\mathbb{R}}$, it vanishes for i large enough. Consequently e_i belongs to $V_{/\mathrm{tor}}$ for i large enough, and v is the limit of a sequence in $V_{/\mathrm{tor}} + K$. □

Together with the equality:

$$E^{\mathrm{vect}} = \nu_{\overline{E}, \mathbb{R}}(\overline{E_{/\mathrm{tor}} + K}),$$

this implies:

COROLLARY 2.3.7. *The inverse image in $E_{\mathbb{R}}$ of E^{vect} by $\nu_{\overline{E}\mathbb{R}}$ is:*

$$\nu^{-1}_{\overline{E}, \mathbb{R}}(E^{\mathrm{vect}}) = \overline{E_{/\mathrm{tor}} + K}$$

The density of $V_{/\mathrm{tor}} + K$ in $V_{\mathbb{R}}$ may be rephrased as the equality:

$$\overline{V}^{\mathrm{vect}} = 0. \tag{2.3.10}$$

The admissible surjective morphism $\nu_{\overline{E}}$ is actually characterized by this property of its kernel $\overline{V}$:

PROPOSITION 2.3.8. *Let $f : \overline{E} := (E, \|.\|) \to \overline{F}$ be an admissible surjective morphism in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, with range $\overline{F}$ an object of $\overline{\mathbf{Vect}}_{\mathbb{Z}}$. If $\overline{\ker f} := (\ker f, \|.\|_{|\ker f_{\mathbb{R}}})$ satisfies:*

$$\overline{\ker f}^{\mathrm{vect}} = 0, \tag{2.3.11}$$

then there exists a unique isometric isomorphism:

$$\iota : \overline{E}^{\mathrm{vect}} \xrightarrow{\sim} \overline{F}$$

such that:

$$f = \iota \circ \nu_{\overline{E}}. \tag{2.3.12}$$

PROOF. Consider the admissible short exact sequence:

$$0 \longrightarrow \overline{\ker f} \hookrightarrow \overline{E} \xrightarrow{f} \overline{F} \longrightarrow 0.$$

Every element ξ of $E^\vee := \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Z})$ such that $\xi_{\mathbb{R}} : E_{\mathbb{R}} \to \mathbb{R}$ is continuous on $(E_{\mathbb{R}}, \mathrm{Vect.Vect})$ vanishes on $\ker f$ because of 2.3.11, and therefore factorizes through F. Conversely, for any η in $F^\vee = \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Z})$, the linear form

$$f^\vee_{\mathbb{R}}(\eta_{\mathbb{R}}) := \eta_{\mathbb{R}} \circ f_{\mathbb{R}} : E_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is continuous on $(E_{\mathbb{R}}, \|.\|)$. Consequently, according to Corollary 2.3.3, there exists a unique isomorphism of $\mathbb{Z}$-modules:

$$i : F^\vee \xrightarrow{\sim} E^{\mathrm{vect}\vee}$$

such that:

$$\nu^\vee_{\overline{E}} \circ i = f^\vee.$$

Its transpose defines an isomorphism:

$$\iota := i^\vee : E^{\mathrm{vect}} \xrightarrow{\sim} F$$

which satisfies (2.3.12). Finally ι is isometric as a consequence of (2.3.12) and of the admissibility of f and $\nu_{\overline{E}}$. □

2.3.1.5. *An example: Euclidean seminorms of rank 1.* Let E be a finitely generated $\mathbb{Z}$-module of rank $n \geq 2$.

Let $\mathcal{Q}(E_{\mathbb{R}})$ be the cone of Euclidean semipositive quadratic forms on $E_{\mathbb{R}}$, and let $\mathcal{Q}(E_{\mathbb{R}})_1$ be the set of semipositive quadratic forms of rank 1 on $E_{\mathbb{R}}$. The corresponding seminorms on $E_{\mathbb{R}}$ are precisely the ones of the form:[15]

$$|\xi| : v \longmapsto |\xi(v)|$$

for some $\xi \in E^\vee_{\mathbb{R}} \setminus \{0\}$. To any such ξ, we may attach the object in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$:

$$\overline{E}_\xi := (E, |\xi|).$$

The $\mathbb{Z}$-module $E^\vee$ and the $\mathbb{Q}$-vector space $E^\vee_{\mathbb{Q}}$ will be identified to subsets of $E^\vee_{\mathbb{R}}$. Any non-zero element ξ of $E^\vee_{\mathbb{R}}$ (resp. of $E^\vee_{\mathbb{Q}}$) defines an element $[\xi]$ of $\mathbb{P}(E)(\mathbb{R}) \simeq (E^\vee_{\mathbb{R}} \setminus \{0\})/\mathbb{R}^*_+$ (resp. of its subset $\mathbb{P}(E)(\mathbb{Q}) \simeq (E^\vee_{\mathbb{Q}} \setminus \{0\})/\mathbb{Q}^*_+.$). Observe that, for every ξ in $E^\vee_{\mathbb{R}} \setminus \{0\}$, its class $[\xi]$ belongs to $\mathbb{P}(E)(\mathbb{Q})$ if and only if

$$\mathbb{R}\xi \cap E^\vee \neq \{0\}.$$

When this holds, there exists a unique element $t(\xi)$ in $\mathbb{R}^*_+$ such that $t(\xi)\xi$ is a primitive element in $E^\vee \setminus \{0\}$, or equivalently such that:

$$\mathbb{R}\xi \cap E^\vee = \mathbb{Z}\, t(\xi)\xi.$$

---

[15]The set $\mathcal{Q}(E_{\mathbb{R}})_1$ is the union of the extremal rays (minus the origin) of of the closed convex cone $\mathcal{Q}(E_{\mathbb{R}})$. It is a real analytic submanifold of $S^2 E^\vee_{\mathbb{R}}$, and the map

$$E^\vee_{\mathbb{R}} \setminus \{0\} \longrightarrow \mathcal{Q}(E_{\mathbb{R}})_1, \quad \xi \longmapsto |\xi|^2$$

is an étale covering of degree 2.

PROPOSITION 2.3.9. *With the above notation, for every* $\xi \in E_{\mathbb{R}}^{\vee} \setminus \{0\}$, *we have:*

$$\overline{E}_{\xi}^{\mathrm{vect}} = 0 \quad \textit{if} \;\; [\xi] \in \mathbb{P}(E)(\mathbb{R}) \setminus \mathbb{P}(E)(\mathbb{Q}).$$

Moreover when $[\xi]$ *belongs to* $\mathbb{P}(E)(\mathbb{Q})$, *the "vectorization" of* $\overline{E}_{\xi}$ *is the surjective admissible morphism:*

$$\nu_{\overline{E}_{\xi}} = t(\xi)\xi : \overline{E}_{\xi} \longrightarrow \overline{E}_{\xi}^{\mathrm{vect}} := (\mathbb{Z}, t(\xi)^{-1}|.|).$$

This is an easy consequence of Corollary 2.3.3, and we leave the details to the reader.

2.3.2. The vectorization functor $.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_X \longrightarrow \overline{\mathbf{Vect}}_X$**.** In this subsection, we extends the construction of the vectorization functor from $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ to $\overline{\mathbf{Vect}}_{\mathbb{Z}}$ we construct a vectorization functor:

$$.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_X \longrightarrow \overline{\mathbf{Vect}}_X$$

where X is $\operatorname{Spec} \mathcal{O}_K$ for K an arbitrary number field. This construction will be compatible with the previous construction where X is $\operatorname{Spec} \mathbb{Z}$ via the direct image functors:

$$\pi_* : \overline{\mathbf{Coh}}_X \longrightarrow \overline{\mathbf{Coh}}_{\mathbb{Z}} \quad \text{ and } \pi_* : \overline{\mathbf{Vect}}_X \longrightarrow \overline{\mathbf{Vect}}_Z.$$

Namely, for every object $\overline{E}$ in $\overline{\mathbf{Coh}}_X$ we will have a canonical isomorphism in $\overline{\mathbf{Vect}}_{\mathbb{Z}}$:

$$(\pi_* \overline{E})^{\mathrm{vect}} \xrightarrow{\sim} \pi_*(\overline{E}^{\mathrm{vect}}).$$

2.3.2.1. *The Hermitian vector bundle* $\overline{E}^{\mathrm{vect}}$ *associated to* $\overline{E}$ *in* $\overline{\mathbf{Coh}}_X$. Let $\overline{E} := (E, (\|.\|_{\sigma})_{\sigma : K \hookrightarrow \mathbb{C}}$ be an object in $\overline{\mathbf{Coh}}_X$. Recall that its direct image is defined as the object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$:

$$\pi_* \overline{E} := (\pi_* E, \|.\|)$$

where $\pi_* E$ denotes E seen as a $\mathbb{Z}$-module, and where, for every $x = (x_{\sigma})_{\sigma : K \hookrightarrow \mathbb{C}}$ in

$$(\pi_* E)_{\mathbb{C}} = E \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} E_{\sigma},$$

its Hermitian seminorm $\|x\|$ is defined by:

$$\|x\|^2 := \sum_{\sigma : K \hookrightarrow \mathbb{C}} \|x_{\sigma}\|_{\sigma}^2.$$

Applied to $\pi_* \overline{E}$, the construction of paragraph 2.3.1.1 produces a Euclidean lattice $(\pi_* \overline{E})^{\mathrm{vect}}$ and an admissible short exact sequence in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$:

$$0 \longrightarrow \overline{V} := \overline{\ker \nu_{\pi_* \overline{E}}} \hookrightarrow \pi_* \overline{E} \xrightarrow{\nu_{\pi_* \overline{E}}} (\pi_* \overline{E})^{\mathrm{vect}} \longrightarrow 0.$$

PROPOSITION 2.3.10. *With the above notation, the* $\mathbb{Z}$*-submodule* $V := \ker \nu_{\pi_* \overline{E}}$ *of* $\pi_* E$ *is an* $\mathcal{O}_K$*-submodule of* E.

PROOF. For every $a \in \mathcal{O}_K$, the morphism of $\mathbb{Z}$-modules:

$$[a_E] : \pi_* E \longrightarrow \pi_* E, \quad v \longmapsto av$$

defines an endomorphism of $\pi_* \overline{E}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. Indeed the $\mathbb{C}$-linear endomorphism $[a]_{\mathbb{C}} := [a] \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{C}}$ of $(\pi_* E)_{\mathbb{C}} \simeq \bigoplus_{\sigma : K \hookrightarrow \mathbb{C}} E_{\sigma}$ satisfies, for every $x = (x_{\sigma})_{\sigma : K \hookrightarrow \mathbb{C}}$ in $(\pi_* E)_{\mathbb{C}}$:

$$\begin{aligned} \|[a_E]_{\mathbb{C}} x\|^2 = \|(\sigma(a) x_{\sigma})_{\sigma : K \hookrightarrow \mathbb{C}}\|^2 = \sum_{\sigma : K \hookrightarrow \mathbb{C}} |\sigma(a)|^2 \, |x_{\sigma}|^2 \leq \max_{\sigma : K \hookrightarrow \mathbb{C}} |\sigma(a)|^2 & \sum_{\sigma : K \hookrightarrow \mathbb{C}} |x_{\sigma}|^2 \\ & = \max_{\sigma : K \hookrightarrow \mathbb{C}} |\sigma(a)|^2 \, \|x\|^2. \end{aligned}$$

Consequently we may consider the endomorphism:

$$[a_E]^{\mathrm{vect}} : (\pi_* \overline{E})^{\mathrm{vect}} \longrightarrow (\pi_* \overline{E})^{\mathrm{vect}}$$

of the object $(\pi_*\overline{E})^{\mathrm{vect}}$ of $\overline{\mathbf{Vect}}_{\mathbb{Z}}$. It satisfies:

$$[a_E]^{\mathrm{vect}} \circ \nu_{\pi_*\overline{E}} = \nu_{\pi_*\overline{E}} \circ [a_E],$$

and therefore $[a]$ maps $V = \ker \nu_{\pi_*\overline{E}}$ into itself. $\square$

We may endow the $\mathbb{Z}$-module $(\pi_* E)^{\mathrm{vect}}$ underlying $(\pi_*\overline{E})^{\mathrm{vect}}$ with the structure of $\mathcal{O}_K$-module that makes the isomorphism defined by $\nu_{\pi_*\overline{E}}$:

$$E/V \xrightarrow{\sim} (\pi_* E)^{\mathrm{vect}}$$

an isomorphism of $\mathcal{O}_K$-modules. Then the short exact sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow V \hookrightarrow \pi_* E \xrightarrow{\nu_{\pi_*\overline{E}}} (\pi_* E)^{\mathrm{vect}} \longrightarrow 0$$

becomes a short exact sequence of $\mathcal{O}_K$-modules, which we shall denote by:

$$0 \longrightarrow V := \ker \nu_{\overline{E}} \hookrightarrow \pi_* E \xrightarrow{\nu_{\overline{E}}} E^{\mathrm{vect}} \longrightarrow 0.$$

In turn, this short exact sequence of $\mathcal{O}_K$-modules gives rise to an admissible short exact sequence in $\overline{\mathbf{Coh}}_X$:

$$\begin{aligned} 0 \longrightarrow \overline{\ker \nu_{\overline{E}}} = (V, (\|.\|_{\sigma|E_\sigma})_{\sigma:K\hookrightarrow\mathbb{C}}) \hookrightarrow \pi_*\overline{E} := (E, (\|.\|_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}}) &\xrightarrow{\nu_{\overline{E}}} \overline{E}^{\mathrm{vect}} \\ &:= (E^{\mathrm{vect}}, (\|.\|_\sigma^{\mathrm{vect}})_{\sigma:K\hookrightarrow\mathbb{C}}) \longrightarrow 0, \end{aligned}$$

where the Hermitian norms defining $\overline{\ker \nu_{\overline{E}}}$ and $\pi_*\overline{E}$ are deduced by restriction and by quotient from the ones defining $\overline{E}$.

The compatibility between the constructions of quotients and of direct sums of Hermitian norms implies that, for every $v = (v_\sigma)_{\sigma:K\hookrightarrow\mathbb{C}}$ in $E^{\mathrm{vect}}_{\mathbb{C}} \simeq \bigoplus_{\sigma:K\hookrightarrow\mathbb{C}} E^{\mathrm{vect}}_\sigma$, the following equality holds:

$$(\|v\|^{\mathrm{vect}})^2 = \sum_{\sigma:K\hookrightarrow\mathbb{C}} \|v_\sigma\|_\sigma^2.$$

In other words, the tautological isomorphism:

$$\pi_* E^{\mathrm{vect}} \simeq (\pi_* E)^{\mathrm{vect}}$$

defines an isometric isomorphism of Euclidean lattices:

$$\pi_*\overline{E}^{\mathrm{vect}} \simeq (\pi_*\overline{E})^{\mathrm{vect}}. \tag{2.3.13}$$

Observe also that, with the notation of the proof of Proposition 2.3.10, for every $a \in \mathcal{O}_K$, the endomorphism $[a_E]$ of $\pi_*\overline{E}$ satisfies:

$$[a_E]^{\mathrm{vect}} = [a_{E^{\mathrm{vect}}}], \tag{2.3.14}$$

where $[a_{E^{\mathrm{vect}}}]$ denotes the multiplication by a in E^{vect}.

2.3.2.2. *The functor* $.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_X \longrightarrow \overline{\mathbf{Vect}}_X$. Recall that to a morphism:

$$f : \overline{E} \longrightarrow \overline{F}$$

in $\overline{\mathbf{Coh}}_X$ (resp. in $\overline{\mathbf{Coh}}_X^{\leq 1}$) is functorially attached a morphism:

$$\pi_* f : \pi_*\overline{E} \longrightarrow \pi_*\overline{F}$$

in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ (resp. in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$). By construction, $\pi_* f$ is the morphism of $\mathcal{O}_K$-module $f : E \to F$ seen as a morphism of $\mathbb{Z}$-modules.

With this notation, for every object $\overline{E}$ in $\overline{\mathbf{Coh}}_X$, the morphisms $\nu_{\pi_*\overline{E}}$ and $\pi_*\nu_{\overline{E}}$ coincides once their ranges are identified by means of (2.3.13).

PROPOSITION 2.3.11. *Every morphism $f : \overline{E} \longrightarrow \overline{F}$ in $\overline{\mathbf{Coh}}_X$ (resp. in $\overline{\mathbf{Coh}}_X^{\leq 1}$), where $\overline{F}$ is an object of $\overline{\mathbf{Vect}}_X$, uniquely factorizes as*

$$f = \tilde{f} \circ \nu_{\overline{E}}, \tag{2.3.15}$$

with $\tilde{f} : \overline{E}^{\mathrm{vect}} \longrightarrow \overline{F}$ a morphism in $\overline{\mathbf{Vect}}_X$ (resp. in $\overline{\mathbf{Vect}}_X^{\leq 1}$).

Observe that the factorization of $\pi_* f : \pi_* \overline{E} \to \pi_* \overline{F}$:

$$\pi_* f = \pi_* \tilde{f} \circ \pi_* \nu_{\overline{E}} = \pi_* \tilde{f} \circ \nu_{\pi_* \overline{E}} \tag{2.3.16}$$

deduced from (2.3.15) by direct image necessarily coincides with its factorization in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ (resp. in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$) constructed in Proposition 2.3.2.

PROOF. Consider the factorization:

$$\pi_* f = (\pi_* f)^{\sim} \circ \nu_{\pi_* \overline{E}}$$

of $\pi_* f : \pi_* \overline{E} \to \pi_* \overline{F}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ provided by Proposition 2.3.2. To establish the proposition, it is sufficient to prove that the morphism of $\mathbb{Z}$-modules:

$$(\pi_* f)^{\sim} : E \longrightarrow F$$

is actually $\mathcal{O}_K$-linear.

To achieve this, observe that for every $a \in \mathcal{O}_K$, the $\mathcal{O}_K$-linearity of $f : E \to F$ implies the equality of morphisms in $\overline{\mathbf{Coh}}_X$ with source $\pi_* \overline{E}$ and range $\pi_* \overline{F}$:

$$\pi_* f \circ [a_E] = [a_F] \circ \pi_* f.$$

Using the adjunction property of the functor $.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow \overline{\mathbf{Vect}}_{\mathbb{Z}}$, this implies the relations:

$$(\pi_* f)^{\sim} \circ [a_E]^{\mathrm{vect}} = (\pi_* f \circ [a_E])^{\sim} = ([a_F] \circ \pi_* f)^{\sim} = [a_F] \circ (\pi_* f)^{\sim}.$$

Together with (2.3.14), this implies:

$$(\pi_* f)^{\sim} \circ [a_{E^{\mathrm{vect}}}] = [a_F] \circ (\pi_* f)^{\sim},$$

and establishes the $\mathcal{O}_K$-linearity of $(\pi_* f)^{\sim}$. □

From Proposition 2.3.11, we immediately deduce the following extension of Corollary 2.3.4:

COROLLARY 2.3.12. *For every morphism $f : \overline{E} \to \overline{F}$ in $\overline{\mathbf{Coh}}_X$, there exists a unique morphism*

$$f^{\mathrm{vect}} : \overline{E}^{\mathrm{vect}} \longrightarrow \overline{F}^{\mathrm{vect}}$$

in $\overline{\mathbf{Vect}}_X$ such that the following diagram is commutative:

$$\begin{array}{ccc} \overline{E} & \xrightarrow{f} & \overline{F} \\ {\scriptstyle \nu_{\overline{E}}}\big\downarrow & & \big\downarrow{\scriptstyle \nu_{\overline{F}}} \\ \overline{E}^{\mathrm{vect}} & \xrightarrow{f^{\mathrm{vect}}} & \overline{F}^{\mathrm{vect}}. \end{array}$$

Moreover the morphism:

$$\pi_* f^{\mathrm{vect}} : \pi_* \overline{E}^{\mathrm{vect}} \longrightarrow \pi_* \overline{F}^{\mathrm{vect}}$$

coincides with $(\pi_* f)^{\mathrm{vect}}$ when $\pi_* \overline{E}^{\mathrm{vect}}$ and $\pi_*(\overline{E}^{\mathrm{vect}})$ (resp. $\pi_* \overline{F}^{\mathrm{vect}}$ and $\pi_*(\overline{F}^{\mathrm{vect}})$) are identified by means of the isomorphism (2.3.13).

As in paragraph 2.3.1.2 above when $X = \operatorname{Spec}\mathbb{Z}$, Corollary 2.3.12 implies that assigning to an object $\overline{E}$ (resp. a morphism f) in $\overline{\mathbf{Coh}}_X$ its "vectorization" $\overline{E}^{\mathrm{vect}}$ (resp. the morphism f^{vect}) defines a functor:

$$.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_X \longrightarrow \overline{\mathbf{Vect}}_X,$$

which restricts to a functor:

$$.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_X^{\leq 1} \longrightarrow \overline{\mathbf{Vect}}_X^{\leq 1},$$

and Proposition 2.3.11 establishes also that these functors are left adjoints to the inclusion functors $\overline{\mathbf{Vect}}_X \hookrightarrow \overline{\mathbf{Coh}}_X$ and $\overline{\mathbf{Vect}}_X^{\leq 1} \hookrightarrow \overline{\mathbf{Coh}}_X^{\leq 1}$.

2.3.2.3. *The objects $\overline{E}$ in $\overline{\mathbf{Coh}}_X$ such that $\overline{E}^{\mathrm{vect}} = 0$.* The characterizations in Proposition 2.3.5 of the objects $\overline{E}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ such that $\overline{E}^{\mathrm{vect}}$ is zero immediately provides a characterization of the objects $\overline{E}$ in $\overline{\mathbf{Coh}}_X$ such that $\overline{E}^{\mathrm{vect}} = 0$, thanks to the isomorphism (2.3.13). The following straightforward consequence of Proposition 2.3.5 will also be useful when constructing invariants on $\overline{\mathbf{Coh}}_X$:

PROPOSITION 2.3.13. *For every object $\overline{E}$ of $\overline{\mathbf{Coh}}_X$, the following conditions are equivalent:*

(i) $\overline{E}^{\mathrm{vect}} = 0$*;*
(ii) there exists $N \in \mathbb{N}$, and for every $\delta \in \mathbb{R}$, a surjective morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$:

$$\mathcal{O}_X(\delta)^{\oplus N} \longrightarrow \overline{E}.$$

Observe also that any object $\overline{E} := (E, (\|.\|_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}})$ of $\overline{\mathbf{Coh}}_X$ such that, for some field embedding $\sigma : K \hookrightarrow \mathbb{C}$:

$$\|.\|_\sigma = 0 \tag{2.3.17}$$

satisfies:

$$\overline{E}^{\mathrm{vect}} = 0.$$

Indeed (2.3.17) implies that every morphism $f : \overline{E} \to \overline{F}$ in $\overline{\mathbf{Coh}}_X$ with $\overline{F}$ in $\overline{\mathbf{Vect}}_X$ satisfies $f_\sigma = 0$, and therefore vanishes.

Corollary 2.3.3 and Proposition 2.3.8 also admit straightforward generalizations concerning Hermitian coherent sheaves over $X = \operatorname{Spec}\mathcal{O}_K$ instead of Hermitian coherent sheaves over $\operatorname{Spec}\mathbb{Z}$.

2.4. Hermitian Quasinorms and Complex Topological Vector Spaces

2.4.1. Hermitian Quasinorms on Complex Vector Spaces.

2.4.1.1. To develop a suitable duality formalism concerning complex vector spaces endowed with seminorms that possibly vanish on some non zero vectors, we need to consider generalizations of norms and seminorms which are allowed to take the value $+\infty$.

DEFINITION 2.4.1. Let V be a complex vector space. A *quasinorm* on V is a map

$$||.|| : V \longmapsto [0, +\infty]$$

such that the following conditions are satisfied:

(i) $\forall \lambda \in \mathbb{C}, \forall v \in V, ||\lambda v|| = \lambda ||v||$;
(ii) $\forall v, w \in V, ||v + w|| \leq ||v|| + ||w||$.

In (ii), we use the convention: $0.(+\infty) = 0$.

Given V and $||.||$ as in Definition 2.4.1, we may consider the vector subspace V_0 (resp. V_{fin}) of V of elements v such that $\|v\| = 0$ (resp. such that $||v|| < +\infty$). We say that that the quasinorm $||.||$ is *finite* when $V_{\mathrm{fin}} = V$, or equivalently when $||.||$ is a seminorm, and that $\|.\|$ is *definite* when $V_0 = 0$. A quasinorm is both finite and definite if and only if it is a norm.

Clearly the data of a quasinorm $||.||$ on a complex vector space V is equivalent to the data of two vector subspaces $V_0 \subset V_{\text{fin}}$ of V, and of a norm $||.||^{\tilde{}}$ on the quotient V_{fin}/V_0: if p denotes the quotient map from V_{fin} to V_{fin}/V_0, one attaches to $(V_0, V_{\text{fin}}, ||.||^{\tilde{}})$ the quasinorm $||.||$ defined by:

$$||v|| := +\infty \quad \text{if } v \in V \setminus V_{\text{fin}},$$

and:

$$||v|| := ||p(v)||^{\tilde{}} \quad \text{if } v \in V_{\text{fin}}.$$

Finally a quasinorm $\|.\|$ on a complex vector space V will be called a *Hermitian quasinorm* when the seminorm defined by its restriction to V_{fin} is a Hermitian seminorm, or equivalently with the above notation, when the norm $||.||^{\tilde{}})$ on V_{fin}/V_0 is a Hermitian norm. This holds if and only if, besides conditions (i) and (ii) in Definition 2.4.1, the quasinorm $\|.\|$ also satisfies:

(iii) $\forall v, w \in V, ||v+w||^2 + ||v-w||^2 = 2||v||^2 + 2||w||^2.$

Let us emphasize that our use of the terminology *quasinorm* is not standard.[16]

The definitions and the results of this section admit straightforward variants, where complex vector spaces are replaced by real vector spaces, and Hermitian (semi)norms by Euclidean (semi)norms.

2.4.1.2. The following example is a basic instance of the construction by duality of Hermitian quasinorms. This construction will be extended to the infinite dimensional setting in the next subsections.

Example 2.4.2. Let W be a finite dimensional $\mathbb{C}$-vector space, and let:

$$W^\vee := \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$$

be the dual vector space.

To any Hermitian seminorm $\|.\|$ on W, we may attach the lower semicontinuous function:

$$\|.\|^\vee := W^\vee \longrightarrow [0, +\infty]$$

defined by:

$$\|\xi\|^\vee := \sup_{x \in W, \|x\| \leq 1} |\xi(x)|.$$

It is actually a definite Hermitian quasinorm on $W^\vee$. Indeed, if we let:

$$K := \{x \in W \mid \|x\| = 0\}$$

and

$$K^\perp := \{\xi \in W^\vee \mid \xi_{|K} = 0\},$$

then, for any $\xi \in W^\vee$, we have:

$$\|\xi\|^\vee < +\infty \Longleftrightarrow \xi \in K^\perp. \tag{2.4.1}$$

The transpose of the quotient map $V \to V/K$ defines an isomorphism of $\mathbb{C}$-vector spaces $(W/K)^\vee \simeq K^\perp$, and using this identification, the restriction $\|.\|_{|K^\perp}$ coincides with the norm dual to the Hermitian norm on W/K deduced from the seminorm $\|.\|$.

Moreover, the seminorm $\|.\|$ on W may be recovered from the quasinorm $\|.\|^\vee$ on $W^\vee$. Indeed, for every $x \in W$, we have:

$$\|x\| = \sup_{\xi \in W^\vee, \|\xi\|^\vee \leq 1} |\langle \xi, x \rangle|.$$

It is straightforward that this construction establishes a bijection between Hermitian seminorms on W and definite Hermitian quasinorms on $W^\vee$.

[16]In [**GR84**, Section 10.1.1], Grauert and Remmert call *protonorm* a definite quasinorm that defines a complete norm on V_{fin}.

2.4.1.3. The maximum of two Hermitian quasinorms on some complex vector space is not always a Hermitian quasinorms.[17] However, Hermitian quasinorms are stable under the operation of "filtrant supremum":

Proposition 2.4.3. *Let $(I, \preceq)$ be a directed set, and let $(\|.\|_i)_{i\in I}$ be a family of Hermitian quasinorms on V such that:*

$$i \preceq i' \Longrightarrow \|.\|_i \leq \|.\|_{i'}$$

for any $(i, i') \in I^2$.

Then the map :

$$\|.\| : V \longrightarrow [0, +\infty]$$

defined by:

$$\|v\| := \sup_{i\in I} \|v\|_i \tag{2.4.2}$$

is a Hermitian quasinorm on V.

This directly follows from the observation that the quasinorm $\|.\|$ defined by (2.4.2) is the pointwise limit of the quasinorms $\|.\|_i$ when i runs over the directed set $(I, \preceq)$.

2.4.1.4. A vector space V equipped with a Hermitian quasinorm $||.||$ as above is endowed with a natural topology where a basis of neighborhood of 0 is given by the open balls $||x|| < r$, $r > 0$. The closure of any singleton $\{x\}$ is $x + V_0$, and for any $x \in V$, $x + V_{\rm fin}$ is open and closed, and a connected component of V. This topology endows the additive group $(V, +)$ with the structure of a topological group ; it endows the complex vector space V with the structure of a topological vector space precisely when the quasinorm $||.||$ is finite.

Let V and W be two complex vector spaces, equipped with quasinorms $\|.\|_V$ and $\|.\|_W$. As above, we let:

$$V_{\rm fin} := \{v \in V \mid \|v\|_V < +\infty\} \quad \text{and} \quad W_{\rm fin} := \{w \in W \mid \|w\|_W < +\infty\}.$$

The following proposition is a straightforward consequence of the definitions.

Proposition 2.4.4. *For every $\mathbb{C}$-linear map $\varphi : V \to W$, the following equality holds*[18] *in $[0, +\infty]$:*

$$\begin{aligned} |||\varphi||| &:= \inf\left\{\lambda \in \mathbb{R}_+ \mid \forall v \in V, \|\varphi(v)\|_W \leq \lambda\|v\|_W\right\} \\ &= \inf\left\{\lambda \in \mathbb{R}_+ \mid \forall v \in V_{\rm fin}, \|\varphi(v)\|_W \leq \lambda\|v\|_W\right\}. \end{aligned} \tag{2.4.3}$$

Moreover the following conditions are equivalent:

(i) the map φ is continuous when V and W are equipped with the topology defined by the quasinorms $\|.\|_V$ and $\|.\|_W$;
(ii) $\varphi(V_{\rm fin}) \subseteq W_{\rm fin}$, and the map $\varphi_{|V_{\rm fin}} : V_{\rm fin} \to W_{\rm fin}$ is continuous when $V_{\rm fin}$ (resp. $W_{\rm fin}$) is equipped wiith the seminorm $\|.\|_{V|V_{\rm fin}}$ (resp. $\|.\|_{W|W_{\rm fin}}$);
(iii) $|||\varphi||| < +\infty$.

Observe that, when the continuity conditions (i)-(ii) are satisfied, the infimum in (2.4.3) is actually a minimum, and that the *operator (quasi)norm*:

$$|||.||| : \mathrm{Hom}_{\mathbb{C}}(V, W) \longrightarrow [0, +\infty]$$

defined by (2.4.3) is indeed a quasinorm on the complex vector space $\mathrm{Hom}_{\mathbb{C}}(V, W)$.

---

[17]Actually, on any complex vector space V of dimension > 1, there exists two Hermitian norms $\|.\|_1$ and $\|.\|_2$ such that the norm $\max(\|.\|_1, \|.\|_2)$ is not Hermitian.

[18]Recall that the infimum of the empty subset of $\mathbb{R}_+$ is $+\infty$.

2.4.1.5. Let V' be a vector subspace of a complex vector space V equipped with a Hermitian quasinorm $||.||$. Then the restriction of $||.||$ to V' is clearly a Hermitian quasinorm, the *induced quasinorm*, and we have

$$V'_0 = V_0 \cap V' \text{ and } V'_{\text{fin}} = V_{\text{fin}} \cap V'.$$

Consider the quotient map $\pi : V \to V/V'$.

PROPOSITION 2.4.5. *We can define a Hermitian quasinorm $||.||'$ on V/V', the* quotient quasinorm deduced from $||.||$, *by the formula*

$$||w||' = \inf_{v\in\pi^{-1}(w)} ||v||$$

for any $w \in V/V'$. Moreover, we have

$$(V/V')_{\text{fin}} = \pi(V_{\text{fin}}) = (V_{\text{fin}} + V')/V'$$

and

$$(V/V')_0 = \overline{V'}/V'$$

where $\overline{V'}$ denotes the closure of V' in V. It coincides with $V' + (\overline{V' \cap V_{\text{fin}}})$, and therefore $(V/V')_0$ is also the image $\pi(\overline{V' \cap V_{\text{fin}}})$ of the closed subspace $\overline{V' \cap V_{\text{fin}}}$ of V_{fin}.

PROOF. Clearly, $||.||'$ is a quasinorm on V/V' – namely, it satisfies (i) and (ii) in Definition 2.4.1. To prove that this quasinorm is Hermitian – namely, that it satisfies (iii) – we may argue as follows.

Let us first assume that $||.||$ is a Hermitian norm on V. When the normed space $(V, ||.||)$ is complete – i.e., when it is a Hilbert space – and V' is closed in V, we may consider the orthogonal V'' of V' in V. It is a closed vector subspace of V. Moreover, $V = V' \oplus V''$, and the isomorphism of complex vector spaces

$$\pi_{|V''} : V'' \longrightarrow V/V'$$

is an isometry between $(V'', ||.||_{|V''})$ and $(V/V', ||.||')$. As a consequence, $(V/V', ||.||')$ is a Hilbert space.

When $(V, ||.||)$ is possibly non-complete, and V' is an arbitrary complex subspace of V, consider the completion $(\widehat{V}, ||.||^\wedge)$ of the normed space $(V, ||.||)$ and the closure $\widehat{V'}$ of V' in $\widehat{V}$. The inclusion $V \hookrightarrow \widehat{V}$ defines a $\mathbb{C}$-linear map

$$j : V/V' \longrightarrow \widehat{V}/\widehat{V'}$$

with kernel

$$\ker j = \overline{V'}/V'.$$

Indeed, we have $\overline{V'} = V \cap \widehat{V'}$. Moreover, if $(||.||^\wedge)'$ denotes the Hilbertian norm on $\widehat{V}/\widehat{V'}$ defined as the quotient quasinorm of $||.||\wedge$, one easily checks the relation

$$(||j(w)||^\wedge)' = ||w||'$$

for every $w \in V/V'$.

This shows that $||.||'$ is a Hermitian seminorm on V/V' and that

$$(V/V')_0 = \overline{V'}/V'.$$

In the general case, the proof can be completed by a formal reduction to the situation where $||.||$ is a norm, which we leave to the reader. □

DEFINITION 2.4.6. Let $\varphi : V \to W$ be a linear map between quasinormed Hermitian vector spaces. Then φ is *admissible* if φ induces an isometry between $V/\operatorname{Ker}\varphi$, endowed with the quotient quasinorm, to $\varphi(V) \subset W$, endowed with the induced quasinorm.

For instance, the admissible injections are exactly the injective maps that induce an isometry onto their image.

2.4.2. Definite Hermitian quasinorms and Hilbert subspaces in Fréchet spaces.

2.4.2.1. In this subsection, we analyze the continuity properties of Hermitian quasinorms on a complex vector space that is equipped with a topology of Fréchet space.

PROPOSITION 2.4.7. *Let F be a complex Fréchet space, and let $\|.\|$ be a definite Hermitian quasinorm on F. Let us consider its $\mathbb{C}$-vector subspace:*

$$H := F_{\text{fin}} = \{x \in F \mid \|x\| < +\infty\}.$$

Then two of following three conditions imply the third one:

(i) The inclusion map $i : H \hookrightarrow F$ is continuous from the pre-Hilbert space $(H, \|.\|)$ to the Fréchet space F.
(ii) The normed $\mathbb{C}$-vector space $(H, \|.\|)$ is complete, or in other words, is a Hilbert space.
(iii) The function $\|.\| : F \to [0, +\infty]$ is lower semicontinuous on F equipped with its Fréchet topology.

Observe that, according to the very definition of lower semicontinuity, condition (iii) is equivalent to each of the following two conditions:

(iii)' For every $r \in \mathbb{R}_+$, the ball

$$\overline{B}_{\|.\|}(0, r) := \{v \in H \mid \|v\| \leq r\}$$

is closed in the Fréchet space F.

(iii)" If a sequence $(v_n)_{n\in\mathbb{N}}$ in F admits a limit v in the Fréchet space F, and if

$$\|v_n\| \leq 1 \quad \textit{for every } n \in \mathbb{N}, \tag{2.4.4}$$

then $\|v\| \leq 1$.

PROOF OF PROPOSITION 2.4.7. Let us assume that (i) and (ii) are satisfied, and let us consider a converging sequence $(v_n)_{n\in\mathbb{N}}$ of limit v in F such (2.4.4) is satisfied. As it is a bounded sequence in the Hilbert space $(H, \|.\|)$, it is weakly convergent in this Hilbert space. Moreover its weak limit w, like the v_n, belongs to the closed ball $\overline{B}_{\|.\|}(0, 1)$. Moreover the continuity of the inclusion morphism i implies that the sequence $(v_n)_{n\in\mathbb{N}}$ converges to w in F equipped with its weak topology. This implies that $w = v$, and finally establishes the upper bound $\|v\| \leq 1$.

Let us assume that (i) and (iii) are satisfied, and let us consider a Cauchy sequence $(v_n)_{n\in\mathbb{N}}$ in H. According to the continuity of i, it possesses a limit v in the Fréchet space F. Condition (iii)' shows that v belongs to H. Moreover, applied to the sequences $(v_{n+k} - v_k)_{n\in\mathbb{N}}$, which converges to $v - v_k$ in F, Condition (iii)' also shows that, for every $k \in \mathbb{N}$,

$$\|v_k - v\| \leq \sup_{n\in\mathbb{N}} \|v_{n+k} - v_k\|.$$

This converges to zero when k goes to infinity, and therefore $(v_n)_{n\in\mathbb{N}}$ converges to v in $(H, \|.\|)$.

Let us finally show that (ii) and (iii) imply (i). By Banach's closed graph theorem, to establish the continuity of i when (ii) is satisfied, it is enough to prove that, if a sequence $(x_n)_{n\in\mathbb{N}}$ in H admits a limit x in F and satisfies:

$$\lim_{n\to+\infty} \|x_n\| = 0,$$

then $x = 0$.

To achieve this, observe that, with the above notation, for any $k \in \mathbb{N}$, the sequence $(x_{k+n})_{n\in\mathbb{N}}$ is contained in $\overline{B}_{\|.\|}(0, r_k)$, where $r_k := \sup_{n\in\mathbb{N}} \|x_{k+n}\|$ goes to 0 when k grows to infinity. Therefore, when moreover (iii) is satisfied, its limit x also belongs to $\overline{B}_{\|.\|}(0, r_k)$, and therefore to the intersection:

$$\bigcap_{k\in\mathbb{N}} \overline{B}_{\|.\|}(0, r_k) = \{0\}. \qquad \square$$

2.4.2.2. Examples of pairs (H, F) of a Hilbert space H continuously embedded in a Fréchet space F — or equivalently of a definite Hermitian quasinorm on a Fréchet space F that satisfies the conditions (i)-(iii) in Proposition 2.4.8 — naturally occur in various areas of analysis and geometry. A noteworthy instance is the Hilbert space $H^1((0,1))$, of distributions with L^2 derivative on the interval $(0,1)$, embedded in the Banach space $\mathcal{C}([0,1])$ of continuous functions on $[0,1]$. The pair $(H^1((0,1)), \mathcal{C}([0,1]))$ plays a key role in the construction of the Brownian motion.

Examples of a Fréchet space F and of a definite Hermitian quasinorm $\|.\|$ on F that satisfy exactly one of the condition (i)-(iii) are easily constructed.

For instance, an example where (i) only is satisfied is obtained by taking for F some infinite dimensional Hilbert space, with Hilbert norm $\|.\|_0$, and by taking for H a vector subspace not closed in F, and by defining:

$$\|x\| := \begin{cases} \|x\|_0 & \text{if } x \in H, \\ +\infty & \text{if } x \in F \setminus H. \end{cases}$$

Examples where (iii) only is satisfied may be constructed by using that an infinite dimensional Hilbert space admit some non-continuous automorphisms when seen as a $\mathbb{C}$-vector space.

Finally, an example where (iii) only is satisfied is given by a Fréchet space F admitting a continuous Hermitian norm $\|.\|$ such that $(F, \|.\|)$ is not complete.[19]

2.4.3. Definite Hermitian quasinorms and Hilbert spaces in $\mathbb{C}^{\mathbb{N}}$. When the Fréchet space F is $\mathbb{C}^{\mathbb{N}}$ equipped with the topology of simple convergence of sequences, conditions (i) and (ii) in Proposition 2.4.7 not only imply condition (iii), but actually are equivalent to condition (iii).

This will follow from Proposition 2.4.7 combined with the description of lower semi-continuous definite Hermitian quasinorms on $\mathbb{C}^{\mathbb{N}}$ in Proposition 2.4.8 below, which constitutes an infinite dimensional generalization of Example 2.4.2.

2.4.3.1. The vector space $\mathbb{C}^{\mathbb{N}}$ may be identified to the vector space $\mathbb{C}^{(\mathbb{N})\vee}$, dual of the vector space $\mathbb{C}^{(\mathbb{N})}$ of sequences with finite supports, by means of the pairing $\langle \cdot, \cdot \rangle$ defined by

$$\langle \xi, x \rangle := \sum_{n \in \mathbb{N}} \xi_n x_n$$

for any $\xi = (\xi_n)_{n \in \mathbb{N}}$ in $\mathbb{C}^{\mathbb{N}}$ and any $x = (x_n)_{n \in \mathbb{N}}$ in $\mathbb{C}^{(\mathbb{N})}$.

The Fréchet topology on $\mathbb{C}^{\mathbb{N}}$ coincides with the topology defined by this duality, namely the $\sigma(\mathbb{C}^{\mathbb{N}}, \mathbb{C}^{(\mathbb{N})})$-topology (*cf.* [**Bou81**, Section II.6.2]). The vector space $\mathbb{C}^{(\mathbb{N})}$ may be identified, by means of the pairing $\langle \cdot, \cdot \rangle$, with the topological dual of the Fréchet space $\mathbb{C}^{\mathbb{N}}$. Moreover the strong topology on this dual coincides with the inductive topology[20] on $\mathbb{C}^{(\mathbb{N})}$, and the Fréchet space $\mathbb{C}^{\mathbb{N}}$ may also be identified with the strong dual of $\mathbb{C}^{(\mathbb{N})}$ equipped with its inductive topology.

PROPOSITION 2.4.8. *For any Hermitian seminorm $\|.\|$ on $\mathbb{C}^{(\mathbb{N})}$, the function:*

$$\|.\|^\vee : \mathbb{C}^{\mathbb{N}} \longrightarrow [0, +\infty]$$

defined by the equality:

$$\|\xi\|^\vee := \sup \left\{ |\langle \xi, x \rangle|, x \in \mathbb{C}^{(\mathbb{N})} \text{ and } \|x\| \leq 1 \right\} \tag{2.4.5}$$

is a definite Hermitian quasinorm on $\mathbb{C}^{\mathbb{N}}$, and is lower semicontinuous when $\mathbb{C}^{\mathbb{N}}$ is equipped with its Fréchet space topology.

[19] For instance, we may take for F some infinite dimensional Hilbert space, and for $\|.\|$ a Hermitian norm on F compact with respect to the Hilbert norm of F.

[20] See 1.4.1 above. The $\mathbb{C}$-vector space $\mathbb{C}^{(\mathbb{N})}$ equipped with its inductive topology may be identified with the inductive limit, in the category of locally convex topological $\mathbb{C}$-vector spaces, of its finite-dimensional vector subspaces equipped with their canonical (Hausdorff locally convex) topology.

Moreover, for any (x, ξ) *in* $\mathbb{C}^{(\mathbb{N})} \times \mathbb{C}^{\mathbb{N}}$, *the inequality:*

$$|\langle \xi, x\rangle| \leq \|\xi\|^{\vee}\, \|x\| \tag{2.4.6}$$

holds when $(\|x\|, \|\xi\|^{\vee}) \neq (0, +\infty)$.

Conversely, any lower semicontinuous definite Hermitian quasinorm $|||.|||$ *on* $\mathbb{C}^{\mathbb{N}}$ *is of the form* $\|.\|^{\vee}$ *for a unique Hermitian semi-norm* $\|.\|$ *on* $\mathbb{C}^{(\mathbb{N})}$, *which satisfies, for every* $x \in \mathbb{C}^{(\mathbb{N})}$*:*

$$\|x\| = \sup \left\{ |\langle \xi, x\rangle|, \xi \in \mathbb{C}^{\mathbb{N}} \text{ and } |||\xi||| \leq 1 \right\}. \tag{2.4.7}$$

As usual, to define the right-hand side of (2.4.6), we use the convention: $+\infty.0 = 0$. If $\|.\| = 0$, then $\|\xi\|^{\vee} = +\infty$ for every non-zero ξ in $\mathbb{C}^{\mathbb{N}}$. In particular, the inequality (2.4.6) does *not* hold for a pair (x, ξ) such that $\langle \xi, x\rangle \neq 0$.

COROLLARY 2.4.9. *For any Hermitian seminorm* $\|.\|$ *on* $\mathbb{C}^{(\mathbb{N})}$, *when equipped with the restriction of the quasinorm* $\|.\|^{\vee}$ *defined by* (2.4.5), *the complex vector space:*

$$H := \left\{ \xi \in \mathbb{C}^{\mathbb{N}} \mid \|\xi\|^{\vee} < +\infty \right\}$$

becomes a Hilbert space.

PROOF. The validity of (2.4.6) when ξ belongs to H implies that the inclusion morphism $i : H \hookrightarrow \mathbb{C}^{\mathbb{N}}$ is continuous from the pre-Hilbert space $(H, \|.\|^{\vee}_{|H})$ to the Fréchet space $\mathbb{C}^{\mathbb{N}}$. According to Proposition 2.4.7, $(H, \|.\|^{\vee}_{|H})$ is therefore a Hilbert space. □

2.4.3.2. PROOF OF PROPOSITION 2.4.8. (1) For every $n \in \mathbb{N}$, let us consider the exhaustive filtration $(F_n)_{n\in\mathbb{N}}$ of $\mathbb{C}^{\mathbb{N}}$ by the subspaces F_n defined by:

$$(x_k)_{k\in\mathbb{N}} \in F_n \Longleftrightarrow \forall\, k \in \mathbb{N}_{\geq n}, x_k = 0.$$

Let us also consider the "closed unit ball" in $(\mathbb{C}^{(\mathbb{N})}, \|.\|)$,

$$B := \{x \in \mathbb{C}^{(\mathbb{N})} \mid \|x\| \leq 1\},$$

and its intersection with F_n,

$$B_n := B \cap F_n.$$

For any x in $\mathbb{C}^{\mathbb{N}}$ and any n in $\mathbb{N}$, we let:

$$\|\xi\|^{\vee}_n := \sup_{x\in B_n} |\langle \xi, x\rangle| \in [0, +\infty].$$

Then, for any n, $\|.\|^{\vee}_n$ is clearly a Hermitian quasinorm on $\mathbb{C}^{\mathbb{N}}$. Actually it is the "pull-back" by the projection

$$p_n : \mathbb{C}^N \longrightarrow \mathbb{C}^n, \quad (\xi_k)_{k\in\mathbb{N}} \longmapsto (\xi_0, \cdots, \xi_{n-1})$$

of some Hermitian quasinorm on $\mathbb{C}^n$, deduced by duality from the Hermitian semi-norm $\|.\|_{|F_n}$ on $F_n \simeq \mathbb{C}^n$. Moreover, for any ξ in $\mathbb{C}^{\mathbb{N}}$, the sequence $(\|\xi\|^{\vee}_n)_{n\in\mathbb{N}})$ is non-increasing and satisfies:

$$\|\xi\|^{\vee} = \sup_{n\in\mathbb{N}} \|\xi\|^{\vee}_n = \lim_{n\to+\infty} \|\xi\|^{\vee}_n.$$

This makes clear that $\|.\|$ is lower semicontinuous Hermitian quasinorm on $\mathbb{C}^{\mathbb{N}}$. Moreover it is definite. Indeed, for any ξ in $\mathbb{C}^{\mathbb{N}} \setminus \{0\}$, there exists $x \in \mathbb{C}^{(\mathbb{N})}$ such that $\langle \xi, x\rangle \neq 0$; then

$$\tilde{x} := \max(1, \|x\|)^{-1} x$$

satisfies $\|\tilde{x}\| \leq 1$, and therefore:

$$\|\xi\|^{\vee} \geq |\langle \xi, \tilde{x}\rangle| = \max(1, \|x\|)^{-1} |\langle \xi, x\rangle| > 0.$$

(2) The inequality (2.4.6) holds when $\|x\| = 1$ by the very definition of $\|\xi\|^{\vee}$. By homogeneity, it remains true for any x in $\mathbb{C}^{(\mathbb{N})}$ such that $\|x\| \neq 0$.

To prove that (2.4.6) also holds when $\|\xi\|^\vee < +\infty$, we are left to show that it holds when moreover $\|x\| = 0$; or in other words, we are left to establish the implication:

$$\|x\| = 0 \quad \text{and} \quad \|\xi\|^\vee < +\infty \Longrightarrow \langle \xi, x\rangle = 0. \tag{2.4.8}$$

Observe that, when $\|.\| = 0$, the condition $\|\xi\|^\vee < +\infty$ implies $\xi = 0$. Therefore, to establish this implication, we may also assume that there exists y in $\mathbb{C}^{(\mathbb{N})}$ such that $\|y\| \neq 0$. Then, for any ε in $\mathbb{R}_+^*$, we have:

$$\|x + \varepsilon y\| = \varepsilon \|y\| \neq 0,$$

and therefore:

$$|\langle \xi, x\rangle + \varepsilon \langle \xi, y\rangle| = |\langle \xi, x + \varepsilon y\rangle| \leq \|\xi\|^\vee \, \|x + \varepsilon y\| = \|\xi\|^\vee \varepsilon \|y\|.$$

By taking the limit when ε goes to 0, we derive the vanishing of $\langle \xi, x\rangle = 0$.

(3) Let $|||.|||$ be a lower continuous definite Hermitian quasinorm on $\mathbb{C}^{\mathbb{N}}$.

Let us consider the following subset of $\mathbb{C}^{\mathbb{N}}$:

$$\overline{B} := \overline{B}_{|||.|||}(0,1) := \{\xi \in H \mid |||\xi||| \leq 1\}.$$

It is convex and balanced; moreover, it is closed in $\mathbb{C}^{\mathbb{N}}$ since $|||.|||$ is lower semi-continuous. Therefore, if we define the polar of $\overline{B}$ as

$$\overline{B}^\circ := \left\{ x \in \mathbb{C}^{(\mathbb{N})} \mid \forall \xi \in \overline{B}, |\langle \xi, x\rangle| \leq 1 \right\},$$

and its bipolar as

$$\overline{B}^{\circ\circ} := \left\{ \xi \in \mathbb{C}^{\mathbb{N}} \mid \forall x \in \overline{B}^\circ, |\langle \xi, x\rangle| \leq 1 \right\},$$

we have:

$$\overline{B}^{\circ\circ} = \overline{B}; \tag{2.4.9}$$

see [**Bou81**, II.6.3, Corollaire 3 and II.8.3, Proposition 2].

Let us define a function:

$$\|.\| : \mathbb{C}^{(\mathbb{N})} \longrightarrow [0, +\infty]$$

by the equation (2.4.7). For every x in $\mathbb{C}^{(\mathbb{N})}$ and any $t \in \mathbb{R}_+^*$, we have:

$$\|x\| := \sup_{\xi \in \overline{B}} |\langle \xi, x\rangle| \leq t \iff x \in t\overline{B}^\circ.$$

In particular, $\|x\| = +\infty$ if and only if $\mathbb{R}_+^* x \cap \overline{B}^\circ = \emptyset$. Since $\overline{B}^\circ$ is convex and balanced, this holds precisely when the complex vector subspace of $\mathbb{C}^{(\mathbb{N})}$ generated by $\overline{B}^\circ$ does not contain x. The existence of a point x such that $\|x\| = +\infty$ would therefore imply the existence of some ξ in $\mathbb{C}^{\mathbb{N}} \setminus \{0\}$ whose kernel would contain $\overline{B}^\circ$. Then the line $\mathbb{C}.\xi$ would be contained in $\overline{B}^{\circ\circ}$, hence in $\overline{B}$ by (2.4.9). This would contradict the definiteness of $|||.|||$.

This shows that $\|.\|$ takes its values in $[0, +\infty)$, and is therefore a seminorm on $\mathbb{C}^{(\mathbb{N})}$.

For any V in the set $\mathcal{F}(\mathbb{C}^{\mathbb{N}})$ of finite dimensional complex vector subspaces of $\mathbb{C}^{\mathbb{N}}$, we may consider the map:

$$\|.\|_V : \mathbb{C}^{(\mathbb{N})} \longrightarrow [0, +\infty]$$

defined by:

$$\|x\|_V := \sup_{\xi \in \overline{B} \cap V} |\langle \xi, x\rangle|.$$

Let us introduce

$$V^\perp := \{x \in \mathbb{C}^{(\mathbb{N})} \mid \forall \xi \in V, \langle \xi, x\rangle = 0\},$$

and

$$V_{\text{fin}} := \{\xi \in V \mid |||\xi||| < +\infty\}.$$

Then $\mathbb{C}^{\mathbb{N}}/V^{\perp}$ may be identified with the dual $V^{\vee}$, the inclusion $V_{\mathrm{fin}} \hookrightarrow V$ defines by duality a surjective linear map $V^{\vee} \twoheadrightarrow V_{\mathrm{fin}}^{\vee}$, and we may consider the composite map:

$$p_V := \mathbb{C}^{(\mathbb{N})} \twoheadrightarrow \mathbb{C}^{(\mathbb{N})}/V^{\perp} \xrightarrow{\sim} V^{\vee} \twoheadrightarrow V_{\mathrm{fin}}^{\vee}.$$

The restriction $|||.|||_{|V_{\mathrm{fin}}}$ of $|||.|||$ to the finite-dimensional space V_{fin} is a Hermitian norm. By duality, it defines a Hermitian norm $|||.|||_{|V_{\mathrm{fin}}}^{\vee}$ on $V_{\mathrm{fin}}^{\vee}$, and it directly follows from the definitions that, for any x in $\mathbb{C}^{(\mathbb{N})}$,

$$\|x\|_V = |||p_V(x)|||_{|V_{\mathrm{fin}}}^{\vee}.$$

This shows that $\|.\|_V$ is a Hermitian semi-norm on $\mathbb{C}^{(\mathbb{N})}$.

Ordered by inclusion, $\mathcal{F}(\mathbb{C}^{\mathbb{N}})$ is a directed poset. It is straightforward that, for any V and V' in $\mathcal{F}(\mathbb{C}^{\mathbb{N}})$, we have:

$$V \subseteq V' \Longrightarrow \|.\|_V \leq \|.\|_{V'},$$

and that, for any $x \in \mathbb{C}^{(\mathbb{N})}$,

$$\|x\| := \sup_{V \in \mathcal{F}(\mathbb{C}^{\mathbb{N}})} \|x\|_V.$$

The stability of Hermitian quasinorms under the operation of "filtrant supremum" stated in Proposition 2.4.3 therefore implies that the seminorm $\|.\|$ is Hermitian. Moreover, the equality (2.4.9) implies (indeed is equivalent to) the relation:

$$|||.||| = \|.\|^{\vee}. \tag{2.4.10}$$

The unicity of the Hermitian seminorm $\|.\|$ on $\mathbb{C}^{(\mathbb{N})}$ that satisfies (2.4.10) follows again from an argument involving bipolars. Indeed, if we let

$$C := \{x \in \mathbb{C}^{(\mathbb{N})} \mid \|x\| \leq 1\},$$

the relation (2.4.10) is equivalent to the equality:

$$\overline{B} = C^{\circ} := \{\xi \in \mathbb{C}^{\mathbb{N}} \mid \forall x \in C, |\langle \xi, x\rangle \leq 1\}.$$

Moreover, according to [**Bou81**, II.6.3, Corollaire 3 and II.8.3, Proposition 2] applied in $\mathbb{C}^{(\mathbb{N})}$ equipped with the inductive topology, we have:

$$C^{\circ\circ} = C.$$

Therefore, if (2.4.10) holds, then we have:

$$C = \overline{B}^{\circ},$$

and consequently $\|.\|$ necessarily satisfies (2.4.7). □

2.4.3.3. For later reference, we spell out the following consequence of the previous results in this subsection.

Corollary 2.4.10. *Let $(H, |.|)$ be a complex Hilbert space, and let $i : H \hookrightarrow \mathbb{C}^{\mathbb{N}}$ be an injective $\mathbb{C}$-linear map, continuous from $(H, |.|)$ to $\mathbb{C}^{\mathbb{N}}$ equipped with its Fréchet space topology.*

Then the function:

$$\|.\|_{H,i} : \mathbb{C}^{\mathbb{N}} \longrightarrow [0, +\infty]$$

defined by the equality:

$$\|\xi\|_{H,i} := \begin{cases} |i^{-1}(\xi)| & \text{if } \xi \in i(H) \\ +\infty & \text{if } \xi \in \mathbb{C}^{\mathbb{N}} \setminus i(H) \end{cases}$$

is a lower continuous definite Hermitian quasinorm on the Fréchet space $\mathbb{C}^{\mathbb{N}}$.

Conversely, any lower continuous definite Hermitian quasinorm of $\mathbb{C}^{\mathbb{N}}$ is of the form $\|.\|_{H,i}$ for a complex Hilbert space $(H, |.|)$ and a continuous injective $\mathbb{C}$-linear map i as above, which are uniquely determined (up to a unique isomorphism).

PROOF. By duality, the map i determines a $\mathbb{C}$-linear map

$$i^\vee : \mathbb{C}^{(\mathbb{N})} \longrightarrow H^\vee$$

from $\mathbb{C}^{(\mathbb{N})}$ to the Hilbert space $(H^\vee, |.|^\vee)$ dual of $(H, |.|)$. Since i is injective, its transpose $i^\vee$ has a dense image, and the double transpose $i^{\vee\vee}$ may be identified with i. This implies that a linear form $\xi \in \mathbb{C}^{\mathbb{N}}$ on $\mathbb{C}^{(\mathbb{N})}$ belongs to the image of i if and and only if it factorizes through $i^\vee$ and defines a continuous linear form on $(H^\vee, \|.\|^\vee)$, and that the following equality holds:

$$\|\xi\|_{H,i} = \sup\left\{ |\langle \xi, x\rangle|; x \in \mathbb{C}^{(\mathbb{N})} \text{ and } |i^\vee(x)|^\vee \leq 1 \right\}. \tag{2.4.11}$$

This expression for $\|.\|_{H,i}$ establishes its lower semi-continuity. It actually shows that $\|.\|_{H,i}$ coincides with the Hermitian quasinorm $\|.\|^\vee$ dual of the Hermitian semi-norm on $\mathbb{C}^{(\mathbb{N})}$ defined as:

$$\|.\| : \mathbb{C}^{(\mathbb{N})} \longrightarrow [0, +\infty), \quad x \longmapsto |i^\vee(x)|^\vee.$$

The last assertion of the proposition follows from Proposition 2.4.8 and Corollary 2.4.9. □

Finally, we may formulate an analogue, valid in the infinite dimensional setting, of the equivalence (2.4.1) in Example 2.4.2.

PROPOSITION 2.4.11. *Consider a Hermitian seminorm $\|.\|$ on $\mathbb{C}^{(\mathbb{N})}$ and the dual quasinorm $\|.\|^\vee$ on $\mathbb{C}^{\mathbb{N}}$ defined by* (2.4.5). *If we let:*

$$K := \{x \in \mathbb{C}^{(\mathbb{N})} \mid \|x\| = 0\},$$

then the closure $\overline{H}$ in the Fréchet space $\mathbb{C}^{\mathbb{N}}$ of its vector subspace:

$$H := \left\{\xi \in \mathbb{C}^{\mathbb{N}} \mid \|\xi\|^\vee < +\infty\right\}$$

satisfies the equality:

$$\overline{H} = K^\perp := \left\{\xi \in \mathbb{C}^{\mathbb{N}} \mid \forall x \in K, \langle \xi, x\rangle = 0\right\}.$$

In particular, H is dense in $\mathbb{C}^{\mathbb{N}}$ if and only if the seminorm $\|.\|$ is actually a norm.

Proposition 2.4.11 follows from a straightforward duality argument that we leave to the reader.

2.4.4. The equivalence of categories $\cdot^\vee : \mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]} \xrightarrow{\sim} \mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$.

2.4.4.1. The results of the preceding subsection may be rephrased in terms of the categories $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$ and $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$, and of their variants $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]\leq 1}$ and $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]\leq 1}$, defined as follows.

The categories $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$ and $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]\leq 1}$ admit as objects the pairs $(W, \|.\|)$ where W is a $\mathbb{C}$-vector space of at most countable dimension. If $(W, \|.\|)$ and $(W', \|.\|')$ are two such pairs, a morphism:

$$\varphi : (W, \|.\|) \longrightarrow (W', \|.\|')$$

in the category $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$ (resp. in $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]\leq 1}$) is a $\mathbb{C}$-linear map $\varphi : W \to W'$ whose operator norm:

$$|||\varphi||| := \sup_{w \in W, \|w\| \leq 1} \|\varphi(w)\|'$$

satisfies the inequality:

$$|||\varphi||| < +\infty \quad (\text{resp. } |||\varphi||| \leq 1).$$

The categories $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$ and $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]\leq 1}$ admit as objects the pairs $(F, \|.\|)$ where F is complex Fréchet space isomorphic to $\mathbb{C}^I$ for some at most countable index set I, and $\|.\|$ is a lower semicontinuous definite quasinorm over F. If $(F, \|.\|)$ and $(F', \|.\|')$ are two such pairs, a morphism:

$$\psi : (F, \|.\|) \longrightarrow (F', \|.\|')$$

in the category $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$ (resp. in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]\leq 1}$) is a continuous $\mathbb{C}$-linear map between Fréchet spaces $\psi : F \to F'$ whose operator quasinorm with respect to $\|.\|$ and $\|.\|'$:

$$|||\psi||| := \inf\left\{\lambda \in \mathbb{R}_+ \mid \forall v \in F, \|\psi(v)\|' \leq \lambda \|v\|\right\}$$

satisfies the inequality:

$$|||\psi||| < +\infty \quad (\text{resp. } |||\psi||| \leq 1).$$

Observe that the isomorphisms in the category $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]\leq 1}$ (resp. in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]\leq 1}$) are precisely isometric isomorphims between objects of $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$ (resp. of $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$). Namely a morphism $\varphi : (W, \|.\|) \to (W', \|.\|')$ (resp. $\psi : (F, \|.\|) \to (F', \|.\|')$) as above is an isomorphism in $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]\leq 1}$ (resp. in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]\leq 1}$) if and only if φ (resp. ψ) establishes a $\mathbb{C}$-linear bijection between W and W' (resp. a $\mathbb{C}$-linear homeomorphism between F and F') and if, for every $x \in W$ (resp. every $\xi \in F$), the following equality holds:

$$\|\varphi(x)\|' = \|x\| \quad (\text{resp. } \|\psi(\xi)\|' = \|\xi\|).$$

2.4.4.2. To an object $(W, \|.\|)$ in $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$, we may attach the dual object in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$ defined by:

$$(W, \|.\|)^\vee := (W^\vee, \|.\|^\vee), \tag{2.4.12}$$

where $W^\vee$ is the Fréchet space:

$$W^\vee := \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$$

of linear forms on W equipped with the $\sigma(W^\vee, W)$-topology and where $\|.\|^\vee$ is the lower continuous definite quasinorm on $W^\vee$ defined by the equality:

$$\|\xi\|^\vee := \sup\left\{|\langle \xi, x\rangle|, x \in W \text{ and } \|x\| \leq 1\right\}.$$

Conversely, to an object $(F, \|.\|)$ of $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$ we may attach the dual object in $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$ defined by:

$$(F, \|.\|)^\vee := (F^\vee, \|.\|^\vee), \tag{2.4.13}$$

where:

$$F^\vee := \mathrm{Hom}_{\mathbb{C}}^{\mathrm{cont}}(F, \mathbb{C})$$

denotes the $\mathbb{C}$-vector space of continuous linear forms on F, and where $\|.\|^\vee$ is the Hermitian seminorm on $F^\vee$ defined by the following equality, for every $x \in F^\vee$:

$$\|x\|^\vee := \sup\left\{|\langle x, \xi\rangle|, \xi \in F \text{ and } \|\xi\| \leq 1\right\}.$$

According to Example 2.4.2 and Proposition 2.4.8, these two constructions are well-defined, and are inverse of each other. Namely, for every object $(W, \|.\|)$ (resp. $(F, \|.\|)$) of $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$ (resp. of $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$), we have a canonical biduality isomorphism:

$$(W, \|.\|)^{\vee\vee} \xrightarrow{\sim} (W, \|.\|) \quad (\text{resp. } (F, \|.\|)^{\vee\vee} \xrightarrow{\sim} (F, \|.\|))$$

in $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]\leq 1}$ (resp. in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]\leq 1}$).

Consider two objects $(W, \|.\|)$ and $(W', \|.\|')$ in $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$, and the dual objects $(F, \|.\|^\vee) := (W, \|.\|)^\vee$ and $(F', \|.\|'^\vee) := (W', \|.\|')^\vee$ in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$. Then the transpose map establish a canonical isomorphism:

$$.^\vee : \mathrm{Hom}_{\mathbb{C}}(W, W') \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}^{\mathrm{cont}}(F', F).$$

It sends a $\mathbb{C}$-linear map $\varphi : W \to W'$ to the continous $\mathbb{C}$-linear map:

$$\varphi^\vee := \cdot \circ \varphi : F' := \mathrm{Hom}_{\mathbb{C}}(W', \mathbb{C}) \longrightarrow F := \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C}).$$

Its inverse sends a continuous $\mathbb{C}$-linear map $\psi : F' \to F$ to the map:

$$\psi^\vee := \cdot \circ \psi : W := \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{C}}(F, \mathbb{C}) \longrightarrow W' := \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{C}}(F', \mathbb{C}).$$

Moreover, with the previous notation, a straightforward duality argument shows that the operator norm $|||\varphi|||$ of φ (with respect to $\|.\|$ and $\|.\|'$) and the operator norm $|||\varphi^\vee|||$ of $\varphi^\vee$ (with respect to $\|.\|^\vee$ and $\|.\|'^\vee$) satisfy:

$$|||\varphi^\vee||| = |||\varphi|||. \tag{2.4.14}$$

This implies that we may define duality functors:

$$\cdot^\vee : \mathbf{ind}\overline{\mathbf{Vect}}^{[0]}_{\mathbb{C}} \xrightarrow{\sim} \mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]}_{\mathbb{C}} \quad \text{and} \quad \cdot^\vee : \mathbf{ind}\overline{\mathbf{Vect}}^{[0]\leq 1}_{\mathbb{C}} \xrightarrow{\sim} \mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]\leq 1}_{\mathbb{C}}$$

$$(\text{resp. } \cdot^\vee : \mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]}_{\mathbb{C}} \xrightarrow{\sim} \mathbf{ind}\overline{\mathbf{Vect}}^{[0]}_{\mathbb{C}} \quad \text{and} \quad \cdot^\vee : \mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]\leq 1}_{\mathbb{C}} \xrightarrow{\sim} \mathbf{ind}\overline{\mathbf{Vect}}^{[0]\leq 1}_{\mathbb{C}})$$

by the assignment (2.4.12) (resp. (2.4.13)) at the level of objects, and by sending a morphism φ in $\mathbf{ind}\overline{\mathbf{Vect}}^{[0]}_{\mathbb{C}}$ or $\mathbf{ind}\overline{\mathbf{Vect}}^{[0]\leq 1}_{\mathbb{C}}$ (resp. ψ in $\mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]}_{\mathbb{C}}$ or $\mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]\leq 1}_{\mathbb{C}}$) to its transpose $\varphi^\vee$ (resp. $\psi^\vee$).

Using Example 2.4.2 and Proposition 2.4.8, one easily checks that these duality functors define adjoint equivalences:

$$\cdot^\vee : \mathbf{ind}\overline{\mathbf{Vect}}^{[0]}_{\mathbb{C}} \leftrightarrows \mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]}_{\mathbb{C}} : \cdot^\vee$$

and:

$$\cdot^\vee : \mathbf{ind}\overline{\mathbf{Vect}}^{[0]\leq 1}_{\mathbb{C}} \leftrightarrows \mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]\leq 1}_{\mathbb{C}} : \cdot^\vee.$$

2.4.4.3. The following observation is a straightforward consequence of Hahn-Banch theorem. We spell it out for later reference.

Proposition 2.4.12. *Let $\overline{F} := (F, \|.\|)$ be an object of $\mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]}_{\mathbb{C}}$, and $\overline{F}^\vee := (F^\vee, \|.\|)^\vee$ be its dual in $\mathbf{ind}\overline{\mathbf{Vect}}^{[0]}_{\mathbb{C}}$.*

For every closed $\mathbb{C}$-vector subspace F' of the Fréchet space F, the pair $\overline{F}' := (F', \|.\|_{|F'})$ is an object of $\mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]}_{\mathbb{C}}$. Moreover its dual $\overline{F}'^\vee := (F'^\vee, \|.\|^\vee_{|F'})$ in $\mathbf{ind}\overline{\mathbf{Vect}}^{[0]}_{\mathbb{C}}$ may be described as follows.

The topological dual $F'^\vee$ fits into a short exact sequence of $\mathbb{C}$-vector spaces:

$$0 \longrightarrow F'^\perp \hookrightarrow F^\vee \xrightarrow{\cdot_{|F'}} F'^\vee \longrightarrow 0,$$

where:

$$F'^\perp := \{\xi \in F^\vee \mid \xi_{|F'} = 0\},$$

and where $\cdot_{|F'}$ denotes the restriction map $(\xi \mapsto \xi_{|F'})$, and the Hermitian seminorm $\|.\|^\vee_{|F'}$ on $F'^\vee$ may be identified with the quotient seminorm deduced from the seminorm $\|.\|^\vee$ on $F^\vee$.

2.5. The Duality Functors $\cdot^\vee : \mathbf{q}\overline{\mathbf{Coh}}_X \to \mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]}_X$ and $\cdot^\vee : \mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]}_X \to \mathbf{ind}\overline{\mathbf{Vect}}^{[0]}_X$

2.5.1. The categories $\mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]}_X$ and $\mathbf{pro}\overline{\mathbf{Vect}}^{[\infty]\leq 1}_X$.

2.5.1.1. Consider an object $(E, (\|.\|_x)_{x \in X(\mathbb{C})})$ of the category $\mathbf{CTC}_{\mathcal{O}_K}$, namely a topological module over the ring $\mathcal{O}_K$, equipped with the discrete topology, that is isomorphic to the limit $\varprojlim_n E_n$ of a projective system

$$E_0 \twoheadleftarrow E_1 \twoheadleftarrow \cdots E_n \twoheadleftarrow E_{n+1} \twoheadleftarrow \cdots \tag{2.5.1}$$

of surjective morphisms between finitely generated projective $\mathcal{O}_K$-modules, equipped with the discrete topology; see [**Bos20b**, Chapter 4] and Subsection 1.2.2 above.

For every embedding $x \in X(\mathbb{C})$ of K in $\mathbb{C}$, we may introduce the completed tensor product:

$$\widehat{E}_x := \widehat{E}\widehat{\otimes}_{\mathcal{O}_K,x}\mathbb{C};$$

see [**Bos20b**, 4.2.1]. It is a complex Fréchet space, isomorphic to $\mathbb{C}^n$ for some n in $\mathbb{N}$ or to $\mathbb{C}^{\mathbb{N}}$; see [**Bos20b**, 4.2.4 and 5.1.3]. When $\widehat{E}$ is the limit $\varprojlim_n E_n$ of the projective system (2.5.1), this Fréchet space may be identified with the limit $\varprojlim_n E_{n,x}$ of the projective system of finite dimensional $\mathbb{C}$-vector spaces:

$$E_{0,x} \twoheadleftarrow E_{1,x} \twoheadleftarrow \cdots E_{n,x} \twoheadleftarrow E_{n+1,x} \twoheadleftarrow \cdots$$

deduced from (2.5.1) by the extension of scalars $x : \mathcal{O}_K \to \mathbb{C}$.[21]

Moreover any morphism:

$$f : \widehat{E} \longrightarrow \widehat{F}$$

in $\mathbf{CTC}_{\mathcal{O}_K}$ — that is, any continuous $\mathcal{O}_K$-linear map between topological $\mathcal{O}_K$-modules as above — induces, by extension of scalars, a continuous $\mathbb{C}$-linear map between Fréchet spaces:

$$f_x : \widehat{E}_x := \widehat{E}\widehat{\otimes}_{\mathcal{O}_K,x}\mathbb{C} \longrightarrow \widehat{F}_x := \widehat{F}\widehat{\otimes}_{\mathcal{O}_K,x}\mathbb{C}.$$

2.5.1.2. The following definition is a generalization of the definition of the pro-Hermitian vector bundles and of their morphisms, which play a central role in [**Bos20b**]

DEFINITION 2.5.1. A *generalized pro-Hermitian vector bundle* over the arithmetic curve X is a pair:

$$\widehat{\overline{E}} := (\widehat{E}, (\|.\|_x)_{x \in X(\mathbb{C})}),$$

where $\widehat{E}$ is an object of $\mathbf{CTC}_{\mathcal{O}_K}$, and where $(\|.\|_x)_{x\in X(\mathbb{C})}$ is a family, invariant under complex conjugation, of lower continuous definite Hermitian quasinorms on the complex Fréchet spaces $(\widehat{E}_x)_{x \in X(\mathbb{C})}$.

If $\widehat{\overline{E}}_1 := (\widehat{E}_1, (\|.\|_{1,x})_{x\in X(\mathbb{C})})$ and $\widehat{\overline{E}}_2 := (\widehat{E}_2, (\|.\|_{x,2})_{x\in X(\mathbb{C})})$ are two generalized pro-Hermitian vector bundles over X, a *morphism of generalized pro-Hermitian vector bundles*:

$$\psi : \widehat{\overline{E}}_1 \longrightarrow \widehat{\overline{E}}_2$$

is a morphism $\psi : \widehat{E}_1 \to \widehat{E}_2$ in $\mathbf{CTC}_{\mathcal{O}_K}$ — that is, a continuous $\mathcal{O}_K$-linear map — such that, for every $x \in X(\mathbb{C})$, the continuous $\mathbb{C}$-linear map:

$$\psi_x : \widehat{E}_{1,x} := \widehat{E}_1\widehat{\otimes}_{\mathcal{O}_K,x}\mathbb{C} \longrightarrow \widehat{E}_{2,x} := \widehat{E}_2\widehat{\otimes}_{\mathcal{O}_K,x}\mathbb{C}$$

are continuous when $\widehat{E}_{1,x}$ and $\widehat{E}_{2,x}$ are equipped with the topology defined by the quasinorms $\|.\|_{1,x}$ and $\|.\|_{2,x}$, or in other words, when their operator quasinorms $|||\psi_x|||$ with respect to $\|.\|_{1,x}$ and $\|.\|_{2,x}$ — which are defined by the relations:

$$|||\psi_x||| := \inf\left\{\lambda \in \mathbb{R}_+ \mid \forall v \in \widehat{E}_{1,x}, \|\psi(v)\|_{2,x} \le \lambda \|v\|_{1,x}\right\}$$

— satisfy the estimates:

$$|||\psi_x||| < +\infty. \tag{2.5.2}$$

[21] The finite dimensional $\mathbb{C}$-vector spaces $E_{n,x}$ are endowed with their canonical topology of Hausdorff complex topological vector space.

Equivalently, if we introduce the following objects in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$:

$$\overline{\widehat{E}}_{1,x} := (\widehat{E}_{1,x}, \|.\|_{1,x}) \quad \text{and} \quad \overline{\widehat{E}}_{2,x} := (\widehat{E}_{2,x}, \|.\|_{2,x}),$$

condition (2.5.2) means that ψ_x defines a morphism in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$:

$$\psi_x : \overline{\widehat{E}}_{1,x} \longrightarrow \overline{\widehat{E}}_{2,x}.$$

The generalized pro-Hermitian vector bundles over X and their morphisms, as defined above, constitue an $\mathcal{O}_K$-linear category, that we will denote by $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$.

The subcategory of $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$ whose objects are the generalized pro-Hermitian vector bundles over X and whose morphisms are the morphisms $\psi : \overline{\widehat{E}}_1 \to \overline{\widehat{E}}_2$ as defined above of operator quasinorms at most one — namely those morphisms such that:

$$|||\psi_x|||_x \leq 1 \quad \text{for every } x \in X(\mathbb{C})$$

— will be denoted by $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1}$.

Here again, the isomorphisms in the category $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1}$ are precisely the isometric isomorphisms, in the obvious sense, between generalized pro-Hermitian vector bundles over X.

2.5.1.3. The pro-Hermitian vector bundles over X introduced in [**Bos20b**, Chapter 5] may be identified with the generalized pro-Hermitian vector bundles $\overline{\widehat{E}} := (\widehat{E}, (\|.\|_x)_{x\in X(\mathbb{C})})$, defined as in Definition 2.5.1 above, that satisfy the following additional condition: *for every* $x \in X(\mathbb{C})$*, the vector subspace:*

$$\overline{E}_{x,\mathrm{fin}} := \{v \in \overline{E}_x \mid \|v\|_x < +\infty\}$$

is dense in the Fréchet space $\overline{E}_x$.

This directly follows from Corollary 2.4.10. Actually the categories $\mathbf{pro}\overline{\mathbf{Vect}}_X$ and $\mathbf{pro}\overline{\mathbf{Vect}}_X^{\leq 1}$ introduced in [**Bos20b**, section 5.4] are naturally equivalent to the full subcategories of $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$ and $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1}$ whose objects satisfy this density condition.

2.5.1.4. When X is $\operatorname{Spec}\mathbb{Z}$, we write $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[\infty]}$ and $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[\infty]\leq 1}$ instead of $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$ and $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1}$, and the objects in these categories will be called *generalized pro-Euclidean vector bundles.*

2.5.2. The duality functor $\cdot^\vee : \mathbf{q}\overline{\mathbf{Coh}}_X \to \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$.

2.5.2.1. Let $\overline{E} := (E, (|.\|_x)_{x\in X(\mathbb{C})})$ be Hermitian quasi-coherent sheaf over X. In order to define the dual object $\overline{E}^\vee$ in the category $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$, we need some preliminary results comparing the completed tensor products $(E^\vee)_x := E^\vee \widehat{\otimes}_{\mathcal{O}_K,x}\mathbb{C}$ of the dual $\mathcal{O}_K$-module $E^\vee$, and the duals $(E_x)^\vee$ of the complex vector spaces E_x.

To the countably generated $\mathcal{O}_K$-module E, we may attach the dual $\mathcal{O}_K$-module:

$$E^\vee := \mathrm{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K).$$

As shown in Theorem 1.2.2, equipped with the topology of pointwise convergence,[22] it is an object of $\mathbf{CTC}_{\mathcal{O}_K}$. For every embedding $x \in X(\mathbb{C})$ of K in $\mathbb{C}$, we may also consider the completed tensor product:

$$(E^\vee)_x := E^\vee \widehat{\otimes}_{\mathcal{O}_K,x}\mathbb{C};$$

see [**Bos20b**, 4.2.1]. It is a complex Fréchet space, isomorphic to $\mathbb{C}^n$ for some n in $\mathbb{N}$ or to $\mathbb{C}^{\mathbb{N}}$; see [**Bos20b**, 4.2.4 and 5.1.3].

[22] the ring $\mathcal{O}_K$ being equipped with the discrete topology.

Actually, we may choose an exhaustive filtration of E by finite generated $\mathcal{O}_K$-submodules:

$$E_0 \subseteq E_1 \subseteq \cdots E_n \subseteq E_{n+1} \subseteq \cdots , \tag{2.5.3}$$

and introduce the projective system defined by the dual $\mathcal{O}_K$-modules and the transposes of the inclusion maps in (2.5.3):

$$E_0^\vee \longleftarrow E_1^\vee \longleftarrow \cdots E_n^\vee \longleftarrow E_{n+1}^\vee \longleftarrow \cdots ,$$

and we have a canonical isomorphism of topological $\mathcal{O}_K$-modules:

$$E^\vee \simeq \varprojlim_n E_n^\vee,$$

where the finitely generated projective $\mathcal{O}_K$-modules $E_n^\vee$ are equipped with the discrete topology.

For every $n \in \mathbb{N}$, we may introduce the following $\mathcal{O}_K$-submodule of $E_n^\vee$:

$$E_n^\sim := \mathrm{im}\,(E^\vee \longrightarrow E_n^\vee).$$

These modules are finitely generated and projective, and fits into a "projective subsystem" of (2.5.3):

$$E_0^\sim \twoheadleftarrow E_1^\sim \twoheadleftarrow \cdots E_n^\sim \twoheadleftarrow E_{n+1}^\sim \twoheadleftarrow \cdots . \tag{2.5.4}$$

By construction, the limit of (2.5.4) is canonically isomorphic to the one of (2.5.3). This defines an isomorphism of topological $\mathcal{O}_K$-modules:

$$E^\vee \simeq \varprojlim_n E_n^\sim. \tag{2.5.5}$$

2.5.2.2. Let $x \in X(\mathbb{C})$ be an imbedding of K in $\mathbb{C}$.

The $\mathbb{C}$-vector space $E_x := E \otimes_{\mathcal{O}_K,x} \mathbb{C}$ admits the following exhaustive filtration by finite dimensional vector subspaces:

$$E_{0,x} \subseteq E_{1,x} \subseteq \cdots E_{n,x} \subseteq E_{n+1,x} \subseteq \cdots ,$$

We may consider the dual projective system defined by the transpose maps of the inclusion maps, which indeed are surjective:

$$E_{0,x}^\vee \twoheadleftarrow E_{1,x}^\vee \twoheadleftarrow \cdots E_{n,x}^\vee \twoheadleftarrow E_{n+1,x}^\vee \twoheadleftarrow \cdots ,$$

where:

$$E_{n,x}^\vee := \mathrm{Hom}_{\mathbb{C}}(E_{n,x}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(E_n \otimes_{\mathcal{O}_K,x} \mathbb{C}, \mathbb{C}) \simeq \mathrm{Hom}_{\mathcal{O}_K}(E_n, \mathcal{O}_K) \otimes_{\mathcal{O}_K,x} \mathbb{C} = (E_n^\vee)_x,$$

and we have a canonical isomorphism of topological $\mathbb{C}$-vector spaces:

$$(E_x)^\vee := \mathrm{Hom}_{\mathbb{C}}(E_x, \mathbb{C}) \simeq \varprojlim_n E_{n,x}^\vee, \tag{2.5.6}$$

when $(E_x)^\vee$ is equipped with its natural structure of Fréchet space, and the finite dimensional $\mathbb{C}$-vector spaces $E_{n,x}^\vee$ with their canonical Hausdorff topology.

We may also consider the completed tensor product:

$$(E^\vee)_x := E^\vee \widehat{\otimes}_{\mathcal{O}_K,x} \mathbb{C}.$$

Using the realization (2.5.5) of $E^\vee$ as the limit of a projective systems of *surjective* morphisms between finitely generated projective $\mathcal{O}_K$-modules, we get an isomorphism:

$$(E^\vee)_x \simeq \varprojlim_n E_{n,x}^\sim,$$

where:

$$E_{n,x}^\sim := E_n^\sim \otimes_{\mathcal{O}_K,x} \mathbb{C}.$$

The injection of $\mathcal{O}_K$-modules $E_n^\sim \hookrightarrow E_n^\vee$ define injections of $\mathbb{C}$-vector spaces $E_{n,x}^\sim \hookrightarrow E_{n,x}^\vee$, which in turn define an injection of projective limits:

$$(E^\vee)_x \simeq \varprojlim_n E_{n,x}^\sim \hookrightarrow \varprojlim_n E_{n,x}^\vee \simeq (E_x)^\vee.$$

This shows that the canonical map:

$$E^\vee \otimes_{\mathcal{O}_K,x} \mathbb{C} := \mathrm{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K) \otimes_{\mathcal{O}_K,x} \mathbb{C} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathcal{O}_K,x}, \mathbb{C}) =: (E_x)^\vee$$

uniquely extends to a continuous injection with closed image:

$$(E^\vee)_x := E^\vee \widehat{\otimes}_{\mathcal{O}_K,x} \mathbb{C} \hookrightarrow (E_x)^\vee,$$

which actually establishes a homeomorphism between $(E^\vee)_x$ and its image in $(E_x)^\vee$.

2.5.2.3. For every $x \in X(\mathbb{C})$, we may consider the dual:

$$(E_x, \|.\|_x)^\vee := (E_x^\vee, \|.\|_x^\vee)$$

in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$ of the object $(E_x, \|.\|_x)$ of $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$.

Equipped with the restriction:

$$\|.\|_x^\sim := \|.\|_{x|(E^\vee)_x}$$

of the quasinorm $\|.\|_x^\vee$, the Fréchet spaces $(E^\vee)_x$ also define an object $((E^\vee)_x, \|.|_x^\sim)$ of $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$, and we define the dual in $\mathbf{pro}\overline{\mathbf{Vect}}_{X}^{[\infty]}$ of the object $(E, (\|.\|_x)_{x\in X(\mathbb{C})})$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ as:

$$\overline{E}^\vee := (E, (\|.\|_x)_{x\in X(\mathbb{C})})^\vee := (E^\vee, (\|.\|_x^\sim)_{x\in X(\mathbb{C})}).$$

Moreover, for every morphism of Hermitian quasi-coherent sheaves over X:

$$\varphi : \overline{E}_1 \longrightarrow \overline{E}_2,$$

the transpose morphism:

$$\varphi^\vee : E_2^\vee \longrightarrow E_1^\vee$$

of the underlying morphism $\varphi : E_1 \to E_2$ of $\mathcal{O}_K$-modules defines a morphism in $\mathbf{pro}\overline{\mathbf{Vect}}_{X}^{[\infty]}$:

$$\varphi^\vee : \overline{E}_2^\vee \longrightarrow \overline{E}_1^\vee. \tag{2.5.7}$$

Indeed, for every $x \in X(\mathbb{C})$, the continuous morphism:

$$(\varphi^\vee)_x : (E_2^\vee)_x \longrightarrow (E_1^\vee)_x$$

between completed tensor products is the restriction of the morphism:

$$(\varphi_x)^\vee : (E_{2,x})^\vee \longrightarrow (E_{1,x})^\vee$$

deduced by transposition from the $\mathbb{C}$-linear map $\varphi_x : E_{1,x} \to E_{2,x}$, and the map $(\varphi_x)^\vee$ defines the morphism in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$:

$$(\varphi_x)^\vee : \overline{E}_{2,x}^\vee \longrightarrow \overline{E}_{1,x}^\vee$$

dual of the morphism in $\mathbf{ind}\overline{\mathbf{Vect}}_{X}^{[0]}$:

$$\varphi_x : \overline{E}_{1,x} \longrightarrow \overline{E}_{2,x}.$$

The construction of the dual morphism (2.5.7) is obviously compatible with composition, and thus defines a contravariant functor:

$$.^\vee : \mathbf{q}\overline{\mathbf{Coh}}_X \to \mathbf{pro}\overline{\mathbf{Vect}}_{X}^{[\infty]}.$$

Moreover the equality (2.4.14) between the operator norm of a morphism in $\mathbf{ind}\overline{\mathbf{Vect}}_{X}^{[0]}$ and the one of its dual shows that it also defines a functor:

$$.^\vee : \mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1} \to \mathbf{pro}\overline{\mathbf{Vect}}_{X}^{[\infty]\leq 1}.$$

2.5.3. The duality functor $\cdot^\vee : \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]} \longrightarrow \mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$. A construction similar, but simpler, to the one in the previous subsection allows one to construct a duality functor from $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$ to $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$.

Consider a generalized pro-Hermitian vector bundle over X:

$$\overline{\widehat{E}} := (\widehat{E}, (\|.\|_x)_{x \in X(\mathbb{C})}).$$

The dual of the topological $\mathcal{O}_K$-module $\widehat{E}$ in $\mathbf{CTC}_{\mathcal{O}_K}$, namely:

$$\widehat{E}^\vee := \mathrm{Hom}^{\mathrm{top}}_{\mathcal{O}_K}(\widehat{E}, \mathcal{O}_K),$$

is an object of $\mathbf{CP}_{\mathcal{O}_K}$, that is a countably generated projective $\mathcal{O}_K$-module. Besides, for every every field embedding $x \in X(\mathbb{C})$ of K in $\mathbb{C}$, we may consider the Fréchet space $\widehat{E}_x := \widehat{E}\widehat{\otimes}_{\mathcal{O}_K,x}\mathbb{C}$, the object $\overline{\widehat{E}}_x := (\widehat{E}_x, \|.\|_x)$ of $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$, and its dual in $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$:

$$\overline{\widehat{E}}_x^\vee := ((\widehat{E}_x)^\vee, \|.\|_x^\vee).$$

Moreover, for every field embedding $x \in X(\mathbb{C})$ of K in $\mathbb{C}$, the $\mathbb{C}$-vector space $\widehat{E}_x^\vee := \widehat{E}^\vee \otimes_{\mathcal{O}_K,x} \mathbb{C}$ may be identified with the topological dual $(\widehat{E}_x)^\vee$ of $\widehat{E}_x$, and therefore may be endowed with the Hermitian seminorm $\|.\|_x^\vee$.

Consequently we may define the dual of $\overline{\widehat{E}}$ as the following object in $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$:

$$\overline{\widehat{E}}^\vee := (\overline{E}^\vee, (\|.\|_x^\vee)_{x \in X(\mathbb{C})}).$$

Moreover, for every morphism of generalized pro-Hermitian vector bundles:

$$\psi : \overline{\widehat{E}}_1 \longrightarrow \overline{\widehat{E}}_2,$$

the transpose morphism:

$$\psi^\vee : \widehat{E}_2^\vee \longrightarrow \widehat{E}_1^\vee$$

of the underlying morphism $\psi : \widehat{E}_1 \to \widehat{E}_2$ of topological $\mathcal{O}_K$-modules defines a morphism in $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$:

$$\psi^\vee := \overline{\widehat{E}}_2^\vee \longrightarrow \overline{\widehat{E}}_1^\vee.$$

Actually, for every $x \in X(\mathbb{C})$, the associated morphism in $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0]}$:

$$(\psi^\vee)_x : \overline{\widehat{E}}_{2,x}^\vee \longrightarrow \overline{\widehat{E}}_{1,x}^\vee$$

is the dual of the morphism in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty]}$:

$$\psi_x : \overline{\widehat{E}}_{1,x} \longrightarrow \overline{\widehat{E}}_{2,x}.$$

This construction of $\psi^\vee$ is compatible with composition, and defines a contravariant functor:

$$\cdot^\vee : \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]} \longrightarrow \mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}.$$

Here again, equality (2.4.14) shows that it also defines a functor:

$$\cdot^\vee : \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1} \longrightarrow \mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]\leq 1}.$$

2.5.4. Biduality and canonical dévissage of Hermitian quasi-coherent sheaves. By combining the adjoint equivalences attached to the duality functors between the categories $\mathbf{CP}_{\mathcal{O}_K}$ and $\mathbf{CTC}_{\mathcal{O}_K}$ and between $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[0](\leq 1)}$ and $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{C}}^{[\infty](\leq 1)}$ discussed in 1.2.2 and 2.4.4.2 above, one easily establishes:

PROPOSITION 2.5.2. *The duality functors constructed in Subsections 2.5.3 define adjoint equivalences:*

$$\cdot^{\vee} : \mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]} \leftrightarrows \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]} : \cdot^{\vee}$$

and:

$$\cdot^{\vee} : \mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]\leq 1} \leftrightarrows \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1} : \cdot^{\vee}.$$

Moreover the following proposition is a straightforward consequence of the definitions and of Proposition 2.4.12:

PROPOSITION 2.5.3. *The composition of the two duality functors:*

$$\cdot^{\vee\vee} : \mathbf{q}\overline{\mathbf{Coh}}_X \xrightarrow{\cdot^{\vee}} \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]} \xrightarrow{\cdot^{\vee}} \mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$$

coincides with the functor:

$$\mathbf{q}\overline{\mathbf{Coh}} \longrightarrow \mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}, \quad \mathcal{F} \longmapsto \mathcal{F}^{\vee\vee}, \quad \varphi \longmapsto \varphi^{\vee\vee}$$

defined in 2.2.4.3 above.

2.5.5. Varia. Various constructions and properties discussed in Section 2.2 concerning Hermitian coherent and quasi-coherent sheaves admit counterparts concerning generalized pro-Hermitian vector bundles.

For instance, if L is a finite extension of the number field K and if

$$f : Y := \operatorname{Spec} \mathcal{O}_L \longrightarrow X := \operatorname{Spec} \mathcal{O}_K$$

denotes the morphism of schemes defined by the inclusion $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$, we may define inverse image functors:

$$f^* : \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]} \longrightarrow \mathbf{pro}\overline{\mathbf{Vect}}_Y^{[\infty]} \quad \text{and} \quad f^* : \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1} \longrightarrow \mathbf{pro}\overline{\mathbf{Vect}}_Y^{[\infty]\leq 1}$$

and direct image functors:

$$f_* : \mathbf{pro}\overline{\mathbf{Vect}}_Y^{[\infty]} \longrightarrow \mathbf{pro}\overline{\mathbf{Vect}}_X[\infty] \quad \text{and} \quad f^* : \mathbf{pro}\overline{\mathbf{Vect}}_Y^{[\infty]\leq 1} \longrightarrow \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1}$$

by mimicking the constructions in Subsection 2.2.3.

If $\overline{V}$ is a Hermitian vector bundle over X, we may define its tensor product $\widehat{\overline{E}} \otimes \overline{V}$ with a generalized pro-Hermitian vector bundles $\widehat{\overline{E}}$, and define functors:

$$\cdot \otimes \overline{V} : \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]} \longrightarrow \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]} \quad \text{and} \quad \cdot \otimes \overline{V} : \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1} \longrightarrow \mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]\leq 1},$$

which are compatible with the duality functors defined above. Namely, for every object $\overline{F}$ (resp. $\widehat{\overline{E}}$) of $\mathbf{q}\overline{\mathbf{Coh}}_X$ (resp. of $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$), we have a canonical isometric isomorphism in $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$ (resp. in $\mathbf{ind}\overline{\mathbf{Vect}}_X^{[0]}$):

$$(\overline{F} \otimes \overline{V})^{\vee} \xrightarrow{\sim} \overline{F}^{\vee} \otimes \overline{V}^{\vee} \quad (\text{resp. } (\widehat{\overline{E}} \otimes \overline{V})^{\vee} \xrightarrow{\sim} \widehat{\overline{E}}^{\vee} \otimes \overline{V}^{\vee}).$$

Moreover the compatibility between duality and direct images expressed by the isometric isomorphism (2.2.19), valid for any Hermitian coherent sheaf $\overline{E}$ over X, extends to Hermitian quasi-coherent sheaves and generalized pro-Hermitian vector bundles, in the form of an isometric isomorphism in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[\infty)}$ (resp. in $\mathbf{ind}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[0]}$):

$$\pi_*(\overline{F}^{\vee} \otimes \overline{\omega}_{\pi}) \xrightarrow{\sim} (\pi_* \overline{F})^{\vee} \quad (\text{resp. } \pi_*(\widehat{\overline{E}}^{\vee} \otimes \overline{\omega}_{\pi}) \xrightarrow{\sim} (\pi_* \widehat{\overline{E}})^{\vee})$$

for every object $\overline{F}$ (resp. $\widehat{\overline{E}}$) of $\mathbf{q}\overline{\mathbf{Coh}}_X$ (resp. of $\mathbf{pro}\overline{\mathbf{Vect}}_X^{[\infty]}$).

Part 2

Positive Invariants of Hermitian Quasi-coherent Sheaves

CHAPTER 3

Quasi-coherent Sheaves over Smooth Projective Curves, First Cohomology Groups, and h^1-Finiteness

3.0.1. This monograph is devoted to the study of invariants of Hermitian quasi-coherent sheaves on arithmetic curves, taking their values in $[0, +\infty]$, that are the arithmetic counterparts of the dimension $h^1(C, \mathcal{F})$ of the first cohomology group of quasi-coherent sheaves $\mathcal{F}$ on a projective curve C over some base field k.

In this chapter, we discuss various properties satisfied by this invariant $h^1(C, \mathcal{F})$, which will play the role of "geometric models" for the properties of invariants of Hermitian quasi-coherent sheaves investigated in the next chapters. These properties provide a conceptual motivation for the introduction of various categories of Hermitian quasi-coherent sheaves defined in terms of such invariants, independently of their applications to Diophantine geometry developed in the sequel.

More specifically, consider a smooth, projective, geometrically connected curve C over some base field k, and denote by $\mathbf{qCoh}_C$ the k-linear abelian category of quasi-coherent $\mathcal{O}_C$-modules. The composition of the k-linear functor:

$$H^1(C, .) : \mathbf{qCoh}_C \longrightarrow \mathbf{Vect}_k, \tag{3.0.1}$$

with values in the category $\mathbf{Vect}_k$ of k-vector spaces, and of the dimension function:

$$\dim_k : \mathbf{Vect}_k \longrightarrow \mathbb{N} \cup \{+\infty\} \tag{3.0.2}$$

defines the invariant:

$$h^1(C, .) : \mathbf{qCoh}_C \longrightarrow \mathbb{N} \cup \{+\infty\}, \quad \mathcal{F} \longmapsto h^1(C, \mathcal{F}) := \dim_k H^1(C, \mathcal{F}).$$

Our goal in this chapter is to investigate which properties of the numerical invariant $h^1(C, .)$ on the abelian category $\mathbf{qCoh}_C$ may be formulated with no mention of the intermediate invariant $H^1(C, .)$ with value in the abelian category $\mathbf{Vect}_k$.

This question, somewhat bizarre from a geometric perspective, is motivated by the classical analogy between functions fields — such as $K := k(C)$ — and number fields.

When dealing with a number field K and with the associated arithmetic curve $X := \operatorname{Spec} \mathcal{O}_K$ and its compactification *Ã la Arakelov*, there are natural counterparts for the category $\mathbf{qCoh}_C$ — namely the category of Hermitian quasi-coherent sheaves over X — and for the invariant $h^1(C, \mathcal{F})$ associated to a coherent sheaf over C — namely the theta invariant $h^1_\theta(\overline{\mathcal{F}})$ of a Hermitian coherent sheaf $\overline{\mathcal{F}}$ over X.[1] However there is presently no established counterpart of the category $\mathbf{Vect}_k$ and of the dimension function $\dim_k$ — not to say of the base field k — which would provide a construction of the theta invariant h^1_θ analogous to the construction of the invariant $h^1(C, .)$ as a composition of (3.0.1) and (3.0.2).

[1] We will study these theta invariants more systematically in Chapter 7. At this stage, let us only recall that, when $\overline{\mathcal{F}}$ is a Hermitian line bundle, the theta invariant $h^1_\theta(\overline{\mathcal{F}})$ already appears, at least implicitly, in Hecke's classical derivation [**Hec17**] of the meromorphic continuation of the Dedekind zeta function ζ_K attached to K and of its functional equation. The analogy between $h^1_\theta(\overline{\mathcal{F}})$ and the invariant $h^1(C, \mathcal{F})$ associated to a line bundle $\mathcal{F}$ over a smooth projective curve C becomes conspicuous when comparing Hecke's proof with the later proof by F.K. Schmidt [**Sch31**] of the same properties for the zeta function attached to the global field $k(C)$ when the base field k is finite.

The question of defining a suitable arithmetic avatar of the category $\mathbf{Vect}_k$, on which would be defined a natural dimension function admitting $[0, +\infty]$ as set of values, and to associate to any Hermitian (quasi-)coherent sheaf $\overline{\mathcal{F}}$ over X some object in this category whose "dimension" would coincide with the theta invariant of $\overline{\mathcal{F}}$, is a fascinating but formidable question.[2]

In this monograph, we follow a less ambitious approach, which bypasses this question by focusing on the *numerical* invariant $h^1_\theta(\overline{\mathcal{F}})$. The investigation of real valued numerical invariants attached to Hermitian vector bundles over a scheme of finite type $\mathcal{X}$ over $\operatorname{Spec}\mathbb{Z}$, analogue to the integral valued numerical of classical algebraic geometry over a field, is actually a central theme in Arakelov geometry.[3] Accordingly, like [**Bos20b**], this monograph may be seen as a contribution to the development of infinite dimensional techniques in Arakelov geometry.

The results in this chapter, which explores the numerical properties of the geometric invariant $h^1(C, \mathcal{F})$, are elementary, in so far as their proofs uses only the basic properties of quasi-coherent and coherent sheaves over a smooth projective curve, such as the Riemann-Roch formula and Serre duality. However it has been a surprise for the authors to discover the validity of several simple (although not completely straightforward) results concerning the cohomology of quasi-coherent sheaves over projective curves, that, to the best of their knowledge, are not present in the literature.[4]

3.0.2. In Section 3.1, we discuss various properties — mainly inequalities — satisfied by the invariant

$$h^1(C, .) : \mathbf{qCoh}_C \longrightarrow \mathbb{N} \cup \{\infty\}$$

and by its restriction:

$$h^1(C, .) : \mathbf{Coh}_C \longrightarrow \mathbb{N}$$

to coherent $\mathcal{O}_C$-modules. These properties constitute the models for the basic properties of "h^1-like invariants" taken as axioms in Chapters 4 and 5.

In Section 3.2, we begin to investigate how the invariant $h^1(C, .)$ on $\mathbf{qCoh}_C$ may be recovered from its restriction to $\mathbf{Coh}_C$: we define two invariants attached to an object $\mathcal{F}$ of $\mathbf{qCoh}_C$, denoted by $\overline{h}^1(C, \mathcal{F})$ and $\underline{h}^1(C, \mathcal{F})$, by taking suitable limits of the invariants $h^1(C, \mathcal{C})$ associated to coherent sheaves $\mathcal{C}$ that are respectively subsheaves or quotient sheaves of $\mathcal{F}$. Basically by construction, they satisfy the inequalities:

$$\underline{h}^1(C, \mathcal{F}) \leq h^1(C, \mathcal{F}) \leq \overline{h}^1(C, \mathcal{F}). \tag{3.0.3}$$

In Theorem 3.2.7, we prove the equality:

$$\underline{h}^1(C, \mathcal{F}) = h^1(C, \mathcal{F}). \tag{3.0.4}$$

and we establish criteria ensuring that a quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$ of countable type $\mathcal{F}$ satisfies the condition:

$$h^1(C, \mathcal{F}) = \overline{h}^1(C, \mathcal{F}) < +\infty. \tag{3.0.5}$$

In view of the equality (3.0.4), one may wonder about the interest of the invariant $\overline{h}^1(C, \mathcal{F})$ and of criteria ensuring (3.0.5), and we want to discuss briefly our motivation for considering them.

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[2]As hinted at by Quillen in the entry dated April 1st, 1983, of its mathematical diary [**Qui**], for constructing such an arithmetic avatar of $\mathbf{Vect}_k$, it is tempting to look for a category of modules over a suitable factor of type II_1, equipped with the real valued dimension function associated to its von Neumann trace. See [**Bor03**] for another perspective on this circle of questions.

[3]When $\mathcal{X}$ is an arithmetic curve, say $\operatorname{Spec}\mathbb{Z}$, Hermitian vector bundles over $\mathcal{X}$ are nothing but Euclidean lattices, and their study is the object of the classical "geometry of numbers". The "Arakelov point of view" on the geometry of numbers is illustrated notably by the work of Stuhler [**Stu76**], Grayson [**Gra84**] Gillet, Mazur, and Soulé [**GMS91**], and McMurray Price [**MP17**].

[4]See for instance Theorem 3.2.7 and the content of Section 3.5 and 3.6.

In the arithmetic situation, where $\mathbf{qCoh}_C$ is replaced by the category $\mathbf{q}\overline{\mathbf{Coh}}_X$ of Hermitian quasi-coherent sheaves over some arithmetic curve X and the invariant $h^1(C, .)$ on $\mathbf{Coh}_C$ by the theta invariant h^1_θ on $\overline{\mathbf{Coh}}_X$, we will be able to define similarly invariants $\overline{h}^1_\theta(\overline{\mathcal{F}})$ and $\underline{h}^1_\theta(\overline{\mathcal{F}})$ attached to some object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$. The invariant $\underline{h}^1_\theta(\overline{\mathcal{F}})$, which according to (3.0.4) should be thought as the right analogue of $h^1(C, \mathcal{F})$, turns out to be delicate to control directly, because of its definition in terms of *quotient* Hermitian coherent sheaves. However we will be able to establish diverse criteria for the validity of the condition:

$$\underline{h}^1_\theta(\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}}) < +\infty, \tag{3.0.6}$$

which will be quite suitable in Diophantine applications.

Establishing these criteria for (3.0.6) in Chapters 4, 5, and 8 will require a substantial amount of work, and Theorem 3.2.7 and its proof may be seen as a (considerably simplified) model, in the geometric situation, for the criteria for (3.0.6) and their derivation in the arithmetic setting.

In Section 3.4, we show that, contrary to the first inequality in (3.0.3), the second one may actually be strict.

In Section 3.5 (resp. in Section 3.6), we investigate the objects of $\mathbf{qCoh}_C$ that are h^1-*finite* (resp. $\overline{h}^1$-*finite*), namely these objects $\mathcal{F}$ such that:

$$h^1(C, \mathcal{F} \otimes L) < +\infty \quad (\text{resp. } h^1(C, \mathcal{F} \otimes L) = \overline{h}^1(C, \mathcal{F} \otimes L) < +\infty)$$

for every line bundle L over C. Here again, our results are models for our later results in the arithmetic situations. For instance, the $\overline{h}^1$-finite objects in $\mathbf{qCoh}_C$ are the analogues of the θ^1-finite objects $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, defined by the validity of the condition:

$$\underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{L}) = \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{L}) < +\infty \tag{3.0.7}$$

for every Hermitian line bundle $\overline{L}$ over X, whose properties will be discussed in detail in Section 8.5.[5]

3.0.3. In this chapter, we use the following notation and conventions.

We denote by C a smooth, projective, geometrically connected curve over some base field k, by

$$\eta := \operatorname{Spec} k(C) \xhookrightarrow{i_\eta} X$$

its generic point and the inclusion morphism, by $\omega_C := \Omega^1_{C/k}$ its dualizing sheaf, and by g its genus.

We shall say "$\mathcal{O}_C$-module" for "sheaf of $\mathcal{O}_C$-modules over C."

Recall that a quasi-coherent $\mathcal{O}_C$-module $\mathcal{G}$ is *torsion* when its stalk $\mathcal{G}_\eta := i^*_\eta \mathcal{G}$ at the generic point $\eta := \operatorname{Spec} k(C)$ of C vanishes, or equivalently if $\mathcal{G}$ is the direct sum $\bigoplus_{x \in C_0} \mathcal{G}_x$ of quasi-coherent subsheaves $\mathcal{G}_x$ supported by the closed points x of C. A quasi-coherent $\mathcal{O}_C$-module $\mathcal{H}$ is *torsion free* when the tautological morphism of quasi-coherent sheaves

$$\mathcal{H} \longrightarrow i_{\eta *} i^*_\eta \mathcal{H}$$

is injective, or equivalently when for any closed point x of C, the k-vector space $\Gamma_{\{x\}}(C, \mathcal{H})$ of sections of $\mathcal{H}$ supported by $\{x\}$ vanishes.

For any quasi-coherent $\mathcal{O}_C$-module $\mathcal{G}$ over C, we define its *torsion subsheaf* $\mathcal{G}_{\text{tor}}$ as the kernel of the tautological morphism:

$$\mathcal{G} \longrightarrow i_{\eta *} i^*_\eta \mathcal{G}.$$

[5] The θ^1-finite objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ will naturally occur in the Diophantine applications of the results established in this monograph. An object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is actually θ^1-finite if and only if $\underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\delta)) = \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\delta)) < +\infty$ for every $\delta \in \mathbb{R}$. Observe that, when $\delta = 0$, this is condition (3.0.6), which expresses that the invariant h^1_θ of $\overline{\mathcal{F}}$ is well-defined and finite. Moreover the Hermitian quasi-coherent sheaves $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\delta)$ are those deduced from $\overline{\mathcal{F}}$ by a joint change of scale of its Hermitian seminorms. In other words, $\overline{\mathcal{F}}$ is θ^1-finite if and only if, after any change of scale, it has a well-defined and finite invariant h^1_θ.

The quotient $\mathcal{O}_C$-module $\mathcal{G}_{/\mathrm{tor}} := \mathcal{G}/\mathcal{G}_{\mathrm{tor}}$ is then torsion free, and fits into a short exact sequence of quasi-coherent $\mathcal{O}_C$-modules:

$$0 \longrightarrow \mathcal{G}_{\mathrm{tor}} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}_{/\mathrm{tor}} \longrightarrow 0.$$

Recall that there exists a canonical isomorphism:

$$\mathrm{Res}_C : H^1(C, \omega_C) \xrightarrow{\sim} k, \tag{3.0.8}$$

which may be defined as follows.

Let $\mathcal{U} := (U_i)_{i \in I}$ be an open covering of C, and let

$$\alpha := (\alpha_{ij})_{(i,j)\in I^2} \in Z^1(\mathcal{U}, \omega_C)$$

be a 1-cocycle. Let i_0 be an element of i such that U_{i_0} is non-empty. Its complement $C \setminus U_{i_0}$ is a finite set of closed points of C. For every $P \in C \setminus U_{i_0}$, we may choose $i(P) \in I$ such that $U_{i(P)}$ contains P, and consider $\alpha_{i_o i(P)} \in \Gamma(U_{i_0} \cap U_{i(P)}, \omega_C)$, which is a meromorphic section of ω_C over C. Its residue $\mathrm{Res}_P \alpha_{i_o i(P)}$ at P belongs to the residue field $k_P := \mathcal{O}_{C,P}/\mathfrak{m}_{C,P}$. The image by Res_C of the class $[\alpha]$ in $H^1(\mathcal{U}, \omega_C)$ — which maps injectively in $H^1(C, \omega_C)$, actually isomorphically if all the U_i are distinct of C, hence affine — satisfies:

$$\mathrm{Res}_C[\alpha] = \sum_{P \in C \setminus U_{i_0}} \mathrm{Tr}_{k_P/k} \mathrm{Res}_P \alpha_{i_0 i(P)}.$$

We shall denote by $\dim_k V$ the "naive" dimension in $\mathbb{N} \cup \{+\infty\}$ of some k-vector space V. In other words, we assign $+\infty$ as the dimension of an infinite dimensional k-vector space V, independently of the actual (infinite) cardinality of its k-bases. With this convention, the following equality holds for any k-vector space V:

$$\dim_k \mathrm{Hom}_k(V, k) = \dim_k V.$$

3.1. Numerical Properties of $h^1(C, \mathcal{F})$

3.1.1. Vanishing on torsion sheaves, monotonicity, and subadditivity. When thinking of properties of the invariant

$$h^1(C, .) : \mathcal{F} \longmapsto h^1(C, \mathcal{F}) := \dim_k H^1(C, \mathcal{F}) \in \mathbb{N} \cup \{+\infty\}$$

of quasi-coherent $\mathcal{O}_C$-modules, the first one that comes to mind is its *additivity*. Namely, for every two quasi-coherent sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ over C, the following inequality holds:

$$h^1(C, \mathcal{F}_1 \oplus \mathcal{F}_2) = h^1(C, \mathcal{F}_1) + h^1(C, \mathcal{F}_2). \tag{3.1.1}$$

After a few moment's thoughts, one realizes that $h^1(C, .)$ satisfies also the following properties:

(i) it vanishes on torsion sheaves; namely, *if a quasi-coherent $\mathcal{O}_C$ module $\mathcal{F}$ is torsion, then:*

$$h^1(C, \mathcal{F}) = 0;$$

(ii) it satisfies the following monotonicity property: *if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of quasi-coherent sheaves over C such that $f_\eta : \mathcal{F}_\eta \to \mathcal{G}_\eta$ is surjective, then*

$$h^1(C, \mathcal{F}) \geq h^1(C, \mathcal{G}).$$

(iii) it is subadditive in short exact sequences; namely, *for any short exact sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

of quasi-coherent sheaves over C, we have:

$$h^1(C, \mathcal{G}) \leq h^1(C, \mathcal{F}) + h^1(C, \mathcal{H}). \tag{3.1.2}$$

These three properties are manifestations of the one-dimensionality of the curve C. Indeed, (i) follows from the fact that any torsion quasi-coherent sheaves over the curve C is a direct sum of quasi-coherent sheaves supported by closed points; (iii) follows from the fact that C has cohomological dimension ≤ 1, which implies the exactness of the diagram:

$$H^1(C,\mathcal{F}) \xrightarrow{\alpha} H^1(C,\mathcal{G}) \xrightarrow{\beta} H^1(C,\mathcal{H}) \longrightarrow 0. \tag{3.1.3}$$

Together with (i), this right exactness of the functor $H^1(C,.)$ also implies (ii).

Observe that the properties (i)–(iii) above formally imply that the invariant $h^1(C,\mathcal{F})$ "does not see torsion". Namely, for any coherent sheaf $\mathcal{F}$ over C, property (iii) applied to the short exact sequence

$$0 \longrightarrow \mathcal{F}_{\mathrm{tor}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{/\mathrm{tor}} \longrightarrow 0$$

and property (ii) applied to the quotient morphism $\mathcal{F} \to \mathcal{F}_{/\mathrm{tor}}$ imply the estimates:

$$h^1(C,\mathcal{F}_{/\mathrm{tor}}) \leq h^1(C,\mathcal{F}) \leq h^1(C,\mathcal{F}_{\mathrm{tor}}) + h^1(C,\mathcal{F}_{/\mathrm{tor}}).$$

Moreover, according to property (i), we have:

$$h^1(C,\mathcal{F}_{\mathrm{tor}}) = 0.$$

This establishes the equality:

$$h^1(C,\mathcal{F}) = h^1(C,\mathcal{F}_{/\mathrm{tor}}).$$

3.1.2. The invariant $\mathrm{rk}_k\alpha^1$. Further inequalities satisfied by the invariant:

$$h^1(C,.) : \mathbf{Coh}_C \longrightarrow \mathbb{N}, \quad \mathcal{F} \longmapsto h^1(C,\mathcal{F}) := \dim_k H^1(C,\mathcal{F})$$

on the category $\mathbf{Coh}_C$ of coherent $\mathcal{O}_C$-modules may be derived by considering the rank of the k-linear map

$$\alpha^1 := H^1(C,\alpha) : H^1(C,\mathcal{F}) \longrightarrow H^1(C,\mathcal{G}) \tag{3.1.4}$$

attached by functoriality of the cohomology to a morphism of coherent $\mathcal{O}_C$-modules[6]

$$\alpha : \mathcal{F} \longrightarrow \mathcal{G}. \tag{3.1.5}$$

3.1.2.1. The rank $\mathrm{rk}_k\alpha^1$ indeed admits a simple expression in terms of the invariant $h^1(C,.)$ of coherent $\mathcal{O}_C$-modules; moreover, for any α as in (3.1.5), $\mathrm{rk}_k\alpha^1$ depends only of the coherent module $\mathcal{G}$ and of the $k(C)$-vector subspace $\mathrm{im}\,\alpha_\eta$ of $\mathcal{G}_\eta$, as shown by the following proposition:

PROPOSITION 3.1.1. *For every morphism of coherent $\mathcal{O}_C$-modules $\alpha : \mathcal{F} \to \mathcal{G}$, the rank of the k-linear map $H^1(C,\alpha)$satisfies the equality:*

$$\mathrm{rk}_k\alpha^1 = h^1(C,\mathcal{G}) - h^1(C,\mathcal{G}/\alpha(\mathcal{F})). \tag{3.1.6}$$

Moreover if two morphisms of coherent $\mathcal{O}_C$-modules

$$\alpha : \mathcal{F} \longrightarrow \mathcal{G} \quad \textit{and} \quad \alpha' : \mathcal{F}' \longrightarrow \mathcal{G}$$

satisfy:

$$\mathrm{im}\,\alpha_\eta = \mathrm{im}\,\alpha'_\eta,$$

then:

$$\mathrm{rk}_k\alpha^1 = \mathrm{rk}_k\alpha'^1. \tag{3.1.7}$$

[6]To be consistent with the notation (3.1.4), the arrows in the diagram (3.1.3) should have been labeled $H^1(C,\alpha)$ and $H^1(C,\beta)$, or α^1 and β^1.

PROOF. The equality (3.1.6) is a consequence of the right exactness of the functor $H^1(C,.)$ applied to the short exact sequence:

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \longrightarrow \mathcal{G}/\alpha(\mathcal{F}) \longrightarrow 0.$$

and of its vanishing on torsion coherent modules.

The equality (3.1.7) follows from (3.1.6) applied to α and α' and from the vanishing of $H^1(C,.)$ on torsion coherent sheaves, which implies that $H^1(C, \mathcal{G}/\alpha(\mathcal{F}))$ and $H^1(C, \mathcal{G}/\alpha'(\mathcal{F}'))$ are both isomorphic to $H^1(C, \mathcal{G}/\alpha(\mathcal{F}) \cap \alpha'(\mathcal{F}'))$. □

To any pair (α, β) of composable morphisms of coherent $\mathcal{O}_C$-modules:

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H},$$

we may attach the following commutative diagram of k-linear maps:

$$\begin{tikzcd} H^1(C,\mathcal{F}) \arrow[r, "\alpha^1"] \arrow[dr, "(\beta\circ\alpha)^1"'] & H^1(C,\mathcal{G}) \arrow[d, "\beta^1"] \\ & H^1(C,\mathcal{H}), \end{tikzcd} \tag{3.1.8}$$

and the following inequality between the ranks of the k-linear maps in (3.1.8) clearly holds:

$$\mathrm{rk}_k(\beta \circ \alpha)^1 \leq \min\left(\mathrm{rk}_k \alpha^1, \mathrm{rk}_k \beta^1\right). \tag{3.1.9}$$

The inequality:

$$\mathrm{rk}_k(\beta \circ \alpha)^1 \leq \mathrm{rk}_k \beta^1,$$

when expressed in terms of the invariant $h^1(C,.)$ by means of Proposition 3.1.1, takes the following form:

$$h^1(C,\mathcal{H}) - h^1(C, \mathcal{H}/\beta(\alpha(\mathcal{H}))) \leq h^1(C,\mathcal{H}) - h^1(C, \mathcal{H}/\beta(\mathcal{G})).$$

It is therefore a special instance of the monotonicity of $h^1(C,.)$, applied to the quotient morphism:

$$\mathcal{H}/\beta(\alpha(\mathcal{F})) \longrightarrow \mathcal{H}/\beta(\mathcal{G}).$$

In turn, the inequality:

$$\mathrm{rk}_k(\beta \circ \alpha)^1 \leq \mathrm{rk}_k \alpha^1$$

may be written:

$$h^1(C,\mathcal{H}) - h^1(C, \mathcal{H}/\beta(\alpha(\mathcal{F}))) \leq h^1(C,\mathcal{G}) - h^1(C, \mathcal{G}/\alpha(\mathcal{F})).$$

It involves the morphism α only through the coherent submodule $\alpha(\mathcal{F})$ and may rephrased as follows:

PROPOSITION 3.1.2. *For every morphism $\beta : \mathcal{G} \to \mathcal{H}$ of coherent $\mathcal{O}_C$-modules and every coherent $\mathcal{O}_C$-submodule $\mathcal{G}' \subseteq \mathcal{G}$, the following inequality holds:*

$$h^1(C,\mathcal{H}) - h^1(C, \mathcal{H}/\beta(\mathcal{G}')) \leq h^1(C,\mathcal{G}) - h^1(C, \mathcal{G}/\mathcal{G}'). \tag{3.1.10}$$

Observe that, applied to $\mathcal{H} = 0$, the inequality (3.1.10) becomes the monotonicity inequality:

$$h^1(C,\mathcal{G}) \geq h^1(C, \mathcal{G}/\mathcal{G}')$$

attached to the quotient morphism from $\mathcal{G}$ to $\mathcal{G}/\mathcal{G}'$. Applied to a monomorphim β and to $\mathcal{G}' := \mathcal{G}$, (3.1.10) becomes the subadditivity inequality:

$$h^1(C,\mathcal{H}) \leq h^1(C,\mathcal{G}) + h^1(C, \mathcal{H}/\beta(\mathcal{G}))$$

associated to the short exact sequence:

$$0 \longrightarrow \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow \mathcal{H}/\beta(\mathcal{G}) \longrightarrow 0.$$

3.1.2.2. The inequality (3.1.10) admits further consequences:

(i) If $\mathcal{F}$ is a coherent $\mathcal{O}_C$-module and if

$$\mathcal{F}'' \subseteq \mathcal{F}' \subseteq \mathcal{F}$$

are two coherent $\mathcal{O}_C$-submodules, then the following inequality holds:

$$h^1(C, \mathcal{F}) + h^1(C, \mathcal{F}'/\mathcal{F}'') \leq h^1(C, \mathcal{F}') + h^1(C, \mathcal{F}/\mathcal{F}''). \tag{3.1.11}$$

This follows from (3.1.10) applied to the inclusion morphism

$$\beta : \mathcal{G} := \mathcal{F}' \hookrightarrow \mathcal{H} := \mathcal{F}$$

and to $\mathcal{G}' := \mathcal{F}''$.

(ii) If $\mathcal{F}$ is a coherent $\mathcal{O}_C$-module and if $\mathcal{F}'$ and $\mathcal{F}''$ are two coherent $\mathcal{O}_C$ submodules of $\mathcal{F}$, then the following inequality holds:

$$h^1(C, \mathcal{F}/\mathcal{F}') + h^1(C, \mathcal{F}/\mathcal{F}'') \leq h^1(C, \mathcal{F}/(\mathcal{F}' + \mathcal{F}'')) + h^1(C, \mathcal{F}/(\mathcal{F}' \cap \mathcal{F}'')). \tag{3.1.12}$$

This follows from (3.1.10) applied to the quotient morphism

$$\beta : \mathcal{G} := \mathcal{F}/(\mathcal{F}' \cap \mathcal{F}'') \longrightarrow \mathcal{H} := \mathcal{F}/\mathcal{F}''$$

and to $\mathcal{G}' := \mathcal{F}'/(\mathcal{F}' \cap \mathcal{F}'')$.

Observe that the inequality (3.1.12) is also a consequence of the additivity (3.1.1) of $h^1(C, .)$ and of its subadditvity (3.1.2) applied to the short exact sequence of $\mathcal{O}_C$-modules:

$$0 \longrightarrow \mathcal{F}/(\mathcal{F}' \cap \mathcal{F}'') \xrightarrow{\Delta} \mathcal{F}/\mathcal{F}' \oplus \mathcal{F}/\mathcal{F}'' \xrightarrow{\delta} \mathcal{F}/(\mathcal{F}' + \mathcal{F}'') \longrightarrow 0, \tag{3.1.13}$$

where the morphisms Δ and δ are defined by the relations:

$$\Delta([x]) := ([x]', [x]'') \quad \text{and} \quad \delta([x']', [x'']'') := [x' - x'']^{\sim},$$

in which we denote by $[s]$, $[s]'$, $[s'']$, and $[s]^{\sim}$ the classes in $\mathcal{F}/(\mathcal{F}' \cap \mathcal{F}'')$, $\mathcal{F}/\mathcal{F}'$, $\mathcal{F}/\mathcal{F}''$, and $\mathcal{F}/(\mathcal{F}' + \mathcal{F}'')$ of a section s of $\mathcal{F}$.

3.2. Recovering $h^1(C, \mathcal{F})$ for $\mathcal{F}$ Quasi-coherent from its Value on Coherent Sheaves

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_C$-module.

3.2.1. The invariant $\overline{h}^1(C, \mathcal{F})$.

3.2.1.1. To $\mathcal{F}$, we may associate the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$ of its coherent $\mathcal{O}_C$-submodules.

If $\mathcal{C}$ and $\mathcal{C}'$ are two elements of $\mathrm{coh}(\mathcal{F})$ such that $\mathcal{C} \subseteq \mathcal{C}'$, we may consider the inclusion morphism

$$i_{\mathcal{C}'\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{C}'$$

and the linear map between finite dimensional k-vector spaces that it induces between first cohomology groups:

$$i^1_{\mathcal{C}'\mathcal{C}} : H^1(C, \mathcal{C}) \longrightarrow H^1(C, \mathcal{C}'). \tag{3.2.1}$$

Moreover, to any $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$, we may also attach the inclusion morphism

$$j_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{F}$$

and the induced k-linear map between cohomology groups:

$$j^1_{\mathcal{C}} : H^1(C, \mathcal{C}) \longrightarrow H^1(C, \mathcal{F}).$$

Clearly, for every two $\mathcal{C}$ and $\mathcal{C}'$ as above, the diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i_{\mathcal{C}'\mathcal{C}}"] \arrow[dr, "j_{\mathcal{C}}"'] & \mathcal{C}' \arrow[d, "j_{\mathcal{C}'}"] \\ & \mathcal{F} \end{tikzcd} \tag{3.2.2}$$

and

$$\begin{tikzcd} H^1(C,\mathcal{C}) \arrow[r, "i^1_{\mathcal{C}'\mathcal{C}}"] \arrow[dr, "j^1_{\mathcal{C}}"'] & H^1(C,\mathcal{C}') \arrow[d, "j^1_{\mathcal{C}'}"] \\ & H^1(C,\mathcal{F}) \end{tikzcd} \tag{3.2.3}$$

are commutative, and consequently the systems of maps $(j_{\mathcal{C}})_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})}$ and $(j^1_{\mathcal{C}})_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})}$ define a map of $\mathcal{O}_C$-modules:

$$j_{\mathcal{F}} : \operatorname*{colim}_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \mathcal{C} \longrightarrow \mathcal{F}$$

and a k-linear map:

$$j^1_{\mathcal{F}} : \operatorname*{colim}_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} H^1(C,\mathcal{C}) \longrightarrow H^1(C,\mathcal{F}),$$

where the sources of $j_{\mathcal{F}}$ and $j^1_{\mathcal{F}}$ are the colimits of the systems of morphisms $i_{\mathcal{C}'\mathcal{C}}$ and $i^1_{\mathcal{C}'\mathcal{C}}$, taken over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$.[7]

The following proposition follows from basic results[8] concerning quasi-coherent sheaves on schemes and colimits of sheaves and their cohomology, which apply since C is a Noetherian scheme.

PROPOSITION 3.2.1. *The maps $j_{\mathcal{F}}$ and $j^1_{\mathcal{F}}$ are isomorphisms.*

We may attach to $\mathcal{F}$ the following invariant in $\mathbb{N}\cup\{+\infty\}$:

$$\overline{h}^1(C,\mathcal{F}) := \liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} h^1(C,\mathcal{C}), \tag{3.2.4}$$

where the superior limit is taken over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$. The following observation is then a straightforward consequence of the fact that $j^1_{\mathcal{F}}$ is an isomorphism of k-vector spaces:

COROLLARY 3.2.2. *For every quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$, the following inequality holds:*

$$h^1(C,\mathcal{F}) \leq \overline{h}^1(C,\mathcal{F}). \tag{3.2.5}$$

The invariant $\overline{h}^1(C,.)$ satisfies formal properties analogous to the ones of $h^1(C,.)$ discussed in 3.1.1 above. Namely one easily establishes the following proposition:

PROPOSITION 3.2.3. *The properties (i), (ii), and (iii) stated in 3.1.1* — namely, the vanishing on torsion quasi-coherent sheaves, the monotonicity, and the subadditivity — *still hold when $h^1(C,.)$ is replaced by $\overline{h}^1(C,.)$.*

We leave the details of the proof to the interested reader, as well as the derivation of the following proposition:

[7] Since C is a Noetherian scheme, the colimit $\mathrm{colim}_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})}\,\mathcal{C}$ may be indifferently taken in the category of presheaves or in the category of sheaves of $\mathcal{O}_C$-modules.

[8] See for instance [**Sta**], Lemma 28.22.3 (= Tag 01PG) and Lemma 20.19.1 (=Tag 01FF).

PROPOSITION 3.2.4. *For every short exact sequence of quasi-coherent $\mathcal{O}_C$-modules:*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0,$$

the following inequality holds in $\mathbb{N} \cup \{+\infty\}$*:*

$$\overline{h}^1(C,\mathcal{F}_1) \leq h^0(C,\mathcal{F}_3) + \overline{h}^1(C,\mathcal{F}_2). \tag{3.2.6}$$

Observe that, as a special case of Proposition 3.2.3, (iii), we obtain the inequality:

$$\overline{h}^1(C,\mathcal{F}_1 \oplus \mathcal{F}_2) \leq \overline{h}^1(C,\mathcal{F}_1) + \overline{h}^1(C,\mathcal{F}_2), \tag{3.2.7}$$

for any two quasi-coherent $\mathcal{O}_C$-modules $\mathcal{F}_1$ and $\mathcal{F}_2$. We do not expect this inequality to be an equality for general quasi-coherent $\mathcal{O}_C$-modules.[9]

3.2.1.2. Recall that a quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$ is said to be *of countable type* when there exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent submodules, or equivalently a cofinal increasing sequence in $(\mathrm{coh}(\mathcal{F}), \subseteq)$.

Then we may consider, among these filtrations, those that satisfy the following condition:

$$\text{the limit } \lim_{i\to+\infty} h^1(C,\mathcal{C}_i) \text{ exists in } \mathbb{N}\cup\{+\infty\}. \tag{3.2.8}$$

Clearly, for any filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ as above, this condition is satisfied by the filtration $(\mathcal{C}_{\iota(n)})_{n\in\mathbb{N}}$ if $\iota : \mathbb{N} \to \mathbb{N}$ is a suitable strictly increasing map. Moreover condition (3.2.8) is satisfied precisely when the sequence $(h^1(C,\mathcal{C}_i))_{i\in\mathbb{N}}$ is either eventually constant, or goes to infinity.

The following proposition is a straightforward but suggestive reformulation of the definition (3.2.4) of $\overline{h}^1(C,\mathcal{F})$ when $\mathcal{F}$ is of countable type:

PROPOSITION 3.2.5. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_C$-module of countable type. For every exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent submodules, we have:*

$$\liminf_{i\longrightarrow+\infty} h^1(C,\mathcal{C}_i) \geq \overline{h}^1(C,\mathcal{F}).$$

Moreover the set of limits $\lim_{i\to+\infty} h^1(C,\mathcal{C}_i)$ where $(\mathcal{C}_i)_{i\in\mathbb{N}}$ runs over the set of exhaustive filtrations of $\mathcal{F}$ by coherent $\mathcal{O}_\mathbb{C}$-submodules that satisfy (3.2.8) *admits $\overline{h}^1(C,\mathcal{F})$ as smallest element.*

According to Proposition 3.2.3, the map

$$\mathrm{coh}(\mathcal{F}) \longrightarrow \mathbb{N}\cup\{+\infty\}, \quad \mathcal{C} \longmapsto \overline{h}^1(C,\mathcal{F}/\mathcal{C})$$

is decreasing. We shall denote its infimum over $\mathrm{coh}(\mathcal{F})$ by $\mathrm{ev}\overline{h}^1(C,\mathcal{F})$. By definition, if $\mathcal{F}$ is countably generated and if $(\mathcal{C}_i)_{i\in\mathbb{N}}$ is an exhaustive filtration of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules, we have:

$$\mathrm{ev}\overline{h}^1(C,\mathcal{F}) = \lim_{i\to+\infty} \overline{h}^1(C,\mathcal{F}/\mathcal{C}_i).$$

Moreover $\mathrm{ev}\overline{h}^1(C,\mathcal{F})$ vanishes if and only if there exists $\mathcal{C} \in \mathrm{coh}(\mathcal{F})$ such that $\overline{h}^1(C,\mathcal{F}/\mathcal{C})$ vanishes.

3.2.2. The invariant $\underline{h}^1(C,\mathcal{F})$. Let $\mathrm{coft}(\mathcal{F})$ be the set of quasi-coherent $\mathcal{O}_C$-submodules $\mathcal{G}$ of $\mathcal{F}$ such that $\mathcal{F}/\mathcal{G}$ is coherent. It is stable under finite intersection, and the partially ordered set $(\mathrm{coft}(\mathcal{F}), \supseteq)$ is a directed set.[10]

To any two elements $\mathcal{G}' \subset \mathcal{G}$ of $\mathrm{coh}(\mathcal{F})$ are attached surjective maps of $\mathcal{O}_C$-modules:

$$\mathcal{F} \longrightarrow \mathcal{F}/\mathcal{G}' \longrightarrow \mathcal{F}/\mathcal{G}',$$

and consequently, using again the monotonicity of $h^1(C,.)$, we get the following inequalities between dimensions of first cohomology groups:

$$h^1(C,\mathcal{F}) \geq h^1(C,\mathcal{F}/\mathcal{G}') \geq h^1(C,\mathcal{F}/\mathcal{G}').$$

[9]Even under some additional countability condition, such as the one introduced in the next paragraph.

[10]The notation "coft" stands for **co**-**f**inite **t**ype.

This observation establishes the following lemma:

LEMMA 3.2.6. *The function*

$$\mathrm{coft}(\mathcal{F}) \longrightarrow \mathbb{N}, \quad \mathcal{G} \longmapsto h^1(C, \mathcal{F}/\mathcal{G})$$

is increasing on the directed set $(\mathrm{coft}(\mathcal{F}), \supseteq)$*. Moreover the element of* $\mathbb{N} \cup \{+\infty\}$ *defined as*

$$\underline{h}^1(C, \mathcal{F}) := \lim_{\mathcal{G} \in \mathrm{coft}(\mathcal{F})} h^1(C, \mathcal{F}/\mathcal{G}) = \sup_{\mathcal{G} \in \mathrm{coft}(\mathcal{F})} h^1(C, \mathcal{F}/\mathcal{G}) \tag{3.2.9}$$

satisfies the inequality:

$$\underline{h}^1(C, \mathcal{F}) \leq h^1(C, \mathcal{F}). \tag{3.2.10}$$

Recall that, for any coherent $\mathcal{O}_C$-module $\mathcal{C}$, the quotient $\mathcal{C}_{/\mathrm{tor}} := \mathcal{C}/\mathcal{C}_{\mathrm{tor}}$ is locally free of finite rank and satisfies:

$$h^1(C, \mathcal{C}) = h^1(C, \mathcal{C}_{/\mathrm{tor}}),$$

as observed in 3.1.1 above. This implies that, for any $\mathcal{G}$ in $\mathrm{coft}(\mathcal{F})$, the saturation $\mathcal{G}^{\mathrm{sat}}$ of $\mathcal{G}$ in $\mathcal{F}$ belongs to the subset[11]

$$\mathrm{scoft}(\mathcal{F}) := \left\{ \tilde{\mathcal{G}} \in \mathrm{coft}(\mathcal{F}) \mid \mathcal{F}/\tilde{\mathcal{G}} \text{ is locally free} \right\}, \tag{3.2.11}$$

of $\mathrm{coft}(\mathcal{F})$ and the following equality holds:

$$h^1(\mathcal{F}/\mathcal{G}) = h^1(\mathcal{F}/\mathcal{G}^{\mathrm{sat}}).$$

The partially ordered set $(\mathrm{scoft}(\mathcal{F}), \supseteq)$ is a directed set — indeed $\mathrm{scoft}(\mathcal{F})$ is stable by finite intersections — and the previous observation show that $\overline{h}^1(C, \mathcal{F})$, which had been defined by (3.2.9), also admits the following expressions:

$$\underline{h}^1(C, \mathcal{F}) := \lim_{\tilde{\mathcal{G}} \in \mathrm{scoft}(\mathcal{F})} h^1(C, \mathcal{F}/\tilde{\mathcal{G}}) = \sup_{\tilde{\mathcal{G}} \in \mathrm{scoft}(\mathcal{F})} h^1(C, \mathcal{F}/\tilde{\mathcal{G}}). \tag{3.2.12}$$

3.2.3. Comparing $\underline{h}^1(C, \mathcal{F})$, $h^1(C, \mathcal{F})$, and $\overline{h}^1(C, \mathcal{F})$. According to (3.2.10) and (3.2.5), the following estimates hold:

$$\underline{h}^1(C, \mathcal{F}) \leq h^1(C, \mathcal{F}) \leq \overline{h}^1(C, \mathcal{F}).$$

Consequently, when $\underline{h}^1(C, \mathcal{F})$ and $\overline{h}^1(C, \mathcal{F})$ coincide, they coincide with $h^1(C, \mathcal{F})$.

This simple observation leads one to ask what are in general the relations between the three invariants $\underline{h}^1(C, \mathcal{F})$, $h^1(C, \mathcal{F})$, and $\overline{h}^1(C, \mathcal{F})$. This question is answered by the following theorem and by the construction in Section 3.4 below of quasi-coherent $\mathcal{O}_C$-modules $\mathcal{F}$ such that $h^1(C, \mathcal{F}) < \overline{h}^1(C, \mathcal{F})$.

THEOREM 3.2.7. *Let* $\mathcal{F}$ *be a quasi-coherent* $\mathcal{O}_C$*-module.*

(1) The following equality holds in $\mathbb{N} \cup \{+\infty\}$*:*

$$\underline{h}^1(C, \mathcal{F}) = h^1(C, \mathcal{F}). \tag{3.2.13}$$

(2) When moreover $\mathcal{F}$ *is of countable type, the following two conditions are equivalent:*

(i) $h^1(C, \mathcal{F}) = \overline{h}^1(C, \mathcal{F}) < +\infty$;

(ii) there exists an exhaustive filtration $(\mathcal{C}_i)_{i \in \mathbb{N}}$ *of* $\mathcal{F}$ *by coherent* $\mathcal{O}_C$*-submodules and* $i_0 \in \mathbb{N}$ *such that:*

$$h^1(C, \mathcal{C}_{i+1}/\mathcal{C}_i) = 0 \quad \text{for every } i \geq i_0.$$

(iii) $\mathrm{ev}\overline{h}^1(C, \mathcal{F}) = 0$.

[11]The notation "scoft" stands for **s**aturated of **co**-**f**inite **t**ype.

Actually, if a quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$ satisfies condition (ii) above, then the sequence $(h^1(C,\mathcal{C}_i))_{i\in\mathbb{N}}$ is constant for $i \geq i_0$, and

$$\overline{h}^1(C,\mathcal{F}) = h^1(C,\mathcal{F}) = h^1(C,\mathcal{C}_{i_0}). \tag{3.2.14}$$

(3) If $\mathcal{F}$ is of countable type and if $\overline{h}^1(C,\mathcal{F})$ is finite, then, for any line bundle L over C such that $\deg_C L > g-1$, we have:

$$h^1(C,\mathcal{F}\otimes L) = \overline{h}^1(C,\mathcal{F}\otimes L) < +\infty. \tag{3.2.15}$$

The proof of Theorem 3.2.7 is presented in Section 3.3. In Section 3.4, we explain how to construct quasi-coherent $\mathcal{O}_C$-modules $\mathcal{F}$ whose invariants $h^1(C,\mathcal{F})$ and $\overline{h}^1(C,\mathcal{F})$ are distinct when the genus g of C is positive.

Theorem 3.2.7, (2) implies that the condition on a quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$ of countable type of satisfying:

$$h^1(C,\mathcal{F}) = \overline{h}^1(C,\mathcal{F}) < +\infty$$

satisfies the following permanence properties:

Corollary 3.2.8. *Let us consider a short exact sequence of quasi-coherent $\mathcal{O}_C$-modules of countable type:*

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{\iota} \mathcal{F}_2 \xrightarrow{p} \mathcal{F}_3 \longrightarrow 0. \tag{3.2.16}$$

1) If $h^1(C,\mathcal{F}_2) = \overline{h}^1(C,\mathcal{F}_2) < +\infty$, then $h^1(C,\mathcal{F}_3) = \overline{h}^1(C,\mathcal{F}_3) < +\infty$.

2) If $h^1(C,\mathcal{F}_1) = \overline{h}^1(C,\mathcal{F}_1) < +\infty$ and $h^1(C,\mathcal{F}_3) = \overline{h}^1(C,\mathcal{F}_3) < +\infty$, then $h^1(C,\mathcal{F}_2) = \overline{h}^1(C,\mathcal{F}_2) < +\infty$.

3) If $\mathcal{F}_3$ is a coherent $\mathcal{O}_C$-module, then $h^1(C,\mathcal{F}_1) = \overline{h}^1(C,\mathcal{F}_1) < +\infty$ if and only if $h^1(C,\mathcal{F}_3) = \overline{h}^1(C,\mathcal{F}_3) < +\infty$.

Proof. According to the equivalence of conditions (i) and (iii) in Theorem 3.2.7, (2), for k in $\{1,2,3\}$ the condition

$$h^1(C,\mathcal{F}_k) = \overline{h}^1(C,\mathcal{F}_k) < +\infty$$

is satisfied if and only if there exists a coherent $\mathcal{O}_C$-submodules $\mathcal{C}_k$ of $\mathcal{F}_k$ such that:

$$\overline{h}^1(C,\mathcal{F}_k/\mathcal{C}_k) = 0.$$

When this holds for $k=2$, then $\mathcal{C}_3 := p(\mathcal{C}_2)$ is a coherent $\mathcal{O}_C$-submodule of $\mathcal{F}$, the morphism p induces a surjective morphism of $\mathcal{O}_C$-modules from $\mathcal{F}_2/\mathcal{C}_2$ onto $\mathcal{F}_3/\mathcal{C}_3$, and the monotonicity of $\overline{h}^1(C,.)$, stated in Proposition 3.2.3, implies the vanishing $\overline{h}^1(C,\mathcal{F}_3/\mathcal{C}_3)$. This establishes 1).

Conversely, let assume that the above condition is satisfied for $k=1$ and for $k=3$. Since the morphism $p:\mathcal{F}_2\to\mathcal{F}_3$ is surjective, there exists a coherent $\mathcal{O}_C$-submodule $\tilde{\mathcal{C}}_3$ of $\mathcal{F}_2$ such that

$$p(\tilde{\mathcal{C}}_3) = \mathcal{C}_3.$$

Then the $\mathcal{O}_C$-submodule

$$\mathcal{C}_2 := \iota(\mathcal{C}_1) + \tilde{\mathcal{C}}_3$$

of $\mathcal{F}_2$ is coherent, and by quotienting the short exact sequence (3.2.16), we get a short exact sequence of quasi-coherent $\mathcal{O}_C$-modules:

$$0 \longrightarrow \mathcal{F}_1/\mathcal{C}_1 \longrightarrow \mathcal{F}_2/\mathcal{C}_2 \longrightarrow \mathcal{F}_3/\mathcal{C}_3 \longrightarrow 0.$$

The subadditivity of $\overline{h}^1(C,.)$, stated in Proposition 3.2.3, implies the vanishing $\overline{h}^1(C,\mathcal{F}_2/\mathcal{C}_2)$. This completes the proof of 2).

Let us finally assume that $\mathcal{F}_3$ is coherent. Then $h^1(C, \mathcal{F}_3) = \overline{h}^1(C, \mathcal{F}_3) < +\infty$, and according to 2), if $h^1(C, \mathcal{F}_1) = \overline{h}^1(C, \mathcal{F}_1) < +\infty$, then $h^1(C, \mathcal{F}_2) = \overline{h}^1(C, \mathcal{F}_2) < +\infty$. Conversely, let us assume that $h^1(C, \mathcal{F}_2) = \overline{h}^1(C, \mathcal{F}_2) < +\infty$. According to implication $(i) \Rightarrow (ii)$ in Theorem 3.2.7, (2), there exists an exhaustive filtration $(\mathcal{C}_{2,i})_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules and $i_0 \in \mathbb{N}$ such that:

$$h^1(C, \mathcal{C}_{2,i+1}/\mathcal{C}_{2,i}) = 0, \quad \text{for every } i \geq i_0.$$

Then the sequence $(\mathcal{C}_{1,i})_{i\in\mathbb{N}}$ of $\mathcal{O}_C$-submodules of $\mathcal{F}_1$ defined by:

$$\mathcal{C}_{1,i} := \iota^{-1}(\mathcal{C}_{2,i}), \quad \text{for every } i \geq 0,$$

constitutes an exhaustive filtration of $\mathcal{F}_1$ by coherent $\mathcal{O}_C$-submodules. Moreover, since $\mathcal{F}_3$ is coherent, there exists $i_1 \in \mathbb{N}$ such that:

$$p(\mathcal{C}_{2,i}) = \mathcal{F}_3, \quad \text{for every } i \geq i_1.$$

Then for every $i \geq i_1$, the map ι induces an isomorphims of $\mathcal{O}_C$-modules from $\mathcal{C}_{1,i+1}/\mathcal{C}_{1,i}$ onto $\mathcal{C}_{2,i+1}/\mathcal{C}_{2,i}$. This shows that:

$$h^1(C, \mathcal{C}_{1,i+1}/\mathcal{C}_{1,i}) = 0, \quad \text{for every } i \geq \max(i_0, i_1).$$

Finally the implication $(ii) \Rightarrow (i)$ in Theorem 3.2.7, (2) shows that $h^1(C, \mathcal{F}_1) = \overline{h}^1(C, \mathcal{F}_1) < +\infty$. This completes the proof of 3). □

3.3. Proof of Theorem 3.2.7

3.3.1. Serre duality for quasi-coherent sheaves on curves. To any sheaf $\mathcal{F}$ of $\mathcal{O}_C$-modules, we may attach the k-vector space

$$\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F}, \omega_C)$$

of morphisms of sheaves of $\mathcal{O}_C$-modules from $\mathcal{F}$ to ω_C. If $\varphi : \mathcal{F} \to \omega_C$ is an element of $\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F}, \omega_C)$ and α is a class in $H^1(C, \mathcal{F})$, we may apply to α the morphism

$$\varphi^1 : H^1(C, \mathcal{F}) \longrightarrow H^1(C, \omega_C)$$

deduced from φ by functoriality of the cohomology. Thus we get a class $\varphi^1(\alpha)$ in the k-vector space $H^1(C, \omega_C)$, itself canonically isomorphic to k.

In this way, we define a k-bilinear map:

$$\langle .,. \rangle_{\mathcal{F}} : \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F}, \omega_C) \times H^1(C, \mathcal{F}) \longrightarrow H^1(C, \omega_C) \xrightarrow[\sim]{\operatorname{Res}_C} k, \quad (\varphi, \alpha) \longmapsto \operatorname{Res}_C \varphi^1(\alpha).$$

When the $\mathcal{O}_C$-module $\mathcal{F}$ is locally free of finite rank, Serre duality asserts that this k-bilinear map is a perfect pairing of finite dimensional k-vector spaces. Serre duality may be extended to arbitrary quasi-coherent $\mathcal{O}_C$-modules in the following guise:

Proposition 3.3.1. *For every quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$ over C, the map*

$$D_{\mathcal{F}} : \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F}, \omega_C) \longrightarrow \operatorname{Hom}_k(H^1(C, \mathcal{F}), k), \quad \varphi \longmapsto \langle \varphi, . \rangle_{\mathcal{F}} := (\alpha \longmapsto \operatorname{Res}_C \varphi^1(\alpha)),$$

is an isomorphism of k-vector spaces.

The construction of the map $D_{\mathcal{F}}$ is natural in $\mathcal{F}$. Namely, for any morphism

$$f : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$$

of $\mathcal{O}_C$-modules, the following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F}_2, \omega_C) \arrow[r, ".\circ f"] \arrow[d, "D_{\mathcal{F}_2}"'] & \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F}_1, \omega_C) \arrow[d, "D_{\mathcal{F}_1}"'] \\
\operatorname{Hom}_k(H^1(C, \mathcal{F}_2), k) \arrow[r, "{}^t f^1 := .\circ f^1"] & \operatorname{Hom}_k(H^1(C, \mathcal{F}_1), k),
\end{tikzcd} \tag{3.3.1}$$

where

$$f^1 : H^1(C, \mathcal{F}_1) \longrightarrow H^1(C, \mathcal{F}_2)$$

denotes the morphism between cohomology groups induced by the morphism of sheaves f. The commutativity of (3.3.1) is indeed a straightforward consequence of the definitions of $D_{\mathcal{F}_1}$ and $D_{\mathcal{F}_2}$ and of the functoriality of the cohomology of $\mathcal{O}_C$-modules.

Proposition 3.3.1 is derived by some standard limit arguments from its special case where $\mathcal{F}$ is locally free of finite rank. Since this proposition plays a key role in the proof of Theorem 3.2.7 (1), we provide some details.

Proof of Proposition 3.3.1. As indicated above, when $\mathcal{F}$ is coherent and locally free, this is a reformulation of the classical Serre duality on smooth projective curves.

When $\mathcal{F}$ is coherent, we may consider the quotient morphism: $p : \mathcal{F} \to \mathcal{F}_{/\mathrm{tor}}$. Its range $\mathcal{F}_{/\mathrm{tor}}$ is coherent and locally free — and therefore $D_{\mathcal{F}_{/\mathrm{tor}}}$ is an isomorphism — and the morphism

$$p^1 : H^1(C, \mathcal{F}) \longrightarrow H^1(C, \mathcal{F}_{/\mathrm{tor}})$$

that it induces between first cohomology groups is an isomorphism. Moreover every morphism of $\mathcal{O}_C$-modules from $\mathcal{F}$ to some line bundle over C vanishes on $\mathcal{F}_{\mathrm{tor}}$, and therefore uniquely factors through $\mathcal{F}_{/\mathrm{tor}}$. This implies that the k-linear map:

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}_{/\mathrm{tor}}, \omega_C) \xrightarrow{.\circ p} \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \omega_C) \tag{3.3.2}$$

is an isomorphism. The commutativity of the naturality diagram (3.3.1) with $\mathcal{F}_1 = \mathcal{F}$, $\mathcal{F}_2 := \mathcal{F}_{/\mathrm{tor}}$, and $f = p$ therefore implies that $D_{\mathcal{F}}$ also is an isomorphism.

Consider now an arbitrary quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$. For every two elements of $\mathrm{coh}(\mathcal{F})$ such that $\mathcal{C} \subseteq \mathcal{C}'$, the commutative diagram (3.3.1) applied to the inclusion morphism $i_{\mathcal{C}'\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{C}'$ reads:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{C}', \omega_C) \arrow[r, ".\circ i_{\mathcal{C}'\mathcal{C}} = ._{|\mathcal{C}}"] \arrow[d, "D_{\mathcal{C}'}"'] & \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{C}, \omega_C) \arrow[d, "D_{\mathcal{C}}"'] \\ \mathrm{Hom}_k(H^1(C, \mathcal{C}'), k) \arrow[r, "{}^t i^1_{\mathcal{C}'\mathcal{C}}"] & \mathrm{Hom}_k(H^1(C, \mathcal{C}), k). \end{tikzcd} \tag{3.3.3}$$

Moreover the maps $D_{\mathcal{C}'}$ and $D_{\mathcal{C}}$ are isomorphisms.

These isomorphisms define an isomorphism of k-vector spaces:

$$\lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} D_{\mathcal{C}} : \lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{C}, \omega_C) \xrightarrow{\sim} \lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \mathrm{Hom}_k(H^1(C, \mathcal{C}), k). \tag{3.3.4}$$

By composition with the canonical isomorphisms:

$$\mathrm{Hom}_{\mathcal{O}_C}(\operatorname*{colim}_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \mathcal{C}, \omega_C) \xrightarrow{\sim} \lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{C}, \omega_C)$$

and

$$\mathrm{Hom}_k(\operatorname*{colim}_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} H^1(C, \mathcal{C}), k) \xrightarrow{\sim} \lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \mathrm{Hom}_k(H^1(C, \mathcal{C}), k)$$

and with the isomorphisms:

$$. \circ j_{\mathcal{F}} : \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \omega_C) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_C}(\operatorname*{colim}_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \mathcal{C}, \omega_C)$$

and:

$${}^t j^1_{\mathcal{F}} := . \circ j^1_{\mathcal{F}} : \mathrm{Hom}_k(H^1(C, \mathcal{F}), k) \xrightarrow{\sim} \mathrm{Hom}_k(\operatorname*{colim}_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} H^1(C, \mathcal{C}), k)$$

deduced from the isomorphisms $j_{\mathcal{F}}$ and $j^1_{\mathcal{F}}$ in Proposition 3.2.1, the isomorphism (3.3.4) defines an isomorphism:

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \omega_C) \xrightarrow{\sim} \mathrm{Hom}_k(H^1(C, \mathcal{F}), k).$$

It a straightforward albeit tedious consequence of the definitions that it coincides with $D_{\mathcal{F}}$, which consequently is an isomorphism. $\square$

3.3.2. Proof of the equality $\underline{h}^1(C,\mathcal{F}) = h^1(C,\mathcal{F})$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_C$-module. To establish part (1) of Theorem 3.2.7, we have to show that the inequality (3.2.10)

$$\underline{h}^1(C,\mathcal{F}) \leq h^1(C,\mathcal{F}).$$

is actually an equality. As Serre duality, in the form established in Proposition 3.3.1, implies the equality of dimensions:

$$h^1(C,\mathcal{F}) = \dim_k \operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F},\omega_C),$$

this will be a consequence of the following lemma:

LEMMA 3.3.2. *For any finite dimensional k-vector subspace V of* $\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F},\omega_C)$, *there exists $\mathcal{G}$ in* $\operatorname{scoft}(\mathcal{F})$ *such that*

$$h^1(C,\mathcal{F}/\mathcal{G}) \geq \dim_k V.$$

PROOF. The subspace V of $\operatorname{Hom}_{\mathcal{O}_C}(\mathcal{F},\omega_C)$ defines a morphism of $\mathcal{O}_C$-modules from $\mathcal{F}\otimes_k V$ to ω_C, or equivalently a morphism of $\mathcal{O}_C$-modules:

$$\varphi : \mathcal{F} \longrightarrow \omega_C \otimes_k V^\vee,$$

where $V^\vee := \operatorname{Hom}_k(V,k)$. By construction, for every $v \in V$, the morphism of $\mathcal{O}_C$-modules defined as the composition:

$$\mathcal{F} \xrightarrow{\varphi} \omega_C \otimes_k V^\vee \xrightarrow{Id\otimes_k v} \omega_C$$

coincides with v itself.

Let us prove that $\mathcal{G} := \ker\varphi$ satisfies the conclusion of Lemma 3.3.2. Clearly $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_C$-submodule of $\mathcal{F}$. Moreover $\mathcal{F}/\mathcal{G} \simeq \operatorname{im}\varphi$ is a quasi-coherent $\mathcal{O}_C$-submodule of $\omega_C \otimes_k V^\vee$, hence is coherent and locally free. This shows that $\mathcal{G}$ is an element of $\operatorname{scoft}(\mathcal{F})$.

The morphism φ defines a k-linear map between cohomology groups:

$$\varphi^1 : H^1(C,\mathcal{F}) \longrightarrow H^1(C,\omega_C\otimes_k V^\vee),$$

the range of which may be identified to $V^\vee$ by means of the isomorphisms:

$$H^1(C,\omega_C\otimes_k V^\vee) \simeq H^1(C,\omega_C)\otimes_k V^\vee \quad \text{and} \quad \operatorname{Res}_C \otimes \operatorname{Id}_{V^\vee} : H^1(C,\omega_C)\otimes_k V^\vee \xrightarrow{\sim} V^\vee.$$

Moreover the k-linear map

$${}^t\varphi^1 : V \longrightarrow \operatorname{Hom}_k(H^1(C,\mathcal{F}),k),$$

defined as the transpose of

$$\varphi^1 : H^1(C,\mathcal{F}) \longrightarrow V^\vee,$$

coincides by construction with the restriction $D_{\mathcal{F}|V}$, and therefore is injective. This proves that φ^1 is surjective.

Since φ factors as

$$\varphi : \mathcal{F} \longrightarrow \operatorname{im}\varphi \hookrightarrow \omega_C \otimes V^\vee,$$

the k-linear map φ^1 factors through

$$H^1(C,\operatorname{im}\varphi) \simeq H^1(C,\mathcal{F}/\mathcal{G}),$$

and therefore:

$$\dim_k H^1(C,\mathcal{F}/\mathcal{G}) \geq \dim_k \operatorname{im}\varphi^1 = \dim_k V. \qquad \square$$

3.3.3. The condition $h^1(C,\mathcal{F}) = \overline{h}^1(C,\mathcal{F}) < +\infty$.

3.3.3.1. *Proof of (i) $\Leftrightarrow$ (ii) in Theorem 3.2.7 (2).* Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_C$-module of countable type such that $N := h^1(C, \mathcal{F})$ is finite.

According to Corollary 3.2.2 and Proposition 3.2.5, the condition

$$\overline{h}^1(C, \mathcal{F}) = N$$

is satisfied if and only if there exists an exhaustive filtration $(\mathcal{C}_i)_{i \in \mathbb{N}}$ by coherent $\mathcal{O}_C$-submodules satisfying the following condition:

$$h^1(C, \mathcal{C}_i) = N \quad \text{for every large enough } i \in \mathbb{N}. \tag{3.3.5}$$

To complete the equivalence $(i) \Leftrightarrow (ii)$ in part (2) of Theorem 3.2.7, we will show that the existence of such a filtration $(\mathcal{C}_i)_{i \in \mathbb{N}}$ satisfying (3.3.5) is equivalent to condition (ii) in Theorem 3.2.7.

For every exhaustive filtration $(\mathcal{C}_i)_{i \in \mathbb{N}}$ of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules, we may consider the inclusion morphisms:

$$\iota_i : \mathcal{C}_i \hookrightarrow \mathcal{C}_{i+1},$$

and the associated k-linear maps:

$$\iota_i^1 : H^1(C, \mathcal{C}_i) \longrightarrow H^1(C, \mathcal{C}_{i+1}).$$

Using again that C has cohomological dimension ≤ 1, we get exact sequences of cohomology groups:

$$H^1(C, \mathcal{C}_i) \xrightarrow{\iota_i^1} H^1(C, \mathcal{C}_{i+1}) \longrightarrow H^1(C, \mathcal{C}_i/\mathcal{C}_{i+1}) \longrightarrow 0,$$

which actually are finite dimensional k-vector spaces. Moreover, as recalled in Proposition 3.2.1, the colimit of the system

$$H^1(C, \mathcal{C}_0) \xrightarrow{\iota_0^1} H^1(C, \mathcal{C}_1) \xrightarrow{\iota_1^1} H^1(C, \mathcal{C}_2) \xrightarrow{\iota_2^1} \dots \xrightarrow{\iota_{i-1}^1} H^1(C, \mathcal{C}_i) \xrightarrow{\iota_i^1} H^1(C, \mathcal{C}_{i+1}) \xrightarrow{\iota_{i+1}^1} \dots \tag{3.3.6}$$

is canonically isomorphic to $H^1(C, \mathcal{F})$.

Let us assume that $(\mathcal{C}_i)_{i \in \mathbb{N}}$ satisfies condition (ii) in Theorem 3.2.7. Then the k-linear maps ι_i^1 are surjective for $i \geq i_0$. The integers $h^1(C, \mathcal{C}_i)$ are therefore decreasing for $i \geq i_0$, hence eventually constant. Consequently the surjective morphisms ι_i^1 are isomorphisms when i is large enough, say when $i \geq i_1$. This implies that, for any $i \geq i_1$, $H^1(C, \mathcal{C}_i)$ is isomorphic to $H^1(C, \mathcal{F})$, and therefore has the same dimension N. This proves that $(\mathcal{C}_i)_{i \in \mathbb{N}}$ satisfies (3.3.5), and therefore establishes the validity of (3.2.14).

Conversely, let us assume that (3.3.5) holds. Then we may apply the following lemma to the system of finite dimensional k-vector spaces (3.3.6). It implies that the maps ι_i^1 are isomorphisms, and therefore that $H^1(C, \mathcal{C}_{i+1}/\mathcal{C}_i)$ vanishes, when i is large enough, and therefore the validity of condition (ii) in Theorem 3.2.7.

Lemma 3.3.3. *Let us consider a direct system, indiced by $\mathbb{N}$, of k-vector spaces:*

$$V_0 \xrightarrow{j_0} V_1 \xrightarrow{j_1} V_2 \xrightarrow{j_2} \dots \xrightarrow{j_{i-1}} V_i \xrightarrow{j_i} V_{i+1} \xrightarrow{j_{i+1}} \dots.$$

If there exists N in $\mathbb{N}$ such that:

$$\dim_k V_i = N \quad \textit{for every } i \in \mathbb{N}, \tag{3.3.7}$$

and

$$\dim_k \operatorname*{colim}_{i \in \mathbb{N}} V_i = N, \tag{3.3.8}$$

then there exists i_0 in $\mathbb{N}$ such that j_i is an isomorphism for every integer $i \geq i_0$.

Proof. Let us assume that (3.3.7) holds and that the conclusion of the lemma is not satisfied, or equivalently that there exists a strictly increasing sequence $(i_n)_{n \in \mathbb{N}}$ such that, for every n in $\mathbb{N}$, j_{i_n} is not an isomorphism, and therefore not injective.

Every finite dimensional k-vector subspace F of $\operatorname{colim}_{i\in\mathbb{N}} V_i$ is contained in the image of V_{i_n} for n large enough. Therefore:

$$\dim_k F \leq \dim_k j_{i_n}(V_{i_n}) \leq N-1.$$

This proves the estimate

$$\dim_k \operatorname*{colim}_{i\in\mathbb{N}} V_i \leq N-1,$$

and implies that (3.3.8) does not hold. □

3.3.3.2. *Proof of* $(ii) \Leftrightarrow (iii)$ *in in Theorem 3.2.7 (2).* Let $(\mathcal{C}_i)_{i\in\mathbb{N}}$ and i_0 be as in condition (ii). Then the sequence $(\mathcal{C}_i/\mathcal{C}_{i_0})_{i\geq i_0}$ is an exhaustive filtration of $\mathcal{F}/\mathcal{C}_{i_0}$ by coherent $\mathcal{O}_C$-submodules. Moreover for any $i \geq i_0$, we have:

$$h^1(C, (\mathcal{C}_{i+1}/\mathcal{C}_{i_0})/(\mathcal{C}_i/\mathcal{C}_{i_0})) = h^1(C, \mathcal{C}_{i+1}/\mathcal{C}_i) = 0.$$

Therefore (3.2.14) applied to $\mathcal{F}/\mathcal{C}_{i_0}$ establishes the equality:

$$\overline{h}^1(C, \mathcal{F}/\mathcal{C}_{i_0}) = h^1(\mathcal{C}_{i_0}/\mathcal{C}_{i_0}) = 0,$$

and therefore the vanishing of $\mathrm{ev}\overline{h}^1(C,\mathcal{F})$. This completes the proof of (iii).

Conversely let us assume that (iii) is satisfied, or equivalently that there exists a coherent $\mathcal{O}_C$-submodule $\mathcal{C}$ of $\mathcal{F}$ such that

$$\overline{h}^1(C, \mathcal{F}/\mathcal{C}) = 0.$$

The implication $(i) \Rightarrow (ii)$ applied to $\mathcal{F}' := \mathcal{F}/\mathcal{C}$ establishes the existence of an exhaustive filtration $(\mathcal{C}'_i)_{i\in\mathbb{N}}$ and of $i_0 \in \mathbb{N}$ such that:

$$h^1(C, \mathcal{C}'_{i+1}/\mathcal{C}'_i) = 0 \quad \text{for every } i \geq i_0.$$

Then the $\mathcal{O}_C$-submodules $\mathcal{C}'_i$ of $\mathcal{F}$ containing $\mathcal{C}$ such that

$$\mathcal{C}_i/\mathcal{C} = \mathcal{C}'_i$$

define an exhaustive filtration of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules that satisfies condition (ii).

3.3.4. Quasi-coherent sheaves of countable type such that $\overline{h}^1(C,\mathcal{F}) < +\infty$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_C$-module of countable type.

The following lemma is a straightforward addition to Proposition 3.2.5:

Proposition 3.3.4. *For every $N \in \mathbb{N}$, the following two conditions are equivalent:*

(i) $\overline{h}^1(C,\mathcal{F}) = N$;
(ii) there exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules that satisfies the following two conditions:

$$h^1(C, \mathcal{C}_i) = N, \quad \textit{for every } i \in \mathbb{N}, \tag{3.3.9}$$

and:

$$h^1(C, \mathcal{C}) \geq N, \quad \textit{for every } \mathcal{C} \textit{ in } \operatorname{coh}(\mathcal{F}) \textit{ containing } \mathcal{C}_0. \tag{3.3.10}$$

Let us recall that the *minimum slope* $\mu_{\min}(\mathcal{C})$ of a coherent $\mathcal{O}_C$-module $\mathcal{C}$ is defined as the infimum of the slopes

$$\mu(Q) := \frac{\deg_C Q}{\operatorname{rk} Q}$$

of the locally free quotients Q of $\mathcal{C}$ of positive rank. It is also the infimum of the slopes $\mu(Q)$ of the coherent quotients Q of $\mathcal{C}$ of positive rank. The minimum slope $\mu_{\min}(\mathcal{C})$ coincides with the minimum slope $\mu_{\min}(\mathcal{C}_{/\mathrm{tor}})$ of the vector bundle $\mathcal{C}_{/\mathrm{tor}}$. It is $+\infty$ if and only if $\mathcal{C}$ is torsion, and satisfies the equality:

$$\mu_{\min}(\mathcal{C} \otimes L) = \mu_{\min}(\mathcal{C}) + \deg_C L \tag{3.3.11}$$

for any line bundle L over C. Moreover the following implications hold:

$$\mu_{\min}(\mathcal{C}) > 2g-2 \Longrightarrow h^1(C,\mathcal{C}) = 0 \Longrightarrow \mu_{\min}(\mathcal{C}) \geq g-1. \tag{3.3.12}$$

LEMMA 3.3.5. *Let us assume that $N := \overline{h}^1(C,\mathcal{F})$ is finite. If $(\mathcal{C}_i)_{i\in\mathbb{N}}$ is an exhaustive filtration of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules satisfying the conditions* (3.3.9) *and* (3.3.10)*, then, for every $i \in \mathbb{N}$, we have:*

$$\mu_{\min}(\mathcal{C}_i/\mathcal{C}_0) \geq g-1. \tag{3.3.13}$$

PROOF. We have to prove that, for every $i \in \mathbb{N}$ and every locally free quotient Q of positive rank of $\mathcal{C}_i/\mathcal{C}_0$, its slope satisfies the lower bound:

$$\mu(Q) \geq g-1,$$

or equivalently, according to the Riemann-Roch formula:

$$\chi(C,Q) := h^0(C,Q) - h^1(C,Q) \geq 0. \tag{3.3.14}$$

To achieve this, observe that any such quotient Q may be written $\mathcal{C}_i/\mathcal{C}$, where $\mathcal{C}$ is a coherent $\mathcal{O}_C$-submodule of $\mathcal{F}$ such that

$$\mathcal{C}_0 \subseteq \mathcal{C} \subseteq \mathcal{C}_i,$$

and consider the short exact sequence of $\mathcal{O}_C$-modules:

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{C}_i \longrightarrow Q \longrightarrow 0.$$

From the associated long exact sequence of cohomology groups, we derive the equalities:

$$\begin{aligned}\chi(C,Q) &= \chi(C,\mathcal{C}_i) - \chi(C,\mathcal{C}) \\ &= h^0(C,\mathcal{C}_i) - h^0(C,\mathcal{C}) + h^1(C,\mathcal{C}) - h^1(C,\mathcal{C}_i).\end{aligned}$$

The difference $h^0(C,\mathcal{C}_i) - h^0(C,\mathcal{C})$ is clearly non-negative, and $h^1(C,\mathcal{C}) - h^1(C,\mathcal{C}_i)$ also, according to (3.3.9) and (3.3.10). This establishes the lower bound (3.3.14). □

To complete the proof of part (3) of Theorem 3.2.7, assume that $N := \overline{h}^1(C,\mathcal{F})$ is finite, and choose an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ as in condition (ii) of Proposition 3.3.4. Then the lower bounds (3.3.13) on the minimal slopes of the quotients $\mathcal{C}_i/\mathcal{C}_0$ are satisfied. These imply the lower bounds:

$$\mu_{\min}(\mathcal{C}_{i+1}/\mathcal{C}_i) \geq g-1, \quad \text{for every } i \in \mathbb{N},$$

since $\mathcal{C}_{i+1}/\mathcal{C}_i$ is a quotient of $\mathcal{C}_{i+1}/\mathcal{C}_0$. Consequently, if L is a line bundle over C such that $\deg_C L > g-1$, then we have, for every $i \in \mathbb{N}$:

$$\mu_{\min}((\mathcal{C}_{i+1}\otimes L)/(\mathcal{C}_i \otimes L)) = \mu_{\min}((\mathcal{C}_{i+1}/\mathcal{C}_i)\otimes L) \geq g-1+\deg_C L > 2g-2,$$

and therefore, according to the implication (3.3.12):

$$h^1(C,(\mathcal{C}_{i+1}\otimes L)/(\mathcal{C}_i\otimes L)) = 0.$$

This shows that the filtration $(\mathcal{C}_i \otimes L)_{i\in\mathbb{N}}$ of $\mathcal{F}\otimes L$ satisfies the condition (ii) of part (2) of Theorem 3.2.7 for $\mathcal{F}\otimes L$ instead of $\mathcal{F}$, and consequently establishes (3.2.15).

3.4. Constructing quasi-coherent $\mathcal{O}_C$-modules $\mathcal{F}$ such that $h^1(C,\mathcal{F}) < \overline{h}^1(C,\mathcal{F})$

In this section, we describe a construction of locally free quasi-coherent $\mathcal{O}_C$-modules of countable type such that $h^1(C,\mathcal{F}) (= \underline{h}^1(C,\mathcal{F}))$ and $\overline{h}^1(C,\mathcal{F})$ are finite and distinct. This construction involve a suitable sequence of extensions by line bundles over C, and is basically dual to the one in [**Bos20b**, Section 9.3]. We therefore allow ourselves to leave a few details to the reader.

3.4.1. The data $(\mathcal{M}_i)_{i\geq 1}$ and $(\beta_i)_{i\geq 2}$, and the $\mathcal{O}_C$-module $\mathcal{F}$. .

Let $(\mathcal{M}_i)_{i\geq 1}$ be a sequence of line bundles over C, and let $(\beta_i)_{i\geq 2}$ be a sequence of cohomology classes:

$$\beta_i \in H^1(C, \mathcal{M}_{i-1} \otimes \mathcal{M}_i^\vee) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{M}_i, \mathcal{M}_{i-1}).$$

From these data, we can construct a diagram:

$$\mathcal{F}_\bullet : \quad \mathcal{F}_0 \xrightarrow{\iota_0} \mathcal{F}_1 \xrightarrow{\iota_1} \mathcal{F}_2 \xrightarrow{\iota_2} \dots \xrightarrow{\iota_{i-1}} \mathcal{F}_i \xrightarrow{\iota_i} \mathcal{F}_{i+1} \xrightarrow{\iota_{i+1}} \dots$$

where, for every $i \in \mathbb{N}$, $\mathcal{F}_i$ is vector bundle of rank i over C and ι_i is an injective morphism of $\mathcal{O}_C$-modules with saturated image, such that the following condition is satisfied: *for every $i \geq 1$, there exists an isomorphism of $\mathcal{O}_C$-modules*

$$p_i : \mathcal{F}_i/\iota_{i-1}(\mathcal{F}_{i-1}) \xrightarrow{\sim} \mathcal{M}_i$$

such that the extension class $\tilde{\beta}_i$ in $\mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{M}_i, \mathcal{M}_{i-1})$ of the 1-extension

$$0 \longrightarrow \mathcal{F}_{i-1} \xrightarrow{\iota_{i-1}} \mathcal{F}_i \xrightarrow{p_i} \mathcal{M}_i \longrightarrow 0 \tag{3.4.1}$$

is sent to β_i by the morphism:

$$p_{i-1} \circ . : \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{M}_i, \mathcal{F}_{i-1}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(\mathcal{M}_i, \mathcal{M}_{i-1}). \tag{3.4.2}$$

This is established by a straightforward inductive construction: starting from $\mathcal{F}_0 := 0$, $\mathcal{F}_1 := \mathcal{M}_1$, $\iota_0 := 0$, and $p_1 := \mathrm{Id}_{\mathcal{M}_1}$, and construct $(\mathcal{F}_k, \iota_{k_1}, p_k)$ by induction on k, by using the surjectivity of the maps (3.4.2), itself a consequence of the vanishing of the functor $\mathrm{Ext}^2_{\mathcal{O}_C}$ on vector bundles over C.

3.4.2. Computation of $h^1(C, \mathcal{F})$ and $\overline{h}^1(C, \mathcal{F})$.. Let us consider the quasi-coherent sheaf over C defined as the colimit of the inductive system $\mathcal{F}_\bullet$:

$$\mathcal{F} := \operatorname{colim} \mathcal{F}_\bullet.$$

This quasi-coherent sheaf $\mathcal{F}$ is locally free. Namely, for every affine open subscheme U in C, the module $\mathcal{F}(U)$ over the Dedekind ring $\mathcal{O}_C(U)$ is projective of infinite countable rank, and therefore free. We will identify the vector bundles $\mathcal{F}_n$ with their image in $\mathcal{F}$.

Let us now assume that the following condition is satisfied: *for every $i \geq 2$, the cup product by β_i defines an isomorphism:*

$$\beta_i \cup . : H^0(C, \mathcal{M}_i) \xrightarrow{\sim} H^1(C, \mathcal{M}_{i-1}).$$

We shall prove that, under this assumption, *the following relations hold:*

$$h^1(C, \mathcal{F}) = 0 \tag{3.4.3}$$

and:

$$\overline{h}^1(C, \mathcal{F}) = \liminf_{k\to+\infty} h^1(C, \mathcal{M}_k). \tag{3.4.4}$$

To achieve this, let us consider the long exact sequence of cohomology groups attached to the short exact sequence of $\mathcal{O}_C$-modules (3.4.1):

$$\begin{aligned} 0 \longrightarrow H^0(C, \mathcal{F}_{i-1}) \xrightarrow{\iota^0_{i-1}} H^0(C, \mathcal{F}_i) \xrightarrow{p^0_i} H^0(C, \mathcal{M}_i) & \\ \xrightarrow{\tilde{\beta}_i \cup .} H^1(C, \mathcal{F}_{i-1}) \xrightarrow{\iota^1_{i-1}} H^1(C, \mathcal{F}_i) \xrightarrow{p^1_i} H^1(C, \mathcal{M}_i) \longrightarrow 0. & \end{aligned} \tag{3.4.5}$$

A straightforward induction, based on the commutativity of the diagram:

$$\begin{tikzcd} H^0(C,\mathcal{M}_i) \arrow[d, "\beta_i\cup."'] \arrow[dr, "\tilde{\beta}_i\cup."] & \\ H^1(C,\mathcal{M}_{i-1}) \arrow[r, "p^1_{i-1}"] & H^1(C,\mathcal{F}_{i-1}), \end{tikzcd}$$

shows that the following properties are satisfied:

- for every $i \geq 1$, the map p^1_i defines an isomorphism:
$$p^1_i : H^1(C,\mathcal{F}_i) \xrightarrow{\sim} H^1(C,\mathcal{M}_i);$$
- for every $i \geq 2$, the morphism $\tilde{\beta}_i \cup .$ in (3.4.5) is an isomorphism, and therefore p^0_i and ι^1_{i-1} vanish and ι^0_{i-1} is an isomorphism.

The vanishing of the maps

$$\iota^1_{i-1} : H^1(C,\mathcal{F}_{i-1}) \longrightarrow H^1(C,\mathcal{F}_i)$$

implies that

$$H^1(C,\mathcal{F}) \simeq \operatorname*{colim}_i H^1(C,\mathcal{F}_i)$$

also vanishes. This completes the proof of (3.4.3).

Similarly the fact that the map ι^0_{i-1} are isomorphisms shows that the inclusion morphism :

$$\mathcal{M}_1 \simeq \mathcal{F}_1 \hookrightarrow \mathcal{F}$$

induces an isomorphism:

$$H^0(C,\mathcal{M}_1) \xrightarrow{\sim} H^0(C,\mathcal{F}).$$

Let $\mathcal{C}$ be an element of $\mathrm{coh}(\mathcal{F})$. There exists a smallest $n \in \mathbb{N}$ such that $\mathcal{C} \subseteq \mathcal{F}_n$. If $\mathcal{C} \neq 0$, then $n \geq 1$ and the composite morphism

$$\mathcal{C} \longrightarrow \mathcal{F}_n \xrightarrow{p_n} \mathcal{M}_n$$

is non-zero; therefore its cokernel is a torsion coherent $\mathcal{O}_C$-module, and the following inequality holds:

$$h^1(C,\mathcal{C}) \geq h^1(C,\mathcal{M}_n).$$

Consequently we obtain the estimate:

$$\overline{h}^1(C,\mathcal{F}) = \liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} h^1(C,\mathcal{C}) \geq \liminf_{k\to+\infty} h^1(C,\mathcal{M}_n). \tag{3.4.6}$$

Moreover, since the maps p^1_i, $i \geq 1$, are isomorphisms, we also have:

$$h^1(C,\mathcal{F}_i) = h^1(C,\mathcal{M}_i),$$

and therefore:

$$\overline{h}^1(C,\mathcal{F}) = \liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} h^1(C,\mathcal{C}) \leq \liminf_{k\to+\infty} h^1(C,\mathcal{F}_n) = \liminf_{k\to+\infty} h^1(C,\mathcal{M}_n). \tag{3.4.7}$$

The equality (3.4.4) follows from (3.4.6) and (3.4.7).

3.4.3. Application: constructing $\mathcal{F}$ such that $h^1(C,\mathcal{F}) = 0$ and $\overline{h}^1(C,\mathcal{F}) = 1$. The previous construction allows one to construct $\mathcal{F}$ such that:

$$h^1(C,\mathcal{F}) < \overline{h}^1(C,\mathcal{F})$$

when the genus g of C is positive,[12] at least when the base field k is algebraically closed or is a finite field of cardinality larger than some function of g.

Indeed assume that $\mathcal{M}$ is a line bundle over C such that

$$h^0(C,\mathcal{M}) = h^1(C,\mathcal{M}) = 1.$$

Such a line bundle exists for instance when $g = 1$ (take $\mathcal{M} = \mathcal{O}_C$), and when $g > 1$ and when C and the smooth locus of the theta divisor in $\mathrm{Pic}^{g-1}_{C/k}$ possess a k-rational point. A straightforward application of Serre duality establishes the existence of β in $H^1(C,\mathcal{O}_C)$ such that the map

$$\beta \cup . : H^0(C,\mathcal{M}) \longrightarrow H^1(C,\mathcal{M})$$

is nonzero, hence an isomorphism. The construction above applied to $\mathcal{M}_i := \mathcal{M}$ (resp. $\beta_i := \beta$) for every $i \geq 1$ (resp. for every $i \geq 2$) produces a quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$ such that:

$$h^0(C,\mathcal{F}) = 1, \quad h^1(C,\mathcal{F}) = 0, \quad \text{and} \quad \overline{h}^1(C,\mathcal{F}) = 0.$$

3.5. h^1-Finiteness of Quasi-coherent Sheaves

3.5.1. h^1-finiteness: definition and first properties.

LEMMA 3.5.1. *Let $\mathcal{F}$ be some quasi-coherent sheaf over C. If L is a line bundle over C that is generated by its global section (for instance, if $\deg_C L \geq 2g$), then the following implication holds:*

$$h^1(C,\mathcal{F}) < +\infty \Longrightarrow h^1(C,\mathcal{F}\otimes L) < +\infty. \tag{3.5.1}$$

PROOF. Let us assume that $s_1, \ldots, s_N$ are elements of $\Gamma(C,L)$ such that

$$\sum_{i=1}^{N} \mathcal{O}_C s_i = L.$$

We may consider the morphism

$$\sigma := (s_1, \ldots, s_n) : \mathcal{O}_C^{\oplus N} \longrightarrow L$$

of sheaves of $\mathcal{O}_C$-modules. It is surjective and its kernel $\ker\sigma$ is locally free of rank $N-1$. From the short exact sequence of locally free coherent sheaves

$$0 \longrightarrow \ker\sigma \hookrightarrow \mathcal{O}_C^{\oplus N} \xrightarrow{\sigma} L \longrightarrow 0,$$

we deduce a short exact sequence of quasi-coherent sheaves:

$$0 \longrightarrow \mathcal{F}\otimes\ker\sigma \hookrightarrow \mathcal{F}^{\oplus N} \xrightarrow{\sigma} \mathcal{F}\otimes L \longrightarrow 0,$$

and finally, since C has cohomological dimension 1, a surjective morphism of k-vector spaces:

$$H^1(C,\mathcal{F})^{\oplus N} \xrightarrow{\sigma^1} H^1(C,\mathcal{F}\otimes L) \longrightarrow 0.$$

This establishes the implication (3.5.1). □

[12] Observe that, according to Theorem 3.2.7, (3), quasi-coherent $\mathcal{O}_C$-modules $\mathcal{F}$ of countable type such that $h^1(C,\mathcal{F})$ and $\overline{h}^1(C,\mathcal{F})$ are finite and distinct do not exist when the genus $g = 0$.

Let D be some divisor of positive degree on C. Lemma 3.5.1 shows that, for any quasi-coherent sheaf $\mathcal{F}$ over C, the following conditions are equivalent.

h^1-**Fin**$_1$: For any line bundle L over C,

$$h^1(C, \mathcal{F} \otimes L^\vee) < +\infty.$$

h^1-**Fin**$_2$: For any $n \in \mathbb{N}$,

$$h^1(C, \mathcal{F} \otimes \mathcal{O}_C(-nD)) < +\infty.$$

h^1-**Fin**$_3$: There exists a sequence L_n of line bundles over C such that

$$\lim_{n \to +\infty} \deg_C L_n = +\infty,$$

and, for any $n \in \mathbb{N}$,

$$h^1(C, \mathcal{F} \otimes L_n^\vee) < +\infty.$$

When they are satisfied, we shall say that $\mathcal{F}$ is *h^1-finite.*

PROPOSITION 3.5.2. *Let us consider a short exact sequence of quasi-coherent $\mathcal{O}_C$-modules:*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0.$$

1) If $\mathcal{F}_2$ is h^1-finite, then $\mathcal{F}_3$ is h^1-finite.

2) If $\mathcal{F}_1$ and $\mathcal{F}_3$ are h^1-finite, then $\mathcal{F}_2$ is h^1-finite.

3) If $\mathcal{F}_3$ is coherent, then $\mathcal{F}_1$ is h^1-finite if and only if $\mathcal{F}_2$ is h^1-finite.

PROOF. This is a straightforward consequence of the exact sequences of cohomology groups, where L denotes an arbitrary line bundle over C:

$$H^0(C, \mathcal{F}_3 \otimes L^\vee) \longrightarrow H^1(C, \mathcal{F}_1 \otimes L^\vee) \longrightarrow H^1(C, \mathcal{F}_2 \otimes L^\vee) \longrightarrow H^1(C, \mathcal{F}_3 \otimes L^\vee) \longrightarrow 0. \qquad \square$$

Any torsion quasi-coherent sheaf over C is clearly h^1-finite. Together with Proposition 3.5.2, 1) and 2), this implies:

COROLLARY 3.5.3. *A quasi-coherent sheaf $\mathcal{F}$ over C is h^1-finite if and only if $\mathcal{F}/\mathcal{F}_{\mathrm{tor}}$ is h^1-finite.* $\square$

Any coherent sheaf over C is clearly h^1-finite. Examples of torsion free and non-coherent h^1–finite quasi-coherent sheaves are provided by the following straightforward proposition:

PROPOSITION 3.5.4. *1) For any open affine subscheme U of C and any quasi-coherent sheaf $\mathcal{F}$ over U, its direct image $i_{U*}\mathcal{F}$ by the inclusion morphism $i_U : U \hookrightarrow C$ is h^1-finite.*

2) Let $(L_n)_{n \in N}$ be a sequence of line bundles over C. The quasi-coherent sheaf $\bigoplus_{n \in \mathbb{N}} L_n$ over C is h^1-finite if and only if $\lim_{n \to +\infty} \deg_C L_n = +\infty$. $\square$

3.5.2. Global sections and h^1-finiteness. Let $\mathcal{F}$ be some quasi-coherent sheaf over C.

3.5.2.1. *Criteria of h^1-finiteness.* For any line bundle M over C, we may consider the tautological morphism of quasi-coherent sheaves:

$$\eta_M : \Gamma(C, \mathcal{F} \otimes M^\vee) \otimes_k M \longrightarrow \mathcal{F}.$$

For any open subscheme U of C and any $s \in \Gamma(C, \mathcal{F} \otimes M^\vee)$ and any $t \in \Gamma(U, M)$, this morphism maps $s \otimes_k t$ to the section $s_{|U} \otimes t$ of $\mathcal{F} \otimes M^\vee \otimes M \simeq \mathcal{F}$ over U. Recall that $\mathcal{F}$ is said to be *generated by its global sections* when the morphism of sheaves

$$\eta_{\mathcal{O}_C} : \Gamma(C, \mathcal{F}) \otimes_k \mathcal{O}_C \longrightarrow \mathcal{F}$$

is surjective. It is straightforward that the sheaf

$$\mathcal{F}_M := \operatorname{im} \eta_M$$

is the largest quasi-coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \otimes M^\vee$ is generated by its global sections.

As before, we denote by D some divisor of positive degree over C.

PROPOSITION 3.5.5. *The following conditions are equivalent:*

h^1-**Fin** *: The quasi-coherent sheaf $\mathcal{F}$ is h^1-finite.*

h^1-$\mathbf{Fin}_{1'}$ *: For any line bundle M over C, the quasi-coherent sheaf* $\operatorname{coker} \eta_M := \mathcal{F}/\mathcal{F}_M$ *over C is coherent.*

h^1-$\mathbf{Fin}_{2'}$ *: For any $n \in \mathbb{N}$, $\mathcal{F}/\mathcal{F}_{\mathcal{O}(nD)}$ is coherent.*

h^1-$\mathbf{Fin}_{3'}$ *: There exists a sequence $(L_n)_{n\geq 0}$ of line bundles over C such that*

$$\lim_{n\to+\infty} \deg_C L_n = +\infty,$$

and, for any $n \in \mathbb{N}$, the sheaf $\mathcal{F}/\mathcal{F}_{L_n}$ is coherent.

3.5.2.2. *Proof of Proposition 3.5.5.* We divide the proof of Proposition 3.5.5 in a succession of lemmas.

LEMMA 3.5.6. *Let M and N be two line bundles over C. If $\mathcal{F}/\mathcal{F}_M$ is coherent and if $H^1(C,N) = 0$, then*

$$\dim_k H^1(C, \mathcal{F} \otimes M^\vee \otimes N) = \dim_k H^1(C, (\mathcal{F}/\mathcal{F}_M) \otimes M^\vee \otimes N)) < +\infty.$$

In particular, if $\mathcal{F}$ is generated by its global sections, then $H^1(C, \mathcal{F} \otimes N) = 0$.

PROOF. From the exact sequence

$$\Gamma(C, \mathcal{F} \otimes M^\vee) \otimes_k N \xrightarrow{\eta_M \otimes Id_{M^\vee \otimes N}} \mathcal{F} \otimes M^\vee \otimes N \longrightarrow (\mathcal{F}/\mathcal{F}_M) \otimes M^\vee \otimes N \longrightarrow 0,$$

we deduce an exact sequence of cohomology groups

$$H^1(C, \Gamma(C, \mathcal{F} \otimes M^\vee) \otimes_k N) \longrightarrow H^1(C, \mathcal{F} \otimes M^\vee \otimes N) \longrightarrow H^1(C, (\mathcal{F}/\mathcal{F}_M) \otimes M^\vee \otimes N) \longrightarrow 0.$$

As C is Noetherian, the cohomology group $H^1(C, \Gamma(C, \mathcal{F} \otimes M^\vee) \otimes_k N)$ may be identified with $\Gamma(C, \mathcal{F} \otimes M^\vee) \otimes_k H^1(C,N)$, and therefore vanishes. Consequently,

$$H^1(C, \mathcal{F} \otimes M^\vee \otimes N) \xrightarrow{\sim} H^1(C, (\mathcal{F}/\mathcal{F}_M) \otimes M^\vee \otimes N).$$

When $\mathcal{F}$ is generated by its global sections, if we let $M = \mathcal{O}_C$, then $\mathcal{F}_M = \mathcal{F}$, and these cohomology groups vanish. $\square$

Let us consider the diagonal embedding of C into $C \times C := C \times_{\operatorname{Spec} k} C$, and the two projections of $C \times C$ to C:

$$C \xhookrightarrow{\Delta} C \times C \xrightarrow{p_1, p_2} C,$$

and let us denote by Δ_C the diagonal $\Delta(C)$ in $C \times C$. For any quasi-coherent sheaf $\mathcal{G}$ over $C \times C$, we shall write $\mathcal{G}(-\Delta_C)$ for $\mathcal{G} \otimes \mathcal{O}_{C\otimes C}(-\Delta_C)$.

For any two quasi-coherent sheaves $\mathcal{G}_1$ and $\mathcal{G}_2$ over C, we may introduce their "external tensor product", namely the quasi-coherent sheaf over $C \times C$:

$$\mathcal{G}_1 \boxtimes \mathcal{G}_2 := p_1^* \mathcal{G}_1 \otimes p_2^* \mathcal{G}_2.$$

For any line bundle M over C, the restriction of $M \otimes M^\vee$ to Δ_C is canonically isomorphic to $\mathcal{O}_{\Delta_C}$. This isomorphism determines a short exact sequence of coherent sheaves over the surface $C \times C$:

$$0 \longrightarrow M \boxtimes M^\vee(-\Delta_C) \longrightarrow M \boxtimes M^\vee \longrightarrow \mathcal{O}_{\Delta_C} \longrightarrow 0.$$

From this short exact sequence, by applying the functor $\cdot \otimes p_2^* F$, we derive a short exact sequence of quasi-coherent sheaves over $C \times C$:

$$0 \longrightarrow M \boxtimes (F \otimes M^\vee)(-\Delta_C) \longrightarrow M \boxtimes (F \otimes M^\vee) \longrightarrow \Delta_* F \longrightarrow 0. \tag{3.5.2}$$

Indeed, the left exactness holds in (3.5.2) because the map $p_1 : \Delta_C \longrightarrow C$ is an isomorphism, and *a fortiori* flat.

LEMMA 3.5.7. *For any line bundle M over C, if the quasi-coherent sheaf $Rp_{1*}(M \boxtimes (F \otimes M^\vee)(-\Delta_C))$ is coherent, then* $\operatorname{coker} \eta_M$ *is coherent.*

PROOF. From (3.5.2), we derive a long exact sequences of higher direct images along p_1:

$$p_{1*}(M \boxtimes (F \otimes M^\vee)) \longrightarrow p_{1*}\Delta_* F \longrightarrow Rp_{1*}(M \boxtimes (F \otimes M^\vee)(-\Delta_C)). \tag{3.5.3}$$

We have canonical isomorphisms:

$$p_{1*}(M \boxtimes (F \otimes M^\vee)) \xrightarrow{\sim} \Gamma(C, M^\vee) \otimes_k M \quad \text{and} \quad p_{1*}\Delta_* F \xrightarrow{\sim} F,$$

and the first arrow in the exact sequence (3.5.3) may be identified with η_M. The quasi-coherent sheaf $\operatorname{coker} \eta_M$ is therefore isomorphic to some subsheaf of $Rp_{1*}(M \boxtimes (F \otimes M^\vee)(-\Delta_M))$, and thus is coherent if the latter is. □

LEMMA 3.5.8. *Let M and N be two line bundles over C. Let us assume that there exists a vector bundle E over C and a surjective morphism of sheaves of $\mathcal{O}_{C\times C}$-modules*

$$\alpha : p_1^* E \longrightarrow p_2^*(N \otimes M^\vee)(-\Delta_C).$$

If moreover

$$\dim_k H^1(C, \mathcal{F} \otimes N^\vee) < +\infty, \tag{3.5.4}$$

then the sheaf $Rp_{1*}(M \boxtimes (F \otimes M^\vee)(-\Delta_M))$ is coherent.

Observe that the existence of some vector bundle E and of some morphism α as above is assured when the line bundle $N \otimes M^\vee$ is positive enough. For instance, it holds with

$$E := p_{1*}(p_2^*(N \otimes M^\vee)(-\Delta_C))$$

and with α the tautological morphism if $\deg_C N \geq \deg_C M + 2g + 1$.

PROOF. By taking the tensor product of the morphism α by the identity morphism of $M \boxtimes (\mathcal{F} \otimes N^\vee)$, we get an exact sequence of coherent sheaves over $C \times C$:

$$(E \otimes M) \boxtimes (\mathcal{F} \otimes N^\vee) \xrightarrow{\alpha \otimes Id_{M \boxtimes (\mathcal{F} \otimes N^\vee)}} (M \boxtimes (\mathcal{F} \otimes M^\vee))(-\Delta_C) \longrightarrow 0. \tag{3.5.5}$$

Observe that, for any quasi-coherent sheaf $\mathcal{G}$ over $C \times C$, the higher direct images $R^i p_{1*}\mathcal{G}$ vanish for every $i > 1$. Indeed, we may write C as the union $U \cup V$ of two affine open subschemes U and V, and therefore $C \times C$ as the union of the two open subschemes $C \times U$ and $C \times V$, the restriction of p_1 to both of them is an affine morphism.

Applied to $\mathcal{G} := \ker(\alpha \otimes Id_{M \boxtimes (\mathcal{F} \otimes N^\vee)})$, this shows that the exact sequence (3.5.5) determines, by considering higher direct images by p_1, a short exact sequence of quasi-coherent sheaves over C:

$$R^1 p_{1*}\big((E \otimes M) \boxtimes (\mathcal{F} \otimes N^\vee)\big) \longrightarrow R^1 p_{1*}\big(M \boxtimes (\mathcal{F} \otimes M^\vee)(-\Delta_C)\big) \longrightarrow 0.$$

Moreover, we have a canonical isomorphisms of quasi-coherent sheaves over C:

$$R^1 p_{1*}(E \otimes M) \boxtimes (\mathcal{F} \otimes N^\vee) \xrightarrow{\sim} H^1(C, \mathcal{F} \otimes N^\vee) \otimes_k E \otimes M.$$

When the finiteness condition (3.5.4) holds, the quasi-coherent sheaf $H^1(C, \mathcal{F} \otimes N^\vee) \otimes_k E \otimes M$, and therefore its quotient $R^1 p_{1*}\big(M \boxtimes (\mathcal{F} \otimes M^\vee)(-\Delta_C)\big)$ is coherent. □

PROOF OF PROPOSITION 3.5.5. The implications h^1-$\mathbf{Fin}_{1'} \Rightarrow h^1$-$\mathbf{Fin}_{2'} \Rightarrow h^1$-$\mathbf{Fin}_{3'}$ are clear.

To prove the implication h^1-$\mathbf{Fin}_{3'} \Rightarrow h^1$-$\mathbf{Fin}_{3}$, choose some line bundle N over C such that $H^1(C, N) = 0$ — for instance, any line bundle of degree $> 2g - 2$. If $(L_n)_{n\in\mathbb{N}}$ is a sequence of line bundles over C as in Condition h^1-$\mathbf{Fin}_{3'}$, Lemma 3.5.6 shows that the line bundles $L'_n := L_n \otimes N^\vee$ — the degrees of which go to infinity with N — satisfy

$$\dim_k H^1(C, \mathcal{F} \otimes L'^\vee_n) < +\infty.$$

To prove the implication h^1-$\mathbf{Fin}_1 \Rightarrow h^1$-$\mathbf{Fin}_{1'}$, observe that, for any two line bundles M and N over C, Lemma 3.5.7 together with Lemma 3.5.8 and the subsequent observation establish the following implication:

$$\dim_k H^1(C, \mathcal{F} \otimes N^\vee) < +\infty \quad \text{and} \quad \deg_C N \geq \deg_C M + 2g + 1 \Longrightarrow \operatorname{coker} \eta_M \text{ is coherent.} \qquad \square$$

3.5.2.3. *Stability under tensor product of h^1-finiteness.* As an application of the criteria for h^1-finiteness established in Proposition 3.5.5, we may prove:

PROPOSITION 3.5.9. *The tensor product $\mathcal{F} \otimes \mathcal{G}$ of any two h^1-finite quasi-coherent sheaves over C is h^1-finite.*

The proof will rely on Proposition 3.5.5 combined with Lemma 3.5.6 and with the following observation:

LEMMA 3.5.10. *For any h^1-finite quasi-coherent sheaf $\mathcal{F}$ over C and any coherent sheaf $\mathcal{G}$ over C, the quasi-coherent sheaf $\mathcal{F} \otimes \mathcal{G}$ is also h^1-finite.*

PROOF. This clearly holds when $\mathcal{G}$ is a line bundle. Assertions 2) and 1) in Proposition 3.5.2 imply that it also holds when $\mathcal{G}$ is any finite sum of line bundles, and then when $\mathcal{G}$ is any quasi-coherent quotient of such a direct sum, that is, for any coherent sheaf $\mathcal{G}$. $\square$

PROOF OF PROPOSITION 3.5.9. Let us choose some line bundle N over C such that:

$$H^1(C, N) = 0.$$

Let $\mathcal{F}$ and $\mathcal{G}$ be h^1-finite quasi-coherent sheaves over C, and let L be some line bundle over C.

As $\mathcal{F}$ and $\mathcal{G}$ satisfy Condition h^1-$\mathbf{Fin}_{1'}$, we may consider the exact sequence of quasi-coherent sheaves over C

$$0 \longrightarrow \mathcal{F}_L \hookrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}_L \longrightarrow 0 \tag{3.5.6}$$

and

$$0 \longrightarrow \mathcal{G}_{\mathcal{O}_C} \hookrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}_{\mathcal{O}_C} \longrightarrow 0. \tag{3.5.7}$$

The quasi-coherent sheaves $\mathcal{F}_L \otimes L^\vee$ and $\mathcal{G}_{\mathcal{O}_C}$ are generated by their global sections. Moreover, the quotients sheaves $\mathcal{F}/\mathcal{F}_L$ and $\mathcal{G}/\mathcal{G}_{\mathcal{O}_C}$ are coherent, and therefore, according to Proposition 3.5.2, 3), $\mathcal{F}_L$ and $\mathcal{G}_{\mathcal{O}_C}$ are h^1-finite.

By applying the functor $\cdot \otimes \mathcal{G} \otimes L^\vee \otimes N$ (resp. $\mathcal{F}_L \otimes L^\vee \otimes N \otimes \cdot$) to the exact sequence (3.5.6) (resp. to (3.5.7)), we get exact sequences of quasi-coherent sheaves:

$$\mathcal{F}_L \otimes \mathcal{G} \otimes L^\vee \otimes N \xrightarrow{\alpha} \mathcal{F} \otimes \mathcal{G} \otimes L^\vee \otimes N \longrightarrow \mathcal{F}/\mathcal{F}_L \otimes \mathcal{G} \otimes L^\vee \otimes N \longrightarrow 0 \tag{3.5.8}$$

and

$$\mathcal{F}_L \otimes \mathcal{G}_{\mathcal{O}_C} \otimes L^\vee \otimes N \xrightarrow{\beta} \mathcal{F}_L \otimes \mathcal{G} \otimes L^\vee \otimes N \longrightarrow \mathcal{F}_L \otimes \mathcal{G}/\mathcal{G}_{\mathcal{O}_C} \otimes L^\vee \otimes N \longrightarrow 0. \tag{3.5.9}$$

The quasi-coherent sheaf $\mathcal{F}_L \otimes L^\vee \otimes \mathcal{G}_{\mathcal{O}_C}$ is generated by its global sections, and therefore, by the last assertion in Lemma 3.5.6,

$$H^1(C, \mathcal{F}_L \otimes L^\vee \otimes \mathcal{G}_{\mathcal{O}_C} \otimes N) = 0. \tag{3.5.10}$$

Using that C has cohomological dimension 1, we deduce from (3.5.10):

$$H^1(C, \operatorname{im} \beta) = 0$$

and then, from (3.5.9), an isomorphism between first cohomology groups:

$$H^1(C, \mathcal{F}_L \otimes \mathcal{G} \otimes L^\vee \otimes N) \xrightarrow{\sim} H^1(C, \mathcal{F}_L \otimes \mathcal{G}/\mathcal{G}_{\mathcal{O}_C} \otimes L^\vee \otimes N). \tag{3.5.11}$$

Moreover, according to Lemma 3.5.10, the quasi-coherent sheaf $\mathcal{F}_L \otimes \mathcal{G}/\mathcal{G}_{\mathcal{O}_C}$ is h^1-finite. Together with (3.5.11), this shows:

$$\dim_k H^1(C, \mathcal{F}_L \otimes L^\vee \otimes \mathcal{G} \otimes N) < +\infty.$$

In turn, this implies:

$$\dim_k H^1(C, \operatorname{im} \alpha) < +\infty. \tag{3.5.12}$$

By Lemma 3.5.10 again, the quasi-coherent sheaf $\mathcal{F}/\mathcal{F}_L \otimes \mathcal{G}$ is also h^1-finite. Combined with the exact sequence of first cohomology groups

$$H^1(C, \operatorname{im} \alpha) \longrightarrow H^1(C, \mathcal{F} \otimes \mathcal{G} \otimes L^\vee \otimes N) \longrightarrow H^1(C, (\mathcal{F}/\mathcal{F}_L) \otimes \mathcal{G} \otimes L^\vee \otimes N)$$

deduced from (3.5.8) and with (3.5.12), this implies:

$$\dim_k H^1(C, \mathcal{F} \otimes \mathcal{G} \otimes L^\vee \otimes N) < +\infty.$$

Since L is arbitrary, this shows that $\mathcal{F} \otimes \mathcal{G}$ is h^1-finite. □

3.6. $\overline{h}^1$-Finiteness of Quasi-coherent Sheaves

3.6.1. $\overline{h}^1$-finiteness: definition and first properties.

3.6.1.1. Â Let $(L_\alpha)_{\alpha \in A}$ be a family of line bundles over C such that

$$\sup_{\alpha \in A} \deg_C L_\alpha = +\infty. \tag{3.6.1}$$

The conditions h^1-$\mathbf{Fin}_1$, h^1-$\mathbf{Fin}_2$, and h^1-$\mathbf{Fin}_3$ defining the h^1-finite quasi-coherent $\mathcal{O}_C$-modules are clearly equivalent to the following one:

$$\text{for every } \alpha \in A, \quad h^1(C, \mathcal{F} \otimes L_\alpha^\vee) < +\infty. \tag{3.6.2}$$

In particular, the condition (3.6.2) is independent of the family $(L_\alpha)_{\alpha \in A}$ satisfying (3.6.1).

Similarly the conditions h^1-$\mathbf{Fin}_{1'}$, h^1-$\mathbf{Fin}_{2'}$, and h^1-$\mathbf{Fin}_{3'}$ in Proposition 3.5.5 are equivalent to the condition:

$$\text{for every } \alpha \in A, \text{ the } \mathcal{O}_C\text{-module } \mathcal{F}/\mathcal{F}_{L_\alpha} \text{ is coherent.} \tag{3.6.3}$$

This condition may be reformulated as follows:

(3.6.4)

for every $\alpha \in A$, there exists $\mathcal{G}$ in $\operatorname{coft}(\mathcal{F})$ such that $\mathcal{G} \otimes L_\alpha^\vee$ is generated by its global sections.

3.6.1.2. Observe that Lemma 3.5.1 still holds when $h^1(C,.)$ is replaced by $\overline{h}^1(C,.)$.[13] This implies that, for any quasi-coherent $\mathcal{O}_C$-modules, the conditions $\overline{h}^1$-$\mathbf{Fin}_1$, $\overline{h}^1$-$\mathbf{Fin}_2$, and $\overline{h}^1$-$\mathbf{Fin}_3$ obtained by replacing $h^1(C,.)$ by $\overline{h}^1(C,.)$ in h^1-$\mathbf{Fin}_1$, h^1-$\mathbf{Fin}_2$, and h^1-$\mathbf{Fin}_3$, are still equivalent, and also equivalent to the condition:

$$\text{for every } \alpha \in A, \quad \overline{h}^1(C, \mathcal{F} \otimes L_\alpha^\vee) < +\infty \tag{3.6.5}$$

for every family $(L_\alpha)_{\alpha \in A}$ of line bundles over C satisfying (3.6.1) as above.

When these equivalent conditions are satisfied, we shall say that $\mathcal{F}$ is $\overline{h}^1$*-finite.*

Clearly a $\overline{h}^1$-finite quasi-coherent $\mathcal{O}_C$-module is h^1-finite. Moreover, for any family $(L_\alpha)_{\alpha \in A}$ as above, we immediately derive from Theorem 3.2.7 (3):

[13] Indeed, with trivial modifications, its proof remains valid for $\overline{h}^1(C,.)$ instead of $h^1(C,.)$. It now relies on the monotonicity and subadditivity properties of $\overline{h}^1(C,.)$ stated in Proposition 3.2.3.

PROPOSITION 3.6.1. *A quasi-coherent $\mathcal{O}_C$-module of countable type $\mathcal{F}$ is $\overline{h}^1$-finite if and only it satisfies:*

$$\text{for every } \alpha \in A, \quad h^1(C, \mathcal{F} \otimes L_\alpha^\vee) = \overline{h}^1(C, \mathcal{F} \otimes L_\alpha^\vee) < +\infty. \tag{3.6.6}$$

From the monotonicity and and subadditivity properties of $\overline{h}^1(C,.)$ stated in Propositions 3.2.3 and 3.2.4, we also obtain the following analogue of Proposition 3.5.2:

PROPOSITION 3.6.2. *Let us consider a short exact sequence of quasi-coherent $\mathcal{O}_C$-modules of countable type:*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0.$$

1) If $\mathcal{F}_2$ is h^1-finite, then $\mathcal{F}_3$ is $\overline{h}^1$-finite.

2) If $\mathcal{F}_1$ and $\mathcal{F}_3$ are h^1-finite, then $\mathcal{F}_2$ is $\overline{h}^1$-finite.

3) If $\mathcal{F}_3$ is coherent, then $\mathcal{F}_1$ is h^1-finite if and only if $\mathcal{F}_2$ is $\overline{h}^1$-finite.

3.6.2. Filtrations by coherent submodules and $\overline{h}^1$-finiteness. The $\overline{h}^1$-finiteness of quasi-coherent $\mathcal{O}_C$-modules of countable type may actually be characterized in terms of the existence of suitable filtrations by coherent submodules.

THEOREM 3.6.3. *For every quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$ of countable type, the following conditions are equivalent:*

(i) $\mathcal{F}$ is $\overline{h}^1$-finite;

(ii) for every $\alpha \in A$, there exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules such that

$$h^1(C, (\mathcal{C}_{i+1}/\mathcal{C}_i) \otimes L_\alpha^\vee) = 0 \quad \text{for every large enough } i \in \mathbb{N};$$

(iii) there exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules such that, for every $\alpha \in A$,

$$h^1(C, (\mathcal{C}_{i+1}/\mathcal{C}_i) \otimes L_\alpha^\vee) = 0 \quad \text{for every large enough } i \in \mathbb{N};$$

(iv) for every D in $\mathbb{N}$, there exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules such that

$$\mu_{\min}(\mathcal{C}_{i+1}/\mathcal{C}_i) \geq D \quad \text{for every large enough } i \in \mathbb{N}; \tag{3.6.7}$$

(v) there exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules such that

$$\lim_{i\to+\infty} \mu_{\min}(\mathcal{C}_{i+1}/\mathcal{C}_i) = +\infty. \tag{3.6.8}$$

The conditions (ii) to (iv), characterizing $\overline{h}^1$-finiteness in terms of *coherent filtrations*, should be compared to conditions (3.6.3) and (3.6.4), which characterize h^1-finiteness in terms of *coherent quotients.*

The proof of Theorem 3.6.3 will rely on the characterization in Theorem 3.2.7, (2), of the quasi-coherent $\mathcal{O}_C$-modules of countable type $\mathcal{F}$ such that

$$h^1(C, \mathcal{F}) = \overline{h}^1(C, \mathcal{F}) < +\infty,$$

and on the following lemma concerning the minimal slopes of subquotients of exhaustive filtrations.

LEMMA 3.6.4. *Let $\mathcal{F}$ be a $\mathcal{O}_C$-module of countable type, and let $(\mathcal{C}_i)_{i\in\mathbb{N}}$ and $(\mathcal{C}'_i)_{i\in\mathbb{N}}$ be two exhaustive filtrations of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules.*

If D' is an integer such that:

$$\mu_{\min}(\mathcal{C}'_{i+1}/\mathcal{C}'_i) \geq D' \quad \text{for every large enough } i \in \mathbb{N}, \tag{3.6.9}$$

then, for every A in $\mathbb{N}$, there exists two integers $N \geq A$ and $M \geq A$ in $\mathbb{N}$ such that

$$\mathcal{C}_N \subseteq \mathcal{C}'_M \quad \textit{and} \quad \mu_{\min}(\mathcal{C}'_M/\mathcal{C}_N) \geq D'. \tag{3.6.10}$$

Proof of Lemma 3.6.4. We may choose B in $\mathbb{N}$ such that:

$$\mu_{\min}(\mathcal{C}'_{i+1}/\mathcal{C}'_i) \geq D' \quad \text{for every integer } i \geq B. \tag{3.6.11}$$

As the filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ is exhaustive and $\mathcal{C}'_B$ is coherent, we may find an integer $N \geq A$ such that $\mathcal{C}'_B \subseteq \mathcal{C}_N$. Similarly, as the filtration $(\mathcal{C}'_i)_{i\in\mathbb{N}}$ is exhaustive and $\mathcal{C}_N$ is coherent, we may find $M \geq \max(A, B)$ such that $\mathcal{C}_N \subseteq \mathcal{C}'_M$.

Then $\mathcal{C}'_M/\mathcal{C}_N$ is a quotient of $\mathcal{C}'_M/\mathcal{C}'_B$, and therefore:

$$\mu_{\min}(\mathcal{C}'_M/\mathcal{C}_N) \geq \mu_{\min}(\mathcal{C}'_M/\mathcal{C}'_B). \tag{3.6.12}$$

Recall that, for every coherent $\mathcal{O}_C$-module $\mathcal{D}$ and every filtration

$$0 = \mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \cdots \subseteq \mathcal{D}_A = \mathcal{D}$$

of $\mathcal{D}$ by coherent $\mathcal{O}_C$-modules, we have:

$$\mu_{\min}(\mathcal{D}) \geq \inf_{1\leq i\leq A} \mu_{\min}(\mathcal{D}_i/\mathcal{D}_{i-1}). \tag{3.6.13}$$

Applied to the coherent $\mathcal{O}_C$-module $\mathcal{D} = \mathcal{C}'_M/\mathcal{C}'_B$ equipped with the filtration:

$$0 = \mathcal{C}'_B/\mathcal{C}'_B \subseteq \mathcal{C}'_{B+1}/\mathcal{C}'_B \subseteq \cdots \subseteq \mathcal{C}'_M/\mathcal{C}'_B,$$

the lower bound (3.6.13) becomes:

$$\mu_{\min}(\mathcal{C}'_M/\mathcal{C}'_B) \geq \inf_{B<i\leq M} \mu_{\min}(\mathcal{C}'_i/\mathcal{C}'_{i-1}). \tag{3.6.14}$$

.

From (3.6.11), (3.6.12), and (3.6.14), the lower bound in (3.6.10) follows. □

Proof of Theorem 3.6.3. The equivalence $(i) \Leftrightarrow (ii)$ follows from the criterion (3.6.6) for $\overline{h}^1$-finiteness combined with part (2) of Theorem 3.2.7.

The implications $(iii) \Rightarrow (ii)$ and $(v) \Rightarrow (iii)$ are clear. Moreover, using the properties (3.3.11) and (3.3.12) of the minimum slope of a coherent $\mathcal{O}_C$-module, we derive the implications:

$$\mu_{\min}(\mathcal{C}_{i+1}/\mathcal{C}_i) > \deg L_\alpha + 2g - 2 \Longrightarrow h^1(C, (\mathcal{C}_{i+1}/\mathcal{C}_i) \otimes L_\alpha^\vee) = 0 \Longrightarrow \mu_{\min}(\mathcal{C}_{i+1}/\mathcal{C}_i) \geq \deg L_\alpha + g - 1.$$

These establish the equivalences $(ii) \Leftrightarrow (iv)$ and $(iii) \Leftrightarrow (v)$.

To complete the proof, we are left to prove the implication $(iv) \Rightarrow (v)$. To achieve this, consider for each $D \in \mathbb{N}$ an exhaustive filtration $(\mathcal{C}_i^D)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent submodules such that $\mu_{\min}(\mathcal{C}_{i+1}^D/\mathcal{C}_i^D) \geq D$ for every large enough $i \in \mathbb{N}$, and let us construct a filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ as in (v).

Firstly we choose a strictly increasing map $\alpha : \mathbb{N} \to \mathbb{N}$ such that $(\mathcal{C}^D_{\alpha(D)})_{D\in\mathbb{N}}$ is an exhaustive filtration of $\mathcal{F}$; any α that grows fast enough will do.

Then, by induction on D in $\mathbb{N}$, we construct another family $(\tilde{\mathcal{C}}_i^D)_{i\in\mathbb{N}}$, $D \in \mathbb{N}$, of exhaustive filtrations of $\mathcal{F}$ by coherent submodules and a strictly increasing map $A : \mathbb{N} \to \mathbb{N}$ such that the following conditions are satisfied:

(1) for every $D \in \mathbb{N}$, $\tilde{\mathcal{C}}^D_{A(D)+1} \supseteq \mathcal{C}^D_{\alpha(D)}$;
(2) for every $D \in \mathbb{N}$ and any $i \geq A(D)$, $\mu_{\min}(\tilde{\mathcal{C}}^D_{i+1}/\tilde{\mathcal{C}}^D_i) \geq D$;
(3) for every $D \in \mathbb{N}_{>0}$ and every $i \in \{0, \ldots, A(D)\}$, $\tilde{\mathcal{C}}_i^D = \tilde{\mathcal{C}}_i^{D-1}$.

Indeed we may define $(\tilde{\mathcal{C}}_i^0)_{i\in\mathbb{N}}$ as $(\mathcal{C}_i^0)_{i\in\mathbb{N}}$. Then (1) and (2) hold when $D=0$ for any large enough $A(0)$. Consider now D in $\mathbb{N}_{>0}$ and assume that $(\tilde{\mathcal{C}}_i^0)_{i\in\mathbb{N}}, \dots, (\tilde{\mathcal{C}}_i^{D-1})_{i\in\mathbb{N}}$ and $A(0) < \cdots < A(D-1)$ satisfying (1), (2), and (3) up to order $D-1$ have already been constructed. Then we construct $(\tilde{\mathcal{C}}_i^D)_{i\in\mathbb{N}}$ by means of Lemma 3.6.4 applied to the filtrations $\mathcal{C}_\bullet := \tilde{\mathcal{C}}_\bullet^{D-1}$ and $\mathcal{C}'_\bullet := \mathcal{C}_\bullet^D$, and to $D' := D$ and $A := \max(\alpha(D), A(D-1)+1)$. Then if we let:

$$(\tilde{\mathcal{C}}_i^D)_{i\in\mathbb{N}} := (\tilde{\mathcal{C}}_0^{D-1}, \dots, \tilde{\mathcal{C}}_N^{D-1}, \mathcal{C}_M^D, \mathcal{C}_{M+1}^D, \dots) \quad \text{and} \quad A(D) := N,$$

then $A(D) > A(D-1)$, and (1), (2), and (3) are clearly satisfied at order D.

Having the filtrations $(\tilde{\mathcal{C}}_i^D)_{i\in\mathbb{N}}$ and the map A at our disposal, we define $(\mathcal{C}_i)_{i\in\mathbb{N}}$ as follows:

$$\mathcal{C}_i := \tilde{\mathcal{C}}_i^i, \quad \text{for every } i \in \mathbb{N}.$$

Observe that, for any given $i \in \mathbb{N}$, the module $\tilde{\mathcal{C}}_i^D$ is independent of D, provided $A(D+1) \geq i$. This last condition is satisfied by $D = i$, and therefore:

$$\tilde{\mathcal{C}}_i^D = \mathcal{C}_i \quad \text{if } i \leq A(D+1). \tag{3.6.15}$$

Notably we have: $\tilde{\mathcal{C}}_{i+1}^i = \mathcal{C}_{i+1}$, since $i+1 \leq A(i+1)$, and therefore: $\mathcal{C}_i \subseteq \mathcal{C}_{i+1}$. This proves that $(\mathcal{C}_i)_{i\in\mathbb{N}}$ is indeed a filtration of $\mathcal{F}$. It is exhaustive, since for every $D \in \mathbb{N}$:

$$\mathcal{C}_{A(D)+1} = \tilde{\mathcal{C}}_{A(D)+1}^D \supseteq \mathcal{C}_{\alpha(D)}^D.$$

Finally, the minimal slope $\mu_{\min}(\mathcal{C}_{i+1}/\mathcal{C}_i)$ goes to infinity with i because, for any D and i in $\mathbb{N}$, the following implication holds:

$$A(D) \leq i < A(D+1) \Longrightarrow \mu_{\min}(\mathcal{C}_{i+1}/\mathcal{C}_i) \geq D. \tag{3.6.16}$$

Indeed, when $i < A(D+1)$, we have: $i+1 \leq A(D+1)$, and therefore, according to (3.6.15) again, $\mathcal{C}_i = \tilde{\mathcal{C}}_i^D$ and $\mathcal{C}_{i+1} = \tilde{\mathcal{C}}_{i+1}^D$. Therefore (3.6.16) follows from condition (2) above. $\square$

CHAPTER 4

Positive invariants of Hermitian Quasi-coherent Sheaves over Arithmetic Curves I: Monotonicity, Subadditivity and φ-Summable Hermitian Quasi-coherent Sheaves

As in Chapter 2, we denote by K a number field, by $\mathcal{O}_K$ its ring of integers, and by X the arithmetic curve $\operatorname{Spec}\mathcal{O}_K$, and by $\overline{\mathbf{Vect}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$ the categories of Hermitian vector bundles and of Hermitian quasi-coherent sheaves over the arithmetic curve X.

4.0.1. This chapter and the next one are devoted to the construction of invariants with values in $[0,+\infty]$ attached to objects of the category $\mathbf{q}\overline{\mathbf{Coh}}_X$, starting from invariants on $\overline{\mathbf{Vect}}_X$ with values in $\mathbb{R}_+$.

As already mentioned, we are interested in invariants that will have the role played in the geometric situation by the invariant

$$h^1(C,\mathcal{F}) := \dim_k H^1(C,\mathcal{F}) \in \mathbb{N}\cup\{+\infty\}$$

attached to a quasi-coherent sheaf $\mathcal{F}$ on a projective curve C over some field k.

In Chapter 3, besides $h^1(C,\mathcal{F})$, we have investigated the invariants $\underline{h}^1(C,\mathcal{F})$ and $\overline{h}^1(C,\mathcal{F})$, the definition of which involve only the knowledge of the invariants $h^1(C,\mathcal{C})$ attached to some coherent sheaves $\mathcal{C}$ over C. In Theorem 3.2.7 notably, we have related these new invariants $\underline{h}^1(C,\mathcal{F})$ and $\overline{h}^1(C,\mathcal{F})$ to the classical invariant $h^1(C,\mathcal{F})$.

In this chapter, we introduce invariants attached to objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ that play the role, in the arithmetic situation, of the invariants $\underline{h}^1(C,.)$ and $\overline{h}^1(C,.)$. We also construct a subcategory of $\mathbf{q}\overline{\mathbf{Coh}}_X$ that may be seen as an analogue of the category of quasi-coherent sheaves over C that appear in Theorem 3.2.7, (2), Condition (iii). Since we are dealing with real valued invariants, instead of integer valued ones, our constructions have necessarily a more analytic flavor than the ones in Chapter 3.

The "h^1-like" invariant of Hermitian vector bundles and Hermitian quasi-coherent sheaves that is our main object of study in this monograph the *θ-invariant* h^1_θ. It is the invariant firstly defined on $\overline{\mathbf{Vect}}_\mathbb{Z}$ by the formula:

$$h^1_\theta(\overline{E}) := \log\left(\operatorname{covol}(\overline{E})\sum_{v\in E} e^{-\pi\|v\|^2}\right)$$

for every Euclidean lattice $\overline{E} := (E,\|.\|)$, and then more generally on $\overline{\mathbf{Vect}}_X$ by reducing to $\overline{\mathbf{Vect}}_\mathbb{Z}$ by "direct image".[1]

However, instead of focusing on h^1_θ in our constructions of invariants on $\mathbf{q}\overline{\mathbf{Coh}}_X$ extending some invariant on $\overline{\mathbf{Vect}}_X$, in this chapter we follow an axiomatic approach to the constructions of these extensions.

[1]Namely, for every Hermitian vector bundle $\overline{E}$ over X, we will have: $h^1_\theta(\overline{E}) := h^1_\theta(\pi_*\overline{E})$, where $\pi_*\overline{E}$ denotes the Euclidean lattice direct image of $\overline{E}$ by the morphism of arithmetic curves $\pi : X \to \operatorname{Spec}\mathbb{Z}$, defined in Subsection 2.2.3.

The constructions in this chapter actually also apply to some other classical invariants attached to Euclidean lattices, for instance to the invariant:

$$\rho^2 : \overline{\mathbf{Vect}}_{\mathbb{Z}} \longrightarrow \mathbb{R}_+,$$

defined as the square of the covering radius.[2] The properties of the invariant ρ^2 and of its extensions to $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ — notably the comparison of ρ^2 and of the invariant h^1_θ — will indeed constitute one the main themes of this work. These constructions in this chapter would also apply to the invariant:

$$\mathrm{gv} : \overline{\mathbf{Vect}}_{\mathbb{Z}} \longrightarrow \mathbb{R}_+,$$

defined and used in Section 9.1 to bound h^1_θ in terms of ρ^2.

Besides being applicable to invariants as diverse as h^1_θ and ρ^2, a further interest of this axiomatic approach is that it clarifies the central role of a few basic inequalities and continuity properties of invariants on $\overline{\mathbf{Vect}}_X$ when one extend them, by successive limit procedures, to invariants on $\mathbf{q}\overline{\mathbf{Coh}}_X$. Our axiomatic approach might also be easily extended to a more general framework where the category $\mathbf{q}\overline{\mathbf{Coh}}_X$ replaced by suitably defined categories of adelic Hermitian quasi-coherent sheaves.

4.0.2. Let us describe in more detail the content of this chapter.

In Section 4.1, we introduce various basic properties of invariants valued in $[0, +\infty]$ attached to objects of the category $\overline{\mathbf{Coh}}_X$ of Hermitian coherent sheaves over X, or to objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

Three of these properties will play a central role in our constructions. The first of them is the *monotonicity* property $\mathbf{Mon}^1$ and its strengthened variant $\mathbf{Mon}^1_K$. An invariant φ of the objects of $\overline{\mathbf{Coh}}_X$ (resp. of $\mathbf{q}\overline{\mathbf{Coh}}_X$) satisfies $\mathbf{Mon}^1$ when, for any morphism

$$f : \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}}$$

in $\overline{\mathbf{Coh}}_X^{\leq 1}$ (resp. in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$) such that $f(\mathcal{F}) = \mathcal{G}$, the following inequality is satisfied:

$$\varphi(\overline{\mathcal{F}}) \geq \varphi(\overline{\mathcal{G}}). \tag{4.0.1}$$

Condition $\mathbf{Mon}^1_K$ requires the validity of (4.0.1) under the weaker condition $f_K(\mathcal{F}_K) = \mathcal{G}_K$.

Properties $\mathbf{Mon}^1$ and $\mathbf{Mon}^1_K$ are obvious analogues of the monotonicity property satisfied by $h^1(C, .)$ discussed in Subsection 3.1.1.[3] A second property of this type, also crucial in our constructions, is the *subadditivity* property **SubAdd**. A positive numerical invariant φ on $\overline{\mathbf{Coh}}_X$ (resp. on $\mathbf{q}\overline{\mathbf{Coh}}_X$) satisfies **SubAdd** when, for every admissible short exact sequence

$$0 \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{H}} \longrightarrow 0$$

in $\overline{\mathbf{Coh}}_X$ (resp. in $\mathbf{q}\overline{\mathbf{Coh}}_X$), the following inequality holds:

$$\varphi(\overline{\mathcal{G}}) \leq \varphi(\overline{\mathcal{F}}) + \varphi(\overline{\mathcal{H}}).$$

Besides properties $\mathbf{Mon}^1$ and **SubAdd**, directly inspired by the properties of the invariant $h^1(C, .)$ in the geometric situation, a condition of analytic nature will equally play a central role in our constructions, namely the *downward continuity condition* $\mathbf{Cont}^+$.

A positive numerical invariant φ defined and satisfies $\mathbf{Mon}^1$ on $\overline{\mathbf{Coh}}_X$, or more generally on some subcategory $\overline{\mathbf{Sh}}_X$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, is said to satisfy $\mathbf{Cont}^+$ on $\overline{\mathbf{Sh}}_X$ when, for every object $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x \in X(\mathbb{C})})$ of $\overline{\mathbf{Sh}}_X$ (resp. $\mathbf{q}\overline{\mathbf{Coh}}_X$) and for every sequence $((\|.\|_{n,x})_{x \in X(\mathbb{C})})_{n \in \mathbb{N}}$ of Hermitian structures on $\mathcal{F}$, the following relation holds:

$$\lim_{n \to +\infty} \varphi(\mathcal{F}, ((\|.\|_{n,x})_{x \in X(\mathbb{C})}) = \varphi(\overline{\mathcal{F}}),$$

[2]For every Euclidean lattice $\overline{E} := (E, \|.\|)$, the covering radius is defined as the maximum distance of point in the Euclidean space $(E_{\mathbb{R}}, \|.\|)$ to the lattice E: $\rho(\overline{E}) := \max_{x \in E_{\mathbb{R}}} \min_{e \in E} \|x - e\|$. Equivalently $\rho(\overline{E})$ is the minimum of the set of $r \in \mathbb{R}_+$ such that the closed balls of radius r centered at the lattice points $e \in E$ cover $E_{\mathbb{R}}$.

[3]The superscript 1 in $\mathbf{Mon}^1$ and $\mathbf{Mon}^1_K$ is meant to recall it is a property of "h^1-like invariants." We will consider later in Chapter 7 a dual monotonicity property of invariants on $\overline{\mathbf{Vect}}_X$, typical of "$h^0$-like invariants," that will be denoted $\mathbf{StMon}^0$.

when the Hermitian quasi-coherent sheaves $(\mathcal{F}, ((\|.\|_{n,x})_{x\in X(\mathbb{C})})$ define objects of $\overline{\mathbf{Sh}}_X$ and when, for every $x \in X(\mathbb{C})$, the sequence $(\|.\|_{n,x})_{n\in\mathbb{N}}$ of Hermitian norms on $\mathcal{F}_x$ is decreasing and converges pointwise to $\|.\|_x$.[4]

Property $\mathbf{Cont}^+$ appears naturally when one extends to $\overline{\mathbf{Coh}}_X$ monotonic invariants initially defined on $\overline{\mathbf{Vect}}_X$. Moreover this property will allow us to control the dependence on Hermitian seminorms, associated to the Archimedean places, in our constructions of invariants on $\mathbf{q}\overline{\mathbf{Coh}}_X$. Finally it will turn out to be satisfied by these invariants on $\mathbf{q}\overline{\mathbf{Coh}}_X$ in various significant situations.

4.0.3. Our construction of invariants on $\overline{\mathbf{Coh}}_X$ starting from invariants on $\overline{\mathbf{Vect}}_X$ will proceed in two steps.

Firstly, in Section 4.2, we extend $\mathbb{R}_+$-valued invariants from $\overline{\mathbf{Vect}}_X$ to $\overline{\mathbf{Coh}}_X$. Our construction is elementary, and involves essentially the properties $\mathbf{Mon}^1$ and $\mathbf{Cont}^+$ of the given invariant on $\overline{\mathbf{Vect}}_X$. It relies on some basic properties of downward continuous functions on convex cones in topological $\mathbb{R}$-vector spaces.

Then Section 4.3 is devoted to the extension to $\mathbf{q}\overline{\mathbf{Coh}}_X$ of an invariant:

$$\varphi : \overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

satisfying the monotonicity condition $\mathbf{Mon}^1$.

We actually introduce two extensions of φ, its *lower extension* $\underline{\varphi}$ and its *upper extension* $\overline{\varphi}$, the definitions of which are similar to the definitions of the invariants $\underline{h}^1(C, \mathcal{F})$ and $\overline{h}^1(C, \mathcal{F})$ considered in 3.2.2 and 3.2.1 above. Namely, for every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, we define:

$$\underline{\varphi}(\overline{\mathcal{F}}) := \lim_{\mathcal{F}'\in \mathrm{coft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{F}'}) = \sup_{\mathcal{F}'\in \mathrm{coft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{F}'}), \tag{4.0.2}$$

and:

$$\overline{\varphi}(\overline{\mathcal{F}}) := \liminf_{\mathcal{C}\in \mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}}), \tag{4.0.3}$$

where, in (4.0.2), the limit is taken over the directed set $(\mathrm{coft}(\mathcal{F}), \supseteq)$ of $\mathcal{O}_X$-submodules $\mathcal{G}$ of $\mathcal{F}$ such that the quotient $\mathcal{O}_X$-module $\mathcal{F}/\mathcal{G}$ is coherent, and, in (4.0.3), the inferior limit is taken over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$ of coherent $\mathcal{O}_X$-submodules of $\mathcal{F}$.[5] These upper and lower extensions $\underline{\varphi}$ and $\overline{\varphi}$ of φ to $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfy the inequality:

$$\underline{\varphi}(\overline{\mathcal{F}}) \leq \overline{\varphi}(\overline{\mathcal{F}}).$$

The lower invariant $\underline{\varphi}$ is arguably more natural than the upper invariant $\overline{\varphi}$. For instance, the results in Chapter 3 — notably the equality $\underline{h}^1(C, \mathcal{F}) = h^1(C, \mathcal{F})$ established in Subsection3.3.2 — would be in favor of seeing it as the correct extension for "h^1-type invariants". Moreover $\underline{\varphi}$ satisfies better formal properties than the upper invariant $\overline{\varphi}$. However it turns out to be difficult to control the invariant $\underline{\varphi}(\overline{\mathcal{F}})$ attached to some Hermitian quasi-coherent sheaf $\overline{\mathcal{F}}$ over X when the K-vector space $\mathcal{F}_K$ is not finite dimensional, at least without some delicate analysis.

On the contrary, when φ satisfies $\mathbf{Mon}$ and $\mathbf{SubAdd}$, there is a natural class of objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$, the *$\varphi$-summable* Hermitian quasi-coherent sheaves over X, on which the upper invariant $\overline{\varphi}$ takes a finite value and may be easily computed. By definition, an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is φ-summable when there exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by submodules in $\mathrm{coh}(\mathcal{F})$ such

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[4]Since φ satisfies $\mathbf{Mon}^1$, the sequence $(\varphi(\mathcal{F}, ((\|.\|_{n,x})_{x\in X(\mathbb{C})})))_{n\in\mathbb{N}}$ is decreasing and therefore admits a limit in $[0, +\infty]$.

[5]Equivalently, $\mathrm{coh}(\mathcal{F})$ (resp. $\mathrm{coft}(\mathcal{F})$) may be defined as the set of $\mathcal{O}_K$-submodules G of $F := \mathcal{F}(X)$ that are finitely generated (resp. such that F/G is finitely generated).

that

$$\sum_{i=1}^{+\infty} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty.$$

When this holds, we have:

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{k \to +\infty} \varphi(\overline{\mathcal{C}}_k) \in \mathbb{R}_+. \tag{4.0.4}$$

The properties of the subcategory φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ defined by its φ-summable objects and of the invariant $\overline{\varphi}$ on this subcategory are investigated in Section 4.5, which constitutes the core of this chapter.

The constructions and proofs in Section 4.5, notably the proof of the expression (4.0.4) for the upper invariant $\overline{\varphi}(\overline{\mathcal{F}})$ of a φ-summable object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, have a lot more analytic content than the ones in the previous sections. They are very much in the spirit of the derivation of the central results measure theory — with also significant differences, since the role played in measure theory by a σ-algebra of subsets of some set Ω is played here by the category $\mathbf{q}\overline{\mathbf{Coh}}_X$, which admits a much richer structure.

Some of our proofs, concerning notably the permanence properties of φ-summability, are admittedly rather technical. However they establish the validity of a formalism that turn out to be quite flexible in applications — notably in applications to Diophantine geometry in which a key role is played by finiteness conditions of the form:

$$\overline{\varphi}(\overline{\mathcal{F}}) < +\infty,$$

where φ denotes one of the invariants h^1_θ or ρ^2 and $\overline{\mathcal{F}}$ is a Hermitian quasi-coherent sheaf naturally attached to some Diophantine geometric data.[6]

4.0.5. Section 4.4 is devoted to a technical, but useful, construction concerning invariants on $\mathbf{q}\overline{\mathbf{Coh}}_X$ that satisfy the monotonicity condition $\mathbf{Mon}^1$. To any such invariant ψ, we attach a new invariant $\mathrm{ev}\psi : \mathbf{q}\overline{\mathbf{Coh}} \to [0, +\infty]$ defined by the formula:

$$\mathrm{ev}\psi(\overline{\mathcal{F}}) := \lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \psi(\overline{\mathcal{F}/\mathcal{C}}) = \inf_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \psi(\overline{\mathcal{F}/\mathcal{C}}),$$

for every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

Roughly speaking, when ψ also satisfies $\mathbf{SubAdd}$, the vanishing condition:

$$\mathrm{ev}\psi(\overline{\mathcal{F}}) = 0$$

means that the invariant $\psi(\overline{\mathcal{F}})$ may be approximated by the invariant $\psi(\overline{\mathcal{C}})$ attached to the coherent submodules $\mathcal{C} \in \mathrm{coh}(\mathcal{F})$. For this reason, this condition enters naturally at various places in our study of the extensions $\underline{\varphi}$ and $\overline{\varphi}$ to $\mathbf{q}\overline{\mathbf{Coh}}_X$ of an invariant φ on $\overline{\mathbf{Coh}}_X$.

In the final Section 4.6, we discuss the compatibility of the extension procedures developed in the previous sections with the direct images functors:

$$\pi_* : \overline{\mathbf{Coh}}_X^{\leq 1} \longrightarrow \overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1} \quad \text{and} \quad \pi_* : \mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1} \longrightarrow \mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$$

attached to the finite morphism:

$$\pi : X \longrightarrow \operatorname{Spec} \mathbb{Z}.$$

[6]For instance, the global sections of some coherent sheaf on some $\mathcal{O}_K$-scheme of finite type.

4.1. Positive Numerical Invariants on $\overline{\mathbf{Coh}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$

In this section, we introduce various properties of positive numerical invariants attached to objects of $\overline{\mathbf{Coh}}_X$ or $\mathbf{q}\overline{\mathbf{Coh}}_X$, and we establish various elementary implications between these properties.

We denote by $\overline{\mathbf{Sh}}_X$ one of the categories $\overline{\mathbf{Coh}}_X$ or $\mathbf{q}\overline{\mathbf{Coh}}_X$, and accordingly, by $\overline{\mathbf{Sh}}_X^{\leq 1}$ the category $\overline{\mathbf{Coh}}_X^{\leq 1}$ or $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$.

4.1.1. Definition.

Definition 4.1.1. An *invariant of objects of* $\overline{\mathbf{Sh}}_X$ *with values in* $[0,+\infty]$ (resp. *with values in* $\mathbb{R}_+$) is a function φ which assigns to each object of $\overline{\mathbf{Sh}}_X$ an element $\varphi(\overline{\mathcal{F}})$ in $[0,+\infty]$ (resp. in $\mathbb{R}_+$) that depends only on the isomorphism class of $\overline{\mathcal{F}}$ in $\overline{\mathbf{Sh}}_X^{\leq 1}$ and vanishes when $\overline{\mathcal{F}}$ is the zero Hermitian coherent sheaf on X.

In other words, we require φ to satisfy the following condition:

(i) *if there exists an isometric isomorphism* $f : \overline{\mathcal{F}}_1 \xrightarrow{\sim} \overline{\mathcal{F}}_2$ *between two objects* $\overline{\mathcal{F}}_1$ *and* $\overline{\mathcal{F}}_2$ *in* $\overline{\mathbf{Sh}}_X$*, then* $\varphi(\overline{\mathcal{F}}_1) = \varphi(\overline{\mathcal{F}}_2)$;

(ii) *if* $\overline{0}$ *denotes the zero Hermitian coherent sheaf over* X,[7] *then* $\varphi(\overline{0}) = 0$.

To denote such an invariant, we shall use the notation:

$$\varphi : \overline{\mathbf{Sh}}_X \longrightarrow [0,+\infty] \quad (\text{resp. } \mathbb{R}_+).$$

4.1.2. Monotonicity and subadditivity. In this chapter and in the following ones, a major role will be played by the following *monotonicity* and *subadditivity* conditions regarding an invariant $\varphi : \overline{\mathbf{Sh}}_X \longrightarrow [0,+\infty]$:

$\mathbf{Mon}^1$: *For every morphism* $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ *in* $\overline{\mathbf{Sh}}_X^{\leq 1}$ *such that the morphism of* $\mathcal{O}_K$*-modules* $f : \mathcal{F} \to \mathcal{G}$ *is surjective, we have*:

$$\varphi(\overline{\mathcal{F}}) \geq \varphi(\overline{\mathcal{G}}).$$

SubAdd: *For every admissible short exact sequence in* $\overline{\mathbf{Sh}}_X$*:*

$$0 \longrightarrow \overline{\mathcal{E}} \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}} \longrightarrow 0, \tag{4.1.1}$$

the following inequality holds:

$$\varphi(\overline{\mathcal{F}}) \leq \varphi(\overline{\mathcal{E}}) + \varphi(\overline{\mathcal{G}}).$$

Observe that, when both conditions $\mathbf{Mon}^1$ and **SubAdd** hold, then, for every admissible short exact sequence (4.1.1) in $\overline{\mathbf{Sh}}_X$, we have:

$$\varphi(\overline{\mathcal{G}}) \leq \varphi(\overline{\mathcal{F}}) \leq \varphi(\overline{\mathcal{E}}) + \varphi(\overline{\mathcal{G}}). \tag{4.1.2}$$

Proposition 4.1.2. *When the invariant* φ *satisfies the conditions* $\mathbf{Mon}^1$ *and* **SubAdd***, then, for any admissible short exact sequence in* $\overline{\mathbf{Sh}}_X$

$$0 \longrightarrow \overline{\mathcal{F}}' \xrightarrow{i} \overline{\mathcal{F}} \xrightarrow{p} \overline{\mathcal{F}}'' \longrightarrow 0 \tag{4.1.3}$$

and any $\mathcal{O}_K$*-submodule* $\mathcal{G}$ *of* $\mathcal{F}$*, the following inequality holds:*

$$\varphi(\overline{\mathcal{F}/\mathcal{G}}) \leq \varphi(\overline{\mathcal{F}'/i^{-1}(\mathcal{G})}) + \varphi(\overline{\mathcal{F}''/p(\mathcal{G})}). \tag{4.1.4}$$

Moreover, for any object $\overline{\mathcal{E}}$ *in* $\overline{\mathbf{Sh}}_X$ *and any two* $\mathcal{O}_K$*-submodules* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *of* $\mathcal{E}$*, we have:*

$$\varphi(\overline{\mathcal{E}_1 + \mathcal{E}_2}) \leq \varphi(\overline{\mathcal{E}}_1) + \varphi(\overline{\mathcal{E}}_2). \tag{4.1.5}$$

[7] or equivalently the null object in $\overline{\mathbf{Sh}}_X$ and in $\overline{\mathbf{Sh}}_X^{\leq 1}$.

PROOF. To establish (4.1.4), we will rely on the constructions of admissible short exact sequences discussed in 2.2.2.2.

Namely, consider the short exact sequence of $\mathcal{O}_K$-modules induced by (4.1.3):

$$0 \longrightarrow \mathcal{F}'/i^{-1}(\mathcal{G}) \xrightarrow{\tilde{i}} \mathcal{F}/\mathcal{G} \xrightarrow{\tilde{p}} \mathcal{F}''/p(\mathcal{G}) \longrightarrow 0, \tag{4.1.6}$$

and $\overline{\operatorname{im}\tilde{i}}$, the object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ defined by the image of $\tilde{i}$ endowed by the Hermitian metrics induced by the ones of $\overline{\mathcal{F}/\mathcal{G}}$. The morphism

$$\tilde{i} : \overline{\mathcal{F}'/i^{-1}(\mathcal{G})} \longrightarrow \overline{\operatorname{im}\tilde{i}}$$

is surjective and has norms ≤ 1, and the diagram

$$0 \longrightarrow \overline{\operatorname{im}\tilde{i}} \longrightarrow \overline{\mathcal{F}/\mathcal{G}} \xrightarrow{\tilde{p}} \overline{\mathcal{F}''/p(\mathcal{G})} \longrightarrow 0$$

is an admissible short exact sequence in $\overline{\mathbf{Sh}}_X$.

Since φ satisfies $\mathbf{Mon}^1$ and $\mathbf{SubAdd}$, the following two inequalities hold:

$$\varphi(\overline{\operatorname{im}\tilde{i}}) \leq \varphi(\overline{\mathcal{F}'/i^{-1}(\mathcal{G})})$$

and

$$\varphi(\overline{\mathcal{F}/\mathcal{G}}) \leq \varphi(\overline{\operatorname{im}\tilde{i}}) + \varphi(\overline{\mathcal{F}''/p(\mathcal{G})}),$$

and (4.1.4) follows.

To prove (4.1.5), consider the admissible short exact sequence:

$$0 \longrightarrow \overline{\mathcal{E}}_1 \longrightarrow \overline{\mathcal{E}_1 + \mathcal{E}_2} \longrightarrow \overline{(\mathcal{E}_1 + \mathcal{E}_2)/\mathcal{E}_1} \longrightarrow 0.$$

The subadditivity of φ implies the inequality:

$$\varphi(\overline{\mathcal{E}_1 + \mathcal{E}_2}) \leq \varphi(\overline{\mathcal{E}}_1) + \varphi(\overline{(\mathcal{E}_1 + \mathcal{E}_2)/\mathcal{E}_1}).$$

Moreover the obvious surjective morphism of $\mathcal{O}_K$-modules

$$\mathcal{E}_2 \longrightarrow (\mathcal{E}_1 + \mathcal{E}_2)/\mathcal{E}_1$$

defines a morphism in $\overline{\mathbf{Sh}}_X^{\leq 1}$ from $\overline{\mathcal{E}}_2$ to $\overline{(\mathcal{E}_1 + \mathcal{E}_2)/\mathcal{E}_1}$. Therefore, according to the monotonicity of φ, we have:

$$\varphi(\overline{(\mathcal{E}_1 + \mathcal{E}_2)/\mathcal{E}_1}) \leq \varphi(\overline{\mathcal{E}}_2),$$

and (4.1.5) follows. □

4.1.3. Downward continuity. Consider an invariant

$$\varphi : \overline{\mathbf{Sh}}_X \longrightarrow [0, +\infty]$$

that satisfies the monotonicity condition $\mathbf{Mon}^1$.

Let $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x \in X(\mathbb{C})}$ and $\overline{\mathcal{F}}' := (\mathcal{F}, (\|.\|'_x)_{x \in X(\mathbb{C})}$ be two objects in $\overline{\mathbf{Sh}}_X$ with the same underlying $\mathcal{O}_K$-modules $\mathcal{F}$ such that:

$$\|.\|'_x \leq \|.\|_x \quad \text{for every } x \in X(\mathbb{C}).$$

Then the morphism

$$\mathrm{Id}_{\mathcal{F}} : \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}'$$

is a morphism in $\overline{\mathbf{Sh}}_X^{\leq 1}$, surjective on the underlying $\mathcal{O}_K$-modules, and $\mathbf{Mon}^1$ applied to this morphism reads:

$$\varphi(\overline{\mathcal{F}}') \leq \varphi(\overline{\mathcal{F}}).$$

More generally, consider a sequence $(\overline{\mathcal{F}}_n)_{n \in \mathbb{N}}$ of objects

$$\overline{\mathcal{F}}_n := (\mathcal{F}, (\|.\|_{n,x})_{x \in X(\mathbb{C})}$$

in $\overline{\mathbf{Sh}}_X$ with the same underlying $\mathcal{O}_K$-modules $\mathcal{F}$ such that, for every $x \in X(\mathbb{C})$, the sequence of seminorms $(\|.\|_{n,x})_{n\in\mathbb{N}}$ is decreasing. We may define an object

$$\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})}) \tag{4.1.7}$$

in $\overline{\mathbf{Sh}}_X$ by means of the Hermitian seminorms on the $\mathcal{F}_x$, $x \in X(\mathbb{C})$, defined as the pointwise limits:

$$\|v\|_x := \lim_{n\to+\infty} \|v\|_{n,x}, \quad \text{for every } v \in \mathcal{F}_x. \tag{4.1.8}$$

According to the previous observation, the validity of the monotonicity condition $\mathbf{Mon}^1$ implies that $(\varphi(\overline{\mathcal{F}}_n))_{n\in\mathbb{N}}$ is a decreasing sequence in $[\varphi(\mathcal{F}), +\infty]$, and therefore has a well-defined limit which satisfies the inequality:

$$\lim_{n\to+\infty} \varphi(\overline{\mathcal{F}}_n) \geq \varphi(\overline{\mathcal{F}}). \tag{4.1.9}$$

We may therefore introducing the following condition of *downward continuity* on the invariant φ:

$\mathbf{Cont}^+$: *For every sequence $(\overline{\mathcal{F}}_n)_{n\in\mathbb{N}}$ of objects in $\overline{\mathbf{Sh}}_X$ defined by decreasing sequences of seminorms as above, the image by φ of the object $\overline{\mathcal{F}}$ defined by* (4.1.7) *and* (4.1.8) *satisfies:*

$$\lim_{n\to+\infty} \varphi(\overline{\mathcal{F}}_n) = \varphi(\overline{\mathcal{C}}). \tag{4.1.10}$$

Let us emphasize the ambivalent character of this definition: the condition $\mathbf{Cont}^+$ is simultaneously a weak condition, in so far as it deals with decreasing sequences of seminorms only, and a strong condition — at least when $\overline{\mathbf{Sh}}_X$ is the category $\mathbf{q}\overline{\mathbf{Coh}}_X$ or a subcategory[8] containing objects $\overline{\mathcal{F}}$ such that the $\mathbb{C}$-vector spaces $\mathcal{F}_x$, $x \in X(\mathbb{C})$, are infinite dimensional — since only pointwise convergence is required in (4.1.8).

4.1.4. Vanishing on torsion sheaves and antiprojective sheaves.

4.1.4.1. *Vanishing on torsion sheaves.* The invariants investigated in this monograph "do not see torsion". Indeed they satisfy the following conditions:[9]

VT: *For every object $\overline{\mathcal{F}}$ in $\overline{\mathbf{Sh}}_X$ whose underlying $\mathcal{O}_K$-module $\mathcal{F}$ is torsion, we have*:

$$\varphi(\overline{\mathcal{F}}) = 0.$$

NST: *For every object $\mathcal{F} := (\mathcal{F}, \|.\|)$ in $\overline{\mathbf{Sh}}_X$, the following equality holds:*

$$\varphi(\overline{\mathcal{F}}) = \varphi(\overline{\mathcal{F}}_{/\mathrm{tor}}).$$

Clearly, we have:

$$\mathbf{NST} \Longrightarrow \mathbf{VT}.$$

Conversely the estimates (4.1.2) applied to the admissible short exact sequence

$$0 \longrightarrow \mathcal{F}_{\mathrm{tor}} \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}_{/\mathrm{tor}} \longrightarrow 0,$$

introduced in Example 2.2.4 above, establish the implication:

$$\big[\mathbf{Mon}^1, \mathbf{SubAdd} \text{ and } \mathbf{VT}\big] \Longrightarrow \mathbf{NST}.$$

---

[8]The validity of Condition $\mathbf{Cont}^+$ on some categories intermediate between $\overline{\mathbf{Coh}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$ will be considered in this monograph. Its precise meaning in these situations will be specified to avoid any ambiguity; see for instance Propositions 4.5.16 and 5.4.9. As demonstrated by the constructions in Examples 5.4.10 and 6.3.5, interesting invariants seldom satisfy $\mathbf{Cont}^+$ on the whole category $\mathbf{q}\overline{\mathbf{Coh}}_X$.

[9]Note however that there exist natural invariants that satisfy $\mathbf{Mon}^1$, $\mathbf{SubAdd}$ and $\mathbf{Cont}^+$, as well as the conditions $\mathbf{Add}_\oplus$, $\mathbf{StMon}^1_2$, $\mathbf{StMon}^1_3$, and $\mathbf{StMon}^1_4$ introduced in 4.1.5 and 5.2 below, but not $\mathbf{VT}$. Such an invariant may be obtained by considering a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, of residue field $\mathbb{F}_\mathfrak{p} := \mathcal{O}_K/\mathfrak{p}$, and by defining:

$$\varphi : \mathbf{q}\overline{\mathbf{Coh}}_X \to \mathbb{N} \cup \{+\infty\}, \;\; \overline{\mathcal{F}} \mapsto \dim_{\mathbb{F}_\mathfrak{p}} \mathcal{F}_{\mathbb{F}_\mathfrak{p}}.$$

It is often convenient to consider the following strengthening of the monotonicity condition φ:

$\mathbf{Mon}^1_K$: *For every morphism* $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ *in* $\overline{\mathbf{Sh}}_X^{\leq 1}$ *such that the morphism of* K*-vector spaces* $f_K : \mathcal{F}_K \to \mathcal{G}_K$ *is surjective, we have*:

$$\varphi(\overline{\mathcal{F}}) \geq \varphi(\overline{\mathcal{G}}).$$

The following proposition shows that, when the subadditivity condition **SubAdd** holds, the validity of $\mathbf{Mon^1}$ and **VT** (or equivalently **NST**) is equivalent to the one of $\mathbf{Mon}^1_K$:

PROPOSITION 4.1.3. *The following implications hold:*

$$\mathbf{Mon}^1_K \Longrightarrow \mathbf{VT} \tag{4.1.11}$$

and:

$$\big[\mathbf{Mon}^1, \mathbf{SubAdd} \textit{ and } \mathbf{VT}\big] \Longrightarrow \mathbf{Mon}^1_K. \tag{4.1.12}$$

PROOF. To prove (4.1.11), consider an object $\overline{\mathcal{F}}$ of $\overline{\mathbf{Sh}}_X$ whose underlying $\mathcal{O}_K$-module $\mathcal{F}$ is torsion. Then the morphism

$$f : \overline{0} \longrightarrow \overline{\mathcal{F}}$$

from the null object in $\overline{\mathbf{Sh}}_X^{\leq 1}$ induces a surjective K-linear map f_K, since $\mathcal{F}_K$ is the null K-vector space. Therefore, if φ satisfies $\mathbf{Mor}^1_K$, then:

$$\varphi(\overline{\mathcal{F}}) \leq \varphi(\overline{0}) = 0,$$

and consequently $\varphi(\overline{\mathcal{F}})$ vanishes.

To prove (4.1.12), consider a morphism $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ in $\overline{\mathbf{Sh}}_X^{\leq 1}$ such that f_K is surjective. It factors in $\overline{\mathbf{Sh}}_X^{\leq 1}$ as:

$$f = i \circ \tilde{f} : \overline{\mathcal{F}} \xrightarrow{\tilde{f}} \overline{\operatorname{im} f} \overset{i}{\hookrightarrow} \overline{\mathcal{G}},$$

where the injection morphism i induces isometries $(i_x)_{x \in X(\mathbb{C})}$, and we may introduce the admissible short exact sequence in $\overline{\mathbf{Sh}}_X^{\leq 1}$:

$$0 \longrightarrow \overline{\operatorname{im} f} \xrightarrow{i} \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}/\operatorname{im} f} \longrightarrow 0.$$

If φ satisfies $\mathbf{Mon}^1$, we have:

$$\varphi(\overline{\operatorname{im} f}) \leq \varphi(\overline{\mathcal{F}}). \tag{4.1.13}$$

If φ satisfies **SubbAdd**, then we also have:

$$\varphi(\overline{\mathcal{G}}) \leq \varphi(\overline{\operatorname{im} f}) + \varphi(\overline{\mathcal{G}/\operatorname{im} f}), \tag{4.1.14}$$

since $\tilde{f} : \mathcal{F} \to \operatorname{im} f$ is surjective, and if φ satisfies **VT**:

$$\varphi(\overline{\mathcal{G}/\operatorname{im} f}) = 0, \tag{4.1.15}$$

since $\mathcal{G}/\operatorname{im} f$ is torsion. Finally, (4.1.13), (4.1.14) and (4.1.15) imply:

$$\varphi(\overline{\mathcal{G}}) \leq \varphi(\overline{\mathcal{F}}).$$

□

We spell out the following simple observation for later reference.

PROPOSITION 4.1.4. *When* $\mathbf{Mon}^1$, **SubAdd**, *and* **VT** *hold, then, for any admissible surjective morphism* $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ *in* $\overline{\mathbf{Sh}}_X^{\leq 1}$ *such that* $\ker f$ *is torsion, we have:*

$$\varphi(\overline{\mathcal{F}}) = \varphi(\overline{\mathcal{G}}).$$

PROOF. This follows form the estimates (4.1.2) applied to the admissible short exact sequence

$$0 \to \ker f \longrightarrow \overline{\mathcal{F}} \xrightarrow{f} \overline{\mathcal{G}} \longrightarrow 0$$

and from the vanishing of $\varphi(\ker f)$. □

4.1.4.2. *Vanishing on antiprojective sheaves.* The invariants studied in the next chapters happen to vanish on antiprojective modules. Namely, they satisfy the following conditions, formally similar to conditions **VT** and **NST**:

VAp: *For every object $\overline{\mathcal{F}}$ in $\overline{\mathbf{Sh}}_X$ whose underlying $\mathcal{O}_K$-module $\mathcal{F}$ is antiprojective, we have*:

$$\varphi(\overline{\mathcal{F}}) = 0.$$

NSAp: *For every object $\overline{\mathcal{F}} := (\mathcal{F}, \|.\|)$ in $\overline{\mathbf{Sh}}_X$, the following equality holds:*

$$\varphi(\overline{\mathcal{F}}) = \varphi(\overline{\mathcal{F}}^{\vee\vee}).$$

The implications

$$\mathbf{VAp} \Longrightarrow \mathbf{VT} \quad \text{and} \quad \mathbf{NSAp} \Longrightarrow \mathbf{NST}$$

trivially hold. They actually are equivalences when $\overline{\mathbf{Sh}}_X = \overline{\mathbf{Coh}}_X$. Furthermore the implication:

$$\mathbf{NSAp} \Longrightarrow \mathbf{VAp}$$

still holds, and by considering the admissible short exact sequence:

$$0 \longrightarrow \overline{\mathcal{F}}_{\mathrm{ap}} \hookrightarrow \overline{\mathcal{F}} \xrightarrow{\delta_{\mathcal{F}}} \overline{\mathcal{F}}^{\vee\vee} \longrightarrow 0,$$

that defines the "canonical dévissage" of $\overline{\mathcal{F}}$ introduced in Subsection 2.2.4, we also obtain:

$$\left[\mathbf{Mon}^1, \mathbf{SubAdd} \text{ and } \mathbf{VAp}\right] \Longrightarrow \mathbf{NSAp}.$$

4.1.5. Compatibility with direct sums. Most of the invariants investigated in this monograph satisfy one of the following two conditions:

Add$_\oplus$: *For any two objects $\overline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_2$ of $\overline{\mathbf{Sh}}_X$, we have:*

$$\varphi(\overline{\mathcal{F}}_1 \oplus \overline{\mathcal{F}}_2) = \varphi(\overline{\mathcal{F}}_1) + \varphi(\overline{\mathcal{F}}_2).$$

Max$_\oplus$: *For any two objects $\overline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_2$ of $\overline{\mathbf{Sh}}_X$, we have:*

$$\varphi(\overline{\mathcal{F}}_1 \oplus \overline{\mathcal{F}}_2) = \max(\varphi(\overline{\mathcal{F}}_1), \varphi(\overline{\mathcal{F}}_2)).$$

The validity of one of these conditions will often be a straightforward consequence of the definition of the invariant φ. Rather surprisingly, the compatibility with direct sums will play a relatively minor role in our study of positive invariants on $\overline{\mathbf{Coh}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$.

4.1.6. Positive linear combinations of invariants. Starting from some invariants on $\overline{\mathbf{Sh}}_X$, we may define some new ones by considering their products by positive real numbers, their supremum, their sum, etc.

In this subsection, we briefly discuss one of these constructions of invariants, and its compatibility with the properties introduced in the previous subsections. This construction endows the space of invariants on $\overline{\mathbf{Sh}}_X$ satisfying some of these properties with a structure of "convex cone," and displays its similarity with the space of positive measures (resp. of positive Radon measures) on some set equipped with a σ-algebra of subsets (resp. on some Hausdorff topological space).

Consider a family $(\varphi_\alpha)_{\alpha \in A}$ of invariants

$$\varphi_\alpha : \overline{\mathbf{Sh}}_X \longrightarrow [0, +\infty],$$

and a family $(\lambda_\alpha)_{\alpha \in A}$ of elements of $[0, +\infty]$.[10] To these data, we may attach the invariant:

$$\sum_{\alpha \in A} \lambda_\alpha \varphi_\alpha : \overline{\mathbf{Sh}}_X \longrightarrow [0, +\infty], \quad \overline{\mathcal{F}} \longmapsto \sum_{\alpha \in A} \lambda_\alpha \varphi_\alpha(\overline{\mathcal{F}}).$$

[10]We place no restriction on the index set A. Without significant loss of generality, we might require it to be countable.

The following proposition is a simple consequence of the definitions, and the details of its proof will be left to the reader.

PROPOSITION 4.1.5. *Let us keep the above notation.*

(1) If one of the properties $\mathbf{Mon}^1$, $\mathbf{SubAdd}$, $\mathbf{VT}$, $\mathbf{NST}$, $\mathbf{Mon}^1_K$, $\mathbf{VAp}$, $\mathbf{NSAp}$, *or* $\mathbf{Add}_\oplus$ *is satisfied by all the invariants in the family* $(\varphi_\alpha)_{\alpha\in A}$, *then the invariant* $\sum_{\alpha\in A}\lambda_\alpha\varphi_\alpha$ *also satisfies it.*

(2) When $(\lambda_\alpha)_{\alpha\in A}$ *belongs to* $\mathbb{R}_+^A$ *and when, for every object* $\overline{\mathcal{F}}$ *of* $\overline{\mathbf{Sh}}_X$, *we have:*

$$\sum_{\alpha\in A}\lambda_\alpha\varphi_\alpha(\overline{\mathcal{F}}) < +\infty, \tag{4.1.16}$$

then, if φ_α *satisfies* $\mathbf{Cont}^+$ *for every* $\alpha\in A$, *then* $\sum_{\alpha\in A}\lambda_\alpha\varphi_\alpha$ *also satisfies* $\mathbf{Cont}^+$.

When $\overline{\mathbf{Sh}}_X$ is $\overline{\mathbf{Coh}}_X$, the validity of the finiteness condition (4.1.16) for an arbitrary object $\overline{\mathcal{F}}$ of $\overline{\mathbf{Sh}}_X$ may often be derived from its validity for the Hermitian line bundles $\overline{\mathcal{O}}(-\delta)$ thanks to the following criterion, the simple proof of which will also be left to the reader:

PROPOSITION 4.1.6. *Assume that* $\overline{\mathbf{Sh}}_X$ *is* $\overline{\mathbf{Coh}}_X$ *and that the invariants* $(\varphi_\alpha)_{\alpha\in A}$ *satisfy* $\mathbf{Mon}^1$ *and* $\mathbf{SubAdd}$. *Then, for every family* $(\delta_i)_{i\in I}$ *of real numbers such that* $\sup_{i\in I}\delta_i = +\infty$, *the following two conditions are equivalent:*

(i) For every $i\in I$, $\sum_{\alpha\in A}\lambda_\alpha\varphi_\alpha(\overline{\mathcal{O}}(-\delta_i)) < +\infty$.
(ii) For every object $\overline{\mathcal{F}}$ *of* $\overline{\mathbf{Coh}}_X$, $\sum_{\alpha\in A}\lambda_\alpha\varphi_\alpha(\overline{\mathcal{F}}) < +\infty$.

4.1.7. Positive numerical invariants on $\overline{\mathbf{Vect}}_X$ and $\overline{\mathbf{Vect}}_X^{[0]}$. The Definition 4.1.1 of invariants with values in $[0,+\infty]$ still makes sense when the category $\overline{\mathbf{Sh}}$ is the category $\overline{\mathbf{Vect}}_X$ or $\overline{\mathbf{Vect}}_X^{[0]}$. Moreover the conditions $\mathbf{Mon}^1$, $\mathbf{SubAdd}$, $\mathbf{Cont}^+$, $\mathbf{Mon}^1_K$, $\mathbf{Add}_\oplus$ and $\mathbf{Max}_\oplus$ still make sense in this setting.

This terminology concerning properties of invariants on $\overline{\mathbf{Vect}}_X$ or $\overline{\mathbf{Vect}}_X^{[0]}$ will be freely used in the sequel — but of course not the elementary results relating these properties established in the previous subsection, which do not make sense, or whose proof are not valid anymore, when $\overline{\mathbf{Sh}}$ is $\overline{\mathbf{Vect}}_X$ or $\overline{\mathbf{Vect}}_X^{[0]}$.

4.2. Extending Invariants from $\overline{\mathbf{Vect}}_X$ to $\overline{\mathbf{Coh}}_X$

In the sequel, we will be concerned with the construction of suitable extensions to $\mathbf{q}\overline{\mathbf{Coh}}_X$ of invariants initially defined on $\overline{\mathbf{Coh}}_X$. In practice, the invariants we are interested in are often defined on the smaller category $\overline{\mathbf{Vect}}_X$ of $\overline{\mathbf{Coh}}_X$, whose objects $\overline{\mathcal{C}} := (\mathcal{C}, (\|.\|_x)_{x\in X(\mathbb{C})})$ are the ones of $\overline{\mathbf{Coh}}_X$ such that $\mathcal{C}$ is torsion free and the Hermitian seminorms $\|.\|_x$ are Hermitian norms.

In this section we discuss how, under suitable conditions, invariants on $\overline{\mathbf{Vect}}_X$ extend naturally to $\overline{\mathbf{Coh}}_X$ and how the validity for these extensions of the properties introduced in Section 4.1 follows from their validity on $\overline{\mathbf{Vect}}_X$.

4.2.1. Increasing positive functions on convex cones. Let V be a $\mathbb{R}$-vector space. A *convex cone* C in V is a subset of V stable under addition and under multiplication by elements of $\mathbb{R}_+^*$. If C is a convex cone in V, we shall denote by $\mathrm{Inc}(C)$ the set of functions $f : C \to \mathbb{R}_+$ that are increasing, namely that satisfy the following property:

$\mathbf{Inc}_C$: for every $(c_1,c_2)\in C^2$, $c_2 - c_1 \in C \Longrightarrow f(c_2)\geq f(c_1)$.

When V is a topological $\mathbb{R}$-vector space, we shall denote by $\mathrm{IncCont}^+(C)$ the set of functions f in $\mathrm{Inc}(C)$ that moreover satisfy the following condition of "downward continuity":

$\mathbf{Cont}^+_C$: for every $c\in C$, $\lim_{\substack{c'\in c+C\\ c'\to c}} f(c') = f(c)$.

PROPOSITION 4.2.1. *Let V be a topological $\mathbb{R}$-vector space, and let Q be a non-empty open convex cone in V and $\overline{Q}$ its closure in V.*

(1) For every function $g : Q \to \mathbb{R}_+$, the function

$$\tilde{g} : \overline{Q} \longrightarrow \mathbb{R}_+, \quad \overline{c} \longmapsto \inf_{\overline{c}+Q} g \tag{4.2.1}$$

belongs to $\mathrm{IncCont}^+(\overline{Q})$ *and is upper semicontinuous.*

(2) The restriction map

$$\mathrm{IncCont}^+(\overline{Q}) \longrightarrow \mathrm{IncCont}^+(Q), \quad f \longmapsto f_{|Q}$$

is a bijection. Its inverse sends $g \in \mathrm{IncCont}^+(Q)$ to $\tilde{g}$.

Observe that $\overline{Q}$ is a closed convex cone in V, and that the following inclusion between subsets of V holds[11]:

$$\overline{Q} + Q \subseteq Q. \tag{4.2.2}$$

Consequently the infimum in the definition (4.2.1) is well-defined in $\mathbb{R}_+$, since $\overline{c} + Q$ is contained in Q.

PROOF. (1) Consider a function $g : Q \to \mathbb{R}_+$, and let us show that $\tilde{g}$ defined by (4.2.1) satisfies conditions $\mathbf{Inc}_{\overline{Q}}$ and $\mathbf{Cont}^{+}_{\overline{Q}}$:

For every c_1 in $\overline{Q}$ and c_2 in $c_1 + \overline{Q}$, we have:

$$c_2 + Q \subseteq c_1 + \overline{Q} + Q \subseteq c_1 + Q,$$

and therefore:

$$\tilde{g}(c_2) \geq \tilde{g}(c_1). \tag{4.2.3}$$

This proves that $\tilde{g}$ satisfies $\mathbf{Inc}_{\overline{Q}}$.

For every $\overline{c}$ in $\overline{Q}$, this implies the inequality:

$$\tilde{g}(\overline{c}) \leq \inf_{\tilde{c} \in \overline{c}+\overline{Q}} \tilde{g}(\tilde{c}). \tag{4.2.4}$$

Moreover, for every neighborhood U of $\overline{c}$ in V and every c in $\overline{c} + Q$, there exists a neighborhood B of 0 in V such that:

$$\overline{c} + B \subseteq U \quad \text{and} \quad c - B \subseteq \overline{c} + Q.$$

Then, for every $\tilde{c} \in (\overline{c} + B) \cap \overline{Q}$, we have:

$$c - \tilde{c} \in c - \overline{c} - B \subseteq Q,$$

and therefore:

$$\tilde{g}(\tilde{c}) \leq g(c).$$

This establishes the lower bound:

$$\sup_{(\overline{c}+B)\cap\overline{Q},} \tilde{g} \leq g(c).$$

Consequently, we have:

$$\limsup_{\tilde{c} \in \overline{Q}, \tilde{c} \to \overline{c}} \tilde{g}(\tilde{c}) \leq \inf_{c \in \overline{c}+Q} g(c) =: \tilde{g}(\overline{c}), \tag{4.2.5}$$

[11] Indeed if $(\overline{c}, c)$ belongs to $\overline{Q} \times Q$, then there exists a neighborhood B of 0 in V such that $c + B \subseteq Q$; then $\overline{c} - B$ is a neighborhood of $\overline{c}$ and thus intersects Q; consequently for some $b \in B$, we have: $\overline{c} - b \in Q$, and thus: $\overline{c} + c = (\overline{c} - b) + (c + b) \in Q + Q \subset Q$. Actually, using that 0 is in the closure of Q when Q is non-empty, one easily establishes the equalities: $Q + Q = \overline{Q} + Q = Q$.

and a fortiori:

$$\limsup_{\tilde{c}\in\overline{c}+\overline{Q},\tilde{c}\to\overline{c}} \tilde{g}(\tilde{c}) \leq \tilde{g}(\overline{c}).$$

Together with (4.2.4), this proves the equality:

$$\lim_{\tilde{c}\in\overline{c}+\overline{Q},\tilde{c}\to\overline{c}} \tilde{g}(\tilde{c}) = \tilde{g}(\overline{c}),$$

and shows that $\tilde{g}$ satisfies $\mathbf{Cont}^+_Q$.

The inequality (4.2.5) also establishes the upper semicontinuity of $\tilde{g}$.

(2) If g satisfies $\mathbf{Inc}_Q$, then $g \leq \tilde{g}_{|Q}$. If moreover g satisfies $\mathbf{Cont}^+_Q$, we may choose $v \in Q$, and for any $c \in Q$, we have:

$$g(c) = \lim_{n\to+\infty} g(c + n^{-1}v) \geq \inf_{c+Q} g = \tilde{g}(c).$$

This shows that, if g belongs to $\mathrm{IncCont}^+(Q)$, then $\tilde{g}_{|Q} = g$ since g satisfies $\mathbf{Cont}^+_Q$.

Moreover, if f belongs to $\mathrm{IncCont}^+(\overline{Q})$, then $\widetilde{f_{|Q}}$ and f coincide on Q, according to the previous observation applied to $g := f_{|Q}$. Since Q is dense in $\overline{Q}$, and both $\widetilde{f_{|Q}}$ and f satisfy $\mathbf{Cont}^+_{\overline{Q}}$, this implies the equality: $\widetilde{f_{|Q}} = f$. □

COROLLARY 4.2.2. *With the notation of Proposition 4.2.1, for every function f in* $\mathrm{Inc}(Q)$, *the following conditions are equivalent:*

(i) f satisfies $\mathbf{Cont}^+_Q$, namely for every $c \in Q$:

$$\lim_{\substack{c'\in c+Q\\ c'\to c}} f(c') = f(c);$$

(ii) for every $c \in Q$,

$$\lim_{\substack{c'\in c+\overline{Q}\\ c'\to c}} f(c') = f(c); \tag{4.2.6}$$

(iii) for every $c \in Q$,

$$\inf_{c'\in c+Q} f(c') = f(c).$$

PROOF. The implications $(ii) \Rightarrow (i) \Rightarrow (iii)$ are straightforward. When (iii) holds, the function $\tilde{f}$ belongs to $\mathrm{IncCont}^+(\overline{Q})$ and extends f, and (ii) holds as a special case of condition $\mathbf{Cont}^+_{\overline{Q}}$ satisfied by $\tilde{f}$. □

Observe that this corollary notably shows that, for every function f in $\mathrm{Inc}(Q)$, if for every $c \in Q$, there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $c + Q$ such that

$$\inf_{n\to\in\mathbb{N}} f(c_n) = f(c)$$

— in particular when there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $c + Q$ such that

$$\lim_{n\to+\infty} f(c_n) = f(c)$$

— then the function f satisfies the conditions of "downward continuity" (4.2.6).

4.2.2. Quotient seminorms and downward continuity. In this subsection, we consider a surjective $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) map between $\mathbb{R}$-vector spaces (resp. $\mathbb{C}$-vector spaces):

$$p : V \longrightarrow W,$$

and we study the continuity properties of the map which associates to a seminorm $\|.\|$ on V its quotient seminorm $\|.\|^\sim$ on W, defined as:

$$\|.\|^\sim : W \longrightarrow \mathbb{R}_+, \quad w \longmapsto \inf_{v \in p^{-1}(w)} \|v\|.$$

PROPOSITION 4.2.3. *With the above notation, let $(\|.\|_n)_{n\in\mathbb{N}}$ be a sequence of seminorms over V such that the limit*

$$\|v\| := \lim_{n\to+\infty} \|v\|_n$$

exists in $\mathbb{R}_+$ for every $v \in V$.

Then $\|.\|$ is a seminorm on V, and the quotient seminorms $(\|.\|_n^\sim)_{n\in\mathbb{N}}$ and $\|.\|^\sim$ satisfy the following inequality, for every $w \in W$:

$$\limsup_{n\to+\infty} \|w\|_n^\sim \leq \|w\|^\sim. \tag{4.2.7}$$

When moreover:

$$\|.\|_{n+1} \leq \|.\|_n \quad \textit{for every } n \in \mathbb{N}, \tag{4.2.8}$$

we have, for every $w \in W$:

$$\lim_{n\to+\infty} \|w\|_n^\sim = \|w\|^\sim. \tag{4.2.9}$$

PROOF. The pointwise limit $\|.\|$ of the seminorms $\|.\|_n$ is clearly a seminorm. Moreover, for every w in W, we have:

$$\limsup_{n\to+\infty} \|w\|_n^\sim = \limsup_{n\to+\infty} \inf_{v\in p^{-1}(w)} \|v\|_n \leq \inf_{v\in p^{-1}(w)} \limsup_{n\to+\infty} \|v\|_n = \inf_{v\in p^{-1}(w)} \|v\| = \|w\|^\sim.$$

This proves (4.2.7).

When (4.2.8) holds, then, for every $n \in \mathbb{N}$, we have:

$$\|.\| \leq \|.\|_{n+1} \leq \|.\|_n,$$

and therefore:

$$\|.\|^\sim \leq \|.\|_{n+1}^\sim \leq \|.\|_n^\sim.$$

For every w in W, this implies the existence of the limit $\lim_{n\to+\infty} \|w\|_n^\sim$ and the inequality:

$$\|w\|^\sim \leq \lim_{n\to+\infty} \|w\|_n^\sim.$$

Together with (4.2.7), this establishes (4.2.9). □

In brief, Proposition 4.2.3 asserts that the map

$$\|.\| \longmapsto \|.\|^\sim \tag{4.2.10}$$

is upper semicontinuous, and "downward continuous."

In this section, we will use this downward continuity in the special case when V, and therefore W, is finite dimensional, and when the seminorms $(\|.\|_n)$ are Euclidean (resp. Hermitian) seminorms. In this situation, the map (4.2.10) restricted to the open cone of Euclidean (resp. Hermitian) norms is easily seen to be continuous by a simple duality argument.[12]

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[12]Indeed, if we denote by $\|.\|^\vee$ the dual norm on $V^\vee$ (resp. on $W^\vee$) of a norm $\|.\|$ on V (resp. on W), this follows from the equality: $\|.\|^\sim = (\|.\|^\sim_{|W^\vee})^\vee$, where $W^\vee$ is identified to a subspace of $V^\vee$ by means of the injective map $p^\vee : W^\vee \to V^\vee$.

However, even in this restricted setting, the map (4.2.10) is in general not continuous, as demonstrated by the following example.

EXAMPLE 4.2.4. Consider the map:

$$p : V := \mathbb{R}^2 \longrightarrow W := \mathbb{R}, \quad (x, y) \longmapsto x.$$

To any $\xi = (\xi_1, \xi_2) \in \mathbb{R}^{2\vee}$, we may attach the seminorm:

$$\|.\|_\xi : \mathbb{R}^2 \longrightarrow \mathbb{R}_+, \quad x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto |\xi(x)| := |\xi_1 x_1 + \xi_2 x_2|.$$

If $\xi_2 \neq 0$, then $p(\ker \xi) = \mathbb{R}$, and therefore:

$$\|.\|_\xi^\sim = 0.$$

Besides, for every $\xi_1 \in \mathbb{R}$ and every $x \in \mathbb{R}^2$, we have:

$$\|x\|_{(\xi_1, 0)} = |\xi_1 x_1|.$$

This shows that, while the seminorm $\|.\|_\xi$ on $\mathbb{R}^2$ is a continuous function of $\xi \in \mathbb{R}^{2\vee}$, the quotient seminorm $\|.\|_\xi^\sim$ does *not* depend continuously of ξ at every point of $\mathbb{R}^* \times \{0\}$.

4.2.3. Extending invariants from $\overline{\mathbf{Vect}}_X$ to $\overline{\mathbf{Vect}}_X^{[0]}$. Recall that we denote by $\overline{\mathbf{Vect}}_X^{[0]}$ the full subcategory of $\overline{\mathbf{Coh}}_X$ whose objects $\overline{\mathcal{C}} := (\mathcal{C}, (\|.\|_x)_{x \in X(\mathbb{C})})$ are those of $\overline{\mathbf{Coh}}_X$ such that $\mathcal{C}$ is torsion free (hence a vector bundle over X). In this subsection, we describe how invariants on $\overline{\mathbf{Vect}}_X$ may be extended to $\overline{\mathbf{Vect}}_X^{[0]}$ by relying on the results in of the previous subsections 4.2.1 and 4.2.2.

4.2.3.1. Let us begin by some preliminary observations.

Let $\mathcal{C}$ be a vector bundle over X.

We shall denote by $\mathrm{Herm}(\mathcal{C})$ the $\mathbb{R}$-vector spaces of "generalized Hermitian structures" on $\mathcal{C}$: its elements are the families $(h_x)_{x \in X(\mathbb{C})}$, where h_x is a Hermitian form on $\mathcal{C}_x$, that are invariant under complex conjugation.[13]

The subset $\mathrm{Herm}(\mathcal{C})^{>0}$ of $\mathrm{Herm}(\mathcal{C})$ whose elements are the families $(h_x)_{x \in X(\mathbb{C})}$ such that h_x is positive definite for every $x \in X(\mathbb{C})$ is an open convex cone in $\mathrm{Herm}(\mathcal{C})$. Its closure in the finite dimensional $\mathbb{R}$-vector space[14] $\mathrm{Herm}(\mathcal{C})$ is the closed cone $\mathrm{Herm}(\mathcal{C})^{\geq 0}$ the elements of which are the families $(h_x)_{x \in X(\mathbb{C})}$ in $\mathrm{Herm}(\mathcal{C})$ such that the h_x are positive semi-definite Hermitian forms.

The elements $(h_x)_{x \in X(\mathbb{C})}$ of $\mathrm{Herm}(\mathcal{C})^{>0}$ (resp. of $\mathrm{Herm}(\mathcal{C})^{\geq 0}$) may be identified with the Hermitian structures $(\|.\|_x)_{x \in X(\mathbb{C})}$ on $\mathcal{C}$, defined by a family $(\|.\|_x)_{x \in X(\mathbb{C})}$ of seminorms (resp. of norms) invariant under complex conjugation on the complex vector spaces $\mathcal{C}_x$, as considered in Section 2.2.5. In this identification, the families $(h_x)_{x \in X(\mathbb{C})}$ and $(\|.\|_x)_{x \in X(\mathbb{C})}$ of Hermitian forms and of Hermitian seminorms are related by the equality:

$$h_x(v) = \|v\|_x^2,$$

for every $x \in X(\mathbb{C})$ and every v in $\mathcal{C}_x$.

Through this identification, the topology of pointwise convergence on the Hermitian structures $(\|.\|_x)_{x \in X(\mathbb{C})}$, which appears in the formulation of the downward continuity condition in 4.1.3 (see for instance (4.1.8)), coincides with the topology induced on $\mathrm{Herm}(\mathcal{C})^{\geq 0}$ by the canonical topology on the finite dimensional $\mathbb{R}$-vector space $\mathrm{Herm}(\mathcal{C})$. This follows from the finite dimensionality of the $\mathbb{C}$-vector spaces $(\mathcal{C}_x)_{x \in X(\mathbb{C})}$.

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[13]Namely, if we denote by $: \mathcal{C}_x := \mathcal{C} \otimes_x \mathbb{C} \simeq \mathcal{C}_{\overline{x}} \simeq \mathcal{C} \otimes_{\overline{x}} \mathbb{C}$ the canonical antilinear isomorphism defined by $\overline{v \otimes_x \lambda} := v \otimes_{\overline{x}} \overline{\lambda}$, then we have: $h_x(v) = h_{\overline{x}}(\overline{v})$, for every field embedding $x \in X(\mathbb{C})$ and every $v \in \mathcal{C}_x$.

[14]endowed with its canonical topology of Hausdorff topological vector space.

Moreover the order relation $\leq$ on $\mathrm{Herm}(\mathcal{C})^{\geq 0}$ deduced from its convex cone structure — namely the relation $\leq$ defined by:

$$h_1 \leq h_2 \Longleftrightarrow h_2 - h_1 \in \mathrm{Herm}(\mathcal{C})^{\geq 0}$$

— coincides with the natural ordering $\leq$ of Hermitian structures, defined by:

$$(\|.\|'_x)_{x\in X(\mathbb{C})} \leq (\|.\|_x)_{x\in X(\mathbb{C})} \Longleftrightarrow \text{for all } x \in X(\mathbb{C}) \text{ and } v \in \mathcal{C}_x,\ \|v\|'_x \leq \|v\|_x. \tag{4.2.11}$$

Similarly the relation $<$ on $\mathrm{Herm}(\mathcal{C})^{>0}$ defined by:

$$h_1 < h_2 \Longleftrightarrow h_2 - h_1 \in \mathrm{Herm}(\mathcal{C})^{>0}$$

coincides with the relation $<$ on Hermitian structures defined by:

$$(\|.\|'_x)_{x\in X(\mathbb{C})} < (\|.\|_x)_{x\in X(\mathbb{C})} \Longleftrightarrow \text{for all } x \in X(\mathbb{C}) \text{ and } v \in \mathcal{C}_x,\ \|v\|'_x < \|v\|_x.$$

4.2.3.2. The finite dimensionality of the complex vector spaces $\mathcal{C}_x$, $x \in X(\mathbb{C})$ also admits the following consequence.

Lemma 4.2.5. *Consider a morphism in* $\overline{\mathbf{Vect}}_X^{[0]\leq 1}$*:*

$$f : \overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_{\mathcal{F},x})_{x\in X(\mathbb{C})})) \longrightarrow \overline{\mathcal{G}} := (\mathcal{G}, (\|.\|_{\mathcal{G},x})_{x\in X(\mathbb{C})})),$$

and a decreasing sequence $((\|.\|^n_{\mathcal{F},x})_{x\in X(\mathbb{C})})_{n\in\mathbb{N}}$ *(resp.* $((\|.\|^n_{\mathcal{G},x})_{x\in X(\mathbb{C})})_{n\in\mathbb{N}}$*) of Hermitian structures on* $\mathcal{F}$ *(resp. on* $\mathcal{G}$*) such that, for every* $x \in X(\mathbb{C})$*, the sequence* $(\|.\|^n_{\mathcal{F},x})_{n\in\mathbb{N}}$ *(resp.* $(\|.\|^n_{\mathcal{G},x})_{n\in\mathbb{N}}$*) converges pointwise to* $\|.\|_{\mathcal{F},x}$ *(resp. to* $\|.\|_{\mathcal{G},x}$*).*

If for every $(x,n) \in X(\mathbb{C}) \times \mathbb{N}$ *the following condition is satisfied:*

$$\|.\|^n_{\mathcal{F},X} > \|.\|_{\mathcal{F},x},$$

then there exist an increasing function $\alpha : \mathbb{N} \to \mathbb{N}$ *such that, for every* $n \in \mathbb{N}$*, the map* $f : \mathcal{F} \to \mathcal{G}$ *defines a morphism in* $\overline{\mathbf{Vect}}_X^{[0]\leq 1}$*:*

$$f := \overline{\mathcal{F}}^n := (\mathcal{F}, (\|.\|^n_{\mathcal{F},x})_{x\in X(\mathbb{C})})) \longrightarrow \overline{\mathcal{G}}^{\alpha(n)} := (\mathcal{G}, (\|.\|^{\alpha(n)}_{\mathcal{G},x})_{x\in X(\mathbb{C})})).$$

This follows from an elementary compactness argument, which we leave to the reader.

4.2.3.3. Consider an invariant:

$$\psi : \overline{\mathbf{Vect}}_X \longrightarrow \mathbb{R}_+$$

that satisfies the monotonicity condition $\mathbf{Mon}^1$.

For every vector bundle $\mathcal{C}$ over X, the function:

$$g_{\mathcal{C}} : \mathrm{Herm}(\mathcal{C})^{>0} \longrightarrow \mathbb{R}_+$$

defined by:

$$g_{\mathcal{C}}((\|.\|^2_x)_{x\in X(\mathbb{C})}) := \psi((\mathcal{G}, (\|.\|_x)_{x\in X(\mathbb{C})})))$$

is increasing. Namely it satisfies the condition $\mathbf{Inc}_Q$ introduced in 4.2.1, when Q denotes the convex cone $\mathrm{Herm}(\mathcal{C})^{>0}$ in the $\mathbb{R}$-vector space $V := \mathrm{Herm}(\mathcal{C})$. Indeed, if two Hermitian structures $(\|.\|'_x)_{x\in X(\mathbb{C})}$ and $(\|.\|_x)_{x\in X(\mathbb{C})}$ on $\mathcal{C}$ satisfy the condition (4.2.11), then the identity map of $\mathcal{C}$ defines a surjective morphism in $\overline{\mathbf{Vect}}_X^{\leq 1}$:

$$\mathrm{Id}_{\mathcal{C}} : (\mathcal{C}, (\|.\|_x)_{x\in X(\mathbb{C})}) \longrightarrow (\mathcal{C}, (\|.\|'_x)_{x\in X(\mathbb{C})}),$$

and therefore, according to $\mathbf{Mon}^1$:

$$\psi((\mathcal{C}, (\|.\|'_x)_{x\in X(\mathbb{C})})) \leq \psi((\mathcal{C}, (\|.\|_x)_{x\in X(\mathbb{C})})).$$

Moreover the invariant ψ satisfies the downward continuity condition $\mathbf{Cont}^+$ precisely when, for every $\mathcal{C}$ as above, the function $g_{\mathcal{C}}$ satisfies the following condition: *for every* $c \in \mathrm{Herm}(\mathcal{C})^{>0}$*, if a sequence* $(c_n)_{n\in\mathbb{N}}$ *in* $\mathrm{Herm}(\mathcal{C})^{>0}$ *satisfies:*

$$c_{n+1} - c_n \in \mathrm{Herm}(\mathcal{C})^{\geq 0} \quad \textit{for all } n \in \mathbb{N} \quad \textit{and} \quad \lim_{n\to+\infty} c_n = c,$$

then:

$$\lim_{n\to+\infty} g_{\mathcal{C}}(c_n) = g_{\mathcal{C}}(c).$$

This condition is immediately seen to be equivalent with the conditions (i)–(iii) in Corollary 4.2.2, and to the one in the subsequent remark, applied to $V := \mathrm{Herm}(\mathcal{C})$, $Q := \mathrm{Herm}(\mathcal{C})^{>0}$, and $f := g_{\mathcal{C}}$.[15]

Consequently, when the invariant ψ satisfies the downward continuity condition $\mathbf{Cont}^+$ on $\overline{\mathbf{Vect}}_X$, Proposition 4.2.1 shows that for every vector bundle $\mathcal{C}$ over X, the function $g_{\mathcal{C}}$ admits a unique extension:

$$\widetilde{g}_{\mathcal{C}} : \mathrm{Herm}(\mathcal{C})^{\geq 0} = \overline{Q} \longrightarrow \mathbb{R}_+$$

that satisfies the conditions $\mathbf{Inc}_{\overline{Q}}$ and $\mathbf{Cont}^+_{\overline{Q}}$. Then, for every object $\overline{\mathcal{C}} := (\mathcal{C}, (\|.\|_x)_{x\in X(\mathbb{C})})$ in $\overline{\mathbf{Vect}}^{[0]}_X$, we may define:

$$\widetilde{\psi}(\overline{\mathcal{C}}) := \widetilde{g}_{\mathcal{C}}((\|.\|^2_x)_{x\in X(\mathbb{C})}).$$

In this way, we have defined an invariant:

$$\widetilde{\psi} : \overline{\mathbf{Vect}}^{[0]}_X \longrightarrow \mathbb{R}_+$$

that extends the invariant ψ.

In concrete terms, for every object $\overline{\mathcal{C}} := (\mathcal{C}, (\|.\|_x)_{x\in X(\mathbb{C})})$ in $\overline{\mathbf{Vect}}^{[0]}_X$, $\widetilde{\psi}(\overline{\mathcal{C}})$ may be defined as the limit:

$$\widetilde{\psi}(\overline{\mathcal{C}}) := \lim_{n\to+\infty} \psi((\mathcal{C}, (\|.\|^{[n]}_x)_{x\in X(\mathbb{C})})) \tag{4.2.12}$$

where $((\|.\|^{[n]}_x)_{x\in X(\mathbb{C})})_{n\in\mathbb{N}}$ is any sequence of Hermitian structures on $\mathcal{C}$ such that, for every $x \in X(\mathbb{C})$, $(\|.\|^{[n]}_x)_{n\in\mathbb{N}}$ is a sequence of Hermitian norms on $\mathcal{C}_x$ converging pointwise to $\|.\|_x$ that satisfies $\|.\|^{[n]}_x \geq \|.\|_x$ for every $n \in \mathbb{N}$.

A straightforward application of Lemma 4.2.5 shows that $\widetilde{\psi}$, like ψ, satisfies the monotonicity condition $\mathbf{Mon}^1$. Moreover the fact that the functions $\widetilde{g}_{\mathcal{C}}$ satisfy $\mathbf{Cont}^+_{\overline{Q}}$ implies that $\widetilde{\psi}$ satisfies $\mathbf{Cont}^+$ on $\overline{\mathbf{Vect}}^{[0]}_X$.

4.2.3.4. The previous considerations establish the first assertion in the following proposition:

PROPOSITION 4.2.6. *For every invariant:*

$$\psi : \overline{\mathbf{Vect}}_X \longrightarrow \mathbb{R}_+$$

satisfying conditions $\mathbf{Mon}^1$ *and* $\mathbf{Cont}^+$*, there exists a unique invariant:*

$$\widetilde{\psi} : \overline{\mathbf{Vect}}^{[0]}_X \longrightarrow \mathbb{R}_+$$

that extends ψ *and satisfies* $\mathbf{Mon}^1$ *and* $\mathbf{Cont}^+$*.*

If moreover ψ *satisfies one of the following conditions:* $\mathbf{Mon}^1_K$, $\mathbf{SubAdd}$, $\mathbf{Add}_\oplus$*, or* $\mathbf{Max}_\oplus$ *on* $\overline{\mathbf{Vect}}_X$*, then* $\widetilde{\psi}$ *satisfies this condition on* $\overline{\mathbf{Vect}}^{[0]}_X$*.*

PROOF. The validity of the second assertion in Proposition 4.2.6 concerning condition $\mathbf{Mon}^1_K$ also follows from Lemma 4.2.5. The one concerning $\mathbf{SubAdd}$ follows from the "downward continuity" of the construction of quotient seminorms established in Proposition 4.2.3, and the ones concerning $\mathbf{Add}_\oplus$, or $\mathbf{Max}_\oplus$ are straightforward. □

[15]Corollary 4.2.2 thus shows that various variants of the condition $\mathbf{Cont}^+$ on the invariant ψ — some of them *a priori* weaker than $\mathbf{Cont}^+$ (for instance condition (iii) in Corollary 4.2.2), some other one *a priori* stronger (for instance condition (ii)) — are actually equivalent to it.

With the notation of Proposition 4.2.6, the non-negative real number $\psi(\overline{\mathcal{O}}_X(\delta))$ is a non-decreasing function of $\delta \in \mathbb{R}$, and its limit when δ goes to infinity is:

$$\lim_{\delta\to+\infty} \psi(\overline{\mathcal{O}}_X(\delta)) = \widetilde{\psi}(\overline{\mathcal{O}}_X(\infty)), \tag{4.2.13}$$

where:

$$\overline{\mathcal{O}}_X(\infty) := (\mathcal{O}_X, (0)_{x\in X(\mathbb{C})})$$

is the object of $\overline{\mathbf{Vect}}_X^{[0]}$ defined by the structural sheaf $\mathcal{O}_X$ equipped with the zero seminorms.

The previous construction of extensions to $\overline{\mathbf{Vect}}_X^{[0]}$ of positive invariants on $\overline{\mathbf{Vect}}_X$ turns out to be compatible with the vectorization functor $.^{\mathrm{vect}} : \overline{\mathbf{Coh}}_X \to \overline{\mathbf{Vect}}_X$ constructed in Subsection 2.3.2 precisely when the limit (4.2.13) vanishes:

Proposition 4.2.7. *Let $\psi : \overline{\mathbf{Vect}}_X \to \mathbb{R}_+$ be an invariant satisfying conditions $\mathbf{Mon}^1$, $\mathbf{SubAdd}$, and $\mathbf{Cont}^+$, and let $\widetilde{\psi} : \overline{\mathbf{Vect}}_X^{[0]} \to \mathbb{R}_+$ be its extension constructed in Proposition 4.2.6.*

The following conditions are equivalent:

(i) $\widetilde{\psi}(\overline{\mathcal{O}}_X(\infty)) = 0$;

(ii) for every object $\overline{\mathcal{C}}$ in $\overline{\mathbf{Vect}}_X^{[0]}$, the following implication holds:

$$\overline{\mathcal{C}}^{\mathrm{vect}} = \overline{0} \Longrightarrow \widetilde{\psi}(\overline{\mathcal{C}}) = 0;$$

(iii) for every object $\overline{\mathcal{C}}$ in $\overline{\mathbf{Vect}}_X^{[0]}$, the following equality holds:

$$\widetilde{\psi}(\overline{\mathcal{C}}) = \psi(\overline{\mathcal{C}}^{\mathrm{vect}}). \tag{4.2.14}$$

Proof. The subadditivity of ψ shows that, for every $N \in \mathbb{N}$,

$$\psi(\overline{\mathcal{O}}_X(\delta)^{\oplus N}) \leq N\psi(\overline{\mathcal{O}}_X(\delta)).$$

Consequently, when (i) holds, then for every $N \in \mathbb{N}$:

$$\lim_{\delta\to+\infty} \psi(\overline{\mathcal{O}}_X(\delta)^{\oplus N}) = 0.$$

The implication $(i) \Rightarrow (ii)$ follows from this observation, the characterization in Proposition 2.3.5 of the objects in $\overline{\mathbf{Coh}}_X$ the vectorization of which is zero, and the monotonicity of $\widetilde{\psi}$.

As shown in Subsections 2.3.1.1 and 2.3.2, to every object $\overline{C}$ in $\overline{\mathbf{Vect}}_X^{[0]}$ we may associate the admissible short exact sequence in $\overline{\mathbf{Vect}}_X^{[0]}$:

$$0 \longrightarrow \overline{\ker \nu_{\overline{\mathcal{C}}}} \hookrightarrow \overline{\mathcal{C}} \xrightarrow{\nu_{\overline{\mathcal{B}}}} \overline{\mathcal{C}}^{\mathrm{vect}} \longrightarrow 0, \tag{4.2.15}$$

and the object $\overline{\ker \nu_{\overline{\mathcal{C}}}}$ satisfies:

$$\overline{\ker \nu_{\overline{\mathcal{C}}}}^{\mathrm{vect}} = \overline{0}.$$

When $\overline{\mathcal{C}}$ belongs to $\overline{\mathbf{Vect}}_X^{[0]}$, $\overline{\ker \nu_{\overline{\mathcal{C}}}}$ also belongs to $\overline{\mathbf{Vect}}_X^{[0]}$, and the monotonicity and the subadditivity of $\widetilde{\psi}$ on $\overline{\mathbf{Vect}}_X^{[0]}$, applied to the morphism $\nu_{\overline{\mathcal{C}}}$ and to the admissible short exact sequence (4.2.15), imply the estimates:

$$\psi(\overline{\mathcal{C}}^{\mathrm{vect}}) \leq \widetilde{\psi}(\overline{\mathcal{C}}) \leq \widetilde{\psi}(\overline{\ker \nu_{\overline{\mathcal{C}}}}) + \psi(\overline{\mathcal{C}}^{\mathrm{vect}}).$$

When (ii) holds, this implies the equality (4.2.14).

This establishes the implication $(ii) \Rightarrow (iii)$. The converse implication $(iii) \Rightarrow (ii)$ is straightforward since the invariant ψ satisfies the normalization condition $\psi(\overline{0}) = 0$, and the implication $(ii) \Rightarrow (i)$ also, since $\overline{\mathcal{O}}_X(\infty)^{\mathrm{vect}} = \overline{0}$. $\square$

4.2.4. Extending invariants from $\overline{\mathbf{Vect}}_X^{[0]}$ to $\overline{\mathbf{Coh}}_X$. Consider a positive invariant on the category $\overline{\mathbf{Vect}}_X^{[0]}$:

$$\varphi : \overline{\mathbf{Vect}}_X^{[0]} \longrightarrow \mathbb{R}_+.$$

It clearly admits a unique extension to an invariant on $\overline{\mathbf{Coh}}_X$ that satisfies condition **NST**, namely the invariant:

$$\varphi_{\mathrm{nst}} : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

defined by the equality:

$$\varphi_{\mathrm{nst}}(\overline{\mathcal{C}}) := \varphi(\overline{\mathcal{C}}_{/\mathrm{tor}}) \tag{4.2.16}$$

for every object $\overline{\mathcal{C}}$ of $\overline{\mathbf{Coh}}_X$.

PROPOSITION 4.2.8. *With the previous notation, if φ satisfies $\mathbf{Mon}^1$ (resp. $\mathbf{Mon}^1_K$) on $\overline{\mathbf{Vect}}_X^{[0]}$, then φ_{nst} satisfies $\mathbf{Mon}^1$ (resp. $\mathbf{Mon}^1_K$) on $\overline{\mathbf{Coh}}_X$.*

*When φ satisfies conditions $\mathbf{Mon}^1_K$ and **SubAdd** on $\overline{\mathbf{Vect}}_X^{[0]}$, then φ_{nst} satisfies conditions $\mathbf{Mon}^1_K$ and **SubAdd** on $\overline{\mathbf{Coh}}_X$.*

Finally, when φ satisfies $\mathbf{Add}_\oplus$, $\mathbf{Max}_\oplus$, or $\mathbf{Cont}^+$, then φ_{nst} also satisfies this condition.

PROOF. (1) A morphism $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ in $\overline{\mathbf{Coh}}_X$ (resp. in $\overline{\mathbf{Coh}}_X^{\leq 1}$) induces a morphism:

$$f_{/\mathrm{tor}} : \overline{\mathcal{F}}_{/\mathrm{tor}} \longrightarrow \overline{\mathcal{G}}_{/\mathrm{tor}}$$

in $\overline{\mathbf{Vect}}_X^{[0]}$ (resp. in $\overline{\mathbf{Vect}}_X^{[0]\leq 1}$). Moreover $f_{/\mathrm{tor}}$ (resp. $f_{/\mathrm{tor},K} = f_K$) is surjective if and only if f (resp. f_K) is. This implies the first assertion of the proposition concerning $\mathbf{Mon}^1$ and $\mathbf{Mon}^1_K$.

(2) Let us assume that φ satisfies $\mathbf{Mon}^1_K$ and **SubAdd**, and consider an admissible short exact sequence in $\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\mathcal{E}} \xrightarrow{i} \overline{\mathcal{F}} \xrightarrow{p} \overline{\mathcal{G}} \longrightarrow 0.$$

The induced morphisms:

$$i_{/\mathrm{tor}} : \overline{\mathcal{E}}_{/\mathrm{tor}} \longrightarrow \overline{\mathcal{F}}_{/\mathrm{tor}} \quad \text{and} \quad p_{/\mathrm{tor}} : \overline{\mathcal{F}}_{/\mathrm{tor}} \longrightarrow \overline{\mathcal{G}}_{/\mathrm{tor}}$$

are respectively injective isometric and surjective admissible; moreover $\operatorname{im} i_{/\mathrm{tor}}$ is contained in $\ker p_{/\mathrm{tor}}$ and the quotient $(\ker p_{/\mathrm{tor}})/(\operatorname{im} i_{/\mathrm{tor}})$ is torsion.

Consequently we may consider the admissible short exact sequence in $\overline{\mathbf{Vect}}_X^{[0]}$:

$$0 \longrightarrow \overline{\ker p_{/\mathrm{tor}}} \hookrightarrow \overline{\mathcal{F}}_{/\mathrm{tor}} \xrightarrow{p_{/\mathrm{tor}}} \overline{\mathcal{G}}_{/\mathrm{tor}} \longrightarrow 0, \tag{4.2.17}$$

and the morphism:

$$i_{/\mathrm{tor}} : \overline{\mathcal{E}}_{/\mathrm{tor}} \longrightarrow \overline{\ker p_{/\mathrm{tor}}} \tag{4.2.18}$$

in $\overline{\mathbf{Vect}}_X^{[0]\leq 1}$. The latter induces an isomorphism

$$\mathcal{E}_{/\mathrm{tor},K} = \mathcal{E}_K \xrightarrow{\sim} \ker p_K = \ker p_{/\mathrm{tor},K}.$$

The subadditivity of φ applied to (4.2.17) establishes the inequality:

$$\varphi(\overline{\mathcal{F}}_{/\mathrm{tor}}) \leq \varphi(\overline{\ker p_{/\mathrm{tor}}}) + \varphi(\overline{\mathcal{G}}_{/\mathrm{tor}}). \tag{4.2.19}$$

Moreover the monotonicity condition $\mathbf{Mon}^1_K$ applied to the morphism (4.2.18) implies:

$$\varphi(\overline{\ker p_{/\mathrm{tor}}}) \leq \varphi(\overline{\mathcal{E}}_{/\mathrm{tor}}). \tag{4.2.20}$$

From (4.2.19) and (4.2.20), we deduce:

$$\varphi(\overline{\mathcal{F}}_{/\mathrm{tor}}) \leq \varphi(\overline{\mathcal{E}}_{/\mathrm{tor}}) + \varphi(\overline{\mathcal{G}}_{/\mathrm{tor}}).$$

This establishes the inequality:

$$\varphi_{\mathrm{nst}}(\overline{\mathcal{F}}) \leq \varphi_{\mathrm{nst}}(\overline{\mathcal{E}}) + \varphi_{\mathrm{nst}}(\overline{\mathcal{G}}_{/\mathrm{tor}})$$

and shows that φ_{nst} satisfies **SubAdd**.

(3) The last assertion concerning $\mathbf{Add}_{\oplus}$, $\mathbf{Max}_{\oplus}$ and $\mathbf{Cont}^+$ is straightforward. □

Using Proposition 4.1.3, we immediately derive the following consequence from Proposition 4.2.8:

COROLLARY 4.2.9. *For every invariant $\varphi : \overline{\mathbf{Vect}}_X^{[0]} \longrightarrow \mathbb{R}_+$, the following three conditions are equivalent:*

(i) φ satisfies $\mathbf{Mon}^1_K$ and **SubAdd** *on $\overline{\mathbf{Vect}}_X^{[0]}$;*
(ii) φ_{nst} satisfies $\mathbf{Mon}^1_K$ and **SubAdd** *on $\overline{\mathbf{Coh}}_X$;*
(iii) φ_{nst} satisfies $\mathbf{Mon}^1$ and **SubAdd** *on $\overline{\mathbf{Coh}}_X$.*

4.2.5. Extending invariants from $\overline{\mathbf{Vect}}_X$ to $\overline{\mathbf{Coh}}_X$. By applying successively the constructions in subsections 4.2.3 and 4.2.4 to an invariant ψ on $\overline{\mathbf{Vect}}_X$ that satisfies some suitable conditions of monotonicity, subadditivity, and downward continuity, we may extend it to an invariant on $\overline{\mathbf{Coh}}_X$.

Indeed we may extend it firstly to an invariant $\widetilde{\psi}$ on $\overline{\mathbf{Vect}}_X^{[0]}$ by the limit construction (4.2.12), and then to an invariant $\widetilde{\psi}_{\mathrm{nst}}$ on $\overline{\mathbf{Coh}}_X$, by forcing the condition **NST** by means of definition (4.2.16), namely by setting:

$$\widetilde{\psi}_{\mathrm{nst}}(\overline{\mathcal{C}}) := \widetilde{\psi}(\overline{\mathcal{C}}_{/\mathrm{tor}})$$

for every object $\overline{\mathcal{C}}$ of $\overline{\mathbf{Coh}}_X$.

For later reference, we gather in the following scholium various properties of the construction of $\widetilde{\psi}_{\mathrm{nst}}$ which directly follows from Propositions 4.2.6 and 4.2.7 and Corollary 4.2.9 :

SCHOLIUM 4.2.10. *For every invariant $\psi : \overline{\mathbf{Vect}}_X \to \mathbb{R}_+$ that satisfies conditions $\mathbf{Mon}^1_K$,* **SubAdd** *and $\mathbf{Cont}^+$, the invariant*

$$\widetilde{\psi}_{\mathrm{nst}} : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

is its unique extension to $\overline{\mathbf{Coh}}_X$ that satisfies $\mathbf{Mon}^1$, **SubAdd**, *$\mathbf{Cont}^+$, and* **NST**. *If ψ satisfies $\mathbf{Add}_{\oplus}$, then $\widetilde{\psi}_{\mathrm{nst}}$ satisfies it also.*

When moreover:

$$\lim_{\delta \to +\infty} \psi(\overline{\mathcal{O}}_X(\delta)) = 0,$$

the invariant $\widetilde{\psi}_{\mathrm{nst}}$ is also "compatible with vectorization"; namely it satisfies the relation:

$$\widetilde{\psi}_{\mathrm{nst}}(\overline{\mathcal{C}}) = \psi(\overline{\mathcal{C}}^{\mathrm{vect}})$$

for every object $\overline{\mathcal{C}}$ of $\overline{\mathbf{Coh}}_X$.

4.2.6. Invariants on $\overline{\mathbf{Coh}}_X$ small on Hermitian coherent sheaves generated by small sections. Let us consider an invariant

$$\varphi : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

that satisfies the monotonicity condition $\mathbf{Mon}^1$.

The following property already appears implicitly in Proposition 4.2.7 and its proof, and will play a key role in the construction of invariants on $\mathbf{q}\overline{\mathbf{Coh}}_X$ that vanish on antiprojective objects in Subsection 4.3.5 below.

DEFINITION 4.2.11. We shall say that the invariant φ *is small on Hermitian coherent sheaves generated by small sections* when, for any nonnegative integer N, there exists a function $C(N, .)$ from $\mathbb{R}^+_*$ to $\mathbb{R}_+$ such that:

(i) $\lim_{\varepsilon \to 0} C(N, \varepsilon) = 0$;
(ii) for any object $\overline{\mathcal{C}}$ of $\overline{\mathbf{Coh}}_X$ such that the $\mathcal{O}_K$-module $\mathcal{C}(X)$ admits a family of generators $(m_i)_{1 \leq i \leq N}$ such that:
$$\sup_{1 \leq i \leq N} \|m_i\|_{\pi_* \overline{\mathcal{C}}} < \varepsilon,$$
we have:
$$\varphi(\overline{\mathcal{C}}) \leq C(N, \varepsilon).$$

When φ satisfies **NST**, a straightforward approximation argument using the monotonicity of φ shows that (ii) holds for any $\overline{\mathcal{C}}$ in $\overline{\mathbf{Coh}}_X$ if it holds for any Hermitian vector bundle on X, that is, when $\mathcal{C}(X)$ is torsion-free and $||.||_{\pi_* \overline{\mathcal{C}}}$ is a Euclidean *norm*.

A simple variant of the arguments in the proof of Proposition 4.2.7 establishes the following proposition. We leave the details of its proof to the interested reader.

PROPOSITION 4.2.12. *Consider an invariant* $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ *that satisfies* $\mathbf{Mon}^1_K$.

1) The following two conditions are equivalent:

(i) π *is small on Hermitian coherent sheaves generated by small sections;*
(ii) for every positive integer N, $\lim_{\delta \to +\infty} \varphi(\overline{\mathcal{O}}_X(\delta)^{\oplus N}) = 0$.

2) When conditions (i) and (ii) are satisfied, then the following implication holds for every object $\overline{\mathcal{C}}$ *of* $\overline{\mathbf{Coh}}_X$*:*
$$\overline{\mathcal{C}}^{\mathrm{vect}} = 0 \Longrightarrow \varphi(\overline{\mathcal{C}}) = 0.$$
In particular, φ *satisfies* **VT**.

3) When φ *satisfies* **SubAdd**, *conditions* (i)-(ii) *are equivalent to:*

(iii) $\lim_{\delta \to +\infty} \varphi(\overline{\mathcal{O}}_X(\delta)) = 0$.

4) When φ *satisfies* **SubAdd** *and* $\mathbf{Cont}^+$, *conditions* (i)-(iii) *are equivalent to:*

(iv) $\varphi(\overline{\mathcal{O}}_X(\infty)) = 0$.

4.3. Lower and Upper Extensions to $\mathbf{q}\overline{\mathbf{Coh}}_X$ of Invariants on $\overline{\mathbf{Coh}}_X$

In this section, we consider an invariant of Hermitian coherent sheaves over X:
$$\varphi : \overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty],$$
and we assume that it satisfies condition $\mathbf{Mon}^1$.

4.3.1. Definitions.

Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

DEFINITIONS 4.3.1. We shall denote by $\mathrm{coh}(\mathcal{F})$ the set of coherent $\mathcal{O}_X$-submodules of $\mathcal{F}$ (or equivalently of finitely generated $\mathcal{O}_K$-submodules of $\mathcal{F}(X)$), and by $\mathrm{coft}(\mathcal{F})$ the set of quasi-coherent $\mathcal{O}_X$-submodules $\mathcal{F}'$ of $\mathcal{F}$ such that the quotient $\mathcal{O}_X$-module $\mathcal{F}/\mathcal{F}'$ is coherent[16] (or equivalently the set of $\mathcal{O}_K$-submodules $\mathcal{F}'(X)$ of $\mathcal{F}(X)$ such that the $\mathcal{O}_K$-module $\mathcal{F}(X)/\mathcal{F}'(X)$ is finitely generated).

For any $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$, we may consider the seminormed Hermitian coherent sheaf $\overline{\mathcal{C}}$ over X and its invariant $\varphi(\overline{\mathcal{C}})$ in $[0, +\infty]$.

For any $\mathcal{F}'$ in $\mathrm{coft}(\mathcal{F})$, we may consider the quotient seminormed Hermitian coherent sheaf $\overline{\mathcal{F}/\mathcal{F}'}$ over X, and its invariant $\varphi(\overline{\mathcal{F}/\mathcal{F}'})$ in $[0, +\infty]$. If $\mathcal{F}''$ is an element of $\mathrm{coft}(\mathcal{F})$ contained in $\mathcal{F}'$, then we may consider the quotient morphism
$$q_{\mathcal{F}''\mathcal{F}'} : \overline{\mathcal{F}/\mathcal{F}''} \longrightarrow \overline{\mathcal{F}/\mathcal{F}'}.$$

[16] The notation "coft" stands for **co-f**inite **t**ype.

It is surjective and has norm at most 1, and therefore:

$$\varphi(\overline{\mathcal{F}/\mathcal{F}'}) \leq \varphi(\overline{\mathcal{F}/\mathcal{F}''}). \tag{4.3.1}$$

The partially ordered sets $(\mathrm{coh}(\mathcal{F}), \subseteq)$ and $(\mathrm{coft}(\mathcal{F}), \supseteq)$ are directed sets, and we may introduce the following definitions.

Definitions 4.3.2. We denote by

$$\underline{\varphi} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

and

$$\overline{\varphi} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

the *upper* and *lower extensions* of φ, namely the invariants defined by the following formulas:

$$\underline{\varphi}(\overline{\mathcal{F}}) := \lim_{\mathcal{F}' \in \mathrm{coft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{F}'}) = \sup_{\mathcal{F}' \in \mathrm{coft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{F}'}), \tag{4.3.2}$$

and

$$\overline{\varphi}(\overline{\mathcal{F}}) := \liminf_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}}), \tag{4.3.3}$$

where, as above, $\overline{\mathcal{F}}$ is an arbitrary Hermitian quasi-coherent sheaf over X.

The existence of the limit in (4.3.2) and its equality with the right-hand side of (4.3.2) follows from the estimates (4.3.1).

Consider the exhaustive filtrations of $\mathcal{F}$ by coherent $\mathcal{O}_X$-submodules, namely the sequences $(\mathcal{C}_i)_{i \in \mathbb{N}}$ in $\mathrm{coh}(\mathcal{F})$ such that

$$\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \cdots \subseteq \mathcal{C}_i \subseteq \mathcal{C}_{i+1} \subseteq \cdots \quad \text{and} \quad \bigcup_{i \in \mathbb{N}} \mathcal{C}_i = \mathcal{F}.$$

These are precisely the ordered sequences that are cofinal in the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$. Among these exhaustive filtrations, we may consider those that satisfy the condition:

$$\text{the limit } \lim_{i \to +\infty} \varphi(\overline{\mathcal{C}}_i) \text{ exists in } [0, +\infty]. \tag{4.3.4}$$

The following statement is a straightforward but useful reformulation of the definition of $\overline{\varphi}(\overline{\mathcal{F}})$ as an inferior limit over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$.

Proposition 4.3.3. *For every exhaustive filtration $(\mathcal{C}_i)_{i \in \mathbb{N}}$ of $\mathcal{F}$ by coherent $\mathcal{O}_X$-submodules, we have:*

$$\liminf_{i \to +\infty} \varphi(\overline{\mathcal{C}}_i) \geq \overline{\varphi}(\overline{\mathcal{F}}).$$

Moreover, the set of the limits $\lim_{i \to +\infty} \varphi(\overline{\mathcal{C}}_i)$, where $(\mathcal{C}_i)_{i \in \mathbb{N}}$ runs over the exhaustive filtrations of $\mathcal{F}$ by coherent $\mathcal{O}_X$-submodules that satisfy (4.3.4) admits $\overline{\varphi}(\overline{\mathcal{F}})$ as its smallest element.

Observe that in general the directed set $(\mathrm{coft}(\mathcal{F}), \supseteq)$, which occurs in the definition of $\underline{\varphi}(\overline{\mathcal{F}})$, does not admit any cofinal increasing sequence. (It is already the case when $\mathcal{O}_K = \mathbb{Z}$ and $\mathcal{F} = \mathbb{Z}^{(\mathbb{N})}$.)

4.3.2. Basic inequalities and additivity. Calling $\underline{\varphi}$ and $\overline{\varphi}$ the lower and upper extensions of φ is justified by the following proposition:

Proposition 4.3.4. *For any object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following inequality holds:*

$$\underline{\varphi}(\overline{\mathcal{F}}) \leq \overline{\varphi}(\overline{\mathcal{F}}). \tag{4.3.5}$$

If moreover $\overline{\mathcal{F}}$ is an object of $\overline{\mathbf{Coh}}_X$, then:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \overline{\varphi}(\overline{\mathcal{F}}) = \varphi(\overline{\mathcal{F}}). \tag{4.3.6}$$

Proof. Let $\mathcal{F}'$ be an element of $\mathrm{coft}(\mathcal{F})$. There exists $\mathcal{C}'$ in $\mathrm{coh}(\mathcal{F})$ such that the composition

$$\mathcal{C}' \hookrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}'$$

is a surjective morphism. Then, for any $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$ containing $\mathcal{C}'$, the morphism

$$\overline{\mathcal{C}} \hookrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}/\mathcal{F}'},$$

is surjective of norm at most one, and therefore:

$$\varphi(\overline{\mathcal{F}/\mathcal{F}'}) \leq \varphi(\overline{\mathcal{C}}).$$

This proves the inequality

$$\varphi(\overline{\mathcal{F}/\mathcal{F}'}) \leq \inf_{\mathcal{C} \in \mathrm{coh}(\mathcal{F}),\, \mathcal{C} \supseteq \mathcal{C}'} \varphi(\overline{\mathcal{C}}),$$

which immediately implies (4.3.5).

When $\mathcal{F}$ is a coherent, then $\{0\}$ (resp. $\mathcal{F}$) is the largest element of the directed set $(\mathrm{coft}(\mathcal{F}), \supseteq)$ (resp. $(\mathrm{coh}(\mathcal{F}), \subseteq)$), and (4.3.6) follows. $\square$

The lower extension $\underline{\varphi}$ is compatible with direct sums if φ is:

Proposition 4.3.5. *If φ satisfies $\mathbf{Add}_\oplus$ (resp. $\mathbf{Max}_\oplus$) on $\overline{\mathbf{Coh}}_X$, then $\underline{\varphi}$ satisfies $\mathbf{Add}_\oplus$ (resp. $\mathbf{Max}_\oplus$) on $\mathbf{q}\overline{\mathbf{Coh}}_X$.*

We do not expect a similar property to hold in general for the upper extension $\overline{\varphi}$.

Proof. Let us consider two objects $\overline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_2$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, and let

$$\iota_i : \mathcal{F}_i \hookrightarrow \mathcal{F}_1 \oplus \mathcal{F}_2, \quad i = 1, 2$$

be the inclusion morphisms.

For any $(\mathcal{G}_1, \mathcal{G}_2)$ in $\mathrm{coh}(\mathcal{F}_1) \times \mathrm{coh}(\mathcal{F}_2)$, the direct sum $\mathcal{G}_1 \oplus \mathcal{G}_2$ belongs to $\mathrm{coh}(\mathcal{F}_1 \oplus \mathcal{F}_2)$, and the following equality holds:

$$\varphi(\overline{(\mathcal{F}_1 \oplus \mathcal{F}_2)/(\mathcal{G}_1 \oplus \mathcal{G}_2)}) = \varphi(\overline{\mathcal{F}_1/\mathcal{G}_1} \oplus \overline{\mathcal{F}_1/\mathcal{G}_2}).$$

Moreover for any $\mathcal{G}$ in $\mathrm{coh}(\mathcal{F}_1 \oplus \mathcal{F}_2)$, its inverse image $\mathcal{G}_1 := \iota_1^{-1}(\mathcal{G})$ (resp. $\mathcal{G}_2 := \iota_2^{-1}(\mathcal{G})$) is an element of $\mathrm{coh}(\mathcal{F}_1)$ (resp. of $\mathrm{coh}(\mathcal{F}_2)$), the direct sum $\mathcal{G}_1 \oplus \mathcal{G}_2$ is contained in $\mathcal{G}$, and therefore, by the monotonicity of φ, we have:

$$\varphi(\overline{(\mathcal{F}_1 \oplus \mathcal{F}_2)/\mathcal{G}}) \leq \varphi(\overline{(\mathcal{F}_1 \oplus \mathcal{F}_2)/(\mathcal{G}_1 \oplus \mathcal{G}_2)}).$$

These observations establish the equality:

$$\sup_{\mathcal{G} \in \mathrm{coh}(\mathcal{F}_1 \oplus \mathcal{F}_2)} \varphi(\overline{(\mathcal{F}_1 \oplus \mathcal{F}_2)/\mathcal{G}}) = \sup_{\mathcal{G}_1 \in \mathrm{coh}(\mathcal{F}_1), \mathcal{G}_2 \in \mathrm{coh}(\mathcal{F}_2)} \varphi(\overline{\mathcal{F}_1/\mathcal{G}_1} \oplus \overline{\mathcal{F}_2/\mathcal{G}_2}),$$

which immediately implies the proposition. $\square$

4.3.3. Monotonicity, lower semicontinuity, and countable additivity.

Proposition 4.3.6. *The invariants $\underline{\varphi}$ and $\overline{\varphi}$ on $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfy condition $\mathbf{Mon}^1$.*

Proof. Let $f : \overline{\mathcal{F}} \to \overline{\mathcal{F}}'$ be a morphism in $\overline{\mathbf{Sh}}^{\leq 1}$ such that $f : \mathcal{F} \to \mathcal{F}'$ is a surjective morphism of $\mathcal{O}_K$-modules.

For every $\mathcal{G}$ in $\mathrm{coft}(\mathcal{F})$, the $\mathcal{O}_K$-module $f(\mathcal{G})$ belongs to $\mathrm{coft}(\mathcal{F}')$ since the morphism f induces a surjective morphism of $\mathcal{O}_K$-modules

$$\tilde{f} : \mathcal{F}/\mathcal{G} \longrightarrow \mathcal{F}'/f(\mathcal{G}).$$

Moreover the map

$$\mathrm{coft}(\mathcal{F}) \longrightarrow \mathrm{coft}(\mathcal{F}'), \quad \mathcal{G} \longmapsto f(\mathcal{G})$$

is surjective; indeed, for every $\mathcal{G}'$ in $\mathrm{coft}(\mathcal{F}')$, $\mathcal{G} := f^{-1}(\mathcal{G}')$ belongs to $\mathrm{coft}(\mathcal{F})$, and $f(\mathcal{G}) = \mathcal{G}'$ because $f : \mathcal{F} \to \mathcal{F}'$ is surjective.

Observe also that, with the above notation, the map $\tilde{f}$ defines a morphism

$$\tilde{f} : \overline{\mathcal{F}/\mathcal{G}} \longrightarrow \overline{\mathcal{F}'/f(\mathcal{G})}$$

in $\overline{\mathbf{Sh}}^{\leq 1}$. Since φ satisfies $\mathbf{Mon}^1$, this implies the inequality:

$$\varphi(\overline{\mathcal{F}/\mathcal{G}}) \geq \varphi(\overline{\mathcal{F}'/f(\mathcal{G})}).$$

This establishes the estimate:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \sup_{\mathcal{G}\in\mathrm{coft}\mathcal{F}} \varphi(\overline{\mathcal{F}/\mathcal{G}}) \geq \sup_{\mathcal{G}\in\mathrm{coft}\mathcal{F}} \varphi(\overline{\mathcal{F}'/f(\mathcal{G})}) = \sup_{\mathcal{G}'\in\mathrm{coft}\mathcal{F}'} \varphi(\overline{\mathcal{F}'/\mathcal{G}'}) = \underline{\varphi}(\overline{\mathcal{F}}'),$$

and shows that $\underline{\varphi}$ satisfies $\mathbf{Mon}^1$.

The map

$$\mathrm{coh}(\mathcal{F}) \longrightarrow \mathrm{coh}(\mathcal{F}'), \quad \mathcal{C} \longmapsto f(\mathcal{C})$$

is surjective and order preserving, and therefore:

$$\liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \varphi(\overline{f(\mathcal{C})}) = \liminf_{\mathcal{C}'\in\mathrm{coh}(\mathcal{F}')} \varphi(\overline{\mathcal{C}}').$$

Moreover, as φ satisfies $\mathbf{Mon}^1$, we have:

$$\varphi(\overline{\mathcal{C}}) \geq \varphi(\overline{f(\mathcal{C})})$$

for every $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$, and therefore:

$$\overline{\varphi}(\overline{\mathcal{F}}) = \liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}}) \geq \liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \varphi(\overline{f(\mathcal{C})}) = \liminf_{\mathcal{C}'\in\mathrm{coh}(\mathcal{F}')} \varphi(\overline{\mathcal{C}}') = \overline{\varphi}(\overline{\mathcal{F}}').$$

This proves that $\overline{\varphi}$ also satisfies $\mathbf{Mon}^1$. □

The invariants $\underline{\varphi}$ and $\overline{\varphi}$ satisfy the following lower semicontinuity property:

PROPOSITION 4.3.7. *Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$. For every exhaustive filtration $(\mathcal{F}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by $\mathcal{O}_X$-submodules, the following inequalities holds:*

$$\underline{\varphi}(\overline{\mathcal{F}}) \leq \liminf_{i\to+\infty} \underline{\varphi}(\overline{\mathcal{F}}_i) \quad \textit{and} \quad \overline{\varphi}(\overline{\mathcal{F}}) \leq \liminf_{i\to+\infty} \overline{\varphi}(\overline{\mathcal{F}}_i). \tag{4.3.7}$$

PROOF. Let $\mathcal{G}$ be an element of $\mathrm{coft}(\mathcal{F})$. For every $i \in \mathbb{N}$, $\mathcal{G}_i := \mathcal{G} \cap \mathcal{F}$ belongs to $\mathrm{coft}(\mathcal{F}_i)$. Indeed the injection morphism $\mathcal{F}_i \hookrightarrow \mathcal{F}$ defines an injective map of $\mathcal{O}_X$-modules

$$\iota_i : \mathcal{F}_i/\mathcal{G}_i \hookrightarrow \mathcal{F}/\mathcal{G},$$

the image of which is the image of $\mathcal{F}_i$ in $\mathcal{F}/\mathcal{G}$. Since $\mathcal{F}/\mathcal{G}$ is finitely generated and the filtration $(\mathcal{F}_i)_{i\in\mathbb{N}}$ is exhaustive, the morphism ι_i is surjective when i is large enough. Moreover it defines a morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$:

$$\iota_i : \overline{\mathcal{F}_i/\mathcal{G}_i} \longrightarrow \overline{\mathcal{F}/\mathcal{G}}.$$

Therefore, according to $\mathbf{Mon}^1$, the following inequality holds for i large enough:

$$\varphi(\overline{\mathcal{F}/\mathcal{G}}) \leq \varphi(\overline{\mathcal{F}_i/\mathcal{G}_i}).$$

Consequently,

$$\varphi(\overline{\mathcal{F}/\mathcal{G}}) \leq \liminf_{i\to+\infty} \varphi(\overline{\mathcal{F}_i/\mathcal{G}_i}) \leq \liminf_{i\to+\infty} \underline{\varphi}(\overline{\mathcal{F}}_i),$$

and finally we get:

$$\underline{\varphi}(\overline{\mathcal{F}}) := \sup_{\mathcal{G}\in\mathrm{coft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{G}}) \leq \liminf_{i\to+\infty} \underline{\varphi}(\overline{\mathcal{F}}_i).$$

To establish the upper bound

$$\overline{\varphi}(\overline{\mathcal{F}}) \leq \liminf_{i \to +\infty} \overline{\varphi}(\overline{\mathcal{F}}_i),$$

it is enough to show that, for every strictly increasing map $\iota : \mathbb{N} \to \mathbb{N}$ such that

$$\overline{\varphi}(\overline{\mathcal{F}}_{\iota(k)}) < t \quad \text{for every } k \in \mathbb{N}, \tag{4.3.8}$$

we have:

$$\overline{\varphi}(\overline{\mathcal{F}}) \leq t. \tag{4.3.9}$$

Since $\mathcal{F}$ is countably generated, we may choose a sequence $(f_k)_{k \in \mathbb{N}}$ of elements of the $\mathcal{O}_K$-module $\mathcal{F}$ that generates the $\mathcal{O}_K$-module $\mathcal{F}$ such that f_k belongs to $\mathcal{F}_{\iota(k)}$ for every $k \in \mathbb{N}$.

When (4.3.8) holds, for every $k \in \mathbb{N}$ there exist arbitrary large submodules $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F}_{\iota(k)})$ such that $\overline{\varphi}(\mathcal{C}) < t$. Consequently we may construct inductively an increasing sequence $(\mathcal{C}_k)_{k \in \mathbb{N}}$ of submodules $\mathcal{C}_k$ in $\mathrm{coh}(\mathcal{F}_{\iota(k)})$ such that, for every $k \in \mathbb{N}$:

$$f_k \in \mathcal{C}_k \quad \text{and} \quad \varphi(\overline{\mathcal{C}}_k) < t.$$

By construction, the filtration $(\mathcal{C}_k)_{k \in \mathbb{N}}$ is exhaustive and satisfies:

$$\liminf_{k \to +\infty} \overline{\varphi}(\overline{\mathcal{C}}_k) \leq t.$$

This establishes (4.3.9). □

Proposition 4.3.8. *Let us assume that φ satisfies $\mathbf{Add}_{\oplus}$. Then, for every countable family $(\overline{\mathcal{F}}_i)_{i \in I}$ of objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following equality holds in $[0, +\infty]$:*

$$\underline{\varphi}\Big(\bigoplus_{i \in I} \overline{\mathcal{F}}_i\Big) = \sum_{i \in I} \underline{\varphi}(\overline{\mathcal{F}}_i). \tag{4.3.10}$$

Moreover for every countable family $(\overline{\mathcal{C}}_i)_{i \in I}$ of objects in $\overline{\mathbf{Coh}}_X$, the following equality holds in $[0, +\infty]$:

$$\overline{\varphi}\Big(\bigoplus_{i \in I} \overline{\mathcal{C}}_i\Big) = \sum_{i \in I} \varphi(\overline{\mathcal{C}}_i). \tag{4.3.11}$$

Proof. Let us write I as the union of an increasing sequence $(I_k)_{k \in \mathbb{N}}$ of finite subsets. According to the lower semicontinuity of $\underline{\varphi}$ established in Proposition 4.3.7, we have:

$$\underline{\varphi}\Big(\bigoplus_{i \in I} \overline{\mathcal{F}}_i\Big) \leq \liminf_{k \to +\infty} \underline{\varphi}\Big(\bigoplus_{i \in I_k} \overline{\mathcal{F}}_i\Big). \tag{4.3.12}$$

Moreover, for every $k \in \mathbb{N}$, we may consider the projection morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$:

$$p_k : \bigoplus_{i \in I} \overline{\mathcal{F}}_i \longrightarrow \bigoplus_{i \in I_k} \overline{\mathcal{F}}_i.$$

Since $\underline{\varphi}$ satisfies $\mathbf{Mon}^1$, the existence of this morphism implies the inequality:

$$\underline{\varphi}\Big(\bigoplus_{i \in I} \overline{\mathcal{F}}_i\Big) \geq \underline{\varphi}\Big(\bigoplus_{i \in I_k} \overline{\mathcal{F}}_i\Big).$$

Consequently,

$$\underline{\varphi}\Big(\bigoplus_{i \in I} \overline{\mathcal{F}}_i\Big) \geq \limsup_{k \to +\infty} \underline{\varphi}\Big(\bigoplus_{i \in I_k} \overline{\mathcal{F}}_i\Big). \tag{4.3.13}$$

From (4.3.12) and (4.3.13), we derive the equality:

$$\underline{\varphi}\Big(\bigoplus_{i \in I} \overline{\mathcal{F}}_i\Big) = \lim_{k \to +\infty} \underline{\varphi}\Big(\bigoplus_{i \in I_k} \overline{\mathcal{F}}_i\Big).$$

Besides, the additivity of $\underline{\varphi}$ established in Proposition 4.3.5 implies:

$$\lim_{k\to+\infty}\underline{\varphi}(\bigoplus_{i\in I_k}\overline{\mathcal{F}}_i) = \lim_{k\to+\infty}\sum_{i\in I_k}\underline{\varphi}(\overline{\mathcal{F}}_i) = \sum_{i\in I}\underline{\varphi}(\overline{\mathcal{F}}_i).$$

This completes the proof of (4.3.10). A similar but simpler argument, using only the additivity of φ on $\overline{\mathbf{Coh}}_X$, establishes the validity of (4.3.11). □

COROLLARY 4.3.9. *For every countable family $(\overline{\mathcal{C}}_i)_{i\in I}$ of objects in $\overline{\mathbf{Coh}}_X$, we have:*

$$\underline{\varphi}(\bigoplus_{i\in I}\overline{\mathcal{C}}_i) = \overline{\varphi}(\bigoplus_{i\in I}\overline{\mathcal{C}}_i).$$

4.3.4. Subadditivity.

PROPOSITION 4.3.10. *If φ satisfies* **SubAdd** *on $\overline{\mathbf{Coh}}_X$, then $\underline{\varphi}$ satisfies* **SubAdd** *on $\mathbf{q}\overline{\mathbf{Coh}}_X$; moreover, for any object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and any coherent submodule $\mathcal{C}$ of $\mathcal{F}$, we have:*

$$\overline{\varphi}(\overline{\mathcal{F}}) \leq \varphi(\overline{\mathcal{C}}) + \overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}}). \tag{4.3.14}$$

If φ satisfies **SubAdd** *and* $\mathbf{Cont}^+$ *on $\overline{\mathbf{Coh}}_X$, then $\overline{\varphi}$ satisfies* **SubAdd** *on $\mathbf{q}\overline{\mathbf{Coh}}_X$.*

PROOF. Assume that φ satisfies **SubAdd** on $\overline{\mathbf{Coh}}_X$.

(1) To prove that $\underline{\varphi}$ also satisfies **SubAdd**, let us consider an admissible short exact sequence

$$0 \longrightarrow \overline{\mathcal{E}} \xrightarrow{i} \overline{\mathcal{F}} \xrightarrow{p} \overline{\mathcal{G}} \longrightarrow 0$$

in $\mathbf{q}\overline{\mathbf{Coh}}_X$, and let us establish the inequality:

$$\underline{\varphi}(\overline{\mathcal{F}}) \leq \underline{\varphi}(\overline{\mathcal{E}}) + \underline{\varphi}(\overline{\mathcal{G}}). \tag{4.3.15}$$

To achieve this, consider an element $\mathcal{F}'$ of $\operatorname{coft}(\mathcal{F})$. As discussed in paragraph 2.2.2.2, to $\mathcal{F}'$ we may associate an admissible sort exact sequence:

$$0 \longrightarrow \overline{\mathcal{E}/\mathcal{E}'}^{\sim} \xrightarrow{\tilde{i}} \overline{\mathcal{F}/\mathcal{F}'} \xrightarrow{\tilde{p}} \overline{\mathcal{G}/\mathcal{G}'} \longrightarrow 0, \tag{4.3.16}$$

where $\mathcal{E}' := i^{-1}(\mathcal{F}')$ and $\mathcal{G}' := p(\mathcal{F}')$, in such a way that the identity map induces a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$:

$$\mathrm{Id}_{\mathcal{E}/\mathcal{E}'} : \overline{\mathcal{E}/\mathcal{E}'} \longrightarrow \overline{\mathcal{E}/\mathcal{E}'}^{\sim}.$$

The $\mathcal{O}_X$-modules $\mathcal{E}'$ and $\mathcal{G}'$ belong to $\operatorname{coft}(\mathcal{E})$ and $\operatorname{coft}(\mathcal{G})$ respectively, and (4.3.16) is an admissible short exact sequence in $\overline{\mathbf{Coh}}_X$. Using the subadditivity and the monotonicity of φ and the definition of $\underline{\varphi}$, we consequently obtain:

$$\varphi(\overline{\mathcal{F}/\mathcal{F}'}) \leq \varphi(\overline{\mathcal{E}/\mathcal{E}'}^{\sim}) + \varphi(\overline{\mathcal{G}/\mathcal{G}'}) \leq \varphi(\overline{\mathcal{E}/\mathcal{E}'}) + \varphi(\overline{\mathcal{G}/\mathcal{G}'}) \leq \underline{\varphi}(\overline{\mathcal{E}}) + \underline{\varphi}(\overline{\mathcal{G}}).$$

By taking the supremum over $\mathcal{F}'$ in $\operatorname{coft}(\mathcal{F})$, this establishes (4.3.15).

(2) Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and let $\mathcal{C}$ be an element of $\operatorname{coh}(\mathcal{F})$.

For any $\mathcal{C}'$ in $\operatorname{coh}(\mathcal{F})$ containing $\mathcal{C}$, we may consider the admissible short exact sequence in $\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}}' \longrightarrow \overline{\mathcal{C}/\mathcal{C}'} \longrightarrow 0.$$

By using **SubAdd**, we derive the inequality:

$$\varphi(\overline{\mathcal{C}}') \leq \varphi(\overline{\mathcal{C}}) + \varphi(\overline{\mathcal{C}'/\mathcal{C}}),$$

and by taking the inferior limit over $\mathcal{C}'$ in the directed set $(\operatorname{coh}(\mathcal{F}), \subseteq)$, we obtain (4.3.14).

(3) Let us finally assume that φ also satisfies **Cont**$^+$, and let us return to the notation of part (1) of this proof. To establish that $\overline{\varphi}$ satisfies **SubAdd**, we have to establish the inequality:

$$\overline{\varphi}(\overline{\mathcal{F}}) \leq \overline{\varphi}(\overline{\mathcal{E}}) + \overline{\varphi}(\overline{\mathcal{G}}). \tag{4.3.17}$$

Equivalently, we have to show that, for any $\varepsilon \in \mathbb{R}_+^*$ and any $\mathcal{B}'$ in $\mathrm{coh}(\mathcal{F})$, there exists $\mathcal{B}$ in $\mathrm{coh}(\mathcal{F})$ containing $\mathcal{B}'$ such that the following inequality is satisfied:

$$\varphi(\overline{\mathcal{B}}) < \overline{\varphi}(\overline{\mathcal{E}}) + \overline{\varphi}(\overline{\mathcal{G}}) + \varepsilon. \tag{4.3.18}$$

To achieve this, observe that, for any $\varepsilon \in \mathbb{R}_+^*$, the following properties hold:

(i) for any $\mathcal{A}'$ in $\mathrm{coh}(\mathcal{E})$, there exists $\mathcal{A}$ in $\mathrm{coh}(\mathcal{E})$ containing $\mathcal{A}'$ and satisfying the inequality:

$$\varphi(\overline{\mathcal{A}}) \leq \overline{\varphi}(\overline{\mathcal{E}}) + \varepsilon/3;$$

(ii) for any $\mathcal{C}'$ in $\mathrm{coh}(\mathcal{G})$, there exists $\mathcal{C}$ in $\mathrm{coh}(\mathcal{G})$ containing $\mathcal{C}'$ and satisfying the inequality:

$$\varphi(\overline{\mathcal{C}}) \leq \overline{\varphi}(\overline{\mathcal{G}}) + \varepsilon/3. \tag{4.3.19}$$

For any $\mathcal{B}'$ in $\mathrm{coh}(\mathcal{F})$, its image $\mathcal{C}' = p(\mathcal{B}')$ belongs to $\mathrm{coh}(\mathcal{G})$. Let $\mathcal{C}$ be an element of $\mathrm{coh}(\mathcal{G})$ such that (4.3.19) holds. Since p is surjective, we may find $\mathcal{B}''$ in $\mathrm{coh}(\mathcal{F})$ containing $\mathcal{B}'$ such that $\mathcal{C} = p(\mathcal{B}'')$. A straigthforward induction based on (i) above allows us to construct an exhaustive filtration $(\mathcal{E}_n)_{n\geq 0}$ on $\mathcal{E}$ by submodules in $\mathrm{coh}(\mathcal{E})$ such that, for any $n \geq 0$,

$$\varphi(\overline{\mathcal{E}_n + i^{-1}(\mathcal{B}'')}) \leq \overline{\varphi}(\overline{\mathcal{E}}) + \varepsilon/3. \tag{4.3.20}$$

We may consider the admissible short exact sequences:

$$0 \longrightarrow \overline{\mathcal{E}_n + i^{-1}(\mathcal{B}'')} \xrightarrow{\ i\ } \overline{i(\mathcal{E}_n) + \mathcal{B}''} \xrightarrow{\ p\ } \overline{p(\mathcal{B}'')_n} \longrightarrow 0$$

where $\overline{p(\mathcal{B}'')_n}$ is defined as $p(\mathcal{B}'') = \mathcal{C}$ equipped with the quotient seminorms deduced from the ones of $\overline{i(\mathcal{E}_n) + \mathcal{B}''}$. The subadditivity **SubAdd** of φ applied to the exact sequences above shows that

$$\varphi(\overline{i(\mathcal{E}_n) + \mathcal{B}''}) \leq \varphi(\overline{\mathcal{E}_n + i^{-1}(\mathcal{B}'')}) + \varphi(\overline{p(\mathcal{B}'')_n}), \tag{4.3.21}$$

for every $n \geq 0$. Moreover, according to Proposition 2.2.6, the sequences of seminorms defining the Hermitian coherent sheaves $\overline{p(\mathcal{B}'')_n}$ are decreasing and converge pointwise toward the seminorms defining $\overline{\mathcal{C}}$. Since φ satisfies **Cont**$^+$, this implies:

$$\lim_{n\to+\infty} \varphi(\overline{p(\mathcal{B}'')_n}) = \varphi(\overline{\mathcal{C}}). \tag{4.3.22}$$

The relations (4.3.19)-(4.3.22) imply that, when the integer n is large enough, then

$$\mathcal{B} := i(\mathcal{E}_n) + \mathcal{B}''$$

satisfies (4.3.18). □

4.3.5. Compatibility with killing torsion and antiprojective modules.

When the invariant φ satisfies **VT**, namely vanishes on torsion sheaves, then so does its upper extension $\overline{\varphi}$, and a fortiori its lower extension $\underline{\varphi}$, as a straigthforward consequence of their definition.

To study how the properties **NST** and **NSAp** introduced in 4.1.4 are inherited by $\underline{\varphi}$ and $\overline{\varphi}$, it is convenient to introduce the following definition.

Definition 4.3.11. For any object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, we denote by $\mathrm{scoft}(\mathcal{F})$ the subset of $\mathrm{coft}(\mathcal{F})$ consisting of the quasi-coherent $\mathcal{O}_X$-submodules $\mathcal{F}'$ of $\mathcal{F}$ such that the quotient $\mathcal{O}_X$-module $\mathcal{F}/\mathcal{F}'$ is coherent and torsion free, hence locally free.[17]

[17]The notation "scoft" stands for **s**aturated of **co**-**f**inite **t**ype.

Observe that if $\mathcal{G}$ is an element of $\mathrm{coft}(\mathcal{F})$, then its saturation $\mathcal{G}^{\mathrm{sat}}$ in $\mathcal{F}$ — namely the $\mathcal{O}_K$-submodule of $\mathcal{F}$ containing $\mathcal{G}$ such that

$$\mathcal{G}^{\mathrm{sat}}/\mathcal{G} = (\mathcal{F}/\mathcal{G})_{\mathrm{tor}}$$

— is an element of $\mathrm{scoft}(\mathcal{F})$. Moreover the map

$$\mathrm{coft}(\mathcal{F}) \longrightarrow \mathrm{scoft}(\mathcal{F}), \quad \mathcal{G} \longmapsto \mathcal{G}^{\mathrm{sat}}$$

coincides with the identity map on $\mathrm{scoft}(\mathcal{F})$, and for any $\mathcal{G}$ in $\mathrm{coft}(\mathcal{F})$, we have a canonical identification:

$$\overline{\mathcal{F}/\mathcal{G}^{\mathrm{sat}}} \simeq \overline{\mathcal{F}/\mathcal{G}}_{/\mathrm{tor}}.$$

Recall that that the property for an invariant on $\overline{\mathbf{Coh}}_X$ of being small on Hermitian coherent sheaves generated by small sections has been introduced in 4.2.6 above.

PROPOSITION 4.3.12. *Let us assume that φ satisfies* **NST** *on* $\overline{\mathbf{Coh}}_X$*.*

For any object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, we have:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \sup_{\mathcal{G}\in \mathrm{scoft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{G}}). \tag{4.3.23}$$

The invariant $\underline{\varphi}$ satisfies **NSAp***, and a fortiori* **NST***, on* $\mathbf{q}\overline{\mathbf{Coh}}_X$*.*

Moreover, $\overline{\varphi}$ satisfies **NST** *on* $\mathbf{q}\overline{\mathbf{Coh}}_X$*, and if φ is small on Hermitian coherent sheaves generated by small sections, $\overline{\varphi}$ satisfies* **VAp** *on* $\mathbf{q}\overline{\mathbf{Coh}}_X$*.*

PROOF. Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

For any $\mathcal{G}$ in $\mathrm{coft}(\mathcal{G})$, the quotient $\overline{\mathcal{F}/\mathcal{G}}$ is an object of $\overline{\mathbf{Coh}}_X$, and therefore:

$$\varphi(\overline{\mathcal{F}/\mathcal{G}}) = \varphi(\overline{\mathcal{F}/\mathcal{G}}_{/\mathrm{tor}}).$$

Thanks to the observations above on the saturation operation, this implies:

$$\sup_{\mathcal{G}\in \mathrm{coft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{G}}) = \sup_{\mathcal{G}\in \mathrm{coft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{G}^{\mathrm{sat}}}) = \sup_{\mathcal{G}\in \mathrm{scoft}\mathcal{F}} \varphi(\overline{\mathcal{F}/\mathcal{G}}).$$

This establishes the expression (4.3.23) for $\underline{\varphi}(\overline{\mathcal{F}})$. Applied to $\overline{\mathcal{F}}^{\vee\vee}$, this expression becomes:

$$\underline{\varphi}(\overline{\mathcal{F}}^{\vee\vee}) = \sup_{\mathcal{G}'\in \mathrm{scoft}(\mathcal{F}^{\vee\vee})} \varphi(\overline{\mathcal{F}^{\vee\vee}/\mathcal{G}'}). \tag{4.3.24}$$

Moreover, if $\mathcal{G}$ is an element of $\mathrm{scoft}(\mathcal{F})$, the $\mathcal{O}_K$-module $\mathcal{F}/\mathcal{G}$ is projective, and therefore the quotient morphism from $\mathcal{F}$ to $\mathcal{F}/\mathcal{G}$ vanishes on the antiprojective module $\mathcal{F}_{\mathrm{ap}}$ and factors through the biduality morphism $\delta_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{\vee\vee}$. This shows that we have a bijection:

$$\mathrm{scoft}(\mathcal{F}^{\vee\vee}) \xrightarrow{\sim} \mathrm{scoft}(\mathcal{F}), \quad \mathcal{G}' \longmapsto \delta_{\mathcal{F}}^{-1}(\mathcal{G}').$$

For any $\mathcal{G}'$ in $\mathrm{scoft}(\mathcal{F}^{\vee\vee})$ of image $\mathcal{G} := \delta_{\mathcal{F}}^{-1}(\mathcal{G}')$ in $\mathrm{scoft}(\mathcal{F})$, we have a canonical isometric isomorphism induced by $\delta_{\mathcal{F}}$:

$$\overline{\mathcal{F}/\mathcal{G}} \xrightarrow{\sim} \overline{\mathcal{F}^{\vee\vee}/\mathcal{G}'},$$

and therefore:

$$\varphi(\overline{\mathcal{F}/\mathcal{G}}) = \varphi(\overline{\mathcal{F}^{\vee\vee}/\mathcal{G}'}).$$

This shows that the right-hand sides of (4.3.23) and (4.3.24) coincides. This establishes the equality:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \underline{\varphi}(\overline{\mathcal{F}}^{\vee\vee}),$$

and proves that $\underline{\varphi}$ satisfies **NSAp** on $\mathbf{q}\overline{\mathbf{Coh}}_X$.

For any submodule $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$, we may consider the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_{\mathrm{tor}} \arrow[r] \arrow[d, hook] & \overline{\mathcal{C}} \arrow[r] \arrow[d, hook] & \overline{\mathcal{C}}_{/\mathrm{tor}} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{F}_{\mathrm{tor}} \arrow[r] & \overline{\mathcal{F}} \arrow[r] & \overline{\mathcal{F}}_{/\mathrm{tor}} \arrow[r] & 0,
\end{tikzcd}$$

where the vertical arrows are isometric injections. The module $\mathcal{C}_{/\mathrm{tor}}$ may be identified to an element of $\mathrm{coh}(\mathcal{F}_{/\mathrm{tor}})$, and the map

$$\mathrm{coh}(\mathcal{F}) \longrightarrow \mathrm{coh}(\mathcal{F}_{/\mathrm{tor}}), \quad \mathcal{C} \longmapsto \mathcal{C}_{/\mathrm{tor}}$$

is surjective and order preserving.

Moreover, since φ satisfies **NST**, we have:

$$\varphi(\overline{\mathcal{C}}) = \varphi(\overline{\mathcal{C}}_{/\mathrm{tor}}).$$

Consequently,

$$\overline{\varphi}(\overline{\mathcal{F}}) = \liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}}) = \liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}}_{/\mathrm{tor}}) = \liminf_{\mathcal{C}'\in\mathrm{coh}(\mathcal{F}_{/\mathrm{tor}})} \varphi(\overline{\mathcal{C}}') = \overline{\varphi}(\mathcal{F}_{/\mathrm{tor}}),$$

and therefore $\overline{\varphi}$ also satisfies **VAp**.

Assume finally that φ is small on Hermitian coherent sheaves generated by small section, and consider an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that $\mathcal{F}(X)$ is antiprojective. Let F be a finite subset of $\mathcal{F}(X)$, and let $C(N, .)$ be some functions as in Definition 4.2.11. According to Proposition 2.2.9, for every $\varepsilon \in \mathbb{R}_+^*$, there exists $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$ such that $\mathcal{C}(X)$ contains F and

$$\varphi(\overline{\mathcal{C}}) \leq C(2|F|, \varepsilon). \tag{4.3.25}$$

Since the right-hand side of (4.3.25) goes to zero with ε, this implies the vanishing of $\overline{\varphi}(\overline{\mathcal{F}}) := \liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}})$. $\square$

4.4. The Construction ev. Eventually Vanishing Invariants

Let us consider an invariant of Hermitian quasi-coherent sheaves over X:

$$\psi : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty],$$

and let us assume that it satisfies the monotonicity condition $\mathbf{Mon}^1$.

4.4.1. Definitions.

Definitions 4.4.1. We denote by

$$\mathrm{ev}\psi : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

the invariant defined by the following formula:

$$\mathrm{ev}\psi(\overline{\mathcal{F}}) := \lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \psi(\overline{\mathcal{F}/\mathcal{C}}) = \inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \psi(\overline{\mathcal{F}/\mathcal{C}}), \tag{4.4.1}$$

where $\overline{\mathcal{F}}$ is an arbitrary Hermitian quasi-coherent sheaf over X.

We say that the invariant ψ is *eventually vanishing* on $\overline{\mathcal{F}}$ when

$$\mathrm{ev}\psi(\overline{\mathcal{F}}) = 0.$$

The limit in (4.4.1) is taken over $\mathcal{C}$ in the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$. Here again, the existence of the limit and its equality with the right-hand side of (4.4.1) follows from the monotonicity $\mathbf{Mon}^1$ of ψ, which shows that, for every elements $\mathcal{C}$ and $\mathcal{C}'$ in $\mathrm{coh}(\mathcal{F})$, the following implication holds:

$$\mathcal{C} \subseteq \mathcal{C}' \Longrightarrow \psi(\overline{\mathcal{F}/\mathcal{C}}) \geq \psi(\overline{\mathcal{F}/\mathcal{C}'}).$$

Clearly, for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, we have:

$$\mathrm{ev}\psi(\overline{\mathcal{F}}) \leq \psi(\overline{\mathcal{F}}). \tag{4.4.2}$$

4.4.2. Permanence properties of the construction ev.

PROPOSITION 4.4.2. *The invariant* $\mathrm{ev}\psi$ *satisfies the condition* $\mathbf{Mon}^1$. *If moreover* ψ *satisfies* **VT** *(resp.* **SubAdd***), then* $\mathrm{ev}\psi$ *also satisfies* **VT** *(resp.* **SubAdd***).*

PROOF. (1) Consider a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X$,

$$f : \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}},$$

such that $f : \mathcal{F} \to \mathcal{G}$ is surjective. Then, for every $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$, the image $f(\mathcal{C})$ belongs to $\mathrm{coh}(\mathcal{G})$, and f induces a morphism from $\mathcal{F}/\mathcal{C}$ onto $\mathcal{G}/f(\mathcal{C})$ that, seen as a morphism from $\overline{\mathcal{F}/\mathcal{C}}$ to $\overline{\mathcal{G}/f(\mathcal{C})}$, has norms ≤ 1. Since ψ satisfies $\mathbf{Mon}^1$, this implies the inequality:

$$\psi(\overline{\mathcal{F}/\mathcal{C}}) \geq \psi(\overline{\mathcal{G}/f(\mathcal{C})}).$$

Consequently,

$$\psi(\overline{\mathcal{F}}) := \inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \psi(\overline{\mathcal{F}/\mathcal{C}}) \geq \inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \psi(\overline{\mathcal{G}/f(\mathcal{C})}) \geq \inf_{\mathcal{C}'\in\mathrm{coh}(\mathcal{G})} \psi(\overline{\mathcal{G}/\mathcal{C}'}) =: \mathrm{ev}\psi(\mathcal{G}).$$

This proves that $\mathrm{ev}\psi$ satisfies $\mathbf{Mon}^1$.

(2) When ψ satisfies **VT**, then so does $\mathrm{ev}\psi$, as a straightforward consequence of the inequality (4.4.2).

(3) Let us assume that ψ satisfies **SubAdd**, and let us consider an admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\mathcal{F}}' \xrightarrow{i} \overline{\mathcal{F}} \xrightarrow{p} \overline{\mathcal{F}}'' \longrightarrow 0. \tag{4.4.3}$$

For any two elements $\mathcal{C}'$ and $\mathcal{C}''$ of $\mathrm{coh}(\mathcal{F}')$ and $\mathrm{coh}(\mathcal{F}'')$ respectively, there exists $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$ mapped onto $\mathcal{C}''$ by p. After possibly replacing $\mathcal{C}$ by $i(\mathcal{C}')+\mathcal{C}$, we may assume that $\mathcal{C}$ contains $i(\mathcal{C}')$, or equivalently, that $i^{-1}(\mathcal{C})$ contains $\mathcal{C}'$. Then, according to the monotonicity of ψ, we have:

$$\psi(\overline{\mathcal{F}'/i^{-1}(\mathcal{C})}) \leq \psi(\overline{\mathcal{F}'/\mathcal{C}'}).$$

Moreover, since ψ satisfies $\mathbf{Mon}^1$ and **SubAdd**, according to Proposition 4.1.2, we have:

$$\psi(\overline{\mathcal{F}/\mathcal{C}}) \leq \psi(\overline{\mathcal{F}'/i^{-1}(\mathcal{C})}) + \psi(\overline{\mathcal{F}''/p(\mathcal{C})}).$$

This proves the inequality:

$$\psi(\overline{\mathcal{F}/\mathcal{C}}) \leq \psi(\overline{\mathcal{F}'/\mathcal{C}'}) + \psi(\overline{\mathcal{F}''/\mathcal{C}''}).$$

By taking the infimum over $(\mathcal{C}',\mathcal{C}'')$ in $\mathrm{coh}(\mathcal{F}') \times \mathrm{coh}(\mathcal{F}'')$, this implies:

$$\mathrm{ev}\psi(\overline{\mathcal{F}}) \leq \mathrm{ev}\psi(\overline{\mathcal{F}}') + \mathrm{ev}\psi(\overline{\mathcal{F}}''),$$

and establishes that $\mathrm{ev}\psi$ satisfies **SubAdd**. □

Proposition 4.4.2 implies permanence properties for objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ with eventually vanishing invariant ψ, namely:

COROLLARY 4.4.3. *If* $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ *is a morphism in* $\mathbf{q}\overline{\mathbf{Coh}}^{\leq 1}$ *such that* f *is surjective, and if the invariant* ψ *is eventually vanishing on* $\overline{\mathcal{F}}$, *then it is eventually vanishing on* $\overline{\mathcal{G}}$.

Let us assume that, besides $\mathbf{Mon}^1$, *the invariant* ψ *also satisfies* **SubAdd**. *Then for every admissible short exact sequence in* $\mathbf{q}\overline{\mathbf{Coh}}_X$,

$$0 \longrightarrow \overline{\mathcal{F}}' \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}'' \longrightarrow 0$$

if the invariant ψ *is eventually vanishing on* $\overline{\mathcal{F}}'$ *and* $\overline{\mathcal{F}}''$, *then* ψ *is also eventually vanishing on* $\overline{\mathcal{F}}$.

Moreover, for every object $\overline{\mathcal{E}}$ *in* $\mathbf{q}\overline{\mathbf{Coh}}_X$ *and any two* $\mathcal{O}_K$*-submodules* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *of* $\mathcal{E}$*, if* ψ *is eventually vanishing on* $\overline{\mathcal{E}}_1$ *and* $\overline{\mathcal{E}}_2$*, it is also eventually vanishing on* $\overline{\mathcal{E}_1+\mathcal{E}_2}$*.*

4.4.3. Approximating quotients by quotients by coherent subsheaves. If $\overline{\mathcal{F}}$ is an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and $\mathcal{G}$ is a submodule of $\mathcal{F}$, then we may consider the map:

$$\mathrm{coh}(\mathcal{G}) \longrightarrow [0,+\infty], \quad \mathcal{C} \longmapsto \psi(\overline{\mathcal{F}/\mathcal{C}}).$$

Since ψ satisfies $\mathbf{Mon}^1$, it is decreasing and bounded below by $\psi(\overline{\mathcal{F}/\mathcal{G}})$. Consequently:

$$\lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{G})} \psi(\overline{\mathcal{F}/\mathcal{C}}) = \inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{G})} \psi(\overline{\mathcal{F}/\mathcal{C}}) \geq \psi(\overline{\mathcal{F}/\mathcal{G}}). \tag{4.4.4}$$

PROPOSITION 4.4.4. *Let us assume that, besides* $\mathbf{Mon}^1$*, the invariant* ψ *also satisfies* $\mathbf{SubAdd}$*. With the above notation, if*

$$\mathrm{ev}\psi(\overline{\mathcal{G}}) = 0,$$

then

$$\lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{G})} \psi(\overline{\mathcal{F}/\mathcal{C}}) = \inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{G})} \psi(\overline{\mathcal{F}/\mathcal{C}}) = \psi(\overline{\mathcal{F}/\mathcal{G}}). \tag{4.4.5}$$

PROOF. For any $\mathcal{C}$ in $\mathrm{coh}(\mathcal{G})$, the subadditivity of ψ applied to the admissible short exact sequence

$$0 \longrightarrow \overline{\mathcal{G}/\mathcal{C}} \longrightarrow \overline{\mathcal{F}/\mathcal{C}} \longrightarrow \overline{\mathcal{F}/\mathcal{G}} \longrightarrow 0$$

establishes the inequality:

$$\psi(\overline{\mathcal{F}/\mathcal{C}}) \leq \psi(\overline{\mathcal{G}/\mathcal{C}}) + \psi(\overline{\mathcal{F}/\mathcal{G}}). \tag{4.4.6}$$

The last equality in (4.4.5) follows by taking the infimum over $\mathcal{C}$ in $\mathrm{coh}(\mathcal{G})$ in (4.4.6) and by using (4.4.4). $\square$

4.5. Upper Extensions and φ-Summable Hermitian Quasi-coherent Sheaves

In this section, we consider an invariant:

$$\varphi : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

that satisfies the monotonicity and subadditivity conditions $\mathbf{Mon}^1$ and $\mathbf{SubAdd}$.

Recall that, according to Propositions 4.3.4 and 4.3.6, the invariant:

$$\overline{\varphi} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0,+\infty]$$

coincides with φ on $\overline{\mathbf{Coh}}_X$ and satisfies $\mathbf{Mon}^1$. Moreover the following implication is a straightforward consequence of the estimate (4.3.14) in Proposition 4.3.10:

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}) < +\infty \Longrightarrow \overline{\varphi}(\overline{\mathcal{F}}) < +\infty. \tag{4.5.1}$$

4.5.1. Filtration by coherent subsheaves and φ-summable objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

4.5.1.1. Our main result concerning general positive invariants on $\overline{\mathbf{Coh}}_X$ and their extensions to $\mathbf{q}\overline{\mathbf{Coh}}_X$ is the following theorem, the proof of which is given in Subsection 4.5.2.

THEOREM 4.5.1. *Let* $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})})$ *be an object of* $\mathbf{q}\overline{\mathbf{Coh}}_X$*, and let* $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ *be an exhaustive filtration of* $\mathcal{F}$ *by elements of* $\mathrm{coh}(\mathcal{F})$*. If* $\mathcal{C}_\bullet$ *satisfies the condition:*

$$\mathbf{Sum}_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet) : \quad \sum_{i=0}^{+\infty} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty, \tag{4.5.2}$$

then the limit $\lim_{i\to+\infty} \varphi(\overline{\mathcal{C}}_i)$ *exists in* $\mathbb{R}_+$*, and moreover:*

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{i\to+\infty} \varphi(\overline{\mathcal{C}}_i) \quad \textit{and} \quad \mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}) = 0. \tag{4.5.3}$$

By convention, in (4.5.2), we let $\mathcal{C}_{-1} := 0$. We shall use the same convention in (4.5.4) and (4.5.5) below.

Theorem 4.5.1 provides a convenient tool to establish that the upper extension $\overline{\varphi}$ takes a finite value on some object of $\mathbf{q}\overline{\mathbf{Coh}}_X$. It may be seen as an axiomatic version of a dual form of the results in [**Bos20b**, 7.3.1 and 7.4], extended to a more general framework where no projectivity is required. This theorem will play a central role in this monograph when φ is the theta invariant h^1_θ. However the scope of Theorem 4.5.1 is not restricted to h^1_θ: it may notably be applied to the invariant ρ^2, the square of the covering radius, investigated in Chapter 6.

As mentioned in the introduction to the present chapter, the formulation and the proof of Theorem 4.5.1, and the ones of Proposition 4.5.17 as well, are reminiscent of basic constructions in measure and capacity theory, as developed for instance in [**Sak37**] and [**Cho54**]. However, while these constructions concern numerical functions attached to *subsets* of a given set X and their compatibility with the operations of (countable) union and intersection, here we deal with invariants attached to *objects in* the *categories* $\overline{\mathbf{Coh}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$ and with their compatibility, not only with the operation of taking intersections or sums of subobjects of a given object, but also with morphisms in these categories and the construction of quotients.

4.5.1.2. Motivated by Theorem 4.5.1, it is natural to introduce the following definitions:

DEFINITIONS 4.5.2. Let $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})})$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

An exhaustive filtration $\mathcal{C}_\bullet$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ is called a *φ-summable filtration of $\overline{\mathcal{F}}$* when it satisfies the condition $\mathbf{Sum}_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet)$, that is when

$$\Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet) := \sum_{i=0}^{+\infty} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty. \tag{4.5.4}$$

The Hermitian quasi-coherent sheaf $\overline{\mathcal{F}}$ is called *φ-summable* when there exists a φ-summable exhaustive filtration of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$.

We will denote by φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ and φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ the full subcategories of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ whose objects are the φ-summable Hermitian quasi-coherent sheaves over X. Every object of $\overline{\mathbf{Coh}}_X$ is clearly φ-summable, and the categories $\overline{\mathbf{Coh}}_X$ and $\overline{\mathbf{Coh}}_X^{\leq 1}$ appear as full subcategories of φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ and φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$.

Observe that, if an exhaustive filtration $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ satisfies the summability condition $\mathbf{Sum}_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet)$, then for every strictly increasing map $\iota : \mathbb{N} \to \mathbb{N}$, the filtration $\mathcal{C}_{\iota(\bullet)} := (\mathcal{C}_i)_{\iota(i)\in\mathbb{N}}$ satisfies the same conditions. Indeed, as a consequence of the subaddivity **SubAdd** of φ, we have:[18]

$$\varphi(\overline{\mathcal{C}_{\iota(k)}/\mathcal{C}_{\iota(k-1)}}) \leq \sum_{i=\iota(k-1)+1}^{\iota(k)} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) \quad \text{for every } k \in \mathbb{N}, \tag{4.5.5}$$

and therefore:

$$\Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_{\iota(\bullet)}) \leq \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet).$$

Observe also that, for every $n \in \mathbb{N}$, the filtration $\mathcal{C}_{\bullet+n} := (\mathcal{C}_{i+n})_{i\in\mathbb{N}}$ satisfies:

$$\Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_{\bullet+n}) = \varphi(\overline{\mathcal{C}}_n) + \sum_{i=n+1}^{+\infty} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}); \tag{4.5.6}$$

[18]By convention, we let $\iota(-1) := -1$.

Therefore, if $\mathcal{C}_\bullet$ is any φ-summable exhaustive filtration of $\mathcal{F}$ by submodules $\mathrm{coh}(\mathcal{F})$, the conclusion of Theorem 4.5.1 implies:

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{n\to+\infty} \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_{\bullet+n}). \tag{4.5.7}$$

If we denote by $\mathrm{CohFil}(\mathcal{F})$ the set of exhaustive filtrations of $\mathcal{F}$ by submodules in $\mathrm{coh}(\mathcal{F})$, we easily deduce from Theorem 4.5.1:

COROLLARY 4.5.3. *For every object in φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$, the following equality holds:*

$$\overline{\varphi}(\overline{\mathcal{F}}) = \inf_{\mathcal{C}_\bullet \in \mathrm{CohFil}(\mathcal{F})} \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet).$$

PROOF. The subadditivity of φ shows that for every $\mathcal{C}_\bullet$ in $\mathrm{CohFil}(\mathcal{F})$ and every k in $\mathbb{N}$, we have:

$$\varphi(\overline{\mathcal{C}}_k) \leq \sum_{i=0}^{k} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) \leq \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet).$$

Together with (4.5.3), this establish the upper bound:

$$\overline{\varphi}(\overline{\mathcal{F}}) \leq \inf_{\mathcal{C}_\bullet \in \mathrm{CohFil}(\mathcal{F})} \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet).$$

The opposite inequality follows from (4.5.7). □

4.5.1.3. As a first illustration of the significance of φ-summable objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$, we may show that, restricted to these objects, the upper extension $\overline{\varphi}$ inherits the additivity property $\mathbf{Add}_\oplus$ from φ.

COROLLARY 4.5.4. *Assume moreover that φ satisfies $\mathbf{Add}_\oplus$ on $\overline{\mathbf{Coh}}_X$. Then, if two objects $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}'$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ are φ-summable, then their direct sum $\overline{\mathcal{F}} \oplus \overline{\mathcal{F}}'$ is φ-summable, and the following equality holds:*

$$\overline{\varphi}(\overline{\mathcal{F}} \oplus \overline{\mathcal{F}}') = \overline{\varphi}(\overline{\mathcal{F}}) \oplus \overline{\varphi}(\overline{\mathcal{F}}'). \tag{4.5.8}$$

PROOF. Let $\mathcal{C}_\bullet$ and $\mathcal{C}'_\bullet$ be exhaustive filtrations of $\mathcal{F}$ and $\mathcal{F}'$ by coherent submodules that satisfy $\mathbf{Sum}_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet)$ and $\mathbf{Sum}_\varphi(\overline{\mathcal{F}}', \mathcal{C}'_\bullet)$ respectively. Then $\tilde{\mathcal{C}}_\bullet := (\mathcal{C}_i \oplus \mathcal{C}'_i)_{i\in\mathbb{N}}$ is an exhaustive filtration of $\mathcal{F} \oplus \mathcal{F}'$ by coherent submodules, and the validity of $\mathbf{Add}_\oplus$ implies, for every $i \in \mathbb{N}$:

$$\varphi(\overline{\tilde{\mathcal{C}}_i/\tilde{\mathcal{C}}_{i-1}}) = \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \oplus \overline{\mathcal{C}'_i/\mathcal{C}'_{i-1}}) = \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) + \varphi(\overline{\mathcal{C}'_i/\mathcal{C}'_{i-1}}) \tag{4.5.9}$$

and

$$\varphi(\overline{\tilde{\mathcal{C}}}_i) = \varphi(\overline{\mathcal{C}}_i \oplus \overline{\mathcal{C}'_i}) = \varphi(\overline{\mathcal{C}}_i) + \varphi(\overline{\mathcal{C}'_i}). \tag{4.5.10}$$

The relations (4.5.9) show that the condition $\mathbf{Sum}_\varphi(\overline{\mathcal{F}} \oplus \overline{\mathcal{F}}', \tilde{\mathcal{C}}_\bullet)$ is satisfied. This already establishes that $\overline{\mathcal{F}} \oplus \overline{\mathcal{F}}'$ is φ-summable. Moreover Theorem 4.5.1, applied to $\overline{\mathcal{F}}$ and $\mathcal{C}_\bullet$, $\overline{\mathcal{F}}'$ and $\mathcal{C}'_\bullet$, and $\overline{\mathcal{F}} \oplus \overline{\mathcal{F}}'$ and $\tilde{\mathcal{C}}_\bullet$ shows that:

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{i\to+\infty} \varphi(\overline{\mathcal{C}}_i), \quad \overline{\varphi}(\overline{\mathcal{F}}) = \lim_{i\to+\infty} \varphi(\overline{\mathcal{C}}_i), \quad \text{and} \quad \overline{\varphi}(\overline{\mathcal{F}} \oplus \overline{\mathcal{F}}') = \lim_{i\to+\infty} \varphi(\overline{\tilde{\mathcal{C}}}_i).$$

Together with (4.5.10), this establishes (4.5.8). □

When the additivity property $\mathbf{Add}_\oplus$ is satisfied, one may easily construct objects in φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ that do not belong to $\overline{\mathbf{Coh}}_X$ by means of countable direct sums.

COROLLARY 4.5.5. *Let us assume that φ satisfies $\mathbf{Add}_\oplus$. For every sequence $(\overline{\mathcal{D}}_i)_{i\in\mathbb{N}}$ of objects in $\overline{\mathbf{Coh}}_X$, of direct sum $\overline{\mathcal{F}} := \bigoplus_{i\in\mathbb{N}} \overline{\mathcal{D}}_i$, the following two conditions are equivalent:*

(i) $\overline{\mathcal{F}}$ *is φ-summable;*
(ii) $\sum_{i\in\mathbb{N}} \varphi(\overline{\mathcal{D}}_i) < +\infty$.

Recall that, with the notation of Corollary 4.5.5, the following equalities hold:

$$\overline{\varphi}(\overline{\mathcal{F}}) = \underline{\varphi}(\overline{\mathcal{F}}) = \sum_{i\in\mathbb{N}} \varphi(\overline{\mathcal{D}}_i),$$

as shown in Proposition 4.3.8.

PROOF. When (i) holds, then $\overline{\varphi}(\bigoplus_{i\in\mathbb{N}} \overline{\mathcal{D}}_i)$ is finite and therefore (ii) holds. Conversely, when (ii) holds, then the submodules

$$\mathcal{C}_i := \bigoplus_{0\leq k\leq i} \mathcal{D}_k$$

define an exhaustive filtration $\mathcal{C}_\bullet$ of $\mathcal{F} = \bigoplus_{i\in\mathbb{N}} \overline{\mathcal{D}}_i$ by finitely generated submodules that is φ-summable, since the subquotient $\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}$ may be identified with $\overline{\mathcal{D}}_i$ for every $i \in \mathbb{N}$, and consequently:

$$\Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet) = \sum_{i\in\mathbb{N}} \varphi(\overline{\mathcal{D}}_i). \qquad \square$$

It is actually possible to establish a permanence property of φ-summability under countable direct sums that encompasses the previous two corollaries:

PROPOSITION 4.5.6. *Let us moreover assume that φ satisfies the condition $\mathbf{Add}_\oplus$ on $\overline{\mathbf{Coh}}_X$. Then for every countable family $(\overline{\mathcal{G}}_i)_{i\in I}$ of objects of φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$, the following equality holds in $[0, +\infty]$:*

$$\overline{\varphi}(\bigoplus_{i\in I} \overline{\mathcal{G}}_i) = \sum_{i\in I} \overline{\varphi}(\overline{\mathcal{G}}_i). \tag{4.5.11}$$

Moreover $\bigoplus_{i\in I} \overline{\mathcal{G}}_i$ is φ-summable if and only if (4.5.11) *is finite.*

PROOF. For any finite subset F of I, we may apply property $\mathbf{Mon}^1$ — that is satisfied by $\overline{\varphi}$ according to Proposition 4.3.6 — to the projection:

$$\overline{\mathcal{G}} := \bigoplus_{i\in I} \overline{\mathcal{G}}_i \longrightarrow \bigoplus_{i\in F} \overline{\mathcal{G}}_i.$$

Together with the additivity of $\overline{\varphi}$ on φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ established in Corollary 4.5.4, this establishes the following inequality:

$$\overline{\varphi}(\overline{\mathcal{G}}) \geq \overline{\varphi}(\bigoplus_{i\in F} \overline{\mathcal{G}}_i) = \sum_{i\in F} \overline{\varphi}(\overline{\mathcal{G}}_i),$$

and consequently, since the finite subset I is arbitrary:

$$\overline{\varphi}(\overline{\mathcal{G}}) \geq \sum_{i\in I} \overline{\varphi}(\overline{\mathcal{G}}_i).$$

This shows that, if $\sum_{i\in I} \overline{\varphi}(\overline{\mathcal{G}}_i)$ is infinite, then $\overline{\mathcal{G}}$ is not φ-summable and (4.5.11) holds.

Let us now assume that $\sum_{i\in I} \overline{\varphi}(\overline{\mathcal{G}}_i)$ is finite, and let us choose $(\varepsilon_i)_{i\in I}$ in $\mathbb{R}_+^{*I}$. For every $i \in I$, we may choose a φ-summable exhaustive filtration $\mathcal{C}_{i,\bullet}$ of $\overline{\mathcal{G}}_i$ of $\mathcal{G}_i$ by coherent submodules such that:

$$\Sigma_\varphi(\overline{\mathcal{G}}_i, \mathcal{C}_{i,\bullet}) \leq \overline{\varphi}(\overline{\mathcal{G}}_i) + \varepsilon_i.$$

After replacing $\mathcal{C}_{i,\bullet}$ by $\mathcal{C}_{i,\bullet+i}$, we may also assume that:

$$\mathcal{C}_{i,k} = 0 \quad \text{if } k < i.$$

Then, if we let:

$$\tilde{\mathcal{C}}_k := \bigoplus_{i\in I} \mathcal{C}_{i,k}$$

for every $k \in \mathbb{N}$, then $\tilde{\mathcal{C}}_\bullet := (\tilde{\mathcal{C}}_k)_{k\in\mathbb{N}}$ is an exhausting filtration of $\mathcal{G}$ by coherent submodules. Moreover, from the additivity of φ on $\overline{\mathbf{Coh}}_X$, we derive:

$$\Sigma_\varphi(\overline{\mathcal{G}}, \tilde{\mathcal{C}}_\bullet) = \sum_{k\in\mathbb{N}, i\in I} \varphi(\overline{\mathcal{C}_{i,k}/\mathcal{C}_{i,k-1}}) = \sum_{i\in I} \Sigma_\varphi(\overline{\mathcal{G}}_i, \mathcal{C}_{i,\bullet}) \leq \sum_{i\in I} \overline{\varphi}(\overline{\mathcal{G}}_i) + \sum_{i\in I} \varepsilon_i.$$

Since $(\varepsilon_i)_{i\in I}$ may be chosen so that $\sum_{i\in I} \varepsilon_i$ is finite, this implies that $\overline{\mathcal{G}}$ is φ-summable. It also establishes the inequality:

$$\overline{\varphi}(\overline{\mathcal{G}}) \leq \sum_{i\in I} \overline{\varphi}(\overline{\mathcal{G}}_i) + \sum_{i\in I} \varepsilon_i.$$

As $(\varepsilon_i)_{i\in I}$ may be chosen so that $\sum_{i\in I} \varepsilon_i$ is arbitrarily small, this completes the proof of (4.5.11). □

4.5.2. Proof of Theorem 4.5.1. Let us consider an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and exhaustive filtration $\mathcal{C}_\bullet$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ such that

$$\Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet) := \sum_{i=0}^{+\infty} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty.$$

LEMMA 4.5.7. *The limit* $\lim_{i\to+\infty} \varphi(\overline{\mathcal{C}}_i)$ *exists in* $\mathbb{R}_+$ *and, for any* $i \in \mathbb{N}$, *satisfies the inequality*

$$\lim_{k\to+\infty} \varphi(\overline{\mathcal{C}}_k) \leq \varphi(\overline{\mathcal{C}}_i) + \sum_{j=i+1}^{+\infty} \varphi(\overline{\mathcal{C}_j/\mathcal{C}_{j-1}}). \tag{4.5.12}$$

PROOF. According to the subadditivity of φ, applied to the admissible short exact sequence

$$0 \longrightarrow \overline{\mathcal{C}}_{i-1} \longrightarrow \overline{\mathcal{C}}_i \longrightarrow \overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \longrightarrow 0,$$

the following estimates holds for every $i \in \mathbb{N}$:

$$\varphi(\overline{\mathcal{C}}_i) \leq \varphi(\overline{\mathcal{C}}_{i-1}) + \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}). \tag{4.5.13}$$

Therefore, if we let:

$$x_i := \varphi(\overline{\mathcal{C}}_i) - \sum_{j=0}^{i} \varphi(\overline{\mathcal{C}_j/\mathcal{C}_{j-1}}),$$

then the sequence $(x_i)_{i\in\mathbb{N}}$ is decreasing and lies in the interval

$$[-\sum_{j=0}^{+\infty} \varphi(\overline{\mathcal{C}_j/\mathcal{C}_{j-1}}), +\infty).$$

Consequently, it admits a limit l in this interval. This shows that the sequence $(\varphi(\overline{\mathcal{C}}_i))_{i\in\mathbb{N}}$ admits

$$l + \sum_{j=0}^{+\infty} \varphi(\overline{\mathcal{C}_j/\mathcal{C}_{j-1}})$$

as a limit in $\mathbb{R}_+$.

The estimates (4.5.13) also imply the following upper bounds, valid for every $(i,k) \in \mathbb{N}^2$ such that $i < k$:

$$\varphi(\overline{\mathcal{C}}_k) \leq \varphi(\overline{\mathcal{C}}_i) + \sum_{j=i+1}^{k} \varphi(\overline{\mathcal{C}_j/\mathcal{C}_{j-1}}). \tag{4.5.14}$$

The estimates (4.5.12) follows by letting k go to infinity. □

LEMMA 4.5.8. *For every $\mathcal{C}$ in* $\mathrm{coh}(\mathcal{F})$*, there exists $\iota(\mathcal{C}) \in \mathbb{N}$ such that $\mathcal{C} \subseteq \mathcal{C}_{\iota(\mathcal{C})}$. The sequence $(\varphi(\overline{\mathcal{C}_k/\mathcal{C}}))_{k\geq\iota(\mathcal{C})}$ admits a limit $l(\mathcal{C})$ in $\mathbb{R}_+$. Moreover $l(\mathcal{C})$ is a decreasing function of $\mathcal{C}$ in the directed set* $(\mathrm{coh}(\mathcal{F}), \subseteq)$ *and satisfies:*

$$\lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{C})} l(\mathcal{C}) = \inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{C})} l(\mathcal{C}) = 0. \tag{4.5.15}$$

PROOF. The existence of $\iota(\mathcal{C})$ follows from the exhaustive character of the filtration $\mathcal{C}_\bullet$. The sequence $(\mathcal{C}_k/\mathcal{C})_{k\geq\iota(\mathcal{C})}$ is an exhaustive filtration of $\mathcal{F}/\mathcal{C}$ by submodules in $\mathrm{coh}(\mathcal{F}/\mathcal{C})$, and the existence in $\mathbb{R}_+$ of the limit

$$l(\mathcal{C}) := \lim_{k\to+\infty} \varphi(\overline{\mathcal{C}_k/\mathcal{C}})$$

follows from Lemma 4.5.7 applied to $\overline{\mathcal{F}/\mathcal{C}}$ and to the filtration $(\mathcal{C}_k/\mathcal{C})_{k\geq\iota(\mathcal{C})}$ instead of $\overline{\mathcal{F}}$ and $(\mathcal{C}_k)_{k\in\mathbb{N}}$. Indeed, for every $k > \iota(\mathcal{C})$, the subquotient $\overline{(\mathcal{C}_k/\mathcal{C})/(\mathcal{C}_{k-1}/\mathcal{C}))}$ may be identified to $\overline{\mathcal{C}_k/\mathcal{C}_{k-1}}$, and consequently the summability condition $\mathbf{Sum}_\varphi(\overline{\mathcal{F}/\mathcal{C}}, (\mathcal{C}_k/\mathcal{C})_{k\geq\iota(\mathcal{C})})$ is satisfied.

Moreover if two submodules $\mathcal{C}$ and $\mathcal{C}'$ in $\mathrm{coh}(\mathcal{F})$ satisfy $\mathcal{C} \subseteq \mathcal{C}'$, then the monotonicity of φ implies the inequality:

$$\varphi(\overline{\mathcal{C}_k/\mathcal{C}}) \geq \varphi(\overline{\mathcal{C}_k/\mathcal{C}'}),$$

for every $k \geq \max(\iota(\mathcal{C}), \iota(\mathcal{C}'))$. Taking the limit when k goes to infinity, this establishes the inequality:

$$l(\mathcal{C}) \geq l(\mathcal{C}'),$$

and shows that $l(\mathcal{C})$ admits a limit when $\mathcal{C}$ runs over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$:

$$\lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{C})} l(\mathcal{C}) = \inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{C})} l(\mathcal{C}) \in \mathbb{R}_+. \tag{4.5.16}$$

When $\mathcal{C} = \mathcal{C}_i$ for some $i \in \mathbb{N}$, we may choose $\iota(\mathcal{C}) := i$, and the upper bound (4.5.12), with $\mathcal{C}_k$ replaced by $\mathcal{C}_k/\mathcal{C}_i$, takes the form:

$$\lim_{k\to+\infty} \varphi(\overline{\mathcal{C}_k/\mathcal{C}_i}) \leq \sum_{j=i+1}^{+\infty} \varphi(\overline{\mathcal{C}_j/\mathcal{C}_{j-1}}). \tag{4.5.17}$$

When i goes to infinity, the right-hand side of (4.5.17) goes to zero, and therefore:

$$\lim_{i\to+\infty} l(\mathcal{C}_i) = 0. \tag{4.5.18}$$

This establishes the vanishing of (4.5.16), and completes the proof of (4.5.15). □

The subadditivity of φ, applied to the admissible short exact sequence

$$0 \longrightarrow \overline{\mathcal{C}} \longrightarrow \overline{\mathcal{C}}_k \longrightarrow \overline{\mathcal{C}_k/\mathcal{C}} \longrightarrow 0,$$

shows that the following inequality holds, for every $\mathcal{C} \in \mathrm{coh}(\mathcal{F})$ and every integer $k \geq \iota(\mathcal{C})$:

$$\varphi(\overline{\mathcal{C}}_k) \leq \varphi(\overline{\mathcal{C}}) + \varphi(\overline{\mathcal{C}_k/\mathcal{C}}).$$

By taking the limit when k goes to infinity, this establishes the estimate:

$$\lim_{k\to+\infty} \varphi(\overline{\mathcal{C}}_k) \leq \varphi(\overline{\mathcal{C}}) + l(\mathcal{C}), \tag{4.5.19}$$

valid for every $\mathcal{C} \in \mathrm{coh}(\mathcal{F})$.

From (4.5.15) and (4.5.19), we get:

$$\lim_{k\to+\infty} \varphi(\overline{\mathcal{C}}_k) \leq \liminf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}}).$$

This establishes the equality:

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{k\to+\infty} \varphi(\overline{\mathcal{C}}_k). \tag{4.5.20}$$

In turn, (4.5.20) applied to applied to $\overline{\mathcal{F}/\mathcal{C}}$ and to the filtration $(\mathcal{C}_k/\mathcal{C})_{k\geq\iota(\mathcal{C})}$ instead of $\overline{\mathcal{F}}$ and $(\mathcal{C}_k)_{k\in\mathbb{N}}$, shows that:

$$\overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}}) = l(\mathcal{C}). \tag{4.5.21}$$

According to (4.5.15), the limit of (4.5.21) when $\mathcal{C}$ runs over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$ vanishes. This establishes the vanishing of $\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}})$ and completes the proof of Theorem 4.5.1.

4.5.3. Permanence properties for φ-summable Hermitian quasi-coherent sheaves.

PROPOSITION 4.5.9. *If $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ such that the morphism of $\mathcal{O}_K$-modules $f : \mathcal{F} \to \mathcal{G}$ is surjective, and if $\overline{\mathcal{F}}$ is φ-summable, then $\overline{\mathcal{G}}$ is φ-summable.*

We shall actually establish the following more precise result, which will be used in the next chapter when investigating the strong monotonicity properties of the upper extension $\overline{\varphi}$:

PROPOSITION 4.5.10. *Under the hypotheses of Proposition 4.5.9, for any exhaustive filtration $\mathcal{C}_\bullet = (\mathcal{C}_i)_{i\in\mathbb{N}}$ by elements of $\mathrm{coh}(\mathcal{F})$, the image filtration $f(\mathcal{C}_\bullet) := (f(\mathcal{C}_i))_{i\in\mathbb{N}}$ is an exhaustive filtration of $\mathcal{G}$ by submodules in $\mathrm{coh}(\mathcal{G})$ and satisfies:*

$$\Sigma_\varphi(\overline{\mathcal{G}}, f(\mathcal{C}_\bullet)) \leq \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet). \tag{4.5.22}$$

When moreover the surjective morphism f from $\overline{\mathcal{F}}$ to $\overline{\mathcal{G}}$ is admissible and φ satisfies $\mathbf{Cont}^+$, if we let $\mathcal{H} := \ker f$, then:

$$\overline{\varphi}(\overline{\mathcal{G}}) = \lim_{i\to+\infty} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_i\cap\mathcal{H}}). \tag{4.5.23}$$

PROOF. Let $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ be a morphism as in Proposition 4.5.9, and let $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ be an exhaustive filtration of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$.

Then $f(\mathcal{C}_\bullet)$ is clearly an exhaustive filtration of $\mathcal{G}$ by submodules in $\mathrm{coh}(\mathcal{G})$. Moreover, for every $i \in \mathbb{N}$, the morphism f induces a morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$,

$$f_i : \overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \longrightarrow \overline{f(\mathcal{C}_i)/f(\mathcal{C}_{i-1})},$$

which is surjective on the underlying $\mathcal{O}_K$-modules. As φ satisfies $\mathbf{Mon}^1$ on $\overline{\mathbf{Coh}}_X$, this implies the inequality (4.5.22). In particular, if $\mathcal{C}_\bullet$ is φ-summable, the filtration $f(\mathcal{C}_\bullet)$ also is.

This notably establishes Proposition 4.5.9, and also the equality:

$$\overline{\varphi}(\overline{\mathcal{G}}) = \lim_{i\to+\infty} \varphi(\overline{f(\mathcal{C}_i)}) \tag{4.5.24}$$

by applying Theorem 4.5.1 to $\overline{\mathcal{G}}$ equipped with the filtration $f(\mathcal{C}_\bullet)$.

Let us now assume that f is admissible and that φ satisfies $\mathbf{Cont}^+$.

For every $i \in \mathbb{N}$, the morphism f induces a morphism

$$\overline{\mathcal{C}_i/(\mathcal{C}_i\cap\mathcal{H})} \longrightarrow \overline{f(\mathcal{C}_i)}$$

in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ that is an isomorphism between the underlying $\mathcal{O}_K$-modules. Since φ satisfies $\mathbf{Mon}^1$, this implies the inequality:

$$\varphi(\overline{\mathcal{C}_i/(\mathcal{C}_i\cap\mathcal{H})}) \geq \varphi(\overline{f(\mathcal{C}_i)}).$$

Combined with (4.5.24), this establishes the lower bound:

$$\liminf_{i\to+\infty} \varphi(\overline{\mathcal{C}_i/(\mathcal{C}_i\cap\mathcal{H})}) \geq \overline{\varphi}(\overline{\mathcal{G}}). \tag{4.5.25}$$

Observe that, for any two integer $j \geq i \geq 0$, the map f induces a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$,

$$\overline{(\mathcal{C}_i + \mathcal{C}_j\cap\mathcal{H})/(\mathcal{C}_j\cap\mathcal{H})} \longrightarrow \overline{f(\mathcal{C}_i)},$$

that is an isomorphism between the underlying $\mathcal{O}_K$-modules; using $\mathbf{Mon}^1$ again, this implies the inequality:

$$\varphi(\overline{(\mathcal{C}_i + \mathcal{C}_j \cap \mathcal{H})/(\mathcal{C}_j \cap \mathcal{H})}) \geq \varphi(\overline{f(\mathcal{C}_i)}).$$

Actually, the sequence $(\varphi(\overline{(\mathcal{C}_i + \mathcal{C}_j \cap \mathcal{H})/\mathcal{C}_j \cap \mathcal{H}}))_{j \geq i}$ is decreasing, again by $\mathbf{Mon}^1$, and since f is admissible and φ satisfies $\mathbf{Cont}^+$, it satisfies:

$$\lim_{j \to +\infty} \varphi(\overline{(\mathcal{C}_i + \mathcal{C}_j \cap \mathcal{H})/(\mathcal{C}_j \cap \mathcal{H})}) = \varphi(\overline{f(\mathcal{C}_i)}), \tag{4.5.26}$$

as a consequence of Proposition 2.2.6, applied with $\mathcal{E} := \mathcal{H}$, $\mathcal{F}' := \mathcal{C}_i$, and $\mathcal{E}_n := \mathcal{C}_n \cap \mathcal{H}$.

Moreover the subadditivity of φ, applied to the admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$

$$0 \longrightarrow \overline{(\mathcal{C}_i + \mathcal{C}_j \cap \mathcal{H})/(\mathcal{C}_j \cap \mathcal{H})} \longrightarrow \overline{\mathcal{C}_j/(\mathcal{C}_j \cap \mathcal{H})} \longrightarrow \overline{\mathcal{C}_j/(\mathcal{C}_i + \mathcal{C}_j \cap \mathcal{H})} \longrightarrow 0,$$

leads to the inequality:

$$\varphi(\overline{\mathcal{C}_j/(\mathcal{C}_j \cap \mathcal{H})}) \leq \varphi(\overline{(\mathcal{C}_i + \mathcal{C}_j \cap \mathcal{H})/(\mathcal{C}_j \cap \mathcal{H})}) + \varphi(\overline{\mathcal{C}_j/(\mathcal{C}_i + \mathcal{C}_j \cap \mathcal{H})}).$$

The monotonicity and the subadditivity of φ also imply:

$$\varphi(\overline{\mathcal{C}_j/(\mathcal{C}_i + \mathcal{C}_j \cap \mathcal{H})}) \leq \varphi(\overline{\mathcal{C}_j/\mathcal{C}_i}) \leq \sum_{i < k \leq j} \varphi(\overline{\mathcal{C}_k/\mathcal{C}_{k-1}}).$$

Thus we have:

$$\varphi(\overline{\mathcal{C}_j/(\mathcal{C}_j \cap \mathcal{H})}) \leq \varphi(\overline{(\mathcal{C}_i + \mathcal{C}_j \cap \mathcal{H})/(\mathcal{C}_j \cap \mathcal{H})}) + \sum_{i < k \leq j} \varphi(\overline{\mathcal{C}_k/\mathcal{C}_{k-1}}).$$

From this inequality and from (4.5.26), by letting j go to infinity, we derive the following inequality, valid for any $i \in \mathbb{N}$ when f is admissible:

$$\limsup_{j \to +\infty} \varphi(\overline{\mathcal{C}_j/\mathcal{C}_j \cap \mathcal{H}}) \leq \varphi(\overline{f(\mathcal{C}_i)}) + \sum_{k=i+1}^{+\infty} \varphi(\overline{\mathcal{C}_k/\mathcal{C}_{k-1}}).$$

Finally, by letting i go to infinity and using (4.5.24), we obtain:

$$\limsup_{j \to +\infty} \varphi(\overline{\mathcal{C}_j/\mathcal{C}_j \cap \mathcal{H}}) \leq \overline{\varphi}(\overline{\mathcal{G}}),$$

which completes the proof of (4.5.23). □

REMARK 4.5.11. For future reference, observe that the proof of the inequality (4.5.25) actually establishes the following result: *for every invariant* $\varphi : \overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$ *satisfying* $\mathbf{Mon^1}$ *and for every morphism* $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ *in* $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ *such that* $\operatorname{im} f = \mathcal{G}$*, the following inequality holds:*

$$\overline{\varphi}(\overline{\mathcal{G}}) \leq \liminf_{\mathcal{C} \in \operatorname{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \ker f)}).$$

PROPOSITION 4.5.12. *If* $\overline{\mathcal{F}}'$ *and* $\overline{\mathcal{F}}''$ *are* φ*-summable objects in* $\mathbf{q}\overline{\mathbf{Coh}}_X$ *and if*

$$0 \longrightarrow \overline{\mathcal{F}}' \xrightarrow{i} \overline{\mathcal{F}} \xrightarrow{p} \overline{\mathcal{F}}'' \longrightarrow 0$$

is an admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$*, then* $\overline{\mathcal{F}}$ *is* φ*-summable.*

This proposition immediately follows from the second part of the next lemma.

LEMMA 4.5.13. *Consider an admissible short exact sequence in* $\mathbf{q}\overline{\mathbf{Coh}}_X$,

$$0 \longrightarrow \overline{\mathcal{F}}' \xrightarrow{i} \overline{\mathcal{F}} \xrightarrow{p} \overline{\mathcal{F}}'' \longrightarrow 0,$$

and an exhaustive filtration $\mathcal{C}'_\bullet := (\mathcal{C}'_k)_{k \in \mathbb{N}}$ *(resp.* $\mathcal{C}''_\bullet := (\mathcal{C}''_k)_{k \in \mathbb{N}}$*) be an exhaustive filtration of* $\mathcal{F}'$ *(resp.* $\mathcal{F}''$*) by elements of* $\operatorname{coh}(\mathcal{F}'')$ *(resp. of* $\operatorname{coh}(\mathcal{F}'')$*).*

(i) *For every sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$ *in* $\mathbb{R}_+^*$, *there exists a strictly increasing map* $\iota : \mathbb{N} \to \mathbb{N}$ *and an exhaustive filtration* $\mathcal{C}_\bullet := (\mathcal{C}_k)_{k\in\mathbb{N}}$ *of* $\mathcal{F}$ *by elements of* $\mathrm{coh}(\mathcal{F})$ *such that the following two conditions are satisfied for every* $k \in \mathbb{N}$*:*

$$i^{-1}(\mathcal{C}_k) = \mathcal{C}'_{\iota(k)} \text{ and } p(\mathcal{C}_k) = \mathcal{C}''_k \tag{4.5.27}$$

and:

$$\varphi(\overline{\mathcal{C}_k/\mathcal{C}_{k-1}}) \leq \varphi(\overline{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}}) + \varphi(\overline{\mathcal{C}''_k/\mathcal{C}''_{k-1}}) + \varepsilon_k. \tag{4.5.28}$$

(ii) *For every* $\varepsilon \in \mathbb{R}_+^*$, *there exists a strictly increasing map* $\iota : \mathbb{N} \to \mathbb{N}$ *and an exhaustive filtration* $\mathcal{C}_\bullet := (\mathcal{C}_k)_{k\in\mathbb{N}}$ *of* $\mathcal{F}$ *by elements of* $\mathrm{coh}(\mathcal{F})$ *such that condition* (4.5.27) *is satisfied for every* $k \in \mathbb{N}$, *and moreover:*

$$\Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet) \leq \Sigma_\varphi(\overline{\mathcal{F}}', \mathcal{C}'_{\iota(\bullet)}) + \Sigma_\varphi(\overline{\mathcal{F}}'', \mathcal{C}''_\bullet) + \varepsilon. \tag{4.5.29}$$

Proof. Since p is surjective, for every $k \in \mathbb{N}$, we may find $\tilde{\mathcal{C}}_k$ in $\mathrm{coh}(\mathcal{F})$ such that

$$p(\tilde{\mathcal{C}}_k) = \mathcal{C}_k.$$

After possibly replacing $\tilde{\mathcal{C}}_k$ by $\sum_{i=0}^k \tilde{\mathcal{C}}_i$, we may assume that the sequence $(\tilde{\mathcal{C}}_k)_{k\in\mathbb{N}}$ is increasing. The submodules $i^{-1}(\tilde{\mathcal{C}}_k)$ of $\mathcal{F}'$ are coherent; consequently there exists a strictly increasing map $\iota_0 : \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$, the following inclusion holds:

$$i^{-1}(\tilde{\mathcal{C}}_k) \subseteq \mathcal{C}'_{\iota_0(k)}.$$

Then, if we let, for $k \in \mathbb{N}$:

$$\mathcal{C}_k := i(\mathcal{C}'_{\iota_0(k)}) + \tilde{\mathcal{C}}_k,$$

then $(\mathcal{C}_k)_{k\in\mathbb{N}}$ is a filtration of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$. This filtration is exhaustive — indeed $\bigcup_{k\in\mathbb{N}} \mathcal{C}_k$ contains

$$\bigcup_{k\in\mathbb{N}} i(\mathcal{C}'_k) = i(\mathcal{F}'),$$

and its image by p is

$$\bigcup_{k\in\mathbb{N}} p(\mathcal{C}_k) = \bigcup_{k\in\mathbb{N}} i(\mathcal{C}''_k) = \mathcal{F}''$$

— and satisfies the conditions (4.5.27) by construction.

Let us show that, for any given sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_+^*$, the estimates (4.5.28) also are satisfied for a suitable choice of the map ι.

The short exact sequences

$$0 \longrightarrow \mathcal{C}'_{\iota(k)} \xrightarrow{i} \mathcal{C}_k \xrightarrow{p} \mathcal{C}''_k \longrightarrow 0$$

induce short exact sequences relating the successive subquotients of the filtrations $\mathcal{C}'_\bullet$, $\mathcal{C}_\bullet$, and $\mathcal{C}''_\bullet$:

$$0 \longrightarrow \mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)} \xrightarrow{i_k} \mathcal{C}_k/\mathcal{C}_{k-1} \xrightarrow{p_k} \mathcal{C}''_k/\mathcal{C}''_{k-1} \longrightarrow 0,$$

where, by convention, we let: $\mathcal{C}''_{\iota(-1)} = \mathcal{C}_{-1} = \mathcal{C}''_{-1} = 0$.

Let us denote by $\overline{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}}$, $\overline{\mathcal{C}_k/\mathcal{C}_{k-1}}$, and $\overline{\mathcal{C}''_k/\mathcal{C}''_{k-1}}$ the objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ defined by these subquotients equiped with the Hermitian seminorms induced by the ones on $\overline{\mathcal{F}}'$, $\overline{\mathcal{F}}$, and $\overline{\mathcal{F}}''$. We shall also denote by $\overline{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}}^\sim$ and $\overline{\mathcal{C}''_k/\mathcal{C}''_{k-1}}^\sim$ the objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ defined by $\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k)}$ and $\mathcal{C}''_k/\mathcal{C}''_{k-1}$ equipped with the Hermitian seminorms such that the following diagram is an admissible short exact in $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}}^\sim \xrightarrow{i_k} \overline{\mathcal{C}_k/\mathcal{C}_{k-1}} \xrightarrow{p_k} \overline{\mathcal{C}''_k/\mathcal{C}''_{k-1}}^\sim \longrightarrow 0.$$

In other words, these seminorms are the ones that make $i_{k,x}$ and the transpose of $p_{k,x}$ isometries for every $x \in X(\mathbb{C})$.

Since φ satisfies **SubAdd**, the following inequality holds:

$$\varphi(\overline{\mathcal{C}_k/\mathcal{C}_{k-1}}) \leq \varphi(\overline{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}}^{\sim}) + \varphi(\overline{\mathcal{C}''_k/\mathcal{C}''_{k-1}}^{\sim}). \tag{4.5.30}$$

Moreover the morphism:

$$\mathrm{Id}_{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}} : \overline{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}} \longrightarrow \overline{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}}^{\sim}$$

has norms ≤ 1, and therefore, as φ satisfies $\mathbf{Mon}^1$, this implies the upper bound:

$$\varphi(\overline{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}}^{\sim}) \leq \varphi(\overline{\mathcal{C}'_{\iota(k)}/\mathcal{C}'_{\iota(k-1)}}). \tag{4.5.31}$$

We are going to show that, if the values $\iota(k)$ of ι are successively chosen sufficiently large, then we also have:

$$\varphi(\overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}}^{\sim}) \leq \varphi(\overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}}) + \varepsilon_k. \tag{4.5.32}$$

Together with (4.5.30) and (4.5.31), this will establish (4.5.28).

To achieve this, observe that $\overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}}$ and $\overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}}^{\sim}$ may be described by means of the following admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\tilde{\mathcal{C}}_{k-1} + i(\mathcal{F}')} \hookrightarrow \overline{\tilde{\mathcal{C}}_k + i(\mathcal{F}')} \longrightarrow \overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}} \longrightarrow 0,$$

and

$$0 \longrightarrow \overline{\tilde{\mathcal{C}}_{k-1} + i(\mathcal{C}_a)} \hookrightarrow \overline{\tilde{\mathcal{C}}_k + i(\mathcal{C}_a)} \longrightarrow \overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}}^{\sim} \longrightarrow 0, \tag{4.5.33}$$

where $a = \iota(k)$. It follows from Proposition 2.2.6 that the Hermitian seminorms of $\overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}}^{\sim}$ defined by the admissible short exact sequences (4.5.33) decrease when the integer a increases, and that they converge to the Hermitian seminorms of $\overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}}$ when a goes to infinity.[19] Since φ satisfies $\mathbf{Cont}^+$, this implies that $\varphi(\overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}}^{\sim})$ converges to $\varphi(\overline{\mathcal{C}''_{\iota(k)}/\mathcal{C}''_{\iota(k-1)}})$ when a goes to infinity, and therefore that (4.5.32) holds if $a = \iota(k)$ is chosen large enough.

Part (2) follows from part (1) applied to the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ defined by: $\varepsilon_k := 2^{-k-1}\varepsilon$. □

COROLLARY 4.5.14. *For any object $\overline{\mathcal{E}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ and any two $\mathcal{O}_K$-submodules $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{E}$, if $\overline{\mathcal{E}}_1$ and $\overline{\mathcal{E}}_2$ are φ-summable, then $\overline{\mathcal{E}_1 + \mathcal{E}_2}$ also is φ-summable.*

PROOF. This follows from the existence of the surjective morphism

$$\overline{\mathcal{E}}_2 \longrightarrow \overline{(\mathcal{E}_1 + \mathcal{E}_2)/\mathcal{E}_1}$$

in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ and of the admissible short exact sequence

$$0 \longrightarrow \overline{\mathcal{E}}_1 \longrightarrow \overline{\mathcal{E}_1 + \mathcal{E}_2} \longrightarrow \overline{(\mathcal{E}_1 + \mathcal{E}_2)/\mathcal{E}_1} \longrightarrow 0,$$

to which we may apply Propositions 4.5.9 and 4.5.12. □

COROLLARY 4.5.15. *Let us assume that φ satisfies condition* **VT**, *namely that it vanishes on torsion modules, or equivalently*[20] *condition $\mathbf{Mon}^1_K$. Then every torsion object in $\mathbf{q}\overline{\mathbf{Coh}}_X$ is φ-summable. More generally, if $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ such that the K-linear map $f_K : \mathcal{F}_K \to \mathcal{G}_K$ is surjective and if $\overline{\mathcal{F}}$ is φ-summable, then $\overline{\mathcal{G}}$ is φ-summable.*

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[19] Apply Proposition 2.2.6 to $\overline{\mathcal{E}} := \overline{\tilde{\mathcal{C}}_k + i(\mathcal{F}')}$, $\overline{\mathcal{F}} := \overline{\mathcal{F}}$, $\overline{\mathcal{G}} := \overline{\mathcal{F}''/\mathcal{C}''_{k-1}}$, $\mathcal{E}_n := \tilde{\mathcal{C}}_{k-1} + i(\mathcal{C}'_n)$, and $\mathcal{F}' := \tilde{\mathcal{C}}_k$.

[20] by Proposition 4.1.3.

PROOF. (1) If $\mathcal{F}$ is an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that the $\mathcal{O}_K$-module $\mathcal{F}$ is torsion, then for every exhaustive filtration $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$, the subquotients $\mathcal{C}_i/\mathcal{C}_{i-1}$ are torsion coherent modules, and therefore:

$$\varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) = 0$$

since φ satisfies $\mathbf{VT}$; consequently the condition $\mathbf{Sum}_\varphi(\mathcal{C}_\bullet)$ is verified, and $\overline{\mathcal{F}}$ is φ-summable.

(2) Let $\overline{\mathcal{G}}$ be a φ-summable object in $\mathbf{q}\overline{\mathbf{Coh}}_X$, and let $\mathcal{F}$ be some $\mathcal{O}_K$-submodule of $\mathcal{G}$ such that the quotient $\mathcal{O}_K$-module $\mathcal{G}/\mathcal{F}$ is torsion. Then we may consider the associated admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}/\mathcal{F}} \longrightarrow 0.$$

We have just shown that $\overline{\mathcal{G}/\mathcal{F}}$ is φ-summable. Therefore, according to Proposition 4.5.12, if $\overline{\mathcal{F}}$ is φ-summable, then $\overline{\mathcal{G}}$ also is.

This establishes the second assertion of the proposition when f is injective and isometric with torsion cokernel.

(3) In general, a morphism f such that f_K is surjective factors as

$$\overline{\mathcal{F}} \xrightarrow{f} \overline{\mathrm{im}\, f} \hookrightarrow \overline{\mathcal{G}},$$

where the quotient $\mathcal{G}/\mathrm{im}\, f$ is torsion. If moreover $\overline{\mathcal{G}}$ is φ-summable, then $\overline{\mathrm{im}\, f}$ is φ-summable by part (2), and $\overline{\mathcal{F}}$ is φ-summable by Proposition 4.5.9. $\square$

4.5.4. Downward continuity of $\overline{\varphi}$ on φ-summable Hermitian quasi-coherent sheaves.

PROPOSITION 4.5.16. *If φ satisfies $\mathbf{Cont}^+$ on $\overline{\mathbf{Coh}}_X$, then $\overline{\varphi}$ satisfies $\mathbf{Cont}^+$ on φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$.*

In other words, consider a sequence $(\overline{\mathcal{F}}_n)_{n\in\mathbb{N}}$ of objects

$$\overline{\mathcal{F}}_n := (\mathcal{F}, (\|.\|_{n,x})_{x\in X(\mathbb{C})})$$

in $\mathbf{q}\overline{\mathbf{Coh}}_X$ with the same underlying $\mathcal{O}_K$-modules $\mathcal{F}$ such that, for every $x \in X(\mathbb{C})$, the sequence of seminorms $(\|.\|_{n,x})_{n\in\mathbb{N}}$ is decreasing, and let us denote by

$$\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})}) \tag{4.5.34}$$

the object in $\mathbf{q}\overline{\mathbf{Coh}}_X$ attached to the Hermitian seminorms on the $\mathcal{F}_x$ defined as the pointwise limits:

$$\|.\|_x := \lim_{n\to+\infty} \|.\|_{n,x}. \tag{4.5.35}$$

Proposition 4.5.16 asserts that if $\overline{\mathcal{F}}_0$ is φ-summable, and therefore all the $\overline{\mathcal{F}}_n$ and $\overline{\mathcal{F}}$ by Proposition 4.5.9, then the decreasing sequence $(\overline{\varphi}(\overline{\mathcal{F}}_n))_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ satisfies:

$$\lim_{n\to+\infty} \overline{\varphi}(\overline{\mathcal{F}}_n) = \overline{\varphi}(\overline{\mathcal{F}}).$$

PROOF. We have to show that, for any given ε in $\mathbb{R}_+^*$, the estimate

$$\overline{\varphi}(\overline{\mathcal{F}}_n) < \overline{\varphi}(\overline{\mathcal{F}}) + \varepsilon$$

is satisfied when n is large enough. To achieve this, choose an exhaustive filtration $\mathcal{C}_\bullet$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ such that:

$$\Sigma_\varphi(\overline{\mathcal{F}}_0, \mathcal{C}_\bullet) := \sum_{i=0}^{+\infty} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}_0) < +\infty,$$

where $\mathcal{C}_{-1} := 0$ and where, for every $i \in \mathbb{N}$ and every $n \in \mathbb{N} \cup \{\sqcup\}$, we denote by $\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}_n$ the object of $\overline{\mathbf{Coh}}_X$ defined as $\mathcal{C}_i/\mathcal{C}_{i-1}$ equipped with the Hermitian seminorms deduced of the ones on $\overline{\mathcal{F}}_n$.

For every $i \in \mathbb{N}$, the sequence $(\varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}_n))_{n\in\mathbb{N}}$ is decreasing with limit $\varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}})$. Indeed the Hermitian seminorms on $\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}$ are the limits of the decreasing sequences of Hermitian seminorms on $\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}_n$ when n goes to infinity, and φ satisfies the condition $\mathbf{Cont}^+$ on $\overline{\mathbf{Coh}}_X$. Consequently $(\Sigma_\varphi(\overline{\mathcal{F}}_n, \mathcal{C}_\bullet))_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathbb{R}_+$, of limit $\Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet)$. In particular, when n is large enough, we have:

$$\Sigma_\varphi(\overline{\mathcal{F}}_n, \mathcal{C}_\bullet) < \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet) + \varepsilon/2.$$

Besides, as observed in (4.5.7) above, we also have:

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{k\to+\infty} \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_{\bullet+k}).$$

Consequently, after possibly replacing the filtration $\mathcal{C}_\bullet$ by $\mathcal{C}_{\bullet+k}$ with k large enough, we may assume:

$$\Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet) < \overline{\varphi}(\overline{\mathcal{F}}) + \varepsilon/2.$$

Therefore, when n is large enough, we have:

$$\overline{\varphi}(\mathcal{F}_n) \leq \Sigma_\varphi(\overline{\mathcal{F}}_n, \mathcal{C}_\bullet) < \overline{\varphi}(\overline{\mathcal{F}}) + \varepsilon. \qquad \square$$

4.5.5. General $\overline{\varphi}$-summable filtrations. By mimicking the proof of Theorem 4.5.1 and using properties of $\overline{\varphi}$ and $\mathrm{ev}\overline{\varphi}$ established in the previous paragraphs, we may prove the following continuity property of the invariant $\overline{\varphi}$ with respect to suitable exhaustive filtrations, which complements the general lower semicontinuity of $\overline{\varphi}$ established in Proposition 4.3.7.

PROPOSITION 4.5.17. *Let us moreover assume that φ satisfies $\mathbf{Cont}^+$ on $\overline{\mathbf{Coh}}_X$.*

Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and let $\mathcal{F}_\bullet := (\mathcal{F}_i)_{i\in\mathbb{N}}$ be an exhaustive filtration of $\mathcal{F}$ by $\mathcal{O}_K$-submodules. If the subquotients[21] *$\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}$ satisfy the condition of eventual vanishing:*

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}) = 0 \quad \textit{for every } i \in \mathbb{N}, \tag{4.5.36}$$

and the summability condition:

$$\mathbf{Sum}_{\overline{\varphi}}(\mathcal{F}_\bullet): \quad \sum_{i=0}^{+\infty} \overline{\varphi}(\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}) < +\infty,$$

then the sequence $(\overline{\varphi}(\overline{\mathcal{F}}_i))_{i\in\mathbb{N}}$ converges in $\mathbb{R}_+$, and we have:

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{i\to+\infty} \overline{\varphi}(\overline{\mathcal{F}}_i) \tag{4.5.37}$$

and

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}) = 0. \tag{4.5.38}$$

Observe that the vanishing conditions (4.5.36) are satisfied if the subquotients $\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}$ are φ-summable. It is plausible that, when the subquotients $\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}$ are φ-summable and the summability condition $\mathbf{Sum}_{\overline{\varphi}}(\mathcal{F}_\bullet)$ holds, then $\overline{\mathcal{F}}$ is actually φ-summable. In Section 5.6.1, Proposition 5.6.3, we will prove that it is indeed the case when the invariant φ satisfies a strengthened form of the subadditivity condition $\mathbf{SubAdd}$.

PROOF. Since the invariant φ satisfies the downward continuity condition $\mathbf{Cont}^+$, its upper extension $\overline{\varphi}$ also satisfies $\mathbf{SubAdd}$, as shown in Proposition 4.3.10. The subbaditivity of $\overline{\varphi}$ applied to the admissible short exact sequences

$$0 \longrightarrow \overline{\mathcal{F}}_{i-1} \longrightarrow \overline{\mathcal{F}}_i \longrightarrow \overline{\mathcal{F}_i/\mathcal{F}_{i-1}} \longrightarrow 0$$

shows that the following estimates holds for every $i \in \mathbb{N}$:

$$\overline{\varphi}(\overline{\mathcal{F}}_i) \leq \overline{\varphi}(\overline{\mathcal{F}}_{i-1}) + \overline{\varphi}(\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}). \tag{4.5.39}$$

Using these estimates, a straightforward variant of the proof of Lemma 4.5.7 establishes the following lemma:

[21] Hre again, by convention, we let: $\mathcal{F}_{-1} = 0$.

LEMMA 4.5.18. *The limit* $\lim_{i\to+\infty}\overline{\varphi}(\overline{\mathcal{F}}_i)$ *exists in* $\mathbb{R}_+$ *and, for any* $i\in\mathbb{N}$*, satisfies the inequality*

$$\lim_{k\to+\infty}\overline{\varphi}(\overline{\mathcal{F}}_k)\leq\varphi(\overline{\mathcal{F}}_i)+\sum_{k=i+1}^{+\infty}\varphi(\overline{\mathcal{F}_k/\mathcal{F}_{k-1}}).\tag{4.5.40}$$

Using Lemma 4.5.18 and a variant of the proof of Lemma 4.5.8, one then establishes the following lemma:

LEMMA 4.5.19. *For every* $\mathcal{C}$ *in* $\mathrm{coh}(\mathcal{F})$*, there exists* $\iota(\mathcal{C})\in\mathbb{N}$ *such that* $\mathcal{C}\subseteq\mathcal{F}_{\iota(\mathcal{C})}$*. The sequence* $(\varphi(\overline{\mathcal{F}_k/\mathcal{C}}))_{k\geq\iota(\mathcal{C})}$ *admits a limit* $l(\mathcal{C})$ *in* $\mathbb{R}_+$*. Moreover* $l(\mathcal{C})$ *is a decreasing function of* $\mathcal{C}$ *in the directed set* $(\mathrm{coh}(\mathcal{F}),\subseteq)$ *and satisfies:*

$$\lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{C})}l(\mathcal{C})=\inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{C})}l(\mathcal{C})=0.$$

Let us only indicate the minor modifications of the proof of Lemma 4.5.8 required for the derivation of Lemma 4.5.19.

The existence of the limit $l(\mathcal{C}):=\lim_{k\to+\infty}\overline{\varphi}(\overline{\mathcal{F}_K/\mathcal{C}})$ follows from Lemma 4.5.18 applied to $\overline{\mathcal{F}/\mathcal{C}}$ and to the filtration $(\mathcal{F}_k/\mathcal{C})_{k\geq\iota(\mathcal{C})}$ instead of $\overline{\mathcal{F}}$ and $\mathcal{F}_\bullet$. This limit satisfies the upper bound:

$$l(\mathcal{C})\leq\overline{\varphi}(\overline{\mathcal{F}_i/\mathcal{C}})+\sum_{k=i+1}^{+\infty}\overline{\varphi}(\overline{\mathcal{F}_k/\mathcal{F}_{k-1}})\tag{4.5.41}$$

for every $i\in\mathbb{N}$ such that $\mathcal{C}\subseteq\mathcal{F}_i$. The monotonicity of $\overline{\varphi}$ implies that the function

$$l:\mathrm{coh}(\mathcal{F})\longrightarrow\mathbb{R}_+$$

is decreasing, and therefore admits a limit over the directed set $(\mathrm{coh}(\mathcal{F}),\subseteq)$:

$$\lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})}l(\mathcal{C})=\inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})}l(\mathcal{C})\in\mathbb{R}_+.\tag{4.5.42}$$

Moreover the inequality (4.5.41) shows that, for every $i\in\mathbb{N}$:

$$\inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})}l(\mathcal{C})\leq\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}_i)+\sum_{k=i+1}^{+\infty}\overline{\varphi}(\overline{\mathcal{F}_k/\mathcal{F}_{k-1}}).\tag{4.5.43}$$

According to Corollary 4.4.3, the eventual vanishing of $\overline{\varphi}$ on the subquotients $\overline{\mathcal{F}_j/\mathcal{F}_{j-1}}$ implies:

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}_i})=0\quad\text{for every } i\in\mathbb{N},\tag{4.5.44}$$

by a straightforward induction argument. Together with (4.5.43) and the equality,

$$\lim_{i\to+\infty}\sum_{k=i+1}^{+\infty}\overline{\varphi}(\overline{\mathcal{F}_k/\mathcal{F}_{k-1}})=0,$$

this establishes the vanishing of (4.5.42):

$$\lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{C})}l(\mathcal{C})=\inf_{i\in\mathbb{N}}\inf_{\mathcal{C}\in\mathrm{coh}(\mathcal{C}_i)}l(\mathcal{C})=0,\tag{4.5.45}$$

and completes the proof of Lemma 4.5.19.

The subadditivity of $\overline{\varphi}$, applied to the admissible short exact sequence

$$0\longrightarrow\overline{\mathcal{C}}\longrightarrow\overline{\mathcal{F}}_k\longrightarrow\overline{\mathcal{F}_k/\mathcal{C}}\longrightarrow 0,$$

shows that the following inequality holds, for every $\mathcal{C}\in\mathrm{coh}(\mathcal{F})$ and every integer $k\geq\iota(\mathcal{C})$:

$$\overline{\varphi}(\overline{\mathcal{F}}_k)\leq\varphi(\overline{\mathcal{C}})+\varphi(\overline{\mathcal{F}_k/\mathcal{C}}).$$

By taking the limit when k goes to infinity, this establishes the estimate:

$$\lim_{k\to+\infty}\overline{\varphi}(\overline{\mathcal{F}}_k)\leq\varphi(\overline{\mathcal{C}})+l(\mathcal{C}),\tag{4.5.46}$$

valid for every $\mathcal{C} \in \mathrm{coh}(\mathcal{F})$.

From (4.5.45) and (4.5.46), we get:

$$\lim_{k \to +\infty} \overline{\varphi}(\overline{\mathcal{F}}_k) \leq \liminf_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}}).$$

This establishes the inequality:

$$\lim_{k \to +\infty} \overline{\varphi}(\overline{\mathcal{F}}_k) \leq \overline{\varphi}(\overline{\mathcal{F}}).$$

Together with Proposition 4.3.7, this completes the proof of (4.5.37).

Finally for any element $\mathcal{C}$ of $\mathrm{coh}(\mathcal{F})$, the equality (4.5.37) applied to $\overline{\mathcal{F}/\mathcal{C}}$ equipped with the filtration $\mathcal{F}_\bullet/\mathcal{C}$ shows that:

$$l(\mathcal{C}) := \lim_{k \to +\infty} \overline{\varphi}(\overline{\mathcal{F}_k/\mathcal{C}}) = \overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}}),$$

and consequently (4.5.45) establishes the vanishing (4.5.38) of $\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}})$. □

4.6. Compatibility with Direct Images

In practice invariants attached to Euclidean (quasi-)coherent sheaves — that is to Hermitian (quasi-)coherent sheaves over the arithmetic curve $X = \operatorname{Spec} \mathbb{Z}$ — play a special role. Most invariants attached to Hermitian (quasi-)coherent sheaves over a general arithmetic curve X are may indeed be constructed from invariants of Euclidean (quasi-)coherent sheaves by using the direct images functors:

$$\pi_* : \overline{\mathbf{Coh}}_X^{\leq 1} \longrightarrow \overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$$

and:

$$\pi_* : \mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1} \longrightarrow \mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$$

attached to the finite morphism:

$$\pi : X \longrightarrow \operatorname{Spec} \mathbb{Z}.$$

In this section, we briefly discuss this construction and its compatibility with the constructions investigated in this chapter.

4.6.1. The invariant $\pi^*\varphi$.

Definition 4.6.1. If φ is an invariant on $\overline{\mathbf{Vect}}_{\mathbb{Z}}$ (resp. $\overline{\mathbf{Coh}}_{\overline{\operatorname{Spec}\mathbb{Z}}}$, resp. $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, resp. $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$) with values in $\mathbb{R}_+$ (resp. in $[0, +\infty]$), we denote by $\pi^*\varphi$ the invariant on $\overline{\mathbf{Vect}}_X$ (resp. $\overline{\mathbf{Coh}}_{\overline{X}}$, resp. $\overline{\mathbf{Coh}}_X$, resp. $\mathbf{q}\overline{\mathbf{Coh}}_X$) defined by:

$$\pi^*\varphi(\overline{\mathcal{F}}) := \varphi(\pi_*\overline{\mathcal{F}}). \tag{4.6.1}$$

The following permanence properties of the construction of $\pi^*\varphi$ from φ are straightforward consequences of the definition.

Proposition 4.6.2. *For every invariant $\psi : \overline{\mathbf{Vect}}_{\mathbb{Z}} \to \mathbb{R}_+$ that satisfies conditions $\mathbf{Mon}^1_K$, $\mathbf{SubAdd}$, and $\mathbf{Cont}^+$, the invariant $\pi^*\psi : \overline{\mathbf{Vect}}_X \to \mathbb{R}_+$ also satisfies these conditions, and the invariants $\pi^*\widetilde{\psi}_{\mathrm{nst}}$ and $(\pi^*\psi)^\sim_{\mathrm{nst}}$ on $\overline{\mathbf{Coh}}_X$ coincide.*

When moreover:

$$\lim_{\delta \to +\infty} \psi(\overline{\mathcal{O}}_{\mathbb{Z}}(\delta)) = 0,$$

we also have:

$$\lim_{\delta \to +\infty} \pi^*\psi(\overline{\mathcal{O}}_X(\delta)) = 0.$$

PROPOSITION 4.6.3. *Let φ be an invariant of objects of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ (resp. $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$) with values in $[0, +\infty]$, and let $\pi^*\varphi$ the invariant of objects of $\overline{\mathbf{Coh}}_X$ (resp. $\mathbf{q}\overline{\mathbf{Coh}}_X$) defined by* (4.6.1).

If $\mathbf{P}$ is any of the following properties: $\mathbf{Mon}^1$, $\mathbf{SubAdd}$, $\mathbf{Cont}^+$, $\mathbf{VT}$, $\mathbf{NST}$, $\mathbf{VAp}$, $\mathbf{NSAp}$, $\mathbf{Add}_{\oplus}$, or $\mathbf{Max}_{\oplus}$, then the following implication holds:

$$\varphi \text{ satisfies } \mathbf{P} \Longrightarrow \pi^*\varphi \text{ satisfies } \mathbf{P}. \tag{4.6.2}$$

If φ satisfies $\mathbf{Mon}^1_{\mathbb{Q}}$, then $\pi^*\varphi$ satisfies $\mathbf{Mon}^1_K$.

If φ is small on Euclidean coherent sheaves generated by small sections, then $\pi^*\varphi$ is small on Hermitian coherent sheaves over X generated by small sections.

4.6.2. Lower and upper extensions and direct images. Let us consider an invariant of Euclidean coherent sheaves

$$\varphi_{\mathbb{Z}} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow [0, +\infty]$$

that satisfies the monotonicity condition $\mathbf{Mon}^1$. Then the invariant

$$\varphi_X := \pi^*\varphi_{\mathbb{Z}} : \overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

satisfies also $\mathbf{Mon}^1$, and we may consider the lower and upper extensions of $\varphi_{\mathbb{Z}}$ and φ_X, namely:

$$\underline{\varphi}_{\mathbb{Z}}, \overline{\varphi}_{\mathbb{Z}} : \mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow [0, +\infty],$$

and:

$$\underline{\varphi}_X, \overline{\varphi}_X : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty].$$

PROPOSITION 4.6.4. *With the above notation, for every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following inequalities hold:*

$$\underline{\varphi}_X(\overline{\mathcal{F}}) \leq \underline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}) \leq \overline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}) \leq \overline{\varphi}(\overline{\mathcal{F}}). \tag{4.6.3}$$

PROOF. For every $\mathcal{G}$ in $\mathrm{coft}(\mathcal{F})$, $\pi_*\mathcal{G}$ is an element of $\mathrm{coft}(\pi_*\mathcal{F})$, the quotient $\overline{\pi_*\mathcal{F}/\pi_*\mathcal{G}}$ is canonically isomorphic to $\pi_*\overline{\mathcal{F}/\mathcal{G}}$ and therefore:

$$h^1_\theta(\overline{\mathcal{F}/\mathcal{G}}) = h^1_\theta(\overline{\pi_*\mathcal{F}/\pi_*\mathcal{G}}). \tag{4.6.4}$$

By considering the supremum of (4.6.4) over $\mathcal{G}$ in $\mathrm{coft}(\mathcal{F})$, this implies the first inequality in (4.6.3).

The second inequality in (4.6.3) has been established in Proposition 4.3.4.

For every $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$, $\pi_*\mathcal{C}$ is an element of $\mathrm{coh}(\pi_*\mathcal{F})$ and satisfies:

$$h^1_\theta(\overline{\pi_*\mathcal{C}}) = h^1_\theta(\overline{\mathcal{C}}).$$

Moreover the image of the map so-defined:

$$\pi_* : \mathrm{coh}(\overline{\mathcal{F}}) \longrightarrow \mathrm{coh}(\pi_*\mathcal{F})$$

is cofinal in the directed set $(\mathrm{coh}(\pi_*\mathcal{F}), \subseteq)$. These observations imply the last inequality in (4.6.3). $\square$

COROLLARY 4.6.5. *Every object $\overline{\mathcal{F}}$ such that*

$$\underline{\varphi}_X(\overline{\mathcal{F}}) = \overline{\varphi}_X(\overline{\mathcal{F}}) < +\infty$$

satisfies also:

$$\underline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}) = \overline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}) < +\infty.$$

Moreover $\varphi(\overline{\mathcal{F}}) := \underline{\varphi}_X(\overline{\mathcal{F}}) = \overline{\varphi}_X(\overline{\mathcal{F}})$ and $\varphi_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}) := \underline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}) = \overline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}})$ coincide.

BEWEIS. Klar. $\square$

PROPOSITION 4.6.6. *Let us assume that, besides condition* $\mathbf{Mon}^1$, *the invariant* $\varphi_{\mathbb{Z}}$ *(and therefore also* φ_X*) satisfies condition* $\mathbf{NST}$. *Then, for every object* $\overline{\mathcal{F}}$ *of* $\mathbf{q}\overline{\mathbf{Coh}}_X$, *the following equality holds:*

$$\underline{\varphi}_X(\overline{\mathcal{F}}) = \underline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}). \tag{4.6.5}$$

PROOF. According to Proposition 4.3.12, the following equalities hold:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \sup_{\mathcal{G}\in \mathrm{scoft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{G}}) \tag{4.6.6}$$

and:

$$\underline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}) = \sup_{\mathcal{G}'\in \mathrm{scoft}(\pi_*\mathcal{F})} \varphi_{\mathbb{Z}}(\overline{\pi_*\mathcal{F}/\mathcal{G}'}). \tag{4.6.7}$$

The Euclidean quasi-coherent sheaves $\overline{\pi_*\mathcal{F}/\pi_*\mathcal{G}}$ is canonically isomorphic to $\pi_*\overline{\mathcal{F}/\mathcal{G}}$, and therefore $\varphi_{\mathbb{Z}}(\overline{\pi_*\mathcal{F}/\pi_*\mathcal{G}})$ and $\varphi(\overline{\mathcal{F}/\mathcal{G}})$ coincide. Consequently the equality (4.6.6) may be also be written:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \sup_{\mathcal{G}\in \mathrm{scoft}(\mathcal{F})} \varphi_{\mathbb{Z}}(\overline{\pi_*\mathcal{F}/\pi_*\mathcal{G}}) \tag{4.6.8}$$

Moreover, as shown in Corollary 1.3.6, the direct image functor π_* defines a map

$$\pi_* : \mathrm{scoft}(\mathcal{F}) \longrightarrow \mathrm{scoft}(\pi_*\mathcal{F}),$$

the image of which is cofinal in the directed set $(\mathrm{scoft}(\pi_*\mathcal{F}), \supseteq)$. Together with the expressions (4.6.7) and (4.6.8) for $\underline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}})$ and $\underline{\varphi}(\overline{\mathcal{F}})$, this establishes their equality. □

4.6.3. The construction ev **and direct images.** Let us consider an invariant of Euclidean quasi-coherent sheaves

$$\psi_{\mathbb{Z}} : \mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow [0, +\infty]$$

that satisfies the monotonicity condition $\mathbf{Mon}^1$. Then the invariant

$$\psi_X := \pi^*\psi_{\mathbb{Z}} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

satisfies also $\mathbf{Mon}^1$, and we may apply the construction ev both to $\psi_{\mathbb{Z}}$ and ψ_X and define:

$$\mathrm{ev}\psi_{\mathbb{Z}} : \mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow [0, +\infty] \quad \text{ and } \quad \psi_X := \pi^*\psi_{\mathbb{Z}} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty].$$

PROPOSITION 4.6.7. *With the above notation, for every object* $\overline{\mathcal{F}}$ *of* $\mathbf{q}\overline{\mathbf{Coh}}_X$, *the following equality holds:*

$$\mathrm{ev}\psi_X(\overline{\mathcal{F}}) = \mathrm{ev}\psi_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}).$$

PROOF. This follows from the definitions of $\mathrm{ev}\psi_X(\overline{\mathcal{F}})$ and $\mathrm{ev}\psi_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}})$ as limits over the directed sets $(\mathrm{coh}(\mathcal{F}), \subseteq)$ and $(\mathrm{coh}(\pi_*\mathcal{F}), \subseteq)$, and from the fact that the "inclusion map"

$$\pi_* : \mathrm{coh}(\mathcal{F}) \longrightarrow \mathrm{coh}(\pi_*\mathcal{F})$$

is trivially order preserving, of image cofinal in $(\mathrm{coh}(\pi_*\mathcal{F}), \subseteq)$ since the $\mathcal{O}_K$-module generated by some finitely generated $\mathbb{Z}$-submodule of $\mathcal{F}(X)$ is finitely generated. □

4.6.4. φ**-summability and direct images.** Let us consider an invariant of Euclidean coherent sheaves

$$\varphi_{\mathbb{Z}} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow [0, +\infty]$$

that satisfies the monotonicity and subadditivity conditions $\mathbf{Mon}^1$ and $\mathbf{SubAdd}$. Then the invariant

$$\varphi_X := \pi^*\varphi_{\mathbb{Z}} : \overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

satisfies also $\mathbf{Mon}^1$ and $\mathbf{SubAdd}$, and we may introduce the categories $\varphi_\Sigma - \mathbf{q}\overline{\mathbf{Coh}}\mathbb{Z}$ and $\varphi_\Sigma - \mathbf{q}\overline{\mathbf{Coh}}_X$ of $\varphi_{\mathbb{Z}}$-summable and φ_X-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$.

PROPOSITION 4.6.8. *If $\overline{\mathcal{F}}$ is a φ_X-summable object in $\mathbf{q}\overline{\mathbf{Coh}}_X$, then $\pi_*\overline{\mathcal{F}}$ is a $\varphi_{\mathbb{Z}}$-summable object in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and the following equality holds:*

$$\overline{\varphi}_X(\overline{\mathcal{F}}) = \overline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}}). \tag{4.6.9}$$

PROOF. Let us assume that $\overline{\mathcal{F}}$ is φ_X-summable, and let us consider an exhaustive filtration $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ that satisfies the summability condition:

$$\mathbf{Sum}_{\varphi_X}(\overline{\mathcal{F}}, \mathcal{C}_\bullet): \quad \sum_{i=0}^{+\infty} \varphi_X(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty. \tag{4.6.10}$$

Then $\pi_*\mathcal{C}_\bullet := (\pi_*\mathcal{C}_i)_{i\in\mathbb{N}}$ is an exhaustive filtration of $\pi_*\mathcal{F}$ by elements of $\mathrm{coh}(\pi_*\mathcal{F})$. Moreover for every $i \in \mathbb{N}$, we have:

$$\varphi_{\mathbb{Z}}(\overline{\pi_*\mathcal{C}_i/\pi_*\mathcal{C}_{i-1}}) = \varphi_{\mathbb{Z}}(\pi_*\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) = \varphi_X(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}).$$

Therefore the summability condition

$$\mathbf{Sum}_{\varphi_{\mathbb{Z}}}(\pi_*\overline{\mathcal{F}}, \pi_*\mathcal{C}_\bullet): \quad \sum_{i=0}^{+\infty} \varphi_{\mathbb{Z}}(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty$$

is satisfied, and $\pi_*\overline{\mathcal{F}}$ is $\varphi_{\mathbb{Z}}$-summable.

Moreover, according to Theorem 4.5.1, $\overline{\varphi}_X(\overline{\mathcal{F}})$ and $\overline{\varphi}_{\mathbb{Z}}(\pi_*\overline{\mathcal{F}})$ satisfy:

$$\overline{\varphi}_X(\overline{\mathcal{F}}) = \lim_{i\to+\infty} \varphi_X(\overline{\mathcal{C}}_i)$$

and:

$$\overline{\varphi}_Z(\pi_*\overline{\mathcal{F}}) = \lim_{i\to+\infty} \varphi_{\mathbb{Z}}(\pi_*\overline{\mathcal{C}}_i),$$

and therefore coincide. $\square$

CHAPTER 5

Positive Invariants of Hermitian Quasi-coherent Sheaves over Arithmetic Curves II: Rank of Morphisms and Strong Monotonicity

As in the previous chapters, we denote by K a number field, by $\mathcal{O}_K$ its ring of integers, and by X the arithmetic curve $\operatorname{Spec} \mathcal{O}_K$.

5.0.1. In this chapter, we pursue the study of positive invariants on the category $\overline{\mathbf{Coh}}_X$ and of their extensions to $\mathbf{q}\overline{\mathbf{Coh}}_X$ initiated in Chapter 4.

As before, we are interested in invariants that share the formal properties of the invariant:

$$h^1(C, .) : \mathbf{Coh}_C \longrightarrow \mathbb{N}, \quad \mathcal{F} \longmapsto h^1(C, \mathcal{F}) := \dim_k H^1(C, \mathcal{F}) \tag{5.0.1}$$

on the category $\mathbf{Coh}_C$ of coherent $\mathcal{O}_C$-modules over a smooth projective curve over some base field k. Among these properties, there are some "obvious" ones, like the monotonicity $\mathbf{Mon}^1$ or the subadditivity **SubAdd**, which are straightforward consequences of the vanishing of the cohomology of coherent $\mathcal{O}_C$-modules in degree > 1. In Chapter 4, we focused on the consequences of these properties for an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$, and we showed that, combined with a mild continuity assumption, the downward continuity condition $\mathbf{Cont}^+$, they allow us to define some natural lower and upper extension $\underline{\varphi}$ and $\overline{\varphi}$ to $\mathbf{q}\overline{\mathbf{Coh}}_X$ and to introduce a natural subcategory φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_\mathbb{Z}$ of φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$, on which the upper-extension $\overline{\varphi}$ may be computed be a simple limit procedure.

In this chapter, we introduce some further property of an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$, the *strong monotonicity*. We show that when φ satisfies it, its extensions $\underline{\varphi}$ and $\overline{\varphi}$ and the φ-summable objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ are especially well-behaved.

A concise way to define the strong monotonicity of the invariant φ is as follows.

Recall that φ satisfies the monotonicity condition $\mathbf{Mon^1}$ when, for every morphism $f : \overline{\mathcal{E}} \to \overline{\mathcal{F}}$ in $\overline{\mathbf{Coh}}_X^{\leq 1}$, the following implication holds:

$$f(\mathcal{E}) = \mathcal{F} \Longrightarrow \varphi(\overline{\mathcal{E}}) \geq \varphi(\overline{\mathcal{F}}). \tag{5.0.2}$$

This condition may be reformulated as asserting that, for every object $\overline{\mathcal{E}} := (\mathcal{E}, (\|.\|_x)_{x \in X(\mathbb{C})})$ of $\overline{\mathbf{Coh}}_X$ and every $\mathcal{O}_K$-submodule $\mathcal{E}'$ of $\mathcal{E}$, the following inequality holds:

$$\varphi(\overline{\mathcal{E}}) - \varphi(\overline{\mathcal{E}/\mathcal{E}'}) \geq 0, \tag{5.0.3}$$

and moreover, that $\varphi(\overline{\mathcal{E}})$ is an increasing function of the seminorms $(\|.\|_x)_{x \in X(\mathbb{C})}$.

We shall say that φ is strongly monotonic when, for every $\overline{\mathcal{E}}$ and $\mathcal{E}'$ as above and every morphism $f : \overline{\mathcal{E}} \longrightarrow \overline{\mathcal{F}}$ in $\overline{\mathbf{Coh}}_X^{\leq 1}$, the following inequality holds:

$$\varphi(\overline{\mathcal{E}}) - \varphi(\overline{\mathcal{E}/\mathcal{E}'}) \geq \varphi(\overline{\mathcal{F}}) - \varphi(\overline{\mathcal{F}/\mathcal{F}'}), \tag{5.0.4}$$

where $\mathcal{F}' := f(\mathcal{E}')$ denote the image of $\mathcal{E}'$ by f.

Applied with $\mathcal{E}' = \mathcal{E}$, the strong monotonicity condition (5.0.4) implies the monotonicity (5.0.2). The inequality (5.0.3) is also the special case of (5.0.4) when $\overline{\mathcal{F}} = \overline{0}$. Thus the strong monotonicity condition (5.0.4) constitutes a strengthening of the monotonicity condition $\mathbf{Mon^1}$, which involves arbitrary morphisms in $\overline{\mathbf{Coh}}_X^{\leq 1}$.

5.0.2. The first two sections of this chapter are devoted to diverse formulations of the strong monotonicity condition.

In Section 5.1, in the spirit of the work of McMurray Price [**MP17**],[1] we associate its *φ-rank* to every monotonic invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$. It is the map rk^1_φ that attaches to a morphism $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ in $\overline{\mathbf{Coh}}_X^{\leq 1}$ the non-negative real number:

$$\mathrm{rk}^1_\varphi(f) := \varphi(\overline{\mathcal{G}}) - \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}).$$

If the invariant φ is thought of as an arithmetic counterpart of the geometric invariant:

$$h^1(C, .) : \mathbf{Coh}_C \longrightarrow \mathbb{N}, \quad \mathcal{F} \longmapsto h^1(C, \mathcal{F}) := \dim_k H^1(C, \mathcal{F})$$

on the category $\mathbf{Coh}_C$ of coherent $\mathcal{O}_C$-modules over a smooth projective curve C defined over some base field k, then $\mathrm{rk}^1_\varphi f$ is the analogue of the rank $\mathrm{rk}_k H^1(C, f)$ of the k-linear map:

$$H^1(C, f) : H^1(C, \mathcal{F}) \longrightarrow H^1(C, \mathcal{G})$$

associated to a morphism $f : \mathcal{F} \longrightarrow \mathcal{G}$ in $\mathbf{Coh}_C$.

The strong monotonicity of φ is defined in Section 5.2 by the compatibility of the φ-rank rk^1_φ with the composition of morphisms. Namely a monotonic invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ is strictly monotonic when the inequality:

$$\mathrm{rk}^1_\varphi(g \circ f) \leq \min(\mathrm{rk}^1_\varphi f, \mathrm{rk}^1_\varphi g) \tag{5.0.5}$$

is satisfied for any two composable morphisms f and g in $\overline{\mathbf{Coh}}_X^{\leq 1}$. We refer the reader to Subsection 3.1.2 for a discussion of the properties of the invariant $h^1(C, .)$ on $\overline{\mathbf{Coh}}_C$ which sheds some light on the equivalence of the formulations (5.0.4) and (5.0.5) of the strong monotonicity of φ.

The strong monotonicity of the invariant φ actually implies a strengthened form of the subadditivity **SubAdd** of φ, and some other remarkable properties as well, the *submodularity* and *metric monotonicity* properties, which we also discuss in Section 5.2. The particular significance of the submodularity condition for establishing that an invariant on $\overline{\mathbf{Coh}}_X$ is strongly monotonic confirms the analogy between our formalism and the one of measure and capacity theory; see Proposition 5.2.9 and 5.2.2.3 below.

5.0.3. Examples of non-zero strictly monotonic invariants are easily constructed. For instance, the rank:

$$\mathrm{rk} : \overline{\mathbf{Coh}} \longrightarrow \mathbb{N}, \quad \overline{\mathcal{F}} \longmapsto \mathrm{rk}\mathcal{F} := \dim_K \mathcal{F}_K,$$

and, for any non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, of residue field $\mathbb{F}_\mathfrak{p} := \mathcal{O}_K/\mathfrak{p}$, the dimension of the reduction modulo $\mathfrak{p}$:

$$\mathrm{rk}_{\cdot\mathbb{F}_\mathfrak{p}} : \overline{\mathcal{F}} \longmapsto \dim_{\mathbb{F}_\mathfrak{p}} \mathcal{F}_{\mathbb{F}_\mathfrak{p}}$$

are strongly monotonic. Indeed the right-exactness of the functors $. \otimes_{\mathcal{O}_K} K$ and $. \otimes_{\mathcal{O}_K} \mathbb{F}_\mathfrak{p}$ implies that the associated φ-ranks are respectively:

$$\mathrm{rk}^1_\varphi(f) = \mathrm{rk}_K f_K$$

and

$$\mathrm{rk}^1_\varphi(f) := \mathrm{rk}_{\mathbb{F}_\mathfrak{p}} f_{\mathbb{F}\mathfrak{p}};$$

they clearly satisfy the inequality (5.0.5).

However the existence of strictly monotonic invariants vanishing on torsion sheaves distinct from a multiple of the rank rk is not clear a priori. For instance, as will be shown in Chapter 6,

[1] MacMurray Price works with invariants that are counterparts of the invariant $h^0(C, .)$ on $\mathbf{Coh}_C$ — where C is a smooth projective curve over some field — instead of $h^1(C, .)$.

the invariant ρ^2, defined as the square of the covering radius, is *not* strictly monotonic, although it satisfies $\mathbf{Mon}^1_K$ and $\mathbf{SubAdd}$.

It turns out that the invariant:

$$h^1_\theta : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow \mathbb{R}_+$$

defined by the relation:

$$h^1_\theta(\overline{E}) := \log \sum_{v \in E^\vee} e^{-\pi \|v\|^2_{\overline{E}^\vee}}$$

is strongly monotonic on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. This is a non-trivial result, whose proof relies on some remarkable estimates satisfied by the theta series of Euclidean lattices that have been established by Banaszczyk[2] and by Regev and Stephens-Davidowitz.[3] A proof that the invariant h^1_θ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ is strictly monotonic will be presented in detail in Chapter 7. In Section 5.3, taking this fact for granted, we discuss various constructions of strongly monotonic invariants on $\overline{\mathbf{Coh}}_X$.

5.0.4. The last three sections of this chapter are devoted to the consequences of the strong monotonicity of some invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ concerning its lower and upper extensions $\underline{\varphi}$ and $\overline{\varphi}$ and the category φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ of φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$.

Notably in Sections 5.4 and 5.5, we show that, when φ is strongly monotonic, the φ-rank rk^1_φ admits some natural lower and upper extensions $\underline{\mathrm{rk}}^1_\varphi$ and $\overline{\mathrm{rk}}^1_\varphi$ defined on the morphisms of $\mathbf{q}\overline{\mathbf{Coh}}_X$. These extensions satisfy properties similar to the ones of rk^1_φ, which imply that $\underline{\varphi}$ and $\overline{\varphi}$ are strongly monotonic on suitable subcategories of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

Finally in Section 5.6, under the assumption of strong monotonicity on φ, we establish some important complements to the results on φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ presented in Section 4.5. For instance we have shown in Theorem 4.5.1 that, when φ satisfies $\mathbf{Mon}^1$ and $\mathbf{SubAdd}$, a φ-summable object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfies:

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}) = 0. \tag{5.0.6}$$

In Subsection 5.6.1, we show that, when φ is strictly monotonic, the vanishing condition (5.0.6) conversely implies the φ-summability of $\overline{\mathcal{F}}$, and we derive some remarkable permanence properties of the φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$. Finally in Subsection 5.6.3, we consider the objects of $\overline{\mathcal{F}}$ such that:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \overline{\varphi}(\overline{\mathcal{F}}) < +\infty. \tag{5.0.7}$$

We show that, when φ is strongly monotonic, they satisfy (5.0.6) and therefore are φ-summable.

These results in Section 5.6 are the most advanced results in our axiomatic investigation of the extensions to $\mathbf{q}\overline{\mathbf{Coh}}_X$ of invariants defined on $\overline{\mathbf{Coh}}_X$ pursued in Chapter 4 and in this one. When comparing these results and the properties of the invariants $\underline{h}^1(C,.) = h^1(C,.)$ and $\overline{h}^1(C,.)$ investigated in Chapter 3, notably in Theorem 3.2.7, it is natural to ask for simple sufficient criteria on some object $\overline{\mathcal{F}}$ of φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ that would imply the validity of (5.0.7).

Our axiomatic formalism does not appear to cover such criteria. However when φ is the θ-invariant h^1_θ, we will be able to complete the results of Section 5.6 by the converse implication:

$$\text{If } \overline{\mathcal{F}} \text{ is } \varphi\text{-summable, then for every } \delta \in \mathbb{R}^*_+,\ \underline{\varphi}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\delta)) = \overline{\varphi}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\delta)) < +\infty.$$

It will be established in Section 8.4, by means of a detailed analysis of the invariant h^1_θ and of its extensions $\underline{h}^1_\theta$ and $\overline{h}^1_\theta$.

As mentioned in Subsection 5.3, the various strong monotonicity properties considered in this chapter are compatible with the construction of invariants by direct images discussed in Section 4.6.

[2]See for instance [**Ban00**, Section 4]

[3]See [**RSD17a**]. This work has been motivated by a question of MacMurray Price; see also [**MP17**].

We leave it to the reader to formulate the compatibility with direct images of the rank rk^1_φ and of its extensions $\underline{\mathrm{rk}}^1_\varphi$ and $\overline{\mathrm{rk}}^1_\varphi$.

5.1. The φ-Rank rk^1_φ Associated to a Monotonic Invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$

In this section, we consider an invariant of Hermitian quasi-coherent sheaves over X with values in $\mathbb{R}_+$:

$$\varphi : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+,$$

satisfying the monotonicity condition $\mathbf{Mon^1}$ introduced in 4.1.2 above.

5.1.1. The φ-rank $\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}})$.

5.1.1.1. *Geometric preliminary.* In Section 3.1, we considered the numerical invariant

$$h^1(C, \mathcal{F}) := \dim_k H^1(C, \mathcal{F})$$

attached to a coherent sheaf $\mathcal{F}$ over a smooth projective curve C defined over some base field k, and we investigated various inequalities satisfied by this invariant. It turned out to be useful to introduce the rank $\mathrm{rk}_k H^1(C, \alpha)$ of the morphism of finite dimensional k-vector spaces:

$$H^1(C, \alpha) : H^1(C, \mathcal{F}) \longrightarrow H^1(C, \mathcal{G})$$

associated to a morphism of coherent $\mathcal{O}_C$-modules:

$$\alpha : \mathcal{F} \longrightarrow \mathcal{G}.$$

We notably observed the relation:

$$\mathrm{rk}_k H^1(C, \alpha) = h^1(C, \mathcal{G}) - h^1(C, \mathcal{G}/\mathrm{im}\, \alpha), \tag{5.1.1}$$

and the fact that $\mathrm{rk}_k H^1(C, \alpha)$ only depends on $\mathcal{G}$ and on the image $\mathrm{im}\, \alpha_\eta$ of α at the generic point η of C. Moreover, in combination with the expression (5.1.1) for the rank of the induced morphism between cohomology groups, the fact that, for every pair (α, β) of composable morphisms of coherent $\mathcal{O}_C$-modules:

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H},$$

the rank of the composed morphism:

$$H^1(C, \beta \circ \alpha) = H^1(C, \beta) \circ H^1(C, \alpha)$$

satisfies the (obvious) upper-bound:

$$\mathrm{rk}_k H^1(C, \beta \circ \alpha) \leq \min(\mathrm{rk}_k H^1(C, \beta), \mathrm{rk}_k H^1(C, \alpha))$$

was shown to lead to non trivial inequalities involving the invariant $h^1(C, .)$ on $\mathbf{Coh}_C$; see Subsection 3.1.2, notably Proposition 3.1.2, and (3.1.11) and (3.1.11).

5.1.1.2. *The φ-rank of a morphism in $\overline{\mathbf{Coh}}_X$.* In Chapter 4, we investigated positive invariants on $\overline{\mathbf{Coh}}_X$ and properties of those that are formally analogue to the one satisfied by the invariant

$$h^1(C, .) : \mathbf{Coh}_C \longrightarrow \mathbb{N}.$$

From this perspective, the discussion in 5.1.1.1 suggests to associate to any morphism

$$f : \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}}$$

in $\overline{\mathbf{Coh}}_X^{\leq 1}$ its φ-rank defined as the non-negative real number:

$$\mathrm{rk}^1_\varphi(f) := \varphi(\overline{\mathcal{G}}) - \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}), \tag{5.1.2}$$

where

$$\mathrm{im}\, f := f(\mathcal{F}),$$

and to study its properties.

The right-hand side of (5.1.2) is clearly independent of the Hermitian structure on $\overline{\mathcal{F}}$. It makes sense when $\mathcal{F}$ is an object of $\mathbf{Coh}_X$, $\overline{\mathcal{G}}$ an object in $\overline{\mathbf{Coh}}_X$, and $f : \mathcal{F} \to \mathcal{G}$ a morphism of $\mathcal{O}_K$-modules from $\mathcal{F}$ to the $\mathcal{O}_K$-module $\mathcal{G}$ underlying $\overline{\mathcal{G}}$. Such data may will be summarized by the sentence: *$f : \mathcal{F} \to \overline{\mathcal{G}}$ is a morphism in $\ddot{\mathbf{Coh}}_X$*. Using this terminology, we may introduce the following definition:

DEFINITION 5.1.1. The *φ-rank* of a morphism $f : \mathcal{F} \longrightarrow \overline{\mathcal{G}}$ in $\ddot{\mathbf{Coh}}_X$, is the real number:

$$\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) := \varphi(\overline{\mathcal{G}}) - \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}).$$

When no confusion may arise, we will write $\mathrm{rk}^1_\varphi(f)$ instead of $\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}})$.

The above definition in terms of morphisms in $\ddot{\mathbf{Coh}}_X$ does not actually enlarge the scope of the previous definition concerning morphisms in $\overline{\mathbf{Coh}}^{\leq 1}_X$. Indeed for any morphism $f : \mathcal{F} \to \overline{\mathcal{G}}$ in $\ddot{\mathbf{Coh}}_X$, if we let $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x \in X(\mathbb{C})})$ for some large enough Hermitian norms $(\|.\|_x)_{x\in X(\mathbb{C})}$ on the $\mathbb{C}$-vector spaces $(\mathcal{F}_x)_{x\in X(\mathbb{C})}$, then the map f becomes a morphism $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ in $\overline{\mathbf{Coh}}^{\leq 1}_X$. However working with morphisms in $\ddot{\mathbf{Coh}}_X$ makes the formalism of the φ-rank more flexible, and its formal properties clearer.

We will often use the (obvious) fact that the φ-rank of a morphism in $\ddot{\mathbf{Coh}}_X$ is unchanged when it is replaced by the inclusion morphism of its image. Namely, with the notation of Definition 5.1.1, we have:

$$\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) = \mathrm{rk}^1_\varphi(\iota : \mathrm{im}\, f \to \overline{\mathcal{G}}),$$

where $\iota : \mathrm{im}\, f \hookrightarrow \mathcal{G}$ denotes the inclusion morphism.

One easily sees that, if the invariant φ "does not see torsion" — that is, if it satisfies the condition **NST** introduced in 4.1.4 above — then the φ-rank $\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}})$ depends only on $\mathcal{G}$ and the K-vector subspace $f_K(\mathcal{F}_K)$ of $\mathcal{G}_K$. This is analogue to the fact, recalled in 5.1.1.1, that $\mathrm{rk}_k H^1(C, \alpha)$ depends only of $\mathcal{G}$ and $\mathrm{im}\, \alpha_\eta$.

5.1.2. Basic properties of rk^1_φ.

5.1.2.1. For every morphism of coherent $\mathcal{O}_C$-modules $f : \mathcal{F} \to \mathcal{G}$ as in 5.1.1.1 above, the rank of the k-linear map:

$$H^1(C, f) : H^1(C, \mathcal{F}) \longrightarrow H^1(C, \mathcal{G})$$

is clearly non-negative and bounded from above by $\max(h^1(C, \mathcal{F}), h^1(C, \mathcal{G}))$.

The φ-rank $\mathrm{rk}^1_\varphi(f)$ defined above satisfies similar properties:

PROPOSITION 5.1.2. *For any morphism $f : \mathcal{F} \to \overline{\mathcal{G}}$ in $\ddot{\mathbf{Coh}}_X$, the following estimates hold:*

$$0 \leq \mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) \leq \varphi(\overline{\mathcal{G}}). \tag{5.1.3}$$

When moreover the invariant φ is subadditive,[4] *for any morphism $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ in $\overline{\mathbf{Coh}}^{\leq 1}_X$, we also have:*

$$\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) \leq \varphi(\overline{\mathcal{F}}). \tag{5.1.4}$$

PROOF. The fact that $\mathrm{rk}^1_\varphi(f)$ is nonnegative is a direct consequence of the monotonicity of φ. The inequality

$$\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) \leq \mathrm{rk}^1_\varphi(\overline{G})$$

is obvious as $\varphi(\overline{\mathcal{G}/\mathrm{im}\, f})$ is nonnegative.

[4] Namely when φ satisfies the condition **SubAdd** introduced in 4.1.2 above.

Assume that φ is subadditive and that $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ is a morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$. Then, from the admissible exact sequence in $\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\operatorname{im} f} \longrightarrow \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}/\operatorname{im} f} \longrightarrow 0$$

we derive the estimate:

$$\varphi(\overline{\mathcal{G}}) \leq \varphi(\overline{\mathcal{G}/\operatorname{im} f}) + \varphi(\overline{\operatorname{im} f}).$$

Besides, the monotonicity of φ applied to the surjective morphism

$$\overline{\mathcal{F}} \longrightarrow \overline{\operatorname{im} f}$$

in $\overline{\mathbf{Coh}}_X^{\leq 1}$ implies the inequality:

$$\varphi(\overline{\operatorname{im} f}) \leq \overline{\mathcal{F}}.$$

This establishes the estimate:

$$\varphi(\overline{\mathcal{G}}) \leq \varphi(\overline{\mathcal{G}/\operatorname{im} f}) + \varphi(\overline{\mathcal{F}}),$$

or equivalently:

$$\operatorname{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) \leq \varphi(\overline{\mathcal{F}}).$$

$\square$

5.1.2.2. *Morphisms and diagrams in* $\overset{....}{\mathbf{Coh}}_X$, $\overset{....}{\mathbf{Coh}}{}_X^{\leq 1}$, $\mathbf{q}\overset{....}{\mathbf{Coh}}_X$, $\mathbf{q}\overset{....}{\mathbf{Coh}}{}_X^{\leq 1}$. When investigating the properties of the φ-rank $\operatorname{rk}^1_\varphi$, we will consider diagrams involving simultaneously morphisms in $\overset{....}{\mathbf{Coh}}_X$ as defined above, morphisms of coherent $\mathcal{O}_X$-modules, and morphisms in $\overline{\mathbf{Coh}}_X^{\leq 1}$. It will be convenient to talk of *(commutative) diagrams in* $\overset{....}{\mathbf{Coh}}_X$ or *in* $\overset{....}{\mathbf{Coh}}{}_X^{\leq 1}$ when dealing with such diagrams.

For instance, we shall say that

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \overline{\mathcal{H}}$$

is a diagram in $\overset{....}{\mathbf{Coh}}_X$ to mean that $f : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules and that $g : \mathcal{G} \to \overline{\mathcal{H}}$ is a morphism in $\overset{....}{\mathbf{Coh}}_X$ as defined in 5.1.1.2 above. Similarly, we shall say that

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}$$

is a diagram in $\overset{....}{\mathbf{Coh}}{}_X^{\leq 1}$ to mean that $f : \mathcal{F} \to \overline{\mathcal{G}}$ is a morphism in $\overset{....}{\mathbf{Coh}}_X$ and that $g : \overline{\mathcal{G}} \to \overline{\mathcal{H}}$ is a morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$.

In the same vein, by *a commutative diagram in* $\overset{....}{\mathbf{Coh}}{}_X^{\leq 1}$ *of the form*:

$$\begin{tikzcd}
\mathcal{F}_1 \arrow[r, "f_1"] \arrow[d, "p"] & \overline{\mathcal{G}}_1 \arrow[d, "q"] \\
\mathcal{F}_2 \arrow[r, "f_2"] & \overline{\mathcal{G}}_2,
\end{tikzcd}$$

we shall mean the data of a morphism:

$$q : \overline{\mathcal{G}}_1 \longrightarrow \overline{\mathcal{G}}_2$$

in $\overline{\mathbf{Coh}}_X^{\leq 1}$ and of a diagram:

$$\begin{tikzcd}
\mathcal{F}_1 \arrow[r, "f_1"] \arrow[d, "p"] & \mathcal{G}_1 \\
\mathcal{F}_2 \arrow[r, "f_2"] & \mathcal{G}_2.
\end{tikzcd}$$

in $\mathbf{Coh}_X$, such that the following diagram in $\mathbf{Coh}_X$ is commutative:

$$\begin{tikzcd} \mathcal{F}_1 \arrow[r, "f_1"] \arrow[d, "p"] & \mathcal{G}_1 \arrow[d, "q"] \\ \mathcal{F}_2 \arrow[r, "f_2"] & \mathcal{G}_2. \end{tikzcd}$$

Similarly we may consider *morphisms in* $\mathbf{q}\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}_X$, and *(commutative) diagrams in* $\mathbf{q}\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}_X$ *and in* $\mathbf{q}\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}{}_X^{\leq 1}$, by replacing coherent by countably generated quasi-coherent $\mathcal{O}_X$-modules and $\overline{\mathbf{Coh}}_X$ by $\mathbf{q}\overline{\mathbf{Coh}}_X$ in the above discussion.

5.1.2.3. In the geometric framework of 5.1.1.1, for any two composable morphisms

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

in $\mathbf{Coh}_C$, the rank of the k-linear map

$$H^1(C, g \circ f) = H^1(C, g) \circ H^1(C, f) : H^1(C, \mathcal{F}) \longrightarrow H^1(C, \mathcal{H})$$

clearly satisfies the following inequality:

$$\mathrm{rk} H^1(C, g \circ f) \leq \min(\mathrm{rk} H^1(C, g), \mathrm{rk} H^1(C, f)). \tag{5.1.5}$$

The following proposition asserts that an analogue of the inequality:

$$\mathrm{rk} H^1(C, g \circ f) \leq \mathrm{rk} H^1(C, g)$$

is satisfied by the φ-rank rk^1_φ. The analogue of the inequality:

$$\mathrm{rk} H^1(C, g \circ f) \leq \mathrm{rk} H^1(C, f) \tag{5.1.6}$$

will be investigated in the next sections: it will define the *strong monotonicity condition* which constitutes the subject of this chapter.

PROPOSITION 5.1.3. *(1) For any diagram in* $\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}_X$ *of the form:*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \overline{\mathcal{H}},$$

the following inequality holds:

$$\mathrm{rk}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \leq \mathrm{rk}^1_\varphi(g : \mathcal{G} \to \overline{\mathcal{H}}). \tag{5.1.7}$$

(2) If moreover the morphism of $\mathcal{O}_K$*-modules* $f : \mathcal{F} \longrightarrow \mathcal{G}$ *is surjective, or if* φ *satisfies condition* **NST** *and* $f_K : \mathcal{F}_K \longrightarrow \mathcal{G}_K$ *is a surjective morphism of* K*-vector spaces, then we have:*

$$\mathrm{rk}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) = \mathrm{rk}^1_\varphi(g : \mathcal{G} \to \overline{\mathcal{H}}). \tag{5.1.8}$$

PROOF. (1) By definition of the φ-rank, we have:

$$\mathrm{rk}^1_\varphi(g) = \varphi(\overline{\mathcal{H}}) - \varphi(\overline{\mathcal{H}/\mathrm{im}\, g})$$

and:

$$\mathrm{rk}^1_\varphi(g \circ f) = \varphi(\overline{\mathcal{H}}) - \varphi(\overline{\mathcal{H}/\mathrm{im}\,(g \circ f)}),$$

so that the inequality:

$$\mathrm{rk}^1_\varphi(g \circ f) \leq \mathrm{rk}^1_\varphi(g)$$

is equivalent to the inequality:

$$\varphi(\overline{\mathcal{H}/\mathrm{im}\, g}) \leq \varphi(\overline{\mathcal{H}/\mathrm{im}\,(g \circ f)}).$$

The latter follows from the monotonicity of φ applied to the quotient morphism:

$$\overline{\mathcal{H}/\mathrm{im}\,(g \circ f)} \longrightarrow \overline{\mathcal{H}/\mathrm{im}\, g}.$$

(2) If f is surjective, then $\operatorname{im}(g \circ f) = \operatorname{im} g$ and the equality (5.1.8) follows from the definition of the φ-rank. If $f_K : \mathcal{F}_K \longrightarrow \mathcal{G}_K$ is a surjective morphism of K-vector spaces, then the surjection

$$\overline{\mathcal{H}/\operatorname{im}(g \circ f)} \longrightarrow \overline{\mathcal{H}/\operatorname{im} g}$$

is an isomorphism modulo torsion. If moreover φ satisfies condition **NST**, this implies the equality:

$$\varphi(\overline{\mathcal{H}/\operatorname{im} g}) = \varphi(\overline{\mathcal{H}/\operatorname{im}(g \circ f)}),$$

and (5.1.8) follows again from the definition of the φ-rank. $\square$

The next two propositions have a more technical character.

PROPOSITION 5.1.4. *Assume that the invariant φ is subadditive. For any diagram in $\overset{\cdots}{\mathbf{Coh}}{}_X^{\leq 1}$ of the form:*

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}},$$

the following inequality holds:

$$\operatorname{rk}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \geq \operatorname{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) + \operatorname{rk}^1_\varphi(g : \mathcal{G} \to \overline{\mathcal{H}}) - \varphi(\overline{\mathcal{G}}). \tag{5.1.9}$$

PROOF. Using the definition of the φ-rank, the inequality (5.1.9) may be written:

$$\varphi(\overline{\mathcal{H}}) - \varphi(\overline{\mathcal{H}/\operatorname{im}(g \circ f)}) \geq \Big[\varphi(\overline{\mathcal{G}}) - \varphi(\overline{\mathcal{G}/\operatorname{im} f})\Big] + \Big[\varphi(\overline{\mathcal{H}}) - \varphi(\overline{\mathcal{H}/\operatorname{im} f})\Big] - \varphi(\overline{\mathcal{G}}),$$

and is equivalent to:

$$\varphi(\overline{\mathcal{H}/\operatorname{im}(g \circ f)}) \leq \varphi(\overline{\mathcal{G}/\operatorname{im} f}) + \varphi(\overline{\mathcal{H}/\operatorname{im} g}). \tag{5.1.10}$$

Consider the admissible short exact sequence in $\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\operatorname{im}(g)/\operatorname{im}(g \circ f)} \longrightarrow \overline{\mathcal{H}/\operatorname{im}(g \circ f)} \longrightarrow \overline{\mathcal{H}/\operatorname{im}(g)} \longrightarrow 0$$

where $\overline{\operatorname{im}(g)/\operatorname{im}(g \circ f)}$ denotes the $\mathcal{O}_K$-module $\operatorname{im}(g)/\operatorname{im}(g \circ f)$ equipped with the Hermitian structure induced by the one of $\overline{\mathcal{H}/\operatorname{im}(g \circ f)}$. The subadditivity of φ implies:

$$\varphi(\overline{\mathcal{H}/\operatorname{im}(g \circ f)}) \leq \varphi(\overline{\operatorname{im}(g)/\operatorname{im}(g \circ f)}) + \varphi(\overline{\mathcal{H}/\operatorname{im} g}). \tag{5.1.11}$$

Considering the surjective morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$ induced by g:

$$\overline{\mathcal{G}/\operatorname{im}(f)} \longrightarrow \overline{\operatorname{im}(g)/\operatorname{im}(g \circ f)},$$

we obtain by monotonicity of φ:

$$\varphi(\overline{\mathcal{G}/\operatorname{im} f}) \geq \varphi(\overline{\operatorname{im}(g)/\operatorname{im}(g \circ f)}). \tag{5.1.12}$$

The inequality (5.1.10) follows from (5.1.11) and (5.1.12). $\square$

We may apply Proposition 5.1.4 to control the φ-rank of a morphism in terms of the φ-rank of its composition with a quotient morphism.

PROPOSITION 5.1.5. *Assume that the invariant φ is subadditive. For every commutative diagram in $\overset{\cdots}{\mathbf{Coh}}{}_X^{\leq 1}$:*

$$\begin{tikzcd}
\mathcal{F}_1 \arrow[r, "f_1"] \arrow[d, "p"] & \overline{\mathcal{G}}_1 \arrow[d, "q"] \\
\mathcal{F}_2 \arrow[r, "f_2"] & \overline{\mathcal{G}}_2,
\end{tikzcd}$$

in which $q : \overline{\mathcal{G}}_1 \to \overline{\mathcal{G}}_2$ is an admissible surjection, the following inequality holds:

$$\operatorname{rk}^1_\varphi(f_1 : \mathcal{F}_1 \to \overline{\mathcal{G}}_1) \leq \operatorname{rk}^1_\varphi(f_2 : \mathcal{F}_2 \to \overline{\mathcal{G}}_2) + \varphi(\overline{\ker q}).$$

Proof. Using Proposition 5.1.3 (1) and Proposition 5.1.4, we obtain:

$$\begin{aligned}\mathrm{rk}^1_\varphi(f_2 : \mathcal{F}_2 \to \overline{\mathcal{G}}_2) \geq \mathrm{rk}^1_\varphi(f_2 \circ p : \mathcal{F}_1 \to \overline{\mathcal{G}}_2) &= \mathrm{rk}^1_\varphi(q \circ f_1 : \mathcal{F}_1 \to \overline{\mathcal{G}}_2) \\ &\geq \mathrm{rk}^1_\varphi(f_1 : \mathcal{F}_1 \to \overline{\mathcal{G}}_1) + \mathrm{rk}^1_\varphi(q : \mathcal{G}_1 \to \overline{\mathcal{G}}_2) - \varphi(\overline{\mathcal{G}}_1).\end{aligned}$$

Since q is surjective, we have:

$$\mathrm{rk}^1_\varphi(q : \mathcal{G}_1 \to \overline{\mathcal{G}}_2) = \varphi(\overline{\mathcal{G}}_2).$$

Moreover, since q is an admissible surjection, the subadditivity of φ implies the inequality:

$$\varphi(\overline{\mathcal{G}}_1) \leq \varphi(\overline{\mathcal{G}}_2) + \varphi(\overline{\ker q}),$$

which completes the proof. □

5.2. The Strong Monotonicity Condition $\mathbf{StMon^1}$

5.2.1. The condition $\mathbf{StMon^1}$: first formulations. As already indicated in 5.1.2.3 above, the following definition highlights the invariants φ on $\overline{\mathbf{Coh}}_X$ whose associated φ-rank satisfies an analogue of the inequality (5.1.6):

$$\mathrm{rk} H^1(C, g \circ f) \leq \mathrm{rk} H^1(C, f),$$

or equivalently by Proposition 5.1.3, of the inequality (5.1.5):

$$\mathrm{rk} H^1(C, g \circ f) \leq \min(\mathrm{rk} H^1(C, g), \mathrm{rk} H^1(C, f)).$$

Definition 5.2.1. We say that an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfies the *strong monotonicity condition* $\mathbf{StMon^1}$ when it satisfies the monotonicity condition $\mathbf{Mon^1}$ and when, for any diagram of the form:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}$$

in $\overset{....}{\mathbf{Coh}}{}_X^{\leq 1}$, the following inequality holds:

$$\mathrm{rk}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \leq \mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}). \tag{5.2.1}$$

Condition $\mathbf{StMon^1}$ may be rephrased as follows in terms of inequalities involving values of the invariant φ.

Proposition 5.2.2. *For every invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$, the following three conditions are equivalent:*

(i) The invariant φ satisfies the strong monotonicity condition $\mathbf{StMon}^1$.

(ii) For every diagram in $\overset{....}{\mathbf{Coh}}{}_X^{\leq 1}$ of the form:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}, \tag{5.2.2}$$

the following inequality holds:

$$\varphi(\overline{\mathcal{H}}) + \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}) \leq \varphi(\overline{\mathcal{G}}) + \varphi(\overline{\mathcal{H}/\mathrm{im}\,(g \circ f)}). \tag{5.2.3}$$

(iii) For every morphism $f : \overline{\mathcal{E}} \to \overline{\mathcal{F}}$ in $\overline{\mathbf{Coh}}_X^{\leq 1}$ and every coherent $\mathcal{O}_X$-submodule $\mathcal{E}'$ of $\mathcal{E}$, of image $\mathcal{F}' := f(\mathcal{E}')$ in $\mathcal{F}$, the following inequality holds:

$$\mathrm{rk}^1_\varphi(f \circ \iota : \mathcal{E}' \longrightarrow \overline{\mathcal{F}}) := \varphi(\overline{\mathcal{F}}) - \varphi(\overline{\mathcal{F}/\mathcal{F}'}) \leq \mathrm{rk}^1_\varphi(\iota : \mathcal{E}' \longrightarrow \overline{\mathcal{E}}) := \varphi(\overline{\mathcal{E}}) - \varphi(\overline{\mathcal{E}/\mathcal{E}'}). \tag{5.2.4}$$

In (5.2.4), we have denoted by ι denotes the inclusion morphism from $\mathcal{E}'$ to $\mathcal{E}$.

PROOF. Applied to a diagram in $\ddddot{\mathbf{Coh}}_X^{\leq 1}$ of the form

$$\mathcal{G}' \hookrightarrow \overline{\mathcal{G}} \longrightarrow \overline{0},$$

where $\mathcal{G}'$ is some $\mathcal{O}_K$-submodule of $\mathcal{G}$, the inequality (5.2.3) reads:

$$\varphi(\overline{\mathcal{G}/\mathcal{G}'}) \leq \varphi(\overline{\mathcal{G}}).$$

Moreover, applied to $\overline{\mathcal{F}} = \overline{0}$, the inequality (5.2.4) reads:

$$0 \leq \varphi(\overline{\mathcal{E}}) - \varphi(\overline{\mathcal{E}/\mathcal{E}'}).$$

This shows that both conditions (ii) and (iii) imply the monotonicity of φ, and therefore that the φ-rank rk^1_φ is well-defined with values in $\mathbb{R}_+$.

To prove the equivalence $(i) \Leftrightarrow (ii)$, observe that, for every diagram in $\ddddot{\mathbf{Coh}}_X^{\leq 1}$ of the form (5.2.2), the definition of the φ-rank implies the following equalities:

$$\varphi(\overline{\mathcal{H}}) + \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}) = \mathrm{rk}^1_\varphi(g \circ f) + \varphi(\overline{\mathcal{H}/\mathrm{im}\,(g \circ f)}) + \varphi(\overline{\mathcal{G}/\mathrm{im}\, f})$$

and

$$\varphi(\overline{\mathcal{G}}) + \varphi(\overline{\mathcal{H}/\mathrm{im}\,(g \circ f)}) = \mathrm{rk}^1_\varphi(f) + \varphi(\overline{\mathcal{H}/\mathrm{im}\,(g \circ f)}) + \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}),$$

and consequently the estimates (5.2.1) and (5.2.3) are equivalent.

The inequality (5.2.4) is the special instance of the inequality (5.2.1) applied to the diagram

$$\mathcal{E}' \xrightarrow{\iota} \overline{\mathcal{E}} \xrightarrow{f} \overline{\mathcal{F}}.$$

This establishes the implication $(i) \Rightarrow (iii)$.

Conversely let us assume that condition (iii) holds, and consider a diagram in $\ddddot{\mathbf{Coh}}_X^{\leq 1}$ of the form (5.2.2). Let us consider the image $\mathcal{G}' := f(\mathcal{F})$ of f and the inclusion morphism $\iota : \mathcal{G}' \hookrightarrow \mathcal{G}$. According to (iii) applied to the morphism $g : \overline{\mathcal{G}} \to \overline{\mathcal{H}}$ and to the submodule $\mathcal{G}'$ of $\mathcal{G}$, the following inequality holds:

$$\mathrm{rk}^1_\varphi(g \circ \iota : \mathcal{G}' \to \overline{\mathcal{H}}) \leq \mathrm{rk}^1_\varphi(\iota : \mathcal{G}' \to \overline{\mathcal{G}}).$$

Moreover, we have:

$$\mathrm{rk}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) = \mathrm{rk}^1_\varphi(g \circ \iota : \mathcal{G}' \to \overline{\mathcal{H}})$$

and

$$\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) = \mathrm{rk}^1_\varphi(\iota : \mathcal{G}' \to \overline{\mathcal{G}}),$$

since $g \circ f$ and $g \circ \iota$ (resp. f and ι) have the same image. This establishes the inequality:

$$\mathrm{rk}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \leq \mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}),$$

and completes the proof of the implication $(iii) \Rightarrow (i)$. □

5.2.2. The conditions $\mathbf{StMon}^\mathbf{1}_i$ for $i \in \{1, 2, 3, 4\}$. In this subsection, we try to clarify the meaning of the strong monotonicity condition $\mathbf{StMon^1}$ by spelling out three special instances of the strong monotonicity estimates (5.2.1), (5.2.3), and (5.2.4).

The three conditions thus obtained as consequences of condition $\mathbf{StMon^1}$ will be denoted by $\mathbf{StMon^1_2}$, $\mathbf{StMon^1_3}$, and $\mathbf{StMon^1_4}$. It will be convenient to denote by $\mathbf{StMon^1_1}$ the monotonicity condition $\mathbf{Mon}^1$. Indeed condition $\mathbf{Mon}^1$ was assumed to hold when introducing $\mathbf{StMon^1}$ in Definition 5.2.1, and was also observed to be a "trivial instance" of the estimates (5.2.3) and (5.2.4) in the proof of Proposition 5.2.2.

5.2.2.1. *The strong subadditivity condition* $\mathbf{StMon^1_2}$. Assume that an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfies condition $\mathbf{StMon^1}$. Consider an object $\overline{\mathcal{F}}$ of $\overline{\mathbf{Coh}}_X$, and two submodules $\mathcal{F}'' \subseteq \mathcal{F}' \subseteq \mathcal{F}$ of the $\mathcal{O}_K$-module $\mathcal{F}$ underlying $\overline{\mathcal{F}}$. We may form the diagram in $\ddot{\mathbf{Coh}}{}_X^{\leq 1}$:

$$\mathcal{F}'' \xrightarrow{f} \overline{\mathcal{F}}' \xrightarrow{g} \overline{\mathcal{F}},$$

where f and g are the inclusion morphisms. The inequality (5.2.3) for this diagram may be written as follows:

$$\varphi(\overline{\mathcal{F}}) + \varphi(\overline{\mathcal{F}'/\mathcal{F}''}) \leq \varphi(\overline{\mathcal{F}}') + \varphi(\overline{\mathcal{F}/\mathcal{F}''}). \tag{5.2.5}$$

Motivated by this observation, we introduce the following *strong subadditivity* condition on some invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$:

$\mathbf{StMon^1_2}$: *for every object $\overline{\mathcal{F}}$ of $\overline{\mathbf{Coh}}_X$ and any two $\mathcal{O}_K$-submodules $\mathcal{F}'' \subseteq \mathcal{F}' \subseteq \mathcal{F}$, the inequality* (5.2.5) *is satisfied.*

Observe that, as $\varphi(\overline{0})$ is zero, the inequality (5.2.5) with $\mathcal{F}'' = \mathcal{F}'$ becomes the subadditivity inequality

$$\varphi(\overline{\mathcal{F}}) \leq \varphi(\overline{\mathcal{F}}') + \varphi(\overline{\mathcal{F}/\mathcal{F}'}),$$

which enters in the subadditivity condition **SubAdd** introduced in 4.1.2 above. This establishes the implication

$$\mathbf{StMon^1_2} \Longrightarrow \mathbf{SubAdd}$$

and justifies the terminology "strong subadditivity" for condition $\mathbf{StMon^1_2}$.

PROPOSITION 5.2.3. *For every invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfying*[5] $\mathbf{Mon^1}$*, the following two conditions are equivalent:*

(i) the invariant φ satisfies the strong subadditivity condition $\mathbf{StMon^1_2}$;

(ii) for any diagram in $\ddot{\mathbf{Coh}}{}_X^{\leq 1}$ of the form:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}$$

where g is an admissible injection, the following inequality holds:

$$\mathrm{rk}^1_\varphi(g \circ f) \leq \mathrm{rk}^1_\varphi(f).$$

PROOF. By the definition of an admissible injection, in assertion (ii) we may assume that $\overline{\mathcal{G}}$ is $\overline{\mathcal{H}}'$, where $\mathcal{H}'$ is a submodule of the $\mathcal{O}_K$-module $\mathcal{H}$ underlying $\overline{\mathcal{H}}$, and g the inclusion morhism. Moreover the invariants $\mathrm{rk}^1_\varphi(g \circ f)$ and $\mathrm{rk}^1_\varphi(f)$ are unchanged when replacing $\overline{\mathcal{F}}$ with its image by f, so that we may assume that $\mathcal{F} = \mathcal{H}''$, where $\mathcal{H}''$ is a submodule of the $\mathcal{O}_K$-module $\mathcal{H}'$.

After these reductions, the inequality $\mathrm{rk}^1_\varphi(g \circ f) \leq \mathrm{rk}^1_\varphi(f)$ equivalent to the strong subadditivity condition $\mathbf{StMon^1_2}$ applied to the object $\overline{\mathcal{H}}$ of $\overline{\mathbf{Coh}}_X$ and to the submodules:

$$\mathcal{H}'' \subset \mathcal{H}' \subset \mathcal{H}. \qquad \square$$

5.2.2.2. *The submodularity condition* $\mathbf{StMon^1_3}$. Assume that φ satisfies condition $\mathbf{StMon^1}$. Consider an object $\overline{\mathcal{F}}$ of $\overline{\mathbf{Coh}}_X$ and two $\mathcal{O}_K$-submodules $\mathcal{F}'$ and $\mathcal{F}''$ of $\mathcal{F}$, and consider the diagram in $\ddot{\mathbf{Coh}}{}_X^{\leq 1}$:

$$\mathcal{F}'' \xrightarrow{f} \overline{\mathcal{F}} \xrightarrow{g} \overline{\mathcal{F}/\mathcal{F}'},$$

where f is the inclusion and g the quotient map. The inequality (5.2.3) for this diagram reads as follows:

$$\varphi(\overline{\mathcal{F}/\mathcal{F}'}) + \varphi(\overline{\mathcal{F}/\mathcal{F}''}) \leq \varphi(\overline{\mathcal{F}/(\mathcal{F}' + \mathcal{F}'')}) + \varphi(\overline{\mathcal{F}}). \tag{5.2.6}$$

[5] The monotonicity condition $\mathbf{Mon^1}$ is required in Proposition 5.2.3, and in Proposition 5.2.7 as well, simply because the φ-rank rk^1_φ has been defined only under this monotonicity condition.

When applied to the subspaces $\mathcal{F}'/(\mathcal{F}'\cap\mathcal{F}'')$ and $\mathcal{F}''/(\mathcal{F}'\cap\mathcal{F}'')$ of $\mathcal{F}/(\mathcal{F}'\cap\mathcal{F}'')$, the inequality (5.2.6) becomes:

$$\varphi(\overline{\mathcal{F}/\mathcal{F}'}) + \varphi(\overline{\mathcal{F}/\mathcal{F}''}) \leq \varphi(\overline{\mathcal{F}/(\mathcal{F}'+\mathcal{F}'')}) + \varphi(\overline{\mathcal{F}/(\mathcal{F}'\cap\mathcal{F}'')}). \tag{5.2.7}$$

Inequalities of this kind, relating the values of an invariant of two objects $\mathcal{F}'$ and $\mathcal{F}''$ in some lattice to its values over their greatest lower and least upper bounds (here $\mathcal{F}'\cap\mathcal{F}''$ and $\mathcal{F}'+\mathcal{F}''$), are classically related to as submodularity inequalities.

These observations lead us to introduce the following *submodularity* condition on some invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$:

$\mathbf{StMon}^1_3$: *for every object $\overline{\mathcal{F}}$ of $\overline{\mathbf{Coh}}_X$ and every two submodules $\mathcal{F}'$ and $\mathcal{F}''$ of $\mathcal{F}$, the inequality* (5.2.7) *is satisfied.*

PROPOSITION 5.2.4. *For every invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfying $\mathbf{Mon^1}$, the following two conditions are equivalent:*

(i) the invariant φ satisfies the submodularity condition $\mathbf{StMon^1_3}$;

(ii) for any diagram in $\overset{\cdots\cdot}{\mathbf{Coh}}{}_X^{\leq 1}$ of the form:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}$$

where g is an admissible surjection, the following inequality holds:

$$\mathrm{rk}^1_\varphi(g\circ f) \leq \mathrm{rk}^1_\varphi(f). \tag{5.2.8}$$

PROOF. By the definition of admissible surjections, in statement (ii) we may assume that $\overline{\mathcal{H}} = \overline{\mathcal{G}/\mathcal{G}'}$, where $\mathcal{G}'$ is a submodule of the $\mathcal{O}_K$-module $\mathcal{G}$ underlying $\overline{\mathcal{G}}$.

The invariants $\mathrm{rk}^1_\varphi(g\circ f)$ and $\mathrm{rk}^1_\varphi(f)$ are unchanged when replacing $\overline{\mathcal{F}}$ with its image by f, so that to establish the inequality (5.2.8), we may assume that $\overline{\mathcal{F}} = \overline{\mathcal{G}}''$, where $\mathcal{G}''$ is a submodule of the $\mathcal{O}_K$-module $\mathcal{G}$. After these reductions, (5.2.8) follows from the submodularity inequality (5.2.7) applied to the subspaces $\mathcal{G}'$ and $\mathcal{G}''$ of $\mathcal{G}$ and from the monotonicity of φ, which implies the inequality:

$$\varphi(\overline{\mathcal{G}/(\mathcal{G}'+\mathcal{G}'')}) \leq \varphi(\overline{\mathcal{G}}).$$

This establish the implication $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (ii)$ follows from the fact, discussed above, that the inequality (5.2.7) follows from (5.2.8) applied to the diagram:

$$\mathcal{F}''/(\mathcal{F}'\cap\mathcal{F}'') \xrightarrow{f} \overline{\mathcal{F}/\mathcal{F}'\cap\mathcal{F}''} \xrightarrow{g} \overline{\mathcal{F}/\mathcal{F}'}. \qquad \square$$

The following is a slight refinement of Proposition 5.2.4.

PROPOSITION 5.2.5. *Let $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ be an invariant satisfying $\mathbf{Mon^1}$. For every object $\overline{\mathcal{F}}$ of $\overline{\mathbf{Coh}}_X$, the following two conditions are equivalent:*

(i) for any two $\mathcal{O}_K$-submodules $\mathcal{F}'$ and $\mathcal{F}''$ of $\mathcal{F}$, the inequality (5.2.7) *holds;*

(ii) for any diagram in $\overset{\cdots\cdot}{\mathbf{Coh}}{}_X^{\leq 1}$ of the form:

$$\mathcal{A} \xrightarrow{f} \overline{\mathcal{B}} \xrightarrow{g} \overline{\mathcal{C}}$$

where $\overline{\mathcal{B}}$ is a quotient of $\overline{\mathcal{F}}$ and g is an admissible surjection, the following inequality holds:

$$\mathrm{rk}^1_\varphi(g\circ f) \leq \mathrm{rk}^1_\varphi(f).$$

PROOF. As noted above, the inequality (5.2.7) is equivalent to the inequality $\mathrm{rk}^1_\varphi(g\circ f) \leq \mathrm{rk}^1_\varphi(f)$ for the diagram

$$\mathcal{F}''/(\mathcal{F}'\cap\mathcal{F}'') \xrightarrow{f} \overline{\mathcal{F}/(\mathcal{F}'\cap\mathcal{F}'')} \xrightarrow{g} \overline{\mathcal{F}/\mathcal{F}'},$$

so that (ii) implies (i).

Conversely, assume that (i) holds. Up to isomorphism, any diagram in (ii) may be written as

$$\overline{\mathcal{F}''/\mathcal{G}} \xrightarrow{f} \overline{\mathcal{F}/\mathcal{G}} \xrightarrow{g} \overline{\mathcal{F}/\mathcal{F}'},$$

where $\mathcal{G}$ is an $\mathcal{O}_K$-submodule of $\mathcal{F}$ contained in $\mathcal{O}_K$-submodules $\mathcal{F}'$ and $\mathcal{F}''$. The inequality

$$\mathrm{rk}^1_\varphi(g\circ f) \leq \mathrm{rk}^1_\varphi(f)$$

may be written as

$$\varphi(\overline{\mathcal{F}/\mathcal{F}'}) + \varphi(\overline{\mathcal{F}/\mathcal{F}''}) \leq \varphi(\overline{\mathcal{F}/(\mathcal{F}'+\mathcal{F}'')}) + \varphi(\overline{\mathcal{F}/\mathcal{G}}). \tag{5.2.9}$$

Since $\overline{\mathcal{F}/(\mathcal{F}'\cap\mathcal{F}'')}$ is a quotient of $\overline{\mathcal{F}/\mathcal{G}}$, the monotonicity of φ shows that

$$\varphi(\overline{\mathcal{F}/(\mathcal{F}'\cap\mathcal{F}'')}) \leq \varphi(\overline{\mathcal{F}/\mathcal{G}}),$$

and therefore (5.2.9) follows from (5.2.7). This completes the proof of (ii). □

5.2.2.3. *Submodularity estimates.* Recall that, if E is a set and $\mathcal{P}(E)$ the set of subsets of E, a real valued function I on some subset of $\mathcal{P}(E)$ stable under $\cup$ and $\cap$ is classically said to be *strongly subadditive* when it satisfies the estimates:

$$I(A\cap B) + I(A\cup B) \leq I(A) + I(B).$$

Increasing positive set functions that are strongly subadditive play a central role in measure and capacity theory, and are formally analogous to positive invariants on $\overline{\mathbf{Coh}}_X$ satisfying the monotonicity and submodularity conditions $\mathbf{Mon^1}$ and $\mathbf{StMon^1_3}$. Indeed a monotonic invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfies $\mathbf{StMon^1_3}$ if and only if, for every object $\overline{\mathcal{F}}$ of $\overline{\mathbf{Coh}}_X$, the function

$$I : \mathcal{F}' \longmapsto \varphi(\overline{\mathcal{F}}) - \varphi(\overline{\mathcal{F}/\mathcal{F}'}),$$

defined on the set of $\mathcal{O}_K$-submodules $\mathcal{F}'$ of $\mathcal{F}$, satisfies the following estimates[6]:

$$I(\mathcal{F}_1\cap\mathcal{F}_2) + I(\mathcal{F}_1+\mathcal{F}_2) \leq I(\mathcal{F}_1) + I(\mathcal{F}_2).$$

Using this analogy, the equivalent characterizations of strongly subadditive set functions — established for instance in [**Cho54**] and [**Mey66**, III.2] — translate into the following reformulations of the submodularity condition $\mathbf{StMon^1_3}$.

PROPOSITION 5.2.6. *For every invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ that satisfies $\mathbf{Mon^1}$ and every object $\overline{\mathcal{E}}$ of $\overline{\mathbf{Coh}}_X$, the following conditions are equivalent:*

(i) for any two $\mathcal{O}_K$-submodules $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{E}$, the following submodularity inequality holds:

$$\varphi(\overline{\mathcal{E}/\mathcal{E}_1}) + \varphi(\overline{\mathcal{E}/\mathcal{E}_2}) \leq \varphi(\overline{\mathcal{E}/(\mathcal{E}_1\cap\mathcal{E}_2)}) + \varphi(\overline{\mathcal{E}/(\mathcal{E}_1+\mathcal{E}_2)}). \tag{5.2.10}$$

(ii) for any three $\mathcal{O}_K$-submodules $\mathcal{X}$, $\mathcal{X}'$, and $\mathcal{Y}$ of $\mathcal{E}$ such that $\mathcal{X}\subseteq\mathcal{X}'$, the following inequality holds:

$$\varphi(\overline{\mathcal{E}/\mathcal{X}'}) + \varphi(\overline{\mathcal{E}/(\mathcal{X}+\mathcal{Y})}) \leq \varphi(\overline{\mathcal{E}/\mathcal{X}}) + \varphi(\overline{\mathcal{E}/(\mathcal{X}'+\mathcal{Y})}); \tag{5.2.11}$$

(iii) if $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, and $\mathcal{B}_2$ are $\mathcal{O}_K$-submodules of $\mathcal{E}$ such that $\mathcal{A}_1\subseteq\mathcal{B}_1$ and $\mathcal{A}_2\subseteq\mathcal{B}_2$, then the following inequality holds:

$$\varphi(\overline{\mathcal{E}/(\mathcal{A}_1+\mathcal{A}_2)}) + \varphi(\overline{\mathcal{E}/\mathcal{B}_1}) + \varphi(\overline{\mathcal{E}/\mathcal{B}_2}) \leq \varphi(\overline{\mathcal{E}/(\mathcal{B}_1+\mathcal{B}_2)}) + \varphi(\overline{\mathcal{E}/\mathcal{A}_1}) + \varphi(\overline{\mathcal{E}/\mathcal{A}_2}); \tag{5.2.12}$$

(iv) for any integer $n\geq 2$, if $\mathcal{A}_, \dots, \mathcal{A}_n$ and $\mathcal{B}_1,\dots,\mathcal{B}_n$ are $\mathcal{O}_K$-submodules of $\mathcal{E}$ such that $\mathcal{A}_i\subseteq\mathcal{B}_i$ for $i=1,\dots,n$, then the following inequality holds:

$$\begin{aligned}\varphi(\overline{\mathcal{E}/(\mathcal{A}_1+\dots+\mathcal{A}_n)}) &+ \varphi(\overline{\mathcal{E}/\mathcal{B}_1}) + \dots + \varphi(\overline{\mathcal{E}/\mathcal{B}_n})\\ &\leq \varphi(\overline{\mathcal{E}/(\mathcal{B}_1+\dots+\mathcal{B}_n)}) + \varphi(\overline{\mathcal{E}/\mathcal{A}_1}) + \dots + \varphi(\overline{\mathcal{E}/\mathcal{A}_n}).\end{aligned} \tag{5.2.13}$$

[6]Note the unfortunate but difficultly avoidable terminological inconsistency: the *strong subadditivity* of set functions is analogue to the *submodularity* of invariants on $\overline{\mathbf{Coh}}_X$.

Somewhat surprisingly, contrary to condition (i), conditions (ii)-(iv) involve only sums, and not intersections, of submodules of $\mathcal{E}$.

PROOF. Let us assume that the submodularity inequality (5.2.10) holds for any two submodules $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{E}$. Given submodules $\mathcal{X}$, $\mathcal{X}'$, and $\mathcal{Y}$ of $\mathcal{E}$ such that $\mathcal{X} \subseteq \mathcal{X}'$, consider the diagram

$$\mathcal{Y} \xrightarrow{f} \overline{\mathcal{E}/\mathcal{X}} \xrightarrow{g} \overline{\mathcal{E}/\mathcal{X}'}$$

in $\overline{\mathbf{Coh}}_X^{\leq 1}$, where f and g are induced by the inclusion and quotient map respectively. Proposition 5.2.5 shows that $\mathrm{rk}^1_\varphi(g \circ f) \leq \mathrm{rk}^1_\varphi(f)$, which is equivalent to the inequality (5.2.11). This establishes the implication $(i) \Rightarrow (ii)$.

To prove the converse implication $(ii) \Rightarrow (i)$, observe that the submodularity inequality (5.2.10) follows from (5.2.11) applied to $\mathcal{X} = \mathcal{E}_1 \cap \mathcal{E}_2$, $\mathcal{X}' = \mathcal{E}_2$, and $\mathcal{Y} = \mathcal{E}_1$.

Let us assume that (ii) holds. Then, for $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$, and $\mathcal{B}_2$ as in (iii), the inequality (5.2.11) applied to

$$(\mathcal{X}, \mathcal{X}', \mathcal{Y}) = (\mathcal{A}_1, \mathcal{B}_1, \mathcal{B}_2)$$

and to

$$(\mathcal{X}, \mathcal{X}', \mathcal{Y}) = (\mathcal{A}_2, \mathcal{B}_2, \mathcal{A}_1)$$

becomes:

$$\varphi(\overline{\mathcal{E}/\mathcal{B}_1}) + \varphi(\overline{\mathcal{E}/(\mathcal{A}_1 + \mathcal{B}_2)}) \leq \varphi(\overline{\mathcal{E}/\mathcal{A}_1}) + \varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \mathcal{B}_2)})$$

and

$$\varphi(\overline{\mathcal{E}/\mathcal{B}_2}) + \varphi(\overline{\mathcal{E}/(\mathcal{A}_1 + \mathcal{A}_2)}) \leq \varphi(\overline{\mathcal{E}/\mathcal{A}_2}) + \varphi(\overline{\mathcal{E}/(\mathcal{A}_1 + \mathcal{B}_2)}).$$

Consequently, we have:

$$\varphi(\overline{\mathcal{E}/\mathcal{B}_1}) + \varphi(\overline{\mathcal{E}/\mathcal{B}_2}) + \varphi(\overline{\mathcal{E}/(\mathcal{A}_1 + \mathcal{A}_2)}) \leq \varphi(\overline{\mathcal{E}/\mathcal{A}_1}) + \varphi(\overline{\mathcal{E}/\mathcal{A}_2}) + \varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \mathcal{B}_2)}).$$

This establishes the implication $(ii) \Rightarrow (iii)$.

Let us prove the implication $(iii) \Rightarrow (i)$. The inequality (5.2.12) applied with $\mathcal{A}_1 = \mathcal{E}_1 \cap \mathcal{E}_2$, $\mathcal{B}_1 = \mathcal{E}_1$, and $\mathcal{A}_2 = \mathcal{B}_2 = \mathcal{E}_2$ reads:

$$\varphi(\overline{\mathcal{E}/\mathcal{E}_2}) + \varphi(\overline{\mathcal{E}/\mathcal{E}_1}) + \varphi(\overline{\mathcal{E}/\mathcal{E}_2}) \leq \varphi(\overline{\mathcal{E}/(\mathcal{E}_1 + \mathcal{E}_2)}) + \varphi(\overline{\mathcal{E}/(\mathcal{E}_1 \cap \mathcal{E}_2)}) + \varphi(\overline{\mathcal{E}/\mathcal{E}_2}).$$

This implies (5.2.10).

Let us assume that (iii) holds. Then the inequalities (5.2.13) hold for every $n \geq 2$, as shown by the following induction argument: for any integer $n \geq 2$ and any submodules $\mathcal{A}_{,} \ldots, \mathcal{A}_{n+1}$ and $\mathcal{B}_1, \ldots, \mathcal{B}_{n+1}$ of $\mathcal{E}$ such that $\mathcal{B}_i \subseteq \mathcal{A}_i$ for $i = 1, \ldots, n+1$, the validity of (5.2.13) for the submodules $\mathcal{A}_1 + \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_{n+1}$ and $\mathcal{B}_1 + \mathcal{B}_2, \mathcal{B}_3, \ldots, \mathcal{B}_{n+1}$ establishes the inequality:

$$\begin{aligned}(5.2.14) \quad \varphi(\overline{\mathcal{E}/(\mathcal{A}_1 + \cdots + \mathcal{A}_{n+1})}) &+ \varphi(\overline{\mathcal{E}/\mathcal{B}_1 + \mathcal{B}_2}) + \varphi(\overline{\mathcal{E}/\mathcal{B}_3}) + \cdots + \varphi(\overline{\mathcal{E}/\mathcal{B}_{n+1}}) \\ &\leq \varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \cdots + \mathcal{B}_{n+1})}) + \varphi(\overline{\mathcal{E}/(\mathcal{A}_1 + \mathcal{A}_2)}) + \varphi(\overline{\mathcal{E}/\mathcal{A}_3}) + \cdots + \varphi(\overline{\mathcal{E}/\mathcal{A}_{n+1}}).\end{aligned}$$

Together with (5.2.12), this implies the inequality:

$$\begin{aligned}\varphi(\overline{\mathcal{E}/(\mathcal{A}_1 + \cdots + \mathcal{A}_{n+1})}) &+ \varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \mathcal{B}_2)}) + \varphi(\overline{\mathcal{E}/\mathcal{B}_1}) + \cdots + \varphi(\overline{\mathcal{E}/\mathcal{B}_{n+1}}) \\ &\leq \varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \cdots + \mathcal{B}_{n+1})}) + \varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \mathcal{B}_2)}) + \varphi(\overline{\mathcal{E}/\mathcal{A}_1}) + \cdots + \varphi(\overline{\mathcal{E}/\mathcal{A}_{n+1}})\end{aligned}$$

and therefore the inequality

$$\begin{aligned}(5.2.15) \quad \varphi(\overline{\mathcal{E}/(\mathcal{A}_1 + \cdots + \mathcal{A}_{n+1})}) &+ \varphi(\overline{\mathcal{E}/\mathcal{B}_1}) + \cdots + \varphi(\overline{\mathcal{E}/\mathcal{B}_{n+1}}) \\ &\leq \varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \cdots + \mathcal{B}_{n+1})}) + \varphi(\overline{\mathcal{E}/\mathcal{A}_1}) + \cdots + \varphi(\overline{\mathcal{E}/\mathcal{A}_{n+1}}).\end{aligned}$$

This establishes the implication $(iii) \Rightarrow (iv)$. The converse implication $(iv) \Rightarrow (iii)$ is clear. $\square$

5.2.2.4. *The metric monotonicity condition* $\mathbf{StMon^1_4}$. Assume that φ satisfies the strong monotonicity condition $\mathbf{StMon^1}$. Let $\overline{\mathcal{G}} := (\mathcal{G}, (\|.\|_x)_{x\in X(\mathbb{C})}$ and $\overline{\mathcal{G}}^{\sim} := (\mathcal{G}, (\|.\|^{\sim}_x)_{x\in X(\mathbb{C})}$ be two objects in $\overline{\mathbf{Coh}}_X$ with the same underlying $\mathcal{O}_K$-module $\mathcal{G}$ whose Hermitian metrics satisfy the condition:

$$\|.\|_x \geq \|.\|^{\sim}_x \quad \text{for every } x \in X(\mathbb{C}),$$

or equivalently, such that the identity map $\mathrm{Id}_{\mathcal{G}}$ defines a morphism $\mathrm{Id}_{\mathcal{G}} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}^{\sim}$ in $\overline{\mathbf{Coh}}^{\leq 1}_X$.

Consider a diagram in $\overset{....}{\mathbf{Coh}}_X$:

$$f : \mathcal{F} \longrightarrow \overline{\mathcal{G}},$$

the associated diagram in $\overset{....}{\mathbf{Coh}}{}^{\leq 1}_X$:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{\mathrm{Id}_{\mathcal{G}}} \overline{\mathcal{G}}^{\sim},$$

and the "composition":

$$f := \mathrm{Id}_{\mathcal{G}} \circ f : \mathcal{F} \longrightarrow \overline{\mathcal{G}}^{\sim}.$$

The strong monotonicity of φ yields the inequality:

$$\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}^{\sim}) \leq \mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}). \tag{5.2.16}$$

This establishes the monotonicity of the φ-rank $\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}})$ as a fonction of the Hermitian metrics defining the Hermitian coherent sheaf $\overline{\mathcal{G}}$.

Observe also that, with the above notation, $\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}^{\sim})$ and $\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}})$ are unchanged when f is replaced by the inclusion map of its image in $\mathcal{G}$:

$$\mathcal{G}' := f(\mathcal{F}) \hookrightarrow \mathcal{G}.$$

This shows that the inequality (5.2.16) is equivalent to the following one:

$$\varphi(\overline{\mathcal{G}}^{\sim}) + \varphi(\overline{\mathcal{G}/\mathcal{G}'}) \leq \varphi(\overline{\mathcal{G}/\mathcal{G}'}^{\sim}) + \varphi(\overline{\mathcal{G}}). \tag{5.2.17}$$

This leads us to introducing the following condition of *metric monotonicity* on some invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfying $\mathbf{Mon^1}$:

$\mathbf{StMon^1_4}$: *for any two objects* $\overline{\mathcal{G}}$ *and* $\overline{\mathcal{G}}^{\sim}$ *of* $\overline{\mathbf{Coh}}_X$ *with the same underlying* $\mathcal{O}_K$*-module* $\mathcal{G}$ *such that* $\mathrm{Id}_{\mathcal{G}} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}^{\sim}$ *is a morphism in* $\overline{\mathbf{Coh}}^{\leq 1}_X$ *and for any subobject* $\mathcal{G}'$ *of* $\mathcal{G}$*, the inequality* (5.2.17) *holds.*

The following proposition is a straightforward consequence of the previous discussion:

PROPOSITION 5.2.7. *For every invariant* $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ *satisfying* $\mathbf{Mon^1}$*, the following two conditions are equivalent:*

(i) the invariant φ *satisfies the metric monotonicity condition* $\mathbf{StMon^1_4}$*;*

(ii) for any diagram in $\overset{....}{\mathbf{Coh}}{}^{\leq 1}_X$ *of the form:*

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}$$

where $g : \mathcal{G} \to \mathcal{H}$ *is an isomorphism of* $\mathcal{O}_K$*-modules, the following inequality holds:*

$$\mathrm{rk}^1_\varphi(g \circ f) \leq \mathrm{rk}^1_\varphi(f).$$

5.2.3. Criteria of strong monotonicity. It turns out that the conditions $\mathbf{StMon^1_i}$ for $i \in \{1, 2, 3, 4\}$ provide sufficient conditions for the validity of condition $\mathbf{StMon^1}$.

Firstly as a consequence of Propositions 5.2.3, 5.2.4, and 5.2.7, we may prove:

PROPOSITION 5.2.8. *An invariants* $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ *satisfies condition* $\mathbf{StMon^1}$ *if and only if it satisfies the conditions* $\mathbf{StMon^1_1}$*,* $\mathbf{StMon^1_2}$*,* $\mathbf{StMon^1_3}$*, and* $\mathbf{StMon^1_4}$*.*

PROOF. The direct implication:

$$\mathbf{StMon^1} \Longrightarrow \left[\mathbf{StMon^1_1}, \mathbf{StMon^1_2}, \mathbf{StMon^1_3}, \text{and } \mathbf{StMon^1_4}\right]$$

is clear.

To prove the converse implication, observe that any morphism

$$g : \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{H}}$$

in $\overline{\mathbf{Coh}}_X^{\leq 1}$ factors as

$$g = i \circ g' \circ p : \overline{\mathcal{G}} \xrightarrow{p} \overline{\mathcal{G}}' \xrightarrow{g'} \overline{\mathcal{G}}'' \xrightarrow{i} \overline{\mathcal{H}}$$

where p is an admissible surjection, i is an admissible injection, and g' is bijective at the level of $\mathcal{O}_X$-modules. Consequently, for any diagram in $\overline{\mathbf{Coh}}_X^{\leq 1}$ of the form:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}},$$

the validity of $\mathbf{StMon^1_2}$, of $\mathbf{StMon^1_4}$, and of $\mathbf{StMon^1_1}$ and $\mathbf{StMon^3_1}$ implies the following successive inequalities:

$$\mathrm{rk}^1_\varphi(g \circ f) = \mathrm{rk}^1_\varphi(i \circ g' \circ p \circ f) \leq \mathrm{rk}^1_\varphi(g' \circ p \circ f) \leq \mathrm{rk}^1_\varphi(p \circ f) \leq \mathrm{rk}^1_\varphi(f),$$

according to Propositions 5.2.3, 5.2.7, and 5.2.4. □

When dealing with additive and downward continuous invariants, one may establish the following variant of Proposition 5.2.8 by an argument inspired by the proof of [**MP17**, Lemma 8.4].

PROPOSITION 5.2.9. *Let $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ be an invariant that satisfies the conditions $\mathbf{Add}_\oplus$ of additivity on direct sums and $\mathbf{Cont}^+$ of downward continuity.*[7] *Then φ satisfies the strong monotonicity condition $\mathbf{StMon}^1$ if and only if it satisfies the monotonicity condition $\mathbf{Mon^1} = \mathbf{StMon^1}$ and the submodularity condition $\mathbf{StMon^1_3}$.*

In other words, for invariants satisfying $\mathbf{Mon^1}$, $\mathbf{Add}_\oplus$, and $\mathbf{Cont}^+$, the submodularity condition $\mathbf{StMon^1_3}$ implies both the strong subadditivity $\mathbf{StMon^1_2}$ — and a fortiori the subadditivity $\mathbf{SubAdd}$ — and the metric monotonicity $\mathbf{StMon^1_4}$.

To put in perspective the criterion of strong subadditivity in Proposition 5.2.9, recall that the submodularity for the invariant $h^1(C, .)$ of coherent $\mathcal{O}_C$-modules over of smooth projective curve C follows from its additivity for direct sums and its subadditivity in short exact sequences, as discussed in paragraph 3.1.2, (ii). Therefore one may ask whether a suitable argument would show that an invariant on $\overline{\mathbf{Coh}}_\mathbb{Z}$ satisfying condition $\mathbf{Add}_\oplus$, $\mathbf{Mon}^1$ and $\mathbf{SubAdd}$ would automatically satisfy the submodularity condition $\mathbf{StMon}^1_3$, and therefore $\mathbf{StMon^1}$. This is actually *not* the case: in Chapter 6, we will show that the square of the covering radius defines an invariant satisfying the former conditions but not the latter.[8]

The proof of Proposition 5.2.9 will rely on the following construction of independent interest.

PROPOSITION 5.2.10. *Let $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ be two objects of $\overline{\mathbf{Coh}}_{\overline{X}}$ and let $g : \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{H}}$ be a morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$ of operator norms < 1.*[9] *There exists a unique family of Hermitian metrics $(\|.\|^\sim_x)_{x \in X(\mathbb{C})}$ on the $\mathbb{C}$-vector spaces $(\mathcal{H}_x)_{x\in X(\mathbb{C})}$, invariant under complex conjugation, such that, if we introduce the object of $\overline{\mathbf{Coh}}_{\overline{X}}$:*

$$\overline{\mathcal{H}}^\sim := (\mathcal{H}, (\|.\|^\sim_x)_{x\in X(\mathbb{C})}),$$

[7] See 4.1.5 above

[8] Observe also that, when working with Euclidean lattices, the construction of the short exact sequence (3.1.13) is *not* compatible with the Euclidean structures, and does not produce an admissible short exact sequence of Euclidean lattices.

[9] Recall that this means that, for every $x \in X(\mathbb{C})$, the Hermitian semi-norms $\|.\|_{\overline{\mathcal{F}},x}$ and $\|.\|_{\overline{\mathcal{G}},x}$ are actually Hermitian norms, and satisfy : $\|g_x(v)\|_{\overline{\mathcal{G}},x} < \|v\|_{\overline{\mathcal{F}},x}$ for every non-zero v in $\mathcal{F}_x$.

then the map:

$$p := (g, \mathrm{Id}_{\mathcal{H}}) : \mathcal{G} \oplus \mathcal{H} \longrightarrow \mathcal{H}, \quad (y, z) \longmapsto g(y) + z$$

becomes an admissible surjective morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$ *:*

(5.2.18) $$p : \overline{\mathcal{G}} \oplus \overline{\mathcal{H}}^{\sim} \longrightarrow \overline{\mathcal{H}}.$$

Proof of Proposition 5.2.10. The surjective map p defines an admissible morphism (5.2.18) if and only if, for every $x \in X(\mathbb{C})$, the $\mathbb{C}$-linear map:

$$p_x : \mathcal{G}_x \oplus \mathcal{H}_x \longrightarrow \mathcal{H}_x, \quad (y, z) \longmapsto g_x(y) + Z$$

defines a "coisometry" from $\overline{\mathcal{G}}_x \oplus \overline{\mathcal{H}}_x^{\sim}$ onto $\overline{\mathcal{H}}_x$, or equivalently, if the transpose map:

$$p_x^{\vee} : \mathcal{H}_x \longrightarrow \mathcal{G}_x^{\vee} \oplus \mathcal{H}_x^{\vee}, \quad \xi \longmapsto g_x^{\vee}(\xi) \oplus \xi$$

is an isometry from $\overline{\mathcal{H}}_x^{\vee}$ into $\overline{\mathcal{G}}_x^{\vee} \oplus \overline{\mathcal{H}}_x^{\sim\vee}$. This holds precisely when, for every $\xi \in \mathcal{H}_x^{\vee}$, the following equality holds:

(5.2.19) $$\|g_x^{\vee}(\xi)\|^{\vee 2}_{\overline{\mathcal{G}}_x} + \|\xi\|^{\sim\vee 2}_x = \|\xi\|^{\vee 2}_{\overline{\mathcal{H}}_x}.$$

The morphism $g : \overline{\mathcal{G}} \to \overline{\mathcal{H}}$ has operator norms < 1 if and only if the transpose maps $g_x^{\vee}$ from $\overline{\mathcal{H}}_x^{\vee}$ to $\overline{\mathcal{G}}_x^{\vee}$ have operator norms < 1. When this holds, we may define some Hermitian norms $(\|.\|_x)_{x \in X(\mathbb{C})}$ on the $\mathbb{C}$-vector spaces $(\mathcal{H}_x^{\vee})_{x \in X(\mathbb{C})}$ by the relations:

$$\|\xi\|_x^2 := \|\xi\|^{\vee 2}_{\overline{\mathcal{H}}_x} - \|g_x^{\vee}(\xi)\|^{\vee 2}_{\overline{\mathcal{G}}_x},$$

and the previous discussion shows that the Hermitian norms:

$$(\|.\|_x^{\sim})_{x \in X(\mathbb{C})} := (\|.\|_x^{\vee})_{x \in X(\mathbb{C})},$$

duals of the Hermitian norms $\|.\|_x$, define the unique Hermitian structure $\overline{\mathcal{H}}^{\sim}$ on $\mathcal{H}$ that make (5.2.18) an admissible surjective morphism. □

Proof of Proposition 5.2.9. The direct implication:

$$\mathbf{StMon^1} \Longrightarrow \left[\mathbf{StMon^1_1} \text{ and } \mathbf{StMon^1_3}\right]$$

is clear.

Conversely, assume that φ satisfies $\mathbf{Add}_{\oplus}$, $\mathbf{Cont}^+$, $\mathbf{StMon^1_1}$ and $\mathbf{StMon^1_3}$, and consider a diagram in $\overset{\dots}{\mathbf{Coh}}_X^{\leq 1}$ of the form:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}.$$

To complete the proof of Proposition 5.2.9, we want to prove the inequality:

(5.2.20) $$\mathrm{rk}^1_{\varphi}(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \leq \mathrm{rk}^1_{\varphi}(f : \mathcal{F} \to \overline{\mathcal{G}}).$$

Let us first assume that $\overline{\mathcal{G}}$ and $\overline{\mathcal{H}}$ are objects of $\overline{\mathbf{Coh}}_{\overline{X}}$ and that f has operator norms < 1. Then we may introduce $\overline{\mathcal{H}}^{\sim}$ and p as in Proposition 5.2.10. The morphism of $\mathcal{O}_K$-modules:

$$f' := f \oplus 0 : \mathcal{F} \longrightarrow \mathcal{G} \oplus \mathcal{H}, \quad x \longmapsto f(x) \oplus 0$$

satisfies the relation:

(5.2.21) $$p \circ f' = g \circ f.$$

Moroever we have:

$$\mathrm{im}\, f' = \mathrm{im}\, f \oplus 0,$$

and the quotient of $\overline{\mathcal{G}} \oplus \overline{\mathcal{H}}^{\sim}$ by the image of f' may be identified with $\overline{\mathcal{G}/\mathrm{im}\, f} \oplus \overline{\mathcal{H}}^{\sim}$. Therefore, using $\mathbf{Add}_{\oplus}$, we obtain:

(5.2.22)
$$\mathrm{rk}^1_{\varphi}(f' : \mathcal{F} \to \overline{\mathcal{G}} \oplus \overline{\mathcal{H}}^{\sim}) = \varphi(\overline{\mathcal{G}} \oplus \overline{\mathcal{H}}^{\sim}) - \varphi(\overline{\mathcal{G}/\mathrm{im}\, f} \oplus \overline{\mathcal{H}}^{\sim}) = \varphi(\overline{\mathcal{G}}) - \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}) = \mathrm{rk}^1_{\varphi}(f : \mathcal{F} \to \overline{\mathcal{G}}).$$

Finally, since p is an admissible surjection and φ satisfies $\mathbf{StMon_1^1}$ and $\mathbf{StMon_3^1}$, Proposition 5.2.4 implies the inequality:

$$\mathrm{rk}^1_\varphi(p \circ f') \leq \mathrm{rk}^1_\varphi(f'), \tag{5.2.23}$$

which together with (5.2.21) and (5.2.23) establishes the inequality (5.2.20).

We may now complete the proof of (5.2.20) when $g : \overline{\mathcal{G}} \to \overline{\mathcal{H}}$ is an arbitrary morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$ by an approximation argument.

Indeed we may construct sequences $((\|.\|_{\mathcal{G}_x,n})_{x\in X(\mathbb{C})})_{n\in\mathbb{N}}$ (resp. $((\|.\|_{\mathcal{H}_x,n})_{x\in X(\mathbb{C})})_{n\in\mathbb{N}}$) of families of Hermitian norms on the $\mathbb{C}$-vector spaces $(\mathcal{G}_x)_{x\in X(\mathbb{C})}$ (resp. $(\mathcal{H}_x)_{x\in X(\mathbb{C})}$) such that the following conditions are satisfied:

- for every $n \in \mathbb{N}$, the families $(\|.\|_{\mathcal{G}_x,n})_{x\in X(\mathbb{C})}$ and $(\|.\|_{\mathcal{H}_x,n})_{x\in X(\mathbb{C})}$ are invariant under complex conjugation, and therefore define some objects of $\overline{\mathbf{Coh}}_X$:
$$\overline{\mathcal{G}}_n := (\mathcal{G}, (\|.\|_{\mathcal{G}_x,n})_{x\in X(\mathbb{C})}) \quad \text{and} \quad \overline{\mathcal{H}}_n := (\mathcal{H}, (\|.\|_{\mathcal{G}_x,n})_{x\in X(\mathbb{C})});$$
- for every n, the morphism $g : \overline{\mathcal{G}}_n \to \overline{\mathcal{H}}_n$ has operator norms < 1;
- for every $x \in X(\mathbb{C})$, the sequence of norms $(\|.\|_{\mathcal{G}_x,n})_{n\in\mathbb{N}}$ (resp. $(\|.\|_{\mathcal{H}_x,n})_{n\in\mathbb{N}}$) is decreasing and converges to the norm $\|.\|_{\overline{\mathcal{G}},x}$ (resp. $\|.\|_{\overline{\mathcal{H}},x}$).

For instance, we may choose some families Hermitian norms $(\|.\|^\sim_{\mathcal{G}_x})_{x\in X(\mathbb{C})}$ (resp. $(\|.\|^\sim_{\mathcal{H}_x})_{x\in X(\mathbb{C})}$)) on the $\mathbb{C}$-vector spaces $(\mathcal{G}_x)_{x\in X(\mathbb{C})}$ (resp. $(\mathcal{H}_x)_{x\in X(\mathbb{C})}$), invariant under complex conjugation, such that the morphisms g_x from $(\mathcal{G}_x, \|.\|^\sim_{\mathcal{G}_x})$ to $(\mathcal{H}_x, \|.\|^\sim_{\mathcal{H}_x})$ have operator norms < 1 and define, for every $x \in X(\mathbb{C})$ and $n \in \mathbb{N}$:

$$\|.\|^2_{\mathcal{G}_x,n} := \|.\|^2_{\overline{\mathcal{G}},x} + 2^{-n}\|.\|^{\sim 2}_{\mathcal{G}_x} \quad \text{and} \quad \|.\|^2_{\mathcal{H}_x,n} := \|.\|^2_{\overline{\mathcal{H}},x} + 2^{-n}\|.\|^{\sim 2}_{\mathcal{H}_x}.$$

According to the first part of the proof, applied to the diagrams:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}}_n \xrightarrow{g} \overline{\mathcal{H}}_n,$$

the following inequality holds for every $n \in \mathbb{N}$:

$$\mathrm{rk}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}_n) \leq \mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}_n).$$

Moreover, according to $\mathbf{Cont}^+$, when n goes to infinity, the φ-rank:

$$\mathrm{rk}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}_n) := \varphi(\overline{\mathcal{H}}_n) - \varphi(\overline{\mathcal{H}/\mathrm{im}\, g \circ f}_n)$$

(resp. the φ-rank:

$$\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}_n) := \varphi(\overline{\mathcal{G}}_n) - \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}_n) \quad)$$

converges to:

$$\mathrm{rk}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) := \varphi(\overline{\mathcal{H}}) - \varphi(\overline{\mathcal{H}/\mathrm{im}\, g \circ f}),$$

(resp. to:

$$\mathrm{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) := \varphi(\overline{\mathcal{G}}) - \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}), \quad)$$

and (5.2.20) follows. □

5.2.4. The strong monotonicity of invariants with values in $[0, +\infty]$**.** It is sometimes useful to extend the notion of strong monotonicity introduced in Definition 5.2.1 above so that it makes sense for arbitrary positive invariants:

$$\varphi : \overline{\mathbf{Sh}} \longrightarrow [0, +\infty],$$

taking possibly the value $+\infty$, defined on the category $\overline{\mathbf{Sh}} := \overline{\mathbf{Coh}}_X$ or $\mathbf{q}\overline{\mathbf{Coh}}_X$.

As already observed, the discussion in paragraph 5.1.2.2 concerning diagrams in $\ddot{\overline{\mathbf{Coh}}}_X$ and $\ddot{\overline{\mathbf{Coh}}}_X^{\leq 1}$, immediately extends to this more general setting. This allows us to consider *diagrams in* $\ddot{\overline{\mathbf{Sh}}}$ and *in* $\ddot{\overline{\mathbf{Sh}}}^{\leq 1}$, and to introduce the following definition:

DEFINITION 5.2.11. With the previous notation, we say that φ satisfies the *strong monotonicity condition* $\mathbf{StMon^1}$ when φ satisfies the monotonicity condition $\mathbf{Mon^1}$ and, for every diagram in $\ddot{\mathbf{Sh}}^{\leq 1}$ of the form:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}},$$

the following inequality holds:

$$\varphi(\overline{\mathcal{H}}) + \varphi(\overline{\mathcal{G}/\mathrm{im}\, f}) \leq \varphi(\overline{\mathcal{G}}) + \varphi(\overline{\mathcal{H}/\mathrm{im}\,(g \circ f)}). \tag{5.2.24}$$

According to Proposition 5.2.2, this definition coincides with Definition 5.2.1 when $\overline{\mathbf{Sh}} = \overline{\mathbf{Coh}}_X$ and φ is valued in $\mathbb{R}_+$.

Clearly, in condition (5.2.24), the map f enters only through its image $\mathcal{G}' := f(\mathcal{F})$. Consequently the definition of the strong monotonicity of φ may be rephrased as follows: *for every morphism* $g : \overline{\mathcal{G}} \to \overline{\mathcal{H}}$ *in* $\overline{\mathbf{Sh}}^{\leq 1}$ *and every* $\mathcal{O}_K$*-submodule* $\mathcal{G}'$ *of* $\mathcal{G}$*, of image* $\mathcal{H}' := g(\mathcal{G}')$ *in* $\mathcal{H}$*, we have*:

$$\varphi(\overline{\mathcal{H}}) + \varphi(\overline{\mathcal{G}/\mathcal{G}'}) \leq \varphi(\overline{\mathcal{G}}) + \varphi(\overline{\mathcal{H}/\mathcal{H}'}). \tag{5.2.25}$$

This plays the role, in the present more general setting, of condition (iii) in Proposition 5.2.2.

The conditions $\mathbf{StMon}^\mathbf{1}_i$ introduced in Subsection 5.2.2 still make sense in this more general setting. Namely, we still define $\mathbf{StMon^1_1}$ to be the monotonicity condition $\mathbf{Mon^1}$, and we define $\mathbf{StMon}^\mathbf{1}_i$ for $i \in \{2, 3, 4\}$ as follows:

$\mathbf{StMon^1_2}$: *for every object* $\overline{\mathcal{F}}$ *of* $\overline{\mathbf{Sh}}$ *and any two* $\mathcal{O}_K$*-submodules* $\mathcal{F}'' \subseteq \mathcal{F}' \subseteq \mathcal{F}$*, the following inequality is satisfied:*

$$\varphi(\overline{\mathcal{F}}) + \varphi(\overline{\mathcal{F}'/\mathcal{F}''}) \leq \varphi(\overline{\mathcal{F}}') + \varphi(\overline{\mathcal{F}/\mathcal{F}''}).$$

$\mathbf{StMon^1_3}$: *for every object* $\overline{\mathcal{F}}$ *of* $\overline{\mathbf{Sh}}$ *and every two submodules* $\mathcal{F}'$ *and* $\mathcal{F}''$ *of* $\mathcal{F}$, *the following inequality is satisfied:*

$$\varphi(\overline{\mathcal{F}/\mathcal{F}'}) + \varphi(\overline{\mathcal{F}/\mathcal{F}''}) \leq \varphi(\overline{\mathcal{F}/(\mathcal{F}' + \mathcal{F}'')}) + \varphi(\overline{\mathcal{F}/(\mathcal{F}' \cap \mathcal{F}'')}).$$

$\mathbf{StMon^1_4}$: *for any two objects* $\overline{\mathcal{G}}$ *and* $\overline{\mathcal{G}}^\sim$ *of* $\overline{\mathbf{Coh}}_X$ *with the same underlying* $\mathcal{O}_K$*-module* $\mathcal{G}$ *such that* $\mathrm{Id}_\mathcal{G} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}^\sim$ *is a morphism in* $\overline{\mathbf{Coh}}_X^{\leq 1}$ *and for any subobject* $\mathcal{G}'$ *of* $\mathcal{G}$*, the following inequality is satisfied:*

$$\varphi(\overline{\mathcal{G}}^\sim) + \varphi(\overline{\mathcal{G}/\mathcal{G}'}) \leq \varphi(\overline{\mathcal{G}/\mathcal{G}'}^\sim) + \varphi(\overline{\mathcal{G}}).$$

As before, these three conditions will be referred to as the *strong subadditivity, submodularity* and *metric monotonicity* conditions respectively.

Clearly the implication:

$$\mathbf{StMon^1} \Longrightarrow \left[\mathbf{StMon^1_1}, \mathbf{StMon^1_2}, \mathbf{StMon^1_3}, \text{and } \mathbf{StMon^1_4}\right]$$

still holds. However, in this generality, it is not clear whether the converse implication holds.

The submodularity estimates of paragraph 5.2.2.3 extend to this general setting:

PROPOSITION 5.2.12. *For every invariant* $\varphi : \overline{\mathbf{Sh}} \to [0, +\infty]$ *that satisfies* $\mathbf{Mon^1}$ *and every object* $\overline{\mathcal{E}}$ *of* $\overline{\mathbf{Sh}}$*, the conditions* (i)*–*(iv) *in Proposition 5.2.6 are equivalent.*

PROOF. The proof of Proposition 5.2.6 remains valid with the following some minor modifications.

The proof of $(i) \Rightarrow (ii)$ may be rephrased as follows. Let us assume that the submodularity inequality (5.2.10) holds for any two submodules $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{E}$. For every submodules $\mathcal{X}$, $\mathcal{X}'$, and $\mathcal{Y}$ of $\mathcal{E}$ such that $\mathcal{X} \subseteq \mathcal{X}'$, when applied to $\mathcal{E}_1 = \mathcal{X}'$ and $\mathcal{E}_2 = \mathcal{X} + \mathcal{Y}$, this inequality reads:

$$\varphi(\overline{\mathcal{E}/\mathcal{X}'}) + \varphi(\overline{\mathcal{E}/(\mathcal{X} + \mathcal{Y})}) \leq \varphi(\overline{\mathcal{E}/(\mathcal{X}' + \mathcal{Y})}) + \varphi(\overline{\mathcal{E}/(\mathcal{X}' \cap (\mathcal{X} + \mathcal{Y})}).$$

Moreover, according to the monotonicity of φ, we have:

$$\varphi(\overline{\mathcal{E}/(\mathcal{X}' \cap (\mathcal{X} + \mathcal{Y}))}) \leq \varphi(\overline{\mathcal{E}/\mathcal{X}}),$$

since $\mathcal{X}$ is a submodule of $\mathcal{X}' + (\mathcal{X} \cap \mathcal{Y})$. Consequently the inequality (5.2.11) holds.

The proof of $(iii) \Rightarrow (i)$ is unchanged when $\varphi(\overline{\mathcal{E}/\mathcal{E}_2}) < +\infty$. When $\varphi(\overline{\mathcal{E}/\mathcal{E}_2})$ is infinite, then $\varphi(\overline{\mathcal{E}/(\mathcal{E}_1 \cap \mathcal{E}_2)})$ also is by monotonicity of φ, and (5.2.10) trivially holds.

The proof of $(iii) \Rightarrow (iv)$ is unchanged when $\varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \mathcal{B}_2)})$ is finite. When $\varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \mathcal{B}_2)})$ is infinite, then $\varphi(\overline{\mathcal{E}/(\mathcal{B}_1 + \cdots + \mathcal{B}_{n+1})})$ also is, by monotonicity of φ, and therefore (5.2.15) still holds. □

5.3. The Cone of Strongly Monotonic Invariants

At this stage, the existence of any non-trivial $\mathbb{R}_+$-valued invariant on $\overline{\mathbf{Coh}}_X$ that satisfies conditions $\mathbf{StMon^1}$ and $\mathbf{Cont}^+$, that vanish on torsion modules and are distinct from a multiple of the invariant "rank", is unclear. In Chapter 7, we will show that, when $X = \operatorname{Spec} \mathbb{Z}$, the theta-invariant

$$h^1_\theta : \overline{\mathbf{Vect}}_{\mathbb{Z}} \longrightarrow \mathbb{R}_+,$$

defined by the equality:

$$h^1_\theta(\overline{E}) := \log \theta_{\overline{E}^\vee}(1) = \log \sum_{v \in E^\vee} e^{-\pi \|v\|^2_{\overline{E}^\vee}}$$

where $\overline{E}$ denotes a Euclidean lattice $\overline{E}$ and $\overline{E}^\vee$ its dual, does indeed induce by the construction of 4.2 an invariant h^1_θ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ that satisfies these conditions.

Starting from the invariant h^1_θ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, we may construct some new invariants on $\overline{\mathbf{Coh}}_X$ satisfying $\mathbf{StMon^1}$ and $\mathbf{Cont}^+$ by means of the following three constructions.

(i) *Inverse image.* As discussed in Section 4.6, starting from some invariant $\varphi_{\mathbb{Z}} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \to \mathbb{R}_+$, we define a new one $\pi^*\varphi_{\mathbb{Z}} : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ by the formula:

$$\pi^*\varphi_{\mathbb{Z}}(\overline{\mathcal{F}}) := \varphi(\pi_*\overline{\mathcal{F}}).$$

In this way, starting from the invariant $\varphi_{\mathbb{Z}} := h^1_\theta$ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, one constructs the invariant still denoted by h^1_θ:

$$h^1_\theta : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+, \quad \overline{\mathcal{F}} \longmapsto h^1_\theta(\pi_*\overline{\mathcal{F}}).$$

As observed in 4.6, the construction of $\pi^*\varphi_{\mathbb{Z}}$ from $\varphi_{\mathbb{Z}}$ preserves the validity of the conditions $\mathbf{Mon^1}$, $\mathbf{SubAdd}$, $\mathbf{VT}$, and $\mathbf{Cont}^+$. It is straightforward that it also preserves the validity of the strong monotonicity condition $\mathbf{StMon^1}$ and of each of the conditions $\mathbf{StMon}^{\mathbf{1}}_i$, $i = 2, 3, 4$.

(ii) *Tensor product.* Starting from some invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ and from some object $\overline{\mathcal{F}}$ of $\overline{\mathbf{Vect}}^{[0]}_X$, we may form the new invariant:

$$\varphi_{\overline{\mathcal{F}}} : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+, \quad \overline{\mathcal{C}} \longrightarrow \varphi(\overline{\mathcal{F}} \otimes \overline{\mathcal{C}}).$$

Here again, it is easy to check that if φ satisfies one of the properties $\mathbf{Mon^1}$, $\mathbf{SubAdd}$, $\mathbf{VT}$, $\mathbf{Cont}^+$, or $\mathbf{StMon^1}$, then so does $\varphi_{\overline{\mathcal{F}}}$.

(iii) *Infinite linear combinations with positive coefficients.* Consider a family $(\varphi_\alpha)_{\alpha \in A}$ of invariants

$$\varphi_\alpha : \overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty],$$

and a family $(\lambda_\alpha)_{\alpha \in A}$ of elements of $[0, +\infty]$. As in Proposition 4.1.5, we may consider the invariant

$$\sum_{\alpha \in A} \lambda_\alpha \varphi_\alpha : \overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty].$$

As observed in Proposition 4.1.5, this construction preserves the validity of the conditions $\mathbf{Mon^1}$, $\mathbf{SubAdd}$, $\mathbf{VT}$, as well as $\mathbf{Cont}^+$ when all the λ_α are finite. Again, this construction preserves the validity of the strong monotonicity condition $\mathbf{StMon^1}$ and of each of the conditions $\mathbf{StMon}^{\mathbf{1}}_i$, $i = 2, 3, 4$, understood in the generalized sense of Subsection 5.2.4.

Furthermore, if the invariants $(\varphi_\alpha)_{\alpha\in A}$ satisfy $\mathbf{Mon^1}$ and **SubAdd** and if the condition:

$$\sum_{\alpha\in A}\lambda_\alpha\varphi_\alpha(\overline{\mathcal{O}}(-\delta)) < +\infty \quad \text{for every } \delta\in\mathbb{R},$$

is satisfied, then Proposition 4.1.5 shows that the invariant

$$\varphi := \sum_\alpha \lambda_\alpha\varphi_\alpha$$

takes finite values on $\overline{\mathbf{Coh}}_X$. Moreover we have:

$$\mathrm{rk}^1_\varphi = \sum_{\alpha\in A}\lambda_\alpha \mathrm{rk}^1_{\varphi_\alpha}.$$

By combining the three constructions above, we finally obtain:

Construction 5.3.1. *If $(\overline{\mathcal{F}}_\alpha)_{\alpha\in A}$ is a family of objects of $\overline{\mathbf{Vect}}^{[0]}_X$ and $(\lambda_\alpha)_{\alpha\in A}$ is a family of elements of $\mathbb{R}_+$ satisfying the condition:*

$$\sum_{\alpha\in A}\lambda_\alpha h^1_\theta(\overline{\mathcal{F}}_\alpha\otimes\overline{\mathcal{O}}(-\delta)) := \sum_{\alpha\in A}\lambda_\alpha h^1_\theta(\pi_*\overline{\mathcal{F}}_\alpha\otimes\overline{\mathcal{O}}(-\delta)) < +\infty \quad \textit{for every } \delta\in\mathbb{R}, \tag{5.3.1}$$

then one defines an invariant φ on $\overline{\mathbf{Coh}}_X$ with values in $\mathbb{R}_+$ that satisfies the conditions $\mathbf{StMon}^1$ *and* $\mathbf{Cont}^+$ *by letting:*

$$\varphi(\overline{\mathcal{E}}) := \sum_{\alpha\in A}\lambda_\alpha h^1_\theta(\overline{\mathcal{F}}_\alpha\otimes\overline{\mathcal{E}}) = \sum_{\alpha\in A}\lambda_\alpha h^1_\theta(\pi_*(\overline{\mathcal{F}}_\alpha)\otimes\overline{\mathcal{E}}). \tag{5.3.2}$$

One may ask whether every $\mathbb{R}_+$-valued invariant on $\overline{\mathbf{Coh}}_X$ satisfying $\mathbf{StMon^1}$, **VT**, and $\mathbf{Cont}^+$ belongs to the closure, in the topology of pointwise convergence on invariants, of the cone defined by the sums of positive multiples of the invariant "rank" and of the invariants described in Construction 5.3.1.[10]

5.4. The Lower φ-Rank $\underline{\mathrm{rk}}^1_\varphi$ and the Strong Monotonicity of $\underline{\varphi}$

In this section, we consider an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ that satisfies the condition $\mathbf{Mon}^1$, and we pursue the study of its lower extension:

$$\underline{\varphi} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0,+\infty]$$

initiated in the previous chapter (see Section 4.3) by investigating the consequence on $\underline{\varphi}$ of the strong monotonicity of φ.

We notably show that when φ is strongly monotonic, the φ-rank rk^1_φ admits a natural "lower extension" $\underline{\mathrm{rk}}^1_\varphi$, which attaches an element $\underline{\mathrm{rk}}^1_\varphi(f:\mathcal{F}\to\overline{\mathcal{G}})$ in $[0,+\infty]$ to any morphism $f:\mathcal{F}\to\overline{G}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, and we establish the strong monotonicity of $\underline{\varphi}$ as a consequences of the properties of $\underline{\mathrm{rk}}^1_\varphi$.

5.4.1. The lower φ-rank $\underline{\mathrm{rk}}^1_\varphi$: definitions. Let $\varphi : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$ be an invariant that satisfies the monotonicity condition $\mathbf{Mon}^1$ and the submodularity condition $\mathbf{StMon^1_3}$.

[10] In the terminology of Chapter 8, if the $\mathcal{F}_\alpha$ are θ^1-finite objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfying the convergence conditions (5.3.1), then the formula (5.3.2) still defines some $\mathbb{R}_+$-valued invariant on $\overline{\mathbf{Coh}}_X$ satisfying $\mathbf{StMon}^1$, **VT**, and $\mathbf{Cont}^+$, which is easily proved to belong to the closure of this cone.

5.4.1.1. Consider a morphism in $\mathbf{q}\ddddot{\mathbf{Coh}}_X$:

$$f : \mathcal{F} \longrightarrow \overline{\mathcal{G}}.$$

The inverse image $f^{-1}(\mathcal{G}')$ of any $\mathcal{O}_X$-submodule $\mathcal{G}'$ of $\mathcal{G}$ in $\operatorname{coft}(\mathcal{G})$ belongs to $\operatorname{coft}(\mathcal{F})$. Indeed the morphism induced by f:

$$f_{\mathcal{G}'} : \mathcal{F}/f^{-1}(\mathcal{G}') \longrightarrow \mathcal{G}/\mathcal{G}'$$

is injective, and accordingly $\mathcal{F}/f^{-1}(\mathcal{G}')$ like $\mathcal{G}/\mathcal{G}'$ is coherent. The map $f_{\mathcal{G}'}$ defines a morphism in $\ddddot{\mathbf{Coh}}_X$:

$$f_{\mathcal{G}'} : \mathcal{F}/f^{-1}(\mathcal{G}') \longrightarrow \overline{\mathcal{G}/\mathcal{G}'},$$

and we may consider its φ-rank:

$$\operatorname{rk}^1_\varphi \left(f_{\mathcal{G}'} : \mathcal{F}/f^{-1}(\mathcal{G}') \longrightarrow \overline{\mathcal{G}/\mathcal{G}'} \right) := \varphi(\overline{\mathcal{G}/\mathcal{G}'}) - \varphi(\overline{\mathcal{G}/\operatorname{im} f + \mathcal{G}'}) \in \mathbb{R}_+. \tag{5.4.1}$$

Observe that if $\mathcal{G}''$ is an element of $\operatorname{coft}(\mathcal{G})$ contained in $\mathcal{G}'$, then the morphism

$$f_{\mathcal{G}''} : \mathcal{F}/f^{-1}(\mathcal{G}'') \longrightarrow \mathcal{G}/\mathcal{G}''$$

associated to $\mathcal{G}''$ fits into a commutative diagram in $\mathbf{q}\ddddot{\mathbf{Coh}}_X^{\leq 1}$:

$$\begin{tikzcd}
\mathcal{F}/f^{-1}(\mathcal{G}'') \arrow[r, "f_{\mathcal{G}''}"] \arrow[d, "p"] & \overline{\mathcal{G}/\mathcal{G}''} \arrow[d, "q"] \\
\mathcal{F}/f^{-1}(\mathcal{G}') \arrow[r, "f_{\mathcal{G}'}"] & \overline{\mathcal{G}/\mathcal{G}'}
\end{tikzcd}$$

where p and q denote the quotient morphisms. Since p is surjective and q is an admissible surjection, the submodularity condition $\mathbf{StMon^1_3}$ implies the following inequality:

$$\operatorname{rk}^1_\varphi \left(f_{\mathcal{G}'} : \mathcal{F}/f^{-1}(\mathcal{G}') \longrightarrow \overline{\mathcal{G}/\mathcal{G}'} \right) \leq \operatorname{rk}^1_\varphi \left(f_{\mathcal{G}''} : \mathcal{F}/f^{-1}(\mathcal{G}'') \longrightarrow \overline{\mathcal{G}/\mathcal{G}''} \right).$$

In other words, the function

$$\operatorname{coft}(\mathcal{G}) \longrightarrow \mathbb{R}_+, \quad \mathcal{G}' \longmapsto \operatorname{rk}^1_\varphi \left(f_{\mathcal{G}'} : \mathcal{F}/f^{-1}(\mathcal{G}') \to \overline{\mathcal{G}/\mathcal{G}'} \right)$$

is increasing on the directed set $(\operatorname{coft}(\mathcal{G}), \supseteq)$.

The previous observation leads us to introduce the following definition:

Definition 5.4.1. For every invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ that satisfies the conditions $\mathbf{Mon^1}$ and $\mathbf{StMon^1_3}$, the *lower φ-rank* of a morphism $f : \mathcal{F} \to \overline{\mathcal{G}}$ in $\mathbf{q}\ddddot{\mathbf{Coh}}_X$ is defined as the following limit over $\mathcal{G}'$ in the directed set $(\operatorname{coft}(\mathcal{G}), \supseteq)$:

$$\begin{aligned}
\underline{\operatorname{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) &:= \lim_{\mathcal{G}' \in \operatorname{coft}(\mathcal{G})} \operatorname{rk}^1_\varphi \left(f'_{\mathcal{G}} : \mathcal{F}/f^{-1}(\mathcal{G}') \to \overline{\mathcal{G}/\mathcal{G}'} \right) \\
&= \sup_{\mathcal{G}' \in \operatorname{coft}(\mathcal{G})} \operatorname{rk}^1_\varphi \left(f'_{\mathcal{G}} : \mathcal{F}/f^{-1}(\mathcal{G}') \to \overline{\mathcal{G}/\mathcal{G}'} \right) \in [0, +\infty].
\end{aligned} \tag{5.4.2}$$

When f is a morphism in $\ddddot{\mathbf{Coh}}_X$, $\underline{\operatorname{rk}}^1_\varphi(f)$ clearly coincides with $\operatorname{rk}^1_\varphi(f)$.

The expression for $\underline{\operatorname{rk}}^1_\varphi(f)$ together with the equality (5.4.1) show that it depends only on $\overline{\mathcal{G}}$ and on the submodule $\operatorname{im} f$ of $\mathcal{G}$. In other words, if $\iota : \operatorname{im} f \to \mathcal{G}$ denotes the inclusion morphism, we have:

$$\underline{\operatorname{rk}}^1_\varphi(f : \mathcal{F} \longrightarrow \overline{\mathcal{G}}) = \underline{\operatorname{rk}}^1_\varphi(\iota : \operatorname{im} f \to \overline{\mathcal{G}}).$$

When moreover $\overline{\mathcal{G}}$ is an object of $\overline{\mathbf{Coh}}_X$, this coincides with $\operatorname{rk}^1_\varphi(\iota : \operatorname{im} f \to \overline{\mathcal{G}})$.

From the definitions of $\underline{\mathrm{rk}}^1_\varphi$ and of the lower extension $\underline{\varphi}$ of φ, we immediately derive the following identity, valid for any object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$\mathrm{rk}^1_\varphi(\mathrm{Id}_{\mathcal{F}} : \mathcal{F} \to \overline{\mathcal{F}}) = \underline{\varphi}(\overline{\mathcal{F}}).$$

5.4.1.2. The above definition of the lower rank $\underline{\mathrm{rk}}^1_\varphi f$ of a morphism $f : \mathcal{F} \to \overline{\mathcal{G}}$ in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_X$ admits the following more flexible variant.

Let us introduce the following subset of $\mathrm{coft}(\mathcal{F}) \times \mathrm{coft}(\mathcal{G})$:

$$\mathrm{coft}(f) := \{(\mathcal{F}', \mathcal{G}') \in \mathrm{coft}(\mathcal{F}) \times \mathrm{coft}(\mathcal{G}) \mid f(\mathcal{F}') \subseteq \mathcal{G}'\},$$

and let us define a relation $\supseteq$ on $\mathrm{coft}(f)$ by letting:

$$(\mathcal{F}', \mathcal{G}') \supseteq (\mathcal{F}'', \mathcal{G}'') \overset{\text{def}}{\Longleftrightarrow} [\mathcal{F}' \supseteq \mathcal{F}'' \text{ and } \mathcal{G}' \supseteq \mathcal{G}''],$$

Then $(\mathrm{coft}(f), \supseteq)$ is easily seen to be a directed set, and to any $(\mathcal{F}', \mathcal{G}')$ in $\mathrm{coft}(f)$ is associated a morphism in $\overset{\cdots}{\mathbf{Coh}}_X$:

$$f_{\mathcal{F}'\mathcal{G}'} : \mathcal{F}/\mathcal{F}' \longrightarrow \overline{\mathcal{G}/\mathcal{G}'}$$

induced by f. Its φ-rank is:

$$\mathrm{rk}^1_\varphi\left(f_{\mathcal{F}'\mathcal{G}'} : \mathcal{F}/\mathcal{F}' \to \overline{\mathcal{G}/\mathcal{G}'}\right) = \varphi(\overline{\mathcal{G}/\mathcal{G}'}) - \varphi(\overline{\mathcal{G}/\mathrm{im}\, f + \mathcal{G}'}) = \mathrm{rk}^1_\varphi\left(f_{\mathcal{G}'} : \mathcal{F}/f^{-1}(\mathcal{G}') \to \overline{\mathcal{G}/\mathcal{G}'}\right).$$

Moreover for every $\mathcal{G}'$ in $\mathrm{coft}(\mathcal{G})$, the pair $(f^{-1}(\mathcal{G}'), \mathcal{G}')$ belongs to $\mathrm{coft}(f)$, and the map

$$\mathcal{G}' \longmapsto (f^{-1}(\mathcal{G}'), \mathcal{G}') \quad (\text{resp. } (\mathcal{F}', \mathcal{G}') \longmapsto \mathcal{G}'\ )$$

from $(\mathrm{coft}(\mathcal{G}), \supseteq)$ to $(\mathrm{coft}(f), \supseteq)$ (resp. from $(\mathrm{coft}(f), \supseteq)$ to $(\mathrm{coft}(\mathcal{G}), \supseteq)$) is increasing.

These observations show that $\mathrm{rk}^1_\varphi(f_{\mathcal{F}'\mathcal{G}'})$ defines an increasing function from the directed set $(\mathrm{coft}(f), \supseteq)$ to $\mathbb{R}_+$ and that the lower φ-rank of f coincides with its limit over this directed set:

$$\underline{\mathrm{rk}}^1_\varphi f = \lim_{(\mathcal{F}', \mathcal{G}') \in \mathrm{coft}(f)} \mathrm{rk}^1_\varphi\left(f_{\mathcal{F}'\mathcal{G}'} : \mathcal{F}/\mathcal{F}' \to \overline{\mathcal{G}/\mathcal{G}'}\right) = \sup_{(\mathcal{F}', \mathcal{G}') \in \mathrm{coft}(f)} \mathrm{rk}^1_\varphi\left(f_{\mathcal{F}'\mathcal{G}'} : \mathcal{F}/\mathcal{F}' \to \overline{\mathcal{G}/\mathcal{G}'}\right).$$

5.4.1.3. The following proposition shows that the lower φ-rank $\underline{\mathrm{rk}}^1_\varphi(f)$ coincides with the rank associated to the lower extension $\underline{\varphi}$ of φ, defined by the equality:

$$\mathrm{rk}^1_{\underline{\varphi}}(f : \mathcal{F} \to \overline{\mathcal{G}}) := \underline{\varphi}(\overline{\mathcal{G}}) - \underline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}),$$

when its right-hand side makes sense, namely when $\underline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f})$ is finite.

PROPOSITION 5.4.2. *For every invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfying conditions $\mathbf{Mon^1}$ and $\mathbf{StMon^1_3}$ and for every morphism $f : \mathcal{F} \to \overline{\mathcal{G}}$ in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_X$, the following equality holds:*

$$\underline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) + \underline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}) = \underline{\varphi}(\overline{\mathcal{G}}). \tag{5.4.3}$$

PROOF. According to the definition of rk^1_φ, the following equality holds for every $\mathcal{G}'$ in $\mathrm{coft}(\mathcal{G})$:

$$\mathrm{rk}^1_\varphi\left(f_{\mathcal{G}'} : \mathcal{F}/f^{-1}(\mathcal{G}') \to \overline{\mathcal{G}/\mathcal{G}'}\right) + \varphi(\overline{\mathcal{G}/\mathrm{im}\, f + \mathcal{G}'}) = \varphi(\overline{\mathcal{G}/\mathcal{G}'}). \tag{5.4.4}$$

Moreover the map:

$$\mathrm{coft}(\mathcal{G}) \longmapsto \mathrm{coft}(\mathcal{G}/\mathrm{im}\, f), \quad \mathcal{G}' \longmapsto (\mathrm{im}\, f + \mathcal{G}')/\mathrm{im}\, f$$

is surjective and increasing, and therefore:

$$\lim_{\mathcal{G}' \in \mathrm{coft}(\mathcal{G})} \varphi\left(\overline{\mathcal{G}/\mathrm{im}\, f + \mathcal{G}'}\right) = \lim_{\tilde{\mathcal{G}} \in \mathrm{coft}(\mathcal{G}/\mathrm{im}\, f)} \varphi\left(\overline{(\mathcal{G}/\mathrm{im}\, f)/\tilde{\mathcal{G}}}\right) = \underline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}). \tag{5.4.5}$$

The equality (5.4.3) follows from (5.4.4) and (5.4.5) by taking the limit over $\mathcal{G}'$ in the directed set $(\mathrm{coft}(\mathcal{G}), \supseteq)$. □

Observe that, with the notation of Proposition 5.4.2, the following estimate holds:

$$\underline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) \leq \underline{\varphi}(\overline{\mathcal{G}}),$$

as a straightforward consequence of the definition (5.4.3). The reader will easily show, by a limiting argument similar to the one in the proof of Proposition 5.4.2, that when moreover φ satisfies the subadditivity condition **SubAdd**, we also have:

$$\underline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) \leq \underline{\varphi}(\overline{\mathcal{F}}).$$

5.4.2. First properties of $\underline{\mathrm{rk}}^1_\varphi$. In this subsection, we still consider an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfying the conditions $\mathbf{Mon^1}$ and $\mathbf{StMon^1_3}$.

The following propositions show that the properties of the φ-rank rk^1_φ established in Propositions 5.1.3, 5.1.4, and 5.1.5 are still satisfied by the lower φ-rank of morphisms in $\mathbf{q}\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}_X$.

PROPOSITION 5.4.3. *(1) For any diagram in $\mathbf{q}\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}_X$ of the form:*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \overline{\mathcal{H}}, \tag{5.4.6}$$

the following inequality holds:

$$\underline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \leq \underline{\mathrm{rk}}^1_\varphi(g : \mathcal{G} \to \overline{\mathcal{H}}). \tag{5.4.7}$$

(2) If moreover the morphism of $\mathcal{O}_K$-modules $f : \mathcal{F} \longrightarrow \mathcal{G}$ is surjective, or if φ satisfies condition **NST** *and $f_K : \mathcal{F}_K \longrightarrow \mathcal{G}_K$ is a surjective morphism of K-vector spaces, then we have:*

$$\underline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) = \underline{\mathrm{rk}}^1_\varphi(g : \mathcal{G} \to \overline{\mathcal{H}}). \tag{5.4.8}$$

PROOF. For every $\mathcal{H}'$ in $\mathrm{coft}(\mathcal{H})$, we may consider the diagram in $\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}_X$ induced by (5.4.6):

$$\mathcal{F}/(g \circ f)^{-1}(\mathcal{H}') \xrightarrow{f'} \mathcal{G}/g^{-1}(\mathcal{H}') \xrightarrow{g'} \overline{\mathcal{H}/\mathcal{H}'}, \tag{5.4.9}$$

and apply Proposition 5.1.3 to (5.4.9). Assertion (1) follows by taking the supremum over $\mathcal{H}' \in \mathrm{coft}(\mathcal{H})$. Assertion (2) is a straightforward consequence of the definitions. □

PROPOSITION 5.4.4. *Assume that the invariant φ is also subadditive. For any diagram in $\mathbf{q}\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}{}^{\leq 1}_X$ of the form:*

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}, \tag{5.4.10}$$

the following inequality holds:

$$\underline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{G}}) + \underline{\varphi}(\overline{\mathcal{G}}) \geq \underline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) + \underline{\mathrm{rk}}^1_\varphi(g : \mathcal{G} \to \overline{\mathcal{H}}). \tag{5.4.11}$$

PROOF. The inequality (5.4.11) is clear when $\underline{\varphi}(\overline{\mathcal{G}})$ is $+\infty$.

Let us now assume that $\underline{\varphi}(\overline{\mathcal{G}})$ is finite. Then $\underline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f})$ also is finite by the monotonicity of $\underline{\varphi}$, and according to Proposition 5.4.2, the inequality (5.4.11) is equivalent to the following inequality:

$$\underline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) + \underline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}) \geq \underline{\mathrm{rk}}^1_\varphi(g : \mathcal{G} \to \overline{\mathcal{H}}). \tag{5.4.12}$$

For every $(\mathcal{G}', \mathcal{H}') \in \mathrm{coft}(g)$, we may consider the diagram in $\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}_X$ induced by (5.4.10):

$$\mathcal{F}/f^{-1}(\mathcal{G}') \xrightarrow{f'} \overline{\mathcal{G}/\mathcal{G}'} \xrightarrow{g'} \overline{\mathcal{H}/\mathcal{H}'}. \tag{5.4.13}$$

According to Proposition 5.1.4 applied to (5.4.13), the following inequality holds:

$$\mathrm{rk}^1_\varphi\left(g' \circ f' : \mathcal{F}/f^{-1}(\mathcal{G}') \to \overline{\mathcal{H}/\mathcal{H}'}\right) + \varphi\left(\overline{\mathcal{G}/\mathrm{im}\, f + \mathcal{G}'}\right) \geq \mathrm{rk}^1_\varphi\left(g : \overline{\mathcal{G}/\mathcal{G}'} \to \overline{\mathcal{H}/\mathcal{H}'}\right).$$

Consequently, we have:

$$\underline{\mathrm{rk}}^1_\varphi\left(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}\right) + \underline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}) \geq \mathrm{rk}^1_\varphi\left(g : \overline{\mathcal{G}/\mathcal{G}'}, \to \overline{\mathcal{H}/\mathcal{H}'}\right).$$

and (5.4.12) follows by taking the supremum over $(\mathcal{G}', \mathcal{H}') \in \mathrm{coft}(g)$. $\square$

PROPOSITION 5.4.5. *Assume that the invariant φ is also subadditive. For every commutative diagram in $\mathbf{q}\overset{\cdots\cdot}{\mathbf{Coh}}{}_X^{\leq 1}$:*

$$
\begin{tikzcd}
\mathcal{F}_1 \arrow[r, "f_1"] \arrow[d, "p"] & \overline{\mathcal{G}}_1 \arrow[d, "q"] \\
\mathcal{F}_2 \arrow[r, "f_2"] & \overline{\mathcal{G}}_2,
\end{tikzcd}
\tag{5.4.14}
$$

in which $q : \overline{\mathcal{G}}_1 \to \overline{\mathcal{G}}_2$ is an admissible surjection, the following inequality holds:

$$\underline{\mathrm{rk}}^1_\varphi(f_1 : \mathcal{F}_1 \to \overline{\mathcal{G}}_1) \leq \underline{\mathrm{rk}}^1_\varphi(f_2 : \mathcal{F}_2 \to \overline{\mathcal{G}}_2) + \underline{\varphi}(\overline{\ker q}). \tag{5.4.15}$$

PROOF. Let $\mathcal{G}'_1$ be an element of $\mathrm{coft}(\mathcal{G}_1)$. Since the morphism $q : \mathcal{G}_1 \to \mathcal{G}_2$ is surjective, the submodule $\mathcal{G}'_2 := q(\mathcal{G}'_1)$ of $\mathcal{G}_2$ belongs to $\mathrm{coft}(\mathcal{G}_2)$, and we may consider the commutative diagram in $\overset{\cdots\cdot}{\mathbf{Coh}}{}_X^{\leq 1}$ induced by (5.4.14):

$$
\begin{tikzcd}
\mathcal{F}_1/f_1^{-1}(\mathcal{G}'_1) \arrow[r, "f'_1"] \arrow[d, "p'"] & \overline{\mathcal{G}_1/\mathcal{G}'_1} \arrow[d, "q'"] \\
\mathcal{F}_2/f_2^{-1}(\mathcal{G}'_2) \arrow[r, "f'_2"] & \overline{\mathcal{G}_2/\mathcal{G}'_2}.
\end{tikzcd}
\tag{5.4.16}
$$

The map q' still defines an admissible surjection, and therefore Proposition 5.1.4 may be applied to (5.4.16) and establishes the inequality:

$$\mathrm{rk}^1_\varphi\left(f'_1 : \mathcal{F}_1/f_1^{-1}(\mathcal{G}'_1) \to \overline{\mathcal{G}_1/\mathcal{G}'_1}\right) \leq \mathrm{rk}^1_\varphi\left(f'_2 : \mathcal{F}_2/f_2^{-1}(\mathcal{G}'_2) \to \overline{\mathcal{G}_2/\mathcal{G}'_2}\right) + \varphi\left(\overline{\ker q'}\right). \tag{5.4.17}$$

Observe that the isomorphism of $\mathcal{O}_X$-modules:

$$\ker q/(\mathcal{G}'_1 \cap \ker q) \xrightarrow{\sim} (\mathcal{G}'_1 + \ker q)/\mathcal{G}'_1 = \ker q'$$

defines a morphism in $\overline{\mathbf{Coh}}{}_X^{\leq 1}$:

$$\overline{\ker q/(\mathcal{G}'_1 \cap \ker q)} \longrightarrow \overline{\ker q'}.$$

This shows that $\mathcal{G}'_1 \cap \ker q$ belongs to $\mathrm{coft}(\ker q)$ and that the following inequalities hold:

$$\varphi(\overline{\ker q'}) \leq \varphi(\overline{\ker q/(\mathcal{G}'_1 \cap \ker q)}) \leq \underline{\varphi}(\overline{\ker q}).$$

Therefore the inequality (5.4.17) implies the following inequality:

$$\mathrm{rk}^1_\varphi\left(f'_1 : \mathcal{F}_1/f_1^{-1}(\mathcal{G}'_1) \to \overline{\mathcal{G}_1/\mathcal{G}'_1}\right) \leq \underline{\mathrm{rk}}^1_\varphi\left(f_2 : \mathcal{F}_2 \to \overline{\mathcal{G}_2}\right) + \underline{\varphi}(\overline{\ker q})),$$

and finally (5.4.15) follows by taking the supremum over $\mathcal{G}'_1$ in $\mathrm{coft}(\mathcal{G}_1)$. $\square$

5.4.3. The strong monotonicity of φ and $\underline{\varphi}$.

5.4.3.1. The following proposition and its corollary show that the strong monotonicity of an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ is inherited by its associated lower φ-rank $\underline{\mathrm{rk}}^1_\varphi$ and lower extension $\underline{\varphi}$.

PROPOSITION 5.4.6. *For every invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfying the strong monotonicity condition* $\mathbf{StMon^1}$ *and for every diagram in $\mathbf{q}\overline{\mathbf{Coh}}{}_X^{\leq 1}$ of the form:*

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}, \tag{5.4.18}$$

the following inequality holds in $[0, +\infty]$:

$$\underline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \leq \underline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}). \tag{5.4.19}$$

Proof. For every $\mathcal{H}' \in \operatorname{coft}(\mathcal{H})$, we may consider the following diagram in $\ddddot{\mathbf{Coh}}_X^{\leq 1}$ induced by the diagram (5.4.18):

$$\mathcal{F}/(g\circ f)^{-1}(\mathcal{H}') \xrightarrow{f'} \overline{\mathcal{G}/g^{-1}(\mathcal{H}')} \xrightarrow{g'} \overline{\mathcal{H}/\mathcal{H}'}. \tag{5.4.20}$$

Applied to (5.4.20), the strong monotonicity inequality (5.2.1) reads:

$$\operatorname{rk}^1_\varphi\left(g'\circ f' : \mathcal{F}/(g\circ f)^{-1}(\mathcal{H}') \to \overline{\mathcal{H}/\mathcal{H}'}\right) \leq \operatorname{rk}^1_\varphi\left(f' : \mathcal{F}/(g\circ f)^{-1}(\mathcal{H}') \to \overline{\mathcal{G}/g^{-1}(\mathcal{H}')}\right).$$

Therefore we have:

$$\operatorname{rk}^1_\varphi\left(g'\circ f' : \mathcal{F}/(g\circ f)^{-1}(\mathcal{H}') \to \overline{\mathcal{H}/\mathcal{H}'}\right) \leq \underline{\operatorname{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}),$$

and the inequality (5.4.19) follows by taking the supremum over $\mathcal{H}'$ in $\operatorname{coft}(\mathcal{H})$. $\square$

Corollary 5.4.7. *If an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfies the strong monotonicity condition* $\mathbf{StMon^1}$*, then its lower extension $\underline{\varphi} : \mathbf{q}\overline{\mathbf{Coh}}_X \to [0,+\infty]$ satisfies it also in the sense of definition 5.2.11; namely, for every diagram in $\mathbf{q}\ddddot{\mathbf{Coh}}_X^{\leq 1}$ of the form:*

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}},$$

the following inequality holds:

$$\underline{\varphi}(\overline{\mathcal{H}}) + \underline{\varphi}(\overline{\mathcal{G}/\operatorname{im} f}) \leq \underline{\varphi}(\overline{\mathcal{G}}) + \underline{\varphi}(\overline{\mathcal{H}/\operatorname{im}(g\circ f)}). \tag{5.4.21}$$

In particular, the invariant $\underline{\varphi}$ satisfies conditions $\mathbf{StMon^1_2}$, $\mathbf{StMon^1_4}$ and $\mathbf{StMon^1_3}$.

Proof. According to Proposition 5.4.2 applied to $g\circ f$ and to f, we have:

$$\underline{\varphi}(\overline{\mathcal{H}}) + \underline{\varphi}(\overline{\mathcal{G}/\operatorname{im} f}) = \operatorname{rk}^1_{\underline{\varphi}}(g\circ f) + \underline{\varphi}(\overline{\mathcal{H}/\operatorname{im}(g\circ f)}) + \underline{\varphi}(\overline{\mathcal{G}/\operatorname{im} f})$$

and

$$\underline{\varphi}(\overline{\mathcal{G}}) + \underline{\varphi}(\overline{\mathcal{H}/\operatorname{im}(g\circ f)}) = \operatorname{rk}^1_{\underline{\varphi}}(f) + \underline{\varphi}(\overline{\mathcal{H}/\operatorname{im}(g\circ f)}) + \underline{\varphi}(\overline{\mathcal{G}/\operatorname{im} f}).$$

The inequality (5.4.21) therefore follows from (5.4.19). $\square$

5.4.3.2. It is actually possible to investigate the strong monotonicity properties of the lower extension $\underline{\varphi}$ without introducing the lower φ-rank $\underline{\operatorname{rk}}^1_\varphi$, and to establish directly the following permanence properties satisfied by the conditions $\mathbf{StMon^1_\mathit{i}}$:

Proposition 5.4.8. *For every invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfying the monotonicity condition* $\mathbf{Mon^1}$*, the following implications hold:*

$$\varphi \text{ satisfies } \mathbf{StMon^1_\mathit{i}} \text{ for } i \in \{2,3,4\} \text{ on } \overline{\mathbf{Coh}}_X \Longrightarrow \underline{\varphi} \text{ satisfies } \mathbf{StMon^1_2} \text{ on } \mathbf{q}\overline{\mathbf{Coh}}_X; \tag{5.4.22}$$

$$\varphi \text{ satisfies } \mathbf{StMon^1_3} \text{ on } \overline{\mathbf{Coh}}_X \Longrightarrow \underline{\varphi} \text{ satisfies } \mathbf{StMon^1_3} \text{ on } \mathbf{q}\overline{\mathbf{Coh}}_X; \tag{5.4.23}$$

$$\varphi \text{ satisfies } \mathbf{StMon^1_4} \text{ on } \overline{\mathbf{Coh}}_X \Longrightarrow \underline{\varphi} \text{ satisfies } \mathbf{StMon^1_4} \text{ on } \mathbf{q}\overline{\mathbf{Coh}}_X. \tag{5.4.24}$$

We leave the proof of Proposition 5.4.8 as an exercise for the interested reader.

5.4.4. Metric monotonicity of φ and downward continuity of $\underline{\varphi}$. This subsection, devoted to the downward continuity property of the lower extension $\underline{\varphi}$ of an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$, is logically independent from the content of the previous subsections 5.4.1-5.4.3.

The following proposition establishes that the downward continuity property $\mathbf{Cont}^+$ is inherited from φ by its lower extension $\underline{\varphi}$ restricted to the subcategory of $\mathbf{q}\overline{\mathbf{Coh}}_X$ where it takes finite values, provided that φ satisfies the metric monotonicity property $\mathbf{StMon}^{\mathbf{1}}_4$ on $\overline{\mathbf{Coh}}_X$.

PROPOSITION 5.4.9. *Let $(\overline{\mathcal{F}}_n)_{n\in\mathbb{N}}$ be a sequence of objects*

$$\overline{\mathcal{F}}_n := (\mathcal{F}, (\|.\|_{n,x})_{x\in X(\mathbb{C})})$$

in $\mathbf{q}\overline{\mathbf{Coh}}_X$ with the same underlying $\mathcal{O}_K$-modules $\mathcal{F}$ such that, for every $x \in X(\mathbb{C})$, the sequence of seminorms $(\|.\|_{n,x})_{n\in\mathbb{N}}$ is decreasing, and let

$$\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})}) \tag{5.4.25}$$

be the object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ associated to the Hermitian seminorms $\|.\|_x$ on the $\mathcal{F}_x$, $x \in X(\mathbb{C})$, defined as the pointwise limits:

$$\|.\|_x := \lim_{n\to+\infty} \|.\|_{n,x}. \tag{5.4.26}$$

If an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfies conditions $\mathbf{Mon}^{\mathbf{1}} = \mathbf{StMon}^{\mathbf{1}}_1$, $\mathbf{Cont}^+$ and $\mathbf{StMon}^{\mathbf{1}}_4$, and if the following condition is verified:

$$\text{there exists } n_0 \in \mathbb{N} \text{ such that } \varphi_{n_0}(\mathcal{F}) < +\infty, \tag{5.4.27}$$

then we have:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \lim_{n\to+\infty} \underline{\varphi}(\overline{\mathcal{F}}_n). \tag{5.4.28}$$

Recall that, according to the monotonicity of $\underline{\varphi}$ established in Proposition 4.3.6, the sequence $(\underline{\varphi}(\overline{\mathcal{F}}_n))_{n\in\mathbb{N}}$ is decreasing and satisfies:

$$\underline{\varphi}(\overline{\mathcal{F}}) \leq \lim_{n\to+\infty} \underline{\varphi}(\overline{\mathcal{F}}_n). \tag{5.4.29}$$

The condition (5.4.27) precisely asserts that the right-hand side of (5.4.29) is finite.

PROOF. Let us assume that φ satisfies $\mathbf{Cont}^+$ and $\mathbf{StMon}^{\mathbf{1}}_4$, and for every $\mathcal{O}_K$-submodule $\mathcal{G}$ of $\mathcal{F}$, let us denote by $\overline{\mathcal{F}/\mathcal{G}}_n$ and $\overline{\mathcal{F}/\mathcal{G}}$ the objects of $\overline{\mathbf{Coh}}_X$ defined as the $\mathcal{O}_K$-module $\mathcal{F}/\mathcal{G}$ endowed with the Hermitian metrics deduced by quotient from $(\|.\|_{n,x})_{x\in X(\mathbb{C})}$ and $(\|.\|_x)_{x\in X(\mathbb{C})}$ respectively.

Observe that, for every $x \in X(\mathbb{C})$, if we denote by

$$p_x : \mathcal{F}_x \longrightarrow \mathcal{F}_x/\mathcal{G}_x \simeq (\mathcal{F}/\mathcal{G})_x$$

the quotient map, the following equalities hold for any $v \in (\mathcal{F}/\mathcal{G})_x$:

$$\begin{aligned}\|v\|_{\overline{\mathcal{F}/\mathcal{G}},x} &= \inf_{\tilde{v}\in p_x^{-1}(v)} \|\tilde{v}\|_x = \inf_{\tilde{v}\in p_x^{-1}(v)} \inf_{n\in\mathbb{N}} \|\tilde{v}\|_{n,x} \\ &= \inf_{n\in\mathbb{N}} \inf_{\tilde{v}\in p_x^{-1}(v)} \|\tilde{v}\|_{n,x} = \inf_{n\in\mathbb{N}} \|v\|_{\overline{\mathcal{F}/\mathcal{G}}_n,x} = \lim_{n\to+\infty} \|v\|_{\overline{\mathcal{F}/\mathcal{G}}_n,x}.\end{aligned}$$

Therefore, for any $\mathcal{G}$ in $\mathrm{coft}(\mathcal{F})$, we may apply the continuity condition $\mathbf{Cont}^+$ satisfied by φ to the sequence $(\overline{\mathcal{F}/\mathcal{G}}_n)_{n\in\mathbb{N}}$. Thus we get the equality:

$$\varphi(\overline{\mathcal{F}/\mathcal{G}}) = \lim_{n\to+\infty} \varphi(\overline{\mathcal{F}/\mathcal{G}}_n).$$

Since φ satisfies $\mathbf{StMon}^{\mathbf{1}}_4$, if $\mathcal{G}' \subseteq \mathcal{G}$ are two elements of $\mathrm{coft}(\mathcal{F})$, the difference

$$\varphi(\overline{\mathcal{F}/\mathcal{G}'}_n) - \varphi(\overline{\mathcal{F}/\mathcal{G}}_n) \tag{5.4.30}$$

— which, according to $\mathbf{Mon}^1$, is non-negative — is a decreasing function of $n \in \mathbb{N}$. By considering the infimum of (5.4.30) over $\mathcal{G}' \in \mathrm{coft}(\mathcal{F})$, this shows that, for any $\mathcal{G} \in \mathrm{coft}(\mathcal{F})$, the difference

$$\underline{\varphi}(\overline{\mathcal{F}}_n) - \varphi(\overline{\mathcal{F}/\mathcal{G}_n}) \tag{5.4.31}$$

is a non-negative decreasing function of $n \in \mathbb{N}$.

For any integer $n \geq n_0$ and any $\mathcal{G} \in \mathrm{coft}(\mathcal{F})$, the definition of $\underline{\varphi}(\overline{\mathcal{F}})$ and the decreasing character of the expression (5.4.31) as a function of n imply:

$$\begin{aligned}0 \leq \underline{\varphi}(\overline{\mathcal{F}}_n) - \underline{\varphi}(\overline{\mathcal{F}}) &\leq \left(\underline{\varphi}(\overline{\mathcal{F}}_n) - \varphi(\overline{\mathcal{F}/\mathcal{G}_n})\right) + \left(\varphi(\overline{\mathcal{F}/\mathcal{G}_n}) - \varphi(\overline{\mathcal{F}/\mathcal{G}})\right) \\ &\leq \left(\underline{\varphi}(\overline{\mathcal{F}}_{n_0}) - \varphi(\overline{\mathcal{F}/\mathcal{G}_{n_0}})\right) + \left(\varphi(\overline{\mathcal{F}/\mathcal{G}_n}) - \varphi(\overline{\mathcal{F}/\mathcal{G}})\right).\end{aligned}$$

For any $\varepsilon \in \mathbb{R}_+^*$, we may choose $\mathcal{G} \in \mathrm{coft}(\mathcal{F})$ such that

$$\underline{\varphi}(\overline{\mathcal{F}}_{n_0}) - \varphi(\overline{\mathcal{F}/\mathcal{G}_{n_0}}) < \varepsilon,$$

and then an integer $n_1 \geq n_0$ such that, for any integer $n \in \mathbb{N}$,

$$n \geq n_1 \Longrightarrow \varphi(\overline{\mathcal{F}/\mathcal{G}_n}) - \varphi(\overline{\mathcal{F}/\mathcal{G}}) < \varepsilon.$$

Then, for any $n \in \mathbb{N}$,

$$n \geq n_1 \Longrightarrow 0 \leq \underline{\varphi}(\overline{\mathcal{F}}_n) - \underline{\varphi}(\overline{\mathcal{F}}) < 2\varepsilon,$$

and consequently equality holds in (5.4.29). □

EXAMPLE 5.4.10. Let us emphasize that equality does not hold in general in (5.4.29) without a finiteness assumption on its right-hand side.

Indeed, let us assume that φ satisfies $\mathbf{Add}_\oplus$ and that there exists some object $\overline{\mathcal{C}}$ in $\overline{\mathbf{Coh}}_X$ such that

$$\varphi(\overline{\mathcal{C}}) > 0,$$

and such that the object $\overline{\mathcal{C}}^\sim$ of $\overline{\mathbf{Coh}}_X$ defined by the underlying module $\mathcal{C}$ equipped with the zero Hermitian seminorms satisfies:[11]

$$\varphi(\overline{\mathcal{C}}^\sim) = 0.$$

If for every $n \in \mathbb{N}$, we let:

$$\overline{\mathcal{F}}_n := \bigoplus_{k \in \mathbb{N}} \overline{\mathcal{C}}_{nk}$$

where

$$\begin{aligned}\overline{\mathcal{C}}_{nk} &:= \overline{\mathcal{C}}^\sim \quad \text{if } k < n \\ &:= \overline{\mathcal{C}} \quad \text{if } k \geq n,\end{aligned}$$

then the $\overline{\mathcal{F}}_n$ have the same underlying module $\mathcal{F} = \mathcal{C}^{(\mathbb{N})}$, their Hermitian seminorms are decreasing functions of n, and the limit seminorms vanish. We are in the situation of Proposition 5.4.9, with:

$$\overline{\mathcal{F}} = \overline{\mathcal{C}}^{\sim \oplus \mathbb{N}}.$$

However, according to Proposition 4.3.8, we have:

$$\underline{\varphi}(\overline{\mathcal{F}}_n) = \sum_{k \in \mathbb{N}} \varphi(\overline{\mathcal{C}}_{nk}) = \sum_{0 \leq k < n} \varphi(\overline{\mathcal{C}}^\sim) + \sum_{k \geq n} \varphi(\overline{\mathcal{C}}) = +\infty$$

for every $n \in \mathbb{N}$, and:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \sum_{k \in \mathbb{N}} \varphi(\overline{\mathcal{C}}^\sim) = 0.$$

[11] These conditions are trivially verified by some invariants on $\overline{\mathbf{Coh}}_X$ considered in the sequel, notably by h^1_θ, which indeed satisfies $\mathbf{Mon}^1$, $\mathbf{Add}_\oplus$, $\mathbf{Cont}^+$, and $\mathbf{StMon}^{\mathbf{1}}_{\mathbf{4}}$.

5.5. The Upper φ-Rank $\overline{\mathrm{rk}}^1_\varphi$ and the Strong Monotonicity of $\overline{\varphi}$

In this section, we still consider an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ that satisfies the condition $\mathbf{Mon}^1$, and we pursue the study initiated in Chapter 4 of its upper extension:

$$\overline{\varphi} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

when φ satisfies suitable strong monotonicity conditions.

The content of this section is formally similar to the content of the previous section, with the lower extension $\underline{\varphi}$ replaced by the upper extension $\overline{\varphi}$. Notably we introduce a suitable upper φ-rank $\overline{\mathrm{rk}}^1_\varphi$ and we use it to investigate the strong monotonicity properties of $\overline{\varphi}$.

The properties of the $\overline{\varphi}$ and $\overline{\mathrm{rk}}^1_\varphi$ on the category $\mathbf{q}\overline{\mathbf{Coh}}_X$ are formally less satisfactory than those of $\underline{\varphi}$ and $\underline{\mathrm{rk}}^1_\varphi$. However the "upper extensions" $\overline{\varphi}$ and $\overline{\mathrm{rk}}^1_\varphi$ will play a key role in the development of infinite dimensional geometry of numbers in Chapter 9, and establishing that they inherits suitable strong monotonicity properties from φ will be important in the sequel.

In this perspective, at the end of this section we prove that, when φ satisfies the continuity condition $\mathbf{Cont}^+$ and the strong monotonicity condition $\mathbf{StMon^1}$, the invariant

$$\overline{\varphi} : \varphi_\Sigma\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+,$$

defined by restricting $\overline{\varphi}$ to the subcategory of φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ introduced in 4.5.1, satisfies a convenient form of condition $\mathbf{StMon}^1$.

5.5.1. The upper φ-rank $\overline{\mathrm{rk}}^1_\varphi$: definition and first properties. In this subsection, we denote by $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ an invariant satisfying the monotonicity condition $\mathbf{Mon^1}$.

5.5.1.1. *Definition and remarks.*

DEFINITION 5.5.1. The *upper φ-rank* of a morphism $f : \mathcal{F} \to \overline{\mathcal{G}}$ in $\mathbf{q}\dddot{\mathbf{Coh}}_X$ is defined as the following lower limit over the directed set $(\mathrm{coh}(\mathcal{G}), \subseteq)$:

$$\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) := \liminf_{\mathcal{C} \in \mathrm{coh}(\mathcal{G})} \left[\varphi(\overline{\mathcal{C}}) - \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)})\right] \in [0, +\infty]. \tag{5.5.1}$$

When f is a morphism in $\dddot{\mathbf{Coh}}_X$, $\overline{\mathrm{rk}}^1_\varphi(f)$ clearly coincides with $\mathrm{rk}^1_\varphi(f)$.

One easily checks sees that $\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}})$ satisfies the following two properties, which provide an alternative definition of the upper φ-rank:

(i) For every exhaustive filtration $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of $\mathcal{G}$ by coherent submodules, we have:

$$\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) \leq \liminf_{n \to +\infty} \left[\varphi(\overline{\mathcal{C}}_n) - \varphi(\overline{\mathcal{C}_n/(\mathcal{C}_n \cap \mathrm{im}\, f)})\right].$$

(ii) There exists an exhaustive filtration $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of $\mathcal{G}$ by coherent submodules such that the sequence $\left(\varphi(\overline{\mathcal{C}}) - \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)})\right)_{n\in\mathbb{N}}$ converges in $[0, +\infty]$ and

$$\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) = \lim_{n \to +\infty} \left[\varphi(\overline{\mathcal{C}}_n) - \varphi(\overline{\mathcal{C}_n/(\mathcal{C}_n \cap \mathrm{im}\, f)})\right].$$

Let us formulate a few remarks concerning the definition of $\overline{\mathrm{rk}}^1_\varphi$, which are parallel to the remarks on the definition of the lower φ-rank in Subsection 5.4.1:

(a) Like its lower φ-rank, the upper φ-rank $\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}})$ depends only on $\overline{\mathcal{G}}$ and on the submodule $\mathrm{im}\, f$ of $\mathcal{G}$. In other words, if $\iota : \mathrm{im}\, f \to \mathcal{G}$ denotes the inclusion morphism, we have:

$$\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) = \overline{\mathrm{rk}}^1_\varphi(\iota : \mathrm{im}\, f \to \overline{\mathcal{G}}).$$

When moreover $\overline{\mathcal{G}}$ is an object of $\overline{\mathbf{Coh}}_X$, this coincides with $\mathrm{rk}^1_\varphi(\iota : \mathrm{im}\, f \to \overline{\mathcal{G}})$.

(b) For every $\mathcal{C}$ in $\mathrm{coh}(\mathcal{G})$, there exists $\mathcal{C}'$ in $\mathrm{coh}(\mathcal{F})$ satisfying the condition:

$$f(\mathcal{C}') = \mathcal{C} \cap \mathrm{im}\, f. \tag{5.5.2}$$

When this holds, we may introduce the morphism in $\overset{\cdots}{\mathbf{Coh}}_X$:

$$f_{\mathcal{C}'\mathcal{C}} := f_{|\mathcal{C}'} : \mathcal{C}' \longrightarrow \overline{\mathcal{C}},$$

and its φ-rank satisfies:

$$\mathrm{rk}^1_\varphi(f_{\mathcal{C}\mathcal{C}'}) := \varphi(\overline{\mathcal{C}}) - \varphi(\overline{\mathcal{C}/f(\mathcal{C}')} = \varphi(\overline{\mathcal{C}}) - \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)}).$$

We may introduce the subset $\mathrm{coh}(f)$ of $\mathrm{coh}(\mathcal{F}) \times \mathrm{coh}(\mathcal{G})$ defined as:

$$\mathrm{coh}(f) := \left\{(\mathcal{C}', \mathcal{C}) \in \mathrm{coh}(\mathcal{F}) \times \mathrm{coh}(\mathcal{G}) \mid f(\mathcal{C}') = \mathcal{C} \cap \mathrm{im}\, f\right\},$$

and the order relation $\subseteq$ on $\mathrm{coh}(f)$ defined by:

$$(\mathcal{F}', \mathcal{G}') \subseteq (\mathcal{F}'', \mathcal{G}'') \Longleftrightarrow [\mathcal{F}' \subseteq \mathcal{F}'' \text{ and } \mathcal{G}' \subseteq \mathcal{G}''].$$

Then the ordered set $(\mathrm{coh}(f), \subseteq)$ is a directed set, and the previous observation shows that the upper-rank of f may also be defined as the following lower limit over this directed set:

$$\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) := \liminf_{(\mathcal{C}', \mathcal{C}) \in \mathrm{coh}(f)} \mathrm{rk}^1_\varphi(f_{\mathcal{C}'\mathcal{C}} : \mathcal{C}' \to \overline{\mathcal{C}}).$$

(c) For every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following equality holds as a straightforward consequence of the definitions:

$$\overline{\mathrm{rk}}^1_\varphi(\mathrm{Id}_\mathcal{F} : \mathcal{F} \to \overline{\mathcal{F}}) = \overline{\varphi}(\overline{\mathcal{F}}).$$

5.5.1.2. The upper and lower φ-ranks satisfy the expected comparison estimate, provided the invariant φ satisfies suitable strong monotonicity properties.

PROPOSITION 5.5.2. *For every invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ satisfying the conditions* $\mathbf{Mon^1}$, $\mathbf{StMon^1_3}$ *and* $\mathbf{StMon^1_4}$, *and for every morphism $f : \mathcal{F} \to \overline{\mathcal{G}}$ in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_X$, the following inequality holds:*

$$\underline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) \leq \overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}). \tag{5.5.3}$$

PROOF. Let $\mathcal{G}'$ be an element of $\mathrm{coft}(\mathcal{G})$. Since the quotient $\mathcal{G}/\mathcal{G}'$ is coherent, we may choose an element $\mathcal{C}$ in $\mathrm{coh}(\mathcal{G})$ such that the composition of the inclusion and quotient morphisms:

$$\mathcal{C} \hookrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}'$$

is surjective.

Proposition 5.5.2 is a consequence of the following lemma:

LEMMA 5.5.3. *For every $\mathcal{C}'$ in $\mathrm{coh}(\mathcal{G})$ containing $\mathcal{C}$, the following inequality holds:*

$$\varphi(\overline{\mathcal{G}/\mathcal{G}'}) - \varphi(\overline{\mathcal{G}/(\mathrm{im}\, f + \mathcal{G}')}) \leq \varphi(\overline{\mathcal{C}}') - \varphi(\overline{\mathcal{C}'/(\mathrm{im}\, f \cap \mathcal{C}')}). \tag{5.5.4}$$

Indeed, by taking the lower limit over $\mathcal{C}'$ in the directed set $(\mathrm{coh}(\mathcal{G}), \subseteq)$, from (5.5.4) we derive the inequality:

$$\varphi(\overline{\mathcal{G}/\mathcal{G}'}) - \varphi(\overline{\mathcal{G}/(\mathrm{im}\, f + \mathcal{G}')}) \leq \overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}),$$

and (5.5.3) follows by taking the supremum over $\mathcal{G}'$ in $\mathrm{coft}(\mathcal{G})$. $\square$

Proof of Lemma 5.5.3. Let us consider the composition of the inclusion and quotient morphisms:

$$q : \mathcal{C}' \hookrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}'.$$

It is surjective and defines a morphism $q : \overline{\mathcal{C}}' \to \overline{\mathcal{G}/\mathcal{G}'}$ in $\overline{\mathbf{Coh}}_X^{\leq 1}$ which fits into a commutative diagram in $\ddot{\overline{\mathbf{Coh}}}_X^{\leq 1}$:

$$\begin{tikzcd} \operatorname{im} f \cap \mathcal{C}' \arrow[r, "i"] \arrow[d, "p"] & \overline{\mathcal{C}}' \arrow[d, "q"] \\ (\operatorname{im} f + \mathcal{G}')/\mathcal{G}' \arrow[r, "j"] & \overline{\mathcal{G}/\mathcal{G}'}, \end{tikzcd}$$

where i and j denote the inclusion morphisms.

According to $\mathbf{StMon^1_3}$ and $\mathbf{StMon^1_4}$, the following inequality holds:

$$\mathrm{rk}^1_\varphi \left(j : (\operatorname{im} f + \mathcal{G}')/\mathcal{G}' \to \overline{\mathcal{G}/\mathcal{G}'} \right) \leq \mathrm{rk}^1_\varphi \left(i : \operatorname{im} f \cap \mathcal{C}' \to \overline{\mathcal{C}}' \right).$$

Taking into account the definition of rk^1_φ, this establishes (5.5.4). $\square$

5.5.1.3. We now turn to some simple properties of the upper φ-rank associated to a monotonic invariant $\varphi : \mathbf{Coh}_X \to \mathbb{R}_+$, similar to the properties of the lower φ-rank established in 5.4.1.3 and 5.4.2 above.

Observe that, due to the occurrence of a lower limit in the definition of $\overline{\mathrm{rk}}^1_\varphi$, instead of a limit in the one of $\underline{\mathrm{rk}}^1_\varphi$, the properties of $\overline{\mathrm{rk}}^1_\varphi$ are often weaker than those of $\mathrm{rk}^1_{\underline{\varphi}}$. This is exemplified by the following proposition, to be compared with Proposition 5.4.2.

Proposition 5.5.4. *Let $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ be an invariant satisfying condition* $\mathbf{Mon^1}$.

(1) For every morphism $f : \mathcal{F} \to \overline{\mathcal{G}}$ in $\mathbf{q}\ddot{\overline{\mathbf{Coh}}}_X$, the following inequality holds in $[0, +\infty]$:

$$\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) + \overline{\varphi}(\overline{\mathcal{G}/\operatorname{im} f}) \leq \overline{\varphi}(\overline{\mathcal{G}}). \tag{5.5.5}$$

(2) Moreover the following equality holds:

$$\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) + \overline{\varphi}(\overline{\mathcal{G}/\operatorname{im} f}) = \overline{\varphi}(\overline{\mathcal{G}}) \tag{5.5.6}$$

when one of the following conditions is satisfied:

- *(a)* $\operatorname{im} f$ *lies in* $\operatorname{coft}(\mathcal{G})$ – or equivalently $\overline{\mathcal{G}/\operatorname{im} f}$ is an object of $\overline{\mathbf{Coh}}_X$ – *and φ satisfies* $\mathbf{Cont^+}$;
- *(b) φ satisfies the strong subadditivity condition* $\mathbf{StMon^1_2}$.

Proof. According to Remark 4.5.11 applied to the quotient morphism $\overline{\mathcal{G}} \to \overline{\mathcal{G}/\operatorname{im} f}$, we have:

$$\overline{\varphi}(\overline{\mathcal{G}/\operatorname{im} f}) \leq \liminf_{\mathcal{C} \in \operatorname{coh}(\mathcal{G})} \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \operatorname{im} f)}).$$

Consequently:

$$\begin{aligned} \overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) + \overline{\varphi}(\overline{\mathcal{G}/\operatorname{im} f}) &\leq \liminf_{\mathcal{C} \in \operatorname{coh}(\mathcal{G})} \left[\varphi(\overline{\mathcal{C}}) - \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \operatorname{im} f)}) \right] + \liminf_{\mathcal{C} \in \operatorname{coh}(\mathcal{G})} \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \operatorname{im} f)}) \\ &\leq \liminf_{\mathcal{C} \in \operatorname{coh}(\mathcal{G})} \varphi(\overline{\mathcal{C}}) =: \overline{\varphi}(\overline{\mathcal{G}}). \end{aligned}$$

When $\operatorname{im} f$ belongs to $\operatorname{coft}(\mathcal{G})$, the quotient morphism:

$$q_{\mathcal{C}} : \mathcal{C}/(\mathcal{C} \cap \operatorname{im} f) \longrightarrow \mathcal{G}/\operatorname{im} f$$

becomes an isomorphism when $\mathcal{C} \in \operatorname{coh}(\mathcal{G})$ is large enough. Moreover, for every $v \in (\mathcal{G}/\operatorname{im} f)_\mathbb{R}$, the fonction:

$$\mathcal{C} \longmapsto \| q^{-1}_{\mathcal{C},\mathbb{R}}(v) \|_{\overline{\mathcal{C}/(\mathcal{C} \cap \operatorname{im} f)}},$$

defined for $\mathcal{C}$ large enough in the directed set $(\mathrm{coh}(\mathcal{G}), \subseteq)$, is decreasing and converges to $\|v\|_{\overline{\mathcal{G}/\mathrm{im}\, f}}$. Consequently, when φ also satisfies $\mathbf{Cont}^+$, we have:

$$\overline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}) = \lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{G})} \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)}).$$

Consequently:

$$\begin{aligned} \overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) + \overline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}) &= \liminf_{\mathcal{C} \in \mathrm{coh}(\mathcal{G})} \left[\varphi(\overline{\mathcal{C}}) - \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)})\right] + \lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{G})} \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)}) \\ &= \liminf_{\mathcal{C} \in \mathrm{coh}(\mathcal{G})} \varphi(\overline{\mathcal{C}}) =: \overline{\varphi}(\overline{\mathcal{G}}). \end{aligned}$$

Finally let us assume that φ satisfies $\mathbf{StMon^1_2}$. If $\mathcal{C}$ and $\mathcal{C}'$ are two elements of $\mathrm{coh}(\mathcal{G})$ such that $\mathcal{C} \subseteq \mathcal{C}'$, we may apply the strong subadditivity estimate (5.2.5) to the object $\overline{\mathcal{C}}'$ of $\overline{\mathbf{Coh}}_X$ and to the submodules $\mathcal{C} \cap \mathrm{im}\, f \subseteq \mathcal{C} \subseteq \mathcal{C}'$; it reads:

$$\varphi(\overline{\mathcal{C}}') + \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)}) \leq \varphi(\overline{\mathcal{C}}) + \varphi(\overline{\mathcal{C}'/(\mathcal{C} \cap \mathrm{im}\, f)}). \tag{5.5.7}$$

The map $(\mathcal{C}' \mapsto \mathcal{C}'/(\mathcal{C} \cap \mathrm{im}\, f))$ is order preserving and maps the set of elements $\mathcal{C}' \in \mathrm{coh}(\mathcal{G})$ containing $\mathcal{C}$ onto the set of elements of $\mathrm{coh}(\mathcal{G}/(\mathcal{C} \cap \mathrm{im}\, f))$ containing $\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)$, which is cofinal in $(\mathrm{coh}(\mathcal{G}), \subseteq)$. Therefore, by taking the lower limit over $\mathcal{C}'$ in the directed set $(\mathrm{coh}(\mathcal{G}), \subseteq)$, we derive the following inequality from (5.5.7):

$$\overline{\varphi}(\overline{\mathcal{G}}) + \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)}) \leq \varphi(\overline{\mathcal{C}}) + \overline{\varphi}(\overline{\mathcal{G}/(\mathcal{C} \cap \mathrm{im}\, f)}).$$

Together with the monotonicity of $\overline{\varphi}$, this implies:

$$\overline{\varphi}(\overline{\mathcal{G}}) \leq \overline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}) + \left[\varphi(\overline{\mathcal{C}}) - \varphi(\overline{\mathcal{C}/(\mathcal{C} \cap \mathrm{im}\, f)})\right].$$

By taking the lower limit over $\mathcal{C}$ in the directed set $(\mathrm{coh}(\mathcal{G}), \subseteq)$, this establishes the inequality:

$$\overline{\varphi}(\overline{\mathcal{G}}) \leq \overline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}) + \overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}).$$

Together with (5.5.5), this completes the proof of (5.5.6). □

PROPOSITION 5.5.5. *(1) For any diagram in $\mathbf{q}\ddot{\overline{\mathbf{Coh}}}_X$ of the form:*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \overline{\mathcal{H}},$$

the following inequality holds:

$$\overline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \leq \overline{\mathrm{rk}}^1_\varphi(g : \mathcal{G} \to \overline{\mathcal{H}}). \tag{5.5.8}$$

(2) If moreover the morphism of $\mathcal{O}_K$-modules $f : \mathcal{F} \longrightarrow \mathcal{G}$ is surjective, or if φ satisfies condition **NST** *and $f_K : \mathcal{F}_K \longrightarrow \mathcal{G}_K$ is a surjective morphism of K-vector spaces, then we have:*

$$\overline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) = \overline{\mathrm{rk}}^1_\varphi(g : \mathcal{G} \to \overline{\mathcal{H}}). \tag{5.5.9}$$

PROOF. According to the definition of the upper π-rank and to the monotonicity of φ, we have:

$$\begin{aligned} \overline{\mathrm{rk}}^1_\varphi(g \circ f) &:= \liminf_{\mathcal{G}' \in \mathrm{coh}(\mathcal{G})} \left[\varphi(\overline{\mathcal{G}'}) - \varphi(\overline{\mathcal{G}'/(\mathcal{G}' \cap \mathrm{im}\,(g \circ f))})\right] \\ &\leq \liminf_{\mathcal{G}' \in \mathrm{coh}(\mathcal{G})} \left[\varphi(\overline{\mathcal{G}'}) - \varphi(\overline{\mathcal{G}'/(\mathcal{G}' \cap \mathrm{im}\, g)})\right] =: \overline{\mathrm{rk}}^1_\varphi(g). \end{aligned}$$

This proves (1).

If f is surjective or if φ satisfies condition **NST** and $f_K : \mathcal{F}_K \longrightarrow \mathcal{G}_K$ is a surjective morphism of K-vector spaces, then we have, for any $\mathcal{G}' \in \mathrm{coh}(\mathcal{G})$:

$$\varphi(\overline{\mathcal{G}'/(\mathcal{G}' \cap \mathrm{im}\,(g \circ f))}) = \varphi(\overline{\mathcal{G}'/(\mathcal{G}' \cap \mathrm{im}\,(g))}),$$

which completes the proof of (2). □

5.5.2. The strong monotonicity of φ and $\overline{\varphi}$. In this subsection, we denote by $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ an invariant satisfying the strong monotonicity property $\mathbf{StMon^1}$ and we investigate the strong monotonicity properties of $\overline{\varphi}$ and $\overline{\mathrm{rk}}^1_\varphi$.

5.5.2.1. We begin with a simple consequence of Proposition 5.5.2 and of Proposition 5.5.4 (2), which may be applied according to our assumption of strong monotonicity on φ.

PROPOSITION 5.5.6. *Let $\overline{\mathcal{G}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfying the following condition:*

$$\underline{\varphi}(\overline{\mathcal{G}}) = \overline{\varphi}(\overline{\mathcal{G}}) < +\infty. \tag{5.5.10}$$

Then for every $\mathcal{O}_X$-submodule $\mathcal{G}'$ of $\mathcal{G}$ and every morphism $f : \mathcal{F} \to \overline{\mathcal{G}}$, the following conditions are satisfied:

$$\underline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) = \overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) < +\infty, \tag{5.5.11}$$

and

$$\underline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) = \overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) < +\infty. \tag{5.5.12}$$

The significance of the condition (5.5.10) is discussed at the end of this chapter, in Subsection 5.6.3. This condition will also play an important an important role in the sequel when the invariant φ is the θ-invariant h^1_θ; see notably Section 8.4.

PROOF. According to the inequality $\underline{\varphi} \leq \overline{\varphi}$ and to the monotonicity of $\overline{\varphi}$ and $\underline{\varphi}$, if $\overline{\varphi}(\overline{\mathcal{G}})$ is finite, then $\underline{\varphi}(\overline{\mathcal{G}})$, $\overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'})$, and $\underline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'})$ also are. Consequently, according to Propositions 5.4.2 and 5.5.4, the lower and upper φ-rank of the inclusion morphism $\iota : \mathcal{G}' \hookrightarrow \overline{\mathcal{G}}$ in $\mathbf{q}\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}_X$ are finite and satisfy:

$$\underline{\mathrm{rk}}^1_\varphi(\iota) = \underline{\varphi}(\overline{\mathcal{G}}) - \underline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) \tag{5.5.13}$$

and

$$\overline{\mathrm{rk}}^1_\varphi(\iota) = \overline{\varphi}(\overline{\mathcal{G}}) - \overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}). \tag{5.5.14}$$

Moreover, according to Proposition 5.5.2, the following inequality holds:

$$\underline{\mathrm{rk}}^1_\varphi(\iota) \leq \overline{\mathrm{rk}}^1_\varphi(\iota).$$

Combined with (5.5.13) and (5.5.14), this shows that, if the condition (5.5.10) is satisfied, then:

$$-\underline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) \leq -\overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}).$$

Together with the inequality $\underline{\varphi} \leq \overline{\varphi}$, this establishes the equality in (5.5.11).

According to (5.5.13) and (5.5.14), we also have:

$$\underline{\mathrm{rk}}^1_\varphi(\iota) = \overline{\mathrm{rk}}^1_\varphi(\iota) < +\infty.$$

Applied to $\mathcal{G}' := \mathrm{im}\, f$, this establishes (5.5.12). □

5.5.2.2. Contrary to the lower extensions $\underline{\varphi}$ and $\underline{\mathrm{rk}}^1_\varphi$, the strong monotonicity properties of φ do not immediately transfer to $\overline{\varphi}$ and $\overline{\mathrm{rk}}^1_\varphi$. However, under some additional technical conditions, we may establish the following counterparts of Propositions 5.4.6 and 5.4.7.

PROPOSITION 5.5.7. *For every diagram in $\mathbf{q}\overset{\cdot\cdot\cdot\cdot}{\mathbf{Coh}}^{\leq 1}_X$ of the form:*

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}}, \tag{5.5.15}$$

the following inequality holds in $[0, +\infty]$:

$$\overline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \leq \overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) \tag{5.5.16}$$

when one of the following conditions is satisfied:

(1) *the morphism of $\mathcal{O}_X$-modules $g : \mathcal{G} \to \mathcal{H}$ is bijective;*
(2) $\operatorname{im} f$ *is coherent;*
(3) $\overline{\varphi}(\overline{\mathcal{G}/\operatorname{im} f}) < +\infty$, $\overline{\varphi}(\overline{\mathcal{H}/\operatorname{im} g \circ f}) < +\infty$, φ *satisfies* $\mathbf{Cont}^+$, *and* $\operatorname{ev}\overline{\varphi}(\overline{\operatorname{im} f}) = 0$*;*
(4) $\underline{\varphi}(\overline{\mathcal{H}}) = \overline{\varphi}(\overline{\mathcal{H}}) < +\infty$.

Proposition 5.5.8. *For every morphism $g : \overline{\mathcal{G}} \to \overline{\mathcal{H}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ and every $\mathcal{O}_X$-submodule $\mathcal{G}'$ of $\mathcal{G}$, the following inequality holds in $[0, +\infty]$:*

$$\overline{\varphi}(\overline{\mathcal{H}}) + \overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) \leq \overline{\varphi}(\overline{\mathcal{G}}) + \overline{\varphi}(\overline{\mathcal{H}/g(\mathcal{G}')}), \tag{5.5.17}$$

provided one of the following conditions is satisfied:

(1) *the morphism of $\mathcal{O}_X$-modules $g : \mathcal{G} \to \mathcal{H}$ is an isomorphism;*
(2) *the submodule $\mathcal{G}'$ belongs to* $\operatorname{coh}(\mathcal{G})$*;*
(3) φ *satisfies* $\mathbf{Cont}^+$, *and* $\operatorname{ev}\overline{\varphi}(\overline{\mathcal{G}}') = 0$*;*
(4) $\underline{\varphi}(\overline{\mathcal{H}}) = \overline{\varphi}(\overline{\mathcal{H}}) < +\infty$.

Proof of Propositions 5.5.7 and 5.5.8. We shall establish the two proposition in parallel. With the notation of Proposition 5.5.8, we shall denote by $\iota : \mathcal{G}' \to \mathcal{G}$ the inclusion morphism.

(1) Let us assume that, in Proposition 5.5.7, the morphism g is bijective. Then it induces a bijection:

$$\operatorname{coh}(\mathcal{G}) \xrightarrow{\sim} \operatorname{coh}(\mathcal{G}'), \quad \mathcal{C} \longmapsto g(\mathcal{C}),$$

and therefore:

$$\begin{aligned}\overline{\operatorname{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) &= \liminf_{\mathcal{C}' \in \operatorname{coh}(\mathcal{H})} \left[\varphi(\overline{\mathcal{C}}') - \varphi(\overline{\mathcal{C}'/(\mathcal{C}' \cap \operatorname{im} g \circ f)})\right] \\ &= \liminf_{\mathcal{C} \in \operatorname{coh}(\mathcal{G})} \left[\varphi(\overline{g(\mathcal{C})}) - \varphi(\overline{g(\mathcal{C})/g(\mathcal{C} \cap \operatorname{im} f)})\right].\end{aligned}$$

Moreover, according to $\mathbf{StMon^1_4}$, for every $\mathcal{C} \in \operatorname{coh}(\mathcal{G})$, the following inequality holds:

$$\varphi(\overline{g(\mathcal{C})} - \varphi(\overline{g(\mathcal{C})/g(\mathcal{C} \cap \operatorname{im} f)}) \leq \varphi(\overline{\mathcal{C}})) - \varphi(\overline{\mathcal{C}/g)(\mathcal{C} \cap \operatorname{im} f)}).$$

Together with the definition of $\overline{\operatorname{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}})$:

$$\overline{\operatorname{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) := \liminf_{\mathcal{C} \in \operatorname{coh}(\mathcal{G})} \varphi(\overline{\mathcal{C}})) - \varphi(\overline{\mathcal{C}/g)(\mathcal{C} \cap \operatorname{im} f)}),$$

this establishes (5.5.17).

In Proposition 5.5.8, when g is an isomorphism, by using successively Proposition 5.5.4 (2) applied to $g \circ \iota : \mathcal{F} \to \overline{\mathcal{H}}$, the inequality (5.5.16) when $f = \iota$, and Proposition 5.5.4 (2) applied to $\iota : \mathcal{F} \to \overline{\mathcal{G}}$, we obtain:

$$\begin{aligned}\overline{\varphi}(\overline{\mathcal{H}}) + \overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) &= \overline{\operatorname{rk}}^1_\varphi(g \circ \iota : \mathcal{G}' \to \overline{\mathcal{H}}) + \overline{\varphi}(\overline{\mathcal{H}/\mathcal{H}'}) + \overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) \\ &\leq \overline{\operatorname{rk}}^1_\varphi(\iota : \mathcal{G}' \to \mathcal{G}) + \overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) + \overline{\varphi}(\overline{\mathcal{H}/\mathcal{H}'}) = \varphi(\overline{\mathcal{G}}) + \varphi(\overline{\mathcal{H}/\mathcal{H}'}).\end{aligned} \tag{5.5.18}$$

This establishes the equality (5.5.17).

(2) Let us assume that, in in Proposition 5.5.7, the module $\operatorname{im} f$ is coherent. For any $\mathcal{C}$ in $\operatorname{coh}(\mathcal{G})$ containing $\mathcal{C}$ and any $\mathcal{D}$ in $\operatorname{coh}(\mathcal{H})$ containing $g(\mathcal{C})$, we may consider the diagram:

$$\mathcal{F} \xrightarrow{f} \overline{\mathcal{C}} \xrightarrow{g_{|\mathcal{C}}} \overline{\mathcal{D}}$$

in $\overset{....}{\mathbf{Coh}}{}_X^{\leq 1}$. Since φ satisfies $\mathbf{StMon^1}$, the following inequality holds:

$$\operatorname{rk}^1_\varphi(g_{|\mathcal{C}} \circ f : \mathcal{F} \to \overline{\mathcal{D}}) := \varphi(\overline{\mathcal{D}}) - \varphi(\overline{\mathcal{D}/\operatorname{im} g \circ f}) \leq \operatorname{rk}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{C}}) := \varphi(\overline{\mathcal{C}}) - \varphi(\overline{\mathcal{C}/\operatorname{im} f}). \tag{5.5.19}$$

By taking firstly the lower limit over $\mathcal{D}$ in the directed set $(\operatorname{coh}(\mathcal{H}), \subseteq)$, and then the lower limit over $\mathcal{C}$ in the directed set $(\operatorname{coh}(\mathcal{G}), \subseteq)$, one derives the inequality (5.5.16) from (5.5.19).

When (2) is satisfied in Proposition 5.5.8, by the same reasoning as in part (1) of this proof, the chain of (in)equalities (5.5.18) holds and establishes (5.5.17).

(3) Let us assume that, in Proposition 5.5.8, condition (3) is satisfied. In part (2) of this proof, we have shown that (5.5.17) holds when $\mathcal{G}'$ belongs to $\mathrm{coh}(\mathcal{G})$. In particular, for very $\mathcal{C}'$ in $\mathrm{coh}(\mathcal{G}')$, we have:

$$\overline{\varphi}(\overline{\mathcal{H}}) + \overline{\varphi}(\overline{\mathcal{G}/\mathcal{C}'}) \leq \overline{\varphi}(\overline{\mathcal{G}}) + \overline{\varphi}(\overline{\mathcal{H}/g(\mathcal{C}')}). \tag{5.5.20}$$

Recall that we have shown in Chapter 4 that, when φ satisfies $\mathbf{Mon^1}$, $\mathbf{SubAdd}$, and $\mathbf{Cont}^+$, then $\overline{\varphi}$ satisfies $\mathbf{Mon^1}$ and $\mathbf{SubAdd}$, and $\mathrm{ev}\overline{\varphi}$ satisfies $\mathbf{Mon^1}$; see Propositions 4.3.6, 4.3.10, and 4.4.2. Therefore we have:

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{G}}') = 0,$$

and finally, according to Proposition 4.4.4 applied to $\psi := \overline{\varphi}$, we have:

$$\overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) = \lim_{\mathcal{C}' \in \mathrm{coh}(\mathcal{G}')} \overline{\varphi}(\overline{\mathcal{G}/\mathcal{C}'}) \tag{5.5.21}$$

and:

$$\overline{\varphi}(\overline{\mathcal{H}/g(\mathcal{G}')}) = \lim_{\mathcal{D}' \in \mathrm{coh}(g(\mathcal{G}'))} \overline{\varphi}(\overline{\mathcal{G}/\mathcal{D}'}). \tag{5.5.22}$$

Since the map:

$$\mathrm{coh}(\mathcal{G}') \longrightarrow \mathrm{coh}(g(\mathcal{G}')), \quad \mathcal{C}' \longmapsto g(\mathcal{C}')$$

is surjective and order preserving, the equality (5.5.22) may also be written:

$$\overline{\varphi}(\overline{\mathcal{H}/g(\mathcal{G}')}) = \lim_{\mathcal{C}' \in \mathrm{coh}(\mathcal{G}')} \overline{\varphi}(\overline{\mathcal{G}/g(\mathcal{C}')}). \tag{5.5.23}$$

Using (5.5.21) and (5.5.23), we derive the inequality (5.5.17) from (5.5.20) by taking the limit over $\mathcal{C}'$ in the directed set $(\mathrm{coh}(\mathcal{G}'), \subseteq)$.

When (3) is satisfied in Proposition 5.5.7, we may use Proposition 5.5.4 (2) to express the upper φ-rank of f and $g \circ f$ as:

$$\overline{\mathrm{rk}}^1_\varphi(f : \mathcal{F} \to \overline{\mathcal{G}}) = \overline{\varphi}(\overline{\mathcal{G}}) - \overline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}),$$

and

$$\overline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) = \overline{\varphi}(\overline{\mathcal{H}}) - \overline{\varphi}(\overline{\mathcal{H}/\mathrm{im}\, g \circ f}),$$

and (5.5.16) follows from (5.5.17) applied to $\mathcal{G}' := \mathrm{im}\, f$.

(4) When (4) is satisfied in Proposition 5.5.7, by applying successively Propositions 5.5.6, 5.4.6, and 5.5.2, we obtain the following strengthened form of the estimate (5.5.16):

$$\overline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) = \underline{\mathrm{rk}}^1_\varphi(g \circ f : \mathcal{F} \to \overline{\mathcal{H}}) \leq \underline{\mathrm{rk}}^1_\varphi(f : \mathcal{G} \to \overline{\mathcal{H}}) \leq \overline{\mathrm{rk}}^1_\varphi(f : \mathcal{G} \to \overline{\mathcal{H}}). \tag{5.5.24}$$

When (4) is satisfied in Proposition 5.5.8, by the same reasoning as in part (1) of this proof, the chain of (in)equalities (5.5.18) holds and establishes (5.5.17). □

5.5.2.3. Â Let us spell out some consequences of Propositions 5.5.7 and 5.5.8 concerning the strong monotonicity properties of $\overline{\varphi}$, when as above φ is assumed to satisfy $\mathbf{StMon^1}$.

Recall that, if $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}^{\leq 1}_X$ where $\overline{\mathcal{F}}$ is φ-summable, then $\overline{f(\mathcal{F})}$ also is φ-summable and therefore $\mathrm{ev}\overline{\varphi}(\overline{f(\mathcal{F})}) = 0$, as shown in Proposition 4.5.9 and Theorem 4.5.1.[12] Therefore, as a consequence of Proposition 5.5.8, we obtain:

[12]In the next section, we shall investigate in more details the property of the φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ and their invariant $\overline{\varphi}$ when φ satisfies $\mathbf{StMon^1}$. Notably in Section 5.6 we shall prove that, when this condition holds, the φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ are characterized by the eventual vanishing of $\overline{\varphi}$.

Corollary 5.5.9. *For every pair of composable morphisms in* $\overline{\mathbf{qCoh}}_X^{\leq 1}$*:*

$$\overline{\mathcal{F}} \xrightarrow{f} \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{H}},$$

the following inequality holds:

$$\overline{\varphi}(\overline{\mathcal{H}}) + \overline{\varphi}(\overline{\mathcal{G}/\mathrm{im}\, f}) \leq \overline{\varphi}(\overline{\mathcal{G}}) + \overline{\varphi}(\overline{\mathcal{H}/\mathrm{im}\,(g \circ f)}), \tag{5.5.25}$$

provided one of the following conditions is satisfied:

(1) φ *also satisfies the downward continuity condition* $\mathbf{Cont}^+$*, and* $\overline{\mathcal{F}}$ *is* φ*-summable;*
(2) $\underline{\varphi}(\overline{\mathcal{H}}) = \overline{\varphi}(\overline{\mathcal{H}}) < +\infty$.

Applied to the situation where f and g are injective, Corollary 5.5.9 shows that $\overline{\varphi}$ satisfies the following version of the strong subadditivity condition $\mathbf{StMon^1_2}$:

Corollary 5.5.10. *Let* $\overline{\mathcal{F}}$ *be an object of* $\overline{\mathbf{qCoh}}_X$ *and let* $\mathcal{F}'$ *and* $\mathcal{F}''$ *be two* $\mathcal{O}_X$*-submodules of* $\mathcal{F}$ *such that* $\mathcal{F}'' \subseteq \mathcal{F}'$*. The following inequality holds:*

$$\overline{\varphi}(\overline{\mathcal{F}}) + \overline{\varphi}(\overline{\mathcal{F}'/\mathcal{F}''}) \leq \varphi(\overline{\mathcal{F}}') + \varphi(\overline{\mathcal{F}/\mathcal{F}''}), \tag{5.5.26}$$

provided one of the following conditions is satisfied:

(1) φ *satisfies condition* $\mathbf{Cont}^+$ *and* $\overline{\mathcal{F}}''$ *is* φ*-summable;*
(2) $\underline{\varphi}(\overline{\mathcal{F}}) = \overline{\varphi}(\overline{\mathcal{F}}) < +\infty$.

We may also derive from Corollary 5.5.9 the following version of the submodularity condition $\mathbf{StMon^1_3}$ on $\overline{\varphi}$:

Corollary 5.5.11. *Let* $\overline{\mathcal{F}}$ *be an object of* $\overline{\mathbf{qCoh}}_X$ *and let* $\mathcal{F}'$ *and* $\mathcal{F}''$ *be two* $\mathcal{O}_X$*-submodules of* $\mathcal{F}$*. The following inequality holds:*

$$\overline{\varphi}(\overline{\mathcal{F}/\mathcal{F}'}) + \overline{\varphi}(\overline{\mathcal{F}/\mathcal{F}''}) \leq \varphi(\overline{\mathcal{F}/(\mathcal{F}'+\mathcal{F}'')}) + \varphi(\overline{\mathcal{F}/(\mathcal{F}' \cap \mathcal{F}'')}), \tag{5.5.27}$$

provided one of the following conditions is satisfied:

(1) φ *satisfies condition* $\mathbf{Cont}^+$ *and* $\overline{\mathcal{F}}'$ *or* $\overline{\mathcal{F}}''$ *is* φ*-summable;*
(2) $\underline{\varphi}(\overline{\mathcal{F}}) = \overline{\varphi}(\overline{\mathcal{F}}) < +\infty$.

Proof. We may consider the following diagram in $\overline{\mathbf{qCoh}}_X^{\leq 1}$:

$$\overline{\mathcal{F}''/(\mathcal{F}' \cap \mathcal{F}'')} \xrightarrow{f} \overline{\mathcal{F}/(\mathcal{F}' \cap \mathcal{F}'')} \xrightarrow{g} \overline{\mathcal{F}/\mathcal{F}'},$$

where f and g respectively denote the inclusion and the quotient morphism. Then the inequality (5.5.25) becomes the inequality (5.5.27). Moreover, as observed above, Proposition 4.5.9 and Theorem 4.5.1 show that condition (1), with $\overline{\mathcal{F}}''$ summable, implies condition (1) in Corollary 5.5.9. Finally, according to Proposition 5.5.6, condition (2) implies condition (2) in Corollary 5.5.9.[13] □

Observe finally that the validity of Proposition 5.5.8 under condition (1) precisely asserts that $\overline{\varphi}$ satisfies $\mathbf{StMon^1_4}$, namely:

Corollary 5.5.12. *For any two objects* $\overline{\mathcal{G}}$ *and* $\overline{\mathcal{G}}^\sim$ *of* $\overline{\mathbf{qCoh}}_X$ *with the same underlying* $\mathcal{O}_K$*-module* $\mathcal{G}$ *such that* $\mathrm{Id}_{\mathcal{G}} : \overline{\mathcal{G}} \to \overline{\mathcal{G}}^\sim$ *is a morphism in* $\overline{\mathbf{qCoh}}_X^{\leq 1}$ *and for any subobject* $\mathcal{G}'$ *of* $\mathcal{G}$*, the following inequality is satisfied:*

$$\overline{\varphi}(\overline{\mathcal{G}}^\sim) + \overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}) \leq \overline{\varphi}(\overline{\mathcal{G}/\mathcal{G}'}^\sim) + \overline{\varphi}(\overline{\mathcal{G}}).$$

[13] Clearly condition (2) could be replaced by the weaker condition: $\underline{\varphi}(\overline{\mathcal{F}/\mathcal{F}'}) = \overline{\varphi}((\overline{\mathcal{F}/\mathcal{F}'}) < +\infty$ *or* $\underline{\varphi}(\overline{\mathcal{F}/\mathcal{F}''}) = \overline{\varphi}((\overline{\mathcal{F}/\mathcal{F}''}) < +\infty$.

5.6. Strong Monotonicity of φ and Criteria of φ-Summability

In this section, we consider an invariant $\varphi : \overline{\mathbf{Coh}}_X \to \mathbb{R}_+$ that satisfies the monotonicity and subadditivity conditions $\mathbf{Mon^1}$ and **SubAdd**, and we complete the study of φ-summable Hermitian quasi-coherent sheaves developed in Section 4.5 by establishing diverse criteria of φ-summability valid under some assumption of strong monotonicity on φ, and by deriving some additional permanence properties of φ-summability.

5.6.1. Criteria for φ-summability I: eventual vanishing of $\overline{\varphi}$.

5.6.1.1. The following proposition establishes a partial of converse to Theorem 4.5.1 when φ satisfies not only **SubAdd**, but also the strong subadditivity condition $\mathbf{StMon^{\frac{1}{2}}}$.

PROPOSITION 5.6.1. *Let us assume that φ satisfies condition $\mathbf{StMon^{\frac{1}{2}}}$.*

If $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})}$ is an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}) = 0$$

and if $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ is an exhaustive filtration of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ such that

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{i\to+\infty} \varphi(\overline{\mathcal{C}}_i), \tag{5.6.1}$$

then there exists a strictly increasing map $\iota : \mathbb{N} \to \mathbb{N}$ such that the following condition is satisfied[14]*:*

$$\mathbf{Sum}_\varphi(\mathcal{C}_{\iota(\bullet)}) : \quad \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_{\iota(\bullet)}) := \sum_{i=0}^{\infty} \varphi(\overline{\mathcal{C}_{\iota(i)}/\mathcal{C}_{\iota(i-1)}}) < +\infty.$$

As observed in Proposition 4.3.3, there always exists some exhaustive filtration of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ satisfying (5.6.1). Consequently, Theorem 4.5.1 and Proposition 5.6.1 imply:

COROLLARY 5.6.2. *When φ satisfies condition $\mathbf{StMon^{\frac{1}{2}}}$, an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is φ-summable if and only if $\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}})$ vanishes.*

PROOF OF PROPOSITION 5.6.1. Let $(\mathcal{C}_i)_{i\in\mathbb{N}}$ be an exhaustive fitration of $\mathcal{F}$ be elements of $\mathrm{coh}(\mathcal{F})$ satisfying (5.6.1). For any two non-negative integers i and j such that $i \leq j$ and any $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$ containing $\mathcal{C}_j$, the strong subbadditivity inequality (5.2.5), applied to $\overline{\mathcal{C}}$ and to the submodules

$$\mathcal{C}_i \subseteq \mathcal{C}_j \subseteq \mathcal{C},$$

reads as follows:

$$\varphi(\overline{\mathcal{C}}) + \varphi(\overline{\mathcal{C}_j/\mathcal{C}_i}) \leq \varphi(\overline{\mathcal{C}}_j) + \varphi(\overline{\mathcal{C}/\mathcal{C}_i}).$$

By taking the inferior limit over $\mathcal{C}$ in the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$, this implies the following inequality:

$$\overline{\varphi}(\overline{\mathcal{F}}) + \varphi(\overline{\mathcal{C}_j/\mathcal{C}_i}) \leq \varphi(\overline{\mathcal{C}}_j) + \overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}_i}).$$

(When φ satisfies $\mathbf{StMon^1}$, this estimate is a special case of Corollary 5.5.10, (2).) Consequently, for any i in $\mathbb{N}$, the following estimate holds:

$$\sup_{j\in\mathbb{N}_{\geq i}} \varphi(\overline{\mathcal{C}_j/\mathcal{C}_i}) \leq \sup_{j\in\mathbb{N}_{\geq i}} |\varphi(\overline{\mathcal{C}}_j) - \overline{\varphi}(\overline{\mathcal{F}})| + \overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}_i}). \tag{5.6.2}$$

As a consequence of (5.6.1) and of the vanishing of $\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}})$, the right-hand side of (5.6.2) converges to 0 when i goes to infinity. We may choose a strictly increasing map $\iota : \mathbb{N} \to \mathbb{N}$ such that:

$$\sum_{i\in\mathbb{N}} \left[\sup_{j\in\mathbb{N}_{\geq\iota(i)}} |\varphi(\overline{\mathcal{C}}_j) - \overline{\varphi}(\overline{\mathcal{F}})| + \overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}_{\iota(i)}}) \right] < +\infty,$$

and then the condition $\mathbf{Sum}_\varphi(\mathcal{C}_{\iota(\bullet)})$ is satisfied. ☐

[14] By convention, we let: $\mathcal{C}_{\iota(-1)} = 0$.

5.6.1.2. If we combine the criterion of φ-summability in Corollary 5.6.2 and the result on $\overline{\varphi}$-summable filtrations established in Proposition 4.5.17, we obtain the following criterion of φ-summability in terms of a general $\overline{\varphi}$-summable filtration:

PROPOSITION 5.6.3. *Let us assume that, beside condition* $\mathbf{Mon}^1$, *the invariant* φ *also satisfies conditions* $\mathbf{Cont}^+$ *and* $\mathbf{StMon}_{\mathbf{\frac{1}{2}}}$ *on* $\overline{\mathbf{Coh}}_X$.

Let $\overline{\mathcal{F}}$ *be an object of* $\mathbf{q}\overline{\mathbf{Coh}}_X$ *and let* $\mathcal{F}_\bullet := (\mathcal{F}_i)_{i\in\mathbb{N}}$ *be an exhaustive filtration of* $\mathcal{F}$ *by* $\mathcal{O}_X$*-submodules. If the subquotients*[15] $(\overline{\mathcal{F}_i/\mathcal{F}_{i-1}})_{i\in\mathbb{N}}$ *are* φ*-summable and if the condition*

$$\mathbf{Sum}_{\overline{\varphi}}(\mathcal{F}_\bullet): \quad \Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{F}_\bullet) := \sum_{i=0}^{+\infty} \overline{\varphi}(\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}) < +\infty$$

is satisfied, then $\overline{\mathcal{F}}$ *is* φ*-summable and*

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{i\to+\infty} \overline{\varphi}(\overline{\mathcal{F}}_i). \tag{5.6.3}$$

Proposition 5.6.3 may be understood as a closure property of the construction of the category φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ starting from the invariant:

$$\varphi : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+.$$

Roughly speaking, it asserts that, if we mimick the construction of φ-summable objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$, starting from the invariant:

$$\overline{\varphi} : \varphi_\Sigma\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

instead of φ, we actually do not enlarge the class of summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$.

5.6.2. The φ-finite Hermitian quasi-coherent sheaves. By combining the argument in the proof of Proposition 5.6.1 with a construction *à la* Mittag-Leffler, we may also establish the following proposition.

PROPOSITION 5.6.4. *Let us assume that* φ *satisfies condition* $\mathbf{StMon}_{\mathbf{\frac{1}{2}}}$. *For every object* $\overline{\mathcal{F}}$ *in* $\mathbf{q}\overline{\mathbf{Coh}}_X$, *and every increasing sequence* $(\delta_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *such that* $\lim_{n\to+\infty}\delta_n = +\infty$, *the following conditions are equivalent:*

(i) *For every* $n \in \mathbb{N}$, $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_n)$ *is* φ*-summable.*
(ii) *For every* $\delta \in \mathbb{R}$, $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ *is* φ*-summable.*
(iii) *For every* $n \in \mathbb{N}$, $\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_n)) = 0$.
(iv) *For every* $\delta \in \mathbb{R}$, $\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)) = 0$.
(v) *There exists an exhaustive filtration* $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ *of* $\mathcal{F}$ *by elements of* $\mathrm{coh}(\mathcal{F})$ *such that, for every* $\delta \in \mathbb{R}$, *the following condition is satisfied:*

$$\Sigma_\varphi\left(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta), \mathcal{C}_\bullet\right) := \sum_{i=0}^{\infty} \varphi\left(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{O}}(-\delta)\right) < +\infty. \tag{5.6.4}$$

PROOF. The implications $(ii) \Rightarrow (i)$ and $(iv) \Rightarrow (iii)$ are straightforward, the implication $(v) \Rightarrow (ii)$ follows from the definition of φ-summability, and the implications $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iv)$ from Theorem 4.5.1.

To complete the proof of the proposition, we shall establish the implication $(iii) \Rightarrow (v)$. Let us assume that condition (iii) is satisfied. To construct a filtration $\mathcal{C}_\bullet$ of $\mathcal{F}$ as in (v), let us choose a sequence $(\varepsilon_n)_{n\geq 1}$ of positive real numbers such that:

$$\sum_{n=1}^{+\infty} \varepsilon_n < +\infty.$$

[15] Here again, we let: $\mathcal{F}_{-1} = 0$.

By a straightforward inductive construction, we may find an exhaustive filtration $\mathcal{C}_\bullet$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ such that $\mathcal{C}_0 = 0$ and such that, for every integer $n \geq 1$, the following estimates holds:

$$\overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}_n} \otimes \overline{\mathcal{O}}(-\delta_n)) \leq \varepsilon_n \tag{5.6.5}$$

and

$$\left| \overline{h}^1_\theta \left(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_{n-1}) \right) - \overline{\varphi} \left(\overline{\mathcal{C}}_n \otimes \overline{\mathcal{O}}(-\delta_{n-1}) \right) \right| \leq \varepsilon_n. \tag{5.6.6}$$

Indeed $\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_n))$ vanishes and $\overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_{n-1}))$ is finite.

Recall that, as observed in the proof of Proposition 5.6.1, if $\mathcal{C} \subseteq \mathcal{C}'$ are two submodules in $\mathrm{coh}(\mathcal{F})$, then the following inequality holds:

$$\overline{\varphi}(\overline{\mathcal{F}}) + \varphi(\overline{\mathcal{C}'/\mathcal{C}}) \leq \varphi(\overline{\mathcal{C}}') + \overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}}).$$

By applying this estimate to $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_n)$ and to $\mathcal{C} := \mathcal{C}_n$ and $\mathcal{C}' := \mathcal{C}_{n+1}$, we obtain the following estimate[16] for every $n \in \mathbb{N}$:

$$\begin{aligned} \varphi \left(\overline{\mathcal{C}_{n+1}/\mathcal{C}_n} \otimes \overline{\mathcal{O}}(-\delta_n) \right) &\leq \overline{\varphi} \left(\overline{\mathcal{C}}_{n+1} \otimes \overline{\mathcal{O}}(-\delta_n)) \right) - \overline{\varphi} \left(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_n) \right) + \overline{\varphi} \left(\overline{\mathcal{F}/\mathcal{C}_n} \otimes \overline{\mathcal{O}}(-\delta_n) \right) \\ &\leq \varepsilon_{n+1} + \varepsilon_n. \end{aligned}$$

For every k in $\mathbb{N}$ and every integer $n \geq k$, we have: $\delta_k \leq \delta_n$, and therefore:

$$\varphi \left(\overline{\mathcal{C}_{n+1}/\mathcal{C}_n} \otimes \overline{\mathcal{O}}(-\delta_k) \right) \leq \varphi \left(\overline{\mathcal{C}_{n+1}/\mathcal{C}_n} \otimes \overline{\mathcal{O}}(-\delta_n) \right).$$

Consequently:

$$\sum_{n=k}^{+\infty} \varphi \left(\overline{\mathcal{C}_{n+1}/\mathcal{C}_n} \otimes \overline{\mathcal{O}}(-\delta_k) \right) \leq \sum_{n=k}^{+\infty} \varphi \left(\overline{\mathcal{C}_{n+1}/\mathcal{C}_n} \otimes \overline{\mathcal{O}}(-\delta_n) \right) \leq \sum_{n=k}^{+\infty} (\varepsilon_{n+1} + \varepsilon_n) < +\infty.$$

Finally, for every δ in $\mathbb{R}$, we may choose k such that $\delta_k \geq \delta$. The trivial estimate

$$\Sigma_\varphi \left(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta), \mathcal{C}_\bullet \right) \leq \sum_{i=0}^{k_1} \varphi \left(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) \otimes \overline{\mathcal{O}}(-\delta) \right) + \sum_{i=k}^{\infty} \varphi \left(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{O}}(-\delta_k) \right)$$

shows that (5.6.4) holds. □

DEFINITION 5.6.5. Assume that, besides $\mathbf{Mon}^1$ and $\mathbf{SubAdd}$, the invariant φ satisfies $\mathbf{StMon}^1_2$. Then a Hermitian quasi-coherent sheaf $\overline{\mathcal{F}}$ is called φ-finite when it satisfies the equivalent conditions in Proposition 5.6.4.

We will denote by $\varphi_f\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X$ and $\varphi_f\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ the full subcategories of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ whose objects are the φ-finite Hermitian quasi-coherent sheaves over X.

The categories $\overline{\mathbf{Coh}}_X$ and $\overline{\mathbf{Coh}}_X^{\leq 1}$ are full subcategories of $\varphi_f\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X$ and $\varphi_f\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$, which themselves are full subcategories of $\varphi_\Sigma\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X$ and $\varphi_\Sigma\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$.

The diverse permanence properties of the φ-summability established in Section 4.5 immediately imply similar permanence properties of the φ-finiteness. For later reference, we spell out some of them in the following Scholium.

SCHOLIUM 5.6.6. *Assume that the invariant φ satisfies conditions $\mathbf{Mon}^1_K$, $\mathbf{SubAdd}$, $\mathbf{StMon}^1_2$ and $\mathbf{Cont}^+$ on $\overline{\mathbf{Coh}}_X$.*

1) If $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ such that the K-linear map $f_K : \mathcal{F}_K \to \mathcal{G}_K$ is surjective and if $\overline{\mathcal{F}}$ is φ-finite, then $\overline{\mathcal{G}}$ is φ-finite.

[16] We let $\varepsilon_0 := \overline{\varphi}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_0))$.

2) If $\overline{\mathcal{F}}'$ and $\overline{\mathcal{F}}''$ are φ-finite objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ and if

$$0 \longrightarrow \overline{\mathcal{F}}' \xrightarrow{i} \overline{\mathcal{F}} \xrightarrow{p} \overline{\mathcal{F}}'' \longrightarrow 0$$

is an admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_X$, then $\overline{\mathcal{F}}$ is φ-finite.

3) For any object $\overline{\mathcal{E}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ and any two $\mathcal{O}_K$-submodules $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{E}$, if $\overline{\mathcal{E}}_1$ and $\overline{\mathcal{E}}_2$ are φ-finite, then $\overline{\mathcal{E}_1 + \mathcal{E}_2}$ also is φ-finite.

4) Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and let $\mathcal{F}_\bullet := (\mathcal{F}_i)_{i\in\mathbb{N}}$ be an exhaustive filtration of $\mathcal{F}$ by $\mathcal{O}_K$-submodules. If the subquotient[17] *$\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}$ is φ-finite for every $i \in \mathbb{N}$, and if, for every $\delta \in \mathbb{R}$:*

$$\sum_{i=0}^{+\infty} \overline{\varphi}(\overline{\mathcal{F}_i/\mathcal{F}_{i-1}} \otimes \overline{\mathcal{O}}(-\delta)) < +\infty,$$

then $\overline{\mathcal{F}}$ is φ-finite, and for every $\delta \in \mathbb{R}$:

$$\overline{\varphi}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)) = \lim_{i\to+\infty} \overline{\varphi}(\overline{\mathcal{F}}_i \otimes \overline{\mathcal{O}}(-\delta)).$$

5) When φ also satisfies $\mathbf{Add}_\oplus$ on $\overline{\mathbf{Coh}}_X$, then φ_f-$\mathbf{q}\overline{\mathbf{Coh}}_X$ is stable under direct sums. Moreover, for every countable family $(\overline{\mathcal{G}}_i)_{i\in I}$ of φ-finite Hermitian quasi-coherent sheaves, their direct sum $\bigoplus_{i\in I}$ if φ-finite if and only if, for every $\delta \in \mathbb{R}$,

$$\sum_{i\in I} \overline{\varphi}(\overline{\mathcal{G}}_i \otimes \overline{\mathcal{O}}(-\delta)) < +\infty.$$

Proof. Assertion 1) follows from Corollary 4.5.15, Assertion 2) from Proposition 4.5.12, Assertion 3) from Corollary 4.5.14, Assertion 4) from Proposition 4.5.17, and Assertion 5) from Corollary 4.5.4 and Proposition 4.5.6. $\square$

5.6.3. Criteria for φ-summability II: coincidence of $\underline{\varphi}$ and $\overline{\varphi}$.

5.6.3.1. By combining the characterization of φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ as the ones for which $\overline{\varphi}$ is eventually vanishing established in Corollary 5.6.2 and the version of the submodularity condition $\mathbf{StMon}^{\mathbf{1}}_{\mathbf{4}}$ for $\overline{\varphi}$ established in Corollary 5.5.11, one may prove the following criterion of φ-summability:

Proposition 5.6.7. *Let us assume that φ satisfies the strong monotonicity $\mathbf{StMon}^{\mathbf{1}}$. If an object in $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfies:*

$$\underline{\varphi}(\overline{\mathcal{F}}) = \overline{\varphi}(\overline{\mathcal{F}}) < +\infty, \tag{5.6.7}$$

then:

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}) = 0,$$

and therefore $\overline{\mathcal{F}}$ is φ-summable.

Proof. Let us assume that (5.6.7) holds.

For any ε in $\mathbb{R}_+^*$, we may find $\mathcal{G}$ in $\mathrm{coft}(\mathcal{F})$ such that:

$$\varphi(\overline{\mathcal{F}/\mathcal{G}}) > \underline{\varphi}(\overline{\mathcal{F}}) - \varepsilon = \overline{\varphi}(\overline{\mathcal{F}}) - \varepsilon. \tag{5.6.8}$$

As the quotient $\mathcal{F}/\mathcal{G}$ is finitely generated, for every large enough element $\mathcal{C}$ of $\mathrm{coh}(\mathcal{F})$, the composition of the inclusion and quotient maps

$$\mathcal{C} \hookrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{G}$$

is surjective, and therefore $\mathcal{C}+\mathcal{G} = \mathcal{F}$. Then, according to Corollary 5.5.11 (2) and to the monotonicity of $\overline{\varphi}$, the following inequalities hold:

$$\overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}}) + \varphi(\overline{\mathcal{F}/\mathcal{G}}) \leq \overline{\varphi}(\overline{\mathcal{F}/(\mathcal{C}\cap\mathcal{G})}) \leq \overline{\varphi}(\overline{\mathcal{F}}). \tag{5.6.9}$$

[17] Here again, by convention, we let: $\mathcal{F}_{-1} = 0$.

From (5.6.8) and (5.6.9), we deduce that every large enough $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$ satisfies the estimate:

$$\overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}}) < \varepsilon.$$

This establishes the upper bound:

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}) \leq \varepsilon,$$

and consequently, since ε is arbitrary, the vanishing of $\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}})$. In turn this implies the φ-summability of $\overline{\mathcal{F}}$ by Corollary 5.6.2. ☐

5.6.3.2. At this stage, when comparing the last propositions and the properties of the invariants $\underline{h}^1(C,.) = h^1(C,.)$ and $\overline{h}^1(C,.)$ investigated in Chapter 3, notably in Theorem 3.2.7, it is natural to ask for some converse to the implication:

$$\underline{\varphi}(\overline{\mathcal{F}}) = \overline{\varphi}(\overline{\mathcal{F}}) < +\infty \Longrightarrow \overline{\mathcal{F}}\ \varphi\text{-summable}$$

established in Proposition 5.6.7.

For instance, we may wonder if the following implication holds, for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$:

$$\text{If } \overline{\mathcal{F}} \text{ is } \varphi\text{-summable, then for every } \delta \in \mathbb{R}_+^*,\ \underline{\varphi}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\delta)) = \overline{\varphi}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\delta)) < +\infty. \tag{5.6.10}$$

To put into perspective the implication (5.6.10), we may point out that Minkowski's Theorem admits the following consequence, the proof of which is presented below:

Proposition 5.6.8. *For every Hermitian line bundle $\overline{L}$ over X, there exists implies a non-zero morphism $f : \overline{\mathcal{O}}(\delta) \to \overline{L}$ in $\overline{\mathbf{Vect}}_X^{\leq 1}$, where:*

$$\delta := [K:\mathbb{Q}]^{-1}\left(\widehat{\deg}\,\overline{L} - (1/2)\log|\Delta_K|\right). \tag{5.6.11}$$

Consequently the validity of (5.6.10) implies that *for every φ-summable object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and for every Hermitian line bundle $\overline{L}$ over X such that $\widehat{\deg}\,\overline{L} > (1/2)\log|\Delta_K|$, the following relations hold*:

$$\underline{\varphi}(\overline{\mathcal{F}} \otimes \overline{L}) = \overline{\varphi}(\overline{\mathcal{F}} \otimes \overline{L}) < +\infty.$$

Indeed δ defined by (5.6.11) is positive and $\overline{\mathcal{F}} \otimes \overline{L} \otimes \overline{\mathcal{O}}(-\delta)$ is φ-summable by Proposition 4.5.9. This property may be seen as an analogue, concerning the invariant φ instead of $h^1(C,.)$, of the implication (3) in Theorem 3.2.7.

The formalism of numerical invariants defined over $\mathbf{q}\overline{\mathbf{Coh}}_X$ developed in this chapter and the previous one does not appear to cover criteria for the coincidence of the lower and upper invariants $\underline{\varphi}$ and $\overline{\varphi}$ such as (5.6.10). However when φ is the θ-invariant h^1_θ, we will be able to establish the validity of (5.6.10) by resorting to the specific features of h^1_θ; see Section 8.4, Theorem 8.4.7 and Proposition 8.4.11.

Proof of Proposition 5.6.8. Consider the Euclidean lattice $\pi_*\overline{L}$. Under the isomorphism

$$(\pi_* L)_{\mathbb{C}} := L_K \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{x \in X(\mathbb{C})} L_x, \quad l \otimes_{\mathbb{Q}} \lambda \longmapsto (l \otimes_x \lambda)_{x \in X(\mathbb{C})}, \tag{5.6.12}$$

the real subspace $(\pi_* L)_{\mathbb{R}} := L_K \otimes_{\mathbb{Q}} \mathbb{R}$ gets identified with the fixed points of the complex conjugation:

$$\bigoplus_{x \in X(\mathbb{C})} L_x \longrightarrow \bigoplus_{x \in X(\mathbb{C})} L_x, \quad (l_x)_{x \in X(\mathbb{C})} \longmapsto (\overline{l_{\overline{x}}})_{x \in X(\mathbb{C})}.$$

Moreover the Hermitian metric $\|.\|_{\pi_*\overline{L}}$ on $(\pi_*\overline{L})_{\mathbb{C}}$, expressed in terms of the isomorphism (5.6.12), takes the following form:

$$\|(l_x)_{x \in X(\mathbb{C})}\|^2_{\pi_*\overline{L}} = \sum_{x \in X(\mathbb{C})} \|l_x\|^2_{\overline{L},x}.$$

The rank of $\pi_*\overline{L}$ is $[K:\mathbb{Q}]$, and its Arakelov degree is easily seen to be:

$$\widehat{\deg}\,\pi_*\overline{L} = \widehat{\deg}\,\overline{L} + \widehat{\deg}\,\pi_*\overline{\mathcal{O}}_X(0) = \widehat{\deg}\,\overline{L} - (1/2)\log|\Delta_K|. \tag{5.6.13}$$

The morphisms $f : \overline{\mathcal{O}}(\delta) \to \overline{L}$ in $\overline{\mathbf{Vect}}_X^{\leq 1}$ may be identified by the map $f \mapsto f(1)$ to the set of lattice points of the Euclidean lattice $\pi_*\overline{L}$ that lies in the compact convex subset C of $(\pi_*L)_{\mathbb{R}}$ defined as follows, in terms of the isomorphism (5.6.12):

$$C := \{(l_x)_{x\in X(\mathbb{C})} \in (\pi_*L)_{\mathbb{R}} \mid \forall x \in X(\mathbb{C}), \|l_x\|_{\overline{L},x} \leq e^{-\delta}\}.$$

A choice compatible with complex conjugation of a unit vector in each of the one-dimensional Hermitian $\mathbb{C}$-vector space $\overline{L}_x$, $x \in X(\mathbb{C})$, determines an isomorphism of Hermitian vector spaces between $(\pi_*\overline{L})_{\mathbb{C}}$ and $\mathbb{C}^{X(\mathbb{C})}$ equipped with the Hermitian norm:

$$(z_x)_{x\in X(\mathbb{C})} \longmapsto (\sum_{x\in X(\mathbb{C})} |z_x|^2)^{1/2}.$$

Consequently the Euclidean $\mathbb{R}$-vector space $(\pi_*Lb)_{\mathbb{R}}$ is isomorphic to[18] $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, equipped with the Euclidean norm

$$(x_1, \ldots, x_{r_1}, z_1, \ldots, z_{r_2}) \longmapsto (x_1^2 + \cdots + x_{r_1}^2 + 2|z_1|^2 + \cdots + 2|z_{r_2}|^2)^{1/2},$$

by an isomorphism which maps C to

$$[-e^{-\delta}, e^{-\delta}]^{r_1} \times \overline{D}(0, e^{-\delta})^{r_2}.$$

Therefore the volume of C with respect to the Lebesgue measure of $(\pi_*\overline{L})_{\mathbb{R}}$ is:

$$\mathrm{vol}(C) = (2e^{-\delta})^{r_1}(2\pi e^{-2\delta})^{r_2} = (\pi/2)^{r_2}(2e^{-\delta})^{[K:\mathbb{Q}]}.$$

Clearly it satisfies the lower bound:

$$\mathrm{vol}(C) \geq (2e^{-\delta})^{[K:\mathbb{Q}]},$$

with equality when K is totally real. Using the definition (5.6.11) of δ and the relation (5.6.8), this inequality may also be written as follows:

$$\mathrm{vol}(C) \geq 2^{[K:\mathbb{Q}]} \exp\left(-\widehat{\deg}\,\overline{L} + (1/2)\log|\Delta_K|\right) = 2^{\mathrm{rk}\pi_*\overline{L}}\,\mathrm{covol}(\pi_*\overline{L}).$$

According to Minkowski's First Theorem applied to the Euclidean lattice $\pi_*\overline{L}$ and to compact symmetric convex subset C in $(\pi_*L)_{\mathbb{R}}$, the intersection of C with the set of lattice points of $\pi_*\overline{L}$ is therefore not reduced to 0. ☐

[18] As usual we denote by r_1 (resp. r_2) the number of real (resp. complex) places of K, or equivalently the number of orbit of cardinlaitu 1 (resp. 2) of complex conjugation acting on $X(\mathbb{C})$.

Interlude:

5.7. A Summary

We conclude this part by a summary of the main results established in Chapters 4 and 5.

This summary is written to be understandable with only a knowledge of the definitions of conditions $\mathbf{Mon}^1$ (monotonicity), **SubAdd** (subadditivity), **NST** ("does not see torsion"), **NSAp** ("does not see the antiprojective part"), $\mathbf{Cont}^+$ (downward continuity), $\mathbf{Add}_\oplus$ (additivity), and $\mathbf{StMon}^1$ (strong monotonicity) introduced in Sections 4.1 and 5.2.

Let us emphasize that this summary does *not* reflect the logical dependences in the derivation of these results, and omits the properties of the φ-rank attached to an invariant φ and of its lower and upper extensions, in spite of their conceptual significance and in their role in the proof of the most advanced of these results.

We have also omitted the definition and the properties of the φ-finite objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$, stated in Subsection 5.6.2, that are immediately accessible after reading this summary.

In this Subsection, we consider an invariant:

$$\varphi : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

that satisfies the conditions $\mathbf{StMon^1}$, **NST**, and $\mathbf{Cont}^+$.

5.7.1. The upper extension $\overline{\varphi}$. To the invariant φ, we may associate its upper extension,

$$\overline{\varphi} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

defined by:

$$\overline{\varphi}(\overline{\mathcal{F}}) := \liminf_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}})$$

for every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, where the inferior limit is taken over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$ of coherent $\mathcal{O}_X$-submodules of $\mathcal{F}$, or equivalently of finitely generated $\mathcal{O}_K$-submodules of $\mathcal{F}(X)$ (*cf.* Definitions 4.3.1 and 4.3.2).

The invariant $\overline{\varphi}$ indeed extends φ — namely, for every $\overline{\mathcal{C}}$ in $\overline{\mathbf{Coh}}_X$, we have:

$$\overline{\varphi}(\overline{\mathcal{C}}) = \varphi(\overline{\mathcal{C}})$$

(*cf.* Proposition 4.3.4). Moreover $\overline{\varphi}$ satisfies the conditions $\mathbf{Mon}^1$ and **SubAdd** on $\mathbf{q}\overline{\mathbf{Coh}}_X$ (*cf.* Propositions 4.3.6 and 4.3.10).

When φ is small on Hermitian coherent sheaves generated by small sections (*cf.* Definition 4.2.11), then $\overline{\varphi}$ also satisfies **NSAp** on $\mathbf{q}\overline{\mathbf{Coh}}_X$ (*cf.* 4.1.4.2 and Proposition 4.3.12).

5.7.2. The category φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ of φ-summable Hermitian quasi-coherent sheaves. To any object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, we may also associate:

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}) := \lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}}) = \inf_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \overline{\varphi}(\overline{\mathcal{F}/\mathcal{C}}) \quad (\in [0, +\infty]),$$

where the limit and the infimum are taken over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$ (*cf.* Definitions 4.3.1 and 4.4.1). If moreover $\mathcal{F}_\bullet := (\mathcal{F}_i)_{i\in\mathbb{N}}$ is a filtration of the $\mathcal{O}_K$-module $\mathcal{F}$ underlying $\overline{\mathcal{F}}$, we let:

$$\Sigma_{\overline{\varphi}}(\overline{\mathcal{F}}, \mathcal{F}_\bullet) := \sum_{i=0}^{+\infty} \overline{\varphi}(\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}) \quad (\in [0, +\infty]),$$

where by convention $\mathcal{F}_{-1} = 0$. In particular, if $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ is a filtration of $\mathcal{F}$ by submodules in $\mathrm{coh}(\mathcal{F})$, we have:

$$\Sigma_{\overline{\varphi}}(\overline{\mathcal{F}}, \mathcal{C}_\bullet) = \Sigma_{\varphi}(\overline{\mathcal{F}}, \mathcal{C}_\bullet) := \sum_{i=0}^{+\infty} \varphi(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}).$$

(*cf.* Definition 4.5.2 and Proposition 4.5.17.) An object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is called *φ-summable* when there exists an exhaustive filtration $\mathcal{C}_\bullet$ of $\mathcal{F}$ by submodules in $\mathrm{coh}(\mathcal{F})$ such that

$$\Sigma_\varphi(\overline{\mathcal{F}}, \mathcal{C}_\bullet) < +\infty$$

(*cf.* Definition 4.5.2). When this holds, we have:

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{k\to+\infty} \varphi(\overline{\mathcal{C}}_k) \in \mathbb{R}_+$$

(*cf.* Theorem 4.5.1). Moreover, an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is φ-summable if and only if

$$\mathrm{ev}\overline{\varphi}(\overline{\mathcal{F}}) = 0$$

(*cf.* Corollary 5.6.2).

The objects of $\overline{\mathbf{Coh}}_X$ are clearly φ-summable. Moreover the φ-summable objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfy the following permanence properties:

(1) Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$, and let $\mathcal{F}_\bullet := (\mathcal{F}_i)_{i\in\mathbb{N}}$ be an exhaustive filtration of the $\mathcal{O}_K$-module $\mathcal{F}$ underlying $\overline{\mathcal{F}}$. If the subquotients $\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}$ are φ-summable and if

$$\Sigma_{\overline{\varphi}}(\overline{\mathcal{F}}, \mathcal{F}_\bullet) < +\infty,$$

then $\overline{\mathcal{F}}$ is φ-summable and

$$\overline{\varphi}(\overline{\mathcal{F}}) = \lim_{i\to+\infty} \overline{\varphi}(\overline{\mathcal{F}}_i)$$

(*cf.* Proposition 5.6.3).

(2) Let $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ be a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$. If $\overline{\mathcal{F}}$ is φ-summable and if the K-linear map $f_K : \mathcal{F}_K \to \mathcal{G}_K$ is surjective, then $\overline{\mathcal{G}}$ is φ-summable (*cf.* Corollary 4.5.15).

If we denote by φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ the subcategory of φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$, then the invariant

$$\overline{\varphi} : \varphi_\Sigma\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

satisfies the conditions **Cont** (*cf.* Proposition 4.5.16) and a version of $\mathbf{StMon}^1$ (*cf.* Proposition 5.5.8 and Corollaries 5.5.9 to 5.5.12).

5.7.3. The lower extension $\underline{\varphi}$. To the invariant φ, we may also attach its lower extension

$$\underline{\varphi} : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

defined by:

$$\underline{\varphi}(\overline{\mathcal{F}}) := \lim_{\mathcal{F}'\in\mathrm{coft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{F}'}) = \sup_{\mathcal{F}'\in\mathrm{coft}(\mathcal{F})} \varphi(\overline{\mathcal{F}/\mathcal{F}'}),$$

where the limit is taken over the directed set $(\mathrm{coft}(\mathcal{F}), \supseteq)$, defined by the set $\mathrm{coft}(\mathcal{F})$ of $\mathcal{O}_K$-submodules $\mathcal{G}$ of $\mathcal{F}$ such that the quotient $\mathcal{O}_X$-module $\mathcal{F}/\mathcal{G}$ is coherent (*cf.* Definition 4.3.2).

The lower extension $\underline{\varphi}$ still extends φ, and for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, it satisfies:

$$\underline{\varphi}(\overline{\mathcal{F}}) \leq \overline{\varphi}(\overline{\mathcal{F}})$$

(*cf.* Proposition 4.3.4). The invariant $\underline{\varphi}$ satisfies the conditions $\mathbf{Mon}^1$, **SubAdd**, **NSAp**, and $\mathbf{StMon}^1$ on $\mathbf{q}\overline{\mathbf{Coh}}_X$ (*cf.* Propositions 4.3.6, 4.3.10, 4.3.12, and Corollary 5.4.7). Moreover its restriction to the subcategory of $\mathbf{q}\overline{\mathbf{Coh}}_X$ where it takes finite values satisfies the condition $\mathbf{Cont}^+$ (*cf.* Proposition 5.4.9).

Any object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that

$$\underline{\varphi}(\overline{\mathcal{F}}) = \overline{\varphi}(\overline{\mathcal{F}}) < +\infty$$

is φ-summable (*cf.* Proposition 5.6.7).

5.7.4. Additivity. Let us assume that φ also satisfies the condition $\mathbf{Add}_\oplus$ on $\overline{\mathbf{Coh}}_X$.

Then $\overline{\varphi}$ satisfies $\mathbf{Add}_\oplus$ on φ_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$; more generally, for any countable family $(\overline{\mathcal{F}}_i)_{i\in I}$ of φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that

$$\sum_{i\in I} \overline{\varphi}(\overline{\mathcal{F}}_i) < +\infty,$$

the direct sum

$$\overline{\mathcal{F}} := \bigoplus_{i\in I} \overline{\mathcal{F}}_i$$

is φ-summable and satisfies

$$\overline{\varphi}(\overline{\mathcal{F}}) = \sum_{i\in I} \overline{\varphi}(\overline{\mathcal{F}}_i)$$

(*cf.* Proposition 4.5.6).

Moreover, for any countable family $(\overline{\mathcal{F}}_i)_{i\in I}$ of objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ of direct sum $\mathcal{F}$ as above, we have:

$$\underline{\varphi}(\mathcal{F}) = \sum_{i\in I} \underline{\varphi}(\overline{\mathcal{F}}_i)$$

(*cf.* Proposition 4.3.8).

Part 3

Infinite-Dimensional Geometry of Numbers and Theta Invariants of Hermitian Quasi-coherent Sheaves

CHAPTER 6

Covering Radius of Euclidean Quasi-coherent Sheaves and Elementary Infinite-Dimensional Geometry of Numbers

6.0.1. Our purpose in this monograph is to develop a theory of Hermitian quasi-coherent sheaves suited to Diophantine applications. More specifically, our aim is to develop a theory of invariants attached to Hermitian quasi-coherent sheaves that play the role of the invariant

$$h^1(C,\mathcal{F}) := \dim_k H^1(C,\mathcal{F})$$

attached to a quasi-coherent sheaf $\mathcal{F}$ on a projective curve C over some base field k. We are notably interested in the Hermitian quasi-coherent sheaves $\overline{F}$ on which such "h^1-like invariants" take a finite value.

Consider for instance Hermitian quasi-coherent sheaves over the arithmetic curve $\operatorname{Spec}\mathbb{Z}$, also called Euclidean quasi-coherent sheaves in this monograph. By definition, these are pairs $\overline{F} := (F, \|.\|)$ consisting in a countably generated[1] $\mathbb{Z}$-module M and a Euclidean seminorm $\|.\|$ on the $\mathbb{R}$-vector space $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. The finiteness of the invariants alluded to above (for instance the finiteness of the upper θ-invariant $\overline{h}^1_\theta(\overline{F})$) turns out to mean that, in a sense to be made precise, every point of $F_{\mathbb{R}}$ may be well approximated, in terms of the seminorm $\|.\|$, by some "integral point," namely by some element of the image $M_{/\mathrm{tor}} \simeq M/M_{\mathrm{tor}}$ of the canonical map $M \to M_{\mathbb{R}}$.

In this chapter, we study invariants of a Euclidean quasi-coherent sheaf $\overline{M}$ that are defined in elementary terms, in the spirit of the classical geometry of numbers,[2] and whose finiteness, or eventual vanishing, also expresses the notion that the points of $M_{\mathbb{R}}$ may be well approximated by points of $M_{/\mathrm{tor}}$.

Among those elementary invariants of a Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$, the *covering radius* $\rho(\overline{M})$ will play a central role. It is defined by the following formula, which is a straightforward generalization of the expression of the classical covering radius of Euclidean lattices:

$$\rho(\overline{M}) := \sup_{x\in M_{\mathbb{R}}} \inf_{m\in M_{/\mathrm{tor}}} \|x-m\| \in [0,+\infty]. \tag{6.0.1}$$

The terminology "covering radius" expresses the fact that $\rho(\overline{M})$ is the infimum of the positive real numbers R such that $M_{\mathbb{R}}$ is covered by the union of the open balls of radius R of centers the points m of $M_{/\mathrm{tor}}$, that is such that:

$$M_{\mathbb{R}} = \bigcup_{m\in M_{/\mathrm{tor}}} \Big\{ x\in M_{\mathbb{R}} \mid \|x-m\| < R \Big\}.$$

This chapter is basically devoted to the study of the covering radius ρ and of diverse extensions and variants of it. In the final chapter of this monograph, we shall study the relations of this invariant ρ with the more sophisticated θ-invariants $\underline{h}^1_\theta$ and $\overline{h}^1_\theta$, relations that actually play a key role in the Diophantine applications of θ-invariants. In contrast, the present chapter is of a rather elementary nature, and extends basic properties of diverse simple invariants attached to (finite dimensional) Euclidean lattices in the infinite dimensional framework of Euclidean quasi-coherent sheaves.

[1] or equivalently, countable

[2] which is concerned with (finite dimensional) Euclidean lattices.

In accordance with this elementary character, this chapter has been written to be readable with a minimal knowledge of the material introduced in the previous chapters.[3] To minimize its prerequisites, we have at several places given simple direct proofs of statements that are special cases of results previously established in Chapters 4 and 5.

This chapter has also been written for an audience of diverse background. As for the remaining of this monograph, the readership we firstly have in mind are the (less than ideal) arithmetic geometers with no familiarity with classical geometry of numbers. For this reason, we have discussed in some detail some basic facts of geometry of numbers, for instance concerning Voronoi cells in Section 6.2. However we hope that this chapter will also be of interest to mathematicians or computer scientists with already some expertise in Euclidean lattices. Indeed our "infinite dimensional perspective" has led us to establish some new results concerning Euclidean lattices, for instance the elementary formulae concerning Euclidean lattices of rank 2 in Section 6.2.3, and various estimates between invariants of Euclidean lattices in Section 6.4, notably Corollary 6.4.16.

In spite of its elementary character, two of the central themes of this monograph already appears in this chapter. Namely, in Section 6.4, the replacement, in estimates relating various invariants of objects or morphisms in $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$, of the Euclidean seminorm $\|.\|$ defining a Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$ by a Euclidean seminorm $\|.\|'$ on $M_{\mathbb{R}}$ that is Hilbert-Schmidt relatively to $\|.\|$. And, in Section 6.7, the development of some *relative geometry of numbers*, which concerns not only objects in $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$, but also morphisms between those.

6.0.2. Let us describe in more detail the content of this chapter.

Section 6.1 is devoted to the basic properties of the covering radius ρ, defined by the expression (6.0.1) above. These properties actually demonstrate that the "right" invariant to consider is not the covering radius ρ, but rather its square ρ^2. The latter actually satisfies the properties of monotonicity $\mathbf{Mon}^1$, of subadditivity $\mathbf{SubAdd}$, and of additivity $\mathbf{Add}_{\oplus}$ introduced in Section 4.1.

Section 6.2 discusses covering radii of Euclidean lattices in relation with the geometry of their Voronoi cell. The content of this section will be of limited use in the remaining of this monograph.[4] However it provides some useful geometric insight on covering radius — and actually suggests properties of this invariant that still hold in the infinite dimensional case, and geometric arguments to establish them.[5] The consideration of Voronoi cells, and the precise description of covering radii of Euclidean lattices of rank 2 it leads to, also allows us to show that the invariant ρ^2 does *not* satisfies the strong monotonicity property $\mathbf{StMon}^1$ introduced in Chapter 5.

In Section 6.3, we resume our discussion of general properties of the invariants ρ and ρ^2. We establish their compatibility with the vectorization in $\mathbf{\overline{Coh}}_{\mathbb{Z}}$ and with the canonical dévissage in $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$, and we discuss their continuity properties over $\mathbf{\overline{Coh}}_{\mathbb{Z}}$. We also introduce and we discuss the lower and upper covering radii $\underline{\rho}$ and $\overline{\rho}$ deduced from the invariant ρ on $\mathbf{\overline{Coh}}_{\mathbb{Z}}$ by the constructions introduced in Section 4.3. These are variants of the invariant ρ, which are defined in terms of "finite-rank approximations" of objects of $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$, and turn out to satisfy better properties than the "naive" invariant ρ and to naturally occur in Diophantine applications.

In Section 6.4, we introduce two other basic invariants attached to a Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$, the invariants $\lambda^{[0]}(\overline{M})$ and $\gamma(\overline{M})$. The invariant $\lambda^{[0]}(\overline{M})$ (resp. $\gamma(\overline{M})$) is defined as the infimum of the $R \in \mathbb{R}_+^*$ such that the set of integral points of seminorm less than R,

[3]Notably we work over $\operatorname{Spec} \mathbb{Z}$, and not over an arbitrary arithmetic curve $X = \operatorname{Spec} \mathcal{O}_K$. Accordingly only the categories of Euclidean quasi-coherent sheaves $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ and $\mathbf{q\overline{Coh}}_{\mathbb{Z}}^{\leq 1}$ will be involved, and not the general categories of Hermitian quasi-coherent sheaves $\mathbf{q\overline{Coh}}_X$ and $\mathbf{q\overline{Coh}}_X^{\leq 1}$. From Chapters 4 and 5, we will basically only use the terminology concerning properties of invariants introduced in Section 4.1, and the construction of the upper and lower extensions to $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ of invariants on $\mathbf{\overline{Coh}}_{\mathbb{Z}}$ in Section 4.3.

[4]With the notable exception of the derivation of comparison estimates between θ-invariants and covering radii in Section 9.1.

[5]See for instance the proof of the inequality $\gamma(\overline{M}) \leq 2\rho(\overline{M})$ in 6.4.2 below.

namely:

$$\{m \in M_{/\mathrm{tor}} \mid \|m\| < R\},$$

generates a $\mathbb{R}$-subvector space of $M_{\mathbb{R}}$ dense in $M_{\mathbb{R}}$ for the topology defined by the seminorm $\|.\|$ (resp. generates the $\mathbb{Z}$-module $M_{/\mathrm{tor}}$).

The invariant $\lambda^{[0]}(\overline{M})$ is a generalization, defined for an arbitrary Euclidean quasi-coherent sheaf $\overline{M}$, of the last of the successive minima classically attached to a Euclidean lattice. The invariant $\gamma(\overline{M})$ attached to Euclidean lattices does not seem to be explicitly investigated in the literature, although it appears implicitly in various places.

These invariants $\lambda^{[0]}(\overline{M})$ and $\gamma(\overline{M})$ have definitions that are arguably still more elementary than the one of the covering radius $\rho(\overline{M})$, but they enjoy less satisfactory formal properties. However simple comparison estimates relate the invariants ρ, $\lambda^{[0]}$, and γ. For instance, in Subsection6.4.2, we establish the following inequalities:

$$\lambda^{[0]}(\overline{M}) \leq \gamma(\overline{M}) \leq 2\rho(\overline{M}).$$

Estimates in the opposite direction — for instance an upper bound on $\rho(\overline{M})$ in terms of $\gamma(\overline{M})$ — are easily seen not to be satisfied for $\overline{M}$ an arbitrary object in $\mathbf{qCoh}_{\mathbb{Z}}$. However such estimates do hold when one restricts to Euclidean lattices of some given rank n, but involve some constant depending on n. For instance, one may prove that for any Euclidean lattice $\overline{E}$ of rank n, the following inequality holds:

$$\rho(\overline{E}) \leq \sqrt{n}/2\, \lambda_n(\overline{E}). \tag{6.0.2}$$

It turns out that dimension dependent estimates like (6.0.2) have infinite dimensional counterparts that concern an arbitrary Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$ *together with a second Euclidean seminorm* $\|.\|'$ on the $\mathbb{R}$-vector space $M_{\mathbb{R}}$. These estimates involve the relative trace[6] $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$ of the associated semipositive quadratic forms $\|.\|'^2$ and $\|.\|^2$ on $M_{\mathbb{R}}$ in place of the rank n of a Euclidean lattice.

For instance, with this notation, we will show in 6.4.3 that the following inequality holds:

$$\rho(M, \|.\|') \leq \sqrt{\mathrm{Tr}(\|.\|'^2/\|.\|^2)}/2\ \lambda^{[0]}(M, \|.\|). \tag{6.0.3}$$

When $\overline{M}$ is a Euclidean lattice of rank n and $\|.\|' = \|.\|$, then:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) = n,$$

and the estimate (6.0.3) specializes to the classical estimate (6.0.2).

The consideration of pairs of Euclidean seminorms $\|.\|$ and $\|.\|'$ that satisfy the Hilbert-Schmidt condition:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty \tag{6.0.4}$$

— the validity of which is required for the estimate (6.0.3) to be non-trivial — will be a central feature of the infinite dimensional geometry of numbers developed in this monograph.

Such pairs of seminorms will notably appear in the comparison estimates relating θ-invariants and covering radii established in the final chapter of this monograph, and the role of the Hilbert-Schmidt condition (6.0.4) is to be compared to the one it plays in measure theory on infinite dimensional topological vector spaces, as demonstrated by the classical work of Prokhorov, Sazonov, and Minlos (see Appendix C). Moreover pairs of Euclidean seminorms satisfying condition (6.0.4) naturally occur in Diophantine applications, as a consequence of the nuclearity properties of spaces of sections of analytic coherent sheaves.

Section 6.5 is devoted to the property of eventual vanishing of the invariants ρ, $\lambda^{[0]}$, and γ.

[6]The definition and the basic properties of relative traces of semipositive quadratic forms are recalled in Appendix B.

These invariants indeed satisfy the monotonicity condition $\mathbf{Mon}^1$, and consequently, if φ denotes one of them, we may consider the invariant $\mathrm{ev}\varphi$ defined in Section 4.4 as follows. Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf, and let $\mathrm{coh}(M)$ denote the set of finitely generated submodules of M. The monotonicity of φ implies that the function:

$$\mathrm{coh}(M) \longrightarrow [0, +\infty], \quad C \longmapsto \varphi(\overline{M/C})$$

is non-increasing on the directed set $(\mathrm{coh}(M), \subseteq)$, and we may consider the limit:

$$\mathrm{ev}\varphi(\overline{M}) := \lim_{C \in \mathrm{coh}(M)} \varphi(\overline{M/C}) = \inf_{C \in \mathrm{coh}(M)} \varphi(\overline{M/C}).$$

We say that the invariant φ is eventually vanishing on $\overline{M}$ when $\mathrm{ev}\varphi(\overline{M})$ vanishes.

When φ is one of the invariant ρ, $\lambda^{[0]}$, or γ, the eventual vanishing of φ turns out to admit various equivalent formulations that we establish in Section 6.5. The eventual vanishing of each of these three invariants on some some Euclidean quasi-coherent sheaf $\overline{M}$ expresses that, in a suitable sense, every point of $M_{\mathbb{R}}$ admits arbitrarily good approximations by points in $M_{/\mathrm{tor}}$, up to some finite dimensional error. We also investigate the relations between these three eventual vanishing properties.

In classical geometry of numbers, to a Euclidean lattice $\overline{E}$ of rank n are associated sequences of n invariants, notably its successive minima $(\lambda_1(\overline{E}, \ldots, \lambda_n(\overline{E}))$. In Section 6.6, we discuss analogous constructions of sequences of invariant in our infinite dimensional setting. These sequences are indexed by some integer $i \in \mathbb{N}$, and appear in two guises, depending on whether i represents either the dimension or the codimension of some auxiliary finite dimensional subspace or quotient space. In this way, we introduce some invariants λ_i and $\lambda^{[i]}$, γ_i and $\gamma^{[i]}$, and ρ_i and $\rho^{[i]}$, that generalize the invariants $\lambda^{[0]}$, γ, and ρ investigated in the previous sections.

Finally, in Section 6.7, we introduce a relative variant of the covering radius of objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, associated to morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$. Since the covering radius ρ and its square ρ^2 do not satisfy the strong monotonicity condition $\mathbf{StMon}^1$ introduced in Chapter 5, we do not define it as a "rank invariant," by means of the formalism of Sections 5.1 and 5.2. Instead we directly define the *relative covering radius of a morphism* $f : \overline{M} \to \overline{N}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{Z}$ as:

$$\rho(f) :=:= \sup_{x \in M_{\mathbb{R}}} \inf_{n \in N_{/\mathrm{tor}}} \|f_{\mathbb{R}}(x) - n\| \in [0, +\infty],$$

and we prove that it satisfies properties formally analogous to the ones of the φ-rank attached to a strongly monotonic invariant φ that have been presented in Section 5.2.

By definition, $\rho(f)$ is the supremum over the real vector space $f_{\mathbb{R}}(M_{\mathbb{R}})$ of the distance to $N_{/\mathrm{tor}}$. When f is the identity morphism $\mathrm{Id}_{\overline{M}}$ of some object $\overline{M}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, $\rho(f)$ coincides with the covering radius of $\overline{M}$, and accordingly the relative covering radius $\rho(f)$ is a generalization of the invariant $\rho(\overline{M})$ investigated in the previous sections. Various properties of the covering radius may be extended to the relative covering radius, which like the covering radius admits a lower and upper variant.

The results in Section 6.7 might at this stage appear as rather formal generalizations. However they will play a key role in Chapter 9, in the comparison of covering radii and θ-invariant and in its applications to density theorems.

6.0.3. In this chapter, we will use the following notation.

If $\overline{M} := (M, \|.\|)$ is a Euclidean quasi-coherent sheaf, that is, an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, we denote by $M_{/\mathrm{tor}}$ the quotient M/M_{tor} of M by its torsion submodule, and we identify it to the image of M in $M_{\mathbb{R}}$.

We write $\overline{M}_{\mathbb{R}}$ for the pair $(M_{\mathbb{R}}, \|.\|)$, which defines a "seminormed" real vector space. If m is an element of M, we denote by $m_{\mathbb{R}}$ its image in $M_{/\mathrm{tor}}$ seen as a submodule of $M_{\mathbb{R}}$, and we let:

$$\|m\| := \|m_{\mathbb{R}}\|.$$

Moreover, for every $R \in \mathbb{R}_+^*$, we let:

$$B(\overline{M};R) := \{m \in M \mid \|m\| < R\} \quad \text{and} \quad \overline{B}(\overline{M};R) := \{m \in M \mid \|m\| \leq R\}\,,$$

and:

$$B(\overline{M}_{\mathbb{R}};R) := \{v \in M_{\mathbb{R}} \mid \|v\| < R\} \quad \text{and} \quad \overline{B}(\overline{M}_{\mathbb{R}};R) := \{v \in M_{\mathbb{R}} \mid \|v\| \leq R\}\,.$$

Finally we denote by $\mathrm{coh}(M)$ (resp. $\mathrm{coft}(M)$, resp. $\mathrm{scoft}(M)$) the set of the $\mathbb{Z}$-submodules N of M such that N is finitely generated (resp. M/N is finitely generated, resp. M/N is finitely generated and torsion free).

We also use the basic notions and notation concerning pairs of Euclidean seminorms recalled in Appendices A and B, notably the definitions of boundedness and compactness of a Euclidean seminorm $\|.\|'$ with respect to another one $\|.\|$, of the relative supremum $\sup(\|.\|'/\|.\|)$, and of the relative trace $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$.

6.1. The Covering Radius of Euclidean Quasi-coherent Sheaves I. Definitions and Basic Properties

6.1.1. Definition of the covering radius.

Definition 6.1.1. The *covering radius* of a Euclidean quasi-coherent sheaf $\overline{M} := (M, ||.||)$ is the element of $[0, +\infty]$:

$$\rho(\overline{M}) := \sup_{x \in M_{\mathbb{R}}} \inf_{m \in M_{/\mathrm{tor}}} ||x - m||.$$

Equivalently, $\rho(\overline{M})$ is the supremum over $M_{\mathbb{R}}$ of the function "distance to $M_{/\mathrm{tor}}$":

$$d_{\overline{M}}(., M_{/\mathrm{tor}}) : M_{\mathbb{R}} \longrightarrow \mathbb{R}_+, \quad x \longmapsto \inf_{m \in M_{/\mathrm{tor}}} ||x - m||. \tag{6.1.1}$$

The above definition of $\rho(\overline{M})$ may also be rephrased as the equivalence, for every $R \in \mathbb{R}_+$, of the following two conditions:

(1) $\rho(\overline{M}) \leq R$;
(2) for every $R' \in (R, +\infty)$, $M_{\mathbb{R}} = M_{/\mathrm{tor}} + B(\overline{M}_{\mathbb{R}}; R')$.

When $\overline{M}$ is a Euclidean lattice — namely, when M is a finitely generated and torsion free, and $\|.\|$ is a Euclidean norm — the invariant $\rho(\overline{M})$ is the classical covering radius, traditionally considered in geometry of numbers.[7]

Let us formulate a few simple observations, which are straightforward consequences of the above definition of the covering radius.

Firstly, the covering radius $\rho(M, ||.||)$ is an increasing function of $\|.\|$, and "does not see torsion"; namely, for every Euclidean quasi-coherent sheaf $\overline{M}$, we have:

$$\rho(\overline{M}) = \rho(\overline{M}_{/\mathrm{tor}}).$$

Moreover $\rho(\overline{M})$ vanishes if and only if M_{tor} is dense in $(M_{\mathbb{R}}, \|.\|)$, and is finite when $M_{\mathbb{R}}$ is a finite dimensional $\mathbb{R}$-vector space.

The covering radius $\rho(\overline{M})$ of a Euclidean quasi-coherent sheaf $\overline{M} = (M, \|.\|)$ is a 1-homogeneous function of the seminorm $\|.\|$. Equivalently, for every $\delta \in \mathbb{R}$, we have:

$$\rho(\overline{M} \otimes \overline{\mathcal{O}}(\delta)) = e^{-\delta}\, \rho(\overline{M}). \tag{6.1.2}$$

[7] For classical results on the covering of radius of Euclidean lattices, we refer the reader to [**Cas71**, Chapter XI] where it is noted μ, and to [**CS99**, Section 2.1.2] where it is noted R. See also [**MG02**, Chapters 7 and 8] and [**GMR05**] for properties of the covering radius of Euclidean lattices from the perspective of lattice based cryptography and complexity theory.

When the $\mathbb{Z}$-module M is cyclic with generator m, its covering radius is simply:

$$\rho(\mathbb{Z}m, \|.\|) = \|m\|/2. \tag{6.1.3}$$

Observe finally that, if $\overline{M}_1$ and $\overline{M}_2$ denote two Euclidean quasi-coherent sheaves, then we have:

$$d_{\overline{M}_1\oplus\overline{M}_2}((x_1,x_2), M_{1/\mathrm{tor}} \oplus M_{2/\mathrm{tor}})^2 = d_{\overline{M}_1}(x_1, M_{1/\mathrm{tor}})^2 + d_{\overline{M}_2}(x_2, M_{2/\mathrm{tor}})^2, \tag{6.1.4}$$

for every (x_1, x_2) in $M_{1,\mathbb{R}} \oplus M_{2,\mathbb{R}}$, and consequently the following equality holds:

$$\rho(\overline{M}_1 \oplus \overline{M}_2)^2 = \rho(\overline{M}_1)^2 + \rho(\overline{M}_2)^2. \tag{6.1.5}$$

In other words, the invariant ρ^2 satisfies the additivity property $\mathbf{Add}_\oplus$ introduced in 4.1.5 above.

The equality (6.1.5) is a first instance of the fact, which will be amply confirmed in the this chapter and in Chapter 9, that the square of the covering radius is often a better behaved invariant than the covering radius itself.

6.1.2. Semicontinuity, monotonicity, and countable aditivity properties. The invariant ρ satisfies the following lower semicontinuity property:

PROPOSITION 6.1.2. *Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf. For every increasing family $(M_i)_{i\in\mathbb{N}}$ of submodules of M such that $M = \bigcup_{i\in\mathbb{N}} M_i$, we have:*

$$\rho(\overline{M}) \leq \liminf_{i\to\infty} \rho(\overline{M}_i), \tag{6.1.6}$$

where $\overline{M}_i := (M_i, \|.\|_{M_{i,\mathbb{R}}})$.

PROOF. Let R' be a positive real number such that

$$R' > \liminf_{i\to\infty} \rho(\overline{M}_i).$$

After replacing the sequence $(M_i)_{i\in\mathbb{N}}$ by a suitable subsequence, we may assume that, for every $i \in \mathbb{N}$, we have:

$$R' > \rho(\overline{M}_i),$$

and consequently:

$$M_{i,\mathbb{R}} = M_{i/\mathrm{tor}} + B(\overline{M}_{i,\mathbb{R}}; R').$$

Since $M_\mathbb{R}$ (resp. $M_{/\mathrm{tor}}$, resp. $B(\overline{M}_\mathbb{R}; R')$) is the increasing union of the $M_{i,\mathbb{R}}$ (resp. $M_{i/\mathrm{tor}}$, resp. $B(\overline{M}_{i,\mathbb{R}}; R')$), this implies the equality:

$$M_\mathbb{R} = M_{/\mathrm{tor}} + B(\overline{M}_\mathbb{R}; R').$$

This establishes the estimate $\rho(\overline{M}) \leq R$ and proves (6.1.6). □

EXAMPLE 6.1.3. The inequality in (6.1.6) may be strict. Consider for instance the Euclidean quasi-coherent sheaf $\overline{M}$ where $M := \mathbb{Q}^{(\mathbb{N})}$ — so that $M_\mathbb{R} = \mathbb{R}^{(\mathbb{N})}$ — and where $\|(x_k)_{k\in\mathbb{N}}\| := \sum_{k\in\mathbb{N}} x_k^2$, and define, for every $i \in \mathbb{N}$:

$$M_i := \{(x_k)_{k\in\mathbb{N}} \in \mathbb{Q}^{(\mathbb{N})} \mid \ x_i \in \mathbb{Z} \text{ and } x_k = 0 \text{ if } k > i\}.$$

Then one easily checks that the covering radii of $\overline{M}$ and of the $\overline{M}_i$ satisfy:

$$\rho(\overline{M}) = 0 \quad \text{and} \quad \rho(\overline{M}_i) = 1/2 \quad \text{for every } i \in \mathbb{N}.$$

PROPOSITION 6.1.4. *Let $f : \overline{M} \to \overline{N}$ be a morphism in $\overline{\mathbf{qCoh}}_\mathbb{Z}^{\leq 1}$. If the image of $f_\mathbb{R} : M_\mathbb{R} \to N_\mathbb{R}$ is dense in $\overline{M}_\mathbb{R}$, then:*

$$\rho(\overline{M}) \geq \rho(\overline{N}). \tag{6.1.7}$$

The image of $f_{\mathbb{R}}$ is clearly dense when $f_{\mathbb{R}}$ is surjective, that is when $f_{\mathbb{Q}}$ is surjective, or equivalently when $\operatorname{coker} f := N/f(M)$ is a torsion $\mathbb{Z}$-module. Consequently, as a special case, Proposition 6.1.4 asserts that the invariant ρ on $\overline{\mathbf{qCoh}}_{\mathbb{Z}}^{\leq 1}$, and therefore ρ^2 also, satisfies the monotonicity condition $\mathbf{Mon}^1_{\mathbb{Q}}$ introduced in 4.1.4.

Proof. Let us consider R' in $(\rho(\overline{M}), +\infty)$, and let us choose R'' in $(\rho(\overline{M}), R')$. We have:

$$M_{\mathbb{R}} = M_{/\mathrm{tor}} + B(\overline{M}_{\mathbb{R}}; R''). \tag{6.1.8}$$

Moreover, $f_{\mathbb{R}}(M_{/\mathrm{tor}}) \subseteq N_{/\mathrm{tor}}$, and, since $f_{\mathbb{R}}$ has norm at most one,

$$f_{\mathbb{R}}(B(\overline{M}_{\mathbb{R}}; R'')) \subseteq B(\overline{N}_{\mathbb{R}}; R'').$$

Therefore, applying $f_{\mathbb{R}}$ to both sides of (6.1.8), we get:

$$f_{\mathbb{R}}(M_{\mathbb{R}}) \subseteq N_{/\mathrm{tor}} + B(\overline{N}_{\mathbb{R}}; R'').$$

Since $f_{\mathbb{R}}(N_{\mathbb{R}})$ is dense in $\overline{N}_{\mathbb{R}}$, this implies:

$$N_{\mathbb{R}} = N_{/\mathrm{tor}} + B(\overline{N}_{\mathbb{R}}; R'),$$

and completes the proof of (6.1.7). $\square$

Using Propositions 6.1.2 and 6.1.4, we may extend the additivity property (6.1.5) of ρ^2 to countable direct sums:

Corollary 6.1.5. *For every countable family $(\overline{M}_i)_{i\in I}$ of Euclidean quasi-coherent sheaves, the following equality holds in $[0, +\infty]$:*

$$\rho\Big(\bigoplus_{i\in I} \overline{M}_i\Big)^2 = \sum_{i\in I} \rho(\overline{M}_i)^2.$$

Proof. According to the additivity (6.1.5) of ρ^2, for every finite subset F of I, we have:

$$\rho\Big(\bigoplus_{i\in F} \overline{M}_i\Big)^2 = \sum_{i\in F} \rho(\overline{M}_i)^2. \tag{6.1.9}$$

Moreover, since the projection

$$p_F : \bigoplus_{i\in I} \overline{M}_i \longrightarrow \bigoplus_{i\in F} \overline{M}_i, \quad (x_i)_{i\in I} \longmapsto (x_i)_{i\in F}$$

is surjective of norm ≤ 1, we have, as a special instance of Proposition 6.1.4:

$$\rho(\overline{M}) \geq \rho\Big(\bigoplus_{i\in F} \overline{M}_i\Big).$$

Consequently:

$$\rho(\overline{M})^2 \geq \sum_{i\in F} \rho(\overline{M}_i)^2,$$

and therefore, as F is arbitrary:

$$\rho(\overline{M})^2 \geq \sum_{i\in I} \rho(\overline{M}_i)^2.$$

Consider an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of subsets of I such that $I = \bigcup_{k\in\mathbb{N}} F_k$. By using successively Proposition 6.1.2 and (6.1.9) with $F = F_k$, we also have:

$$\rho(\overline{M})^2 \leq \liminf_{k\to+\infty} \rho\Big(\bigoplus_{i\in F_k} \overline{M}_i\Big)^2 = \liminf_{k\to+\infty} \sum_{i\in F_k} \rho(\overline{M}_i)^2 = \sum_{i\in I} \rho(\overline{M}_i)^2. \qquad \square$$

6.1.3. Covering radius and admissible short exact sequences.

6.1.3.1. The invariant ρ^2 on $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ satisfies the subadditivity condition **SubAdd** introduced in Subsection 4.1.2. Namely, we have:

PROPOSITION 6.1.6. *For any admissible short exact sequence in* $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$*,*

$$0 \longrightarrow \overline{M}' \longrightarrow \overline{M} \xrightarrow{f} \overline{M}'' \longrightarrow 0,$$

the following inequality holds:

$$\rho(\overline{M})^2 \leq \rho(\overline{M}')^2 + \rho(\overline{M}'')^2. \tag{6.1.10}$$

The significance of the subaddivity inequality (6.1.10) when $\overline{M}$, $\overline{M}'$, and $\overline{M}''$ are Euclidean lattices has been emphasized in the work of Shapira and Weiss; see [**SW16**, Section 5.1 and Lemma 5.3]. The subadditivity inequality (6.1.10) for Euclidean lattices is also crucially used, although not stated explicitly, in [**RSD17b**]; see notably the proof of Proposition 6.4 in [**RSD17b**].

As indicated in [**SW16**], the inequality (6.1.10) for Euclidean lattices goes back to the article of Woods [**Woo65**], where a special case is stated in Lemma 2 and is established by an argument valid for general Euclidean lattices.

PROOF. Let us write $\overline{M} = (M, \|.\|)$, $\overline{M}' = (M', \|.\|')$, and $\overline{M}'' = (M'', \|.\|'')$, and let us denote by $\langle ., . \rangle$ the scalar product on $M_{\mathbb{R}}$ that defines the Euclidean seminorm $\|.\|$. Let us consider $x \in M_{\mathbb{R}}$ and two positive real numbers $R' \in (\rho(\overline{M}'), +\infty)$ and $R'' \in (\rho(\overline{M}''), +\infty)$.

We may find $m'' \in M''_{/\mathrm{tor}}$ and $b'' \in M''_{\mathbb{R}}$ such that $p(x) = m'' + b''$ and $\|b''\|'' \leq R''$. Then we may choose m in $M_{/\mathrm{tor}}$ such that $m'' = p_{\mathbb{R}}(m)$ and a sequence $(\tilde{b}_i)_{i\in\mathbb{N}}$ in $p_{\mathbb{R}}^{-1}(b'')$ such that:

$$\lim_{i\to+\infty} \|\tilde{b}_i\| = \|b''\|''.$$

The sequence $(\tilde{b}_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $(M_{\mathbb{R}}, \|.\|)$, and for every y in $M'_{\mathbb{R}} = \ker p_{\mathbb{R}}$, we have:

$$\lim_{i\to+\infty} \langle \tilde{b}_i, y \rangle = 0. \tag{6.1.11}$$

Indeed, we may consider the completions $M^{\mathrm{cpt}}_{\mathbb{R},\|.\|}$ and $M''^{\mathrm{cpt}}_{\mathbb{R},\|.\|'}$ of $(M_{\mathbb{R}}, \|.\|)$ and $(M''_{\mathbb{R}}, \|.\|'')$, and the continuous linear map $f^{\mathrm{cpt}}_{\mathbb{R}} : M^{\mathrm{cpt}}_{\mathbb{R},\|.\|} \to M'^{\mathrm{cpt}}_{\mathbb{R},\|.\|}$ that extends $f_{\mathbb{R}}$. Then the image in $M^{\mathrm{cpt}}_{\mathbb{R},\|.\|}$ of the sequence $(\tilde{b}_i)_{i\in\mathbb{N}}$ converges to the unique element of $f^{\mathrm{cpt}-1}_{\mathbb{R}}([b''])$ orthogonal to $\ker f^{\mathrm{cpt}}_{\mathbb{R}}$. Indeed this kernel coincides with the closure in $M^{\mathrm{cpt}}_{\mathbb{R},\|.\|}$ of (the image) of $M'_{\mathbb{R}}$, and may be identified with the completion $M^{\mathrm{cpt}}_{\mathbb{R},\|.\|}$ of $(M'_{\mathbb{R}}, \|.\|')$.

For every $i \in \mathbb{N}$, we let:

$$x'_i := x - m - \tilde{b}_i.$$

It is an element of $M'_{\mathbb{R}}$, and we may choose $m'_i \in M'_{/\mathrm{tor}}$ and $b'_i \in M'_{\mathbb{R}}$ such that $x'_i = m'_i + b'_i$ and such that $\|b'_i\| = \|b'_i\|'$ satisfies the upper bound $\|b'_i\| \leq R'$.

Then, for every $i \in \mathbb{N}$, we have:

$$x = m + \tilde{b}_i + x'_i = m + m'_i + b'_i + \tilde{b}_i,$$

where $m + m'_i$ belongs to $M_{/\mathrm{tor}}$. Moreover, using that $(b_i)_{i\in\mathbb{N}}$ (resp. $(\tilde{b}_i)_{i\in\mathbb{N}}$) is bounded in $(M'_{\mathbb{R}}, \|.\|)$ (resp. is a Cauchy sequence in $(M_{\mathbb{R}}, \|.\|)$), and (6.1.11), we get:

$$\lim_{i\to+\infty} \langle b'_i, \tilde{b}_i \rangle = 0,$$

and therefore:

$$\lim_{i\to+\infty} (\|b'_i + \tilde{b}_i\|^2 - \|b'_i\|^2) = \lim_{i\to+\infty} \|\tilde{b}_i\|^2 = \|b''\|''^2.$$

This establishes the estimates:

$$\limsup_{i\to+\infty} \|x - (m + m_i)\|^2 = \limsup_{i\to+\infty} \|b'_i + \tilde{b}_i\|^2 \leq \sup_{i\in I} \|b'_i|^2 + \|b''\|''^2 \leq R'^2 + R''^2,$$

and completes the proof of (6.1.10). □

Combined with the semicontinuity and the monotonicity of the covering radius established in Subsection 6.1.2, Proposition 6.1.6 allows us to establish the following countable subadditivity property of ρ^2:

COROLLARY 6.1.7. *Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf and let $(M_i)_{i\in\mathbb{N}}$ be an increasing sequence of $\mathbb{Z}$-submodules of M. For any $i \in \mathbb{N}$, let $\|.\|_i$ be the Euclidean semi-norm on $(M_i/M_{i-1})_{\mathbb{R}} \simeq M_{i,\mathbb{R}}/M_{i-1,\mathbb{R}}$ quotient of $\|.\|_{|M_{i,\mathbb{R}}}$ and let $\overline{M_i/M_{i-1}}$ be the Euclidean quasi-coherent sheaf $(M_i/M_{i-1}, \|.\|_i)$, where by convention $M_{-1} := \{0\}$.*

If the $\mathbb{R}$-vector space $\bigcup_{i\in\mathbb{N}} M_{i,\mathbb{R}}$ is dense in $\overline{M}_{\mathbb{R}}$, then we have:

$$\rho(\overline{M})^2 \leq \sum_{i=0}^{+\infty} \rho(\overline{M_i/M_{i-1}})^2.$$

PROOF. Let us introduce the $\mathbb{Z}$-submodule $M_\infty := \bigcup_{i\in\mathbb{N}} M_i$ of M and the Euclidean quasi-coherent sheaf $\overline{M}_\infty := (M_\infty, \|.\|_{|M_{\infty,\mathbb{R}}})$. Then the following inequalities hold:

$$\rho(\overline{M})^2 \leq \rho(\overline{M}_\infty)^2 \tag{6.1.12}$$

$$\leq \liminf_{k\to\infty} \rho(\overline{M}_k)^2 \tag{6.1.13}$$

$$\leq \liminf_{k\to\infty} \sum_{i=0}^{k} \rho(\overline{M_i/M_{i-1}})^2 = \sum_{i=0}^{+\infty} \rho(\overline{M_i/M_{i-1}})^2. \tag{6.1.14}$$

Indeed (6.1.12) follows from Proposition 6.1.4 applied to the inclusion morphism $\overline{M}_\infty \hookrightarrow \overline{M}$; the inequality (6.1.13) follows from Proposition 6.1.2, and (6.1.14) from Proposition 6.1.6. □

6.1.3.2. The following lifting result highlights the cohomological character of the covering radius:

PROPOSITION 6.1.8. *Consider an admissible short exact sequence in $\overline{\mathbf{qCoh}}_{\mathbb{Z}}$:*

$$0 \longrightarrow \overline{M}' \longrightarrow \overline{M} \xrightarrow{p} \overline{M}'' \longrightarrow 0.$$

For every $m'' \in M''$ and every $\varepsilon \in \mathbb{R}_+^*$, there exists $m \in M$ such that:

$$f(m) = m'' \tag{6.1.15}$$

and

$$\|m\|^2_{\overline{M}} \leq \|m''\|^2_{\overline{M}''} + \rho(\overline{M}')^2 + \varepsilon. \tag{6.1.16}$$

When $\overline{M}$, and therefore $\overline{M}'$ and $\overline{M}''$, are Euclidean lattices, the conclusion of Proposition 6.1.8 still holds when $\varepsilon = 0$. This follows from Proposition 6.1.8 together with a straightforward finiteness argument, or from an easy variant of the proof below.

PROOF. Choose $\tilde{m}$ in M such that $p(\tilde{m}) = m''$, and a sequence $(m_i^\perp)_{i\in\mathbb{N}}$ in $M_{\mathbb{R}}$ such that, for every $i \in \mathbb{N}$:

$$p_{\mathbb{R}}(m_i^\perp) = m''_{\mathbb{R}}$$

and:

$$\lim_{i\to+\infty} \|m_i^\perp\|_{\overline{M}} = \|m''_{\mathbb{R}}\|_{\overline{M}''}. \tag{6.1.17}$$

Then $(m_i^\perp)_{i\in\mathbb{N}}$ is a Cauchy sequence in $(M_{\mathbb{R}}, \|.\|_{\overline{M}})$. Its limit in the completion $M^{\mathrm{cpt}}_{\mathbb{R},\|.\|_{\overline{M}}}$ of $(M_{\mathbb{R}}, \|.\|_{\overline{M}})$ is actually orthogonal to (the image of) $M''_{\mathbb{R}}$.

Moreover, for every $i \in \mathbb{N}$, $\tilde{m}_{\mathbb{R}} - m_i^{\perp}$ belongs to $\ker p_{\mathbb{R}} = M''_{\mathbb{R}}$, and we may choose m''_i in M'' such that:

$$\|\tilde{m}_{\mathbb{R}} - m_i^{\perp} - m''_{i\mathbb{R}}\|^2 \leq \rho(\overline{M}')^2 + \varepsilon/2. \tag{6.1.18}$$

Then, for every $i \in \mathbb{N}$, the difference:

$$m_i := \tilde{m} - m''_i$$

is an element of M which satisfies:

$$p(m_i) = m''.$$

Moreover we have:

$$m_{i\mathbb{R}} = \tilde{m}_{\mathbb{R}} - m''_{i\mathbb{R}} = (\tilde{m}_{\mathbb{R}} - m_i^{\perp} - m''_{i\mathbb{R}}) + m_i^{\perp}.$$

Since $(\tilde{m}_{\mathbb{R}} - m_i^{\perp} - m''_{i\mathbb{R}})_{i\in\mathbb{N}}$ is a bounded sequence in $(M''_{\mathbb{R}}, \|.\|_{\overline{M}''})$ and $(m_i^{\perp})_{i\in\mathbb{N}}$ converges in $M^{\mathrm{cpt}}_{\mathbb{R},\|.\|_{\overline{M}}}$ to a vector orthogonal to the image of $M''_{\mathbb{R}}$, this implies:

$$\lim_{i\to+\infty} \left(\|m_{i\mathbb{R}}\|^2_{\overline{M}} - \|\tilde{m}_{\mathbb{R}} - m_i^{\perp} - m''_{i\mathbb{R}}\|^2_{\overline{M}} - \|m_i^{\perp}\|^2_{\overline{M}}\right) = 0. \tag{6.1.19}$$

Using (6.1.17), (6.1.18), and (6.1.19), we obtain:

$$\limsup_{i\to+\infty} \|m_{i\mathbb{R}}\|^2_{\overline{M}} \leq \rho(\overline{M})^2 + \varepsilon/2 + \|m''\|^2_{\overline{M}''},$$

and, for i large enough, $m := m_i$ satisfies (6.1.15) and (6.1.16). □

6.2. Covering Radius and Voronoi Cells of Euclidean Lattices

In this section, we discuss how the properties of the covering radius of Euclidean lattices are closely related to the geometry of their Voronoi cells.

These relations provide some geometric insight on the properties of the covering radius established in the previous section, and will also play a key role when comparing the covering radius of Euclidean lattices and Euclidean quasi-coherent sheaves to some others of their invariant, notably in Subsection 6.4.2 and Section 9.1.

They are also useful when investigating the covering radius of specific Euclidean lattices. For instance, we shall use the interpretation of the covering radius in terms of Voronoi cells to construct an example showing that, although the invariant ρ^2 on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ satisfies the properties $\mathbf{Mon}^1_{\mathbb{Q}}$ and **SubbAdd** introduced in Chapter 4, it does *not* satisfies the strong monotonicity property $\mathbf{StMon}^1$ investigated in Chapter5.

6.2.1. Voronoi cells of Euclidean lattices.

In this subsection, we recall a few basic facts concerning the Voronoi cells of Euclidean lattices. We refer to [**CS99**, Section 2.1 and Chapter 21] or [**Mar03**, Section 1.8] for more details and additional references concerning this topic.

Let $\overline{M} := (M, \|.\|)$ be a Euclidean lattice of positive rank, and let us denote by $\langle .,. \rangle$ the Euclidean scalar product on $M_{\mathbb{R}}$ that defines $\|.\|$. Its *Voronoi cell* $\mathcal{V}(\overline{M})$ is the set of points x in $M_{\mathbb{R}}$ such that the distance $\min_{m\in M} \|x - m\|$ of x to M is attained at $m = 0$. In other words:

$$\begin{aligned}\mathcal{V}(\overline{M}) :&= \left\{x \in M_{\mathbb{R}} \mid \forall m \in M, \|x\| \leq \|x - m\|\right\} \\ &= \bigcap_{m \in M\setminus\{0\}} \left\{x \in M_{\mathbb{R}} \mid 2\langle m, x\rangle \leq \langle m, m\rangle\right\}.\end{aligned}$$

By considering the tesselation of $M_{\mathbb{R}}$ by the translates $\mathcal{V}(\overline{M}) + m$ of $\mathcal{V}(\overline{M})$ by vectors m in M, one establishes the following facts:

(i) The Voronoi cell $\mathcal{V}(\overline{M})$ is a polytope in $M_{\mathbb{R}}$ (that is, the convex hull of a finite subset of $M_{\mathbb{R}}$). Its set of *facets* — that is, of faces of dimension $\operatorname{rk} M - 1$ — is in bijection with the set F of its *facet* (or *Voronoi-relevant) vectors*, defined as the elements m in $M \setminus \{0\}$ such that the polytope

$$\mathcal{V}(\overline{M}) \cap (\mathcal{V}(\overline{M}) + m)$$

has dimension $\operatorname{rk} M - 1$. The facet associated to $m \in F$ coincides with this intersection, and admits the vector $m/2$ as center of symmetry.

(ii) Let us denote by $\mathcal{V}(\overline{M})_0$ the set of vertices, or equivalently of extremal points or of zero-dimensional faces, of $\mathcal{V}(\overline{M})$. Then the following equalities hold:

$$\rho(\overline{M}) = \max_{x \in \mathcal{V}(\overline{M})} \|x\| = \max_{x \in \mathcal{V}(\overline{M})_0} \|x\|. \tag{6.2.1}$$

Actually the set $M + \mathcal{V}(\overline{M})_0$ of translates of these vertices by M is precisely the set of points in $M_{\mathbb{R}}$ where the distance function

$$d_{\overline{M}}(., M) := M_{\mathbb{R}} \longrightarrow \mathbb{R}_+, \quad v \longmapsto \inf_{m \in M} \|v - m\|$$

achieves a local maximum. These points are the *holes* of the lattices; among them, those at distance $\rho(\overline{M})$ from M are the *deep holes*.

(iii) The set F of facet vectors generates the $\mathbb{Z}$-module M. A vector $f \in M \setminus \{0\}$ is a facet vector if and only if the minimum value of $\|.\|$ on the coset $f + 2M$ is achieved precisely at f and $-f$.

6.2.2. The monotonicity and subadditivity of ρ^2 and the geometry of Voronoi cells. Consider an admissible short exact sequence of Euclidean lattices:

$$0 \longrightarrow \overline{F} \hookrightarrow \overline{E} \xrightarrow{p} \overline{E/F} \longrightarrow 0.$$

According to Propositions 6.1.4 and 6.1.6, the following relations holds between their covering radii:

$$\rho(\overline{E/F})^2 \leq \rho(\overline{E})^2 \leq \rho(\overline{F})^2 + \rho(\overline{E/F})^2. \tag{6.2.2}$$

The following relations involving the Voronoi cells of these Euclidean lattices provide a geometric interpretation of these inequalities.

Proposition 6.2.1. *With the above notation, let*

$$s^{\perp} : (E/F)_{\mathbb{R}} \xrightarrow{\sim} F_{\mathbb{R}}^{\perp} \hookrightarrow E_{\mathbb{R}}$$

be the orthogonal splitting of the surjective $\mathbb{R}$-linear map $p_{\mathbb{R}} : E_{\mathbb{R}} \to (E/F)_{\mathbb{R}}$. The following inclusion holds:

$$s^{\perp}(\mathcal{V}(\overline{E/F})) \subseteq \mathcal{V}(\overline{E}). \tag{6.2.3}$$

Moreover, we have:

$$E + \mathcal{V}(\overline{F}) + s^{\perp}(\mathcal{V}(\overline{E/F})) = E_{\mathbb{R}}. \tag{6.2.4}$$

By definition, $s_{\mathbb{R}}^{\perp}$ is the composition

$$s^{\perp} : (E/F)_{\mathbb{R}} \xrightarrow{\sim} F_{\mathbb{R}}^{\perp} \hookrightarrow E_{\mathbb{R}},$$

where $F_{\mathbb{R}}^{\perp}$ denotes the orthogonal complement of $F_{\mathbb{R}}$ in the Euclidean $\mathbb{R}$-vector space $\overline{E}_{\mathbb{R}}$, and coincides with the adjoint of the $\mathbb{R}$-linear map $p_{\mathbb{R}}$ from $\overline{E}_{\mathbb{R}}$ to $\overline{E/F}_{\mathbb{R}}$ In particular, it is an isometry from $\overline{E/F}_{\mathbb{R}}$ to $\overline{E}_{\mathbb{R}}$, and therefore the inclusion (6.2.3) immediately implies the inequality:

$$\rho(\overline{E/F}) \leq \rho(\overline{E}).$$

Moreover the equality (6.2.4) implies the following relations:

$$\begin{aligned}(6.2.5)\quad \rho(\overline{E})^2 \leq \max_{w\in\mathcal{V}(\overline{F})+s^{\perp}(\mathcal{V}(\overline{E/F}))} \|w\|^2 &= \max_{u\in\mathcal{V}(\overline{F})}\max_{v\in\mathcal{V}(\overline{E/F})}(\|u\|^2+\|v\|^2)\\ &= \max_{u\in\mathcal{V}(\overline{F})}\|u\|^2 + \max_{v\in\mathcal{V}(\overline{E/F})}\|v\|^2 = \rho(\overline{F})^2+\rho(\overline{E/F})^2.\end{aligned}$$

Proof of Proposition 6.2.1. For every z in $E_{\mathbb{R}}/F_{\mathbb{R}}$ and any e in E, we have:

$$s^{\perp}(z).e = z.p(e).$$

When z belongs to $\mathcal{V}(\overline{E/F})$, we have, for every $g\in E/F$:

$$2z.g \leq \|g\|^2_{\overline{E/F}},$$

and therefore, for any $e\in E$:

$$2s^{\perp}(z).e = 2z.p(e) \leq \|p(e)\|^2_{\overline{E/F}} \leq \|e\|^2_{\overline{E}}.$$

This shows that $s^{\perp}(z)$ belongs to $\mathcal{V}(\overline{E})$.

To prove (6.2.4), consider a point x in $E_{\mathbb{R}}$ and its image $p_{\mathbb{R}}(x)$ in $(E/F)_{\mathbb{R}}$. Choose g in E/F such that $\|p(x)-g\|_{\overline{E/F}}$ is the distance from $p(x)$ to E/F in $((E/F)_{\mathbb{R}}, \|.\|_{\overline{E/F}})$, and choose e in E such that $p(e)=g$.

Then $p(x)-g$ belongs to $\mathcal{V}(\overline{E/F})$. Moreover the difference

$$y := (x-e) - s^{\perp}(p(x-e))$$

belongs to $F_{\mathbb{R}}$, and consequently there exists f in F such that $y-f$ belongs to $\mathcal{V}(\overline{F})$.

Finally, we have:

$$x = (e+f)+(y-f)+s^{\perp}(p(x-e)) \in E+\mathcal{V}(\overline{F})+s^{\perp}(\mathcal{V}(\overline{E/F})). \qquad \square$$

6.2.3. The covering radius of Euclidean lattices of rank 2. The second equality in (6.2.1) shows that the determination of the Voronoi cell of some Euclidean lattice allows one to compute its covering radius. See for instance [**CS91**] for an application of this method to the computation of the covering radii of Euclidean lattices associated to root systems.

In this paragraph, we illustrate this method by deriving a formula for the covering radius of a Euclidean lattice of rank 2. To achieve this, it is convenient to use the description of Euclidean lattices of rank 2 by means of obtuse superbase as in [**CS92**] and [**BK10**, Appendix B], which deal with more general Euclidean lattices "of Voronoi first kind".

Recall that an *obtuse superbase* of a Euclidean lattice $\overline{E} := (E, \|.\|)$ of rank 2 is a triple (v_0, v_1, v_2) of E satisfying the following three conditions:

(i) $v_0+v_1+v_2=0$;
(ii) (v_1,v_2) is a $\mathbb{Z}$-basis of E, and therefore (v_2,v_0) and (v_0,v_1) also;
(iii) for any $0\leq i<j\leq 2$, $v_i.v_j\leq 0$.

where we denote by $a.b$ the scalar product of two vectors a and b in the Euclidean vector space $(E_{\mathbb{R}}, \|.\|)$.

To the obtuse superbase (v_0, v_1, v_0), we attach a triple (p_0,p_1,p_2) in $\mathbb{R}^3_+$ by letting:

$$p_k := -v_i.v_j$$

for any permutation (i,j,k) of $(0,1,2)$. The knowledge of (p_0,p_1,p_2) allows one to recover the Euclidean norm $\|.\|$ on $E_{\mathbb{R}}$. Indeed we have:

$$\|v_i\|^2 = -v_i.(v_k+v_j) = p_j+p_k,$$

and by a straightforward computation, this implies that, for every $(x_0, x_1, x_2) \in \mathbb{R}^3$:

$$\|x_0 v_0 + x_1 v_1 + x_2 v_2\|^2 = \sum_{0 \leq i < j \leq 2} (x_i - x_j)^2 p_k. \tag{6.2.6}$$

where in the right-hand side of (6.2.6), for every $0 \leq i < j \leq 2$, we define k by $\{k\} := \{0, 1, 2\} \setminus \{i, j\}$. This notably implies that at most one of the p_i's vanish.

Every Euclidean lattice $\overline{E}$ of rank 2 admits an obtuse superbase as above. Indeed, if (v_1, v_2) is a basis of $\overline{E}$ reduced in the usual sense — namely if it is a $\mathbb{Z}$-basis of E satisfying the conditions:

$$\|v_1\| = \lambda_1(\overline{E}) := \min_{e \in E \setminus \{0\}} \|e\| \quad \text{and} \quad \|v_2\| = \min_{e \in E \setminus \mathbb{Z} v_1} \|e\|,$$

or equivalently:

$$\|v_1\| \leq \|v_2\| \leq \|v_2 \pm v_1\|$$

— then, after possibly replacing v_2 by $-v_2$ to ensure that $v_1.v_2$ be non-positive, the triple $(-v_1 - v_2, v_1, v_2)$ is easily seen to be an obtuse superbase of $\overline{E}$.

Conversely any triple (p_0, p_1, p_2) in $\mathbb{R}^3_+$ such that at most one of the p_i vanishes arises by the above construction from an obtuse superbase of some Euclidean lattice $\overline{E}$ of rank 2: simply take $E := \mathbb{Z}^2$, $v_1 := (1, 0)$, $v_2 := (0, 1)$, and $v_0 := (-1, -1)$, and define a Euclidean norm $\|.\|$ on $E_{\mathbb{R}} = \mathbb{R}^2$ by (6.2.6).

The basic invariants of a Euclidean lattice $\overline{E}$ of rank 2 admit closed expressions in terms of the parameters (p_0, p_1, p_2) associated to some obtuse superbase (v_0, v_1, v_2):

PROPOSITION 6.2.2. *With the above notation, we have:*

$$(\operatorname{covol} \overline{E})^2 = p_0 p_1 + p_1 p_2 + p_2 p_0, \tag{6.2.7}$$

$$\lambda_1(\overline{E})^2 = \min(p_0 + p_1, p_1 + p_2, p_2 + p_0) \tag{6.2.8}$$

and

$$\rho(\overline{E})^2 = \frac{(p_0 + p_1)(p_1 + p_2)(p_2 + p_0)}{4(p_0 p_1 + p_1 p_2 + p_2 p_0)}. \tag{6.2.9}$$

Surprisingly, closed formulae for the covering radius of Euclidean lattices of rank 2 do not seem to appear in the literature. As an application of (6.2.9), the reader will easily check that the dimensionless quotient $\rho(\overline{E})^2 / \operatorname{covol} \overline{E}$ satisfies the lower bound:

$$\frac{\rho(\overline{E})^2}{\operatorname{covol} \overline{E}} \geq \frac{2}{\sqrt{27}}, \tag{6.2.10}$$

and that equality is achieved in (6.2.10) if and only if $\overline{E}$ is a "hexagonal lattice", associated to parameters $p_0 = p_1 = p_2$; compare for instance with [**FT72**, Sections III.2-3].

PROOF. The expression (6.2.7) for $(\operatorname{covol} \overline{E})^2$ follows from the equalities:

$$(\operatorname{covol} \overline{E})^2 = \begin{vmatrix} v_1.v_1 & v_1.v_2 \\ v_2.v_1 & v_2.v_2 \end{vmatrix} = \begin{vmatrix} p_0 + p_2 & -p_0 \\ -p_0 & p_0 + p_1 \end{vmatrix} = p_0 p_1 + p_1 p_2 + p_2 p_0.$$

The value (6.2.8) for $\lambda_1(\overline{E})^2$ is a straightforward consequence of (6.2.6).

When for some permutation (i, j, k) of $(0, 1, 2)$ we have $p_i = 0$, then $\overline{E}$ is the "rectangular lattice" $\overline{\mathbb{Z} v_j} \oplus \overline{\mathbb{Z} v_k}$, its Voronoi cell $\mathcal{V}(\overline{E})$ is the rectangle of vertices $\pm v_j/2 \pm v_k/2$, and therefore:

$$\rho(\overline{E})^2 = (\|v_j\|^2 + \|v_k\|^2)/4 = (p_k + p_j)/4.$$

This establishes (6.2.9) when one the p_i vanishes.

When the p_i are all positive, the Voronoi cell $\mathcal{V}(\overline{E})$ and its vertices $\mathcal{V}(\overline{E})_0$ admit the following description, which is the special instance when $\operatorname{rk} \overline{E} = 2$ of the description of the Voronoi cell of a

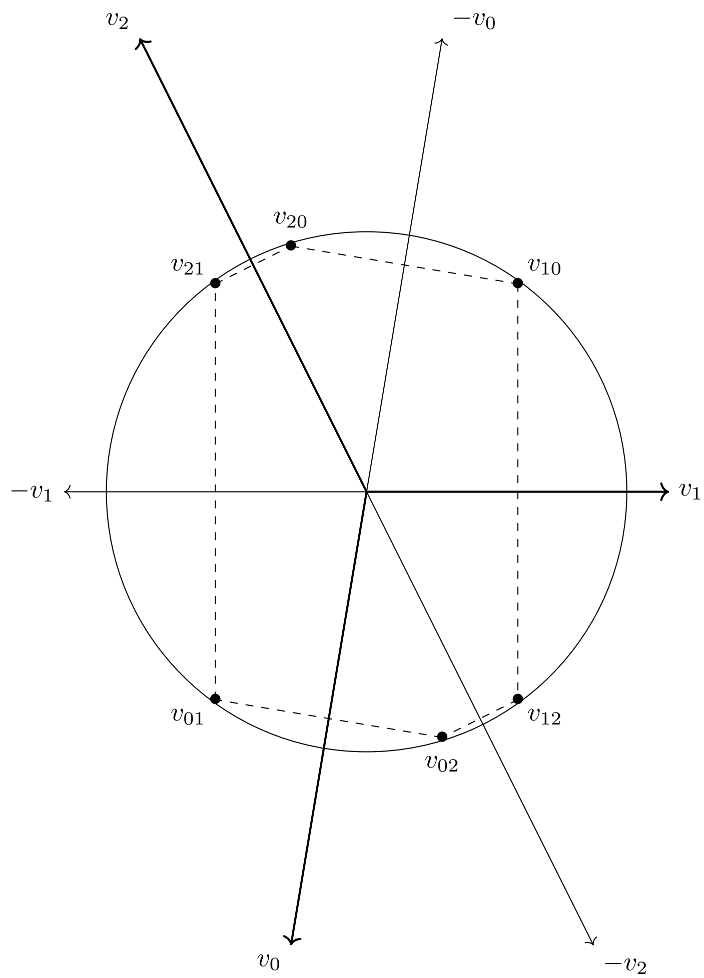

FIGURE 1. The facet vectors and the Voronoi cell of a Euclidean lattice of rank 2

Euclidean lattice $\overline{E}$ of Voronoi's first kind with strictly obtuse superbase in [**BK10**, Section B.3]. We refer the reader to *loc. cit.* for a detailed justification of this description, which actually becomes considerably simpler in this special case.

Firstly the set facets vectors of $\overline{E}$ is

$$\{v_0, v_1, v_2, -v_0, -v_1, -v_2\}.$$

This follows from the characterization of facet vectors recalled in 6.2.1 (iii) above, combined with the expression (6.2.6) for the Euclidean norm of $\overline{E}$. Moreover the pairs of facets with non-empty intersections are the pairs associated to two facet vectors v_i and $-v_j$. Consequently the Voronoi cell $\mathcal{V}(\overline{E})$ is an hexagon, with vertices

$$\mathcal{V}(\overline{E})_0 := \{v_{ij}, 0 \leq i \neq j \leq 2\},$$

where v_{ij} is the intersection of the bisectors of the segments $[0, v_i]$ and $[0, -v_j]$; see Figure 1.

The symmetry property of the facets of $\mathcal{V}(\overline{E})$ recalled in 6.2.1 (i) implies that the six points v_{ij} have the same Euclidean norm. This also follows from a direct computation, which moreover establishes (6.2.9). Indeed for every pair of basis vectors (a, b) in $E_{\mathbb{R}}$, the norm $\|x\|$ of the intersection point x of the bisectors of $[0, a]$ and $[0, b]$ is easily checked to satisfy:

$$\|x\|^2 = \frac{\|a\|^2\|b\|^2\|a-b\|^2}{4(\|a\|^2\|b\|^2 - (a.b)^2)}. \tag{6.2.11}$$

When $a = v_i$ and $b = -v_j$, the right-hand side of (6.2.11) becomes the right-hand side of (6.2.9). □

6.2.4. The invariant ρ^2 is not strongly monotonic. As established in Propositions 6.1.4 and 6.1.6, the invariant ρ^2 satisfies the monotonicity and subadditivity conditions $\mathbf{Mon}^1_{\mathbb{Q}}$ and **SubbAdd** of Chapter 4. At this stage, it is natural to ask whether it also satisfies the strong monotonicity condition $\mathbf{StMon}^1$ studied in Chapter 5. As a special instance of the submodularity inequality (5.2.7), this would imply the following inequality to hold:

$$\rho(\overline{E/F'})^2 + \rho(\overline{E/F''})^2 \leq \rho(\overline{E})^2, \tag{6.2.12}$$

for every Euclidean lattice $\overline{E}$ and for every pair of saturated $\mathbb{Z}$-submodules F' and F'' of E such that $E_{\mathbb{Q}} = F'_{\mathbb{Q}} \oplus F'_{\mathbb{Q}}$.

To investigate the validity of (6.2.12) when E has rank 2 and F' and F'' have rank 1, we may use the explicit expression (6.2.9) for the covering radius of Euclidean lattices of rank 2, combined with the following proposition:

Proposition 6.2.3. *Let us keep the notation of Proposition 6.2.2, and consider a saturated $\mathbb{Z}$-submodule F of rank 1 in E. If ξ denotes the element of $E^\vee$ well-defined up to a sign as the composition:*

$$\xi : E \longrightarrow E/F \xrightarrow{\sim} \mathbb{Z},$$

and if $(\xi_0, \xi_1, \xi_2) := (\xi(v_0), \xi(v_1), \xi(v_2))$, then:

$$\rho(\overline{E/F})^2 = \frac{p_0p_1 + p_1p_2 + p_2p_0}{4(p_0\xi_0^2 + p_1\xi_1^2 + p_2\xi_2^2)}. \tag{6.2.13}$$

Observe that the triples (ξ_0, ξ_1, ξ_2) in $\mathbb{Z}^3$ occurring in this construction are precisely those satisfying the condition:

$$(\xi_0, \xi_1, \xi_2) \text{ are coprime and } \xi_0 + \xi_1 + \xi_2 = 0. \tag{6.2.14}$$

Proof. Let us consider the admissible short exact sequence of Euclidean lattices:

$$0 \longrightarrow \overline{F} \longrightarrow \overline{E} \longrightarrow \overline{E/F} \longrightarrow 0.$$

We obtain the following relation between Arakelov degrees:

$$\widehat{\deg}\, \overline{F} + \widehat{\deg}\, \overline{E/F} = \widehat{\deg}\, \overline{E}. \tag{6.2.15}$$

The vector $f := \xi_2 v_1 - \xi_1 v_2$ is a generator of F. Moreover, according to (6.2.6), we have:

$$\|f\|^2 = p_0(\xi_1 + \xi_2)^2 + p_1\xi_1^2 + p_2\xi_2^2 = p_0\xi_0^2 + p_1\xi_1^2 + p_2\xi_2^2. \tag{6.2.16}$$

Moreover, if η is a generator of E/F, the equality (6.2.15) may be written:

$$\|f\|^2\, \|\eta\|^2_{\overline{E/F}} = (\operatorname{covol} \overline{E})^2.$$

Consequently, we have:

$$\rho(\overline{E/F})^2 = \frac{1}{4}\|\eta\|^2_{\overline{E/F}} = \frac{(\operatorname{covol}\overline{E})^2}{4\|f\|^2}. \tag{6.2.17}$$

Finally (6.2.14) follows from (6.2.7), (6.2.16) and (6.2.17). □

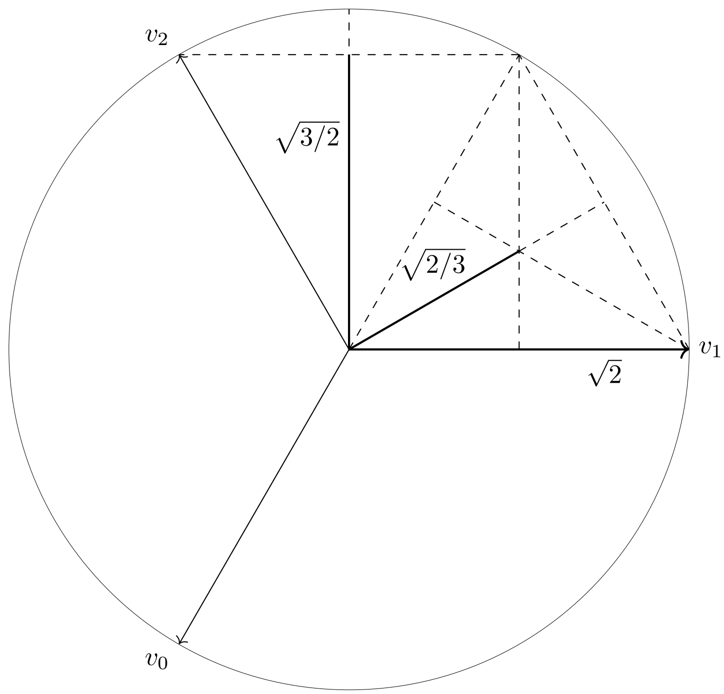

FIGURE 2

With the notation of Propositions 6.2.2 and 6.2.3, if (ξ'_0, ξ'_1, ξ'_2) and $(\xi''_0, \xi''_1, \xi''_2)$ are two non-colinear triples in $\mathbb{Z}^3$ that satisfy condition (6.2.14), we may consider the submodules F' and F'' of E defined as the kernels of the corresponding elements ξ' and ξ'' in $E^\vee$. Then, according to (6.2.9) and (6.2.13), the submodularity inequality (6.2.12) holds if and only if the following estimate is satisfied:

$$\frac{p_0p_1 + p_1p_2 + p_2p_0}{4(p_0{\xi'_0}^2 + p_1{\xi'_1}^2 + p_2{\xi'_2}^2)} + \frac{p_0p_1 + p_1p_2 + p_2p_0}{4(p_0{\xi''_0}^2 + p_1{\xi''_1}^2 + p_2{\xi''_2}^2)} \leq \frac{(p_0 + p_1)(p_1 + p_2)(p_2 + p_0)}{4(p_0p_1 + p_1p_2 + p_2p_0)}. \tag{6.2.18}$$

Rewritten in this form, the submodularity inequality is easily seen *not* to be always true. For instance when

$$p_0 = p_1 = p_2 = 1, \quad (\xi'_0, \xi'_1, \xi'_2) = (1, -1, 0) \quad \text{and} \quad (\xi''_0, \xi''_1, \xi''_2) = (0, 1, -1),$$

the left-hand side of (6.2.18) is $3/4$ and its right-hand side is $2/3$.

This counterexample to the submodularity inequality (6.2.12) may be equivalently described as follows. Consider the Euclidean lattice $\overline{E} := (\mathbb{Z}^2, \|.\|)$, where the Euclidean norm $\|.\|$ is defined by:

$$\|(x, y)\|^2 := 2(x^2 - xy + y^2),$$

and the submodules

$$F' := \mathbb{Z} \times \{0\} \quad \text{and} \quad F'' := \{0\} \times \mathbb{Z}.$$

The associated covering radii satisfy:

$$\rho(\overline{E})^2 = 2/3 \quad \text{and} \quad \rho(\overline{E/F'})^2 = \rho(\overline{E/F''})^2 = 3/8, \tag{6.2.19}$$

and clearly violate (6.2.12).

The relations (6.2.19) may actually be established directly without recourse to Propositions 6.2.2 and 6.2.3, as demonstrated in Figure 2. The reformulation (6.2.18) of the submodularity inequality has the interest to show that, when E has rank 2, it *is* satisfied for most choices of F' and F'', since the left-hand side of "small" for most choices of (ξ'_0, ξ'_1, ξ'_2) and $(\xi''_0, \xi''_1, \xi''_2)$. More specifically, as the reader may check as an elementary exercise, one may deduce the following upper bound from Propositions 6.2.2 and 6.2.3:

COROLLARY 6.2.4. *With the notation of Propositions 6.2.2 and 6.2.3, if F is not $\mathbb{Z}v_0$, $\mathbb{Z}v_1$, or $\mathbb{Z}v_2$ — or equivalently if* $\min_{0\leq i\leq 2}|\xi_i| \geq 1$ *— then:*

$$\rho(\overline{E/F})^2 \leq \rho(\overline{E})^2/3.$$

Consequently, when $\overline{E}$ is a rank 2 Euclidean lattice with a strictly obtuse superbase, the inequality (6.2.12) may be violated only when F' or F'' is one of the three $\mathbb{Z}$-submodules of E generated by the facet vectors of $\overline{E}$.

Observe finally that, when F' and F'' violate (6.2.12), then they *a fortiori* violate the stronger estimate:

$$\rho(\overline{E/F'}) + \rho(\overline{E/F''}) \leq \rho(\overline{E}).$$

Consequently the invariant ρ also is not strongly monotonic.

6.3. The Covering Radius of Euclidean Quasi-coherent Sheaves II. Further Properties

In this section, we investigate in more detail the properties of the invariant ρ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and on $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, motivated by the general formalism developed in Chapter 4.

6.3.1. Compatibility of the covering radius with vectorization and canonical dévissage. In this subsection, we show how the invariant ρ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ (resp. on $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$) is determined by its values on Euclidean lattices (resp. on quasi-coherent Euclidean sheaves with free underlying $\mathbb{Z}$-module). Our results might be obtained as formal consequences of the properties on the invariant ρ established in 6.1 and in 6.3.2 below, and of the general results concerning invariants on $\overline{\mathbf{Coh}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$ of Chapter 4. However we have preferred to provide short direct proofs.

6.3.1.1. *Covering radius and vectorization.* The following proposition reduces the evaluation of the covering radius of Euclidean coherent sheaves to the one of Euclidean lattices.

PROPOSITION 6.3.1. *If $\overline{E}$ is an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and if $\nu_{\overline{E}} : \overline{E} \to \overline{E}^{\mathrm{vect}}$ denotes its "vectorization," as defined in Subsection 2.3.1 above, then the following equality holds:*

$$\rho(\overline{E}) = \rho(\overline{E}^{\mathrm{vect}}). \tag{6.3.1}$$

PROOF. As in 2.3.1, consider the admissible short exact sequence (2.3.8) attached to $\nu_{\overline{E}}$:

$$0 \longrightarrow \overline{V} \hookrightarrow \overline{E} \xrightarrow{\nu_{\overline{E}}} \overline{E}^{\mathrm{vect}} \longrightarrow 0.$$

The last assertion in Proposition 2.3.6 shows that $V_{/\mathrm{tor}}$ is dense in $(V_{\mathbb{R}}, \|.\|_{\overline{V}})$, or equivalently:

$$\rho(\overline{V}) = 0. \tag{6.3.2}$$

Moreover, since ρ^2 satisfies $\mathbf{Mon}^1$ and $\mathbf{SubAdd}$, we have:

$$\rho(\overline{E}^{\mathrm{vect}})^2 \leq \rho(\overline{E})^2 \leq \rho(\overline{E}^{\mathrm{vect}})^2 + \rho(\overline{V})^2. \tag{6.3.3}$$

The equality (6.3.1) follows from (6.3.2) and (6.3.3). □

6.3.1.2. *Covering radius and canonical dévissage.* The following proposition and its corollary establih that invariant $\rho : \mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}} \to [0, +\infty]$ also satisfies the conditions **VAp** and **NSap** introduced in paragraph 4.1.4.2.

PROPOSITION 6.3.2. *Every Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$ such that the $\mathbb{Z}$-module M is antiprojective satisfies:*

$$\rho(\overline{M}) = 0.$$

PROOF. Let us assume that M, and therefore $M_{/\mathrm{tor}}$, is antiprojective, and let us consider an element x of $M_{\mathbb{R}}$. Let us choose a finite family $(f_1, \ldots, f_k)$ of elements of $M_{/\mathrm{tor}}$ such that its $\mathbb{R}$-span $\sum_{i=1}^{k} \mathbb{R} f_i$ contains x.

According to Proposition 2.2.9, for any $\varepsilon > 0$, there exists $(\tilde{f}_1, \ldots, \tilde{f}_{2k})$ in $M_{/\mathrm{tor}}$ such that $\sum_{i=1}^{2k} \mathbb{Z} \tilde{f}_i$ contains $f_1, \ldots, f_k$ and:

$$\max_{1 \leq i \leq 2k} \|\tilde{f}_{2i}\| < \varepsilon.$$

Then x belongs to $\sum_{i=1}^{2k} \mathbb{R} \tilde{f}_i$, and we may write $x = \sum_{i=1}^{2k} t_i \tilde{f}_i$ for a suitable choice of $(t_1, \ldots, t_{2k})$ in $\mathbb{R}^{2k}$.

If we let:

$$m := \sum_{i=1}^{2k} \lfloor t_i \rfloor \tilde{f}_i \quad \text{and} \quad r := \sum_{i=1}^{2k} (t_i - \lfloor t_i \rfloor) \tilde{f}_i,$$

then m belongs to $M_{/\mathrm{tor}}$, and therefore:

$$d_{\|.\|}(x, M_{/\mathrm{tor}}) \leq \|x - m\| = \|r\| \leq \sum_{i=1}^{2k} \|\tilde{f}_i\| \leq 2k\varepsilon.$$

Since ε is arbitrary this proves that $d_{\|.\|}(x, M_{/\mathrm{tor}})$ vanishes. $\square$

More generally, the covering radius of a Euclidean quasi-coherent sheaf is unaltered when killing its antiprojective part:

COROLLARY 6.3.3. *For every Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$, the following equality holds in $[0, +\infty]$:*

$$\rho(\overline{M}) = \rho(\overline{M}^{\vee\vee}). \tag{6.3.4}$$

PROOF. We may consider the canonical dévissage of $\overline{M}$, as defined in Section 2.2.4.

$$0 \longrightarrow \overline{M}_{\mathrm{ap}} \longrightarrow \overline{M} \xrightarrow{\delta_M} \overline{M}^{\vee\vee} \longrightarrow 0.$$

The monotonicity of the covering radius, established in Proposition 6.1.4, implies the inequality:

$$\rho(\overline{M}) \geq \rho(\overline{M}^{\vee\vee}),$$

and its subadditivity, established in Proposition 6.1.6, implies:

$$\rho(\overline{M})^2 \leq \rho(\overline{M}^{\vee\vee})^2 + \rho(\overline{M}_{\mathrm{ap}})^2.$$

Moroever, according to Proposition 6.3.2, $\rho(\overline{M}_{\mathrm{ap}})$ vanishes. $\square$

6.3.2. Continuity properties of the covering radius.

6.3.2.1. *Regularity properties of the invariant $\rho : \overline{\mathbf{Vect}}_{\mathbb{Z}} \to \mathbb{R}_+$.* The classical covering radius, namely the restriction $\rho : \overline{\mathbf{Vect}}_{\mathbb{Z}} \to \mathbb{R}_+$ of the invariant ρ to Euclidean lattices, is easily seen to depend continuously on the Euclidean norms defining Euclidean lattices.

To formulate precisely the regularity property of the invariant $\rho : \overline{\mathbf{Vect}}_{\mathbb{Z}} \to \mathbb{R}_+$, let us fix a finitely generated free $\mathbb{Z}$-module E. To any element $\|.\|$ in the cone $\mathcal{Q}(E_{\mathbb{R}})$ of Euclidean seminorms over $E_{\mathbb{R}}$, we may attach the covering radius $\rho(E, \|.\|)$ of the object $(E, \|.\|)$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. Let us also choose a relatively compact subset Δ of $E_{\mathbb{R}}$ such that $E + \Delta = E_{\mathbb{R}}$.

PROPOSITION 6.3.4. *The function*

$$\rho(E, .) : \mathcal{Q}(E_{\mathbb{R}}) \longrightarrow \mathbb{R}_+$$

is locally Lipschitz on the cone $\mathring{\mathcal{Q}}(E_{\mathbb{R}})$ of Euclidean norms over $E_{\mathbb{R}}$.

Actually, for every compact subset $\mathcal{K}$ of $\overset{\circ}{\mathcal{Q}}(E_{\mathbb{R}})$, there exists a finite subset A of E such that, for every $\|.\|$ in $\mathcal{K}$, the following equality holds:

$$\rho(E, \|.\|) = \sup_{x \in \Delta} \min_{a \in A} \|x - a\|. \tag{6.3.5}$$

We leave the details of the proof as an easy exercise. The locally Lipschitz character of the function $\rho(E, .)$ is also a formal consequence of the fact that $\rho(E, \|.\|)$ is both an increasing and a 1-homogeneous function of the Euclidean norm $\|.\|$. Observe also that the expression (6.3.5) for $\rho(E, \|.\|)$ implies not only that $\rho(E, .)$ is a locally Lipschitz function on $\overset{\circ}{\mathcal{Q}}(E_{\mathbb{R}})$, but also that locally its graph is subanalytic.

6.3.2.2. *The property* $\mathbf{Cont}^+$. In our study of invariants on $\overline{\mathbf{Coh}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}_X$ in Chapter 4, the property of downward continuity $\mathbf{Cont}^+$ plays a central role. In this subsection, we want to discuss this property, as regards the invariant ρ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

Recall the formulation of the property $\mathbf{Cont}^+$.

Consider an object $\overline{M} := (M, \|.\|)$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and a decreasing sequence $(\|.\|_n)_{n \in \mathbb{N}}$ of Euclidean seminorms on $M_{\mathbb{R}}$ such that:

$$\lim_{n \to +\infty} \|v\|_n = \|v\| \quad \text{for every } n \in \mathbb{N}. \tag{6.3.6}$$

Then for any $n \in \mathbb{N}$, we have:

$$\|.\| \leq \|.\|_{n+1} \leq \|.\|_n$$

and

$$\rho(M, \|.\|) \leq \rho(M, \|.\|_{n+1}) \leq \rho(M, \|.\|_n).$$

Consequently the limit $\lim_{n \to +\infty} \rho(M, \|.\|_n)$and satisfies the inequality:

$$\rho(M, \|.\|) \leq \lim_{n \to +\infty} \rho(M, \|.\|_n). \tag{6.3.7}$$

The downward continuity property $\mathbf{Cont}^+$ applied to the invariant ρ and to the Euclidean quasi-coherent sheaves $(\overline{M}_n)_{n \in \mathbb{N}}$ asserts that the estimate (6.3.7) is actually an equality, namely:

$$\rho(M, \|.\|) = \lim_{n \to +\infty} \rho(M, \|.\|_n). \tag{6.3.8}$$

This property is easily seen not hold in full generality.

EXAMPLE 6.3.5. Consider the $\mathbb{Z}$-module $M := \mathbb{Z}^{(\mathbb{N})}$ and the sequence of Euclidean norms $(\|.\|_n)_{n \in \mathbb{N}}$ on $M_{\mathbb{R}}$ defined by:

$$\|(x_i)_{i \in \mathbb{N}}\|_n^2 := \sum_{i \geq n} x_i^2.$$

This sequence of norms is decreasing, and converges point wise to the seminorm $\|.\| := 0$. For every $n \in \mathbb{N}$, we have:

$$\rho(M, \|.\|_n) = +\infty.$$

Moreover:

$$\rho(M, \|.\|) = 0,$$

and (6.3.8) does not hold.

6.3.2.3. *The invariant $\rho : \overline{\mathbf{Coh}}_{\mathbb{Z}} \to \mathbb{Z}$ satisfies* $\mathbf{Cont}^+$. In view of Example 6.3.5, one may wonder whether the equality (6.3.8) is satisfied as soon as the left-hand side of (6.3.8) is finite (that is, if for some $n \in \mathbb{N}$, the covering radius $\rho(M, \|.\|_n)$ is finite). We do not know the answer to this question.[8] However we can prove that (6.3.8) holds under a finite-dimensionality assumption:

PROPOSITION 6.3.6. *With the notation of 6.3.2.2, if the $\mathbb{Q}$-vector space $M_{\mathbb{Q}}$ — or equivalently the $\mathbb{R}$-vector space $M_{\mathbb{R}}$ — is finite dimensional, then:*

$$\rho(M, \|.\|) = \lim_{n\to+\infty} \rho(M, \|.\|_n). \tag{6.3.9}$$

PROOF. Since $M_{/\mathrm{tor}}$ generates the $\mathbb{R}$-vector space $M_{\mathbb{R}}$, when the latter is finite dimensional, it contains a compact subset K such that

$$M_{\mathbb{R}} = M_{/\mathrm{tor}} + K.$$

Consequently, for every Euclidean seminorm $|.|$ on $M_{\mathbb{R}}$, we have:

$$\rho(M, |.|) = \sup_{x\in K} \inf_{m\in M_{/\mathrm{tor}}} |x - m|.$$

The function

$$d_{|.|}(., M_{/\mathrm{tor}}) : M_{\mathbb{R}} \longrightarrow \mathbb{R}_+, \quad x \longmapsto \inf_{m\in M_{/\mathrm{tor}}} |x - m|$$

is 1-Lipschitz on $(M_{\mathbb{R}}, |.|)$, hence continuous on $M_{\mathbb{R}}$. Therefore there exists $x \in K$ such that:

$$\rho(M, |.|) = d_{|.|}(x, M_{/\mathrm{tor}}).$$

Since (6.3.7) holds, to prove (6.3.9), we are left to establish the following inequality:

$$\rho(M, \|.\|) \geq \lim_{n\to+\infty} \rho(M, \|.\|_n). \tag{6.3.10}$$

To achieve this, for every $n \in \mathbb{N}$ choose x_n in K such that:

$$\rho(M, \|.\|_n) = d_{\|.\|_n}(x, M_{/\mathrm{tor}}).$$

Since K is compact, we may assume that the sequence $(x_n)_{n\in\mathbb{N}}$ admits a limit x in K. Let us prove the inequality:

$$d_{\|.\|}(x, M_{/\mathrm{tor}}) \geq \lim_{n\to+\infty} \rho(M, \|.\|_n), \tag{6.3.11}$$

which clearly implies (6.3.10).

To achieve this, let us consider $\varepsilon \in \mathbb{R}_+^*$, and choose $m \in M_{/\mathrm{tor}}$ such that

$$d_{\|.\|}(x, M_{/\mathrm{tor}}) \geq \|x - m\| - \varepsilon.$$

If the integer n is large enough, we also have:

$$\|x - m\| \geq \|x - m\|_n - \varepsilon,$$

since the sequence $(\|.\|_n)_{n\in\mathbb{N}}$ converges pointwise towards $\|.\|$. Then we have:

$$d_{\|.\|}(x, M_{/\mathrm{tor}}) \geq \|x - m\|_n - 2\varepsilon \geq \|x_n - m\|_n - \|x - x_n\|_n - 2\varepsilon \geq \rho(M, \|.\|_n) - \|x - x_n\|_0 - 2\varepsilon.$$

This implies that, for every large enough integer n, the following inequality holds:

$$d_{\|.\|}(x, M_{/\mathrm{tor}}) \geq \lim_{n\to+\infty} \rho(M, \|.\|_n) - 3\varepsilon.$$

As $\varepsilon \in \mathbb{R}_+^*$ is arbitrary, this establishes (6.3.11). □

As a special case of Proposition 6.3.6, we have:

COROLLARY 6.3.7. *Restricted to $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the covering radius ρ satisfies* $\mathbf{Cont}^+$.

[8] Although we would expect that it is negative.

Since ρ also satisfies $\mathbf{Mon}^1$, this implies that the invariant

$$\widetilde{\psi}_{\mathrm{nst}} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow \mathbb{R}_+$$

deduced from the classical covering radius:

$$\psi := \rho : \overline{\mathbf{Vect}}_{\mathbb{Z}} \longrightarrow \mathbb{R}_+$$

by the construction in Subsection 4.2.5 coincides with the invariant

$$\rho : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow \mathbb{R}_+$$

defined by formula (6.0.1).

A related property of the covering radius ρ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ is that *it is small on Euclidean coherent sheaves generated by small sections* in the sense of Definition 4.2.11. This is a straightforward consequence of the criteria in Proposition 4.2.12, and may also be seen directly on the definition.

6.3.2.4. *The upper-semicontinuity of $\rho(E, .) : \mathcal{Q}(E_{\mathbb{R}}) \to \mathbb{R}_+$.* We return to the notation of 6.3.2.1. Namely, we denote by E a finitely generated free $\mathbb{Z}$-module and by $\mathcal{Q}(E_{\mathbb{R}})$ the cone of Euclidean seminorms over $E_{\mathbb{R}}$, and for every $\|.\|$ in $\mathcal{Q}(E_{\mathbb{R}})$, we consider the covering radius $\rho(E, \|.\|)$ in $\mathbb{R}_+$.

PROPOSITION 6.3.8. *The function $\rho(E, .) : \mathcal{Q}(E_{\mathbb{R}}) \longrightarrow \mathbb{R}_+$ is upper semicontinuous.*

PROOF. This follows from Proposition 4.2.1 applied to $Q := \mathring{\mathcal{Q}}(E_{\mathbb{R}})$, $\overline{Q} := \mathcal{Q}(E_{\mathbb{R}})$, and $g := \rho(E, .)$.

We may also recover the upper semicontinuity of $\rho(E, .)$ from the results in this subsection as follows.

Choose some Euclidean norm $\|.\|_0$ on $E_{\mathbb{R}}$. According to the downward continuity of the covering radius established in Proposition 6.3.6, for every seminorm $\|.\|$ in $\mathcal{Q}(E_{\mathbb{R}})$ the following equality holds:

$$\rho(E, \|.\|) = \inf_{n \in \mathbb{N}_{>0}} \rho\big(E, (\|.\|^2 + \|.\|_0^2/n)^{1/2}\big).$$

This expresses the function $\rho(E, .)$ as the infimum of the family of functions

$$\mathcal{Q}(E_{\mathbb{R}}) \longrightarrow \mathbb{R}_+, \quad \|.\| \longmapsto \rho\big(E, (\|.\|^2 + \|.\|_0^2/n)^{1/2}\big),$$

and each of these is continuous according to Proposition 6.3.4. □

It turns out that, when $\mathrm{rk} E \geq 2$, the function $\rho(E, .)$ is *not* continuous on $\mathcal{Q}(E_{\mathbb{R}})$. Its restriction to the subcone $\mathcal{Q}(E_{\mathbb{R}})_1$ of extremal rays of $\mathcal{Q}(E_{\mathbb{R}})$ is already not continuous, as will be demonstrated by Proposition 6.3.9 below.

Let us introduce some further notation. We choose a Euclidean metric $\|.\|_0$ on $E_{\mathbb{R}}$. To this metric is canonically attached a *height function*:

$$\mathrm{ht} : \mathbb{P}(E)(\mathbb{Q}) \longrightarrow \mathbb{R}$$

on $\mathbb{P}(E)(\mathbb{Q}) \simeq \mathbb{P}(E)(\mathbb{Z})$. To a point P in $\mathbb{P}(E)(\mathbb{Z})$ defined by a rank one quotient $E \twoheadrightarrow L$ of E, it associates the real number:

$$\mathrm{ht}(P) := \widehat{\deg}\, \overline{L},$$

where $\overline{L}$ is the rank one Euclidean lattice defined by L equipped with the Euclidean metric on $L_{\mathbb{R}}$ quotient of the Euclidean metric $\|.\|_0$ on $E_{\mathbb{R}}$. Any point P in $\mathbb{P}(E)(\mathbb{Q})$ is the class $[\xi]$ of an element ξ in $E^\vee \setminus \{0\}$ that is primitive[9], and we have:

$$\mathrm{ht}(P) = \log \|\xi\|_0^\vee,$$

where $\|.\|_0^\vee$ denotes the Euclidean norm on $E_{\mathbb{R}}^\vee$ dual to $\|.\|_0$.

[9]Namely, the submodule $\mathbb{Z}\xi$ is saturated in $E^\vee$. Such a primitive ξ is unique up to a sign.

In paragraph 2.3.1.5, to any $\xi \in E_{\mathbb{R}}^{\vee} \setminus \{0\}$, we have associated the semipositive quadratic forms $\xi^2 \in \mathcal{Q}(E_{\mathbb{R}})_1$ and the Euclidean coherent sheaf

$$\overline{E}_{\xi} := (E, |\xi|).$$

When the point $[\xi] \in \mathbb{P}^2(\mathbb{R})$ belongs to $\mathbb{P}^2(\mathbb{Q})$, we have defined $t(\xi) \in \mathbb{R}_+^*$ by the relation:

$$\mathbb{R}\xi \cap E^{\vee} = \mathbb{Z}\, t(\xi)\xi,$$

or equivalently by the fact that $t(\xi)\xi$ is a primitive representative of $[\xi]$ in $E^{\vee} \setminus \{0\}$.

PROPOSITION 6.3.9. *Let ξ be an element of $E_{\mathbb{R}}^{\vee} \setminus \{0\}$.*

If its class $[\xi]$ in $\mathbb{P}(E)(\mathbb{R})$ does not belong to $\mathbb{P}(E)(\mathbb{Q})$, then:

$$\rho(\overline{E}_{\xi}) = 0. \tag{6.3.12}$$

If $[\xi]$ belongs to $\mathbb{P}(E)(\mathbb{Q})$, then:

$$\rho(\overline{E}_{\xi}) = (2t(\xi))^{-1} = e^{-\mathrm{ht}([\xi])}\, \|\xi\|_0^{\vee}/2. \tag{6.3.13}$$

PROOF. This follows from the equality:

$$\rho(\overline{E}_{\xi}) = \rho(\overline{E}_{\xi}^{\mathrm{vect}}),$$

established in Proposition 6.3.1, from the description of $\overline{E}_{\xi}^{\mathrm{vect}}$ in Proposition 2.3.9, and from the relation, when $[\xi]$ belongs to $\mathbb{P}(E)(\mathbb{Q})$:

$$\mathrm{ht}([\xi]) = \log \|t(\xi)\xi\|_0^{\vee} = \log t(\xi) + \log \|\xi\|_0^{\vee}.$$ □

6.3.3. The lower and upper covering radii; ρ^2-summable Euclidean quasicoherent sheaves. To define the covering radius $\rho(\overline{M})$ of a some Euclidean quasi-coherent sheaf with possibly non-finitely generated underlying $\mathbb{Z}$-module M, an alternative to the direct approach in Definition 6.1.1 would be to apply the general construction of invariants on $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ starting from invariants on $\overline{\mathbf{Vect}}_{\mathbb{Z}}$ developed in Chapter 4.

Namely we could start from the invariant ρ on the category $\overline{\mathbf{Vect}}_{\mathbb{Z}}$ of Euclidean lattices (namely from the covering radius as classically studied in geometry of numbers) and firstly extend it to $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ by downward continuity as discussed in Section 4.2. As discussed in paragraph 6.3.2.3 above, this extension coincides with the restriction to $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ of the invariant ρ introduced in Definition 6.1.1.

Then we could introduce its lower and upper extensions to $\mathbf{q}\overline{\mathbf{Coh}}_{Z}$ as defined in Section 4.3. This second step leads one to introduce significant variants of the covering radius ρ, which we discuss in this subsection.

6.3.3.1. *The invariants $\underline{\rho}$ and $\overline{\rho}$.* By specializing to the covering radius the general construction in Section 4.3, to any object $\overline{M}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, we attach its *lower covering radius*:

$$\underline{\rho}(\overline{M}) := \sup_{N \in \mathrm{coft}(M)} \rho(\overline{M/N}), \tag{6.3.14}$$

and its *upper covering radius*:

$$\overline{\rho}(\overline{M}) := \liminf_{C \in \mathrm{coh}(M)} \rho(\overline{C}). \tag{6.3.15}$$

By construction, they coincide with $\rho(\overline{M})$ when $\overline{M}$ is an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. Moreover various properties of the covering radius on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ are inherited by $\underline{\rho}$ and $\overline{\rho}$ on $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$. For instance, they satisfy the 1-homogeneity property (6.1.2), the monotonicity and subadditivity properties $\mathbf{Mon}^1$ and **SubAdd** (see Propositions 4.3.6 and 4.3.10 If), and the property **NSAp** (see Proposition 4.3.23). In particular the relations (6.1.10) and (6.3.4) hold with $\underline{\rho}$ or $\overline{\rho}$ instead of ρ. According to Proposition 4.3.7, it is also the case for the lower semicontinuity estimate (6.1.6).

One may wonder about the relation between the invariants $\underline{\rho}$ and $\overline{\rho}$ so-defined and the "direct" definition of ρ on $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ proposed in Definition 6.1.1. The semi-continuity and monotonicity properties of ρ already lead to the following estimates:

PROPOSITION 6.3.10. *For every Euclidean quasi-coherent sheaf $\overline{M}$, the following inequalities hold:*

$$\underline{\rho}(\overline{M}) \leq \rho(\overline{M}) \leq \overline{\rho}(\overline{M}). \tag{6.3.16}$$

The first inequality in (6.3.16) may actually be strict, as demonstrated by the construction in 6.3.3.4 below. We do not expect equality to hold in general in the second inequality in (6.3.16).

PROOF. For every N in $\mathrm{coft}(M)$, Proposition 6.1.4 applied to the quotient map $\overline{M} \to \overline{M/N}$ establishes the estimate:

$$\rho(\overline{M/N}) \leq \rho(\overline{M}).$$

The first inequality in (6.3.16) follows by taking the supremum over $N \in \mathrm{coft}(M)$.

Moreover, by the very definition of $\overline{\rho}(\overline{M})$, we may choose an exhaustive filtration $(C_i)_{i\in\mathbb{N}}$ of M by submodules in $\mathrm{coh}(M)$ such that:

$$\overline{\rho}(\overline{M}) = \lim_{i\to+\infty} \rho(\overline{C}_i).$$

Then Proposition 6.1.2 applied to the filtration $(C_i)_{i\in\mathbb{N}}$ establishes the second inequality in (6.3.16). □

6.3.3.2. *ρ^2-summable Euclidean quasicoherent sheaves.* Since the invariant ρ^2 satisfies properties $\mathbf{Mon}^1$ and $\mathbf{SubAdd}$ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the whole formalism of the lower and upper extensions developed in Chapter 4 applies to the invariant ρ^2 on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and to its extensions $\underline{\rho}^2$, and $\overline{\rho}^2$. Notably the main theorem of Chapter 4, Theorem 4.5.1, concerning φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, takes the following form when applied to $\varphi = \rho^2$:

PROPOSITION 6.3.11. *Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf, and $(C_i)_{i\in\mathbb{N}}$ exhaustive filtration of M by submodules in* $\mathrm{coh}(M)$.

If the following summability condition holds:

$$\sum_{i\in\mathbb{N}} \rho(\overline{C_i/C_{i-1}})^2 < +\infty, \tag{6.3.17}$$

then the limit $\lim_{i\to+\infty} \rho(\overline{C}_i)$ exists in $\mathbb{R}_+$, and

$$\overline{\rho}(\overline{M}) = \lim_{i\to+\infty} \rho(\overline{C}_i). \tag{6.3.18}$$

Moreover, $\overline{M}$ has eventually vanishing upper covering radius, and therefore eventually vanishing covering radius.[10]

As a special instance of the terminology of Chapter 4, a Euclidean quasi-coherent sheaf $\overline{M}$ such that there exists an exhaustive filtration $(C_i)_{i\in\mathbb{N}}$ of M by submodules in $\mathrm{coh}(M)$ satisfying (6.3.17) will be called *ρ^2-summable.*

Observe that, as a consequence of the 1-homogeneity (6.1.2) of the covering radius on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, if $\overline{M}$ is ρ^2-summable, then $\overline{M} \otimes \overline{\mathcal{O}}(\delta)$ also is ρ^2-summable for every δ in $\mathbb{R}$.

The general results concerning φ-summability in Section 4.5 apply to ρ^2-summability. Notably ρ^2-summability is preserved by surjective morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ (Proposition 4.5.9), and $\overline{\rho}^2$ satisfies the downward continuity property $\mathbf{Cont}^+$ on ρ^2-summable Euclidean quasi-coherent sheaves (Proposition 4.5.16).

[10]See Sections 4.4 and 6.5 for the definitions of these properties of eventual vanishing.

6.3.3.3. *Euclidean quasi-coherent sheaves with positive lower covering radius.* The following proposition clarifies to some extent the meaning of the lower covering radius.

PROPOSITION 6.3.12. *For every Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$, the following two conditions are equivalent:*

(i) $\underline{\rho}(\overline{M}) > 0$;
(ii) there exists an non-zero element ξ in $M^\vee := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ such that the $\mathbb{R}$-linear form

$$\xi_{\mathbb{R}} : M_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is continuous on $(M_{\mathbb{R}}, \|.\|)$.

Condition (ii) precisely asserts the existence of a non-zero morphism

$$\xi : \overline{M} \longrightarrow \overline{\mathcal{O}}_{\mathbb{Z}} := (\mathbb{Z}, |.|)$$

in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, or equivalently that the dual $\widehat{\overline{E}} := \overline{M}^\vee$ of $\overline{M}$ in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[\infty]}$ satisfies:

$$\widehat{E} \cap E_{\mathbb{R}}^{\mathrm{Hilb}} \neq \{0\}.$$

PROOF. We first observe that the proposition holds when $\overline{M}$ is a Euclidean coherent sheaf. Indeed, for every object $\overline{E}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following conditions are successively equivalent:

(a) $\rho(\overline{E}) > 0$;
(b) $\rho(\overline{E}^{\mathrm{vect}}) > 0$;
(c) $\overline{E}^{\mathrm{vect}} \neq 0$;
(d) $\overline{E}^{\mathrm{vect}\vee} \neq 0$;
(e) there exists $\xi \in \mathrm{Hom}(E, \mathbb{Z})$ such that $\xi_{\mathbb{R}} : E_{\mathbb{R}} \to \mathbb{R}$ is continuous on $(E_{\mathbb{R}}, \|.\|)$.

The equivalence (a) $\Leftrightarrow$ (b) follows from Proposition 6.3.1; the equivalences (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) are straightforward, and (d) $\Leftrightarrow$ (e) follows from Proposition 2.3.3.

In general, by the definition of $\underline{\rho}$, a Euclidean quasi-coherent sheaf $\overline{M}$ satisfies (i) if and only if there exists $N \in \mathrm{coft}(M)$ such that:

$$\rho(\overline{M/N}) > 0.$$

When this holds, according to the implication (a) $\Rightarrow$ (d) applied to $\overline{E} := \overline{M/N}$, there exists a non-zero morphism:

$$\tilde{\xi} : \overline{M/N} \longrightarrow \overline{\mathcal{O}}_{\mathbb{Z}}$$

in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, and then its composition with the quotient map:

$$\xi := \overline{M} \longrightarrow \overline{M/N} \xrightarrow{\tilde{\xi}} \overline{\mathcal{O}}_{\mathbb{Z}}$$

is a non-zero morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

Conversely, when (ii) holds, the element ξ defines a non-zero morphism

$$\xi : \overline{M} \longrightarrow \overline{\mathcal{O}}_{\mathbb{Z}}(\delta)$$

in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ if $\delta \in \mathbb{R}$ is chosen large enough. Then $N := \ker \xi$ belong to $\mathrm{coft}(M)$, and ξ factorizes through a non-zero morphism

$$\tilde{\xi} : \overline{M/N} \longrightarrow \overline{\mathcal{O}}_{\mathbb{Z}}(\delta)$$

in $\overline{\mathbf{Vect}}_{\mathbb{Z}}^{\leq 1}$. Consequently, by (a trivial case of) the monotonicity of ρ,

$$\rho(\overline{M/N}) \geq \rho(\overline{\mathcal{O}}(\delta)) = e^{-\delta}/2. \qquad \square$$

6.3.3.4. *An exotic Euclidean quasi-coherent sheaf.* Let us indicate how to construct a Euclidean quasi-coherent sheaf $\overline{M}$ such that:

$$\underline{\rho}(\overline{M}) = 0 \quad \text{and} \quad \rho(\overline{M}) > 0.$$

This construction is a special instance of Banaszczyk's "exotic groups" [**Ban91**, Section 5], and already appears in [**Bos20b**, Section 6.4.4].

Let us equip $\mathbb{R}^{(\mathbb{N})}$ with the usual Euclidean norm $\|.\|$ defined by:

$$\|(x_i)_{i\in\mathbb{N}}\|^2 := \sum_{i\in\mathbb{N}} x_i^2,$$

and let us denote by $(e_n)_{n\in\mathbb{N}}$ the standard basis of $\mathbb{R}^{(\mathbb{N})}$ defined by $e_n := (\delta_{in})_{i\in\mathbb{N}}$, and by $(\xi_n)_{n\in\mathbb{N}}$ the family of linear forms defined as:

$$\xi_n : \mathbb{R}^{(\mathbb{N})} \longrightarrow \mathbb{R}, \quad (x_i)_{i\in\mathbb{N}} \longmapsto x_n.$$

One easily see that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}^{(\mathbb{N})}$ satisfying the following conditions:

$$(f_n)_{n\in\mathbb{N}} \text{ is dense in } (\mathbb{R}^{(\mathbb{N})}, \|.\|) \tag{6.3.19}$$

and

$$\text{for every } n \in \mathbb{N}, \quad f_n \in \bigoplus_{0\leq i<n} \mathbb{Q}e_i. \tag{6.3.20}$$

Then the sequence $(e_n + f_n)_{n\in\mathbb{N}}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}^{(\mathbb{N})}$ and a $\mathbb{R}$-basis of $\mathbb{R}^{(\mathbb{N})}$, and the $\mathbb{Z}$-module

$$M := \bigoplus_{n\in\mathbb{N}} \mathbb{Z}(e_n + f_n)$$

that it generates is free, the real vector space $M_\mathbb{R}$ may be identified with $\mathbb{R}^{(\mathbb{N})}$, and we may consider the Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$.

The following properties are satisfied by this construction:

(i) There is no non-zero continuous linear form ξ on $(\mathbb{R}^{(\mathbb{N})}, \|.\|)$ such that $\xi(M) \subseteq \mathbb{Z}$.

(ii) Every element m in $M \setminus \{0\}$ satisfies:

$$\|m\| \geq 1.$$

In particular, M is a discrete $\mathbb{Z}$-submodule of $(\mathbb{R}^{(\mathbb{N})}, \|.\|)$.

(iii) For every non-zero finite dimensional $\mathbb{Q}$-vector space V of $\mathbb{Q}^{(\mathbb{N})}$, the set

$$\{n \in \mathbb{N} \mid \xi_{n|V} \neq 0\}$$

is finite and non-empty. If $n(V)$ denotes its largest element, then we have:

$$\xi_{n(V)}(V \cap M) \subset \mathbb{Z}.$$

Indeed to prove (i), consider ξ is a continuous linear form on $(\mathbb{R}^{(\mathbb{N})}, \|.\|)$ mapping M to $\mathbb{Z}$. According to (6.3.19), every x in $\mathbb{R}^{(\mathbb{N})}$ may be written as the limit in $(\mathbb{R}^{(\mathbb{N})}, \|.\|)$ of a sequence $(f_{n_i})_{i\in\mathbb{N}}$ for a suitable strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$, and therefore

$$\xi(x) = \lim_{i\to+\infty} \xi(f_{n_i}) = \lim_{i\to+\infty} \xi(e_{n_i} + f_{n_i})$$

is a limit of integers, hence an integer. Consequently $\xi(\mathbb{R}^{(\mathbb{N})})$ is contained in $\mathbb{Z}$. This immediately implies that ξ is zero.

To prove (ii), observe that if $m = (x_i)_{i\in\mathbb{N}}$ is a non-zero element of M, then, according to (6.3.20), the last of its coordinates x_i that is non-zero is an integer. The last assertion (iii) is proved similarly; see [**Bos20b**, Proposition 6.4.6].

According to Proposition 6.3.12, Property (i) is equivalent to the vanishing of $\underline{\rho}(\overline{M})$. Property (ii) implies that the distance function

$$d_{\overline{M}}(., M) : \mathbb{R}^{(\mathbb{N})} \longrightarrow \mathbb{R}_+$$

vanishes only on M, and therefore that its supremum $\rho(\overline{M})$ is positive. More precisely, it satisfies the following estimates:

$$1/2 \leq \rho(\overline{M}) \leq 1.$$

Indeed, every $x \in \mathbb{R}^{(\mathbb{N})}$ may be written as the limit in $(\mathbb{R}^{(\mathbb{N})}, \|.\|)$ of a sequence $(f_{n_i})_{i\in\mathbb{N}}$ as above. Then we have:

$$\lim_{i\to+\infty} \|x - (e_{n_i} + f_{n_i})\| = \lim_{i\to+\infty} \|e_{n_i}\| = 1,$$

and therefore

$$d_{\overline{M}}(x, M) \leq 1.$$

Moreover, one easily establishes the equality:

$$d_{\overline{M}}(e_0/2, M) = 1/2;$$

indeed $f_0 = 0$ and therefore $M \cap \mathbb{R}e_0 = \mathbb{Z}e_0$, and for every $m \in M \setminus \mathbb{R}e_0$, the last non-zero coordinate of $m - e_0/2$ is an integer.

Actually when the f_n are chosen so that f_1, f_2, and f_3 vanish, a similar argument shows the following equality holds:

$$d_{\overline{M}}((e_0 + e_1 + e_2 + e_3)/4, M) = 1.$$

Therefore in this case, $\rho(\overline{M}) = 1$.

6.3.3.5. The results in this subsection show that, when dealing with objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ not in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the upper and lower covering radii $\underline{\rho}$ and $\overline{\rho}$ are arguably more sensible invariants than the original covering radius ρ defined by the "naive formula" (6.0.1).

In particular Proposition 6.3.11 and the subsequent observations show that ρ^2-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ constitute a natural class on which the upper covering radius $\overline{\rho}$ is especially well-behaved.

From a formal perspective, another natural class of objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ to consider are those Euclidean quasi-coherent sheaves such that:

$$\underline{\rho}(\overline{M}) = \overline{\rho}(\overline{M}) < +\infty. \tag{6.3.21}$$

According to Proposition 6.3.10, they satisfy:

$$\rho(\overline{M}) = \underline{\rho}(\overline{M}) = \overline{\rho}(\overline{M}).$$

Both classes are stable under change of scale[11] and contain $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. Unfortunately we do not know how these two classes compare, and actually the possibility to establish positive simple general results on this question does not seem likely.

This is in stark contrast with the situation concerning invariants constructed from the θ-invariants h^1_θ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. Indeed one the main results of this monograph, established in Chapter 8, will be that the objects $\overline{M}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ such that $\overline{M} \otimes \overline{\mathcal{O}}(\delta)$ is h^1_θ-summable for every $\delta \in \mathbb{R}$ are precisely those such that, for every $\delta \in \mathbb{R}$:

$$\underline{h}^1_\theta(\overline{M} \otimes \overline{\mathcal{O}}(\delta)) = \overline{h}^1_\theta(\overline{M} \otimes \overline{\mathcal{O}}(\delta)) < +\infty.$$

The so-defined class of θ^1-finite objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ turns out to be especially flexible and useful in Diophantine applications. The lack of a similar formalism concerning the covering radius limits its use in applications.

[11] that is, under tensoring by $\overline{\mathcal{O}}(\delta)$ for δ in $\mathbb{R}$.

6.4. The Invariants $\lambda^{[0]}(\overline{M})$ and $\gamma(\overline{M})$ and the Covering Radius

6.4.1. The invariants $\lambda^{[0]}(\overline{M})$ and $\gamma(\overline{M})$: definitions and first properties.

6.4.1.1. *Definitions.* This section is devoted to the properties of the invariants $\lambda^{[0]}$ and γ on $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ with values in $[0,+\infty]$ defined as follows:

DEFINITION 6.4.1. For every Euclidean quasi-coherent sheaf $\overline{M}$, we let:

$$\lambda^{[0]}(\overline{M}) := \inf\left\{R\in\mathbb{R}_+^* \mid \text{the } \mathbb{R}\text{-vector space generated by } B(\overline{M}_{/\mathrm{tor}};R) \text{ is dense in } M_{\mathbb{R}}\right\}$$

and

$$\gamma(\overline{M}) := \inf\left\{R\in\mathbb{R}_+^* \mid \ B(\overline{M};R) \text{ generates the } \mathbb{Z}\text{-module } M\right\}.$$

We say that $\overline{M}$ is *generated by bounded sections* when $\gamma(\overline{M})$ is finite, or equivalently when there exists R in $\mathbb{R}_+$ such that $B(\overline{M};R)$ generates the $\mathbb{Z}$-module M.

The invariants $\lambda^{[0]}$ and γ clearly satisfy the following inequality:

$$\lambda^{[0]}(\overline{M}) \leq \gamma(\overline{M}). \tag{6.4.1}$$

Moreover, like the covering radius ρ, they are are 1-homogeneous; namely, for every δ in $\mathbb{R}$, we have:

$$\lambda^{[0]}(\overline{M}\otimes\overline{\mathcal{O}}(\delta)) = e^{-\delta}\,\lambda^{[0]}(\overline{M}) \quad \text{and} \quad \rho(\overline{M}\otimes\overline{\mathcal{O}}(\delta)) = e^{-\delta}\,\rho(\overline{M}).$$

When the $\mathbb{R}$-vector space $M_{\mathbb{R}}$, or equivalently the $\mathbb{Q}$-vector space $M_{\mathbb{Q}}$, is finite dimensional (resp. when the $\mathbb{Z}$-module $M_{/\mathrm{tor}}$ is finitely generated, for instance when $\overline{M}$ is a an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$), then $\lambda^{[0]}(\overline{M})$ (resp. $\gamma(\overline{M})$) is easily seen to be finite.

When $\overline{M}$ is a Euclidean lattice (that is, when the $\mathbb{Z}$-module M is finitely generated and free and the Euclidean seminorm $\|.\|$ is a norm), the invariant $\lambda^{[0]}(\overline{M})$ is finite and coincides with the "ultimate of the successive minima" of $\overline{M}$, namely with $\lambda_n(\overline{M})$ where $n := \mathrm{rk} M$:

$$\lambda^{[0]}(\overline{M}) = \lambda_n(\overline{M}) := \min\left\{R\in\mathbb{R}_+ \mid \ \overline{B}(\overline{M};R) \text{ generates the } \mathbb{R}\text{-vector space } M_{\mathbb{R}}\right\}.$$

Similarly, still assuming that $\overline{M}$ is a Euclidean lattice, the invariant $\gamma(\overline{M})$ is easily seen to be the following minimum:

$$\gamma(\overline{M}) := \min\left\{R\in\mathbb{R}_+^* \mid \ \overline{B}(\overline{M};R) \text{ generates the } \mathbb{Z}\text{-module } M\right\}.$$

The invariants $\lambda^{[0]}(\overline{M})$ and $\gamma(\overline{M})$ attached to some object $\overline{M}$ of $\mathbf{q}\overline{\mathbf{Coh}}$ have arguably a more intuitive definition than its covering radius $\rho(\overline{M})$. However they enjoy less satisfactory formal properties. In this section, we firstly discuss their basic properties, then we establish comparison estimates relating the "naive" invariants $\lambda^{[0]}$ and γ, and the covering radius ρ.

In the remaining of this monograph, these results concerning $\lambda^{[0]}$ and γ will play a role by their consequences concerning diverse eventual vanishing properties of Euclidean quasi-coherent sheaves established in Section 6.5.

6.4.1.2. *Monotonicity.* The invariant γ is easily seen to satisfy the monotonicity property $\mathbf{Mon}^1$, namely:

PROPOSITION 6.4.2. *If $f:\overline{M}_1\to\overline{M}_2$ is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ such that $f(M_1)=M_2$, then:*

$$\gamma(\overline{M}_1) \geq \gamma(\overline{M}_2). \tag{6.4.2}$$

PROOF. To establish the inequality (6.4.2), it is enough to show that, for any $R\in\mathbb{R}_+^*$, if $B(\overline{M}_1;R)$ generates the $\mathbb{Z}$-module M_1, then $B(\overline{M}_2;R)$ generates the $\mathbb{Z}$-module M_2. This follows from the surjectivity of $f: M_1\to M_2$ and from the inclusion $f(B(\overline{M}_1;R))\subseteq B(\overline{M}_2;R)$. $\square$

A similar proof shows that $\lambda^{[0]}$ satisfies the monotonicity property $\mathbf{Mon}^1_{\mathbb{Q}}$:

PROPOSITION 6.4.3. *Let $f : \overline{M}_1 \to \overline{M}_2$ be a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$. If $f_{\mathbb{Q}}(M_{1,\mathbb{Q}}) = M_{2,\mathbb{Q}}$, then:*

$$\lambda^{[0]}(\overline{M}_1) \geq \lambda^{[0]}(\overline{M}_2). \tag{6.4.3}$$

More generally, the estimate (6.4.3) *holds as soon as $f_{\mathbb{R}}(M_{1,\mathbb{R}})$ is dense in $\overline{M}_{2,\mathbb{R}}$.*

6.4.1.3. *Compatibility with direct sums and with admissible short exact sequences.* The invariant γ and $\lambda^{[0]}$ satisfy the condition $\mathbf{Max}_{\oplus}$ introduced in 4.1.5. Indeed we have, as a straightforward consequence of the definitions:

PROPOSITION 6.4.4. *For any two objects $\overline{M}_1$ and $\overline{M}_2$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following equalities hold:*

$$\gamma(\overline{M}_1 \oplus \overline{M}_2) = \max(\gamma(\overline{M}_1), \gamma(\overline{M}_2))$$

and:

$$\lambda^{[0]}(\overline{M}_1 \oplus \overline{M}_2) = \max(\lambda^{[0]}(\overline{M}_1), \lambda^{[0]}(\overline{M}_2)).$$

As a consequence of the lifting result involving the covering radius established in Proposition 6.1.8, the invariant γ and $\lambda^{[0]}$ satisfy the following compatibility with admissible short exact sequences:

PROPOSITION 6.4.5. *For every admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$:*

$$0 \longrightarrow \overline{M}' \xrightarrow{i} \overline{M} \xrightarrow{p} \overline{M}'' \longrightarrow 0,$$

the following estimates hold:

$$\gamma(\overline{M})^2 \leq \max\big(\gamma(\overline{M}')^2, \gamma(\overline{M}'')^2 + \rho(\overline{M}')^2\big). \tag{6.4.4}$$

and:

$$\lambda^{[0]}(\overline{M})^2 \leq \max\big(\lambda^{[0]}(\overline{M}')^2, \lambda^{[0]}(\overline{M}'')^2 + \rho(\overline{M}')^2\big). \tag{6.4.5}$$

PROOF. Let R be a positive real number such that:

$$R^2 > \gamma(\overline{M}')^2 \tag{6.4.6}$$

and:

$$R^2 > \gamma(\overline{M}'')^2 + \rho(\overline{M}')^2. \tag{6.4.7}$$

To prove (6.4.4), it is enough to prove that $B(\overline{M}, R)$ generates the $\mathbb{Z}$-module M.

To achieve this, observe that, according to (6.4.7), we may find R'' and ρ in $\mathbb{R}_+^*$ such that:

$$R^2 = R''^2 + \rho^2, \tag{6.4.8}$$

$$R'' > \gamma(\overline{M}''), \tag{6.4.9}$$

and:

$$\rho > \rho(\overline{M}'). \tag{6.4.10}$$

Together with (6.4.8) and (6.4.10), Proposition 6.1.8 shows that every element of $B(\overline{M}'', R'')$ is the image by p of an element of $B(\overline{M}'', R)$. Moreover (6.4.9) implies that $B(\overline{M}'', R'')$ generates the $\mathbb{Z}$-module M''. Consequently $p(B(\overline{M}, R))$ generates the $\mathbb{Z}$-module M''.

The inequality (6.4.6) implies that $B(\overline{M}', R)$ generates the $\mathbb{Z}$-module M'. Since i maps $B(\overline{M}', R)$ into $B(\overline{M}, R)$, this shows that the $\mathbb{Z}$-module generated by $B(\overline{M}, R)$ contains $\operatorname{im} i = \ker p$. Since this $\mathbb{Z}$-module is mapped onto M'' by p, it coincides with M.

The estimate (6.4.5) is established by a similar argument that we leave to the reader. $\square$

COROLLARY 6.4.6. *If $\overline{E}$ is an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and if $\nu_{\overline{E}} : \overline{E} \to \overline{E}^{\mathrm{vect}}$ denotes its "vectorization," then the following equalities holds:*

$$\gamma(\overline{E}) = \gamma(\overline{E}^{\mathrm{vect}}) \tag{6.4.11}$$

and:

$$\lambda^{[0]}(\overline{E}) = \lambda^{[0]}(\overline{E}^{\mathrm{vect}}). \tag{6.4.12}$$

Moreover, for every Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$, the following equalities hold:

$$\gamma(\overline{M}) = \gamma(\overline{M}^{\vee\vee}) \tag{6.4.13}$$

and

$$\lambda^{[0]}(\overline{M}) = \lambda^{[0]}(\overline{M}^{\vee\vee}). \tag{6.4.14}$$

PROOF. The monotonicity of γ and $\lambda^{[0]}$ imply the inequalities:

$$\gamma(\overline{E}) \geq \gamma(\overline{E}^{\mathrm{vect}}) \quad \text{and} \quad \lambda^{[0]}(\overline{E}) \geq \lambda^{[0]}(\overline{E}^{\mathrm{vect}}).$$

The converse inequalities follow from the estimates (6.4.4) and (6.4.5) applied to the admissible short exact sequence (2.3.8) attached to $\nu_{\overline{E}}$:

$$0 \longrightarrow \overline{V} \hookrightarrow \overline{E} \xrightarrow{\nu_{\overline{E}}} \overline{E}^{\mathrm{vect}} \longrightarrow 0,$$

and from the vanishing of $\rho(\overline{V})$, already observed in (6.3.2) above.

The validity of (6.4.13) and (6.4.14) is established by a similar argument, using the canonical dévissage of $\overline{M}$ introduced in Subsection 2.2.4:

$$0 \longrightarrow \overline{M}_{\mathrm{ap}} \longrightarrow \overline{M} \xrightarrow{\delta_M} \overline{M}^{\vee\vee} \longrightarrow 0,$$

and the vanishing of $\rho(\overline{M}_{\mathrm{ap}})$ established in Proposition 6.3.2. $\square$

However the invariant γ and $\lambda^{[0]}$, even restricted to $\overline{\mathbf{Vect}}_{\mathbb{Z}}$, does not satisfy the subaddivity condition **SubAdd**. Actually, in an admissible short exact sequence of Euclidean lattices:

$$0 \longrightarrow \overline{E} \longrightarrow \overline{F} \longrightarrow \overline{G} \longrightarrow 0,$$

the invariant $\gamma(\overline{F})$ (resp. $\lambda^{[0]}(\overline{F})$) cannot be bounded from above in terms of $\gamma(\overline{E})$ and $\gamma(\overline{G})$ (resp. of $\lambda^{[0]}(\overline{E})$ and $\lambda^{[0]}(\overline{G})$) only as demonstrated by the following example.

EXAMPLE 6.4.7. For every integer $n \geq 2$, let us consider the admissible short exact sequence in $\overline{\mathbf{Vect}}_{\mathbb{Z}}$:

$$0 \longrightarrow \overline{\mathcal{O}}_{\mathbb{Z}}^{\oplus(n-1)} \xrightarrow{i_n} \overline{V}_n \xrightarrow{p_n} \overline{\mathcal{O}}_{\mathbb{Z}} \longrightarrow 0$$

defined by the Euclidean lattice $\overline{V}_n := (\mathbb{Z}^n, \|.\|_n)$, where:

$$\|(x_1, \ldots, x_n)\|_n^2 := \sum_{i=1}^{n-1} (x_i - x_n/2)^2 + x_n^2,$$

and by the morphisms:

$$i_n(x_1, \ldots, x_{n-1}) := (x_1, \ldots, x_{n-1}, 0) \quad \text{and} \quad p_n(x_1, \ldots, x_n) := x_n.$$

It is straightforward that:

$$\gamma(\overline{\mathcal{O}}_{\mathbb{Z}}^{\oplus(n-1)}) = \lambda^{[0]}(\overline{\mathcal{O}}_{\mathbb{Z}}^{\oplus(n-1)}) = 1 \quad \text{and} \quad \gamma(\overline{\mathcal{O}}_{\mathbb{Z}}) = \lambda^{[0]}(\overline{\mathcal{O}}_{\mathbb{Z}}) = 1.$$

Moreover, for every v in $V_n = \mathbb{Z}^n$ whose n-th component is not zero, we have:

$$\|v\|_n^2 \geq (n-1)/2 + 1 = (n+3)/2.$$

This easily implies:

$$\gamma(\overline{V}_n) = \lambda^{[0]}(\overline{V}_n) = \sqrt{(n+3)/2}.$$

6.4.1.4. *Bases with controlled size.* The monotonicity of $\lambda^{[0]}$ (Proposition 6.4.3) and the lifting result in Proposition 6.1.8 also admit the following consequence, which has been used to derive the characterizations of the object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ such that $\overline{E}^{\mathrm{vect}} = 0$ in Proposition 2.3.5.

PROPOSITION 6.4.8. *Let $\overline{E} := (E, \|.\|)$ be an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ with E a free $\mathbb{Z}$-module of finite rank N. For every $\varepsilon \in \mathbb{R}_+^*$, there exists a basis $(e_1, \dots, e_N)$ of E such that, for every $i \in \{1, \dots, N\}$:*

$$\|e_i\| \leq \alpha(N)\big(\lambda^{[0]}(\overline{E}) + \varepsilon\big), \tag{6.4.15}$$

where:

$$\alpha(N) := \begin{cases} 0 & \text{if } N = 0 \\ \sqrt{N+3}/2 & \text{if } N \geq 1. \end{cases}$$

When $\overline{E}$ is a Euclidean lattice, namely when the seminorm $\|.\|$ defining $\overline{E}$ is a norm, the conclusion of Proposition 6.4.8 holds with $\varepsilon = 0$. This follows from Proposition 6.4.8 by a straightforward finiteness argument.

In general, when $\|.\|$ is not a norm, it may happen that its conclusion does not holds when $\varepsilon = 0$. This is for instance the case when $\overline{E}^{\mathrm{vect}} = 0$ (in which case $\lambda^{[0]}(\overline{E}) = \gamma(\overline{E}) = 0$) and $\|.\| \neq 0$. Examples of such objects $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ have been constructed in paragraph 2.3.1.5.

PROOF OF PROPOSITION 6.4.8. The proposition is clear when $N = 0$. In general we proceed by induction on N. So we assume that N is a positive integer and that Proposition 6.4.8 holds when E has rank $N - 1$.

Consider an object $\overline{E} := (E, \|.\|)$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ with E a free $\mathbb{Z}$-module of rank N, and let choose λ in $(\lambda^{[0]}(\overline{E}), +\infty)$. We want to prove the existence of a basis $(e_1, \dots, e_N)$ of E such that, for $i \in \{1, \dots, N\}$:

$$\|e_i\| \leq \alpha(N)\lambda.$$

By the very definition of $\gamma(\overline{E})$, there exists a primitive vector e_N in E such that:

$$\|e_N\| < \lambda. \tag{6.4.16}$$

We may introduce the following admissible short exact sequence in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$:

$$0 \longrightarrow \overline{\mathbb{Z}e_N} \hookrightarrow \overline{E} \xrightarrow{p} \overline{E}' := \overline{E/\mathbb{Z}e_N} \longrightarrow 0. \tag{6.4.17}$$

As observed in (6.1.3), we have:

$$\rho(\overline{\mathbb{Z}e_N}) = \|e_N\|/2 < \lambda/2.$$

Moreover the $\mathbb{Z}$-module E' is free of rank $N - 1$, and according to Proposition 6.4.2,

$$\lambda^{[0]}(\overline{E}') \leq \lambda^{[0]}(\overline{E}) < \lambda.$$

Therefore, by our inductive assumption, there exists a basis $(e'_1, \dots, e'_{N-1})$ of E' such that:

$$\|e'_i\|_{\overline{E}'} \leq \alpha(N-1)\,\lambda \quad \text{for every } i \in \{1, \dots, N-1\}.$$

According to Proposition 6.1.8 applied to the short exact sequence (6.4.17), for every $i \in \{1, \dots, N-1\}$, there exists e_i in E such that:

$$p(e_i) = e'_i \quad \text{and} \quad \|e_i\|^2 \leq \|e'_i\|^2_{\overline{E}'} + (\lambda/2)^2.$$

Then $(e_1, \dots, e_N)$ is a basis of E. Moreover it satisfies:

$$\max_{1 \leq i \leq N} \|e_i\|^2 \leq \max\big(\alpha(N-1)^2 + 1/4, 1\big)\,\lambda^2 = \alpha(N)^2\,\lambda^2. \qquad \square$$

Observe that the conclusion of Proposition 6.4.8 implies the inequality:

$$\gamma(\overline{E}) \leq \sqrt{N+3}/2\, \lambda^{[0]}(\overline{E}).$$

We will establish a slightly sharper bound in Subsection 6.4.4. However this inequality is already basically optimal, as demonstrated by the following example.

EXAMPLE 6.4.9. For every integer $n \geq 4$, denote by $D_n^\vee$ the Euclidean lattice $D_n^\vee$ dual of the root lattice D_n. It may be realized as the lattice $\mathbb{Z}^n + \mathbb{Z}(1/2, \ldots, 1/2)$ inside $D_{n,\mathbb{R}}^\vee = \mathbb{R}^n$ equipped with the standard Euclidean norm.

Using this description, it is straightforward that the Euclidean lattice $D_n^\vee$ satisfies:

$$\gamma(D_n^\vee) = \sqrt{n}/2 \quad \text{and} \quad \lambda_n(D_n^\vee) = 1.$$

In [**CS91**, Section 7], Conway and Sloane describe the Voronoi cell $\mathcal{V}(D_n^\vee)$, and from this description deduce its covering radius:

$$\rho(D_n^\vee) = \begin{cases} \sqrt{n/8} & \text{if } n \geq 4 \text{ is even} \\ \sqrt{(2n-1)/16} & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

6.4.1.5. *The invariants $\underline{\lambda}^{[0]}$ and $\overline{\lambda}^{[0]}$, and $\underline{\gamma}$ and $\overline{\gamma}$.* Let us finally indicate some further properties of the invariants $\lambda^{[0]}$ and γ and of their lower and upper extensions. These properties provide some additional illustration of the general formalism of lower and upper extensions of invariants on $\overline{\mathbf{Coh}}_X$ introduced in Section 4.3. They will not be used in the sequel, and we shall leave the details of their proofs to the interested reader.

Since the invariant $\lambda^{[0]}$ (resp. γ) is monotonic on $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, hence on $\overline{\mathbf{Coh}}_Z$, we may consider the lower and upper extensions $\underline{\lambda}^{[0]}$ and $\overline{\lambda}^{[0]}$ (resp. $\underline{\gamma}$ and $\overline{\gamma}$) to $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ of the restriction of $\lambda^{[0]}$ (resp. of γ) to $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. They are defined by formulae (6.3.14) and (6.3.15) with ρ replaced by $\lambda^{[0]}$ (resp. by γ).

One easily sees that Proposition 6.1.2, which asserts the lower semicontinuity of the covering radius, still holds with $\lambda^{[0]}$ or γ instead of ρ. This implies the inequalities, for every object $\overline{M}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$:

$$\lambda^{[0]}(\overline{M}) \leq \overline{\lambda}^{[0]}(\overline{M}) \quad \text{and} \quad \gamma(\overline{M}) \leq \overline{\gamma}(\overline{M}).$$

The converse inequalities are straightforward consequences of the definition, and therefore we have:

$$\lambda^{[0]}(\overline{M}) = \overline{\lambda}^{[0]}(\overline{M}) \quad \text{and} \quad \gamma(\overline{M}) = \overline{\gamma}(\overline{M}).$$

As a straightforward consequence of the monotonicity of γ, for every object $\overline{M}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ the following inequalities holds:

$$\underline{\lambda}^{[0]}(\overline{M}) \leq \lambda^{[0]}(\overline{M}) \quad \text{and} \quad \underline{\gamma}(\overline{M}) \leq \gamma(\overline{M}). \tag{6.4.18}$$

Moreover Proposition 6.3.12 remains valid with $\underline{\rho}$ replaced by $\underline{\lambda}^{[0]}$ or $\underline{\gamma}$; namely:

$$\underline{\lambda}^{[0]}(\overline{M}) > 0 \Longleftrightarrow \underline{\gamma}(\overline{M}) > 0 \Longleftrightarrow \text{there exists a non-zero } \xi : \overline{M} \to \overline{\mathcal{O}}_{\mathbb{Z}} \text{ in } \mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}.$$

Finally the inequality in (6.4.18) may be strict: the Euclidean quasi-coherent sheaf $\overline{M}$ in Example 6.3.3.4 satisfies:

$$\underline{\lambda}^{[0]}(\overline{M}) = \underline{\gamma}(\overline{M}) = 0 \quad \text{and} \quad 1 \leq \underline{\lambda}^{[0]}(\overline{M}) \leq \gamma(\overline{M}).$$

6.4.2. Comparing $\gamma(\overline{M})$ and $\rho(\overline{M})$. Contrary to the covering radius $\rho(\overline{M})$ and to the last of the successive minima $\lambda^{[0]}(\overline{M})$, the invariant $\gamma(\overline{M})$ of Euclidean lattices does not seem to have been systematically investigated in the classical literature on Euclidean lattices.[12] However some of its basic properties follow directly from some well-known properties of the Voronoi cells of Euclidean lattices.

Indeed from the properties of the Voronoi cell of a Euclidean lattice $\overline{M}$ recalled in 6.2.1 (i) and (ii), we derive that every facet vector f of $\overline{M}$ satisfies:

$$\|f/2\| \leq \rho(\overline{M}).$$

Together with the property (iii) in 6.2.1, this establishes the estimates:

$$\gamma(\overline{M}) \leq \max_{f \in F} \|f\| \leq 2\rho(\overline{M}). \tag{6.4.19}$$

Inspired by this geometric argument, we may extend (part of) the inequality (6.4.19) to arbitrary Euclidean quasi-coherent sheaves:

PROPOSITION 6.4.10. *For every Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$, the following inequality holds in $[0, +\infty]$:*

$$\gamma(\overline{M}) \leq 2\rho(\overline{M}). \tag{6.4.20}$$

Combined with the obvious estimate (6.4.1), this implies:

$$\lambda^{[0]}(\overline{M}) \leq 2\rho(\overline{M}). \tag{6.4.21}$$

If $\overline{M}$ is a Euclidean lattice of rank n, (6.4.21) becomes the upper-bound

$$\lambda_n(\overline{M}) \leq 2\rho(\overline{M}), \tag{6.4.22}$$

on the last of the successive minima of the Euclidean lattice $\overline{M}$. This upper-bound appears in [**Cas71**, X1.3, p. 313, equation (2)], where it is established by a reasoning of a "finite-dimensional" nature,[13] and in [**MG02**, Theorem 7.9].

PROOF OF PROPOSITION 6.4.10. Let m be an element of $M_{/\mathrm{tor}}$ and let r be a positive real number such that $r > \rho(\overline{M})$. To establish (6.4.20), it is enough to show the existence of a finite family $(m_\alpha)_{\alpha \in A}$ of elements of $M_{/\mathrm{tor}}$ such that the following condition are satisfied:

$$m = \sum_{\alpha \in A} m_\alpha \quad \text{and} \quad \|m_\alpha\| \leq 2r \text{ for every } \alpha \in A. \tag{6.4.23}$$

To achieve this, consider a continuous map

$$c : [0, 1] \longrightarrow M_{\mathbb{R}}$$

into the seminormed space $(M_{\mathbb{R}}, \|.\|)$ such that $c(0) = 0$ and $c(1) = m$.[14]

Observe that, for any $t \in [0, 1]$, there exists $m(t)$ in $M_{/\mathrm{tor}}$ such that $\|c(t) - m(t)\| < r$. Using the continuity of c and the compactness of $[0, 1]$, we see that there exists a subdivision

$$t_0 = 0 < t_1 < \cdots < t_n = 1 \tag{6.4.24}$$

---

[12] See [**Cai03**] for some recent work involving the invariant $\gamma(\overline{M})$ and its generalizations $\gamma_i(\overline{M})$ introduced in Subsection 6.6.2 below.

[13] It may be summarized as follows. Let us choose a family $(f_1, \ldots, f_n)$ of vectors in $\frac{1}{2}M$ such that their class $([f_1], \ldots, [f_n])$ in $(\frac{1}{2}M)/M \simeq M \otimes \mathbb{F}_2$ is a $\mathbb{F}_2$-basis of $M \otimes \mathbb{F}_2$. This implies that $(f_1, \ldots, f_n)$ is $\mathbb{Q}$-linearly independent, and therefore a $\mathbb{R}$-basis of $M_{\mathbb{R}}$. Moreover, after possibly translating the f_i by elements in M, we may assume that they satisfy $\|f_i\| \leq \rho(\overline{M})$. Then $(2f_1, \ldots, 2f_n)$ is a $\mathbb{R}$-basis of $M_{\mathbb{R}}$ consisting of vectors in M of norms at most $2\rho(\overline{M})$.

[14] We may choose c defined by $c(t) := tm$, but an arbitrary path will do.

of this interval such that, for any $i \in \{1, \dots, n\}$, the element $m(t)$ may be chosen to be independent of t in $[t_{i-1}, t_i]$. In other words, there exists a subdivision (6.4.24) and a family $(m_i)_{1 \leq i \leq n}$ of elements of $M_{/\mathrm{tor}}$ such that, for every $i \in \{1, \dots, n\}$ and every $t \in [t_{i-1}, t_i]$,

$$\|c(t) - m_i\| < r.$$

Therefore we may write:

$$m = c(1) = (c(1) - m_n) + \sum_{i=1}^{n} (m_i - m_{i-1}) + m(0).$$

The $n+2$ terms of this sum belong to $M_{/\mathrm{tor}}$. Moreover,

$$\|c(1) - m_n\| = \|c(t_n) - m_n\| < r,$$

$$\|m_0\| = \|c(t_0) - m_0\| < r,$$

and, for every $i \in \{1, \dots, n\}$,

$$\|m_i - m_{i-1}\| \leq \|m_i - c(t_{i-1})\| + \|c(t_{i-1}) - m_{i-1})\| < 2r.$$

This establishes the existence of a decomposition of m as in (6.4.23) and completes the proof. $\square$

To derive comparison estimates involving the invariants γ and ρ, it is often useful to use the following observation:

PROPOSITION 6.4.11. *Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf, and let N be a $\mathbb{Z}$-submodule of M and $\overline{N} := (N, \|.\|_{|N_\mathbb{R}})$ the Euclidean quasi-coherent sheaf it defines.*

If $N_\mathbb{R}$ is dense in $\overline{M}_\mathbb{R} := (M_\mathbb{R}, \|.\|)$, then:

$$\gamma(\overline{M}) \leq \max(\gamma(\overline{N}), \rho(\overline{N})). \tag{6.4.25}$$

For instance, together with Proposition 6.4.10 applied to $\overline{N}$, the estimate (6.4.25) implies the following amplified version of Proposition 6.4.10:

COROLLARY 6.4.12. *With the notation of Proposition 6.4.11, the following inequality holds:*

$$\gamma(\overline{M}) \leq 2\rho(\overline{N}).$$

PROOF OF PROPOSITION 6.4.11. For any $\varepsilon \in \mathbb{R}_+^*$, N is generated by $B(\overline{N}; \gamma(\overline{N}) + \varepsilon)$. Moreover, for every $m \in M_{/\mathrm{tor}}$, there exists v in $N_\mathbb{R}$ such that

$$\|m - v\| < \varepsilon.$$

In turn, there exists n in $N_{/\mathrm{tor}}$ such that

$$\|v - n\| < \rho(\overline{N}) + \varepsilon.$$

Then the element $m - n$ of $M_{/\mathrm{tor}}$ satisfies:

$$\|m - n\| < \rho(\overline{N}) + 2\varepsilon.$$

This shows that $B(\overline{N}; \gamma(\overline{N}) + \varepsilon) \cup B(\overline{M}; \rho(\overline{N}) + 2\varepsilon)$, and a fortiori $B(\overline{M}; \max(\gamma(\overline{N}), \rho(\overline{N})) + 2\varepsilon)$, generate the $\mathbb{Z}$-module M. $\square$

6.4.3. Comparing $\rho(\overline{M}')$ and $\lambda^{[0]}(\overline{M})$.

PROPOSITION 6.4.13. *Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf. For any Euclidean semi-norm $\|.\|'$ on $M_{\mathbb{R}}$, we have:*

$$\rho(M, \|.\|') \leq \sqrt{\mathrm{Tr}(\|.\|'^2/\|.\|^2)}/2\; \lambda^{[0]}(M, \|.\|). \tag{6.4.26}$$

In particular, if $\lambda^{[0]}(\overline{M})$ is finite and if $\|.\|'$ is Hilbert-Schmidt relatively to $\|.\|$, then $\rho(M, \|.\|')$ is finite.

PROOF. To prove (6.4.26), we may and will assume that $\lambda^{[0]}(\overline{M})$ is finite.

Let us choose R in $(\lambda^{[0]}(\overline{M}), +\infty)$. By definition of $\lambda^{[0]}(\overline{M})$, there exists a sequence $(m_k)_{k\in\mathbb{N}}$ of elements of $B(\overline{M}; R)$ such that the $\mathbb{R}$-vector space

$$M_{\infty,\mathbb{R}} := \sum_{k\in\mathbb{N}} \mathbb{R} m_k$$

is dense in $(M_{\mathbb{R}}, \|.\|)$. Then we may apply Corollary 6.1.7 to $\overline{M}' := (M, \|.\|')$ and to the sequence $(M_i)_{i\in\mathbb{N}}$ of $\mathbb{Z}$-submodules of M defined by:

$$M_i := \sum_{k=0}^{i} \mathbb{Z} m_k.$$

This establishes the upper-bound:

$$\rho(\overline{M}')^2 \leq \sum_{i=0}^{+\infty} \rho(\overline{M_i/M_{i-1}}')^2. \tag{6.4.27}$$

Clearly, for any integer $i \geq 1$, the $\mathbb{Z}$-module M_i/M_{i-1} is generated by the class $\overline{m}_i$ of m_i. As observed in (6.1.3) above, the covering radius of the cyclic Euclidean coherent sheaf $\overline{M_i/M_{i-1}}' := (M_i/M_{i-1}, \|.\|'_i)$ may be expressed in terms of the norm of this generator:

$$\rho(\overline{M_i/M_{i-1}}') = \|\overline{m}_i\|'_i/2, \tag{6.4.28}$$

and consequently:

$$4\rho(\overline{M}')^2 \leq \sum_{i=0}^{+\infty} \|\overline{m}_i\|_i'^2/4. \tag{6.4.29}$$

Moreover, according to Corollary A.3.4, we have:

$$\sum_{i=0}^{+\infty} \|\overline{m}_i\|_i'^2 \leq \mathrm{Tr}\,(\|.\|'^2/\|.\|^2) \sup_{0\leq i<+\infty} \|m_i\|^2 \leq \mathrm{Tr}\,(\|.\|'^2/\|.\|^2)\, R^2. \tag{6.4.30}$$

The estimate (6.4.26) follows from (6.4.29) and (6.4.30), since R is arbitrary in $(\lambda^{[0]}(\overline{M}), +\infty)$. □

REMARK 6.4.14. For later reference, observe that the proof of Proposition 6.4.13 establishes that, *if a Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$ satisfies:*

$$\lambda^{[0]}(M, \|.\|) < +\infty,$$

and if a Euclidean seminorm $\|.\|'$ on $M_{\mathbb{R}}$ is Hilbert-Schmidt with respect to $\|.\|$, then there exists a $\mathbb{Z}$-submodule N of M such that $N_{\mathbb{R}}$ is dense in $(M_{\mathbb{R}}, \|.\|)$ and that the Euclidean quasi-coherent sheaf $\overline{N}' := (N, \|.\|'_{|N_{\mathbb{R}}}$ is ρ^2-summable.

When $\overline{M}$ is a Euclidean lattice of rank n, we may apply Proposition 6.4.13 with $\|.\|' := \|.\|$. Then we have:

$$\mathrm{Tr}\,(\|.\|'^2/\|.\|^2)) = n,$$

and the upper-bound (6.4.26) becomes:

$$\rho(\overline{M}) \leq \sqrt{n}/2\,\lambda_n(\overline{M}). \tag{6.4.31}$$

This estimate is established in [**MG02**, Theorem 7.9] as an application of the so-called *nearest plane* algorithm, introduced by Babai in [**Bab86**]; see also [**MG02**, Section 2.3]. The introduction of the sequence $\overline{N}_0 = \overline{\{0\}} \hookrightarrow \overline{N}_1 \hookrightarrow \overline{N}_2 \hookrightarrow \dots$ of Euclidean coherent subsheaves of $\overline{M}$ and of the subquotients $(\overline{N_k/N_{k-1}})_{k\geq 1}$ may be seen as an infinite-dimensional avatar of Babai's construction.

Observe also that Example 6.4.9 shows that the estimate (6.4.31), and therefore (6.4.26), is optimal up-to a factor 2.

6.4.4. Comparing $\gamma(\overline{M}')$ and $\lambda^{[0]}(\overline{M})$. By combining Proposition 6.4.10 and Proposition 6.4.13, we obtain that, with the notation of Proposition 6.4.13, the following inequality holds:

$$\gamma(\overline{M}') \leq \sqrt{\mathrm{Tr}\,(\|.\|'^2/\|.\|^2)}\,\lambda^{[0]}(\overline{M}). \tag{6.4.32}$$

It is actually possible to establish a slightly stronger inequality by means of Proposition 6.4.11:

COROLLARY 6.4.15. *Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf. For every Euclidean seminorm $\|.\|'$ over $M_{\mathbb{R}}$, the following estimate holds:*

$$\gamma(\overline{M}') \leq \max\left(\sup(\|.\|'/\|.\|), \sqrt{\mathrm{Tr}\,(\|.\|'^2/\|.\|^2)}/2\right)\,\lambda^{[0]}(\overline{M}). \tag{6.4.33}$$

PROOF. Let us consider a Euclidean seminorm $\|.\|'$ that is Hilbert-Schmidt relatively to $\|.\|$, and R a real number in $(\lambda^{[0]}(\overline{M}), +\infty)$.

Let introduce the $\mathbb{Z}$-submodule N of M generated by $B(\overline{M}; R)$. According to the definition of $\lambda^{[0]}(\overline{M})$, the $\mathbb{R}$-vector space $N_{\mathbb{R}}$ is dense in $\overline{M}_{\mathbb{R}} := (M_{\mathbb{R}}, \|.\|)$, and therefore in $\overline{M}'_{\mathbb{R}} := (M_{\mathbb{R}}, \|.\|')$. Proposition 6.4.11, applied to $\overline{M}'$ and N, shows that:

$$\gamma(\overline{M}') \leq \max(\gamma(\overline{N}'), \rho(\overline{N}')). \tag{6.4.34}$$

where $\overline{N}' := (N, \|.\|'_{|N_R})$. Moreover, by the very definition of the submodule N, we have:

$$\gamma(\overline{N}') \leq \sup(\|.\|'/\|.\|)\,\gamma(\overline{N}) \leq \sup(\|.\|'/\|.\|)\,R, \tag{6.4.35}$$

and, according to Proposition 6.4.13, we have:

$$\rho(\overline{N}') \leq \frac{1}{2}\sqrt{\mathrm{Tr}\,(\|.\|'^2/\|.\|^2)}\,\lambda^{[0]}(\overline{N}) \leq \frac{1}{2}\sqrt{\mathrm{Tr}\,(\|.\|'^2/\|.\|^2)}\,R. \tag{6.4.36}$$

The estimates (6.4.34), (6.4.35), and (6.4.36) imply the upper-bound (6.4.33) on $\gamma(\overline{M}')$. □

Applied to a Euclidean lattice $\overline{M}$ and to the Euclidean (semi)norm $\|.\|' = \|.\|$, Corollary 6.4.15 becomes:

COROLLARY 6.4.16. *For every Euclidean lattice $\overline{M}$ of rank n, the following inequality holds:*

$$\gamma(\overline{M}) \leq \max(1, \sqrt{n}/2)\,\lambda_n(\overline{M}). \tag{6.4.37}$$

From the inequality (6.4.37), we recover that, when $n \leq 4$, the invariants $\gamma(\overline{M})$ and $\lambda_n(\overline{M})$ coincide. This is trivial when $n = 1$, and follows from some specific features of the reduction theory of Euclidean lattices of low rank, which go back to Lagrange, Gauss, and Julia, when $n = 2$, 3, and 4; see for instance [**vdW56**, Â§7].

Moreover for every $n \geq 5$, the inequality

$$\gamma(\overline{M}) \leq (\sqrt{n}/2)\, \lambda_n(\overline{M})$$

may become an equality for some Euclidean lattice $\overline{M}$ of rank n, as demonstrated by the lattice $D_n^\vee$ described in Example 6.4.9.

This shows the constant $\max(1, \sqrt{n}/2)$ in the inequality (6.4.37) is optimal, and accordingly the inequality (6.4.33) is in a sense optimal.

6.5. Euclidean Quasi-coherent Sheaves with Eventually Vanishing ρ^2, $\lambda^{[0]}$, or γ

6.5.1. The invariants $\mathrm{ev}\lambda^{[0]}$, $\mathrm{ev}\gamma$, and $\mathrm{ev}\rho$.

6.5.1.1. As recalled in the introduction of this chapter, if φ is one of the invariant $\lambda^{[0]}$, γ, or ρ, the monotonicity of φ allows one to define a new invariant

$$\mathrm{ev}\varphi : \mathbf{q\overline{Coh}}_{\mathbb{Z}} \longrightarrow [0, +\infty]$$

by the formula:

$$\mathrm{ev}\varphi(\overline{M}) := \lim_{C \in \mathrm{coh}(M)} \varphi(\overline{M/C}) = \inf_{C \in \mathrm{coh}(M)} \varphi(\overline{M/C}),$$

where the limit and the infimum are taken over the directed set $(\mathrm{coh}(M), \subseteq)$ of finitely generated submodules of M; see Section 4.4.

If $(C_i)_{i \in \mathbb{N}}$ is an increasing sequence of finitely generated $\mathbb{Z}$-submodules of M such that

$$\bigcup_{i \in \mathbb{N}} C_i = M,$$

the family $(C_i)_{i \in \mathbb{N}}$ is cofinal in $(\mathrm{coh}(M), \subseteq)$, and therefore:

$$\mathrm{ev}\varphi(\overline{M}) = \lim_{i \to +\infty} \varphi(\overline{M/C_i}).$$

Like φ itself, the invariant $\mathrm{ev}\varphi$ satisfies the monotonicity condition $\mathbf{Mon}^1$, and is 1-homogeneous. We refer to Subsection 4.4.2 for a more complete discussion of the properties of φ inherited by $\mathrm{ev}\varphi$, and recall that the invariant φ is said to be *eventually vanishing* on some Euclidean quasi-coherent sheaf $\overline{M}$ when $\mathrm{ev}\varphi(\overline{M})$ vanishes.

We shall be interested in this eventual vanishing property notably when φ is the covering radius ρ or the invariant γ. When $\mathrm{ev}\rho(\overline{M})$ vanishes, we shall say that $\overline{M}$ *has eventually vanishing covering radius*; when $\mathrm{ev}\gamma(\overline{M})$ vanishes, we shall say that $\overline{M}$ *is eventually generated by small sections.*

6.5.1.2. Let us indicate various relations between by the eventual vanishing and the finiteness of the three invariants $\lambda^{[0]}$, γ and ρ that are consequences of their properties established in Section 6.4.

The estimates relating these three invariants:

$$\lambda^{[0]}(\overline{M}) \leq \gamma(\overline{M}) \leq 2\rho(\overline{M})$$

immediately imply:

$$\mathrm{ev}\lambda^{[0]}(\overline{M}) \leq \mathrm{ev}\gamma(\overline{M}) \leq 2\, \mathrm{ev}\rho(\overline{M}).$$

In turn, this implies:

PROPOSITION 6.5.1. *For every Euclidean quasi-coherent sheaf $\overline{M}$, the following implications hold:*

$$\mathrm{ev}\rho(\overline{M}) = 0 \Longrightarrow \mathrm{ev}\gamma(\overline{M}) = 0 \Longrightarrow \mathrm{ev}\lambda^{[0]}(\overline{M}) = 0. \tag{6.5.1}$$

The following proposition relates the finiteness and the eventual vanishing for each of the invariants $\lambda^{[0]}$, γ, or ρ.

PROPOSITION 6.5.2. *Let φ be any of the invariants $\lambda^{[0]}$, γ, or ρ.*

For every Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$, and for every Euclidean seminorm $\|.\|'$ on $M_{\mathbb{R}}$ that is compact relatively to $\|.\|$, the following implications hold, where $\overline{M}'$ denotes the Euclidean quasi-coherent sheaf $(M, \|.\|')$:

$$\mathrm{ev}\varphi(\overline{M}) < +\infty \Longrightarrow \varphi(\overline{M}) < +\infty, \tag{6.5.2}$$

and:

$$\varphi(\overline{M}) < +\infty \Longrightarrow \mathrm{ev}\varphi(\overline{M}') = 0. \tag{6.5.3}$$

PROOF. When $\mathrm{ev}\varphi(\overline{M})$ is finite, there exists C in $\mathrm{coh}(M)$ such that $\varphi(\overline{M/C})$ is finite. Then Proposition 6.4.5 (resp. Proposition 6.1.6) when φ is $\lambda^{[0]}$ or γ (resp. when φ is ρ) implies the finiteness of $\varphi(\overline{M})$, since $\rho(\overline{C})$ is finite.

Let $(C_i)_{i\in\mathbb{N}}$ be an exhaustive filtration of M by submodules in $\mathrm{coh}(M)$. For every $i \in \mathbb{N}$, we may consider the quotients $\overline{M/C_i}$ and $\overline{M/C_i}'$ of $\overline{M}$ and $\overline{M}'$ and define:

$$\varepsilon_i := \sup(\|.\|_{\overline{M/C_i}'}/\|.\|_{\overline{M/C_i}}).$$

Then we have:

$$\|.\|_{\overline{M/C_i}'} \leq \varepsilon_i \, \|.\|_{\overline{M/C_i}},$$

and, from the 1-homogeneity and the monotonicity of the invariant φ, we deduce the estimates:

$$\varphi(\overline{M/C_i}') \leq \varepsilon_i \, \varphi(\overline{M/C_i}) \leq \varepsilon_i \, \varphi(\overline{M}).$$

Moreover, according to Proposition A.1.2, the compactness of $\|.\|'$ with respect to $\|.\|$ implies:

$$\lim_{i\to+\infty} \varepsilon_i = 0,$$

and therefore, when $\varphi(\overline{M})$ is finite, we have:

$$\mathrm{ev}\varphi(\overline{M}) = \lim_{i\to+\infty} \varphi(\overline{M/C_i}) = 0. \qquad \square$$

The following proposition shows that, provided one replaces the Euclidean seminorm $\|.\|$ defining a Euclidean quasi-coherent sheaf by a Euclidean seminorm $\|.\|'$ that is Hilbert-Schmidt with respect to $\|.\|$, one may establish some converse to the implications (6.5.1).

PROPOSITION 6.5.3. *For every Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$ and for every Euclidean seminorm $\|.\|'$ on $M_{\mathbb{R}}$ that is Hilbert-Schmidt relatively to $\|.\|$, the following implication holds, where $\overline{M}'$ denotes the Euclidean quasi-coherent sheaf $(M, \|.\|')$:*

$$\lambda^{[0]}(\overline{M}) < +\infty \Longrightarrow \mathrm{ev}\rho(\overline{M}') = 0.$$

PROOF. According to Proposition A.4.2, for every Euclidean seminorm $\|.\|'$ over $M_{\mathbb{R}}$ that is Hilbert-Schmidt relatively to $\|.\|$, we may choose a Euclidean seminorm $\|.\|^\sim$ over $M_{\mathbb{R}}$ such that $\|.\|^\sim$ is Hilbert-Schmidt relatively to $\|.\|$, and $\|.\|'$ is compact relatively to $\|.\|^\sim$.

Then Proposition 6.4.13 shows that $\rho(M, \|.\|^\sim)$ is finite, and Proposition 6.5.2 that $\mathrm{ev}\rho(\overline{M}')$ vanishes. $\square$

6.5.2. The invariants $\mathrm{ev}\gamma(\overline{M})$ and $\inf_{N\in\mathrm{coft}(M)} \gamma(\overline{N})$. The following proposition provides an alternative interpretation of the invariant $\mathrm{ev}\gamma$.

PROPOSITION 6.5.4. *For every Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$ the following estimates hold:*

$$\inf_{N\in\mathrm{coft}(M)} \gamma(\overline{N})/2 \leq \mathrm{ev}\gamma(\overline{M}) \leq \inf_{N\in\mathrm{coft}(M)} \gamma(\overline{N}). \tag{6.5.4}$$

A key point in the proof of Proposition 6.5.4 will be the following lemma.

LEMMA 6.5.5. *Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf, and r a positive real number.*

If there exists a finitely generated $\mathbb{Z}$-submodule C of M such that $\gamma(\overline{M/C}) < r$, then there exists a finite subset F of M such that $F \cup B(\overline{M}; 2r)$ generates the $\mathbb{Z}$-module M.

PROOF. Let C be an element of $\mathrm{coh}(M)$ such that $\gamma(\overline{M/C}) < r$.

Let us first assume that the Euclidean seminorm $\|.\|$ is actually a norm. We shall denote by $\langle .,. \rangle$ the scalar product that defines $\|.\|$.

Observe that $\overline{C}_{/\mathrm{tor}} := (C_{/\mathrm{tor}}, \|.\|_{|C_\mathbb{R}})$ is a Euclidean lattice, and in particular its covering radius $\rho(\overline{C}_{/\mathrm{tor}}) = \rho(\overline{C})$ is finite.

Since $C_\mathbb{R}$ is a finite-dimensional $\mathbb{R}$-subvector space of $M_\mathbb{R}$, we may consider the orthogonal projection:

$$p : M_\mathbb{R} \longrightarrow C_\mathbb{R}.$$

If $(e_i)_{1 \leq i \leq n}$ is an orthonormal basis of $C_\mathbb{R}$, the map p may be defined by the equality:

$$p(v) := \sum_{i=1}^{n} \langle v, e_i \rangle e_i, \tag{6.5.5}$$

valid for any v in $M_\mathbb{R}$. Moreover, it satisfies:

$$\|v\|^2 = \|p(v)\|^2 + \|v - p(v)\|^2, \tag{6.5.6}$$

and

$$\|v - p(v)\| = \|[v]\|_{\overline{M/C}}, \tag{6.5.7}$$

where $[v]$ denotes the class of v in $M_\mathbb{R}/C_\mathbb{R} \simeq (M/C)_\mathbb{R}$. In particular, we have:

$$\|v\| \leq \|p(v)\| + \|[v]\|_{\overline{M/C}}.$$

Let us choose two positive real numbers r' and ε such that:

$$\gamma(\overline{M/C}) < r' < r \quad \text{and} \quad \varepsilon + 2r' \leq 2r.$$

Let us choose a family $(\overline{m}_i)_{i \in \mathbb{N}}$ of generators of M/C such that, for every $i \in I$,

$$\|\overline{m}_i\|_{\overline{M/C}} \leq r'.$$

For every $i \in I$, we may choose an element m_i in M such that its class $[m_i]$ in M/C coincides with $\overline{m}_i$. After possibly adding to it a suitable element of C, we may actually assume that the following upper-bound is satisfied:[15]

$$\|p(m_i)\| \leq \rho(\overline{C}). \tag{6.5.8}$$

The compactness of the closed ball $B(\overline{C}_\mathbb{R}; \rho(\overline{C}))$ in $C_\mathbb{R}$ implies the existence of a non-negative integer n and of a finite partition $(I_j)_{1 \leq j \leq n}$ of I in disjoint non-empty subsets I_j such that, for every j in $\{1, \ldots, n\}$, the elements $(p(m_i))_{i \in I_f}$ lie in a subset of diameter at most ε in the normed vector space $(C_\mathbb{R}, \|.\|_{|C_\mathbb{R}})$.

For every j in $\{1, \ldots, n\}$, let us choose an element $\iota(j)$ of I_j, and let us define, for every $i \in I_j$:

$$\tilde{m}_i := m_i - m_{\iota(j)}.$$

Clearly the elements $(m_{\iota(j)})_{1 \leq j \leq n}$ and $(\tilde{m}_i)_{i \in I}$ generates the $\mathbb{Z}$-module M. Moreover, if i belongs to I_j, we have:

$$\|\tilde{m}_i\| \leq \|p(\tilde{m}_i)\| + \|[\tilde{m}_i]\|_{\overline{M/C}} = \|p(m_i) - p(m_{\iota(j)}\| + \|\overline{m}_i - \overline{m}_{\iota(j)}\|_{\overline{M/C}} \leq \varepsilon + 2r' \leq 2r.$$

[15] For simplicity, we still write m_i for the image $m_{i\mathbb{R}}$ of m_i in the additive subgroup $M_{/\mathrm{tor}}$ of $M_\mathbb{R}$.

Therefore, if we define F as the set of the $m_{\iota(j)}$ for $j \in \{1, \ldots, n\}$, then $F \cup B(\overline{M}; 2r)$ generates the $\mathbb{Z}$-module M.

The previous reasoning extends with the following minor modifications to the situation where $\|.\|$ is an arbitrary Euclidean semi-norm:

(1) Since C is finitely generated, its covering radius

$$\rho(\overline{C}_{/\mathrm{tor}}) = \rho(\overline{C}) := \inf\{R \in \mathbb{R}_+^* \mid C_{/\mathrm{tor}} + B(\overline{C}; R) = C_{\mathbb{R}}\} \tag{6.5.9}$$

is still finite, although the infimum in the right-hand side of (6.5.9) is possibly not a minimum.

(2) We may choose a supplement $\tilde{C}_{\mathbb{R}}$ of the $\mathbb{R}$-subvector space

$$\ker \|.\|_{|C_{\mathbb{R}}} := \{v \in C_{\mathbb{R}} \mid \|v\| = 0\}$$

in $C_{\mathbb{R}}$ and an orthonormal basis $(e_i)_{1 \leq i \leq n}$ of the Euclidean vector space $(\tilde{C}_{\mathbb{R}}, \|.\|_{|\tilde{C}_{\mathbb{R}}})$, and define a $\mathbb{R}$-linear map

$$p : M_{\mathbb{R}} \longrightarrow \tilde{C}_{\mathbb{R}} \subseteq C_{\mathbb{R}}$$

by the formula (6.5.5). Then the relations (6.5.6) and (6.5.7) still hold.

(3) We may assume that, instead of (6.5.8), the m_i satisfy the upper-bounds:

$$\|p(m_i)\| \leq \tilde{\rho}, \tag{6.5.10}$$

where $\tilde{\rho}$ denotes an arbitrary real number in the interval $(\rho(\overline{C}), +\infty)$, and use the compactness of the closed ball $B(\tilde{C}_{\mathbb{R}}, \|.\|_{|\tilde{C}_{\mathbb{R}}}; \tilde{\rho})$ instead of the one of $B(C_{\mathbb{R}}, \|.\|_{|C_{\mathbb{R}}}; \rho(\overline{C}))$. □

Proof of Proposition 6.5.4. Observe that, with the notation of Lemma 6.5.5, the $\mathbb{Z}$-submodule N of M generated by $B(\overline{M}; 2r)$ belongs to $\mathrm{coft}(M)$ and satisfies $\gamma(\overline{N}) \leq 2r$. Consequently Lemma 6.5.5 yields the following inequality:

$$\mathrm{ev}\gamma(\overline{M}) := \inf_{C \in \mathrm{coh}(M)} \gamma(\overline{M/C}) \geq \inf_{N \in \mathrm{coft}(M)} \gamma(\overline{N})/2. \tag{6.5.11}$$

Moreover, if N is an element of $\mathrm{coft}(M)$, then there exists C in $\mathrm{coh}(M)$ such that $C + N = M$. Then the morphism

$$f : \overline{N} \hookrightarrow \overline{M} \twoheadrightarrow \overline{M/C}$$

in $\mathbf{q\overline{Coh}}_{\mathbb{Z}}^{\leq 1}(\overline{N}, \overline{M/C})$, defined as the composition of the inclusion morphism (from $\overline{N}$ to $\overline{M}$) and of the quotient map (from $\overline{M}$ to $\overline{M/C}$), satisfies:

$$f(N) = M/C.$$

Consequently, according to Proposition 6.4.2, we have:

$$\gamma(\overline{N}) \geq \gamma(\overline{M/C}).$$

This establishes the inequality:

$$\inf_{N \in \mathrm{coft}(M)} \gamma(\overline{N}) \geq \inf_{C \in \mathrm{coh}(M)} \gamma(\overline{M/C}) =: \mathrm{ev}\gamma(\overline{M}). \tag{6.5.12}$$ □

6.5.3. Euclidean quasi-coherent sheaves with eventually vanishing covering radius, or eventually generated by small sections. The following proposition spells out the geometric meaning of the property, for a Euclidean quasi-coherent sheaf, to have eventually vanishing covering radius.

Proposition 6.5.6. *For every Euclidean quasi-coherent sheaf $\overline{M}$, the following conditions are equivalent:*

(i) the Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$ has eventually vanishing covering radius;

(ii) for every $\eta \in \mathbb{R}_+^*$, there exists a finitely generated $\mathbb{Z}$-submodule C of M such that

$$M_{\mathbb{R}} = C_{\mathbb{R}} + M_{/\mathrm{tor}} + B(\overline{M}_{\mathbb{R}}; \eta);$$

(iii) for every $\eta \in \mathbb{R}_+^*$, there exists a finitely generated $\mathbb{R}$-vector subspace V of $M_{\mathbb{R}}$ such that

$$M_{\mathbb{R}} = V + M_{/\mathrm{tor}} + B(\overline{M}_{\mathbb{R}}; \eta).$$

PROOF. The equivalence of (i) and (ii) is a consequence of the following straightforward observation:

LEMMA 6.5.7. *For every $\mathbb{Z}$-submodule N of M and every $\varepsilon \in \mathbb{R}_+$, the following two conditions are equivalent:*

(i)' the quotient $\overline{M/N}$ satisfies $\rho(\overline{M/N}) \leq \varepsilon$;
(ii)' for every $\eta \in (\varepsilon, +\infty)$, we have: $M_{\mathbb{R}} = N_{\mathbb{R}} + M_{/\mathrm{tor}} + B(\overline{M}_{\mathbb{R}}; \eta)$. □

The implication $(ii) \Rightarrow (iii)$ is clear, and the converse implication $(iii) \Rightarrow (ii)$ follows from the fact that any finite subset of $M_{\mathbb{R}}$ is contained in $C_{\mathbb{R}}$, for a suitable finitely generated $\mathbb{Z}$-submodule C of M. □

In a similar vein, Proposition 6.5.4 leads to criteria for the vanishing of the invariant $\mathrm{ev}\gamma(\overline{M})$ attached to some Euclidean quasi-coherent sheaf $\overline{M}$:

PROPOSITION 6.5.8. *For every Euclidean quasi-coherent sheaf $\overline{M}$, the following conditions are equivalent:*

(i) $\overline{M}$ is eventually generated by small sections;
(ii) for every $\eta \in \mathbb{R}_+^*$, there exists a finite subset F_η of M such that $F_\eta \cup B(\overline{M}; \eta)$ generates the $\mathbb{Z}$-module M;
(iii) there exists A in $\mathbb{N} \cup \{+\infty\}$ and a sequence $(m_k)_{0 \leq k < A}$ of elements of M which generate the $\mathbb{Z}$-module M and satisfies:

$$\lim_{k \to +\infty} \|m_k\| = 0 \quad \text{if } A = +\infty. \tag{6.5.13}$$

The terminology "eventually generated by small sections" used to express the vanishing of $\mathrm{ev}\gamma$ has actually been chosen to reflect the content of conditions (ii) or (iii) in this proposition.

PROOF. Condition (ii) is clearly equivalent to the vanishing of $\inf_{N \in \mathrm{coft}(M)} \gamma(\overline{N})$, and therefore to (i) by Proposition 6.5.4.

Let us assume that (iii) is satisfied. If A is finite, then (ii) is satisfied with F_η the set with elements the m_k, $0 \leq k < A$, for every $\eta \in \mathbb{R}_+^*$. When A is $+\infty$, for every $\eta \in \mathbb{R}_+^*$, we may find $n(\eta) \in \mathbb{N}$ such that:

$$k \geq n(\eta) \Longrightarrow \|m_k\| < \eta,$$

and define F_η as the set with elements the m_k, $0 \leq k < n(\eta)$. Then $F_\eta \cup B(\overline{M}; \eta)$ contains all the m_k, $k \in \mathbb{N}$, and therefore generates the $\mathbb{Z}$-module M. This shows that (ii) is satisfied.

Finally, let us assume that (ii) is satisfied, and let us show that (iii) is satisfied. To achieve this, for every positive integer n, let us choose a finite subset $F_{1/n}$ of M such that $F_{1/n} \cup B(\overline{M}; 1/n)$ generates the $\mathbb{Z}$-module M. After possibly increasing the $F_{1/n}$, we may and will assume that their union $\bigcup_{n \geq 1} F_{1/n}$ generates the $\mathbb{Z}$-module M.

We may choose a finite sequence $(m_1^0, \ldots, m_{n_0}^0)$ that enumerates the elements of F_1, and for every positive integer k, a finite sequence $(m_1^k, \ldots, m_{n_k}^k)$ in $B(\overline{M}, 1/k)$ that, together with $F_{1/k}$, generates a $\mathbb{Z}$-submodule of M containing $F_{1/(k+1)}$. Then the sequence

$$(m_1^0, \cdots, m_{n_0}^0, m_1^1, \cdots, m_{n_1}^1, \cdots, m_1^k, \cdots, m_{n_k}^k, \cdots)$$

defined by concatenation of these finite sequences is easily seen to generate the $\mathbb{Z}$-module M and to satisfy the condition (6.5.13). □

6.6. The Invariants λ_i and $\lambda^{[i]}$, γ_i and $\gamma^{[i]}$, and ρ_i and $\rho^{[i]}$

6.6.1. The successive minima λ_i and $\lambda^{[i]}$. Let $\overline{M}$ be a Euclidean quasi-coherent sheaf.

DEFINITION 6.6.1. The *successive minima* of $\overline{M}$ are the elements $\lambda_i(\overline{M})$ and $\lambda^{[i]}(\overline{M})$ of $[0, +\infty]$ defined as follows, for any nonnegative integer i:

(i) the invariant $\lambda_i(\overline{M}) \in [0, +\infty]$ is the infimum of the set of those $R \in \mathbb{R}_+^*$ such that the $\mathbb{R}$-vector space generated by $B(\overline{M}_{/\mathrm{tor}}; R)$ has dimension at least i or, equivalently, the $\mathbb{Z}$-submodule of $M_{/\mathrm{tor}}$ generated by $B(\overline{M}_{/\mathrm{tor}}; R)$ has rank at least i ;

(ii) the invariant $\lambda^{[i]}(\overline{M}) \in [0, +\infty]$ is the infimum of the set of those $R \in \mathbb{R}_+^*$ such that the closure in $\overline{M}_{\mathbb{R}}$ of the $\mathbb{R}$-vector space generated by $B(\overline{M}_{/\mathrm{tor}}; R)$ has codimension at most i.

When $i = 0$, the invariant $\lambda^{[i]}$ defined in Definition 6.6.1 (ii) coincides with the invariant $\lambda^{[0]}$ introduced in Definition 6.4.1. Definition 6.6.1 would actually make sense when $\overline{M} = (M, \|.\|)$ is a pair consisting of an arbitrary $\mathbb{Z}$-module M and of a pseudonorm $\|.\|$ on $M_{\mathbb{R}}$. Note that $\lambda_i(\overline{M})$ is finite if and only if i is at most the dimension of the real vector space $M_{\mathbb{R}}$.

Given a morphism of $\mathbb{Z}$-modules $f : M \to N$, we may also define a relative version $\lambda(f)$ of the first minimum by the formula:

$$\lambda(f) = \inf \{\|v\| \mid f(v) \neq 0\}.$$

By definition, the invariant $\lambda^{[0]}(\overline{M})$ is the infimum of those positive real numbers R such that the $\mathbb{R}$-vector space generated by $B(\overline{M}_{/\mathrm{tor}}; R)$ is dense in $M_{\mathbb{R}}$, and the following inequalities hold:

$$0 = \lambda_0(\overline{M}) \leq \lambda_1(\overline{M}) \leq \lambda_2(\overline{M}) \leq \dots$$

and

$$\lambda^{[0]}(\overline{M}) \geq \lambda^{[1]}(\overline{M}) \geq \lambda^{[2]}(\overline{M}) \geq \dots$$

If the dimension of $M_{\mathbb{R}}$ is infinite, then, for any nonnegative integers i and j, we have:

$$\lambda_i(\overline{M}) \leq \lambda^{[j]}(\overline{M}).$$

As already observed in 6.4.1.1, when $\overline{M}$ is a Euclidean lattice (that is, when the $\mathbb{Z}$-module M is finitely generated and free and the Euclidean seminorm $\|.\|$ is a norm), the invariant $\lambda^{[0]}(\overline{M})$ is finite and coincides with the "ultimate of the successive minima" of $\overline{M}$, namely with $\lambda_n(\overline{M})$ where $n := \mathrm{rk} M$:

$$\lambda^{[0]}(\overline{M}) = \lambda_n(\overline{M}) := \min \left\{R \in \mathbb{R}_+ \mid \; B(\overline{M}; R) \text{ generates the } \mathbb{R}\text{-vector space } M_{\mathbb{R}}\right\}.$$

This may be generalized as follows.

PROPOSITION 6.6.2. *Let $\overline{M} = (M, \|.\|)$ be a Euclidean quasi-coherent sheaf such that $M_{\mathbb{R}}$ is a finite-dimensional real vector space. For any nonnegative integer i, $\lambda^{[i]}(\overline{M})$ is the infimum of the set of those positive real numbers R such that the $\mathbb{R}$-vector space generated by $B(\overline{M}; R)$ has codimension at most i in $M_{\mathbb{R}}$.*

In other words, if $M_{\mathbb{R}}$ has finite dimension n, then for any integer i between 0 and n, we have:

$$\lambda^{[i]}(\overline{M}) = \lambda_{n-i}(\overline{M}).$$

We start the proof of Proposition 6.6.2 with a lemma.

LEMMA 6.6.3. *Let V be a finite-dimensional real vector space endowed with a Euclidean norm. We denote by d the induced metric on V. Let W be a vector subspace of V, and let G be a $\mathbb{Z}$-submodule of V. Assume that the vector subspace of V generated by G contains W. Then, for any $\varepsilon > 0$, the vector subspace of V generated by those elements g of G with*

$$d(g, W) < \varepsilon$$

contains W.

PROOF. We immediately reduce to the case where W is a line L and G is a lattice in V. If L contains a nonzero element of G, the result is clear. Assume that the intersection $L \cap G$ is reduced to 0. Let n be the dimension of V, and let

$$\pi : V \longrightarrow V/L$$

be the quotient map. Then $\pi(G)$ is a $\mathbb{Z}$-submodule of rank n of the vector space V/L, which has dimension $n-1$. Since $\pi(G)$ generates V/L, the connected component of the identity of the closure of $\pi(G)$ in V/L is a line L' in V/L.

For any $\varepsilon > 0$, we may find elements g of G such that $\pi(g)$ lies in $L' \setminus \{0\}$ and $\pi(g)$ is arbitrarily close to 0 in V/L, i.e., such that $d(g, L)$ is arbitrarily small. Given $\varepsilon > 0$, consider a primitive element g of G such that $\pi(g)$ lies in $L' \setminus \{0\}$ and $d(g, L) < \varepsilon$. Consider a primitive element h of G such that $\pi(h)$ lies in $L' \setminus \{0\}$ and $d(h, L) < d(g, L)$. Since both $\pi(g)$ and $\pi(h)$ lie in L', we may find real numbers λ and μ, not both zero, such that

$$\lambda\pi(g) + \mu\pi(h)h = 0,$$

i.e.:

$$\lambda g + \mu h \in L.$$

Since g is primitive in G and $d(h, L) < d(g, L)$, h does not lie in $\mathbb{Q}g$, so that

$$\lambda g + \mu h \neq 0.$$

This proves that L is contained in the vector subspace of V generated by those $g \in G$ such that $d(g, L) < \varepsilon$. □

PROOF OF PROPOSITION 6.6.2. Write $\mu^{[i]}(\overline{M})$ for the infimum of the set of those positive real numbers R such that the $\mathbb{R}$-vector space generated by $B(\overline{M}; R)$ has codimension at most i in $M_{\mathbb{R}}$. Certainly, we have:

$$\lambda^{[i]}(\overline{M}) \leq \mu^{[i]}(\overline{M}).$$

To prove the reverse inequality, consider the subspace K of $M_{\mathbb{R}}$ consisting of those v with $\|v\| = 0$. Let

$$\pi : M_{\mathbb{R}} \longrightarrow M_{\mathbb{R}}/K$$

be the quotient map. If V is a vector subspace of $M_{\mathbb{R}}$, then the closure of V in $M_{\mathbb{R}}$ is $\pi^{-1}(\pi(V))$. In particular, the closure of V in $M_{\mathbb{R}}$ has codimension at most i in $M_{\mathbb{R}}$ if and only if $\pi(V)$ has codimension at most i in $M_{\mathbb{R}}/K$.

Let R be a positive real number. Lemma 6.6.3 shows that the $\mathbb{R}$-vector space generated by $B(\overline{M}_{/\mathrm{tor}}; R)$ contains K, so that it is closed. Consider a real number R with $R > \lambda^{[i]}(\overline{M})$. Then the closure of the $\mathbb{R}$-vector space V generated by $B(\overline{M}_{/\mathrm{tor}}; R)$ has codimension at most i in $\overline{M}_{\mathbb{R}}$. Since V is closed in $M_{\mathbb{R}}$, this proves that V has codimension at most i in $M_{\mathbb{R}}$, so that:

$$R \geq \mu^{[i]}(\overline{M}).$$

This finishes the proof. □

The invariants $\lambda^{[i]}$ are easily seen to satisfy the monotonicity property $\mathbf{Mon}^1$, namely:

PROPOSITION 6.6.4. *If $f : \overline{M}_1 \to \overline{M}_2$ is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ such that $f(M_1) = M_2$, then, for any nonnegative integer i:*

$$\lambda^{[i]}(\overline{M}_1) \geq \lambda^{[i]}(\overline{M}_2) \tag{6.6.1}$$

PROOF. To establish the inequality (6.6.1), let R be a positive real number, and let V be the closure in $\overline{M}_{1,\mathbb{R}}$ of the $\mathbb{R}$-vector space generated by $B(\overline{M}_1; R)$.

Let W be the closure in $\overline{M}_{2,\mathbb{R}}$ of the $\mathbb{R}$-vector space generated by $B(\overline{M}_2; R)$. Since $f(B(\overline{M}_1; R))$ is contained in $B(\overline{M}_2; R)$, we obtain:

$$f(V) \subseteq W.$$

Since $f : M_1 \to M_2$ is surjective, the codimension of $f(V)$ in $M_{2,\mathbb{R}}$ is bounded above by the codimension of V in $M_{1,\mathbb{R}}$. In particular, if $R > \lambda^{[i]}(\overline{M}_1)$, then V has codimension at most i in $M_{1,\mathbb{R}}$, so that W has codimension at most i in $M_{2,\mathbb{R}}$, and $R > \lambda^{[i]}(\overline{M}_2)$. □

6.6.2. The invariants $\gamma_i(\overline{M})$ and $\gamma^{[i]}(\overline{M})$. Given a Euclidean quasi-coherent sheaf $\overline{M}$, we may also define successive invariants $\gamma^{[i]}(\overline{M})$ and $\gamma_i(\overline{M})$ as follows.

DEFINITION 6.6.5. Let i be a nonnegative integer.

(i) The invariant $\gamma_i(\overline{M})$ is the infimum of the set of those $R \in \mathbb{R}_+^*$ such that the $\mathbb{Z}$-submodule of $M_{/\mathrm{tor}}$ generated by $B(\overline{M}_{/\mathrm{tor}}; R)$ contains a $\mathbb{Z}$-submodule N of $M_{/\mathrm{tor}}$ that is saturated of rank i, namely N has rank i and $M_{/\mathrm{tor}}/N$ is torsion-free ;

(ii) the invariant $\gamma^{[i]}(\overline{M})$ is the infimum of the set of those $R \in \mathbb{R}_+^*$ such that the $\mathbb{Z}$-submodule of $M_{/\mathrm{tor}}$ generated by $B(\overline{M}_{/\mathrm{tor}}; R)$ contains a $\mathbb{Z}$-submodule N of $M_{/\mathrm{tor}}$ that is saturated of corank i, namely, $M_{/\mathrm{tor}}/N$ is free of rank i.

In the case of Euclidean lattices, the invariants γ_i appear in [**Cai03**]. The invariant $\gamma^{[0]}$ coincides with the invariant γ introduced in Definition 6.4.1.

The following inequalities hold:

$$0 = \gamma_0(\overline{M}) \leq \gamma_1(\overline{M}) \leq \gamma_2(\overline{M}) \leq \dots$$

and

$$\gamma^{[0]}(\overline{M}) \geq \gamma^{[1]}(\overline{M}) \geq \gamma^{[2]}(\overline{M}) \geq \dots$$

If the dimension of $M_{\mathbb{R}}$ is infinite, then, for any nonnegative integers i and j, we have:

$$\gamma_i(\overline{M}) \leq \gamma^{[j]}(\overline{M}).$$

If $M_{\mathbb{R}}$ has finite dimension n, then, for any integer i between 0 and n, we have:

$$\gamma^{[i]}(\overline{M}) = \gamma_{n-i}(\overline{M}).$$

Finally, for any nonnegative i, we have:

$$\lambda_i(\overline{M}) \leq \gamma_i(\overline{M}) \tag{6.6.2}$$

and

$$\lambda^{[i]}(\overline{M}) \leq \gamma^{[i]}(\overline{M}). \tag{6.6.3}$$

As in the case of successive minima, the invariants $\gamma^{[i]}$ are easily seen to satisfy the monotonicity property $\mathbf{Mon}^1$ as in the following statement, whose proof is similar to that of Proposition 6.6.4 and left to the reader:

PROPOSITION 6.6.6. *If $f : \overline{M}_1 \to \overline{M}_2$ is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ such that $f(M_1) = M_2$, then, for any nonnegative integer i:*

$$\gamma^{[i]}(\overline{M}_1) \geq \gamma^{[i]}(\overline{M}_2).$$

6.6.3. The successive covering radii $\rho_i(\overline{M})$ and $\rho^{[i]}(\overline{M})$. In similar fashion, the usual covering radius fits into a sequence of *successive covering radii* defined as follows. Let $\overline{M} = (M, \|.\|)$ be a Euclidean quasi-coherent sheaf. If N is a $\mathbb{Z}$-submodule of M, and

$$f : N \hookrightarrow M$$

is the inclusion map, we write $\rho(N, \overline{M})$ for the relative covering radius $\rho(f : N \to \overline{M})$ as defined in Subsection 6.7.1 below; namely:

$$\rho(N, \overline{M}) := \sup_{x \in N_{\mathbb{R}}} \inf_{m \in M_{/\mathrm{tor}}} \|x - m\|.$$

DEFINITION 6.6.7. Let i be a nonnegative integer.

(i) The invariant $\rho_i(\overline{M})$ is the infimum of the set of real numbers of the form $\rho(N, \overline{M})$, where N is a $\mathbb{Z}$-submodule of M such that $N_{\mathbb{R}}$ has dimension i ;
(ii) The invariant $\rho^{[i]}(\overline{M})$ is the infimum of the set of real numbers of the form $\rho(N, \overline{M})$, where N is a $\mathbb{Z}$-submodule of M such that $N_{\mathbb{R}}$ has codimension i in $M_{\mathbb{R}}$.

PROPOSITION 6.6.8. *The following equality holds:*

$$\rho^{[0]}(\overline{M}) = \rho(\overline{M}).$$

For any two nonnegative integers i, j with $i \leq j$, we have:

$$\rho^{[j]}(\overline{M}) \leq \rho^{[i]}(\overline{M})$$

and

$$\rho_i(\overline{M}) \leq \rho_j(\overline{M}).$$

PROOF. By definition, $\rho^{[0]}(\overline{M})$ is the infimum of the quantity $\rho(N, \overline{M})$, where N runs through the $\mathbb{Z}$-submodules of M that generate the real vector space $M_{\mathbb{R}}$. For any such N, we have

$$\rho(N, \overline{M}) = \sup_{x \in N_{\mathbb{R}}} \inf_{m \in M_{/\mathrm{tor}}} \|x - m\| = \sup_{x \in M_{\mathbb{R}}} \inf_{m \in M_{/\mathrm{tor}}} \|x - m\| = \rho(\overline{M}).$$

This proves the equality $\rho^{[0]}(\overline{M}) = \rho(\overline{M})$.

Consider nonnegative integers i and j with $i \leq j$. If N is any $\mathbb{Z}$-submodule of M such that $N_{\mathbb{R}}$ has codimension i in $M_{\mathbb{R}}$, we may find a $\mathbb{Z}$-submodule N' of M contained in N such that $N_{\mathbb{R}}$ has codimension j in $M_{\mathbb{R}}$. In particular, $N'_{\mathbb{R}}$ is contained in $N_{\mathbb{R}}$ and we have:

$$\rho(N', \overline{M}) \leq \rho(N, \overline{M}).$$

Taking the infimum over all such $\mathbb{Z}$-submodules N, we find:

$$\rho^{[j]}(\overline{M}) \leq \rho^{[i]}(\overline{M}).$$

The remaining inequality may be proved similarly. □

If $M_{\mathbb{R}}$ has finite dimension n, then, for any integer i between 1 and n, we have:

$$\rho^{[i]}(\overline{M}) = \rho_{n-i}(\overline{M}).$$

If $M_{\mathbb{R}}$ is infinite-dimensional, then, for any positive integers i and j, we have:

$$\rho_i(\overline{M}) \leq \rho_j(\overline{M}).$$

As with the previous invariants, the successive covering radii $\rho^{[i]}$ are easily seen to satisfy the monotonicity property $\mathbf{Mon}^1$ as in the following statement, whose proof is once again left to the reader:

PROPOSITION 6.6.9. *If $f : \overline{M}_1 \to \overline{M}_2$ is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ such that $f(M_1) = M_2$, then, for any nonnegative integer i:*

$$\rho^{[i]}(\overline{M}_1) \geq \rho^{[i]}(\overline{M}_2).$$

Successive invariants related to the covering radius, namely, the covering minima, were introduced for Euclidean lattices by Kannan and Lovász in [**KL88**]. In Subsection 6.6.5, we will investigate the relationship between covering minima and successive covering radii. As with the other invariants we considered, the covering minima admit a natural generalization to Euclidean quasi-coherent sheaves, that we leave to the reader.

6.6.4. Comparison estimates. The comparison estimates established in Subsection 6.4.2 admit generalizations concerning the successive invariants introduced in the last subsections.

PROPOSITION 6.6.10. *For every Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$ and any nonnegative integer $i \leq \dim M_{\mathbb{R}}$, the following inequality holds in $[0, +\infty]$:*

$$\gamma^{[i]}(\overline{M}) \leq 2\rho^{[i]}(\overline{M}). \tag{6.6.4}$$

In particular:

$$\lambda^{[i]}(\overline{M}) \leq 2\rho^{[i]}(\overline{M}). \tag{6.6.5}$$

Similarly, we have:

$$\gamma_i(\overline{M}) \leq 2\rho_i(\overline{M}). \tag{6.6.6}$$

In particular:

$$\lambda_i(\overline{M}) \leq 2\rho_i(\overline{M}). \tag{6.6.7}$$

PROOF OF PROPOSITION 6.6.10. We will prove (6.6.4) by a variant of the proof of Proposition 6.4.10 and leave it to the reader to adapt the argument to prove (6.6.6). The remaining inequalities follow from the estimates (6.6.2) and (6.6.3).

Let N be a $\mathbb{Z}$-submodule of M such that $N_{\mathbb{R}}$ has codimension i in $M_{\mathbb{R}}$. We need to prove the inequality:

$$\gamma^i(\overline{M}) \leq \rho(N, \overline{M}) := \sup_{x \in N_{\mathbb{R}}} \inf_{m \in M_{/\mathrm{tor}}} \|x - m\|. \tag{6.6.8}$$

We may replace N with $M_{/\mathrm{tor}} \cap N_{\mathbb{R}}$ and assume that N is saturated in $N_{/\mathrm{tor}}$.

Let n be an element of $N_{/\mathrm{tor}}$ and let r be a positive real number such that $r > \rho(N, \overline{M})$. As $N_{\mathbb{R}}$ has codimension i in $M_{\mathbb{R}}$, to establish (6.6.8), it is enough to show the existence of a finite family $(m_\alpha)_{\alpha \in A}$ of elements of $M_{/\mathrm{tor}}$ such that the following condition are satisfied:

$$n = \sum_{\alpha \in A} m_\alpha \quad \text{and} \quad \|m_\alpha\| \leq 2r \text{ for every } \alpha \in A. \tag{6.6.9}$$

To achieve this, we proceed exactly as in the proof of Proposition 6.4.10. Namely we consider a continuous map

$$c : [0, 1] \longrightarrow M_{\mathbb{R}}$$

into the seminormed space $(N_{\mathbb{R}}, \|.\|)$ such that $c(0) = 0$ and $c(1) = n$. For any $t \in [0, 1]$, there exists $m(t)$ in $M_{/\mathrm{tor}}$ such that $\|c(t) - m(t)\| < r$, and consequently there exists a subdivision

$$t_0 = 0 < t_1 < \cdots < t_n = 1$$

of the interval $[0, 1]$ and a family $(m_j)_{1 \leq j \leq n}$ of elements of $M_{/\mathrm{tor}}$ such that, for every $j \in \{1, \ldots, n\}$ and every $t \in [t_{j-1}, t_j]$,

$$\|c(t) - m_j\| < r.$$

Therefore we may write:

$$m = c(1) = (c(1) - m_n) + \sum_{j=1}^{n} (m_j - m_{j-1}) + m(0).$$

The $n+2$ terms of this sum belong to $M_{/\mathrm{tor}}$ and have norm bounded above by $2r$. This establishes the existence of a decomposition of m as in (6.6.9) and completes the proof. □

COROLLARY 6.6.11. *For every Euclidean quasi-coherent sheaf $\overline{M}$, the following equality holds:*

$$\rho_1(\overline{M}) = \frac{1}{2}\lambda_1(\overline{M}).$$

PROOF. By Proposition 6.6.10, we have:

$$\rho_1(\overline{M}) \geq \frac{1}{2}\lambda_1(\overline{M}).$$

Let r be a positive real number with $r > \lambda_1(\overline{M})$. Let m be a nonzero element of $M_{/\mathrm{tor}}$ with

$$\|v\| < r.$$

Then clearly:

$$\rho_1(\overline{M}) \leq \rho(\mathbb{Z}m, \overline{M}) = \sup_{\lambda\in\mathbb{R}} \inf_{v\in M_{/\mathrm{tor}}} \|\lambda m - v\| \leq \frac{1}{2}\|m\| < \frac{1}{2}r.$$ □

6.6.5. Comparison with the covering minima of Kannan-Lovász. Let $\overline{E} = (E, \|.\|)$ be a Euclidean lattice of rank n. In [**KL88**], Kannan and Lovász introduce the sequence of *covering minima* $\mu_i(\overline{E})$ by defining, for any integer i between 0 and n, $\mu_i(\overline{E})$ to be the smallest nonnegative real number R such that, for any affine subspace V of codimension i in the real vector space $E_{\mathbb{R}}$, there exists $v \in V$ and $e \in E$ such that

$$\|v - e\| \leq R.$$

In particular, we have:

$$0 = \mu_0(\overline{E}) \leq \mu_1(\overline{E}) \leq \ldots \leq \mu_n(\overline{E}) = \rho(\overline{E}).$$

It would be possible to generalize those covering minima to invariants μ_i and $\mu^{[i]}$ of arbitrary quasi-coherent Euclidean sheaves along the lines of the preceding paragraphs.

As the following proposition shows, the relationship between the covering minima of $\overline{E}$ and its successive covering radii is not tight, as the $\rho_i(\overline{E})$ are controlled by the smallest nonzero invariant $\mu_1(\overline{E})$. The proof below makes use of the generalized transference inequalities proved in Chapter 9. Since the invariants μ_i are not used in this text outside of this discussion, the proofs of Chapter 9 are independent of the proposition below.

PROPOSITION 6.6.12. *For any integer i between 0 and $n-1$, we have:*

$$\rho_i(\overline{E}) \leq 2n\mu_1(\overline{E}).$$

PROOF. By [**KL88**, Lemma 2.3], we have:

$$\mu_1(\overline{E}) = \frac{1}{2\lambda_1(\overline{E}^\vee)}.$$

As a consequence, we find:

$$\frac{\rho_(\overline{E})}{\mu_1(\overline{E})} = 2\rho_i(\overline{E})\lambda_1(\overline{E}^\vee) \leq 2\rho_i(\overline{E})\lambda_{1+n-i}(\overline{E}^\vee) \leq \frac{n}{\pi} + \frac{4\sqrt{n}}{\pi},$$

where the last inequality is proved in Theorem 9.5.7. This finishes the proof. □

Conversely, there is no nonzero lower bound for the ratio

$$\frac{\rho_{n-1}(\overline{E})}{\mu_1(\overline{E})}$$

as $\overline{E}$ runs through the Euclidean lattices of rank n, as may be seen by considering the norm $\|.\|$ on a lattice of rank n with orthogonal basis $e_1, \ldots, e_n$ with

$$\|e_1\| = \ldots = \|e_{n-1}\| = 1$$

and

$$\|e_n\| = C.$$

Indeed, we have:

$$\rho_{n-1}(\overline{E}) \leq \rho(\mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n, \|.\|) = \frac{1}{2}\sqrt{n-1}$$

and

$$\mu_1(\overline{E}) \geq \frac{C}{2}$$

as may be seen by considering the hyperplane

$$\frac{1}{2}e_n + (\mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n).$$

In spite of the results above regarding the discrepancy between successive covering radii and covering minima, we may obtain a better relationship between those invariants when, instead of considering $\mu_i(\overline{E})$, we allow ourselves to discard some affine subspaces of codimension i:

Proposition 6.6.13. *Let i be an integer between* 0 *and* 1*, and let G denote the grassmannian of codimension i affine subspaces of $E_{\mathbb{R}}$. Then, for any $\varepsilon > 0$, there exists a nonempty Zariski open subset U of G such that, for any $V \in U$, there exists $v \in V$ and $e \in E$ such that*

$$\|v - e\| \leq \rho_i(\overline{E}) + \varepsilon.$$

Proof. Let M be a $\mathbb{Z}$-submodule of E of rank i such that

$$\rho(M, \overline{E}) \leq \rho_i(\overline{E}) + \varepsilon.$$

Then, by construction, any element of $M_{\mathbb{R}}$ is at distance at most $\rho_i(\overline{E}) + \varepsilon$ from E.

Let U be the open subset of G whose elements represent those codimension i affine subspaces of $E_{\mathbb{R}}$ which are not parallel to the dimension i subspace $M_{\mathbb{R}}$ of $E_{\mathbb{R}}$. Let V be an element of U. Then the intersection $V \cap M_{\mathbb{R}}$ is nonempty, so that V contains a point which is at distance at most $\rho_i(\overline{E}) + \varepsilon$ from E. □

6.7. The Relative Covering Radii of a Morphism in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$

6.7.1. Definitions and first properties of the relative covering radius $\rho(f : M \to \overline{N})$.

6.7.1.1. In this section, we study a relative variant of the covering radius, attached to a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, and more generally to the data:

$$f : M \longrightarrow \overline{N}$$

of a countably generated $\mathbb{Z}$-module M, of an object $\overline{N} := (N, \|.\|)$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, and of a morphism f in $\mathrm{Hom}_{\mathbb{Z}}(M, N)$ — that is, to a morphism in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$ in the terminology of Chapter 5 (see 5.1.2.2).

Definition 6.7.1. The *relative covering radius* of a morphism $f : M \to \overline{N}$ in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$ is the element of $[0, +\infty]$:

$$\rho(f : M \to \overline{N}) := \sup_{x \in M_{\mathbb{R}}} \inf_{n \in N_{/\mathrm{tor}}} \|f_{\mathbb{R}}(x) - n\|. \tag{6.7.1}$$

Equivalently $\rho(f : M \to \overline{N})$ is the supremum over the real vector space $f(M_{\mathbb{R}})$ of the function "distance to $N_{/\mathrm{tor}}$," $d(., N_{/\mathrm{tor}})$, defined by (6.1.1) with $\overline{N}$ instead of $\overline{M}$.

Every morphism $f : \overline{M} \to \overline{N}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ induces a morphism in $\mathbf{q}\ddddot{\mathbf{Coh}}_{\mathbb{Z}}$ and we will use the notation $\rho(f : \overline{M} \to \overline{N})$ for the relative covering radius of $f : M \to \overline{N}$. When no ambiguity may arise, we will write $\rho(f)$ instead of $\rho(f : M \to \overline{N})$ or $\rho(f : \overline{M} \to \overline{N})$.

Observe that, for every object $\overline{M}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the identity morphism Id_M of M defines a morphism $\mathrm{Id}_M = M \to \overline{M}$ in $\mathbf{q}\ddddot{\mathbf{Coh}}_{\mathbb{Z}}$ and that, by the very definition (6.7.1) of the relative covering radius, the following equality holds:

$$\rho(\mathrm{Id}_M : M \to \overline{M}) = \rho(\overline{M}).$$

In this sense, the definition of the relative covering radius of a morphism in $\mathbf{q}\ddddot{\mathbf{Coh}}_{\mathbb{Z}}$ extends the definition of the covering radius of an object in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

Let us indicate a few simple properties of the relative covering radius, which are straightforward consequences of its definition (6.7.1).

Firstly, the relative covering radius admits an alternative definition, which turns out to be useful in application. Let $f : M \to \overline{N} := (N, \|.\|)$ be a morphism in $\mathbf{q}\ddddot{\mathbf{Coh}}_{\mathbb{Z}}$, some data as above, and let us denote by:

$$p : N \longrightarrow N/f(M) \quad \text{and} \quad p_{\mathbb{R}} : N_{\mathbb{R}} \longrightarrow (N/f(M))_{\mathbb{R}} \simeq N_{\mathbb{R}}/f_{\mathbb{R}}(M_{\mathbb{R}})$$

the quotient morphisms.

Proposition 6.7.2. *With the notation above, we have:*

$$\rho(f : M \to \overline{N}) = \sup_{y \in N_{\mathbb{R}},\, p_{\mathbb{R}}(y) \in (N/f(M))_{/\mathrm{tor}}} \inf_{n \in N_{/\mathrm{tor}}} \|y - n\|. \tag{6.7.2}$$

Proof. We have the equality of sets:

$$\left\{y \in N_{\mathbb{R}},\, p_{\mathbb{R}}(y) \in (N/f(M))_{/\mathrm{tor}}\right\} = f_{\mathbb{R}}(M_{\mathbb{R}}) + N_{/\mathrm{tor}},$$

so that the right-hand side of (6.7.2) is equal to

$$\sup_{y \in f_{\mathbb{R}}(M_{\mathbb{R}}) + N_{/\mathrm{tor}}} \inf_{n \in N_{/\mathrm{tor}}} \|y - n\|,$$

which clearly equals

$$\sup_{x \in M_{\mathbb{R}}} \inf_{n \in N_{/\mathrm{tor}}} \|f_{\mathbb{R}}(x) - n\| =: \rho(f : M \to \overline{N}). \qquad \square$$

The relative covering radius $\rho(f : M \to \overline{N})$ of a morphism $f : M \to \mathbb{N} := (N, \|.\|)$ is a 1-homogeneous increasing function of $\|.\|$ and is finite if the $\mathbb{Q}$-vector space $M_{\mathbb{Q}}$ (or equivalently the $\mathbb{R}$-vector space $M_{\mathbb{R}}$) is finite dimensional. Moreover, "it does not see torsion." Namely it is unchanged when M (resp. $\overline{N}$) is replaced by $N_{/\mathrm{tor}}$ (resp. by $\overline{N}_{/\mathrm{tor}} := (N_{/\mathrm{tor}}, \|.\|)$), and f by induced morphism

$$f_{/\mathrm{tor}} : M_{/\mathrm{tor}} \to N_{/\mathrm{tor}}.$$

For any two morphisms:

$$f_1 : M_1 \longrightarrow \overline{N}_1 \quad \text{and} \quad f_2 : M_2 \longrightarrow \overline{N}_2$$

in $\mathbf{q}\ddddot{\mathbf{Coh}}_{\mathbb{Z}}$, the following relation — which generalizes the additivity property $\mathbf{Add}_{\oplus}$ of the covering radius stated in (6.1.5) — is satisfied:

$$\rho(f_1 \oplus f_2 : M_1 \oplus M_2 \to \overline{N}_1 \oplus \overline{N}_2)^2 = \rho(f_1 : M_1 \to \overline{N}_1)^2 + \rho(f_2 : M_2 \to \overline{N}_2)^2. \tag{6.7.3}$$

Like (6.1.5), it is a straightforward consequence of (6.1.4) (with $\overline{M}_i$ replaced with $\overline{N}_i$).

By definition, the relative covering radius $\rho(f : M \to \overline{N})$ only depends on $\overline{N}$ and the real vector space $f_{\mathbb{R}}(M_{\mathbb{R}})$ in $N_{\mathbb{R}}$. This generalizes as follows:

PROPOSITION 6.7.3. *Let $f : M \to \overline{N}$ and $f' : M' \to \overline{N}$ be two morphisms in $\mathbf{q\overset{....}{Coh}}_{\mathbb{Z}}$ with the same range $\overline{N} := (N, \|.\|)$. If the closures of $f(M_{\mathbb{R}})$ and $f(M'_{\mathbb{R}})$ in the seminormed vector space $(N_{\mathbb{R}}, \|.\|)$ coincide, then:*

$$\rho(f : M \to \overline{N}) = \rho(f' : M' \to \overline{N}).$$

PROOF. Let $\widetilde{M_{\mathbb{R}}}$ be the closure of $M_{\mathbb{R}}$ in $(N_{\mathbb{R}}, \|.\|)$. Since the function $d(., N_{/\mathrm{tor}})$ is 1-Lipschitz, hence continuous, on the seminormed vector space $(N_{\mathbb{R}}, \|.\|)$, we have:

$$\rho(f : M \to \overline{N}) = \sup_{x \in M_{\mathbb{R}}} d(f_{\mathbb{R}}(x), N_{/\mathrm{tor}}) = \sup_{x \in \widetilde{M_{\mathbb{R}}}} d(f_{\mathbb{R}}(x), N_{/\mathrm{tor}}).$$

This proves that $\rho(f : M \to \overline{N})$ depends only on $\overline{N}$ and on the real vector subspace $\widetilde{M_{\mathbb{R}}}$ of $N_{\mathbb{R}}$. □

6.7.1.2. The properties of semicontinuity, monotonicity, and countable additivity of the covering radius established in Subsection 6.1.2 also generalizes to the relative covering radius.

The following proposition extends Proposition 6.1.2.

PROPOSITION 6.7.4. *Let $f : M \to \overline{N} := (N, \|.\|)$ be a morphism in $\mathbf{q\overset{....}{Coh}}_{\mathbb{Z}}$, and let $(M_i)_{i\in\mathbb{N}}$ and $(N_i)_{i\in\mathbb{N}}$ be increasing families of submodules of M and N respectively such that, for any $i \in \mathbb{N}$.*

$$f_i(M_i) \subseteq N_i.$$

Consider the associated morphisms in $\mathbf{q\overset{....}{Coh}}_{\mathbb{Z}}$:

$$f_i := f_{|M_i} : M_i \longrightarrow \overline{N}_i,$$

where $\overline{N}_i := (N_i, \|.\|_{|N_{i,\mathbb{R}}})$, and assume that $M = \bigcup_{i\in\mathbb{N}} M_i$. Then we have:

$$\rho(f : M \to \overline{N}) \leq \liminf_{i\to\infty} \rho(f_i : M_i \to \overline{N}_i). \tag{6.7.4}$$

PROOF. The proof is a straightforward generalization of the one of Proposition 6.1.2.

Indeed consider a positive real number R with $R > \liminf_{i\to\infty} \rho(f_i : M_i \to \overline{N}_i)$. After replacing the sequence $(f_i)_{i\in\mathbb{N}}$ by a suitable subsequence, we may assume that, for every $i \in \mathbb{N}$, we have:

$$R > \rho(f_i : M_i \to \overline{N}_i),$$

and consequently:

$$f_i(M_{i,\mathbb{R}}) \subset N_{i/\mathrm{tor}} + B(\overline{N}_{i,\mathbb{R}}; R) \subset N_{/\mathrm{tor}} + B(\overline{N}_{\mathbb{R}}; R).$$

Since $M_{\mathbb{R}}$ is the increasing union of the $M_{i,\mathbb{R}}$, this implies the equality:

$$f(M_{\mathbb{R}}) \subset N_{/\mathrm{tor}} + B(\overline{N}_{\mathbb{R}}; R).$$

This establishes the estimate $\rho(f : M \to \overline{N}) \leq R$ and proves (6.7.4). □

The next proposition extends Proposition 6.1.4, which one recovers when $M_i = N_i$ and $f_i = \mathrm{Id}_{M_i}$, $i = 1, 2$.

PROPOSITION 6.7.5. *For every diagram in $\mathbf{q\overset{....}{Coh}}_{\mathbb{Z}}^{\leq 1}$:*

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] & \overline{N}_1 \arrow[d, "g"] \\ M_2 \arrow[r, "f_2"] & \overline{N}_2 \end{tikzcd}$$

such that the closure of $g \circ f_1(M_1)_{\mathbb{R}}$ in $\overline{N}_{2,\mathbb{R}}$ contains $f_2(M_2)_{\mathbb{R}}$, the following inequality holds:

$$\rho(f_1 : M_1 \to \overline{N}_1) \geq \rho(f_2 : M_2 \to \overline{N}_2). \tag{6.7.5}$$

Let us spell out a few special cases of Proposition 6.7.5. When $M_2 = M_1$ and $f_2 = g \circ f_1$, it becomes:

COROLLARY 6.7.6. *For every diagram in* $\mathbf{q}\overset{....}{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$*:*

$$M \xrightarrow{f} \overline{N}_1 \xrightarrow{g} \overline{N}_2,$$

the following inequality holds:

$$\rho(g \circ f : M \to \overline{N}_2) \leq \rho(f : M \to \overline{N}_1). \tag{6.7.6}$$

When $\overline{N}_1 = \overline{N}_2$, $M_1 = N_1$, and $f_1 = \mathrm{Id}_{N_1}$, Proposition 6.7.5 becomes:

COROLLARY 6.7.7. *For every morphism* $f : M \to \overline{N}$ *in* $\mathbf{q}\overset{....}{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$*: the following inequality holds:*

$$\rho(f : M \to \overline{N}) \leq \rho(\overline{N}). \tag{6.7.7}$$

When $M_1 = M_2 = N_1$, $f_1 = \mathrm{Id}_{N_1}$, and $f_2 = g$, Proposition 6.7.5 becomes:

COROLLARY 6.7.8. *For every morphism* $f : \overline{M} \to \overline{N}$ *in* $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$*: the following inequality holds:*

$$\rho(f : M \to \overline{N}) \leq \rho(\overline{M}). \tag{6.7.8}$$

The special case of Proposition 6.7.5 where $\overline{N}_1 = \overline{N}_2$ and $g = \mathrm{Id}_{N_1}$ implies:

COROLLARY 6.7.9. *For every diagram in* $\mathbf{q}\overset{....}{\mathbf{Coh}}_{\mathbb{Z}}$ *of the form:*

$$L \xrightarrow{f} M \xrightarrow{g} \overline{N},$$

the following inequality holds:

$$\rho(g \circ f : L \to \overline{N}) \leq \rho(g : M \to \overline{N}). \tag{6.7.9}$$

PROOF OF PROPOSITION 6.7.5. Let us consider a positive real number R such that:

$$R > \rho(f : M \to \overline{N}_1).$$

We may choose R' and ε in $\mathbb{R}_+^*$ such that:

$$R = R' + \varepsilon \tag{6.7.10}$$

and

$$R' > \rho(f : M \to \overline{N}_1). \tag{6.7.11}$$

According to (6.7.11), we have:

$$f_1(M_1)_{\mathbb{R}} \subseteq N_{1/\mathrm{tor}} + B(\overline{N}_{1\mathbb{R}}, R').$$

Moreover, since $g_{\mathbb{R}}$ has norm at most one, we have:

$$g_{\mathbb{R}}(B(\overline{N}_{1,\mathbb{R}}, R')) \subseteq B(\overline{N}_{2,\mathbb{R}}, R').$$

Consequently the following inclusion holds:

$$g \circ f_1(M_1)_{\mathbb{R}} \subseteq g(N_1)_{/\mathrm{tor}} + B(\overline{N}_{2,\mathbb{R}}, R'),$$

and therefore:

$$g \circ f_1(M_1)_{\mathbb{R}} \subseteq N_{2/\mathrm{tor}} + B(\overline{N}_{2,\mathbb{R}}, R'). \tag{6.7.12}$$

Since the closure of $g \circ f_1(M_1)_{\mathbb{R}}$ in $\overline{N}_{2,\mathbb{R}}$ contains $f_2(M_2)_{\mathbb{R}}$, we also have:

$$f_2(M_2)_{\mathbb{R}} \subseteq g \circ f_1(M_1)_{\mathbb{R}} + B(\overline{N}_{2,\mathbb{R}}, \varepsilon). \tag{6.7.13}$$

Using (6.7.10), (6.7.12), and (6.7.13), we obtain:

$$f_2(M_2)_{\mathbb{R}} \subseteq N_{2/\mathrm{tor}} + B(\overline{N}_{2,\mathbb{R}}, R),$$

and therefore:

$$R \geq \rho(f_2 : M_2 \to \overline{N}_2). \qquad \square$$

Using Proposition 6.7.4 and Corollary 6.7.6, we may extend the additivity of the relative covering radius to countable direct sums.

COROLLARY 6.7.10. *For every countable family* $(f_i : M_i \to \overline{N}_i)_{i \in I}$ *of morphisms in* $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$*, the following equality holds in* $[0, +\infty]$*:*

$$\rho\Big(\bigoplus_{i\in I} f_i : \bigoplus_{i\in I} M_i \to \bigoplus_{i\in I} \overline{N}_i\Big)^2 = \sum_{i\in I} \rho(f_i : M_i \to \overline{N}_i)^2.$$

The proof is a straightforward extension of the proof of Corollary 6.1.5, which we leave to the reader.

6.7.1.3. The continuity properties of the covering radius established in Subsection 6.3.2 also extend to the relative covering radius.

Consider a morphism $f : M \to N$ of countably generated $\mathbb{Z}$-modules. For every Euclidean seminorm $\|.\|$ on $N_{\mathbb{R}}$, we may consider the morphism

$$f : M \longrightarrow (N, \|.\|)$$

in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$ and its relative covering radius:

$$\rho(f : M \to (N, \|.\|)).$$

The following proposition extends the downward continuity property of the covering radius established in Proposition 6.3.6.

PROPOSITION 6.7.11. *If the* $\mathbb{R}$*-vector space* $M_{\mathbb{R}}$ *is finite dimensional, and if* $(\|.\|_n)_{n\in\mathbb{N}}$ *is a decreasing sequence of Euclidean seminorms on* $M_{\mathbb{R}}$*, of pointwise limit* $\|.\|$*, then:*

$$\rho(f : M \to (N, \|.\|)) = \lim_{n\to+\infty} \rho(f : M \to (N, \|.\|_n)).$$

This follows from a straightforward variant of the proof of Proposition 6.3.6 in 6.3.2.3. The arguments in 6.3.2.1 and 6.3.2.4 also establish:

PROPOSITION 6.7.12. *If the* $\mathbb{R}$*-vector space* $N_{\mathbb{R}}$ *is finite dimensional, then the function:*

$$\mathcal{Q}(N_{\mathbb{R}}) \longrightarrow \mathbb{R}_+, \quad \|.\| \longmapsto \rho(f : M \to (N, \|.\|))$$

is upper semicontinuous on the cone $\mathcal{Q}(N_{\mathbb{R}})$ *of Euclidean seminorms over* $N_{\mathbb{R}}$ *and locally Lipschitz on the cone* $\mathring{\mathcal{Q}}(N_{\mathbb{R}})$ *of Euclidean norms over* $N_{\mathbb{R}}$*.*

We leave the details to the reader.

6.7.2. The relative covering radius as a substitute to a rank invariant. As shown above in 6.2.4, neither the covering radius ρ nor its square ρ^2 satisfy the strong monotonicity condition. Consequently, the rank rk_ρ and rk_{ρ^2} attached to morphisms in $\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$ defined by formula (5.1.2) with $\varphi = \rho$ or $\varphi = \rho^2$ do *not* satisfy in general the estimate (5.2.1) which characterizes the strong monotonicity in Definition 5.2.1. However the estimate (6.7.6) in Corollary 6.7.6 precisely asserts that the relative covering radius satisfies the estimate (5.2.1).

The relative covering radius may actually be seen as a suitable substitute for the rank invariant rk_ρ attached to the covering radius ρ. Indeed, besides the estimate (6.7.6), it also satisfies estimates similar to the ones satisfied by rk_φ that have been established in Subsection 5.1.2 as a consequence of the monotonicity and the subadditivity of φ, specialized to $\varphi = \rho$.

For instance the non-negativity of the relative covering radius and the estimate (6.7.7) are the analogues of the estimates (5.1.3), the estimate (6.7.8) is the analogue of (5.1.4), and the estimate (6.7.9) the analogue of (5.1.7). In the same vein, the following proposition is an analogue of Proposition 5.1.5.

PROPOSITION 6.7.13. *For every commutative diagram in* $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{Z}^{\leq 1}$

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] \arrow[d, "p"] & \overline{N}_1 \arrow[d, "q"] \\ M_2 \arrow[r, "f_2"] & \overline{N}_2 \end{tikzcd}$$

where q *is an admissible surjection in* $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$*, the following inequality holds in* $[0, +\infty]$*:*

$$\rho(f_1 : M_1 \to \overline{N}_1) \leq \rho(f_2 : M_2 \to \overline{N}_2) + \rho(\overline{\ker q}). \tag{6.7.14}$$

PROOF. We may assume that both $\rho(f_2 : M_2 \to \overline{N}_2)$ and $\rho(\overline{\ker q})$ are finite. We make the abuse of notation of denoting all seminorms by $\|.\|$.

Let R (resp. R') be a positive real number with $\rho(f_2 : M_2 \to \overline{N}_2) < R$ (resp. $\rho(\overline{\ker q}) < R'$), and consider an element x of $M_{1,\mathbb{R}}$. We may find an element n_2 of $N_{2,/\mathrm{tor}}$ such that

$$\|(f_2 \circ p)_{\mathbb{R}}(x) - n_2\| < R,$$

or equivalently:

$$\|(q \circ f_1)_{\mathbb{R}}(x) - n_2\| < R.$$

Since q is an admissible surjection, we may find an element n_1 of $N_{1,/\mathrm{tor}}$ with $q(n_1) = n_2$, and an element y of $(\ker q)_{\mathbb{R}}$ such that

$$\|f_{1,\mathbb{R}}(x) - y - n_1\| < R.$$

Finally, we may find an element v of $(\ker q)_{/\mathrm{tor}}$ such that

$$\|y - v\| < R'.$$

We obtain:

$$\|f_{1,\mathbb{R}}(x) - (n_1 + v)\| < R + R'.$$

This proves (6.7.14). □

Finally the following proposition may be seen as a comparison estimate between the relative covering radius and the rank invariant rk_ρ.

PROPOSITION 6.7.14. *For every morphism* $f : M \to \overline{N}$ *in* $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$*, the following inequality holds in* $[0, +\infty]$*:*

$$\rho(\overline{N}) \leq \rho(f : M \to \overline{N}) + \rho(\overline{N/f(M)}). \tag{6.7.15}$$

Indeed, it immediately implies:

COROLLARY 6.7.15. *For every morphism* $f : M \to \overline{N}$ *in* $\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$*, the following inequality holds in* $\mathbb{R}_+$*:*

$$\mathrm{rk}_\rho(f : M \to \overline{N}) \leq \rho(f : M \to \overline{N}). \tag{6.7.16}$$

PROOF OF PROPOSITION 6.7.14. We may assume that both $\rho(f : M \to \overline{N})$ and $\rho(\overline{N/f(M)})$ are finite. Let R (resp. R') be a positive real number with $\rho(f : M \to \overline{N}) < R$ (resp. $\rho(\overline{N/f(M)}) < R'$), and consider an element x of $N_{\mathbb{R}}$. We may find an element n of $N_{/\mathrm{tor}}$ with

$$\|\overline{x} - \overline{n}\|_{\overline{N/f(M)}} < R,$$

where $\overline{x}$ and $\overline{n}$ denote the images of x and n in $(N/f(M))_{\mathbb{R}}$ respectively, and $\|.\|_{\overline{N/f(M)}}$ is the quotient seminorm. As a consequence, we may find an element y of $f(M)_{\mathbb{R}}$ such that

$$\|x - n - y\| < R.$$

Finally, we may find an element n' of $N_{/\mathrm{tor}}$ such that

$$\|y - n'\| < R'.$$

We obtain:

$$\|x - (n + n')\| < R + R'.$$

This proves (6.7.15). □

6.7.3. Lower and upper extensions of the relative covering radius. As for the covering radius of objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ which admits variants $\underline{\rho}$ and $\overline{\rho}$ defined in terms of the covering radius of objects in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, we may define some *lower and upper extensions* of the relative covering radius attached to morphisms in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$. In this final subsection, we briefly describe their construction, which is parallel to the one of the lower and upper extensions of the rank invariants as described in Sections 5.4 and 5.5.

Let us consider the data a morphism in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$:

$$f : M \longrightarrow \overline{N}.$$

We define the *lower relative covering radius of* f by the formula:

$$\underline{\rho}(f : M \to \overline{N}) := \sup_{N' \in \mathrm{coft}(N)} \rho(f_{N'} : M \to \overline{N/N'}) \in [0, +\infty], \tag{6.7.17}$$

where $f_{N'}$ denotes the composition of f with the quotient morphism $N \to N/N'$.

Note that the relative covering radius $\rho(f_{N'})$ is equal to the relative covering radius of the morphism in $\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$:

$$M/f^{-1}(N') \longrightarrow \overline{N/N'}$$

induced by f. Moreover, using Corollary 6.7.6, it is readily checked that the relative covering radius

$$\rho(f_{N'} : M \to \overline{N/N'})$$

is an increasing function of N' in the directed set $(\mathrm{coft}(N), \supseteq)$. Accordingly we may also write:

$$\underline{\rho}(f : M \to \overline{N}) = \lim_{N' \in \mathrm{coft}(N)} \rho(f_{N'} : M \to \overline{N/N'}),$$

where the limit is taken over the directed set $(\mathrm{coft}(N), \supseteq)$.

Similarly, we define the *upper relative covering radius of* f by the formula:

$$\overline{\rho}(f : M \to \overline{N}) := \liminf_{C \in \mathrm{coh}(N)} \rho(f_{|f^{-1}(C)} : f^{-1}(C) \to \overline{C}) \in [0, +\infty]. \tag{6.7.18}$$

Again, note that the relative covering radius $\rho(f_{|f^{-1}(C)} : f^{-1}(C) \to \overline{C})$ in the right-hand side of (6.7.18) is equal to the relative covering radius of the "inclusion morphism" in $\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$:

$$C \cap \mathrm{im}\, f \longrightarrow \overline{C}.$$

For every object $\overline{M}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, as a straightforward consequence of the definitions, we have:

$$\underline{\rho}(\mathrm{Id}_M : M \to \overline{M}) = \underline{\rho}(\overline{M}) \quad \text{and} \quad \overline{\rho}(\mathrm{Id}_M : M \to \overline{M}) = \overline{\rho}(\overline{M}).$$

Moreover an analogue of Proposition 6.3.10 holds for the relative covering radius:

PROPOSITION 6.7.16. *For every morphim $f : M \to \overline{N}$ in $\mathbf{q}\overset{\cdots}{\mathbf{Coh}}_{\mathbb{Z}}$, the following inequalities hold:*

$$\underline{\rho}(f : M \to \overline{N}) \leq \rho(f : M \to \overline{N}) \leq \overline{\rho}(f : M \to \overline{N}). \tag{6.7.19}$$

PROOF. Let N' be an element of $\mathrm{coft}(N)$ and let

$$f_{N'} : M \longrightarrow \overline{N/N'}$$

be the associated map. As $f_{N'}$ factors through f, Corollary 6.7.6 shows the inequality:

$$\rho(f_{N'} : M \to \overline{N/N'}) \leq \rho(f : M \to \overline{N}).$$

This proves the first inequality in (6.7.19).

To prove the second inequality, we may assume that $\overline{\rho}(f)$ is finite and consider R a real number with $R > \overline{\rho}(f)$. Consider also an exhaustive filtration $(C_i)_{i\in\mathbb{N}}$ of N by submodules in $\mathrm{coh}(N)$ such that

$$\overline{\rho}(f : M \to \overline{N}) = \lim_{i\to\infty} \rho(f_i : f^{-1}(C_i) \to \overline{C_i}),$$

where f_i is the restriction of f to $f^{-1}(C_i)$.

Let x be an element of $M_{\mathbb{R}}$. For any large enough integer i, we have $R > \rho(f_i : f^{-1}(C_i) \to \overline{C_i})$ and $f_{\mathbb{R}}(x)$ belongs to $C_{i,\mathbb{R}}$. Then we may find $n \in C_{i,/\mathrm{tor}} \subset N_{/\mathrm{tor}}$ such that:

$$\|f_{\mathbb{R}}(x) - n\| < R.$$

This proves the inequality:

$$\rho(f : M \to \overline{N}) \leq R$$

and implies the second inequality in (6.7.19). □

Observe that, for every object $\overline{N} = (N, \|.\|)$ of $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ and any $\mathbb{Z}$-submodule M of N, we may define:

$$\underline{\rho}(M, \overline{N}) := \underline{\rho}(\iota : M \to \overline{N}),$$

where $\iota : M \to N$ denotes the inclusion morphism. Finally we may define "lower variants" $\underline{\rho}^{[i]}(\overline{N})$ of the successive covering radii $\rho^{[i]}(\overline{N})$ by mimicking Definition 6.6.7 (ii). Namely, for every $i \in \mathbb{N}$, $\underline{\rho}^{[i]}(\overline{N})$ is the infimum of the set of real numbers of the form $\rho(M, \overline{N})$, where M is a $\mathbb{Z}$-submodule of N such that $M_{\mathbb{R}}$ has codimension i in $N_{\mathbb{R}}$.

CHAPTER 7

The Theta Invariants of Hermitian Coherent Sheaves over an Arithmetic Curve

7.0.1 In this chapter, the main character of this monograph finally enters, namely the theta invariant $h^1_\theta(\overline{E})$ attached to an object $\overline{E}$ of the category $\overline{\mathbf{Coh}}_X$ of Hermitian coherent sheaves over the arithmetic curve $X := \operatorname{Spec} \mathcal{O}_K$ attached to some number field K.

In the next chapter, we shall study in depth the extensions of the invariant h^1_θ to the category $\mathbf{q}\overline{\mathbf{Coh}}_X$, and investigate the theta invariants $h^1_\theta(\overline{E})$ of Hermitian quasi-coherent sheaves $\overline{E} := (E, (\|.\|_x)_{x \in X(\mathbb{C})})$ over X. In the present chapter, we focus on the construction and the properties of $h^1_\theta(\overline{E})$ when $\overline{E}$ lies in the subcategory $\overline{\mathbf{Coh}}_X$, that is when the $\mathcal{O}_K$-module E is finitely generated.

The invariant $h^1_\theta(\overline{E})$ attached to an object $\overline{E}$ of the category $\overline{\mathbf{Vect}}_\mathbb{Z}$ of Hermitian vector bundles over the arithmetic curve $X = \operatorname{Spec} \mathbb{Z}$ — in other words, to a Euclidean lattice $\overline{E} := (E, \|.\|)$ defined by a finitely generated free $\mathbb{Z}$-module E and a Euclidean norm $\|.\|$ on the real vector space $E_\mathbb{R}$ — is defined by the following expression, which involves the dual Euclidean lattice $\overline{E}^\vee := (E^\vee, \|.\|^\vee)$:

$$h^1_\theta(\overline{E}) := h^0_\theta(\overline{E}^\vee) := \log \sum_{\xi \in E^\vee} e^{-\pi \|\xi\|^2}. \tag{7.0.1}$$

The series in the right-hand side of (7.0.1) is a special value of the theta series classically associated to $\overline{E}^\vee$. This fact is at the origin of the terminology *theta invariant* and of the notation h^0_θ and h^1_θ, introduced in [**Bos20b**].

The invariant h^1_θ on $\overline{\mathbf{Vect}}_\mathbb{Z}$ defined by (7.0.1) is easily seen to satisfy the conditions of monotonicity $\mathbf{Mon}^1_\mathbb{Q}$, of subadditivity **SubAdd**, and of downward continuity $\mathbf{Cont}^+$ introduced at the beginning of Chapter 4, in Subsections 4.1.2–4.1.4 and 4.1.7. As discussed at the end of Chapter 4, in Subsection 4.6.1, one defines an invariant h^1_θ on $\overline{\mathbf{Vect}}_X$ that still satisfies these conditions by letting:

$$h^1_\theta(\overline{E}) := h^1_\theta(\pi_* \overline{E})$$

for every Hermitian vector bundle $\overline{E}$ over the arithmetic curve X, where π denotes as usual he morphism of schemes from X to $\operatorname{Spec} \mathbb{Z}$.

In turn, the invariant h^1_θ on $\overline{\mathbf{Vect}}_X$ may be extended to an invariant on $\overline{\mathbf{Coh}}_X$ by means of the simple constructions discussed in Section 4.2. In this way, we construct an invariant:

$$h^1_\theta : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

that still satisfies Conditions $\mathbf{Mon}^1_K$, **SubAdd**, and $\mathbf{Cont}^+$. Consequently, the constructions of extensions of invariants from $\overline{\mathbf{Coh}}_X$ to $\mathbf{q}\overline{\mathbf{Coh}}_X$ developped in the main part of Chapter 4 — notably the existence of a nice subcategory of "h^1_θ-summable objects" in $\mathbf{q}\overline{\mathbf{Coh}}_X$ — will apply to the invariant h^1_θ on $\overline{\mathbf{Coh}}$.

The invariant h^1_θ on $\overline{\mathbf{Coh}}_X$ actually satisfies the strong monotonicity condition $\mathbf{StrMon}^1$ investigated in Chapter 5, and consequently its extensions to $\mathbf{q}\overline{\mathbf{Coh}}_X$ will satisfy the significantly stronger formalism developed in Chapter 5, notably in Section 5.6.

Our first goal in this chapter is to present a proof of the strong monotonicity of h^1_θ on $\overline{\mathbf{Coh}}_X$. Our proof is directly inspired by the work of Banaszczyk and of Regev and Stephens-Davidowitz, and involve some remarkable estimates established by these authors concerning the function:

$$B_{\overline{E}} : E_{\mathbb{R}} \longrightarrow \mathbb{R}_+^*$$

attached to a Euclidean lattice $\overline{E} := (E, \|.\|)$, that is defined by the following equality, for every $x \in E_{\mathbb{R}}$:

$$B_{\overline{E}}(x) := \frac{\sum_{v\in E} e^{-\pi\|x-v\|^2}}{\sum_{v\in E} e^{-\pi\|v\|^2}}. \tag{7.0.2}$$

This function $B_{\overline{E}}$ may also be expressed as the Fourier transform:

$$B_{\overline{E}} = \mathcal{F}^{-1}(\beta_{\overline{E}^\vee}) \tag{7.0.3}$$

of the following probability measure on the dual vector space $E^\vee_{\mathbb{R}}$:

$$\beta_{\overline{E}^\vee} := \frac{\sum_{\xi\in E^\vee} e^{-\pi\|\xi\|^{\vee 2}}\delta_\xi}{\sum_{\xi\in E^\vee} e^{-\pi\|\xi\|^{\vee 2}}}. \tag{7.0.4}$$

The relation (7.0.3) is an avatar of the Poisson formula and of the functional equation satisfied by the Riemann theta function, and plays a key role in Banaszczyk's seminal paper [**Ban93**] and in the large number of works concerning Euclidean lattice relying on the techniques introduced in [**Ban93**], notably in papers of computed science devoted to lattice based cryptography.[1] For this reason, we call $B_{\overline{E}}$ and $\beta_{\overline{E}^\vee}$ the *Banaszczyk function* and the *Banaszczyk measure* attached to the Euclidean lattice $\overline{E}$.

Beyond their role in the proof of the strong monotonicity of the invariant h^1_θ on $\overline{\mathbf{Coh}}_X$, the function $B_{\overline{E}}$ and the measure $\beta_{\overline{E}^\vee}$ defined by (7.0.2) and (7.0.4) and their generalizations will play a key role in Chapter 8, in a possibly infinite dimensional setting, when studying the invariant h^1_θ attached to Hermitian quasi-coherent sheaves. In the last sections of the present chapter, we establish further properties of Banaszczyk functions and measures — still in a finite dimensional setting — that will play a key role in Chapters 8 and 9.

The general philosophy concerning the invariant h^1_θ on $\overline{\mathbf{Vect}}_{\mathbb{Z}}$, or more generally on $\overline{\mathbf{Coh}}_X$, that comes out of this chapter might be summarized as follows:

(i) The invariant $h^1_\theta(\overline{E})$ attached to an object of $\overline{\mathbf{Coh}}_X$ is a refined analytic invariant which satisfies formal properties similar to the one of the invariant $h^1(C, \mathcal{F})$ attached to some coherent sheaf $\mathcal{F}$ over a smooth projective curve C. This holds at a astonishing level of precision, and for rather subtle reasons, as demonstrated by the work of Banaszczyk and of Regev and Stephens-Davidowitz on which the proof of its strong monotonicity property relies.

(ii) The intuitive geometric meaning of the invariant $h^1_\theta(\overline{E})$ attached to some Euclidean lattice $\overline{E} := (E, \|.\|)$ is that $h^1_\theta(\overline{E})$ is small when, in the Euclidean vector space $(E_{\mathbb{R}}, \|.\|)$, every point is close to some lattice point in E. This point of view will be comforted by the relations between the theta invariant $h^1_\theta(\overline{E})$ and the covering radius $\rho(\overline{E})$ established in Chapter 9.

(iii) The Banaszczyk function $B_{\overline{E}}$, or rather its logarithmic variant $b_{\overline{E}}$, is arguably a still more fundamental invariant of a Euclidean lattice $\overline{E}$ than its theta invariant $h^1_\theta(\overline{E})$.

7.0.2 Let us describe the content of this chapter in more details.

In Section 7.1, we introduce our key analytic tools. We consider a Euclidean lattice $\overline{E} := (E, \|.\|)$, and we introduce the theta function $\theta_{\overline{E}}$ on $E_{\mathbb{R}}$, defined by:

$$\theta_{\overline{E}}(x) := \sum_{v\in E} e^{-\pi\|x-v\|^2},$$

[1] See for instance [**Reg03**], [**MR07**], [**DRSD14**], [**ADRSD15**].

the Banaszczyk function $B_{\overline{E}} := \theta_{\overline{E}}/\theta_{\overline{E}}(0)$ and its logarithmic variant $b_{\overline{E}}$, defined by:

$$B_{\overline{E}} = e^{-\pi b_{\overline{E}}},$$

and the Banaszczyk measure $\beta_{\overline{E}^\vee}$. We also introduce the theta invariants of $\overline{E}$:

$$h^0_\theta(\overline{E}) := \log \theta_{\overline{E}}(0) \quad \text{and} \quad h^1_\theta(\overline{E}) := h^0_\theta(\overline{E}^\vee).$$

The Poisson formula applied to the lattice E in $E_\mathbb{R}$ and to the Gaussian function $e^{-\pi\|\cdot\|^2}$ leads notably to the relation (7.0.3), and to the Poisson-Riemann-Roch formula:

$$h^0_\theta(\overline{E}) - h^1_\theta(\overline{E}) = \widehat{\deg}\, \overline{E}.$$

We also illustrate the significance of Banaszczyk function $B_{\overline{E}}$ by various formulae where it enters naturally, notably by a formula for the theta invariants of an extension of Euclidean lattices that makes transparent the subadditivity of the invariants h^0_θ and h^1_θ on $\overline{\mathbf{Vect}}_\mathbb{Z}$. We finally discuss the relation between the theta functions attached to Euclidean lattices, as considered in this chapter, and the classical complex analytic theta functions, *à la* Riemann.

In Section 7.2, we introduce the invariants h^0_θ and h^1_θ, previously defined on $\overline{\mathbf{Vect}}_\mathbb{Z}$, attached to objects of $\overline{\mathbf{Coh}}_{\overline{X}}$. Up to a finite torsion submodule, the objects of $\overline{\mathbf{Coh}}_{\overline{X}}$ coincide with objects of $\overline{\mathbf{Vect}}_X$, whose theta invariants have been studied in [**Bos20b**, Chapter 2], elaborating on [**vdGS00**]. The content of Section 7.2 may be seen as a minor variation of the results in *loc. cit.*, and accordingly, we have left some details of the proofs to the reader.

Section 7.3 is devoted to the extension of the invariant h^1_θ from $\overline{\mathbf{Coh}X}$ to $\overline{\mathbf{Coh}}_X$, as an application of the general recipe in Section 4.2.

At this stage, we have constructed an invariant:

$$h^1_\theta : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

that satisfies Conditions $\mathbf{Mon}^1_K$, **SubAdd**, and $\mathbf{Cont}^+$, and the next two sections are devoted to proving it also satisfies $\mathbf{StrMon}^1$.

Section 7.4 is devoted to proving several key properties of the Banaszczyk functions $B_{\overline{E}}$ and $b_{\overline{E}}$, and as such constitutes the heart of this chapter.

Firstly, following Banaszczyk [**Ban22**][2] and Regev and Stephens-Davidowitz [**RSD17a**], we prove some general "quadratic inequalities" they satisfied by those. Notably, we show that, for every Euclidean lattice $\overline{E} := (E, \|.\|)$ and every x and y in $E_\mathbb{R}$, the following inequality holds:

$$b_{\overline{E}}(x+y) + b_{\overline{E}}(x-y) \leq 2\big(b_{\overline{E}}(x) + b_{\overline{E}}(y)\big). \tag{7.0.5}$$

This estimate is to be compared with the parallelogram identity satisfied by the squared norm $\|.\|^2$ on $E_\mathbb{R}$:

$$\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big). \tag{7.0.6}$$

Indeed, in various respects, the function $b_{\overline{E}}$ may be seen as a counterpart, concerning the Euclidean lattice $\overline{E}$, of the function $\|.\|^2$ on the Euclidean vector space $\overline{E}_\mathbb{R} := (E_\mathbb{R}, \|.\|)$.

Then, relying on the inequality (7.0.5), we establish a monotonicity property for the Banaszczyk functions. Namely we show that, if $f : \overline{E} \to \overline{F}$ is a morphism in $\overline{\mathbf{Vect}}^{\leq 1}_\mathbb{Z}$, then for every $x \in E_\mathbb{R}$, the following inequality holds for every $x \in E_\mathbb{R}$:

$$b_{\overline{F}}(f_\mathbb{R}(x)) \leq b_{\overline{E}}(x). \tag{7.0.7}$$

[2]As indicated in [**Ban22**], Banaszczyk's derivation of these estimates originally appeared in the preprint version [**Ban92**] of his famous article [**Ban93**].

In the same vein as the analogy between (7.0.5) and (7.0.6), the inequality(7.0.7) should be compared with the estimate:

$$\|f_{\mathbb{R}}(x)\|^2_{\overline{F}} \leq \|x\|^2_{\overline{E}},$$

which expresses that a morphism of $\mathbb{Z}$-modules $f : E \to F$ defines a morphism in $\overline{\mathbf{Vect}}_{\mathbb{Z}}^{\leq 1}$.

In Section 7.5 we use this monotonicity property of the Banaszczyk functions to establish a suitable strong monotonicity property of the invariant h^0_θ on $\overline{\mathbf{Coh}}_{\overline{X}}$, which in turns implies by duality the sought for strong monotonicity of h^1_θ on $\overline{\mathbf{Coh}}_X$.

Actually, in Sections 7.4 and 7.5, we work with Banaszczyk functions associated to data more general than the one of a Euclidean lattice. The data of a Euclidean lattice $\overline{E} := (E, \|.\|)$ is obviously equivalent to the one of the Euclidean vector space $\overline{E}_{\mathbb{R}} := (E_{\mathbb{R}}, \|.\|)$ and of the discrete cocompact subgroup E of $E_{\mathbb{R}}$. To achieve the proof of the strong monotonicity properties of h^0_θ and h^1_θ, we need to use the Banaszczyk function $B_{\overline{V},\Lambda}$ associated to the data of a finite dimensional Euclidean vector space $\overline{V} := (V, \|.\|)$ and of an arbitrary discrete subgroup Λ of V. It is the function of $x \in V$ defined by the following straightforward generalization of (7.0.2):

$$B_{\overline{V},\Lambda}(x) := \frac{\sum_{\lambda \in \Lambda} e^{-\pi \|x-\lambda\|^2}}{\sum_{v \in \Lambda} e^{-\pi \|\lambda\|^2}}. \tag{7.0.8}$$

Banaszczyk himself considers “his” functions in a significantly more general framework; see for instance [**Ban22**] and its references. The generalization (7.0.8) of Banaszczyk functions associated to Euclidean lattices does not explicitly occurs in [**RSD17a**]. However, when spelling out in detail the proofs in [**RSD17a**], one is led to consider it.

Section 7.6 is devoted to some further properties of Banaszczyk functions, and to alternative arguments for deriving special cases of their monotonicity properties established in Section 7.4. Notably, we consider the limit behavior of the functions $B_{\overline{V},\Lambda}$ when Λ varies in decreasing or increasing sequences of discrete subgroups.

In Section 7.7, the construction and the basic properties of the Banaszczyk functions $B_{\overline{E}}$ and $b_{\overline{E}}$ and of the Banaszczyk measure $\beta_{\overline{E}^\vee}$ associated to some Euclidean lattice $\overline{E}$ are extended to the situation when $\overline{E}$ is an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. Here again the key point is the monotonicity property of Banaszczyk functions established in Section 7.4.

In the final section 7.8, we establish some further estimates involving Banaszczyk functions and measures, and morphisms in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$. Notably, by using an argument that goes back to Banaszcyk's paper [**Ban93**], we show that the uniform norm of the Banaszczyk function $b_{\overline{E}}$ is attached to an object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ is comparable with its theta invariant $h^1_\theta(\overline{E})$ when $h^1_\theta(\overline{E})$ is small enough. We also establish some estimates in the opposite direction of the main monotonicity estimate (7.0.7).

Although their derivation is rather straightforward when compared to the one of (7.0.7), these estimates will play an important technical role in the infinite dimensional setting of Chapter 8, notably in Section 8.3. They will also be crucial in Chapter 9, when deriving upper bounds on covering radii of objects and morphims in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ or $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ in terms of their theta invariants and their theta ranks.

7.0.3 In this chapter, we denote by K a number field, by $\mathcal{O}_K$ its ring of integers, and by X the arithmetic curve $\operatorname{Spec} \mathcal{O}_K$, and by $\overline{\mathbf{Vect}}_X$ and $\overline{\mathbf{Coh}}_X$ the categories of Hermitian vector bundles and of Hermitian coherent sheaves over the arithmetic curve X.

Recall that $\overline{\mathbf{Vect}}_X^{[0]}$ (resp. $\overline{\mathbf{Coh}}_{\overline{X}}$) denotes the full subcategory of $\overline{\mathbf{Coh}}_X$ the objects of which are the Hermitian coherent sheaves $(E, (\|.\|_x)_{x \in X(\mathbb{C})}$ such that E is torsion free (resp. such that the Hermitian seminorms $\|.\|_x$ are actually Hermitian norms).

7.1. Theta Functions and Theta Invariants of Euclidean Lattices

7.1.1. Poisson formula, theta functions and theta invariants. Let $\overline{E} := (E, \|.\|)$ be a Euclidean lattice and $\overline{E}^\vee := (E^\vee, \|.\|^\vee)$ the dual Euclidean lattice. As usual, we identify E (resp. $E^\vee$) to a lattice in the $\mathbb{R}$-vector space $E_\mathbb{R}$ (resp. in $E^\vee_\mathbb{R}$).

7.1.1.1. *The Poisson formula.* The Gaussian functions on the Euclidean vector spaces $\overline{E}_\mathbb{R} := (E_\mathbb{R}, \|.\|)$ and $\overline{E}^\vee_\mathbb{R} := (E^\vee_\mathbb{R}, \|.\|^\vee)$, namely the functions:

$$e^{-\pi\|\cdot\|^2} : E_\mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto e^{-\pi\|x\|^2}$$

and

$$e^{-\pi\|\cdot\|^{\vee 2}} : E^\vee_\mathbb{R} \longrightarrow \mathbb{R}, \quad \xi \longmapsto e^{-\pi\|\xi\|^{\vee 2}},$$

are exchanged by Fourier transform. Namely, with the notation introduced in Subsection 0.5.2 in the Introduction, we have the equality of functions in the Schwartz space $\mathcal{S}(E^\vee_\mathbb{R})$:

$$\mathcal{F}_{\overline{E}_\mathbb{R}}(e^{-\pi\|\cdot\|^2}) = e^{-\pi\|\cdot\|^{\vee 2}}. \tag{7.1.1}$$

Consequently, applied to the Gaussian function $f := e^{-\pi\|\cdot\|^2}$ on $V := E_\mathbb{R}$ and to the lattice $\Lambda := E$, the Poisson formula (0.5.5) becomes the following classical result, which plays a central role in this monograph:

THEOREM 7.1.1. *For every $x \in E_\mathbb{R}$, the following equality holds:*

$$\operatorname{covol}(\overline{E}) \sum_{v \in E} e^{-\pi\|x-v\|^2} = \sum_{\xi \in E^\vee} e^{-\pi\|\xi\|^{\vee 2} + 2\pi i\langle \xi, x\rangle}. \tag{7.1.2}$$

Specialized to $x = 0$, the equality (7.1.2) becomes the identity:

$$\operatorname{covol}(\overline{E}) \sum_{v \in E} e^{-\pi\|v\|^2} = \sum_{\xi \in E^\vee} e^{-\pi\|\xi\|^{\vee 2}}, \tag{7.1.3}$$

which will play a particularly important role.

7.1.1.2. *Definitions.* To every Euclidean lattice $\overline{E} := (E, \|.\|)$ as above , we may attach its *theta function*:

$$\theta_{\overline{E}} : E_\mathbb{R} \longrightarrow \mathbb{R},$$

defined by the series in the left-hand side of (7.1.2). Namely, for every $x \in E_\mathbb{R}$, we let:

$$\theta_{\overline{E}}(x) := \sum_{v \in E} e^{-\pi\|x-v\|^2}. \tag{7.1.4}$$

The function $\theta_{\overline{E}}$ is the convolution product $e^{-\pi\|\cdot\|^2} * \delta_E$ of the Gaussian function $e^{-\pi\|\cdot\|^2}$ and of the counting measure δ_E on the lattice E:

$$\delta_E := \sum_{v \in E} \delta_v.$$

It is clearly $\mathbb{R}$-analytic and E-periodic on $E_\mathbb{R}$, and takes its values in $[1, +\infty)$.

We may also consider the following positive measure on $E^\vee_\mathbb{R}$ supported by $E^\vee$:

$$\gamma_{\overline{E}^\vee} := e^{-\pi\|\cdot\|^{\vee 2}} \delta_{E^\vee} = \sum_{\xi \in E^\vee} e^{-\pi\|\xi\|^{\vee 2}} \delta_\xi. \tag{7.1.5}$$

Its total mass is:

$$\gamma_{\overline{E}^\vee}(E^\vee_\mathbb{R}) = \gamma_{\overline{E}^\vee}(E^\vee) = \sum_{\xi \in E^\vee} e^{-\pi\|\xi\|^{\vee 2}} = \theta_{\overline{E}^\vee}(0) \in [1, +\infty).$$

Using this notation, the Poisson formula (7.1.2) may be written as the following equality of functions on $E_{\mathbb{R}}$:

$$\operatorname{covol}(\overline{E})\,\theta_{\overline{E}} = \mathcal{F}^{-1}(\gamma_{\overline{E}^\vee}), \tag{7.1.6}$$

and its special case (7.1.3) as the equality:

$$\operatorname{covol}(\overline{E})\,\theta_{\overline{E}}(0) = \theta_{\overline{E}^\vee}(0). \tag{7.1.7}$$

The equality (7.1.6) may equivalently derived from the distributional version of the Poisson formula:

$$\operatorname{covol}(\overline{E})\,\delta_E = \mathcal{F}^{-1}_{\overline{E}_{\mathbb{R}}}(\delta_{E^\vee}),$$

by taking its convolution product with the Gaussian function:

$$e^{-\pi\|.\|^2} = \mathcal{F}^{-1}_{\overline{E}_{\mathbb{R}}}(e^{-\pi\|.\|^{\vee 2}}).$$

We may finally define the *θ-invariants* of Euclidean lattices as follows:

DEFINITION 7.1.2. For every Euclidean lattice $\overline{E} := (E, \|.\|)$, of dual lattice $\overline{E}^\vee := (E^\vee, \|.\|^\vee)$, we define:

$$h^0_\theta(\overline{E}) := \log \theta_{\overline{E}}(0) = \log \sum_{v\in E} e^{-\pi\|v\|^2} \in \mathbb{R}_+, \tag{7.1.8}$$

and:

$$h^1_\theta(\overline{E}) := h^0_\theta(\overline{E}^\vee) = \log \sum_{\xi\in E^\vee} e^{-\pi\|\xi\|^{\vee 2}} \in \mathbb{R}_+. \tag{7.1.9}$$

The special case (7.1.3) of Poisson formula, or equivalently (7.1.7), may be expressed as follows in terms of these theta invariants and of the Arakelov degree:

COROLLARY 7.1.3. *For every Euclidean lattice $\overline{E}$, the following equality holds:*

$$h^0_\theta(\overline{E}) - h^1_\theta(\overline{E}) = \widehat{\deg}\,\overline{E}. \tag{7.1.10}$$

Observe that the equality (7.1.10) is similar to the Rieman-Roch formula for a vector bundle over a curve of genus $g = 1$. Accordingly we shall sometimes call (7.1.10) the *Poisson-Riemann-Roch* formula for the Euclidean lattice $\overline{E}$.

7.1.2. The functions $B_{\overline{E}}$ and $b_{\overline{E}}$ and the measure $\beta_{\overline{E}^\vee}$.

7.1.2.1. *Definitions.* We may also introduce the E-periodic function on $E_{\mathbb{R}}$:

$$B_{\overline{E}} := \theta_{\overline{E}}(0)^{-1}\,\theta_{\overline{E}}, \tag{7.1.11}$$

and the probability measure deduced from the measure $\gamma_{\overline{E}^\vee}$ by dividing it by its total mass:

$$\beta_{\overline{E}^\vee} := \gamma_{\overline{E}^\vee}(E^\vee_{\mathbb{R}})^{-1}\,\gamma_{\overline{E}^\vee} = \frac{\sum_{\xi\in E^\vee} e^{-\pi\|\xi\|^{\vee 2}}\delta_\xi}{\sum_{\xi\in E^\vee} e^{-\pi\|\xi\|^{\vee 2}}}. \tag{7.1.12}$$

From the Poisson formula (7.1.2), we immediately deduce the following consequence:

COROLLARY 7.1.4. *For every $x \in E_{\mathbb{R}}$, the following identity holds:*

$$B_{\overline{E}}(x) := \frac{\sum_{v\in E} e^{-\pi\|x-v\|^2}}{\sum_{v\in E} e^{-\pi\|v\|^2}} = \frac{\sum_{\xi\in E^\vee} e^{-\pi\|\xi\|^{\vee 2}+2\pi i\langle\xi,x\rangle}}{\sum_{\xi\in E^\vee} e^{-\pi\|\xi\|^{\vee 2}}} =: \mathcal{F}^{-1}(\beta_{\overline{E}^\vee})(x). \tag{7.1.13}$$

In turn, this implies:

COROLLARY 7.1.5. *For every $x \in E_{\mathbb{R}}$, the following inequalities hold:*

$$0 < B_{\overline{E}}(x) \leq 1. \tag{7.1.14}$$

Moreover, we have:

$$B_{\overline{E}}(x) = 1 \Longleftrightarrow x \in E. \tag{7.1.15}$$

As already mentioned in the introduction to this chapter, we will call $B_{\overline{E}}$ and $\beta_{\overline{E}^\vee}$ the *Banaszczyk function* and the *Banaszczy probability measure* associated to the Euclidean lattice $\overline{E}$ and its dual $\overline{E}^\vee$.

According to Corollary 7.1.5, we define a real analytic function:

$$b_{\overline{E}} : E_{\mathbb{R}} \longrightarrow \mathbb{R}_+,$$

which is E-periodic and vanishes precisely on E, by the relation:

$$B_{\overline{E}}(x) = e^{-\pi b_{\overline{E}}(x)}. \tag{7.1.16}$$

As demonstrated by various results in this chapter, the function $b_{\overline{E}}$ is arguably a more natural function to consider than the function $B_{\overline{E}}$ itself.

In the next paragraphs, we present two results concerning the theta invariants of Euclidean lattices where the Banaszczyk function $B_{\overline{E}}$ occurs naturally.

7.1.2.2. *An integral formula for $h^1_\theta(\overline{E})$.* We denote by $\lambda_{\overline{E}_{\mathbb{R}}}$ the Lebesgue measure on the Euclidean vector space $\overline{E}_R := (E_{\mathbb{R}}, \|.\|)$, and also the Haar measure induced by $\lambda_{\overline{E}_{\mathbb{R}}}$ on the compact torus $E_{\mathbb{R}}/E$. If we denote by:

$$q : E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}/E$$

the quotient map, then, for every Borel function $\varphi : E_{\mathbb{R}} \to [0, +\infty]$, the following equality holds:

$$\int_{E_{\mathbb{R}}/E} q_*\varphi([x])\, d\lambda_{\overline{E}_R}([x]) = \int_{E_{\mathbb{R}}} \varphi(x)\, d\lambda_{\overline{E}_{\mathbb{R}}}(x), \tag{7.1.17}$$

where the Borel function $q_*\varphi : E_{\mathbb{R}}/E \to [0, +\infty]$ is defined by the following relation, for every $x \in E_{\mathbb{R}}$ of class $[x] := q(x)$ in $E_{\mathbb{R}}/E$:

$$q_*\varphi([x]) := \sum_{v \in E} \varphi(x - v).$$

By the very definition of the covolume $\operatorname{covol}(\overline{E})$ of the Euclidean lattice $\overline{E}$, we have:

$$\int_{E_{\mathbb{R}}/E} d\lambda_{\overline{E}_{\mathbb{R}}} = \operatorname{covol}(\overline{E}).$$

Consequently, if we denote by $\lambda_{E_{\mathbb{R}/\mathbb{Z}}}$ the Haar probability measure on the compact torus $E_{\mathbb{R}}/E$, the following equality of measure on $E_{\mathbb{R}}/E$ is satisfied:

$$\lambda_{\overline{E}_{\mathbb{R}}} = \operatorname{covol}(\overline{E})\, \lambda_{E_{\mathbb{R}/\mathbb{Z}}}. \tag{7.1.18}$$

By a slight abuse of notation, we shall also denote by $\theta_{\overline{E}}$, $B_{\overline{E}}$ and $b_{\overline{E}}$ the functions on the quotient $E_{\mathbb{R}}/E$ defined from the E-periodic functions $\theta_{\overline{E}}$, $B_{\overline{E}}$ and $b_{\overline{E}}$ on $E_{\mathbb{R}}$.

The Gaussian function $e^{-\pi\|\cdot\|^2}$ on $E_{\mathbb{R}}$ satisfies:

$$\int_{E_{\mathbb{R}}} e^{-\pi\|x\|^2}\, d\lambda_{\overline{E}_{\mathbb{R}}}(x) = 1.$$

Consequently the identity (7.1.17) applied to $e^{-\pi\|\cdot\|^2}$ becomes:

$$\int_{E_{\mathbb{R}}/E} \theta_{\overline{E}}([x])\, d\lambda_{\overline{E}_{\mathbb{R}}}([x]) = 1. \tag{7.1.19}$$

Using (7.1.19) and (7.1.18), we obtain the following expression for the average value of the function $\theta_{\overline{E}}$ on the compact torus $E_{\mathbb{R}}/E$:

$$\int_{E_{\mathbb{R}}/E} \theta_{\overline{E}}([x])\, d\lambda_{E_{\mathbb{R}/\mathbb{Z}}}([x]) = \operatorname{covol}(\overline{E})^{-1}. \tag{7.1.20}$$

Equivalently, we have:

$$\widehat{\deg}\, \overline{E} = \log \int_{E_{\mathbb{R}}/E} \theta_{\overline{E}}([x])\, d\lambda_{E_{\mathbb{R}/\mathbb{Z}}}([x]). \tag{7.1.21}$$

In turn, using the definition (7.1.11) of the function $B_{\overline{E}}$ and the Poisson-Riemann-Roch formula (7.1.10), we deduce from (7.1.21):

$$h^1_\theta(\overline{E}) = -\log \int_{E_{\mathbb{R}}/E} B_{\overline{E}}([x])\, d\lambda_{E_{\mathbb{R}/\mathbb{Z}}}([x]). \tag{7.1.22}$$

The equality (7.1.22) may be written as follows in terms of the function $b_{\overline{E}}$:

$$h^1_\theta(\overline{E}) = -\log \int_{E_{\mathbb{R}}/E} e^{-\pi b_{\overline{E}}([x])}\, d\lambda_{E_{\mathbb{R}/\mathbb{Z}}}([x]). \tag{7.1.23}$$

This expression may be interpreted as a "free energy" in statistical physics; see for instance [**Bos21**, Section 6].

7.1.2.3. *Admissible short exact sequences and theta invariants.* Consider an admissible short exact sequence of Euclidean lattices:

$$\overline{\mathcal{E}}:\ 0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0.$$

According to the additivity of the Arakelov degree, the following equality holds:

$$\widehat{\deg}\, \overline{E} - \widehat{\deg}\, \overline{F} + \widehat{\deg}\, \overline{G} = 0.$$

Together with the Poisson-Riemann-Roch formula (7.1.10), this implies the equality:

$$h_\theta(\overline{\mathcal{E}}) := h^0_\theta(\overline{E}) - h^0_\theta(\overline{F}) + h^0_\theta(\overline{G}) = h^1_\theta(\overline{E}) - h^1_\theta(\overline{F}) + h^1_\theta(\overline{G}).$$

This real number is known to be non-negative. This subadditivity property of the theta invariants h^0_θ and h^1_θ, which plays a central role in the monograph [**Bos20b**], appears in Quillen's notebook [**Qui**] in the entry of 26/04/1973, and is established as Lemma 5.3 in [**Gro01**]; see also [**Bos20b**, Section 2.8].

It is actually possible to write a closed formula for $h_\theta(\overline{\mathcal{E}})$, which involves the function $b_{\overline{E}}$, that makes clear its non-negativity.

To achieve this, recall that, an "arithmetic extension class" $[\overline{\mathcal{E}}]$ in the arithmetic extension group:

$$\widehat{\operatorname{Ext}}^1_{\mathbb{Z}}(G, E) := \operatorname{Hom}_Z(G, E) \otimes \mathbb{R}/\mathbb{Z} \simeq \frac{\operatorname{Hom}_{\mathbb{Z}}(G, E_{\mathbb{R}})}{\operatorname{Hom}_{\mathbb{Z}}(G, E)};$$

is canonically associated to the admissible short exact sequence $\overline{\mathcal{E}}$; see [**BK10**] and [**Bos20b**, Section 1.4]. The class $[\overline{\mathcal{E}}]$ is defined as follows.

We may choose a $\mathbb{Z}$-linear splitting:

$$s^{\mathrm{int}} : G \longrightarrow F$$

of the surjective morphism $p : F \to G$. Besides we may consider the orthogonal splitting

$$s^\perp_{\mathbb{R}} : G_{\mathbb{R}} \longrightarrow F_{\mathbb{R}}$$

of the surjective $\mathbb{R}$-linear map $p_{\mathbb{R}} : F_{\mathbb{R}} \to G_{\mathbb{R}}$, namely its splitting with values in the orthogonal complement $i(E)^\perp_{\mathbb{R}}$ of $i(E)_{\mathbb{R}}$ in the Euclidean vector space $\overline{F}_{\mathbb{R}}$.

The difference $s^{\perp}_{\mathbb{R}} - s^{\rm int}$ maps G into $i(E)_{\mathbb{R}}$, and consequently may be written:

$$s^{\perp}_{\mathbb{R}|G} - s^{\rm int} = i_{\mathbb{R}} \circ T$$

for some uniquely determined T in $\mathrm{Hom}_{\mathbb{Z}}(G, E_{\mathbb{R}})$. The class $[T]$ of T in $\widehat{\mathrm{Ext}}^1_{\mathbb{Z}}(G, E)$ does not depend of the choice of s and defines $[\overline{\mathcal{E}}]$. This class may also be identified to an element of $\mathrm{Hom}_{\mathbb{Z}}(G, E_{\mathbb{R}}/E)$

PROPOSITION 7.1.6 (compare [**Bos20b**], Proposition 2.8.3). *With the above notation, the following equality holds:*

$$h_\theta(\overline{\mathcal{E}}) = -\log \frac{\sum_{g\in G} e^{-\pi\|g\|^2_{\overline{G}} - \pi b_{\overline{E}}([T](g))}}{\sum_{g\in G} e^{-\pi\|g\|^2_{\overline{G}}}}. \tag{7.1.24}$$

The fact that $b_{\overline{E}}$ is non-negative on $E_{\mathbb{R}}$ and vanishes precisely on E implies that $h_\theta(\overline{\mathcal{E}})$ is non-negative and vanishes if and only if the admissible short exact sequence $\overline{\mathcal{E}}$ splits.

PROOF. For every $g \in G$, we have a bijection:

$$E \xrightarrow{\sim} p^{-1}(g), \quad e \longmapsto i(e) + s^{\rm int}(g).$$

Moreover, for every $e \in E$, we have:

$$\begin{aligned} i(e) + s^{\rm int}(g) &= i(e) + s^{\perp}_{\mathbb{R}}(g) - i_{\mathbb{R}} \circ T(g) \\ &= i_{\mathbb{R}}(e - T(g)) + s^{\perp}_{\mathbb{R}}(g), \end{aligned}$$

and consequently:

$$\begin{aligned} \|i(e) + s^{\rm int}(g)\|^2_{\overline{F}} &= \|i_{\mathbb{R}}(e - T(g)) + s^{\perp}_{\mathbb{R}}(g)\|^2_{\overline{F}} \\ &= \|e - T(g)\|^2_{\overline{E}} + \|g\|^2_{\overline{G}}. \end{aligned}$$

Therefore we have:

$$\begin{aligned} \sum_{f\in p^{-1}(g)} e^{-\pi\|f\|^2_{\overline{F}}} &= e^{-\pi\|g\|^2_{\overline{G}}} \sum_{e\in E} e^{-\pi\|e-T(g)\|^2_{\overline{E}}} \\ &= e^{-\pi\|g\|^2_{\overline{G}}}\, \theta_{\overline{E}}(0)\, B_{\overline{E}}(T(g)) \\ &= e^{h^0_\theta(\overline{E}) - \pi\|g\|^2_{\overline{G}} - \pi b_{\overline{E}}([T](g))}. \end{aligned}$$

By summing over $g \in G$ and taking the logarithm, this implies:

$$h^0_\theta(\overline{F}) - h^0_\theta(\overline{E}) = \log \sum_{g\in G} e^{-\pi\|g\|^2_{\overline{G}} - \pi b_{\overline{E}}([T](g))},$$

and (7.1.24) follows. ☐

7.1.3. Relation with classical theta functions. In this monograph, as in [**Bos20b**], we use the terminology *theta invariants* for the invariants attached to Euclidean lattices and their generalizations that are defined in terms of special values of the series (7.1.4). This is in contrast with the seminal article of Banaszczyk [**Ban93**] and with the developments of Banaszczyk's techniques motivated by lattice-based cryptography,[3] where these series appear under the name of "Gaussian functions", associated to "Gaussian-like measures on lattices" or "discrete Gaussian distributions."

Our terminology is dictated by the fact that these series are special instances of the classical theta series, which play a prominent role in the study of elliptic and abelian functions since the beginning of the XIX-th century, notably in the work of Jacobi [**Jac29**] and Riemann [**Rie57**].

[3]See for instance [**MR09**] or [**Pei16**] for surveys and references, and more specifically in relation with the content of this chapter [**DRSD14**], [**ADRSD15**], and [**RSD17a**].

In this subsection, we briefly describe the specific relation between the theta series (7.1.4) and Riemann theta functions.[4]

7.1.3.1. Recall that, for any $g \in \mathbb{N}$, the *Siegel upper halfspace* $\mathfrak{H}_g$ is the open tube of complex symmetric matrixes $\tau = (\tau_{ab})_{1\leq a,b\leq g}$ in $M_g(\mathbb{C})$ whose imaginary part $\mathrm{Im}\tau$ is positive definite. The *Riemann theta function* is the complex analytic function:

$$\theta : \mathfrak{H}_g \times \mathbb{C}^g \longrightarrow \mathbb{C}$$

defined by the series:

$$\theta(\tau, z) := \sum_{n\in\mathbb{Z}^g} e^{\pi i\, {}^t n.\tau.n + 2\pi i\, {}^t n.z}. \tag{7.1.25}$$

The theta function $\theta_{\overline{E}}$ attached to some Euclidean lattice $\overline{E}$, as defined by (7.1.4), coincides up to the normalization factor $\mathrm{covol}\,\overline{E}$ with a special instance of the Riemann theta function defined by the series (7.1.25).

Indeed, let us fix a free $\mathbb{Z}$-module E of finite rank g. To study the theta function $\theta_{\overline{E}}(x)$ as a function of the Euclidean metric $\|.\|$ defining $\overline{E}$, we may introduce the open cone $\overset{\circ}{\mathcal{Q}}(E_{\mathbb{R}}^\vee)$ of positive definite quadratic forms on $E_{\mathbb{R}}^\vee$ and the real analytic function:

$$\Theta_E : \overset{\circ}{\mathcal{Q}}(E_{\mathbb{R}}^\vee) \times E_{\mathbb{R}} \longrightarrow \mathbb{R}_+^*$$

defined by the convergent series:

$$\Theta_E(q, x) := \sum_{\xi\in E^\vee} e^{-\pi q(\xi) + 2\pi i\langle \xi, x\rangle}. \tag{7.1.26}$$

According to the Poisson formula (7.1.2), we have:

$$\theta_{\overline{E}}(x) = (\mathrm{covol}\,\overline{E})^{-1}\,\Theta(\|.\|^{\vee 2}, x).$$

The choice of some $\mathbb{Z}$-basis of E allows one to identify E with $\mathbb{Z}^g$, $E_{\mathbb{R}}$ and $E_{\mathbb{R}}^\vee$ with $\mathbb{R}^g$, and $\overset{\circ}{\mathcal{Q}}(E_{\mathbb{R}}^\vee) = \overset{\circ}{\mathcal{Q}}(\mathbb{R}^g)$ with the cone of positive definite symmetric matrices in $M_g(\mathbb{R})$. This cone embeds in $\mathfrak{H}_g$ by means of the map:

$$\overset{\circ}{\mathcal{Q}}(\mathbb{R}^g) \hookrightarrow \mathfrak{H}_g, \quad Y \longmapsto iY,$$

and, as a consequence of the definitions, the following equality holds for every (Y, x) in $\overset{\circ}{\mathcal{Q}}(\mathbb{R}^g) \times \mathbb{R}^g$:

$$\Theta_{\mathbb{Z}^g}(Y, x) = \theta(iY, x). \tag{7.1.27}$$

In other words, if $\overline{E} = (E, \|.\|)$ is the Euclidean lattice defined by:

$$E = \mathbb{Z}^g \quad \text{and} \quad \|x\|^2 = {}^t x.Y^{-1}.x, \tag{7.1.28}$$

then:

$$\theta_{\overline{E}}(x) = (\det Y)^{1/2}\,\theta(iY, x) \quad \text{for every } x \in E_{\mathbb{R}} \simeq \mathbb{R}^g. \tag{7.1.29}$$

[4] The literature on theta functions is considerable. The reader might refer to [**Igu72**], [**Mum83**], or [**BL04**] for modern treatments, to [**Kem91**] or [**Deb05**] for more concise introductions, and to [**KW20**] for historical references concerning the study of theta functions during the XIX-th century.

7.1.3.2. The function $\theta_{\overline{E}}$ admits another expression in terms of the Riemann theta function, which is a direct consequence of its definition (7.1.4), but takes a slightly less simple form than (7.1.29). Namely, with the notation (7.1.28), we also have:

$$\begin{aligned}\theta_{\overline{E}}(x) &= e^{-\pi\|x\|^2} \sum_{v\in E} e^{-\pi\|v\|^2+2\pi\langle v,x\rangle} \\ &= e^{-\pi {}^t x.Y^{-1}.x} \sum_{n\in\mathbb{Z}^g} e^{-\pi {}^t n.Y^{-1}.n+2\pi {}^t n.Y.x} \\ &=: e^{-\pi {}^t x.Y^{-1}.x}\, \theta(iY^{-1}, -iY^{-1}.x).\end{aligned} \tag{7.1.30}$$

The equality of the two expressions (7.1.29) and (7.1.30) is a special instance of the functional equation satisfied by the Riemann theta function:

$$\theta(-\tau^{-1}, \tau^{-1}.z) = \sqrt{\det(\tau/i)}\, \exp(\pi i\, {}^t z.\tau^{-1}.z)\, \theta(\tau, z), \tag{7.1.31}$$

valid for any (τ, z) in $\mathfrak{H}_g \times \mathbb{C}^g$, where the branch of the square root $\sqrt{\det(\tau/i)}$ is used which has positive value when τ is purely imaginary; see [**Mum83**, section II.5].

Conversely the equality of (7.1.29) and (7.1.30), established above by means of the Poisson formula (7.1.2), implies the functional equation (7.1.31) by analytic continuation.

7.2. The Theta Invariants h^0_θ and h^1_θ on $\overline{\mathbf{Coh}}_{\overline{X}}$

In Section 7.1, we have defined the theta invariants $h^0_\theta(\overline{E})$ and $h^1_\theta(\overline{E})$ associated to a Euclidean lattice $\overline{E}$, that is, to an element of $\overline{\mathbf{Vect}}_{\mathrm{Spec}\,\mathbb{Z}}$. We may easily extend these definitions to arbitrary objects of $\overline{\mathbf{Coh}}_{\overline{\mathrm{Spec}\,\mathbb{Z}}}$ and then, by taking the direct image by the morphism of schemes:

$$\pi : X \longrightarrow \mathrm{Spec}\,\mathbb{Z},$$

to an object of $\overline{\mathbf{Coh}}_{\overline{X}}$, where $X := \mathrm{Spec}\,\mathcal{O}_K$ denotes the arithmetic curve defined by the ring of integers $\mathcal{O}_K$ of an arbitrary number field K.

In this section, we spell out the definitions and the basic properties of the theta invariants in this setting. They may be seen as the arithmetic analogues of the non-negative integers $h^0(C, \mathcal{F})$ and $h^1(C, \mathcal{F})$ attached to a coherent sheaf $\mathcal{F}$ on some smooth projective curve C over some field.

The framework of this section is a minor extension of the one in [**Bos20b**, Chapter 2], devoted to the theta invariants $h^0_\theta(\overline{E})$ and $h^1_\theta(\overline{E})$ attached to objects in $\overline{\mathbf{Vect}}_X$: we now allow the objects $\overline{E}$ to lie in $\overline{\mathbf{Coh}}_{\overline{X}}$, and therefore to have a non-zero (but finite) torsion submodule E_{tor}.

The properties of the theta invariants in this more general context may actually be easily deduced from those in the more restrictive context of *loc. cit.* by considering the object $\overline{E}_{/\mathrm{tor}} := \overline{E/E_{\mathrm{tor}}}$ of $\overline{\mathbf{Vect}}_X$ attached to some object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\overline{X}}$, and we shall leave the proofs of Theorems 7.2.3 and 7.2.4 below as exercises for the reader.[5]

Definition 7.2.1. For every object $\overline{E} := (E, \|.\|)$ in $\overline{\mathbf{Coh}}_{\overline{\mathrm{Spec}\,\mathbb{Z}}}$, we let:

$$h^0_\theta(\overline{E}) := \log \sum_{v\in E} e^{-\pi\|v\|^2} \tag{7.2.1}$$

and:

$$h^1_\theta(\overline{E}) := h^0_\theta(\overline{E}^\vee). \tag{7.2.2}$$

Moreover for every object $\overline{E}$ in $\overline{\mathbf{Coh}}_{\overline{X}}$, we let:

$$h^i_\theta(\overline{E}) := h^i_\theta(\pi_*\overline{E}) \quad \text{for } i = 0 \text{ or } 1. \tag{7.2.3}$$

[5] The most delicate point in these proofs is the subadditivity of h^0_θ and h^1_θ, the proof of which on $\overline{\mathbf{Vect}}_{\mathrm{Spec}\,\mathbb{Z}}$ is subsumed in the one of Proposition 7.1.6, which may be easily extended to allow torsion.

The invariants $h^0_\theta(\overline{E})$ and $h^1_\theta(\overline{E})$ belong to $\mathbb{R}_+$. Moreover $h^0_\theta(\overline{E})$ (resp. $h^1_\theta(\overline{E})$) vanishes if and only if E vanishes (resp. E is torsion).

THEOREM 7.2.2. *For every object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\overline{X}}$, the following equalities hold:*

$$h^0_\theta(\overline{E}) = h^0_\theta(\overline{E}_{/\mathrm{tor}}) + \log|E_{\mathrm{tor}}|, \tag{7.2.4}$$

where E_{tor} denotes the torsion submodule of E and $|E_{\mathrm{tor}}|$ its cardinality, and $\overline{E}_{/\mathrm{tor}} := \overline{E/E_{\mathrm{tor}}}$;

$$h^1_\theta(\overline{E}) = h^1_\theta(\overline{E}_{/\mathrm{tor}}); \tag{7.2.5}$$

$$h^1_\theta(\overline{E}) = h^0_\theta(\overline{E}^\vee \otimes \overline{\omega}_\pi); \tag{7.2.6}$$

and:

$$h^0_\theta(\overline{E}) - h^1_\theta(\overline{E}) = \widehat{\deg}\, \pi_*\overline{E} = \widehat{\deg}\, \overline{E} - (\mathrm{rk} E/2). \log|\Delta_K|. \tag{7.2.7}$$

Indeed (7.2.2) and (7.2.5) follow from the definitions. The equality (7.2.6) follows from the existence of an isometric isomorphism

$$(\pi_*\overline{E})^\vee \xrightarrow{\sim} \pi_*(\overline{E}^\vee \otimes \overline{\omega}_\pi), \tag{7.2.8}$$

and will be referred to as the *Hecke duality formula* because of its implicit occurence in Hecke's paper [**Hec17**] when $\overline{E}$ is a Hermitian line bundle over $X := \operatorname{Spec} \mathcal{O}_K$. The equality (7.2.7) is a straightforward consequence of the Poisson-Riemann-Roch formula for Euclidean lattices in Corollary 7.1.3, and will also be referred to as the Poisson-Riemann-Roch formula.

THEOREM 7.2.3. *(1) The invariant $h^0_\theta(\overline{E})$ of an object $\overline{E} := (E, (\|.\|)_{x\in X(\mathbb{C})})$ of $\overline{\mathbf{Coh}}_{\overline{X}}$ depends $\mathbb{R}$-analytically of the Hermitian norms $\|.\|_x$ on the finite dimensional complex vector spaces $(E_x)_{x\in X(\mathbb{C})}$.*

(2) The invariant h^0_θ is additive over $\overline{\mathbf{Coh}}_{\overline{X}}$. Namely, for any two objects $\overline{E}_1$ and $\overline{E}_2$ in $\overline{\mathbf{Coh}}_{\overline{X}}$, we have:

$$h^0_\theta(\overline{E}_1 \oplus \overline{E}_2) = h^0_\theta(\overline{E}_1) + h^0_\theta(\overline{E}_2). \tag{7.2.9}$$

(3) The invariant h^0_θ is monotonic over $\overline{\mathbf{Coh}}_{\overline{X}}$ in the following sense. For every morphism

$$i : \overline{E}_1 \longrightarrow \overline{E}_2$$

in $\overline{\mathbf{Coh}}^{\leq 1}_{\overline{X}}$ such that the morphism of $\mathcal{O}_K$-module $i : E_1 \longrightarrow E_2$ is injective, we have:

$$h^0_\theta(\overline{E}_1) \leq h^0_\theta(\overline{E}_2). \tag{7.2.10}$$

Moreover equality holds in (7.2.10) if and only i is an isometric isomorphism.

(4) The invariant h^0_θ is subadditive over $\overline{\mathbf{Coh}}_{\overline{X}}$. Namely, for every admissible exact sequence

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0 \tag{7.2.11}$$

in $\overline{\mathbf{Coh}}_{\overline{X}}$, we have:

$$h^0_\theta(\overline{F}) \leq h^0_\theta(\overline{E}) + h^0_\theta(\overline{G}). \tag{7.2.12}$$

Moreover equality holds in (7.2.12) if and only the admissible short exact sequence (7.2.11) is split.[6]

[6]Namely if there exists a morphism $s : \overline{G} \to \overline{F}$ in $\overline{\mathbf{Coh}}^{\leq 1}_{\overline{X}}$ (necessarily isometric) such that $s \circ p = \mathrm{Id}_G$.

THEOREM 7.2.4. *(1) The invariant $h^1_\theta(\overline{E})$ of an object $\overline{E} := (E, (\|.\|)_{x\in X(\mathbb{C})})$ of depends continuously of the Hermitian norms $\|.\|_x$ on the finite dimensional complex vector spaces $(E_x)_{x\in X(\mathbb{C})}$.*

(2) The invariant h^1_θ is additive over $\overline{\mathbf{Coh}}_X$. Namely, for any two objects $\overline{E}_1$ and $\overline{E}_2$ in $\overline{\mathbf{Coh}}_X$, we have:

$$h^1_\theta(\overline{E}_1 \oplus \overline{E}_2) = h^1_\theta(\overline{E}_1) + h^1_\theta(\overline{E}_2). \tag{7.2.13}$$

(3) The invariant h^1_θ is monotonic over $\overline{\mathbf{Coh}}_X$ in the following sense. For every morphism

$$p : \overline{E}_1 \longrightarrow \overline{E}_2$$

in $\overline{\mathbf{Coh}}^{\leq 1}_X$ such that the morphism of K-vector spaces $p : E_{1,K} \longrightarrow E_{2,K}$ is surjective, we have:

$$h^1_\theta(\overline{E}_1) \geq h^1_\theta(\overline{E}_2). \tag{7.2.14}$$

Moreover equality holds in (7.2.14) *if and only the morphism $p_{/\mathrm{tor}} : \overline{E}_{1/\mathrm{tor}} \to \overline{E}_{2/\mathrm{tor}}$ induced by p is an isometric isomorphism.*

(4) The invariant h^1_θ is subadditive over $\overline{\mathbf{Coh}}_X$. Namely, for every admissible exact sequence

$$0 \longrightarrow \overline{E} \xrightarrow{i} \overline{F} \xrightarrow{p} \overline{G} \longrightarrow 0 \tag{7.2.15}$$

in $\overline{\mathbf{Coh}}_X$, we have:

$$h^1_\theta(\overline{F}) \leq h^1_\theta(\overline{E}) + h^1_\theta(\overline{G}). \tag{7.2.16}$$

Moreover equality holds in (7.2.16) *if and only the admissible short exact sequence* (7.2.15) *is split.*

Together with the Poisson-Riemann-Roch formula (7.2.7), the monotonicity properties of h^0_θ and h^1_θ in Theorems 7.2.3 and 7.2.4 imply the following estimates:

COROLLARY 7.2.5. *Let $f : \overline{E}_1 \to \overline{E}_2$ be a morphism in $\overline{\mathbf{Coh}}^{\leq 1}_X$.*

If the morphism of K-vector spaces $f_K : E_{1,K} \to E_{2,K}$ is injective — that is, if $\ker f$ is a torsion $\mathcal{O}_K$-module, or equivalently a finite set — then the following inequality holds:

$$-\log|\ker f| \leq h^0_\theta(\overline{E}_2) - h^0_\theta(\overline{E}_1). \tag{7.2.17}$$

If f_K is surjective — that is, if $\operatorname{coker} f := E_2/f(E_1)$ is a torsion $\mathcal{O}_K$-modules, or equivalently a finite set — then the following inequality holds:

$$h^0_\theta(\overline{E}_2) - h^0_\theta(\overline{E}_1) \leq \widehat{\deg}\,\overline{E}_2 - \widehat{\deg}\,\overline{E}_1 + (1/2)(\operatorname{rk} E_1 - \operatorname{rk} E_2)\log|\Delta_K|. \tag{7.2.18}$$

If f_K is bijective, then the following inequality holds:

$$-\log|\ker f| \leq h^0_\theta(\overline{E}_2) - h^0_\theta(\overline{E}_1) \leq -\log|\ker f| + \log|\operatorname{coker} f| - \sum_{x\in X(\mathbb{C})} \|\det f_x\|_x. \tag{7.2.19}$$

In the last sum in (7.2.19), for every $x \in X(\mathbb{C})$, $\det f_x$ denotes the element of $\det E^\vee_{1,x} \otimes \det E_{2,x}$ defined as the determinant of $f_x : E_{1,x} \to E_{2,x}$, and $\|.\|_x$ denotes the Hermitian norm on the complex line $\det E^\vee_{1,x} \otimes \det E_{2,x}$ deduced from the Hermitian norms $\|.\|_{\overline{E}_{1,x}}$ and $\|.\|_{\overline{E}_{2,x}}$ on $E_{1,x}$ and $E_{2,x}$ by exterior power, duality, and tensor product.

PROOF. The inequality (7.2.17) is a straightforward consequence of the definition of h^0_θ. It also follows from the monotonicity (7.2.10) of h^0_θ applied to the morphism $f_{/\mathrm{tor}} : \overline{E}_{1,/\mathrm{tor}} \to \overline{E}_{2,/\mathrm{tor}}$ induced by f, from by the equality (7.2.4) applied to $\overline{E} = \overline{E}_1$ and $\overline{E} = \overline{E}_2$, and from the exact sequence:

$$0 \longrightarrow \ker f \hookrightarrow E_{1,\mathrm{tor}} \xrightarrow{f} E_{2,\mathrm{tor}},$$

which implies the estimate:

$$|\ker f|^{-1}\,|E_{1,\mathrm{tor}}| \leq |E_{2,\mathrm{tor}}|.$$

When f_K is surjective then its transpose $f_K^\vee : E_{2,K} \to E_{1,K}$ is injective, and therefore we may apply 7.2.4, (3) to the morphism:

$$f^\vee : \overline{E}_2^\vee \to \overline{E}_1^\vee$$

in $\overline{\mathbf{Vect}}_X^{\leq 1}$. Therefore we have:

$$h^1_\theta(\overline{E}_2^\vee) \geq h^1_\theta(\overline{E}_1^\vee),$$

and (7.2.18) follows thanks to the Poisson-Riemann-Roch formula (7.2.7) applied to $\overline{E} = \overline{E}_1$ and $\overline{E} = \overline{E}_2$.

Finally, if f_K is bijective, then:

$$\mathrm{rk}\, E_1 = \mathrm{rk} E_2,$$

and, according to the definition of the Arakelov degree, the following equality holds;

$$\widehat{\deg}\,\overline{E}_2 - \widehat{\deg}\,\overline{E}_1 = -\log|\ker f| + \log|\mathrm{coker}\, f| - \sum_{x\in X(\mathbb{C})} \|\det f_x\|_x. \qquad \square$$

7.3. The Theta Invariant h^1_θ on $\overline{\mathbf{Coh}}_X$

7.3.1. Construction and basic properties. Theorems 7.2.2 and 7.2.4 assert notably that the invariant:

$$h^1_\theta : \overline{\mathbf{Vect}}_X \longrightarrow \mathbb{R}_+$$

satisfies conditions $\mathbf{Mon}^1_K$, **SubAdd** and $\mathbf{Cont}^+$, and condition $\mathbf{Add}_\oplus$ as well. Consequently we may apply to h^1_θ the construction of extensions of invariants from $\overline{\mathbf{Vect}}_X$ to $\overline{\mathbf{Coh}}_X$ described in Section 4.2.

Moreover, if we let:

$$\overline{E} := (\pi_*\overline{\mathcal{O}}_X)^\vee,$$

then, for every $\delta \in \mathbb{R}$, we have:

$$h^1_\theta(\overline{\mathcal{O}}_X(\delta)) = h^0_\theta(\overline{E} \otimes \overline{\mathcal{O}}(-\delta)) = \log \sum_{v\in E} \exp\big(-\pi e^{2\delta}\|v\|^2_{\overline{E}}\big),$$

and therefore:

$$\lim_{\delta\to+\infty} h^1_\theta(\overline{\mathcal{O}}_X(\delta)) = 0.$$

Consequently Scholium 4.2.10 and Proposition 4.2.12 apply, and we obtain:

SCHOLIUM 7.3.1. *There exists a unique invariant:*

$$h^1_\theta : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

that extends the invariant

$$h^1_\theta : \overline{\mathbf{Vect}}_X \longrightarrow \mathbb{R}_+$$

introduced in Definition 7.2.1[7] *and that satisfies* $\mathbf{Mon}^1$, **SubAdd**, $\mathbf{Cont}^+$, *and* **NST**.

This invariant satisfies $\mathbf{Add}_\oplus$ *and is small on Hermitian coherent sheaves generated by small sections. Moreover, for every objects* $\overline{E}$ *of* $\overline{\mathbf{Coh}}_X$*, the following relation holds:*

$$h^1_\theta(\overline{E}) = h^1_\theta(\overline{E}^{\mathrm{vect}}). \tag{7.3.1}$$

[7] restricted to objects $\overline{E}$ of $\overline{\mathbf{Coh}}_{\overline{\mathrm{Spec}\,\mathbb{Z}}}$ or $\overline{\mathbf{Coh}}_{\overline{X}}$ with E torsion free.

The notation h^1_θ for the invariant h^1_θ on $\overline{\mathbf{Coh}}_X$ constructed in Scholium 7.3.1 does not conflict with the notation in Definition 7.2.1. Indeed, as a consequence of $\mathbf{Mon}^1$, **SubAdd**, and **NST**, or as a special case of (7.3.1), the invariant h^1_θ constructed in Scholium 7.3.1 satisfies the equality:

$$h^1_\theta(\overline{E}) = h^1_\theta(\overline{E}_{/\mathrm{tor}}) \tag{7.3.2}$$

for every object $\overline{E}$ of $\overline{\mathbf{Coh}}_X$, and a fortiori for every object of $\overline{\mathbf{Coh}}_{\overline{X}}$.

The compatibility with vectorization (7.3.1) of h^1_θ also implies the following equivalence, for every object $\overline{E}$ of $\overline{\mathbf{Coh}}_X$:

$$h^1_\theta(\overline{E}) = 0 \Longleftrightarrow \overline{E}^{\mathrm{vect}} = 0.$$

We refer the reader to Proposition 2.3.5 for diverse characterizations of the vanishing of $\overline{E}^{\mathrm{vect}}$.

Moreover, according to Proposition 4.6.2, the invariants h^1_θ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and $\overline{\mathbf{Coh}}_X$ are related by the equality:

$$h^1_\theta(\overline{E}) = h^1_\theta(\pi_*\overline{E}), \tag{7.3.3}$$

valid for every object $\overline{E}$ of $\overline{\mathbf{Coh}}_X$.

Finally recall that the invariant $h^1_\theta(\overline{E})$ of some object $\overline{E}$ of $\overline{\mathbf{Coh}}_X$ may be expressed as a limit, in terms of the value of h^1_θ on some objects of $\overline{\mathbf{Coh}}_{\overline{X}}$ or of $\overline{\mathbf{Vect}}_X$. Indeed, as a special instance of the discussion in 4.2.3.3, combined with the equality (7.3.2), we obtain:

SCHOLIUM 7.3.2. *For every object $\overline{E} := (E, (\|.\|_x)_{x\in X(\mathbb{C})})$ in $\overline{\mathbf{Coh}}_X$, the following equality holds:*

$$h^1_\theta(\overline{E}) := \lim_{n\to+\infty} h^1_\theta((E, (\|.\|^{[n]}_x)_{x\in X(\mathbb{C})})) = \lim_{n\to+\infty} h^1_\theta((E_{/\mathrm{tor}}, (\|.\|^{[n]}_x)_{x\in X(\mathbb{C})})), \tag{7.3.4}$$

where $((\|.\|^{[n]}_x)_{x\in X(\mathbb{C})})_{n\in\mathbb{N}}$ is any sequence of Hermitian structures on E such that, for every $x \in X(\mathbb{C})$, $(\|.\|^{[n]}_x)_{n\in\mathbb{N}}$ is a sequence of Hermitian norms on E_x converging pointwise to $\|.\|_x$ that satisfies $\|.\|^{[n]}_x \geq \|.\|_x$ for every $n \in \mathbb{N}$.

7.3.2. The invariant h^0_θ on $\overline{\mathbf{Coh}}_X$ and on $\overline{\mathbf{Vect}}^{[\infty]}_X$ and a duality formula. When $\overline{E}$ is an object of $\overline{\mathbf{Coh}}_{\overline{X}}$, its theta-invariant $h^1_\theta(\overline{E})$ has been defined by (7.2.1), (7.2.2), and (7.2.1) in Definition 7.2.1, and therefore may be expressed in terms of the Euclidean lattice

$$(\pi_*\overline{E})^\vee \simeq \pi_*(\overline{E}^\vee \otimes \overline{\omega}_\pi)$$

by the following closed formula:

$$h^1_\theta(\overline{E}) = h^0_\theta((\pi_*\overline{E})^\vee) = \log \sum_{\xi\in(\pi_*E)^\vee} e^{-\pi\|\xi\|^2_{(\pi_*\overline{E})^\vee}}. \tag{7.3.5}$$

One may wish to extend this expression to arbitrary object $\overline{E}$ in $\overline{\mathbf{Coh}}_X$, defined by Hermitian seminorms that are possibly not Hermitian norms. This requires to extend the definition of the theta invariant h^0_θ so that it makes sense for more general objects than Euclidean lattices or Hermitian vector bundles over X.

7.3.2.1. The real valued invariant h^0_θ on $\overline{\mathbf{Vect}}_X$ admits a straightforward generalization to an invariant:

$$h^0_\theta : \overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty].$$

Namely, for every object $\overline{F} := (F, (\|.\|_x)_{x\in X(\mathbb{C})})$ in $\overline{\mathbf{Coh}}_X$, we let:

$$h^0_\theta(\overline{F}) = \log \sum_{s\in F} e^{-\pi\|s\|^2_{\pi_*\overline{F}}}. \tag{7.3.6}$$

This extension of h^0_θ is natural, in so far as it satisfies a "limit formula" similar to the one satisfied by h^1_θ spelled out in Scholium 7.3.2. Indeed, with the notation of Scholium 7.3.2, one easily sees that the following equality holds in $[0, +\infty]$:

$$h^1_\theta(\overline{F}) := \lim_{n \to +\infty} h^1_\theta((F, (\|.\|_x^{[n]})_{x \in X(\mathbb{C})})).$$

However the following proposition shows that this extension of the invariant h^0_θ to $\overline{\mathbf{Coh}}_X$ has no significant new content.

PROPOSITION 7.3.3. *Let $\overline{F} := (F, (\|.\|_x)_{x \in X(\mathbb{C})})$ be an object of $\overline{\mathbf{Coh}}_X$.*

If $h^0_\theta(\overline{F}) < +\infty$, then, for every $x \in X(\mathbb{C})$, the Hermitian seminorm $\|.\|_x$ is a norm. In other words, $\overline{F}$ is an object of $\overline{\mathbf{Coh}_{\overline{X}}}$.

This is a consequence of the following observation: if $\overline{F} := (F, \|.\|)$ is an object of $\overline{\mathbf{Coh}}_\mathbb{Z}$ and if the Euclidean norm $\|.\|$ on $F_\mathbb{R}$ is not a norm, then, for every $\varepsilon > 0$, the set:

$$\{v \in F \mid \|v\| < \varepsilon\}$$

is infinite. We shall not use Proposition 7.3.3, and we leave the details of its proof to the reader.

7.3.2.2. For our purpose, the relevant extension of the invariant h^0_θ on $\overline{\mathbf{Vect}}_X$ is the invariant:

$$h^0_\theta : \overline{\mathbf{Vect}}_X^{[\infty]} \longrightarrow \mathbb{R}_+$$

on the category of definite Hermitian quasinormed vector bundles over X introduced in Chapter 2, that is still defined by the equality (7.3.6) understood as follows.

An object $\overline{F} := (F, (\|.\|_x)_{x \in X(\mathbb{C})})$ is pair consisting in a finitely generated $\mathcal{O}_K$-module and in a family $(\|.\|_x)_{x \in X(\mathbb{C})}$, invariant under complex conjugations, of definite Hermitian quasinorms on the $\mathbb{C}$-vector spaces $(E_x)_{x \in X(\mathbb{C})}$.

Its direct image $\pi_*\overline{F}$ is the object $(\pi_* F, \|.\|_{\pi_*\overline{F}})$ of $\overline{\mathbf{Vect}}_\mathbb{Z}^{[\infty]}$, where $\pi_* F$ denotes F seen as a $\mathbb{Z}$-module and $\|.\|_{\pi*\overline{F}}$ denotes the definite Hermitian quasinorm on:

$$(\pi_* F)_\mathbb{C} := F \otimes_\mathbb{Z} \mathbb{C} \simeq \bigoplus_{x \in X(\mathbb{C})} F_x$$

such that, for every $v = (v_x)_{x \in \mathbb{C}}$ in $(\pi_* F)_\mathbb{C}$:

$$\|v\|^2_{\pi_*\overline{F}} = \sum_{x \in X(\mathbb{C})} \|v_x\|^2_x.$$

The invariant $h^0_\theta(\overline{F})$ is defined as:

$$h^0_\theta(\overline{F}) = \log \sum_{s \in F} e^{-\pi \|s\|^2_{\pi_*\overline{F}}}, \tag{7.3.7}$$

where the function $(x \mapsto e^{-\pi x^2})$ of $x \in \mathbb{R}_+$ that appears in the right-hand side of (7.3.7) is extended by continuity to $x \in [0, +\infty]$ by letting:

$$e^{-\pi \infty^2} := 0.$$

The sum in the in the right-hand side of (7.3.7) is easily seen to lie in $[1, +\infty)$, and $h^0_\theta(\overline{F})$ to satisfy:

$$h^0_\theta(\overline{F}) = h^0_\theta(\pi_*\overline{F}) \in \mathbb{R}_+.$$

As a straighforward consequence of the definitions, the invariant $h^0_\theta(\overline{F})$ so defined by (7.3.7), seen as a function of the quasinorms $(\|.\|_x)_{x \in X(\mathbb{C})}$, satisfies the following continuity property:

PROPOSITION 7.3.4. *For every $n \in \mathbb{N}$, let $(\|.\|^n_x)_{x\in X(\mathbb{C})}$ be a family, invariant under complex conjugations, of definite Hermitian quasinorms on $\mathbb{C}$-vector spaces $(E_x)_{x\in X(\mathbb{C})}$. Assume that, for every $x \in X(\mathbb{C})$, the sequence $(\|.\|^n_x)_{n\in\mathbb{N}}$ of quasinorms on E_x converges pointwise to $\|.\|_x$ and satisfies:*

$$\|.\|^n_x \geq \|.\|^\sim_x \quad \text{for every } n \in \mathbb{N},$$

where $\|.\|^\sim_x$ denotes some fixed definite quasinorm on E_x. Then we have:

$$h^0_\theta(\overline{F}) = \lim_{n\to+\infty} h^0_\theta(F, (\|.\|^n_x)_{x\in X(\mathbb{C})}).$$

Recall finally that an object $\overline{E} := (E, (\|.\|)_{x\in X(\mathbb{C})})$ of $\overline{\mathbf{Coh}}_X$ admits a dual object $\overline{E}^\vee := (E^\vee, (\|.\|^\vee)_{x\in X(\mathbb{C})})$ in $\overline{\mathbf{Vect}}^{[\infty]}_X$. It is defined by the $\mathcal{O}_K$-module:

$$E^\vee := \mathrm{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K)$$

and by the definite quasinorms $\|.\|_x$ on the $\mathbb{C}$-vector spaces:

$$E^\vee_x \simeq \mathrm{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$$

dual to the seminorms $\|.\|_x$. Moreover the duality isomorphism (7.2.8) still holds in this setting.

7.3.2.3. Using the previous formalism, we may establish the following extension to objects of $\overline{\mathbf{Coh}}_X$ of the expression (7.3.5) for the invariant $h^1_\theta(\overline{E})$ of an object $\overline{E}$ in $\overline{\mathbf{Coh}}_{\overline{X}}$.

PROPOSITION 7.3.5. *For every object $\overline{E}$ of $\overline{\mathbf{Coh}}_X$, the following equalities holds:*

$$h^1_\theta(\overline{E}) = h^0_\theta(\overline{E}^\vee \otimes \overline{\omega}_\pi) = \log \sum_{\xi\in(\pi_* E)^\vee} e^{-\pi\|\xi\|^2_{(\pi_*\overline{E})^\vee}}. \tag{7.3.8}$$

PROOF. This a straightforward consequences of (7.3.5) when the seminorms defining $\overline{E}$ are actually norms, and of the continuity properties of the invariants h^1_θ on $\overline{\mathbf{Coh}}_X$ and h^0_θ on $\overline{\mathbf{Vect}}^{[\infty]}_X$ with respect to the seminorms (resp. quasinorms) defining the objects of $\overline{\mathbf{Coh}}_X$ (resp. of $\overline{\mathbf{Vect}}^{[\infty]}_X$) in Scholium 7.3.2 (resp. in Proposition 7.3.4). □

Remark that, with the notation of Proposition 7.3.5, an element ξ of $(\pi_* E)^\vee$ satisfies

$$\|\xi\|_{(\pi_*\overline{E})^\vee} < +\infty$$

if and only if ξ belongs to the (image of the module underlying) $(\pi_*\overline{E}^{\mathrm{vect}})^\vee$. This follows from the description of the vectorization functor over $\operatorname{Spec}\mathbb{Z}$ by duality in Corollary 2.3.3, and from the compatibility of vectorization with direct images to $\operatorname{Spec}\mathbb{Z}$ discussed in Subsection 2.3.2.

The right-hand side of (7.3.8) therefore coincides with:

$$h^0_\theta((\pi_*\overline{E}^{\mathrm{vect}})^\vee) = h^1_\theta(\overline{E}^{\mathrm{vect}}),$$

and the equality (7.3.8) may be seen as a reformulation of the invariance of h^1_θ under vectorization stated in (7.3.1).

7.3.3. The theta invariant $h^1(E, \|.\|)$ as a function of the seminorm $\|.\|$. In this subsection, we briefly discuss the regularity property of the theta invariant $h^1_\theta(\overline{E})$ of an object $\overline{E}$ of $\overline{\mathbf{Coh}}_X$ as a function of the Hermitian seminorms $(\|.\|_{\overline{E},x})_{x\in X(\mathbb{C})}$ that define $\overline{E}$.

We only consider the situation where X is $\operatorname{Spec}\mathbb{Z}$, since the case of a general arithmetic curve reduces to this one because of the compatibility (7.3.3) of h^1_θ with direct images. Our discussion will be similar to the one concerning the regularity properties of the covering radius in 6.3.2 above.

Let E be a finitely generated free $\mathbb{Z}$-module. The theta invariant h^1_θ defines a function on the cone $\mathcal{Q}(E_{\mathbb{R}})$ of Euclidean seminorms on $E_{\mathbb{R}}$:

$$h^1_\theta(E, .) : \mathcal{Q}(E_{\mathbb{R}}) \longrightarrow \mathbb{R}_+, \quad \|.\| \longmapsto h^1_\theta(E, \|.\|). \tag{7.3.9}$$

PROPOSITION 7.3.6. *The function $h^1_\theta(E, .) : \mathcal{Q}(E_\mathbb{R}) \to \mathbb{R}_+$ is upper semi-continuous, and its restriction to the open cone $\mathring{\mathcal{Q}}(E_\mathbb{R})$ of Euclidean norms on $E_\mathbb{R}$ is $\mathbb{R}$-analytic.*

PROOF. Restricted to $\mathring{\mathcal{Q}}(E_\mathbb{R})$, the function $h^0_\theta(E, .)$ is clearly increasing and $\mathbb{R}$-analytic.

If we choose some element $\|.\|_0$ of $\mathring{\mathcal{Q}}(E_\mathbb{R})$, and if, to an element $\|.\|$ of $\mathcal{Q}(E_\mathbb{R})$ and to a positive integer k, we attach the element $\|.\|_k$ of $\mathring{\mathcal{Q}}(E_\mathbb{R})$ defined by:

$$\|.\|_k^2 := \|.\|^2 + k^{-1}\|.\|_0^2, \tag{7.3.10}$$

then, according to Scholium 7.3.2, the sequence $(h^1_\theta(E, \|.\|_k))_{k\geq 1}$ is decreasing and satisfies:

$$\lim_{k\to+\infty} h^1_\theta(E, \|.\|_k) = h^1_\theta(E, \|.\|). \tag{7.3.11}$$

This exhibits the function $h^0_\theta(E, .)$ on $\mathcal{Q}(E_\mathbb{R})$ as the pointwise limit of a decreasing sequence of continuous functions, and therefore establishes its upper semicontinuity. □

When E has rank at least 2, the function $h^1_\theta(E, .)$ defined in (7.3.9) is not continuous on $\mathcal{Q}(E_\mathbb{R})$. Indeed, as was already shown to be the case for the covering radius in Proposition 6.3.9, the restriction of $h^1_\theta(E, .)$ to the subcone $\mathcal{Q}(E_\mathbb{R})_1$ of extremal rays of $\mathcal{Q}(E_\mathbb{R})$ is already non-continuous. This follows from Proposition 7.3.7 below, which is similar to Proposition 6.3.9 and uses the same notation.

Recall that, in paragraph 2.3.1.5, to any $\xi \in E^\vee_\mathbb{R} \setminus \{0\}$, we have associated the semipositive quadratic forms $\xi^2 \in \mathcal{Q}(E_\mathbb{R})_1$ and the Euclidean coherent sheaf:

$$\overline{E}_\xi := (E, |\xi|).$$

When the point $[\xi] \in \mathbb{P}^2(\mathbb{R})$ belongs to $\mathbb{P}^2(\mathbb{Q})$, we have defined $t(\xi) \in \mathbb{R}^*_+$ by the relation:

$$\mathbb{R}\xi \cap E^\vee = \mathbb{Z}\, t(\xi)\xi,$$

or equivalently by the fact that $t(\xi)\xi$ is a primitive representative of $[\xi]$ in $E^\vee \setminus \{0\}$.

We may assume to have chosen a Euclidean norm $\|.\|_0$ on $E_\mathbb{R}$. To this norm is attached a height function:

$$\mathrm{ht} : \mathbb{P}(E)(\mathbb{Q}) \longrightarrow \mathbb{R}$$

on $\mathbb{P}(E)(\mathbb{Q}) \simeq \mathbb{P}(E)(\mathbb{Z})$. By definition, if a point P in $\mathbb{P}(E)(\mathbb{Q})$ is the class $[\xi]$ of an element ξ in $E^\vee \setminus \{0\}$ that is primitive, then:

$$\mathrm{ht}(P) = \log \|\xi\|_0^\vee,$$

where $\|.\|_0^\vee$ denotes the Euclidean norm on $E^\vee_\mathbb{R}$ dual to $\|.\|_0$.

We may also introduce the function:

$$\theta : \mathbb{R}^*_+ \longrightarrow (1, +\infty)$$

defined by:

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi x n^2}.$$

Observe that, for every $\lambda \in \mathbb{R}^*_+$, we have:

$$h^1_\theta(\mathbb{Z}, \lambda\,|.|) = h^0_\theta(\mathbb{Z}, \lambda^{-1}\,|.|) = \log\theta(\lambda^{-2}), \tag{7.3.12}$$

and that this expression defines an increasing $\mathbb{R}$-analytic diffeomorphism:

$$\mathbb{R}^*_+ \xrightarrow{\sim} \mathbb{R}^*_+, \quad \lambda \longmapsto \log\theta(\lambda^{-2}).$$

PROPOSITION 7.3.7. *Let ξ be an element of $E^\vee_{\mathbb{R}} \setminus \{0\}$.*
If its class $[\xi]$ in $\mathbb{P}(E)(\mathbb{R})$ does not belong to $\mathbb{P}(E)(\mathbb{Q})$, then:

$$h^1_\theta(E, |\xi|) = 0.$$

If $[\xi]$ belongs to $\mathbb{P}(E)(\mathbb{Q})$, then:

$$h^1_\theta(E, |\xi|) = \log \theta(t(\xi)^2) = \log \theta\big(\|\xi\|_0^{\vee -2}\, e^{2\mathrm{ht}([\xi])}\big).$$

PROOF. This follows from (7.3.1), which implies the equality:

$$h^1_\theta(\overline{E}_\xi) = h^1_\theta(\overline{E}_\xi^{\mathrm{vect}}),$$

from the description of $\overline{E}_\xi^{\mathrm{vect}}$ in Proposition 2.3.9, from (7.3.12), and from the relation, when $[\xi]$ belongs to $\mathbb{P}(E)(\mathbb{Q})$:

$$\mathrm{ht}([\xi]) = \log \|t(\xi)\xi\|_0^\vee = \log t(\xi) + \log \|\xi\|_0^\vee. \qquad \square$$

7.4. The Banaszczyk Functions $B_{\overline{V},\Lambda}$ and $b_{\overline{V},\Lambda}$ and their Monotonicity Properties

In this section, we study in more details the Banascczyk functions attached to a Euclidean lattice, or more generally, to a discrete subgroup of a finite dimensional Euclidean vector space. We establish some remarkable estimates and monotonicity properties satisfies by these functions, which are basically due to Banasczyk and to Regev and Stephens-Davidowitz.

The key role of Banascczyk functions in the study of Euclidean lattice is already conspicuous in Banaszczyk's seminal paper [**Ban93**]. Some of the properties discussed in this section were established in the preprint version of this paper [**Ban92**], and used in further works by Banaszczyk and by Banaszczyk and Steglinski; see notably [**Ban00**], [**BS08**], and [**BS17**]. Some of these have been independently established in [**RSD17a**].

The recent paper [**Ban22**] by Banaszczyk investigates these properties in a more general framework, which sheds some light on their significance from the perspective of harmonic analysis.

7.4.1. The functions $B_{\overline{V},\Lambda}$ and $b_{\overline{V},\Lambda}$.

7.4.1.1. To a Euclidean lattice $\overline{E} := (E, \|.\|)$, we have attached the two functions:

$$B_{\overline{E}} : E_{\mathbb{R}} \longrightarrow \mathbb{R}^*_+ \quad \text{and} \quad b_{\overline{E}} : E_{\mathbb{R}} \longrightarrow \mathbb{R}$$

defined by the following relations, for every $x \in E_{\mathbb{R}}$:

$$B_{\overline{E}}(x) := \theta_{\overline{E}}(x)/\theta_{\overline{E}}(0) = \frac{\sum_{v\in E} e^{-\pi\|v-x\|^2}}{\sum_{v\in E} e^{-\pi\|v\|^2}} =: e^{\pi b_{\overline{E}}(x)}. \tag{7.4.1}$$

More generally, if $\overline{V} := (V, \|.\|)$ is a finite dimensional Euclidean vector space and Λ is a discrete subgroup of the additive group $(V, +)$), we define the *Banaszczyk functions* attached to $(\overline{V}, \Lambda)$:

$$B_{\overline{V},\Lambda} : V \longrightarrow \mathbb{R}^*_+ \quad \text{and} \quad b_{\overline{V},\Lambda} : E_{\mathbb{R}} \longrightarrow \mathbb{R}$$

by letting, for every $x \in V$:

$$B_{\overline{V},\Lambda}(x) := \frac{\sum_{v\in\Lambda} e^{-\pi\|v-x\|^2}}{\sum_{v\in\Lambda} e^{-\pi\|v\|^2}} = e^{-\pi b_{\overline{V},\Lambda}(x)}. \tag{7.4.2}$$

In particular, for every Euclidean lattice $\overline{E}$, we have:

$$B_{\overline{E}} = B_{\overline{E_{\mathbb{R}}},E} \quad \text{and} \quad b_{\overline{E}} = b_{\overline{E_{\mathbb{R}}},E}$$

— where as usual $\overline{E}_{\mathbb{R}}$ denotes the Euclidean vector space $(E_{\mathbb{R}}, \|.\|)$ — and for every Euclidean vector space $\overline{V} := (V, \|.\|)$ and every $x \in V$:

$$B_{\overline{V},\{0\}}(x) = e^{-\pi\|x\|^2} \quad \text{and} \quad b_{\overline{V},\{0\}}(x) = \|x\|^2.$$

For typographical reasons, we shall also use the following alternative notation for the values of the Banaszczyk functions attached to $(\overline{V}, \Lambda)$:

$$B_{\overline{V},\Lambda}(x) = B(\overline{V}, \Lambda; x) = B(V, \|.\|, \Lambda; x)$$

and:

$$b_{\overline{V},\Lambda}(x) = b(\overline{V}, \Lambda; x) = b(V, \|.\|, \Lambda; x).$$

PROPOSITION 7.4.1. *If $\overline{V}_1$ (resp. $\overline{V}_2$) is a finite dimensional Euclidean vector space and if Λ_1 (resp. Λ_2) is a discrete subgroup of V_1 (resp. of V_2), the for every $x_1 \in V_1$ and $x_2 \in V_2$, we have:*

$$B(\overline{V}_1 \oplus \overline{V}_2, \Lambda_1 \oplus \Lambda_2; (x_1, x_2)) = B(\overline{V}_1, \Lambda_1; x_1)\, B(\overline{V}_2, \Lambda_2; x_2), \tag{7.4.3}$$

and:

$$b(\overline{V}_1 \oplus \overline{V}_2, \Lambda_1 \oplus \Lambda_2; (x_1, x_2)) = b(\overline{V}_1, \Lambda_1; x_1) + b(\overline{V}_2, \Lambda_2; x_2). \tag{7.4.4}$$

PROOF. This is a straightforward consequence of the definitions and of the identity:

$$e^{-\pi\|(z_1,z_2)\|^2_{\overline{V}_1\oplus\overline{V}_2}} = e^{-\pi\|z_1\|^2_{\overline{V}_1}}\, e^{-\pi\|z_2\|^2_{\overline{V}_2}},$$

valid for every $z_1 \in V_1$ and $z_2 \in V_2$. □

7.4.1.2. Let us consider $\overline{V} := (V, \|.\|)$ and Λ as above.

We shall denote by $\Lambda_{\mathbb{R}}$ the $\mathbb{R}$-vector subspace of V generated by Λ, which indeed may be identified with $\Lambda \otimes \mathbb{R}$, by

$$q : V \longrightarrow V/\Lambda_{\mathbb{R}}$$

the quotient map, by $\|.\|_{V/\Lambda_{\mathbb{R}}}$ the Euclidean norm on $V/\Lambda_{\mathbb{R}}$ defined as the quotient norm of $\|.\|$, and by

$$p^{\perp} : V \longrightarrow \Lambda_{\mathbb{R}}$$

the orthogonal projection from V onto $\Lambda_{\mathbb{R}}$.

PROPOSITION 7.4.2. *With the previous notation, if we introduce the Euclidean lattice:*

$$\overline{\Lambda} := (\Lambda, \|.\|_{|\Lambda_{\mathbb{R}}}),$$

then the following relations hold for every $x \in V$:

$$B_{\overline{V},\Lambda}(x) = e^{-\pi\|q(x)\|^2_{V/\Lambda_{\mathbb{R}}}}\, B_{\overline{\Lambda}}(p^{\perp}(x)), \tag{7.4.5}$$

and

$$b_{\overline{V},\Lambda}(x) = \|q(x)\|^2_{V/\Lambda_{\mathbb{R}}} + b_{\overline{\Lambda}}(p^{\perp}(x)). \tag{7.4.6}$$

PROOF. Observe that, for every x in V and every $v \in \Lambda$, the following equalities holds:

$$\|v - x\|^2 = \|q(x)\|^2_{V/\Lambda_{\mathbb{R}}} + \|v - p^{\perp}(x)\|^2.$$

Consequently:

$$e^{-\pi\|v-x\|^2} = e^{-\pi\|q(x)\|^2_{V/\Lambda_{\mathbb{R}}}}\, e^{-\pi\|v-p^{\perp}(x)\|^2}.$$

The relation (7.4.5) easily follows by summing over $v \in \Lambda$, and finally (7.4.6) by taking the logarithms. □

Proposition 7.4.2 may also be seen as a special case of Proposition 7.4.1 and reduces the study of the Banaszczyk functions attached to pairs $(\overline{V}, \Lambda)$ as above to the ones attached to Euclidean lattices.

PROPOSITION 7.4.3. *For any $(\overline{V},\Lambda)$ as above, the Banaszcyk functions $B_{\overline{V},\Lambda}$ and $b_{\overline{V},\Lambda}$ are real analytic and Λ-periodic.*

Moreover, for every $x \in V$, we have:

$$e^{-\pi\|x\|^2} \leq B_{\overline{V},\Lambda}(x) \leq 1, \tag{7.4.7}$$

and

$$B_{\overline{V},\Lambda}(x) = 1 \Longleftrightarrow x \in \Lambda.$$

Equivalently, we have:

$$0 \leq b_{\overline{V},\Lambda}(x) \leq \|x\|^2, \tag{7.4.8}$$

and

$$b_{\overline{V},\Lambda}(x) = 0 \Longleftrightarrow x \in \Lambda.$$

PROOF. Proposition 7.4.2 shows that the validity of these properties for an arbitrary pair $(\overline{V},\Lambda)$ follows from its validity when $B_{\overline{V},\Lambda}$ and $b_{\overline{V},\Lambda}$ are the Banaszczyk function $B_{\overline{E}}$ and $b_{\overline{E}}$ associated to some Euclidean lattice $\overline{E}$.

In this case, the assertions in Proposition 7.4.3 have already been observed in 7.1.1.2 and 7.1.2.1, with the exception of the equivalent estimates:

$$e^{-\pi\|x\|^2} \leq B_{\overline{V},\Lambda}(x), \tag{7.4.9}$$

and:

$$b_{\overline{V},\Lambda}(x) \leq \|x\|^2.$$

To establish (7.4.9), observe that, for every $x \in V$ and every $v \in \Lambda$, we have:

$$(1/2)\left(e^{-\pi\|v-x\|^2} + e^{-\pi\|v+x\|^2}\right) \geq e^{-(\pi/2)(\|v-x\|^2+\|v+x\|^2)} = e^{-\pi\|v\|^2}\, e^{-\pi\|x\|^2}.$$

Consequently the following estimate holds:

$$\begin{aligned}\sum_{v\in\Lambda} e^{-\pi\|v\|^2} B_{\overline{V},\Lambda}(x) &:= \sum_{v\in\Lambda} e^{-\pi\|v-x\|^2}\\ &= (1/2)\left(\sum_{v\in\Lambda} e^{-\pi\|v-x\|^2} + \sum_{v\in\Lambda} e^{-\pi\|v+x\|^2}\right)\\ &\geq \sum_{v\in\Lambda} e^{-\pi\|v\|^2}\, e^{-\pi\|x\|^2}.\end{aligned}$$

□

The first inequality in (7.4.7) admits the following amplification:

COROLLARY 7.4.4. *With the notation of Proposition 7.4.3, for every $x \in V$, the following inequality holds:*

$$e^{-\pi d_{\overline{V}}(x,\Lambda)^2} \leq B_{\overline{V},\Lambda}(x), \tag{7.4.10}$$

or equivalently:

$$b_{\overline{V},\Lambda}(x) \leq d_{\overline{V}}(x,\Lambda)^2, \tag{7.4.11}$$

where

$$d_{\overline{V}}(x,\Lambda) := \inf_{\lambda\in\Lambda} \|x-\lambda\|$$

denotes the distance from x to Λ in the Euclidean vector space $\overline{V}$.

PROOF. Since $B_{\overline{V},\Lambda}$ is Λ-periodic, the first inequality in (7.4.7) implies that, for every (x,λ) in $V\times\Lambda$, we have:

$$e^{-\pi\|x-\lambda\|^2}\leq B_{\overline{V},\Lambda}(x-\lambda)=B_{\overline{V},\Lambda}(x).$$

The inequality (7.4.10) follows by taking the supremum over λ in Λ, and finally (7.4.11) by taking logarithms. □

As we shall see in Chapter 9, the estimates in Corollary 7.4.4 immediately imply an upper bound on the theta invariant $h^1_\theta(\overline{E})$ of some Euclidean lattice $\overline{E}$ in terms of its covering radius $\rho(\overline{E})$; see Section 9, Proposition 9.1.3.

7.4.1.3. It is possible to describe the behavior of $b_{\overline{E}}$ near zero in $E_{\mathbb{R}}$:

PROPOSITION 7.4.5. *For every Euclidean lattice $\overline{E}:=(E,\|.\|)$, when $x\in E_{\mathbb{R}}$ goes to zero, we have:*

$$b_{\overline{E}}(x)=\|x\|^2-2\pi\frac{\sum_{v\in E}e^{-\pi\|v\|^2}\langle x,v\rangle^2}{\sum_{v\in E}e^{-\pi\|v\|^2}}+O(\|x\|^4). \tag{7.4.12}$$

We have denoted by $\langle.,.\rangle$ the scalar product on $E_{\mathbb{R}}$ defining the Euclidean norm $\|.\|$. The equality (7.4.12) follows from an elementary computation that we leave as an exercise for the reader.

Observe that, if $\lambda_{\overline{E}_{\mathbb{R}}}$ denotes the Lebesgue measure on the Euclidean vector space $\overline{E}_{\mathbb{R}}:=(E_{\mathbb{R}},\|.\|)$, we have, for every $x\in E_{\mathbb{R}}$:

$$\int_{E_{\mathbb{R}}}e^{-\pi\|y\|^2}\langle x,y\rangle^2\,d\lambda_{\overline{E}_{\mathbb{R}}}(y)=(2\pi)^{-1}\|x\|^2. \tag{7.4.13}$$

Consequently, the first two terms in the right-hand side of (7.4.12) coincide, up to a factor 2π, with the squared L^2-norms of the function $\langle x,.\rangle$ with respect to the probability measures $e^{-\pi\|\cdot\|}d\lambda_{\overline{E}_R}$ and $\beta_{\overline{E}}$ on $E_{\mathbb{R}}$ and E respectively, and (7.4.12) may be written as follows:

$$b_{\overline{E}}(x)=2\pi\|\langle x,.\rangle\|^2_{L^2(e^{-\pi\|\cdot\|^2}d\lambda_{\overline{E}_R})}-2\pi\|\langle x,.\rangle^2_{|E}\|_{L^2(\beta_{\overline{E}})}+O(\|x\|^4). \tag{7.4.14}$$

7.4.1.4. Let us emphasize that, according to the results of this subsection, the function $b_{\overline{V},\Lambda}$ on the quotient Lie group V/Λ appears as a counterpart of the function $\|.\|^2$ on the Euclidean vector space $\overline{V}:=(V,\|.\|)$ — a function that is actually the special case of $b_{\overline{V},\Lambda}$ when $\Lambda=\{0\}$.

This analogy is comforted by the fact, established by Banaszcyk in [**Ban22**], that the function $b^{1/2}_{\overline{V},\Lambda}$ is subadditive.

7.4.2. The quadratic inequality satisfied by Banaszczyk functions. The following simple estimate will play a key role in the sequel.

PROPOSITION 7.4.6. *Let $\overline{V}:=(V,\|.\|)$ be a finite dimensional Euclidean vector space and let Λ be a discrete subgroup of V. For every x and y in V, the following inequalities hold:*

$$B_{\overline{V},\Lambda}(x)^2B_{\overline{V},\Lambda}(y)^2\leq B_{\overline{V},\Lambda}(x+y)B_{\overline{V},\Lambda}(x-y). \tag{7.4.15}$$

The estimate (7.4.15) may be reformulated as follows in terms of the function $b_{\overline{V},\Lambda}$:

$$b_{\overline{V},\Lambda}(x+y)+b_{\overline{V},\Lambda}(x-y)\leq 2\big(b_{\overline{V},\Lambda}(x)+b_{\overline{V},\Lambda}(y)\big). \tag{7.4.16}$$

This relation may be compared with the parallelogram law satisfied by the quadratic form $\|.\|^2$:

$$\|x+y\|^2+\|x-y\|^2=2\big(\|x\|^2+\|y\|^2\big),$$

and therefore appears as a further instance of the analogy mentioned in 7.4.1.4.[8]

[8]The parallelogram law also show that the validity of the estimates (7.4.15) and (7.4.16) for the Banaszczyk functions attached to a pair $(\overline{V},\Lambda)$ follows from their validity for the Banaszczyk functions attached to a Euclidean lattice.

Observe also that the estimate (7.4.15) immediately implies:

COROLLARY 7.4.7. *With the notation of Proposition 7.4.6, for every x and y in V, the following inequality holds:*

$$B_{\overline{V},\Lambda}(x)B_{\overline{V},\Lambda}(y) \leq (1/2)\big(B_{\overline{V},\Lambda}(x+y) + B_{\overline{V},\Lambda}(x-y)\big). \tag{7.4.17}$$

From the inequalities (7.4.15), we shall also derive estimates concerning the Hessian of the function $b_{\overline{V},\Lambda}$. Recall that the space $S^2W^\vee$ of quadratic forms, and consequently the space of symmetric bilinear forms[9] $\Gamma^2 W^\vee$, on a finite dimensional $\mathbb{R}$-vector space W is equipped with an order relation $\preceq$ defined by the equivalence:

$$q_1 \preceq q_2 \quad \Longleftrightarrow \quad \text{the quadratic form } q_2 - q_1 \text{ is semipositive.} \tag{7.4.18}$$

COROLLARY 7.4.8. *With the notation of Proposition 7.4.6, for every $x \in V$, we have:*

$$D^2 b_{\overline{V},\Lambda}(x) \preceq D^2 b_{\overline{V},\Lambda}(0). \tag{7.4.19}$$

Up to a factor 2, the quadratic form $D^2 b_{\overline{V},\Lambda}(0)$ is the quadratic form given by the first two terms in the right-hand side of (7.4.12) or of (7.4.13).

This subsection is devoted to the proof of Proposition 7.4.6 and Corollary 7.4.8, directly inspired by the arguments in [**RSD17a**]. We shall follow a formal presentation to shed some light on the algebraic structures that underlie the derivation of the above estimates.

7.4.2.1. *From quadratic relations to lower bounds on Hessians.*

PROPOSITION 7.4.9. *Let $(A, +)$ be an additive group and let $(H, \langle .,. \rangle)$ be a real Hilbert space. If two maps $\varphi : A \to \mathbb{R}$ and $\psi : A \to \mathbb{R}$ satisfy the conditions:*

$$\varphi(x)\varphi(y) = \langle \psi(x+y), \psi(x-y) \rangle, \quad \textit{for every } (x, y) \in A^2, \tag{7.4.20}$$

and

$$\varphi(0) \neq 0, \tag{7.4.21}$$

then the function

$$\beta : A \longrightarrow \mathbb{R}, \quad x \longmapsto \varphi(0)^{-1}\varphi(x)$$

satisfies the estimates:

$$\beta(x)^2\beta(y)^2 \leq \beta(x+y)\beta(x-y), \quad \textit{for every } (x, y) \in A^2, \tag{7.4.22}$$

PROOF. The condition (7.4.20) with $y = 0$ reads:

$$\varphi(x)\varphi(0) = \langle \psi(x), \psi(x) \rangle =: \|\psi(x)\|^2. \tag{7.4.23}$$

Moreover, according to Cauchy-Schwarz inequality, it also implies:

$$\varphi(x)^2\varphi(y)^2 \leq \|\psi(x+y)\|^2 \|\psi(x-y)\|^2. \tag{7.4.24}$$

This establishes the estimates:

$$\varphi(x)^2\varphi(y)^2 \leq \varphi(x+y)\varphi(x-y)\varphi(0)^2, \quad \text{for every } (x, y) \in A^2.$$

When (7.4.21) is satisfied, this may written as (7.4.22). □

[9] If E denotes denotes a vector space over some field k, one denotes by $S^n E$ and $\Gamma^n E$ the k-vector spaces of coinvariants and invariants of $V^{\otimes n}$ under the action of the permutation group $\mathfrak{S}_n$. When the characteristic of k is 0 or $> n$, these spaces may be identified by the composite of the canonical maps $\Gamma^n E \hookrightarrow E^{\otimes n} \twoheadrightarrow S^n E$.

Observe that, besides (7.4.23), the validity also implies the relations:

$$\varphi(y)\varphi(0) = \langle \psi(y), \psi(-y)\rangle. \tag{7.4.25}$$

and:

$$\varphi(x)^2 = \langle \psi(2x), \psi(0)\rangle. \tag{7.4.26}$$

This shows that, when (7.4.20) holds, $\varphi(0)$ vanishes if and only if φ and ψ vanish everywhere on A, and that φ is an even function[10].

COROLLARY 7.4.10. *Let us keep the notation of Proposition 7.4.9, and let us assume that* (7.4.20) *and* (7.4.21) *are satisfied. If* $(A, +)$ *is the underlying group underlying a finite dimensional real vector space* V *and if* φ *is of class* C^2 *and does not vanish on* V*, then, for every* $x \in V$*, the following inequality holds in* $\Gamma^2 V^\vee$*:*

$$D^2 \log \beta(0) \preceq D^2 \log \beta(x). \tag{7.4.27}$$

Observe that, like φ, the function β is of class C^2 and does not vanish on V, and is therefore everywhere positive since $\beta(0) = 1$. Actually one may easily show that this condition of non-vanishing is necessarily satisfied when φ is also assumed to be real analytic.

PROOF. The estimates (7.4.22) may be written:

$$\log \beta(x) + \log \beta(y) \leq (1/2)\left(\log \beta(x+y) + \log \beta(x-y)\right), \quad \text{for every } (x, y) \in V^2. \tag{7.4.28}$$

Since β is even and takes the value 1 at zero, both $\log \beta$ and $D \log \beta$ vanish at zero, and therefore, when $y \in V$ goes to zero in V, we have:

$$\log \beta(x) + \log \beta(y) = \log \beta(x) + (1/2) D^2 \log \beta(0).y^{\otimes 2} + o(\|y\|^2). \tag{7.4.29}$$

Moreover, we also have, when $y \in V$ goes to zero in V:

$$\log \beta(x+y) = \log \beta(x) + D \log \beta(x).y + (1/2) D^2 \log \beta(x).y^{\otimes 2} + o(\|y\|^2),$$

and therefore:

$$(1/2)\left(\log \beta(x+y) + \log \beta(x-y)\right) = \log \beta(x) + (1/2) D^2 \log \beta(x).y^{\otimes 2} + o(\|y\|^2). \tag{7.4.30}$$

The inequality (7.4.27) follows by reporting (7.4.29) and (7.4.30) in (7.4.28). $\square$

7.4.2.2. *Banaszczyk functions and quadratic relations.* Let $\overline{V} := (V, \|.\|)$ be some finite dimensional Euclidean vector space and let Λ be some discrete subgroup of V. To $(\overline{V}, \Lambda)$, we may attach the function

$$\theta_{\overline{V},\Lambda} : \mathbb{R}_+^* \times V \longrightarrow \mathbb{R}$$

defined by:

$$\theta_{\overline{V},\Lambda}(t, x) := \sum_{v \in \Lambda} e^{-\pi t \|x - v\|^2}.$$

Let us denote by $[v]$ the class in $\Lambda_{\mathbb{F}_2} \simeq \Lambda/2\Lambda$ of an element v of Λ. For every c in Λ the coset $\Lambda + c/2$ in $V_{\mathbb{R}}$ depends only of the class $[c]$, and the sum

$$\psi_{[c]}(z) := \sum_{w \in \Lambda + c/2} e^{-\pi/2 \|z - w\|^2} = \theta_{\overline{V},\Lambda}(1/2; z - c) \tag{7.4.31}$$

defines a Λ-periodic function of $z \in V_{\mathbb{R}}$.

The following proposition, combined with Proposition 7.4.9 and Corollary 7.4.10, completes the proof of Proposition 7.4.6 and Corollary 7.4.8.

[10]Namely, it satisfies: $\varphi(-x) = \varphi(x)$ for every $x \in A$.

PROPOSITION 7.4.11. *With the previous notation, for every x and y in $V_{\mathbb{R}}$, the following relation holds:*

$$\theta_{\overline{V},\Lambda}(x)\theta_{\overline{V},\Lambda}(y) = \sum_{\alpha\in\Lambda_{\mathbb{F}_2}} \psi_\alpha(x+y)\psi_\alpha(x-y). \tag{7.4.32}$$

Indeed, we may consider $H := \mathbb{R}^{\Lambda_{\mathbb{F}_2}}$ equipped with the scalar product $\langle .\,,.\rangle$ defined by:

$$\langle (t_\alpha)_{\alpha\in\Lambda_{\mathbb{F}_2}}, (t'_\alpha)_{\alpha\in\Lambda_{\mathbb{F}_2}}\rangle := \sum_{\alpha\in\Lambda_{\mathbb{F}_2}} t_\alpha t'_\alpha,$$

and the map

$$\psi := (\psi_\alpha)_{\alpha\in\Lambda_{\mathbb{F}_2}} : V_{\mathbb{R}} \longrightarrow H.$$

Then Proposition 7.4.11 show that the maps $\varphi := \theta_{\overline{V},\Lambda}$ and ψ on V satisfy the assumptions of Proposition 7.4.9 with $A = V$ and of Corollary 7.4.10, and the estimates (7.4.22) and (7.4.27) are precisely the estimates (7.4.15) and (7.4.19) in Proposition 7.4.6 and Corollary 7.4.8.

PROOF OF PROPOSITION 7.4.11. Observe that we have an exact sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow \Lambda^{\oplus 2} \xrightarrow{i} \Lambda^{\oplus 2} \xrightarrow{p} \Lambda_{\mathbb{F}_2} \longrightarrow 0,$$

where:

$$i(a,b) := (a+b, a-b) \quad \text{and} \quad p(u,v) := [u-v].$$

Therefore, as an application of the parallelogram law, we obtain, for every x and y in $V_{\mathbb{R}}$,:

$$\begin{aligned}\theta_{\overline{V},\Lambda}(x)\theta_{\overline{V},\Lambda}(y) &= \sum_{(a,b)\in\Lambda^2} e^{-\pi\|x-a\|^2-\pi\|y-b\|^2}\\ &= \sum_{(a,b)\in\Lambda^2} e^{-\frac{\pi}{2}\|(x+y)-(a+b)\|^2-\frac{\pi}{2}\|(x-y)-(a-b)\|^2}\\ &= \sum_{(v,w)\in\Lambda^2,[v]=[w]} e^{-\frac{\pi}{2}\|(x+y)-v\|^2-\frac{\pi}{2}\|(x-y)-w\|^2}\\ &= \sum_{\alpha\in\Lambda_{\mathbb{F}_2}} \psi_\alpha(x+y)\psi_\alpha(x-y).\end{aligned}$$

□

7.4.3. Concerning the proof of the quadratic inequality (7.4.15)**.** In striking contrast with its elementary character, the derivation of the quadratic inequality (7.4.15) in Proposition 7.4.9 is closely related to various major results in algebraic and complex analytic geometry and in probability theory.

Thanks to the formula (7.4.5), the existence of a decomposition of the product

$$\theta_{\overline{V},\Lambda}(x)\,\theta_{\overline{V},\Lambda}(y)$$

as a finite sum (7.4.32) for a general pair $(\overline{V},\Lambda)$ follows from its existence when $V = \Lambda_{\mathbb{R}}$, that is when $\theta_{\overline{V},\Lambda}$ is the theta function $\theta_{\overline{E}}$ associated to some Euclidean lattice $\overline{E}$.

In this case, using the dictionary relating theta functions attached to Euclidean lattices and classical theta functions discussed in subsection 7.1.3, the relations (7.4.32) are a special instance of the so-called Riemann relations satisfied by classical theta series; see for instance [**Igu72**, IV.1], where they appear in Theorem 2 under the name "addition formula", or [**Mum83**, section II.6], where they appear under the name of "Riemann identity" (the identity (7.4.32) is actually a special instance of equation (6.6) in *loc. cit.*, applied with $n = 1$, and $\vec{a} = \vec{b} = 0$).

The apparent generality of the hypotheses in Proposition 7.4.9 and Corollary 7.4.10 would make one expect them to have a wide range of possible applications besides the ones to theta functions presented above. However, it follows from a remarkable theorem of Barsotti that, if φ, V, and H

are as in Corollary 7.4.10 with φ real analytic and H finite dimensional, then φ admits a simple expression in terms of theta functions associated to some abelian variety over $\mathbb{R}$, hence in terms of theta functions of Euclidean lattices; see [**Bar70**] and [**Bar89**].

In a more analytic or probabilistic perspective, one should also recall that the simple identity that lies behind the quadratic relations (7.4.32) — namely:

$$e^{-\|x\|^2}e^{-\|y\|^2} = e^{-\|x+y\|^2/2}e^{-\|x-y\|^2/2}$$

— also implies that, if X and Y are two independent identically distributed Gaussian random vectors in some Banach space, then (X,Y) and $(X+Y)/\sqrt{2}, (X-Y)/\sqrt{2})$ have the same distribution: this is precisely the central point of the proof of Fernique's theorem on the strong integrability of the norm of Gaussian random vectors; see [**Fer70**].[11]

7.4.4. The monotonicity of Banaszczyk functions. The quadratic inequality on the Banaszczyk functions established in Proposition 7.4.6 will be used, in the form of its Corollary 7.4.7, to establish the following monotonicity property.

THEOREM 7.4.12. *For $i = 1, 2$, let us consider a finite dimensional Euclidean vector space $\overline{V}_i := (V_i, \|.\|_i)$ and a discrete subgroup Λ_i of V_i.*

If some $\mathbb{R}$-linear map $f : V_1 \to V_2$ satisfies the conditions:

$$f(\Lambda_1) \subseteq \Lambda_2 \quad \text{and} \quad \|f(x)\|_2 \leq \|x\|_1 \text{ for every } x \in V_1,$$

then following inequality holds for every $x \in V_1$:

$$B_{\overline{V}_2,\Lambda_2}(f(x)) \geq B_{\overline{V}_1,\Lambda_1}(x). \tag{7.4.33}$$

The inequality (7.4.33) takes an arguably more pleasant form when expressed in terms of the functions $b_{\overline{V}_i,\Lambda_i}$, namely:

$$b_{\overline{V}_2,\Lambda_2}(f(x)) \leq b_{\overline{V}_1,\Lambda_1}(x).$$

Theorem 7.4.12 will be the key point in the proof of the strong monotonicity properties of the invariants h^0_θ and h^1_θ in Section 7.5, and will play a central in our study of the theta invariants of Hermitian quasi-coherent sheaves in Chapter 8.

Special instances of Theorem 7.4.12 appears in the work of Banasczyk and of Regev and Stephens-Davidowitz quoted at the beginning of this section. Its validity when f is an isometry from $\overline{V}_1$ to $\overline{V}_2$ is notably used in the papers of Banaszczyk and Stegliński. It is also established in [**RSD17a**] when $f : V_1 \to V_2$ is an isomorphism, Λ_1 and Λ_2 are cocompact in V_1 and V_2, and $f(\Lambda_1) = \Lambda_2$.

Taking these special cases for granted, one easily sees that to prove Theorem 7.4.12 in full generality, it is enough to handle the case of a map f associated by an admissible surjective morphism of Euclidean lattices. In Subsection 7.6.2, we will see that the monotonicity of Banasczyk functions in this case may be deduced from its validity when f is a morphism of Euclidean lattices such that $f_\mathbb{R}$ is an isometric isomorphism combined with the description, of independent interest, of the limit of the Banaszczyk functions $B_{\overline{V},\Lambda}$ when Λ varies over an increasing sequence of lattices in V.

In this subsection, we give a streamlined derivation of Theorem 7.4.12, based on the inequality on Banaszczyk functions established in Corollary 7.4.7. In Section 7.6 below, we will present alternative arguments for the monotonicity of Banaszczyk functions, which will provide some further insight on their properties.

[11] See also [**Str11**, Section 8.2.1], where Fernique's proof is presented as "arguably the most singularly beautiful results in the theory of Gaussian measures on a Banach space."

PROOF. We divide the proof of Theorem 7.4.12 in three successive steps.

(1) *Proof of* (7.4.33) *when* f *is an isometry*, that is when it satisfies:

$$\|f(x)\|_2 = \|x\|_1 \text{ for every } x \in V_1.$$

In this case, f is injective, and we have:

$$\sum_{[c]\in\Lambda_2/f(\Lambda_1)} B_{\overline{V}_2,f(\Lambda_1)}(c) = \frac{\sum_{w\in\Lambda_2} e^{-\pi\|w\|_2^2}}{\sum_{w\in f(\Lambda_1)} e^{-\pi\|w\|_2^2}},$$

and, for every $x \in V_1$:

$$B_{\overline{V}_2,f(\Lambda_1)}(f(x)) = B_{\overline{V}_1,\Lambda_1}(x).$$

Consequently, using the estimates (7.4.17), we obtain, for every $x \in V_1$:

$$\begin{aligned}
\sum_{w\in\Lambda_2} e^{-\pi\|f(x)-w\|_2^2} &= \sum_{[c]\in\Lambda_2/f(\Lambda_1)} \sum_{w\in f(\Lambda_1)} e^{-\pi\|f(x)-c-v\|_2^2} \\
&= \sum_{w\in f(\Lambda_1)} e^{-\pi\|w\|_2^2}. \sum_{[c]\in\Lambda_2/f(\Lambda_1)} B_{\overline{V}_2,f(\Lambda_1)}(f(x)-c) \\
&= (1/2) \sum_{w\in f(\Lambda_1)} e^{-\pi\|w\|_2^2}. \sum_{[c]\in\Lambda_2/f(\Lambda_1)} \Big(B_{\overline{V}_2,f(\Lambda_1)}(f(x)-c) + B_{\overline{V}_2,f(\Lambda_1)}(f(x)+c)\Big) \\
&\geq \sum_{w\in f(\Lambda_1)} e^{-\pi\|w\|_2^2} \sum_{[c]\in\Lambda_2/f(\Lambda_1)} B_{\overline{V}_2,f(\Lambda_1)}(f(x))\, B_{\overline{V}_2,f(\Lambda_1)}(c) \\
&= B_{\overline{V}_1,\Lambda_1}(x). \sum_{w\in\Lambda_2} e^{-\pi\|w\|_2^2}.
\end{aligned}$$

This establishes the inequality (7.4.33).

(2) *Proof of* (7.4.33) *when* f *is contracting*, that is when f satisfies:

$$\|f(x)\|_2 < \|x\|_1 \text{ for every } x \in V_1.$$

The map f factorizes as:

$$f = p \circ i,$$

where

$$i : V_1 \longrightarrow V_1 \oplus V_2, \quad x \longmapsto (x, f(x)),$$

and

$$p : V_1 \oplus V_2 \longrightarrow V_2, \quad (x,y) \longmapsto y.$$

Since f is contracting from $(V_1, \|.\|_1)$ to $(V_2, \|.\|_2)$, we define a Euclidean norm $\|.\|'_1$ on V_1 by letting:

$$\|x\|_1'^2 := \|x\|_1^2 - \|f(x)\|_2^2$$

for every $x \in V_1$. Then the map i from the Euclidean vector space $\overline{V}_1$ to the direct sum $\overline{V}'_1 \oplus \overline{V}_2$ of the Euclidean vector spaces $\overline{V}'_1 := (V_1, \|.\|'_1)$ and $\overline{V}_2$ is an isometry. Moreover i maps Λ_1 into $\Lambda_1 \oplus f(\Lambda_1)$.[12] Therefore Step (1) applies to i and shows that, for every $x \in V_1$:

$$B_{\overline{V}'_1\oplus\overline{V}_2,\Lambda_1\oplus f(\Lambda_1)}(x, f(x)) \geq B_{\overline{V}_1,\Lambda_1}(x). \tag{7.4.34}$$

Moreover, according to Proposition 7.4.1, for every $(x,y) \in V_1 \oplus V_2$, we have:

$$B_{\overline{V}'_1\oplus\overline{V}_2,\Lambda_1\oplus f(\Lambda_1)}(x,y) = B_{\overline{V}'_1,\Lambda_1}(x)\, B_{\overline{V}_2,f(\Lambda_1)}(y),$$

and therefore:

$$B_{\overline{V}_2,f(\Lambda_1)}(y) \geq B_{\overline{V}'_1\oplus\overline{V}_2,\Lambda_1\oplus f(\Lambda_1)}(x,y). \tag{7.4.35}$$

[12]This construction and the one in Proposition 5.2.10 are dual of each other.

In other words, Theorem 7.4.12 is valid for the projection p and the pairs $(\overline{V}'_1 \oplus \overline{V}_2, \Lambda_1 \oplus f(\Lambda_1))$ and $(\overline{V}_2, f(\Lambda_1))$.

The inequality (7.4.33) follows from (7.4.34) and (7.4.35).

(3) *Proof of* (7.4.33) *for an arbitrary norm decreasing map* f.

Let us choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^*_+$ such that $\lim_{n\to+\infty} \varepsilon_n = 0$. Then, for every $n \in \mathbb{N}$, we have:

$$e^{-\varepsilon_n}\|f(x)\|_2 < \|x\|_1 \text{ for every } x \in V_1.$$

Therefore, according to Step (2), the following inequality holds for every $x \in V_1$:

$$B(V_2, e^{-\varepsilon_n}\|.\|_2; f(x)) \geq B(V_1, \|.\|_1; x).$$

The inequality (7.4.33) follows by letting n go to infinity. □

7.5. Theta Ranks and Strong Monotonicity

The invariant $h^1_\theta : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$ defined in Sections 7.1 and 7.3 is already known to satisfy diverse conditions introduced in Chapter 4, namely the conditions of monotonicity $\mathbf{Mon}^1$ and of subadditivity **SubAdd**, as well as, for trivial reasons, the conditions **VT** and **Cont**.

This section is devoted to the proof of the following theorem:

THEOREM 7.5.1. *The invariant $h^1_\theta : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$ satisfies the strong monotonicity condition* $\mathbf{StMon}^1$.

The strong monotonicity of an invariant φ on $\overline{\mathbf{Coh}}_X$ has been defined in Chapter 5 in terms of the associated rank invariant rk^1_φ. When $\varphi = h^1_\theta$, this rank invariant becomes the θ^1*-rank*, which to a morphism $f : \overline{E} \longrightarrow \overline{F}$ in $\overline{\mathbf{Coh}}^{\leq 1}_X$ attaches the non-negative real number:

$$\mathrm{rk}^1_\theta f := \mathrm{rk}^1_{h^1_\theta} f = h^1_\theta(\overline{F}) - h^1_\theta(\overline{F/f(E)}) \in \mathbb{R}_+. \tag{7.5.1}$$

As discussed in Subsection 5.1.1, this number does not depend of the actual choice of the Hermitian semi-norms $(\|.\|_{\overline{E},x})_{x\in X(\mathbb{C})}$ defining $\overline{E}$, but only of $\overline{F}$ and of image $f(E)$ of the morphism of $\mathcal{O}_X$-modules $f : E \longrightarrow F$. Actually, since h^1_θ "does not see torsion," it depends only of $\overline{F}$ and of the K-vector subspace $\mathrm{im}\, f_K := f_K(E_K)$ of F_K.

To emphasize this independence, the θ^1-rank (7.5.1) will often be denoted as:

$$\mathrm{rk}^1_\theta(f : E \to \overline{F}).$$

It is defined if E is any coherent $\mathcal{O}_C$-module, if $\overline{F}$ is an object of $\overline{\mathbf{Coh}}_X$, and if $f : E \to F$ is any morphism of $\mathcal{O}_C$-modules — in brief, in the terminology introduced in Section5.1, if $f : E \to \overline{F}$ is a morphism in $\ddot{\mathbf{Coh}}_X$.

According to Proposition 5.2.2, the strong monotonicity of h^1_θ may be expressed as follows:

THEOREM 7.5.2. *Let $f : \overline{E} \to \overline{F}$ be a morphism in $\overline{\mathbf{Coh}}^{\leq 1}_X$ and let E' be a coherent $\mathcal{O}_X$-submodule of E. If we denote by $F' := f(E')$ its image and ι the inclusion morphism from E' to E, the following inequality holds:*

$$\mathrm{rk}^1_\theta(f \circ \iota : E' \to \overline{F}) \leq \mathrm{rk}^1_\theta(\iota : E' \to \overline{E}),$$

or equivalently:

$$h^1_\theta(\overline{F}) - h^1_\theta(\overline{F/F'}) \leq h^1_\theta(\overline{E}) - h^1_\theta(\overline{E/E'}). \tag{7.5.2}$$

This formulation of the strong monotonicity of h^1_θ will be especially convenient for its proof. In Subsection 7.5.2, we will actually establish a slightly stronger property, expressed in terms of the invariant h^0_θ and of the attached θ^0-rank rk^0_θ defined in Subsection 7.5.1, as a consequence of the monotonicity properties of the Banaszczyk functions. The strong monotonicity of h^1_θ will follow by a duality argument.

7.5.1. The θ-rank $\mathrm{rk}^0_\theta(f)$. The θ^1-rank rk^1_θ admits a counterpart, the θ^0-rank rk^0_θ, which plays a role similar to rk^1_θ when one deals with the invariant:

$$h^0_\theta : \overline{\mathbf{Coh}}_{\overline{X}} \longrightarrow \mathbb{R}_+$$

instead of the invariant:

$$h^1_\theta : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+.$$

In this subsection, we define rk^0_θ and we discuss its basic properties. We shall leave some details of the proofs to the reader. Indeed they are often similar to the corresponding arguments concerning rk^1_θ, or more generally the φ-rank rk^1_φ discussed in Section 5.1. Moreover in this monograph the θ^0-rank plays only a technical role in the derivation of the strong monotonicity $\mathbf{StMon}^1$ of h^1_θ.

7.5.1.1. *A geometric analogue.* If C is a smooth, projective, geometrically connected curve C over some base field k. To any coherent $\mathcal{O}_C$-module $\mathcal{F}$, we may attach the dimension of its space of sections:

$$h^0(C,\mathcal{F}) := \dim_k H^0(C,\mathcal{F}) \quad \in \mathbb{N}. \tag{7.5.3}$$

For every morphism $f : \mathcal{F} \longrightarrow \mathcal{G}$ of coherent $\mathcal{O}_C$-modules, we may also consider the rank of the induced k-linear map $f^0 : H^0(C,\mathcal{F}) \longrightarrow H^0(C,\mathcal{G})$. It satisfies the equality:

$$\mathrm{rk}_k f^0 = \dim_k H^0(C,\mathcal{F}) - \dim_k \ker f^0,$$

and may therefore be expressed as the difference:

$$\mathrm{rk}_k f^0 = \dim_k H^0(C,\mathcal{F}) - \dim_k H^0(C,\ker f), \tag{7.5.4}$$

where $\ker f$ denotes the coherent $\mathcal{O}_C$-module kernel of f.

7.5.1.2. *Definition of* rk^0_θ. Motivated by the equality (7.5.4), to a morphism $g : \overline{V} \longrightarrow \overline{W}$ in $\overline{\mathbf{Coh}}^{\leq 1}_{\overline{X}}$, we may attach its θ^0*-rank* defined as:

$$\mathrm{rk}^0_\theta(g) := h^0_\theta(\overline{V}) - h^0_\theta(\overline{\ker g}) \in \mathbb{R}_+. \tag{7.5.5}$$

The positivity of $\mathrm{rk}^0_\theta(g)$ is indeed a consequence of the monotonicity of h^0_θ.

Clearly this rank does not depend on the choice of the Hermitian semi-norms $(\|.\|_{\overline{W},x})_{x\in X(\mathbb{C})}$ defining $\overline{W}$, but only of $\overline{V}$ and of the kernel $\ker g$ the morphism of coherent $\mathcal{O}_X$-modules $g : V \to W$. Accordingly the θ^0-rank rk^0_θ will often be denoted:

$$\mathrm{rk}^0_\theta(g : \overline{V} \to W).$$

It is actually defined when $\overline{V}$ is an object of $\overline{\mathbf{Coh}}_{\overline{X}}$ and $g : V \to W$ is a morphism of coherent $\mathcal{O}_X$-module, and it satisfies the equality:

$$\mathrm{rk}^0_\theta(g : \mathcal{V} \to W) = \mathrm{rk}^0_\theta(g : \mathcal{V} \to \operatorname{im} g).$$

One should beware that the definition of rk^0_θ is "torsion-sensitive": when one "kills" the torsion in W, the θ^0-rank $\mathrm{rk}^0_\theta(g : \overline{V} \to W)$ may change.

More generally, the following proposition is a straightforward consequence of the definition of rk^0_θ and of the monotonicity of h^0_θ.

PROPOSITION 7.5.3. *For every diagram*

$$\overline{V} \xrightarrow{g} W \xrightarrow{h} U,$$

where $\overline{V}$ is an object of $\overline{\mathbf{Coh}}_{\overline{X}}$, and W and U (resp. $g : V \to W$ and h) are coherent $\mathcal{O}_C$-modules (resp. a morphism of $\mathcal{O}_C$-modules), the following inequality holds:

$$\mathrm{rk}^0_\theta(h \circ g : \overline{V} \to W) \leq \mathrm{rk}^0_\theta(g : \overline{V} \to U).$$

7.5.1.3. *The strong monotonicity of h^0_θ.* The following theorem is a counterpart concerning h^0_θ and rk^0_θ of the strong monotonicity property $\mathbf{StMon}^1$ satisfied by the invariant h^1_θ on $\overline{\mathbf{Coh}}_{\overline{X}}$. It will be referred to as the *strong monotonicity* $\mathbf{StMon}^0$ of the invariant h^0_θ on $\overline{\mathbf{Coh}}_X$.

THEOREM 7.5.4. *For every diagram*

$$\overline{V} \xrightarrow{g} \overline{W} \xrightarrow{h} U,$$

where $g : \overline{V} \to \overline{W}$ is a morphism $\overline{\mathbf{Coh}}^{\leq 1}_{\overline{X}}$, and U (resp. $h : W \to U$) is a coherent $\mathcal{O}_C$-modules (resp. a morphism of $\mathcal{O}_C$-modules), the following inequality holds:

$$\mathrm{rk}^0_\theta(h \circ g : \overline{V} \to U) \leq \mathrm{rk}^0_\theta(h : \overline{V} \to W). \tag{7.5.6}$$

Since both sides of (7.5.6) are unchanged when the morphism $g : W \to U$ is replaced by the quotient map $W \to W/W'$ where $W' := \ker h$, this theorem admits the following equivalent formulation:

THEOREM 7.5.5. *For every morphism $g : \overline{V} \to \overline{W}$ in $\overline{\mathbf{Coh}}^{\leq 1}_{\overline{X}}$ and every coherent $\mathcal{O}_X$-submodule W' of W, of inverse image $V' := g^{-1}(W')$ in V, the following inequality holds:*

$$h^0_\theta(\overline{V}) - h^0_\theta(\overline{V}') \leq h^0_\theta(\overline{W}) - h^0_\theta(\overline{W}'). \tag{7.5.7}$$

We defer the proof of Theorems 7.5.4 and 7.5.5 to the next subsection, and we spell out some consequences of these theorems applied with $X = \operatorname{Spec} \mathbb{Z}$, that concern Euclidean lattices.

COROLLARY 7.5.6. *(1) For every Euclidean lattice $\overline{E} := (E, \|.\|)$ and every pair of $\mathbb{Z}$-submodules $E'' \subseteq E'$, the following inequality holds:*

$$h^0_\theta(\overline{E}) - h^0_\theta(\overline{E}') \leq h^0_\theta(\overline{E/E''}) - h^0_\theta(\overline{E'/E''}). \tag{7.5.8}$$

(2) For every Euclidean lattice $\overline{E} := (E, \|.\|)$ and any two $\mathbb{Z}$-submodules F_1 and F_2 of E, the following inequality holds:

$$h^0_\theta(\overline{F}_1) + h^0_\theta(\overline{F}_2) \leq h^0_\theta(\overline{F_1 \cap F_2}) + h^0_\theta(\overline{F_1 + F_2}). \tag{7.5.9}$$

(3) For every Euclidean lattice $\overline{E} := (E, \|.\|)$ and any $\mathbb{Z}$-submodule E' of E, the difference $h^0_\theta(\overline{E}) - h^0_\theta(\overline{E}')$ is a decreasing function of the Euclidean norm $\|.\|$.

Observe that, when $E'' = E'$, the estimate (7.5.8) is nothing but the subadditivity of h^0_θ:

$$h^0_\theta(\overline{E}) \leq h^0_\theta(\overline{E}') + h^0_\theta(\overline{E/E'}).$$

Observe that the right hand side of (7.5.8) admits the following expression in terms of invariants h^0_θ attached to Euclidean lattices:

$$h^0_\theta(\overline{E/E''}) - h^0_\theta(\overline{E'/E''}) = h^0_\theta(\overline{E/E''^{\mathrm{sat}}}) - h^0_\theta(\overline{E'/(E' \cap E''^{\mathrm{sat}})}) + \log[E' \cap E''^{\mathrm{sat}} : E''].$$

The submodularity inequality (7.5.9) on h^0_θ has been established by Regev and Stephens-Davidowitz in [**RSD17a**, Theorem 5.1] at the instigation of McMurray Price; see also [**MP17**].

PROOF OF COROLLARY 7.5.6. Assertion (1) follows from Theorem 7.5.5 applied to the quotient morphism

$$g : \overline{V} := \overline{E} \to \overline{W} := \overline{E/E''}$$

and to the submodule $W' := E'/E''$ of W.

Assertion (2) follows from Theorem 7.5.5 applied to the inclusion morphism

$$g : \overline{V} := \overline{F} \longrightarrow \overline{W} := \overline{F_1 + F_2}$$

and to the submodule $W' := F_2$ of W.

To prove (3), observe that, if a Euclidean norm $\|.\|'$ on $E_{\mathbb{R}}$ satisfies the inequality:

$$\|.\|^{\sim} \leq \|.\|,$$

then the identity map Id_E defines a morphism

$$\mathrm{Id}_E : (E, \|.\|) \to (E, \|.\|^{\sim})$$

in $\mathrm{Vect}_{\mathbb{Z}}^{\leq 1}$. Consequently we may apply Theorem 7.5.5 to this morphism and to the submodule E' of E. Thus we get:

$$h^0_\theta(E, \|.\|) - h^0_\theta(E', \|.\|_{|E'_{\mathbb{R}}}) \leq h^0_\theta(E, \|.\|^{\sim}) - h^0_\theta(E', \|.\|^{\sim}_{|E'_{\mathbb{R}}}). \qquad \square$$

The reader may prove that conversely one may deduce the validity of Theorems 7.5.4 and 7.5.5 when $X = \operatorname{Spec} \mathbb{Z}$ from its various special instances appearing in Corollary 7.5.6; see Subsection 5.2.3 for a similar reasoning concerning the condition $\mathbf{StMon}^1$.

In turn, the validity of Theorems 7.5.4 and 7.5.5 when $X = \operatorname{Spec} \mathbb{Z}$ implies their validity in general according to the compatibility of the invariant h^0_θ with direct images under the morphism π from X to $\operatorname{Spec} \mathbb{Z}$.

7.5.1.4. *θ-ranks and duality.* If $\overline{E}$ is a coherent $\mathcal{O}_C$-module, $\overline{F}$ is an object of $\overline{\mathbf{Coh}}_{\overline{X}}$ and

$$f : E \longrightarrow F$$

is a morphism of $\mathcal{O}_C$-modules, then $\overline{E}^\vee$ is a locally free coherent $\mathcal{O}_X$-module, $\overline{F}^\vee$ is objects of $\overline{\mathbf{Vect}}_X$, and by duality, the morphism f defines a morphism of $\mathcal{O}_X$-modules:

$$f^\vee : F^\vee \longrightarrow E^\vee. \tag{7.5.10}$$

According to the Hecke duality formula (7.2.6), we have:

$$h^1_\theta(\overline{F}) = h^0_\theta(\overline{F}^\vee \otimes \overline{\omega}_\pi) \tag{7.5.11}$$

and:

$$h^1_\theta(\overline{F/f(E)}) = h^1_\theta(\overline{F/f(E)^{\mathrm{sat}}}) = h^0_\theta(\overline{F/f(E)^{\mathrm{sat}}}^\vee \otimes \overline{\omega}_\pi).$$

Moreover the Hermitian vector subbundle $\overline{\ker f^\vee}$ of $\overline{F}^\vee$ is canonically isomorphic to $\overline{F/f(E)^{\mathrm{sat}}}^\vee$. Consequently:

$$h^1_\theta(\overline{F/f(E)}) == h^0_\theta(\overline{\ker f^\vee} \otimes \overline{\omega}_\pi). \tag{7.5.12}$$

From (7.5.11) and (7.5.12), we deduce the following proposition:

PROPOSITION 7.5.7. *For every object $\overline{\mathcal{F}}$ in $\overline{\mathbf{Coh}}_{\overline{X}}$ and any morphism of coherent $\mathcal{O}_X$-modules $f : E \to F$ as above, the following equality holds:*

$$\mathrm{rk}^1_\theta(f : E \to \overline{F}) = \mathrm{rk}^0_\theta(f^\vee \otimes \mathrm{Id}_{\omega_\pi} : \overline{F}^\vee \otimes \overline{\omega}_\pi \to E^\vee \otimes \omega_\pi). \tag{7.5.13}$$

7.5.2. Proof of the strong monotonicity of h^0_θ and h^1_θ.

7.5.2.1. *The rank rk^0_θ and the Banaszcyk functions.* Proposition 7.1.6 already demonstrates the relevance of Banaszczyk functions for establishing the subaddivity of the theta invariants. In the same vein, the following observation will play a key role in the derivation of their strong monotonicity.

PROPOSITION 7.5.8. *For any object $\overline{E} := (E, \|.\|)$ in $\overline{\mathbf{Coh}}_{\operatorname{Spec}\mathbb{Z}}$ and any $\mathbb{Z}$-submodule E' of E, the following equality holds:*

$$h^0_\theta(\overline{E}) - h^0_\theta(\overline{E}') = \log \sum_{[c] \in E/E'} B(\overline{E}_{\mathbb{R}}, E'_{/\mathrm{tor}}; c). \tag{7.5.14}$$

In the right hand-side of (7.5.14), $\overline{E}_{\mathbb{R}}$ denotes the Euclidean vector space $(E_{\mathbb{R}}, \|.\|)$ and $E'_{/\mathrm{tor}}$ the $\mathbb{Z}$-module E'/E'_{tor}, identified to a discrete subgroup of $E_{\mathbb{R}}$. For every $c \in E$, the value $B(\overline{E}_{\mathbb{R}}, E'_{/\mathrm{tor}}; c)$ of the function $B_{\overline{E}_{\mathbb{R}}, E'_{/\mathrm{tor}}}$ at the image, still denoted c, of c in $E_{\mathbb{R}}$ depends only of the class $[c]$ of c in E/E'. This value actually depends only of the class of c in $E/(E' + E_{\mathrm{tor}})$, and the equality (7.5.14) might also be written:

$$h^0_\theta(\overline{E}) - h^0_\theta(\overline{E}') = \log \sum_{[c] \in E/(E'+E_{\mathrm{tor}})} B(\overline{E}_{\mathbb{R}}, E'_{/\mathrm{tor}}; c) + \log |E_{\mathrm{tor}}/(E_{\mathrm{tor}} \cap E')|.$$

In terms of the function $b_{\overline{E}_{\mathbb{R}}, E'}$, the equality (7.5.14) takes the more suggestive form:

$$h^0_\theta(\overline{E}) - h^0_\theta(\overline{E}') = \log \sum_{[c] \in E/E'} \exp\left[-\pi b_{\overline{E}_{\mathbb{R}}, E'_{/\mathrm{tor}}}(c)\right]. \tag{7.5.15}$$

More generally, we may consider an object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\overline{\operatorname{Spec}\mathbb{Z}}}$, a finitely generated $\mathbb{Z}$-module F, and a morphism of $\mathbb{Z}$-modules $f : E \longrightarrow F$. Applied to $E' := \ker f$, the equality (7.5.15) becomes the following expression for the θ^0-rank of f:

$$\mathrm{rk}^0_\theta f = \log \sum_{[c] \in E/\ker f} \exp\left[-\pi b_{\overline{E}_{\mathbb{R}}, (\ker f)_{/\mathrm{tor}}}(c)\right]. \tag{7.5.16}$$

For every element i in $f(E)$, the preimage $f^{-1}(i)$ is a coset c in $E/\ker f$, and (7.5.16) may also be written:

$$\mathrm{rk}^0_\theta f = \log \sum_{i \in f(E)} \exp\left[-\pi b_{\overline{E}_{\mathbb{R}}, (\ker f)_{/\mathrm{tor}}}(f^{-1}(i))\right]. \tag{7.5.17}$$

This expression is formally similar to the one defining h^0_θ:

$$h^0_\theta(\overline{E}) := \log \sum_{v \in E} \exp\big(-\pi \|v\|^2\big),$$

which is actually the instance of (7.5.17) when f is the morphism Id_E.

Proof of Proposition 7.5.8. As usual, for every element v of E, we denote by $\|v\|$ the Euclidean norm of the class of v in $E_{/\mathrm{tor}} := E/E_{\mathrm{tor}}$, identified to a submodule of $E_{\mathbb{R}}$.

For every coset $c + E'$ of E' in E, we have:

$$\sum_{v \in c+E'} e^{-\pi\|v\|^2} = \sum_{v \in E'} e^{-\pi\|v-c\|^2} = B(\overline{E}_{\mathbb{R}}, E'_{/\mathrm{tor}}; c). \sum_{v \in E'} e^{-\pi\|v\|^2}.$$

Consequently, we have:

$$\sum_{v \in E} e^{-\pi\|v\|^2} = \sum_{[c] \in E/E'} \sum_{v \in c+E'} e^{-\pi\|v\|^2} = \sum_{[c] \in E/E'} B(\overline{E}_{\mathbb{R}}, E'_{/\mathrm{tor}}; c). \sum_{v \in E'} e^{-\pi\|v\|^2},$$

and (7.5.14) follows by taking logarithms. □

7.5.2.2. *Proof of the strong monotonicity of h^0_θ.* We may now complete the proof of Theorem 7.5.5.

To achieve this, observe that we may assume that X is $\operatorname{Spec}\mathbb{Z}$. Indeed the general case reduces to this one by means of the direct image operation π_*. Then, with the notation of Theorem 7.5.5, Proposition 7.5.8 implies the relations:

$$h^0_\theta(\overline{V}) - h^0_\theta(\overline{V}') = \log \sum_{[d] \in V/V'} B(\overline{V}_{\mathbb{R}}, V'_{/\mathrm{tor}}; d) \tag{7.5.18}$$

and

$$h^0_\theta(\overline{W}) - h^0_\theta(\overline{W}') = \log \sum_{[c]\in W/W'} B(\overline{W}_{\mathbb{R}}, W'_{/\mathrm{tor}}; c). \tag{7.5.19}$$

Moreover, according to Proposition 7.4.12, applied to the map $g_{\mathbb{R}}$ of norm ≤ 1 from the Euclidean vector space $\overline{V}_{\mathbb{R}}$ to $\overline{W}_{\mathbb{R}}$ which maps $V'_{/\mathrm{tor}}$ to $W'_{/\mathrm{tor}}$, we have:

$$B(\overline{V}_{\mathbb{R}}, V'_{/\mathrm{tor}}; d) \leq B(\overline{W}_{\mathbb{R}}, W'_{/\mathrm{tor}}; g(d)) \quad \text{for every } d \in V'. \tag{7.5.20}$$

The inequality (7.5.7) follows from (7.5.18), (7.5.19), (7.5.20), and the injectivity of the map

$$\tilde{g} : V/V' \longrightarrow W/W'$$

induced by $g : V \to W$.

7.5.2.3. *Proof of the strong monotonicity of h^1_θ.* We may finally derive the strong monotonicity of h^1_θ, in its formulation in Theorem 7.5.2, from Theorem 7.5.5.

Here again by considering direct images by π,, it is enough to prove Theorem 7.5.2 when X is $\operatorname{Spec}\mathbb{Z}$. Moreover a straightforward approximation argument using the downward continuity $\mathbf{Cont}^+$ of h^1_θ and Proposition 4.2.3 shows that, to establish the inequality (7.5.2) in Theorem 7.5.2, we may assume that $\overline{E}$ and $\overline{F}$ are objects of $\overline{\mathbf{Coh}}_{\operatorname{Spec}\mathbb{Z}}$, or equivalently that the Euclidean seminorms $\|.\|_{\overline{E}}$ and $\|.\|_{\overline{F}}$ defining $\overline{E}$ and $\overline{F}$ are actually norms.

When this holds, the duals $\overline{V} := \overline{F}^\vee$ and $\overline{W} := \overline{E}^\vee$ are objects in $\overline{\mathbf{Vect}}_{\mathbb{Z}}$, that is Euclidean lattices, and according to the definition of h^1_θ, we have:

$$h^1_\theta(\overline{F}) = h^0_\theta(\overline{V}) \quad \text{and} \quad h^1_\theta(\overline{E}) = h^0_\theta(\overline{W}).$$

The $\mathbb{Z}$-submodule

$$W' := E'^\perp = \{\xi \in E^\vee \mid \xi_{|E'} = 0\}$$

of $W := E^\vee$ may be identified to $(E/E')^\vee$, and the Euclidean lattice $\overline{W}'$ to $\overline{E/E'}^\vee$. In particular, we have:

$$h^1_\theta(\overline{E/E'}) = h^0_\theta(\overline{W}').$$

Moreover the transpose $g := f^\vee$ of f defines a morphism in $\overline{\mathbf{Coh}}^{\leq 1}_{\operatorname{Spec}\mathbb{Z}}$ from $\overline{V}$ to $\overline{W}$, and the inverse image of W' by g,

$$V' := g^{-1}(W') = (f^\vee)^{-1}(E'^\perp),$$

is the $\mathbb{Z}$-submodule:

$$f(E')^\perp := \{\eta \in F^\vee \mid \eta_{|f(E')} = 0\}.$$

Consequently the dual of $\overline{F/f(E')}$ may be identified with $\overline{V}'$, and therefore:

$$h^1_\theta(\overline{F/f(E')}) = h^0_\theta(\overline{V}').$$

With these choices of $g : \overline{V} \to \overline{W}$ and of W' the inequality (7.5.7) in Theorem 7.5.5 becomes the inequality (7.5.2).[13]

[13]We might as well have used the duality formula in Proposition 7.5.7.

7.6. Further Properties of Banaszczyk Functions and Alternative Derivations of their Monotonicity Properties

In the previous section, we have tried to give a proof of the monotonicity of Banaszczyk functions (Theorem 7.4.12) that is as direct as possible. It turns out that the Banaszczyk functions satisfies further remarkable properties that lead to alternative derivations of some special case of this monotonicity.

In this section we discuss some of these properties, which moreover establish some intriguing links between Banaszczyk functions and diverse classical topics related to Euclidean lattices. The results of this section will not be used in the next chapters, and it could be skipped with no inconvenience.

7.6.1. The Banaszczyk function $B_{(E,\|.\|)}$ as a function of $\|.\|$ and the heat equation.

7.6.1.1. A special instance of Theorem 7.4.12 is the following property of Banasczyk functions of Euclidean lattices:

Corollary 7.6.1. *For every finitely generated free $\mathbb{Z}$-module E and every x in $E_{\mathbb{R}}$, considered as a function of the Euclidean norm $\|.\|$ on $E_{\mathbb{R}}$, the Banaszczyk function $B_{(E,\|.\|)}(x) = B(E_{\mathbb{R}}, \|.\|, E; x)$ is decreasing.*

This corollary may be stated in terms of the real analytic function

$$\Theta_E : \mathring{\mathcal{Q}}(E_{\mathbb{R}}^{\vee}) \times E_{\mathbb{R}} \longrightarrow \mathbb{R}_+^*$$

introduced in paragraph 7.1.3.1 above.

Corollary 7.6.2. *Let E be a finitely generated free $\mathbb{Z}$-module. The function*

$$\mathcal{B}_E : \mathring{\mathcal{Q}}(E_{\mathbb{R}}^{\vee}) \times E_{\mathbb{R}} \longrightarrow \mathbb{R}_+^*$$

defined by:

$$\mathcal{B}_E(q, x) := \Theta_E(q, x)/\Theta_E(q, 0)$$

is increasing in the first variable. Namely, for every pair (q_1, q_2) of elements of $\mathring{\mathcal{Q}}(E_{\mathbb{R}}^{\vee})$, the following implication holds[14]*:*

$$q_1 \preceq q_2 \Longrightarrow \mathcal{B}_E(q_1, x) \leq \mathcal{B}_E(q_2, x) \quad \textit{for every } x \in E_{\mathbb{R}}. \tag{7.6.1}$$

Indeed, with the notation of Corollary 7.6.2, for every Euclidean lattice $\overline{E} := (E, \|.\|)$ and for every $x \in E_{\mathbb{R}}$, we have:

$$B_{\overline{E}}(x) = \mathcal{B}_E(\|.\|^{\vee 2}, x)$$

and

$$b_{\overline{E}}(x) = -\pi^{-1} \log \mathcal{B}_E(\|.\|^{\vee 2}, x). \tag{7.6.2}$$

where $\|.\|^{\vee}$ denotes the norm on $E_{\mathbb{R}}^{\vee}$ dual to the norm $\|.\|$ on $E_{\mathbb{R}}$.

7.6.1.2. If f is a function of class C^1 on the product $\mathring{\mathcal{Q}}(E_{\mathbb{R}}^{\vee}) \times E_{\mathbb{R}}$, we shall denote by $D_1 f(q, x)$ or $D_q f(q, x)$ (resp. $D_2 f(q, x)$ or $D_x f(q, x)$) its differential with respect to the first (reps. second) variable in $\mathring{\mathcal{Q}}(E_{\mathbb{R}}^{\vee})$ (resp. in $E_{\mathbb{R}}$) evaluated at (q, x). It is an element of $(S^2 E_{\mathbb{R}})^{\vee}$ (resp. of $E_{\mathbb{R}}^{\vee}$).

Similarly, if f is of class C^2, we shall denote by $D_2^2 f(q, x)$ or $D_x^2 f(q, x)$ its second differential, or Hessian, in the second variable at (q, x). It is an element of $\Gamma^2(E_{\mathbb{R}}^{\vee}) \simeq (S^2 E_{\mathbb{R}})^{\vee}$,[15] and the Taylor

[14] The notation $q_1 \preceq q_2$ has been introduced in (7.4.18).

[15] Recall that, if V is a vector space, one denotes by $S^k V$ and $\Gamma^k V$ the spaces of coinvariants and invariants of $V^{\otimes k}$ under the action of the permutation group $\mathfrak{S}_k$. The canonical isomorphism $(V^{\otimes k})^{\vee} \simeq V^{\vee \otimes k}$ induces an indentification $(S^k V)^{\vee} \simeq \Gamma^k V^{\vee}$.

expansion at order 2 of $f(q,x)$ as a function of x takes the form:

$$f(q,x+h) = f(q,x) + D_x f(q,x).h + (1/2)D_x^2 f(q,x).h^{\otimes 2} + o(\|h\|^2)$$

when $h \in E_{\mathbb{R}}$ goes to zero.

Somewhat abusively, we shall denote by $D_x f(q,0)$ and $D_x^2 f(q,0)$ the value of the first and second differentials $D_x f(q,x)$ and $D_x^2 f(q,x)$ when $x=0$.

7.6.1.3. Using this notation, Corollary 7.6.2 may rephrased as follows, in term of the differential $D_1\mathcal{B}_E$ of $\mathcal{B}_E$ with respect with the first variable:

COROLLARY 7.6.3. *Let E be a finitely generated free $\mathbb{Z}$-module. For every q in $\mathring{\mathcal{Q}}(E_{\mathbb{R}}^\vee)$ and every δq in $S^2E_{\mathbb{R}}$, the following implication holds:*

$$\delta q \succeq 0 \Longrightarrow D_q\mathcal{B}(q,x).\delta q \geq 0 \quad \text{for every } x \in E_{\mathbb{R}}. \tag{7.6.3}$$

In the next paragraphs, we present an alternative derivation of this corollary, which will not depend on Theorem above, but will directly rely on Corollary 7.4.8, itself a direct consequence of the "parallelogram inequality" (7.4.16) satisfied by the function $b_{\overline{E}}$. This alternative derivations follow the arguments in [**RSD17a**], with a few variations intended to clarify their relations with some classical results on the Riemann theta function $\theta(\tau,z)$.

The crucial point in this alternative derivation is the following result.

PROPOSITION 7.6.4. *The function Θ_E satisfies the following linear partial differential equation:*

$$D_x^2\Theta_E(q,x) = 4\pi D_q\Theta_E(q,x); \tag{7.6.4}$$

The relation (7.6.4) is an equality of real analytic functions on $\mathring{\mathcal{Q}}(E_{\mathbb{R}}^\vee) \times E_{\mathbb{R}}$ with values in $(S^2E_{\mathbb{R}})^\vee \simeq \Gamma^2(E_{\mathbb{R}}^\vee)$.

A consequence of this relation is the well-known fact that, for every Euclidean lattice $\overline{E} := (E, \|.\|)$, the function:

$$p : \mathbb{R}_+^* \times E_{\mathbb{R}}/E \longrightarrow \mathbb{R}$$

defined by:

$$p(t,x) := t^{-\mathrm{rk}E/2} \sum_{v \in E} e^{-\pi\|x-v\|^2/t}$$

satisfies the heat equation:

$$\frac{\partial p(t,x)}{\partial t} = \frac{1}{4\pi}\Delta_x p(t,x),$$

where Δ denotes the Laplace operator on the compact torus $E_{\mathbb{R}}/E$ endowed with the flat Riemannian metric defined by the Euclidean norm $\|.\|$.

PROOF. It is a straightforward consequence of the definition of Θ_E by means of the series (7.1.26), and of the fact that, for every $\xi \in E^\vee$, the function $\exp(-\pi q(\xi) + 2\pi i\langle \xi, x\rangle)$ satisfies:

$$D_x^2 \exp(-\pi q(\xi) + 2\pi i\langle \xi, x\rangle)(\delta x_1, \delta x_2) = (2\pi i)^2 \langle \xi, \delta x_1\rangle\langle \xi, \delta x_2\rangle \exp(-\pi q(\xi) + 2\pi i\langle \xi, x\rangle)$$

for every $(\delta x_1, \delta x_2) \in E_{\mathbb{R}}^2$, and:

$$D_q \exp(-\pi q(\xi) + 2\pi i\langle \xi, x\rangle)(\delta q) = -\pi\, \delta q(\xi) \exp(-\pi q(\xi) + 2\pi i\langle \xi, x\rangle),$$

for every $\delta q \in S^2E_{\mathbb{R}}$. □

The partial differential equation (7.6.4) satisfied by Θ_E may be reformulated as a partial differential equation satisfied by $\log \mathcal{B}_E$.

COROLLARY 7.6.5. *The function $\log\mathcal{B}_E$ satisfies the following partial differential equation:*

$$4\pi D_q \log\mathcal{B}_E(q,x) = D_x^2 \log\mathcal{B}_E(q,x) - D_x^2 \log\mathcal{B}_E(q,0) + (D_x \log\mathcal{B}_E(q,x))^{\otimes 2}. \tag{7.6.5}$$

PROOF. Observe that, if f is a positive function of class C^2 of one variable, we have:

$$(\log f)'' = (f'/f)' = f''/f - (f'/f)^2 = f''/f - (\log f)'^2.$$

Therefore, we have:

$$D_x \log \mathcal{B}_E(q,x) = D_x \log \Theta_E(q,x) \tag{7.6.6}$$

and

$$D_x^2 \log \mathcal{B}_E(q,x) = D_x^2 \log \Theta_E(q,x) = \Theta_E(q,x)^{-1} D_x^2 \Theta_E(q,x) - (D_x \log \Theta_E(q,x))^{\otimes 2}. \tag{7.6.7}$$

Since $\Theta_E(q,x)$ is an even function of x, we have:

$$D_x \log \Theta_E(q,0) = 0,$$

and therefore, when $x = 0$, the equality (7.6.7) takes the form:

$$D_x^2 \log \mathcal{B}_E(q,0) = \Theta_E(q,0)^{-1} D_x^2 \Theta_E(q,0). \tag{7.6.8}$$

Moreover, we have:

$$\begin{aligned} D_q \log \mathcal{B}_E(q,x) &= D_q \log \Theta(q,x) - D_q \log \Theta(q,0) \\ &= \Theta(q,x)^{-1} D_q \Theta(q,x) - \Theta(q,0)^{-1} D_q \Theta(q,0). \end{aligned} \tag{7.6.9}$$

Combined with the heat equation (7.6.4), this becomes:

$$4\pi D_q \log \mathcal{B}_E(q,x) = \Theta(q,x)^{-1} D_x^2 \Theta(q,x) - \Theta(q,0)^{-1} D_x^2 \Theta(q,0). \tag{7.6.10}$$

The relation (7.6.5) follows from this equation together with (7.6.6), (7.6.7), and (7.6.8). □

From Corollaries 7.4.8 and 7.6.5, we obtain a new proof of Corollary 7.6.3. Indeed, with the notation of Corollary 7.6.3, if $\overline{E} = (E, \|.\|)$ denotes the Euclidean lattice such that $\|.\|^{\vee 2} = q$, then according to (7.6.2) and (7.6.5), we have for every $v \in E_{\mathbb{R}}$:

$$4\pi D_q \log \mathcal{B}_E(q,x).v^2 = \pi \left(D^2 b_{\overline{E}}(0) - D^2 b_{\overline{E}}(x)\right)(v,v) + (Db_{\overline{E}}(x).v)^2.$$

According to Corollary 7.4.8, this is non-negative. Since the convex cone $\{\delta q \in S^2 E_{\mathbb{R}} \mid \delta q \succeq 0\}$ is the convex hull of the set $\{v^2; v \in E_{\mathbb{R}}\}$, this establishes (7.6.3).

7.6.1.4. Using the expression (7.1.27) of Θ_E in terms of the Riemann theta function, the generalized heat equation (7.6.4) appears as a special instance of the classical heat equation satisfied by Riemann theta function θ on $\mathfrak{H}_g \times \mathbb{C}^g$, namely of the partial differential equations:

$$\frac{\partial^2 \theta(\tau,z)}{\partial z_a \partial z_b} = 2\pi i (1 + \delta_{ab}) \frac{\partial \theta(\tau,z)}{\partial \tau_{ab}}, \quad \text{for every } 1 \leq a \leq b \leq g, \tag{7.6.11}$$

where $\theta(\tau,z)$ is considered as a function of the variables $z = (z_a)_{1 \leq a \leq g}$ and $(\tau_{ab})_{1 \leq a \leq b \leq g}$.

The relations (7.6.11) directly follow from the definition of the Riemann theta function by the series (7.1.25), and may equivalently be formulated as the identity:

$$\sum_{1 \leq a,b \leq g} \delta\tau_{ab} \frac{\partial^2 \theta(\tau,z)}{\partial z_a \partial z_b} = 4\pi i \frac{d}{dt} \theta(\tau + t\,\delta\tau, z)_{|t=0},$$

valid for every $(\tau,z) \in \mathfrak{H} \times \mathbb{C}^g$ and every symmetric matrix $\delta\tau = (\delta\tau_{ab})_{1 \leq a,b \leq g}$ in $M_g(\mathbb{C})$; see for instance [**Igu72**, Section V.3].

7.6.2. The Banaszczyk function $B_{\overline{V},\Lambda}$ as a function of the closed subgroup Λ. In this subsection, we briefly discuss the continuity properties of the Banasczyk function $B_{\overline{V},\Lambda}$ as a function of the subgroup Λ, notably the limit behavior of $B_{\overline{V},\Lambda}$ when Λ varies in some decreasing or increasing sequence of discrete subgroups.

To formulate these properties, it is convenient to define the Banasczyk function $B_{\overline{V},\Lambda}$ in a more general context than the one considered till now, namely when Λ is an arbitrary closed subgroup of V.[16]

7.6.2.1. *The Banaszczyk function $B_{\overline{V},\Lambda}$ associated to a closed subgroup Λ.* To formulate the limit behavior of $B_{\overline{V},\Lambda}$ when Λ varies in an increasing sequence of discrete subgroups of V, it is useful to define the B. functions $B_{\overline{V},\Lambda}$ in a slightly more general context, namely when Λ is a arbitrary closed subgroup of V.

Definition 7.6.6. Let $\overline{V} := (V, \|.\|)$ be a finite dimensional Euclidean vector space, and let Λ be a closed subgroup of V. The Banaszczyk function

$$B_{\overline{V},\Lambda} : V \longrightarrow \mathbb{R}_+^*$$

is the function defined by following equality:

$$B_{\overline{V},\Lambda}(x) := \frac{\int_\Lambda e^{-\pi\|v-x\|^2}\, d\mu(v)}{\int_\Lambda e^{-\pi\|v\|^2}\, d\mu(v)}, \tag{7.6.12}$$

where we denote by μ a Haar measure on the group Λ.

The right-hand side of (7.6.12) is clearly independent of the choice of the Haar measure μ. It is easily seen to define a real analytic function of $x \in V$.

Actually, the construction of the function $B_{\overline{V},\Lambda}$ associated to a general closed subgroup Λ of V easily reduces to the construction of the Banaszczyk function associated to some discrete subgroup.

Indeed, with the notation of Definition 7.6.6, the closed subgroup Λ of V is a Lie group. Its connected component Λ° is the largest $\mathbb{R}$-subvector space of Λ, and the exact sequence

$$0 \longrightarrow \Lambda^\circ \longrightarrow \Lambda \longrightarrow \Lambda/\Lambda^\circ \longrightarrow 0$$

describes the commutative Lie group Λ as a (necessarily split) extension of a discrete free finitely generated abelian group by a vector group.

Proposition 7.6.7. *With the previous notation, if we denote by $r : V \to V/\Lambda^\circ$ the quotient map, and by $\overline{V/\Lambda^\circ}$ the Euclidean vector space defined by the quotient V/Λ° endowed with the norm quotient of the Euclidean norm $\|.\|$ over V, then the following equality holds for every $x \in V$:*

$$B_{\overline{V},\Lambda}(x) = B_{\overline{V/\Lambda^\circ},\Lambda/\Lambda^\circ}(r(x)). \tag{7.6.13}$$

Proof. The Euclidean vector space $\overline{V}$ may be identified with the direct sum $\overline{V/\Lambda^\circ} \oplus \overline{\Lambda^\circ}$, where $\overline{\Lambda^\circ}$ denotes the Euclidean vector space $(\Lambda^\circ, \|.\|_{|\Lambda^\circ}$. Moreover this identification maps Λ to $\Lambda/\Lambda^\circ \oplus \Lambda^\circ$. Using this identification, (7.6.13) easily follows. □

Moreover Proposition 7.4.2 also easily extends to the present setting. Indeed a straightforward variant of the proof of Proposition 7.4.2 establishes the following proposition.

Proposition 7.6.8. *With the previous notation, let* $\mathrm{Vect}(\Lambda)$ *be the $\mathbb{R}$-vector subspace of V generated by Λ and* $\overline{\mathrm{Vect}(\Lambda)}$ *the Euclidean vector space* $(\mathrm{Vect}(\Lambda), \|.\|_{|\mathrm{Vect}(\Lambda)})$*, let $q : V \to V/\mathrm{Vect}(\Lambda)$ be the quotient map and $\|.\|_{V/\mathrm{Vect}(\Lambda)}$ be the Euclidean norm on $V/\mathrm{Vect}(\Lambda)$ defined as the quotient norm of $\|.\|$, and let $p^\perp : V \longrightarrow \mathrm{Vect}(\Lambda)$ be the orthogonal projection from V onto* $\mathrm{Vect}(\Lambda)$.

[16] This extended definition is a special instance of a general construction of Banaszczyk; see [**Ban22**].

Then the following relation holds for every $x \in V$:

$$B_{\overline{V},\Lambda}(x) = e^{-\pi \|q(x)\|^2_{V/\mathrm{Vect}(\Lambda)}}\, B_{\overline{\mathrm{Vect}(\Lambda)},\Lambda}(p^\perp(x)). \tag{7.6.14}$$

Using Propositions 7.6.7 and 7.6.8, we may write the function $B_{\overline{V},\Lambda}$ as the product of a Gaussian function and of the Banaszczyk function of a Euclidean lattice — namely the one defined by the lattice Λ/Λ° in the Euclidean vector subspace $\overline{\mathrm{Vect}(\Lambda)/\Lambda^\circ}$ of $\overline{V/\Lambda^\circ}$, where $\mathrm{Vect}(\Lambda)$ denotes the $\mathbb{R}$-vector subspace of V generated by Λ. Indeed, with the notation of these propositions, we obtain:

$$B_{\overline{V},\Lambda}(x) = e^{-\pi \|q(x)\|^2_{V/\mathrm{Vect}(\Lambda)}})B_{\overline{\mathrm{Vect}(\Lambda)/\Lambda^\circ},\Lambda/\Lambda^\circ}(r \circ p^\perp(x)).$$

Using Proposition 7.6.7, one easily extends the compatibility of Banaszczyk functions with direct sum in Proposition 7.4.1 and their monotonicity functions established in Theorem 7.4.12 to the Banaszczyk functions $B_{\overline{V},\Lambda}$ associated to general closed subgroups. For instance, we have:

THEOREM 7.6.9. *For $i = 1, 2$, let us consider a finite dimensional $\mathbb{R}$-vector space V_i, a Euclidean norm $\|.\|_i$, and a closed subgroup Λ_i of V_i.*

If some $\mathbb{R}$-linear map $f : V_1 \to V_2$ satisfies the conditions:

$$f(\Lambda_1) \subseteq \Lambda_2 \quad \text{and} \quad \|f(x)\|_2 \leq \|x\|_1 \text{ for every } x \in V_1,$$

then following inequality holds for every $x \in V_1$:

$$B(V_2, \|.\|_2, \Lambda_2; f(x)) \geq B(V_1, \|.\|_1, \Lambda_1; x). \tag{7.6.15}$$

COROLLARY 7.6.10. *For any finite dimensional vector space V and any $x \in V$, the Banaszczyk function $B(V, \|.\|, \Lambda; x)$ is a decreasing function of $\|.\|$ in the convex cone $\overset{\circ}{\mathcal{Q}}(V)$ of Euclidean norms over V, and an increasing function of Λ in the space $\mathcal{SG}(V)$ of closed subgroups of V.*

The expression (7.1.13) for the Banaszczyk function $B_{\overline{E}}$ of a Euclidean lattice in terms of its dual $\overline{E}^\vee$ also admits an extension to the Banaszczyk function $B_{\overline{V},\Lambda}$ attached to an arbitrary closed subgroup Λ of V. For simplicity, we shall spell it out in the situation when Λ is cocompact in V, or equivalently when $\mathrm{Vect}(\Lambda) = V$.

When this holds, the closed subgroup:

$$\Lambda^\perp := \left\{\xi \in V^\vee \mid \xi(\Lambda) \subseteq \mathbb{Z}\right\}$$

of V is discrete, and the quotient topological group $V^\vee/\Lambda^\perp$ may be identified with the Pontrjagin dual $\widehat{\Lambda}$ by the map

$$V^\vee/\Lambda^\perp \xrightarrow{\sim} \widehat{\Lambda}, \quad [\xi] \longmapsto \left(\lambda \mapsto e^{2\pi i \langle \xi, \lambda\rangle}\right).$$

Observe also that, when Λ is a lattice in V, $\Lambda^\perp$ may be identified with the dual of Λ as a $\mathbb{Z}$-module:

$$\Lambda^\vee := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$$

by the map:

$$\Lambda^\perp \xrightarrow{\sim} \Lambda^\vee, \quad \xi \longmapsto \xi_{|\Lambda}.$$

More generally, any $\xi \in V^\perp$ vanishes on Λ°, and — still assuming that $\mathrm{Vect}(\Lambda) = V$ — the discrete group $\Lambda^\perp$ may be identified with the dual of the $\mathbb{Z}$-module Λ/Λ° by means of the map:

$$\Lambda^\perp \xrightarrow{\sim} (\Lambda/\Lambda^\circ)^\vee, \quad \xi \longmapsto \left([\lambda] \mapsto \langle \xi, \lambda\rangle\right). \tag{7.6.16}$$

PROPOSITION 7.6.11. *Let $\overline{V} := (V, \|.\|)$ be a finite dimensional Euclidean vector space, $\overline{V}^\vee := (V^\vee, \|.\|^\vee)$ its dual, and Λ a closed subgroup of V such that $\mathrm{Vect}(\Lambda) = V$. Then the following equality holds for every $x \in V$:*

$$B_{\overline{V},\Lambda}(x) = \frac{\sum_{\xi \in \Lambda^\perp} e^{-\pi \|\xi\|^{\vee 2} + 2\pi i \langle \xi, x\rangle}}{\sum_{\xi \in \Lambda^\perp} e^{-\pi \|\xi\|^{\vee 2}}}. \tag{7.6.17}$$

PROOF. When Λ is a lattice in V, this is a reformulation of the second equality in (7.1.13). One reduces to this situation by means of Proposition 7.6.7 and of the identification (7.6.16). $\square$

7.6.2.2. *The limit behavior of Banaszczyk functions attached to a decreasing sequence of subgroups.*

PROPOSITION 7.6.12. *Let $\overline{V} := (V, \|.\|)$ be a finite dimensional Euclidean vector space, and let let $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of closed subgroups of V. If the sequence $(\Lambda_n)_{n\in\mathbb{N}}$ is decreasing and if*

$$\Lambda := \bigcap_{n\in\mathbb{N}} \Lambda_n$$

is the closed subgroup of V defined as its intersection, then the following equality holds for every $x \in V$:

$$B(\overline{V}, \Lambda; x) = \lim_{n\to+\infty} B(\overline{V}, \Lambda_n; x). \tag{7.6.18}$$

PROOF. Firstly assume that the subgroups Λ_n are discrete. Then, for every $n \in \mathbb{N}$ and every $x \in V$, we have:

$$B_{\overline{V},\Lambda_n}(x) = \frac{\sum_{v\in\Lambda_n} e^{-\pi\|x-v\|^2}}{\sum_{v\in\Lambda_n} e^{-\pi\|v\|^2}}.$$

The subgroup Λ also is discrete, and

$$B_{\overline{V},\Lambda}(x) = \frac{\sum_{v\in\Lambda} e^{-\pi\|x-v\|^2}}{\sum_{v\in\Lambda} e^{-\pi\|v\|^2}}.$$

It is straightforward, that for every $x \in V$, the sequence $\left(\sum_{v\in\Lambda_n} e^{-\pi\|x-v\|^2}\right)_{n\in\mathbb{N}}$ is decreasing and satisfies:

$$\lim_{n\to+\infty} \sum_{v\in\Lambda_n} e^{-\pi\|x-v\|^2} = \sum_{v\in\Lambda} e^{-\pi\|x-v\|^2};$$

in particular:

$$\lim_{n\to+\infty} \sum_{v\in\Lambda_n} e^{-\pi\|v\|^2} = \sum_{v\in\Lambda} e^{-\pi\|v\|^2}.$$

The relation (7.6.18) immediately follows.

In general, the decreasing sequence $(\Lambda_n^\circ)_{n\in\mathbb{N}}$ of vector subspaces of V is eventually constant, and thanks to Proposition 7.6.7, one reduces the proof of Proposition 7.6.12 to the case where the subgroups Λ_n are discrete. $\square$

This easy result allows one to derive, by simple formal arguments, the validity for Banasczyk functions associated to arbitrary pairs $(\overline{V}, \Lambda)$, with Λ a discrete subgroup of V, of various results concerning Banasczyk functions of Euclidean lattices.

7.6.2.3. *The limit behavior of Banaszczyk functions attached to an increasing sequence of subgroups.*

PROPOSITION 7.6.13. *Let $(\Lambda_n)_{n\in\mathbb{N}}$ be a sequence of closed subgroups of V. If $(\Lambda_n)_{n\in\mathbb{N}}$ is increasing and if*

$$\Lambda := \overline{\bigcup_{n\in\mathbb{N}} \Lambda_n}$$

is the closed subgroup of V defined as the closure of its subgroup $\bigcup_{n\in\mathbb{N}} \Lambda_n$, then, for every $x \in V$, the following equality holds:

$$B_{\overline{V},\Lambda}(x) = \lim_{n\to+\infty} B_{\overline{V},\Lambda_n}(x). \tag{7.6.19}$$

PROOF. The increasing sequence $(\mathrm{Vect}(\Lambda_n))_{n\in\mathbb{N}}$ of vector subspaces of V is eventually constant, and may therefore be assumed to be constant. Then, using Proposition 7.6.8, one reduces the proof of Proposition 7.6.13 to the case where $\mathrm{Vect}(\Lambda_n) = V$ for every $n \in \mathbb{N}$. In this case, we also have $\mathrm{Vect}(\Lambda) = V$, and according to Proposition 7.6.11, for every $x \in V$ and every $n \in \mathbb{N}$, the following equalities hold:

$$B_{\overline{V},\Lambda_n}(x) = \frac{\sum_{\xi\in\Lambda_n^\perp} e^{-\pi\|\xi\|^{\vee 2}+2\pi i\langle\xi,x\rangle}}{\sum_{\xi\in\Lambda_n^\perp} e^{-\pi\|\xi\|^{\vee 2}}}, \tag{7.6.20}$$

and

$$B_{\overline{V},\Lambda}(x) = \frac{\sum_{\xi\in\Lambda^\perp} e^{-\pi\|\xi\|^{\vee 2}+2\pi i\langle\xi,x\rangle}}{\sum_{\xi\in\Lambda^\perp} e^{-\pi\|\xi\|^{\vee 2}}}. \tag{7.6.21}$$

Moreover the sequence $(\Lambda_n^\perp)_{n\in\mathbb{N}}$ of discrete subgroups of $V^\vee$ is decreasing, and by Pontryagin duality, it satisfies:

$$\bigcap_{n\in\mathbb{N}} \Lambda_n^\perp = \Lambda^\perp.$$

This implies the relations:

$$\lim_{n\to+\infty} \sum_{\xi\in\Lambda_n^\perp} e^{-\pi\|\xi\|^{\vee 2}+2\pi i\langle\xi,x\rangle} = \sum_{\xi\in\Lambda^\perp} e^{-\pi\|\xi\|^{\vee 2}+2\pi i\langle\xi,x\rangle}$$

and

$$\lim_{n\to+\infty} \sum_{\xi\in\Lambda_n^\perp} e^{-\pi\|\xi\|^{\vee 2}} = \sum_{\xi\in\Lambda^\perp} e^{-\pi\|\xi\|^{\vee 2}},$$

and finally the equality (7.6.19) according to (7.6.20) and (7.6.21). □

7.6.2.4. *Application to the monotonicity of Banaszczyk functions.* As indicated in the discussion following the statement of the monotonicity property of Banaszczyk functions in Theorem 7.4.12, by relying on Proposition 7.6.13 it is possible to derive the special case of Theorem 7.4.12 when f is an admissible surjective morphism of Euclidean lattices from its special case when f is a morphism of Euclidean lattices such that $f_\mathbb{R}$ is an isometric isomorphism.

Indeed, for every admissible short exact sequence of Euclidean lattices:

$$\overline{\mathcal{E}}: \quad 0 \longrightarrow \overline{G} \xrightarrow{i} \overline{E} \xrightarrow{p} \overline{F} \longrightarrow 0,$$

we may perform the following constrcution.

For any positive integer N, we may consider the lattice

$$E_N := N^{-1}i(G) + E$$

inside the Euclidean vector space $(E_\mathbb{R}, \|.\|_{\overline{E}})$. It defines a Euclidean lattice $\overline{E}_N$, whose underlying $\mathbb{R}$-vector space $E_{N,\mathbb{R}}$ coincides with $E_\mathbb{R}$.

Moreover we may choose a strictly increasing sequence $(N_n)_{n\in\mathbb{N}}$ of positive integers such that N_n divides N_{n+1} for every $n \in \mathbb{N}$. Then $(E_{N(n)})_{n\in\mathbb{N}}$ is an increasing sequences of discrete subgroups of $E_\mathbb{R}$, and we may apply Proposition 7.6.13 to the Euclidean $\mathbb{R}$-vector space $\overline{V} := \overline{E}_\mathbb{R}$ and to the sequence of subgroups $(\Lambda_n) := (E_{N(n)})$. Then we have:

$$\Lambda := \overline{\bigcup_{n\in\mathbb{N}} E_{N(n)}} = i_\mathbb{R}(G_\mathbb{R}) + E,$$

and therefore, according to Proposition 7.6.13,

$$\lim_{n\to+\infty} B(\overline{E}_{N(n)}; x) = B(\overline{E}_\mathbb{R}, \Lambda; x). \tag{7.6.22}$$

Moreover, we have :

$$\Lambda^\circ = i_{\mathbb{R}}(G_{\mathbb{R}}),$$

the Euclidean lattice $\overline{\Lambda/\Lambda^\circ}$ may be identified with $\overline{F}$ and the quotient map

$$r : E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}/\Lambda^\circ$$

with the map

$$p_{\mathbb{R}} : E_{\mathbb{R}} \longrightarrow F_{\mathbb{R}}.$$

Consequently, according to Proposition 7.6.7, for every $x \in E_{\mathbb{R}}$, we have:

$$B(\overline{E}_{\mathbb{R}}, \Lambda; x) = B(\overline{F}; p_{\mathbb{R}}(x)). \tag{7.6.23}$$

Moreover the monotonicity of the Banaszczyk functions stated in Theorem 7.4.12, applied to the morphism of Euclidean lattices $\overline{E} \to \overline{E}_N$ defined by the inclusion of lattices $E \hookrightarrow E_N$ (and the identity morphism of $E_{\mathbb{R}}$,) implies that, for every $N \in \mathbb{N}$ and every $x \in E_{\mathbb{R}}$, the following inequality holds:

$$B(\overline{E}; x) \leq B(\overline{E}_N; x). \tag{7.6.24}$$

Finally the conjunction of (7.6.22), (7.6.23), and (7.6.24) implies the estimate:

$$B(\overline{E}; x) \leq B(\overline{F}; p_{\mathbb{R}}(x)),$$

namely the monotonicity of Banaszczyk functions applied to the surjective admissible morphism of Euclidean lattices $p : \overline{E} \to \overline{F}$.

7.6.2.5. Recall that the space $\mathcal{SG}(V)$ of closed subgroups of the topological group $(V, +)$ is equipped with a natural topology, which makes it a metrizable compact space; see for instance [**Bou63**, Chap. VIII, Â§ 5]. Both in Proposition 7.6.12 and in Proposition 7.6.13, the sequence $(\Lambda_n)_{n\in\mathbb{N}}$ converges to Λ in this topology, and it is natural to expect that the continuity properties of $B_{\overline{V},\Lambda}$ as a function of Λ established in these propositions hold in much greater generality.

For instance, one might expect that the map:

$$\mathcal{SG}(V) \longrightarrow C^\infty(V, \mathbb{R}), \quad \Lambda \longmapsto B_{\overline{V},\Lambda}$$

is continuous when $C^\infty(V, \mathbb{R})$ is equipped with its natural topology of Fréchet space.

As will be discussed in the next section, it is actually possible to extend the definition of $B_{\overline{V},\Lambda}$ to the situation where the Euclidean norm $\|.\|$ defining the Euclidean vector space $\overline{V} := (V, \|.\|)$ is replaced by a Euclidean seminorm. The regularity properties of $B_{\overline{V},\Lambda}$ as a function both of $\|.\|$ and Λ seem to be an especially interesting and delicate issue, as already demonstrated by the construction in Proposition 7.3.7.

7.7. The Banaszczyk function $B_{\overline{E}}$ and the measure $\beta_{\overline{E}^\vee}$ associated to an object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$

7.7.1. Definitions and first properties of $B_{\overline{E}}$.

7.7.1.1. *Definition of* $B_{\overline{E}}$. Let us consider a finitely generated $\mathbb{Z}$-module E, the convex cone $\mathcal{Q}(E_{\mathbb{R}})$ of semipositive quadratic forms on the finite dimensional $\mathbb{R}$-vector space $E_{\mathbb{R}}$ and $\mathring{\mathcal{Q}}(E_{\mathbb{R}})$ its interior, the cone of positive definite quadratic forms on $E_{\mathbb{R}}$. As usual, we shall identify $E_{/\mathrm{tor}} := E/E_{\mathrm{tor}}$ with a discrete cocompact subgroup of $E_{\mathbb{R}}$.

For every $x \in E_{\mathbb{R}}$, the function

$$\mathring{\mathcal{Q}}(E_{\mathbb{R}}) \longrightarrow (0, 1], \quad \|.\|^2 \longmapsto B(E_{\mathbb{R}}, \|.\|, E) \tag{7.7.1}$$

is clearly $\mathbb{R}$-analytic. Moreover, as observed in Corollary 7.6.1, it is decreasing.

DEFINITION 7.7.1. For every Euclidean seminorm $\|.\|$ on $E_{\mathbb{R}}$ and every $x \in E_{\mathbb{R}}$, we let:

$$B(E_{\mathbb{R}}, \|.\|, E_{/\mathrm{tor}}; x) := \sup \left\{ B(E_{\mathbb{R}}, \|.\|', E_{/\mathrm{tor}}; x); \|.\|'^2 \in \mathring{\mathcal{Q}}(E_{\mathbb{R}}) \text{ and } \|.\|' \geq \|.\| \right\} \in (0,1]. \tag{7.7.2}$$

We shall also use the following notation:

$$B_{(E,\|.\|)}(x) := B(E_{\mathbb{R}}, \|.\|, E_{/\mathrm{tor}}; x). \tag{7.7.3}$$

In this way we attach a Banaszczyk function $B_{\overline{E}}$ over $E_{\mathbb{R}}$ to any object $\overline{E}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. Clearly this construction "does not see torsion"; namely, if $\overline{E}_{/\mathrm{tor}} = (E/E_{\mathrm{tor}}, \|.\|)$, then:

$$B_{\overline{E}} = B_{\overline{E}_{/\mathrm{tor}}}.$$

Observe that, according to Proposition 4.2.1, the function

$$\mathcal{Q}(E_{\mathbb{R}}) \longrightarrow (0,1], \quad \|.\|^2 \longmapsto B(E_{\mathbb{R}}, \|.\|, E),$$

which extends the function (7.7.1) to the closed cone $\mathcal{Q}(E_{\mathbb{R}})$ satisfies the following condition of downward continuity: *for any decreasing sequence $(\|.\|_n)_{n\in\mathbb{N}}$ of Euclidean seminorms on $E_{\mathbb{R}}$, of limit $\|.\|$, and any $x \in \mathbb{R}$, the following equality holds:*

$$\lim_{n\to+\infty} B_{(E,\|.\|_n)}(x) = B_{(E,\|.\|)}(x). \tag{7.7.4}$$

7.7.1.2. *Properties of $B_{\overline{E}}$.* Using the definition (7.7.1) of $B_{\overline{E}}$ and the downward continuity property (7.7.4), various properties of the Banaszczyk functions of Euclidean lattices immediately extend to the Banaszczyk functions of arbitrary Euclidean coherent sheaves. Notably from (7.4.7), (7.1.22), and Theorem 7.4.12, we deduce:

PROPOSITION 7.7.2. *For every object $\overline{E} := (E, \|.\|)$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, we have:*

$$e^{-\pi\|x\|^2} \leq B_{\overline{E}}(x) \leq 1 \quad \textit{for every } x \in E_{\mathbb{R}}, \tag{7.7.5}$$

and

$$h^1_\theta(\overline{E}) = -\log \int_{E_{\mathbb{R}}/E_{/\mathrm{tor}}} B_{\overline{E}}(x)\, d\lambda_{E_{\mathbb{R}/\mathbb{Z}}}(x). \tag{7.7.6}$$

Moreover, for every morphism $f : \overline{E} \to \overline{F}$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ and every $x \in E_{\mathbb{R}}$, the following inequality holds:

$$B_{\overline{F}}(f_{\mathbb{R}}(x)) \geq B_{\overline{E}}(x). \tag{7.7.7}$$

In (7.7.6), we denote by $\lambda_{E_{\mathbb{R}/\mathbb{Z}}}$ the normalized Haar measure on the compact torus $E_{\mathbb{R}}/E_{/\mathrm{tor}} \simeq E \otimes \mathbb{R}/\mathbb{Z}$.

As before, we may introduce the function

$$b_{\overline{E}} := \pi^{-1} \log B_{\overline{E}}^{-1},$$

and (7.7.7) may also be written as:

$$b_{\overline{F}}(f_{\mathbb{R}}(x)) \leq b_{\overline{E}}(x). \tag{7.7.8}$$

7.7.2. The measure $\beta_{\overline{E}^\vee}$. Compatibility with vectorization. Let $\overline{E} := (E, \|.\|)$ be an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

In paragraph 2.3.1, we have introduced its "vectorization":

$$\nu_{\overline{E}} : \overline{E} \longrightarrow \overline{E}^{\mathrm{vect}}.$$

It is an admissible surjective morphism in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ whose range $\overline{E}^{\mathrm{vect}} := (E^{\mathrm{vect}}, \|.\|_{\overline{E}^{\mathrm{vect}}})$ is a Euclidean lattice. Heuristically $\overline{E}^{\mathrm{vect}}$ is "the best possible approximation of $\overline{E}$ among Euclidean lattices." To $\nu_{\overline{E}}$ is associated the underlying $\mathbb{R}$-linear map:

$$\nu_{\overline{E},\mathbb{R}} : E_{\mathbb{R}} \longrightarrow E_{\mathbb{R}}^{\mathrm{vect}}$$

and its transpose:

$$\nu_{\overline{E},\mathbb{R}*}^{\vee} : E_{\mathbb{R}}^{\mathrm{vect}\vee} \longrightarrow E_{\mathbb{R}}^{\vee}.$$

Recall also that the object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ admits a dual object $\overline{E}^\vee := (E^\vee, \|.\|^\vee)$ in $\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[\infty]}$, defined by the free $\mathbb{Z}$-module $E^\vee := \mathrm{Hom}_{\mathbb{Z}}(E, \mathbb{Z})$ and the definite Euclidean quasinorm $\|.\|^\vee =: \|.\|_{\overline{E}^\vee}$ on $E_{\mathbb{R}}^\vee$ dual to the Euclidean seminorm $\|.\|$ on $E_{\mathbb{R}}$.

According to the description of vectorization in terms of duality in Corollary 2.3.3, the map $\nu_{\overline{E},\mathbb{R}*}^{\vee}$ is an isometry from the Euclidean vector space $(E_{\mathbb{R}}^{\mathrm{vect}\vee}, \|.\|_{\overline{E}^{\mathrm{vect}\vee}})$ into $(E_{\mathbb{R}}^\vee, \|.\|^\vee)$. In particular, $\nu_{\overline{E},\mathbb{R}}^{\vee}$ is injective and its image is contained in:

$$E_{\mathbb{R},f}^\vee := \{\xi \in E_{\mathbb{R}}^\vee \mid \|\xi\|^\vee < +\infty\}.$$

Moreover it establishes a bijection:

$$\nu_{\overline{E}}^{\vee} : E^{\mathrm{vect}\vee} \xrightarrow{\sim} E^\vee \cap E_{\mathbb{R},f}^\vee. \tag{7.7.9}$$

The definition (7.1.5) of the measure $\gamma_{\overline{E}^\vee}$ still makes sense in the present situation:

$$\gamma_{\overline{E}^\vee} := e^{-\pi\|.\|^{\vee 2}}\,\delta_{E^\vee} = \sum_{\xi \in E^\vee} e^{-\pi\|\xi\|^{\vee 2}}\delta_\xi,$$

where, as before:

$$e^{-\pi\|\xi\|^{\vee 2}} := 0 \quad \text{if } \|\xi\|^\vee = +\infty.$$

Using the isometric bijection (7.7.9), we also have:

$$\gamma_{\overline{E}^\vee} = \sum_{\xi \in E^\vee \cap E_{\mathbb{R},f}^\vee} e^{-\pi\|\xi\|^{\vee 2}}\delta_\xi = \sum_{\xi \in E^\vee \cap E_{\mathbb{R},f}^\vee} e^{-\pi\|\xi\|^{\vee 2}}\delta_{\nu_{\overline{E}}^\vee(\xi')}.$$

This establishes the equality of measures on $E_{\mathbb{R}}^\vee$ supported by $E^\vee \cap E_{\mathbb{R},f}^\vee$:

$$\gamma_{\overline{E}^\vee} = \nu_{\overline{E}*}^\vee \gamma_{\overline{E}^{\mathrm{vect}\vee}}. \tag{7.7.10}$$

In particular, the measures $\gamma_{\overline{E}^\vee}$ and $\gamma_{\overline{E}^{\mathrm{vect}\vee}}$ have the same total mass in $[1, +\infty))$ and as already observed in 7.3.2.3, we recover the invariance of h_θ^1 under vectorization:

$$e^{h_\theta^1(\overline{E})} = \sum_{\xi \in E^\vee \cap E_{\mathbb{R},f}^\vee} e^{-\pi\|\xi\|^{\vee 2}} = \gamma_{\overline{E}^\vee}(E_{\mathbb{R}}^\vee) = \gamma_{\overline{E}^{\mathrm{vect}\vee}}(E_{\mathbb{R}}^\vee) = \sum_{\xi \in E^\vee \cap E_{\mathbb{R},f}^\vee} e^{-\pi\|\xi\|^{\vee 2}} = e^{h_\theta^1(\overline{E}^{\mathrm{vect}\vee})}. \tag{7.7.11}$$

Finally we define the Banaszczyk measure $\beta_{\overline{E}^\vee}$ by means of formula (7.1.12), namely as the probability measure:

$$\beta_{\overline{E}^\vee} := \gamma_{\overline{E}^\vee}(E_{\mathbb{R}}^\vee)^{-1}\,\gamma_{\overline{E}^\vee}.$$

PROPOSITION 7.7.3. *Let $\overline{E} := (E, \|.\|)$ be an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$.*

(1) If $(\|.\|_n)_{n\in\mathbb{N}}$ is a decreasing sequence of Euclidean seminorms on $E_{\mathbb{R}}$ converging pointwise to $\|.\|$, then the sequence of probability measures $(\beta_{\overline{E}_n^\vee})_{n\in\mathbb{N}}$ in $\mathcal{M}^1_+(E^\vee_{\mathbb{R}})$ associated to the Euclidean coherent sheaves $\overline{E}_n := (E, \|.\|_n)$ converges to $\beta_{\overline{E}^\vee}$ in the topology of narrow convergence.

Moreover the following equality of probability measures on $E^\vee_{\mathbb{R}}$ is satisfied:

$$\beta_{\overline{E}^\vee} = \nu^\vee_{\overline{E},\mathbb{R}*}\beta_{\overline{E}^{\mathrm{vect}\vee}}. \tag{7.7.12}$$

(2) The following equality of functions on $E_{\mathbb{R}}$ is satisfied:

$$B_{\overline{E}} = \mathcal{F}^{-1}_{E_{\mathbb{R}}}(\beta_{\overline{E}^\vee}). \tag{7.7.13}$$

Moreover, for every $x \in E_{\mathbb{R}}$, the following equality holds:

$$B_{\overline{E}}(x) = B_{\overline{E}^{\mathrm{vect}}}(\nu_{\overline{E}}(x)). \tag{7.7.14}$$

PROOF. (1) If $(\|.\|_n)_{n\in\mathbb{N}}$ is a decreasing sequence of Euclidean seminorms on $E_{\mathbb{R}}$ converging pointwise to $\|.\|$, then the sequence $(\|.\|_n^\vee)_{n\in\mathbb{N}}$ of dual quasinorms on $E^\vee_{\mathbb{R}}$ is increasing and converges pointwise to $\|.\|^\vee$. This immediately implies the convergence, in the topology of narrow convergence, of the sequence of measures $(\gamma_{\overline{E}_n^\vee})_{n\in\mathbb{N}}$ to $\gamma_{\overline{E}^\vee}$, and therefore of $(\beta_{\overline{E}_n^\vee})_{n\in\mathbb{N}}$ to $\beta_{\overline{E}^\vee}$.

The relation (7.7.12) is a straightforward consequence of (7.7.10) and of the definitions of $\beta_{\overline{E}^\vee}$ and $\beta_{\overline{E}^{\mathrm{vect}\vee}}$.

(2) We may choose a a decreasing sequence of Euclidean norms on $E_{\mathbb{R}}$ converging pointwise to $\|.\|$ that converges pointwise to $\|.\|$. Then, as a special instance of (7.7.4), we have, for every $x \in E_{\mathbb{R}}$:

$$\lim_{n\to+\infty} B_{\overline{E}_n}(x) = B_{\overline{E}}(x). \tag{7.7.15}$$

Moreover, according to Corollary 7.1.4 applied to the Euclidean lattice $\overline{E}_{n/\mathrm{tor}}$, we have:

$$B_{\overline{E}_n}(x) = \mathcal{F}^{-1}(\beta_{\overline{E}_n^\vee})(x). \tag{7.7.16}$$

Finally the convergence of of $(\beta_{\overline{E}_n^\vee})_{n\in\mathbb{N}}$ to $\beta_{\overline{E}^\vee}$ implies the equality:

$$\lim_{n\to+\infty} \mathcal{F}^{-1}(\beta_{\overline{E}_n^\vee})(x) = \mathcal{F}^{-1}(\beta_{\overline{E}^\vee})(x). \tag{7.7.17}$$

The relations (7.7.15)-(7.7.17) imply the equality:

$$B_{\overline{E}}(x) = \mathcal{F}^{-1}_{E_{\mathbb{R}}}(\beta_{\overline{E}^\vee})(x).$$

This establishes (7.7.13).

The equality (7.7.14) follows from (7.7.12) and from the relations:

$$B_{\overline{E}} = \mathcal{F}^{-1}_{E_{\mathbb{R}}}(\beta_{\overline{E}^\vee}) \quad \text{and} \quad B_{\overline{E}^{\mathrm{vect}}} = \mathcal{F}^{-1}_{E_{\mathbb{R}}}(\beta_{\overline{E}^{\mathrm{vect}\vee}}). \qquad \square$$

According to (7.7.13) and to the definitions of $\gamma_{\overline{E}^\vee}$ and $\beta_{\overline{E}^\vee}$, $B_{\overline{E}}(x)$ admits the following expression, for every x in $E_{\mathbb{R}}$:

$$B_{\overline{E}}(x) = e^{-h^1_\theta(\overline{E})} \sum_{\xi\in E^\vee\cap E^\vee_{\mathbb{R},f}} e^{-\pi\|\xi\|^2_{\overline{E}^\vee}+2\pi i\xi(x)} = e^{-h^1_\theta(\overline{E})} \sum_{\xi\in E^\vee\cap E^\vee_{\mathbb{R},f}} e^{-\pi\|\xi\|^2_{\overline{E}^\vee}} \cos(2\pi\xi(x)). \tag{7.7.18}$$

Using that, when $\overline{E}$ is a Euclidean lattice, the Banaszczyk function $B_{\overline{E}}$ is real analytic on $E_{\mathbb{R}}$ and takes the value 1 precisely over E, together with the description of $\nu^{-1}_{\overline{E},\mathbb{R}}(E^{\mathrm{vect}})$ in Corollary 2.3.7, we immediately derive from the equality (7.7.14).

COROLLARY 7.7.4. *The function $B_{\overline{E}}$ is real analytic of $E_{\mathbb{R}}$. Moreover, for every $x \in E_{\mathbb{R}}$, the equality $B_{\overline{E}}(x) = 1$ holds if and only if x belongs to $\overline{E_{/\mathrm{tor}} + K}$, the closure in $E_{\mathbb{R}}$ of the sum of the lattice $E_{/\mathrm{tor}}$ and the vector subspace $K := \|.\|^{-1}(0)$.*

7.8. A Lower Bound on $B_{\overline{E}}$ and the Inverse Monotonicity of $B_{\overline{E}}$ and $\beta_{\overline{E}^\vee}$

In this subsection we some additional estimates concerning the functions $B_{\overline{E}}$ and the measures $\beta_{\overline{E}^\vee}$. Notably we show that the uniform norm of the Banaszczyk function $b_{\overline{E}}$ is comparable with the theta invariant $h^1_\theta(\overline{E})$, provided $h^1_\theta(\overline{E})$ is small enough. We also show that, besides the basic monotonicity estimate (7.7.7), the Banaszcyk functions $B_{\overline{E}}$ satisfy estimates in the opposite direction.

7.8.1. A lower bound on $B_{\overline{E}}$. The following lower bound on the Banaszczyk function $B_{\overline{E}}$ already appears in Banaszczyk's seminal paper [**Ban93**], at least when $\overline{E}$ is a Euclidean lattice.

PROPOSITION 7.8.1. *Let $\overline{E}$ be an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. For every x in $E_{\mathbb{R}}$, we have:*

$$\begin{aligned} 1 + B_{\overline{E}}(x) &= e^{-h^1_\theta(\overline{E})} \sum_{\xi \in E^\vee \cap E^\vee_{\mathbb{R},f}} e^{-\pi \|\xi\|^2_{\overline{E}^\vee}} (1 + \cos(2\pi \langle \xi, x \rangle)) \\ &= 2e^{-h^1_\theta(\overline{E})} \sum_{\xi \in E^\vee \cap E^\vee_{\mathbb{R},f}} e^{-\pi \|\xi\|^2_{\overline{E}^\vee}} \cos^2(\pi \langle \xi, x \rangle)). \end{aligned} \tag{7.8.1}$$

Consequently:

$$B_{\overline{E}}(x) \geq 2e^{-h^1_\theta(\overline{E})} - 1. \tag{7.8.2}$$

PROOF. The formulae (7.8.1) immediately follow from the expressions (7.7.11) and (7.7.18) for $e^{h^1_\theta(\overline{E})}$ and $B_{\overline{E}}(x)$. The inequality (7.8.2) immediately follows. □

The inequality (7.8.2) is non-trivial only when its right-hand side is positive, namely when:

$$h^1_\theta(\overline{E}) < \log 2. \tag{7.8.3}$$

When (7.8.3) holds, the estimate (7.8.2) may be rephrased as follows:

$$\sup_{x \in E_{\mathbb{R}}} b_{\overline{E}}(x) \leq \pi^{-1} \log(2e^{-h^1_\theta(\overline{E})} - 1)^{-1}. \tag{7.8.4}$$

7.8.2. The function $\kappa(x) := \pi^{-1} \log(2e^{-x} - 1)^{-1}$. When dealing with estimates such as (7.8.4), it is convenient to introduce the function:

$$\kappa : [0, \log 2) \longrightarrow \mathbb{R}_+, \quad x \longmapsto \pi^{-1} \log(2e^{-x} - 1)^{-1}$$

It satisfies $\kappa(0) = 0$, and defines an increasing convex $\mathbb{R}$-analytic diffeomorphism. Moreover, when x goes to 0:

$$\pi^{-1} \log(2e^{-x} - 1)^{-1} = \frac{2}{\pi} x + O(x^2).$$

The convexity of κ implies the following upper bound:

$$x \in [0, 1/2] \Longrightarrow \kappa(x) \leq 2x\, \kappa(1/2) \leq x. \tag{7.8.5}$$

Indeed we have:

$$2\pi\kappa(1/2) = 2 \log(2e^{-1/2} - 1)^{-1} = 3.092... \leq \pi.$$

It will also be convenient to let:

$$\kappa(x) := +\infty \quad \text{if } x \in [\log 2, +\infty).$$

Then the inequality (7.8.4) may be reformulated as:

$$\sup_{x \in E_{\mathbb{R}}} b_{\overline{E}}(x) \leq \kappa(h^1_\theta(\overline{E})),$$

and implies:

$$\sup_{x \in E_{\mathbb{R}}} b_{\overline{E}}(x) \leq h^1_\theta(\overline{E}) \quad \text{if } h^1_\theta(\overline{E}) \leq 1/2. \tag{7.8.6}$$

This upper bound on $\sup_{x\in E_{\mathbb{R}}} b_{\overline{E}}(x)$ should be compared with the following consequence of the expression (7.7.6) for the average value of $B_{\overline{E}}$:

$$\inf_{x\in E_{\mathbb{R}}} B_{\overline{E}}(x) \leq e^{-h^1_\theta(\overline{E})},$$

which may be rephrased as the following lower bound on $\sup_{x\in E_{\mathbb{R}}} b_{\overline{E}}(x)$:

$$\sup_{x\in E_{\mathbb{R}}} b_{\overline{E}}(x) \geq \pi^{-1} h^1_\theta(\overline{E}).$$

For later reference, we gather these estimates on the supremum of the function $b_{\overline{E}}$ in the following scholium:

SCHOLIUM 7.8.2. *For every object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following estimates hold:*

$$\pi^{-1} h^1_\theta(\overline{E}) \leq \sup_{x\in E_{\mathbb{R}}} b_{\overline{E}}(x) \leq \kappa(h^1_\theta(\overline{E})). \tag{7.8.7}$$

7.8.3. Inverse monotonicity of $B_{\overline{E}}$.

PROPOSITION 7.8.3. *Let $f : \overline{E} \to \overline{F}$ be a morphism in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$. The following inequality between positive measures on $E^\vee_{\mathbb{R}}$ is satisfied:*

$$f^\vee_{\mathbb{R}*}\gamma_{\overline{F}^\vee} \leq e^{h^1_\theta(\overline{F/f(E)})}\, \gamma_{\overline{E}^\vee}, \tag{7.8.8}$$

or equivalently:

$$f^\vee_{\mathbb{R}*}\beta_{\overline{F}^\vee} \leq e^{h^1_\theta(\overline{F/f(E)}) + h^1_\theta(\overline{E}) - h^1_\theta(\overline{F})}\, \beta_{\overline{E}^\vee}. \tag{7.8.9}$$

Moreover, for every $x \in E_{\mathbb{R}}$, the following inequality holds:

$$1 + B_{\overline{F}}(f_{\mathbb{R}}(x)) \leq e^{h^1_\theta(\overline{F/f(E)}) + h^1_\theta(\overline{E}) - h^1_\theta(\overline{F})}\, (1 + B_{\overline{E}}(x)). \tag{7.8.10}$$

As usual, we have denoted by $f_{\mathbb{R}} : E_{\mathbb{R}} \to F_{\mathbb{R}}$ the $\mathbb{R}$-linear map induced by $f : E \to F$, and by $f^\vee_{\mathbb{R}} : E^\vee_{\mathbb{R}} \to F^\vee_{\mathbb{R}}$ its transpose.

PROOF. (1) A straightforward approximation argument show that, to establish (7.8.8), we may assume that the Euclidean seminorms $\|.\|_{\overline{E}}$ and $\|.\|_{\overline{F}}$ defining $\overline{E}$ and $\overline{F}$ are actually Euclidean norms. Then $\overline{E}^\vee$ and $\overline{F}^\vee$ are Euclidean lattices, the transpose $f^\vee$ of f defines a morphism

$$f^\vee : \overline{E}^\vee \longrightarrow \overline{F}^\vee$$

in $\overline{\mathbf{Vect}}_{\mathbb{Z}}^{\leq 1}$, and the measures $\gamma_{\overline{E}^\vee}$ and $\gamma_{\overline{F}^\vee}$ satisfy:

$$\gamma_{\overline{E}^\vee} := \sum_{\xi\in E^\vee} e^{-\pi\|\xi\|^2_{\overline{E}^\vee}}\, \delta_\xi \quad \text{and} \quad \gamma_{\overline{F}^\vee} := \sum_{\eta\in F^\vee} e^{-\pi\|\eta\|^2_{\overline{F}^\vee}}\, \delta_\eta.$$

Consequently we have:

$$f^\vee_{\mathbb{R}*}\gamma_{\overline{F}^\vee} = \sum_{\eta\in F^\vee} e^{-\pi\|\eta\|^2_{\overline{F}^\vee}}\, \delta_{f^\vee(\eta)},$$

and the inequality of measures (7.8.8) is equivalent to the validity of the following estimate for every $\xi \in E^\vee$:

$$\sum_{\eta\in f^{\vee -1}(\xi)} e^{-\pi\|\eta\|^2_{\overline{F}^\vee}} \leq e^{h^1_\theta(\overline{F/f(E)})}\, e^{-\pi\|\xi\|^2_{\overline{E}^\vee}}; \tag{7.8.11}$$

here $f^{\vee -1}(\xi)$ denotes the preimage of ξ by the $\mathbb{Z}$-linear map $f^\vee : F^\vee \to E^\vee$.

To prove (7.8.11), we may and shall assume that ξ belongs to $f^\vee(F^\vee)$, and choose $\eta_0 \in F^\vee$ such that $\xi = f^\vee(\eta_0)$. We may also consider the orthogonal projection $\eta^\perp$ of η_0 onto the orthogonal $(\ker f^\vee_{\mathbb{R}})^\perp$ of $\ker f^\vee_{\mathbb{R}}$ in the Euclidean vector space $(F^\vee_{\mathbb{R}}, \|.\|_{\overline{F}^\vee})$. This orthogonal projection $\eta^\perp$ is the element of smallest norm $\|\eta^\perp\|_{\overline{F}^\vee}$ in $f^{\vee -1}_{\mathbb{R}}(\xi)$.

We have a bijection:

$$\ker f^\vee \xrightarrow{\sim} f^{\vee -1}(\xi), \quad \varepsilon \longmapsto \varepsilon + \eta_0,$$

and if we introduce the element

$$\delta := \eta^\perp - \eta_0 \in \ker f^\vee_{\mathbb{R}},$$

then, for every $\varepsilon \in \ker f^\vee$, the following equality holds:

$$\|\varepsilon + \eta_0\|^2_{\overline{F}^\vee} = \|\varepsilon - \delta + \eta^\perp\|^2_{\overline{F}^\vee} = \|\varepsilon - \delta\|^2_{\overline{F}^\vee} + \|\eta^\perp\|^2_{\overline{F}^\vee}.$$

Moreover:

$$\|\eta^\perp\|^2_{\overline{F}^\vee} \geq \|f^\vee_{\mathbb{R}}(\eta^\perp)\|^2_{\overline{E}^\vee} = \|\xi\|^2_{\overline{E}^\vee}.$$

Consequently the following inequality holds:

$$\sum_{\eta \in f^{\vee -1}(\xi)} e^{-\pi\|\eta\|^2_{\overline{F}^\vee}} \leq e^{-\pi\|\xi\|^2_{\overline{E}^\vee}} \sum_{\varepsilon \in \ker f^\vee} e^{-\pi\|\varepsilon - \delta\|^2_{\overline{F}^\vee}}. \tag{7.8.12}$$

Moreover, according to the inequality (7.1.14) applied to $\overline{\ker f^\vee}$ and δ in the role of $\overline{E}$ and x, we have:

$$\sum_{\varepsilon \in \ker f^\vee} e^{-\pi\|\varepsilon - \delta\|^2_{\overline{F}^\vee}} \leq \sum_{\varepsilon \in \ker f^\vee} e^{-\pi\|\varepsilon\|^2_{\overline{F}^\vee}} =: e^{h^0_\theta(\overline{\ker f^\vee})}. \tag{7.8.13}$$

From (7.8.12) and (7.8.13), we derive the inequality:

$$\sum_{\eta \in f^{\vee -1}(\xi)} e^{-\pi\|\eta\|^2_{\overline{F}^\vee}} \leq e^{h^0_\theta(\overline{\ker f^\vee})}\, e^{-\pi\|\xi\|^2_{\overline{E}^\vee}}.$$

To complete the proof of (7.8.11), simply observe that $\overline{F/f(E)}^\vee$ is canonically isomorphic to $\overline{\ker f^\vee}$, and therefore:

$$h^1_\theta(\overline{F/f(E)}) = h^0_\theta(\overline{\ker f^\vee}).$$

(2) For any $x \in E_{\mathbb{R}}$, we may consider the non-negative function:

$$E^\vee_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad \xi \longmapsto 2\cos^2(\pi\langle \xi, x\rangle).$$

According to Proposition 7.8.1, its integral against the measure $\gamma_{\overline{E}^\vee}$ is:

$$2\int_{E_{\mathbb{R}}} \cos^2(\pi\langle\xi, x\rangle)\, d\gamma_{\overline{E}^\vee}(\xi) = e^{h^1_\theta(\overline{E})}\,(1 + B_{\overline{E}}(x)). \tag{7.8.14}$$

Similarly, according to Proposition 7.8.1 applied to $\overline{F}$, its integral against the measure $f^\vee_{\mathbb{R}*}\gamma_{\overline{F}^\vee}$ is:

$$\begin{aligned} 2\int_{E_{\mathbb{R}}} \cos^2(\pi\langle\xi, x\rangle)\, f^\vee_{\mathbb{R}*}\, d\gamma_{\overline{F}^\vee}(\xi) &= 2\int_{F_{\mathbb{R}}} \cos^2(\pi\langle f^\vee_{\mathbb{R}}(\eta), x\rangle)\, d\gamma_{\overline{F}^\vee}(\eta) \\ &= 2\int_{F_{\mathbb{R}}} \cos^2(\pi\langle\eta), f_{\mathbb{R}}(x)\rangle)\, d\gamma_{\overline{F}^\vee}(\eta) = e^{h^1_\theta(\overline{F})}\,(1 + B_{\overline{F}}(f_{\mathbb{R}}(x))). \end{aligned} \tag{7.8.15}$$

The estimate (7.8.10) therefore follows from the inequality of measures (7.8.8). □

When $f = 0$, the inequality (7.8.10) becomes the lower bound (7.8.2) on $B_{\overline{E}}$. More generally, from (7.8.10) we may derive the following lower bound on $B_{\overline{E}}$:

Corollary 7.8.4. *Let $\overline{E} := (E, \|.\|)$ be an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and let E' be a $\mathbb{Z}$-submodule of E. For every $x' \in E'_{\mathbb{R}}$, the following inequality holds:*

$$B_{\overline{E}}(x') \geq 2e^{-h^1_\theta(\overline{E}) + h^1_\theta(\overline{E/E'})} - 1. \tag{7.8.16}$$

Proof. This follows from Proposition 7.8.3 applied to the quotient morphism f from $\overline{E}$ to $\overline{F} := \overline{E/E'}$, and to x in $E'_{\mathbb{R}} := \ker f_{\mathbb{R}}$. Indeed, with this choice of f and x, we have:

$$\overline{F/f(E)} = 0 \quad \text{and} \quad B_{\overline{F}}(f_{\mathbb{R}}(x)) = B_{\overline{F}}(0) = 1.$$

□

COROLLARY 7.8.5. *For every admissible short exact sequence in* $\overline{\mathbf{Coh}}_{\mathbb{Z}}$*:*

$$0 \longrightarrow \overline{G} \xrightarrow{i} \overline{E} \xrightarrow{p} \overline{F} \longrightarrow 0, \tag{7.8.17}$$

the following inequality between positive measures on $E_{\mathbb{R}}^{\vee}$ *is satisfied:*

$$p_{\mathbb{R}*}^{\vee} \beta_{\overline{F}^{\vee}} \leq e^{h_{\theta}^{1}(\overline{G})} \, \beta_{\overline{E}^{\vee}}, \tag{7.8.18}$$

and for every $x \in E_{\mathbb{R}}$*, the following inequality holds:*

$$1 + B_{\overline{F}}(p_{\mathbb{R}}(x)) \leq e^{h_{\theta}^{1}(\overline{G})} \, (1 + B_{\overline{E}}(x)). \tag{7.8.19}$$

PROOF. This follows from Proposition 7.8.3 applied to $f = p$, which implies the vanishing of $F/f(E)$, from the relations:

$$\beta_{\overline{E}^{\vee}} = e^{-h_{\theta}^{1}(\overline{E})} \quad \text{and} \quad \beta_{\overline{F}^{\vee}} = e^{-h_{\theta}^{1}(\overline{F})},$$

and from the subadditivity of h_{θ}^{1}, which implies the inequality:

$$h_{\theta}^{1}(\overline{E}) - h_{\theta}^{1}(\overline{F}) \leq h_{\theta}^{1}(\overline{G}).$$

□

7.8.4. Formulation in terms of θ^1-ranks. In Corollary 7.8.4, if we denote by $i : E' \to E$ the inclusion morphism, then we may consider its θ^1-rank, as defined in Sections 5.1 and 7.5:

$$\mathrm{rk}_{\theta}^{1}(i : E' \to \overline{E}) := \mathrm{rk}_{h_{\theta}^{1}}(i : E' \to \overline{E}) := h_{\theta}^{1}(\overline{E}) - h_{\theta}^{1}(\overline{E/E'}),$$

and the lower bound (7.8.16) may equivalently be stated as follows:

$$B_{\overline{E}}(x') \geq 2e^{-\mathrm{rk}_{\theta}^{1}(i:E' \to \overline{E})} - 1, \tag{7.8.20}$$

or equivalently:

$$b_{\overline{E}}(x') \leq \kappa(\mathrm{rk}_{\theta}^{1}(i : E' \to \overline{E})). \tag{7.8.21}$$

In the same vein, in Proposition 7.8.3, if we introduce the θ^1-rank of f:

$$\mathrm{rk}_{\theta}^{1}(f : E \to \overline{F}) := h_{\theta}^{1}(\overline{F}) - h_{\theta}^{1}(\overline{F/f(E)}),$$

then the inequality (7.8.10) may be reformulated as follows:

$$e^{\mathrm{rk}_{\theta}^{1}(f:E \to \overline{F})} \, (1 + B_{\overline{F}}(f_{\mathbb{R}}(x))) \leq e^{h_{\theta}^{1}(\overline{E})} \, (1 + B_{\overline{E}}(x)). \tag{7.8.22}$$

CHAPTER 8

The Theta Invariants of Hermitian Quasi-coherent Sheaves over an Arithmetic Curve

We denote by K a number field, by $\mathcal{O}_K$ its ring of integers, and by X the arithmetic curve $\operatorname{Spec} \mathcal{O}_K$.

8.0.1. In this chapter, we develop a theory of theta invariants attached to objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$.

To every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, we may attach an invariant $h^0_\theta(\overline{\mathcal{F}})$ in $[0, +\infty]$ by a straightforward extension of the definition of the invariant $h^0_\theta(\overline{\mathcal{F}})$ attached to objects in $\overline{\mathbf{Vect}}_X$ and in $\overline{\mathbf{Coh}}_X$.

Extending to $\mathbf{q}\overline{\mathbf{Coh}}_X$ the invariant h^1_θ, previously defined on $\overline{\mathbf{Coh}}_X$ with values in $\mathbb{R}_+$, requires more care and relies on the general machinery developed in Chapters 4 and 5.

Firstly, using the monotonicity and subadditivity of h^1_θ, the constructions of Chapter 4 provides two natural extensions of h^1_θ to the category $\mathbf{q}\overline{\mathbf{Coh}}_X$ with values in $[0, +\infty]$, namely the lower extension $\underline{h}^1_\theta$ and the upper extension $\overline{h}^1_\theta$, which satisfy the inequality

$$\underline{h}^1_\theta(\overline{\mathcal{F}}) \leq \overline{h}^1_\theta(\overline{\mathcal{F}})$$

for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$.

In Chapter 7, we have shown that the invariant h^1_θ is not only monotonic and subadditive on $\overline{\mathbf{Coh}}_X$, but is also strongly monotonic, as a consequence of the works of Banaszczyk and of Regev and Stephens-Davidowitz on Euclidean lattices. This implies that the full strength of the results in Chapter 5 applies to $\underline{h}^1_\theta$ and $\overline{h}^1_\theta$.

In this way, the axiomatic approach of Chapters 4 and 5, together with the "advanced theory" of the invariant h^1_θ on $\overline{\mathbf{Coh}}_X$ developed in Chapter 7, allows us to define a full subcategory θ^1_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ of θ^1-*summable* objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$, on which the invariant $\overline{h}^1_\theta$ takes finite values and satisfy suitable properties of strong monotonicity and of compatiblity with exhaustive filtrations. In particular, there is a good notion of θ^1-rank associated to morphisms in θ^1_Σ-$\mathbf{q}\overline{\mathbf{Coh}}^{\leq 1}_X$.

At this stage, it is natural to consider the objects $\overline{\mathcal{F}}$ that satisfy the condition:

$$\underline{h}^1_\theta(\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}}) < +\infty. \tag{8.0.1}$$

When this holds, we shall say that $\overline{\mathcal{F}}$ *admits a well-defined and finite invariant* $h^1_\theta(\overline{\mathcal{F}})$.

Indeed the results of Chapter 3, concerning the invariant $h^1(C, \mathcal{F})$ attached to a quasi-coherent sheaf over a smooth projective curve $\mathcal{F}$, and the properties of the elementary invariants attached to objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ in Chapter 6, indicate that, when an object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfies (8.0.1), then the "right definition" of the invariant h^1_θ attached to $\mathcal{F}$ has to be:

$$h^1_\theta(\overline{\mathcal{F}}) := \underline{h}^1_\theta(\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}}). \tag{8.0.2}$$

It is actually possible to complete our earlier results concerning the invariants $\underline{h}^1_\theta$, $\overline{h}^1_\theta$, and the category θ^1_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ by a complete description of the objects $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfying condition (8.0.1). Indeed we shall prove that (8.0.1) holds precisely when $\overline{\mathcal{F}}$ is θ^1-summable and furthermore satisfies an additional "tameness condition" concerning the Banaszcyzk measure $\beta_{\pi_*\overline{\mathcal{F}}^\vee}$ that is associated to the object $\pi_*\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ by extending to the infinite dimensional setting the construction in Chapter 7 of the Banaszczyk measure $\beta_{\overline{E}^\vee}$ associated to an object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

We shall also prove that if for some $\varepsilon \in \mathbb{R}_+^*$, the object $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\varepsilon)$ is θ^1-summable, then $\overline{\mathcal{F}}$ satisfies (8.0.1). In particular its invariant $\overline{h}^1_\theta(\overline{\mathcal{F}})$ may be defined by (8.0.2). This implies that, for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following conditions are equivalent:

(i) for every δ in $\mathbb{R}$, $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ is θ^1-summable;

(ii) for every δ in $\mathbb{R}$, $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ has a well-defined and finite invariant $h^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta))$.

The objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfying these conditions will be called *θ^1-finite.* Condition (i) is indeed the special instance when $\varphi = h^1_\theta$ of the condition of φ-finiteness defined in Subsection 5.6.2 by the equivalent conditions in Proposition 5.6.4. When $\varphi = h^1_\theta$, these conditions turn out to be equivalent to the very natural condition (ii).

An object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is θ^1-finite if and only if the object $\pi_*\overline{\mathcal{F}} := (\mathcal{F}, \|.\|)$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ is θ^1-finite. In turn this holds precisely, when for every Euclidean seminorm $\|.\|'$ on $F_{\mathbb{R}}$ such that $\|.\|'/\|.\|$ is bounded, the object $(\mathcal{F}, \|.\|')$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ has a well-defined and finite invariant $h^1_\theta(\mathcal{F}, \|.\|')$.

The θ^1-finite objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ are the arithmetic counterparts of the $\overline{h}^1$-finite quasi-coherent sheaves of countable type over a projective curve C considered in Section 3.5. They turn out to be the objects of interest in the applications of the invariant h^1_θ and of its finiteness properties to Diophantine geometry.

8.0.2. Let us describe the content of this chapter in more details.

In Section 8.1, we discuss the definition and the properties of the invariant h^0_θ on $\mathbf{q}\overline{\mathbf{Coh}}_X$. In striking contrast with the invariant h^1_θ investigated in the sequel, these are straightforward extensions of the results concerning h^0_θ on $\overline{\mathbf{Coh}}_{\overline{X}}$ discussed in Section 7.2. We also show that every objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that $h^0_\theta(\overline{\mathcal{F}})$ is finite is the direct sum of a torsion coherent $\mathcal{O}_X$-module (or equivalently of a finite $\mathcal{O}_K$-module) and of an object of the subcategory[1] $\mathbf{ind}\overline{\mathbf{Vect}}_X$ already investigated in [**Bos20b**].

In Section 8.2, we spell out the main results of Chapters 4 and 5 concerning the extensions $\underline{\varphi}$ and $\overline{\varphi}$ to $\mathbf{q}\overline{\mathbf{Coh}}_X$ of suitable invariants φ on $\overline{\mathbf{Coh}}_X$ when specialized to $\varphi = h^1_\theta$. This section should be readable with a limited knowledge of the content of Chapters 4 and 5. Their summary in Section 5.7, together with the content of Subsection 5.6.2 on φ-finiteness, should constitute a sufficient background. Some of the results of Section 8.2 already appear in dual form in [**Bos20b**, Chapters 6 and 7]. However the theory in *loc. cit.* deals with objects of $\mathbf{ind}\overline{\mathbf{Vect}}_X$ only, and was developed without recourse to the strong monotonicity of h^1_θ. For these reasons, our results in Section 8.2 are considerably more general than the ones in *loc. cit.*.[2]

Section 8.3 contains the most important technical results in this chapter. In this section, we extend to Euclidean quasi-coherent sheaves — that is, to objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ — the constructions of the Banaszczyk function $B_{\overline{E}}$ and of the Banaszczyk measure $\beta_{\overline{E}^\vee}$ developed in Section 7.7 when $\overline{E}$ is an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. It is striking that $B_{\overline{E}}$ and $\beta_{\overline{E}^\vee}$ make sense when $\overline{E}$ is replaced by an *arbitrary* Euclidean quasi-coherent sheaf $\overline{F} := (F, \|.\|)$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

The measure $\beta_{\overline{F}^\vee}$ is defined as a probability measure on the Fréchet space

$$F_{\mathbb{R}}^\vee := \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{R}) \simeq \mathrm{Hom}_{\mathbb{R}}(F_{\mathbb{R}}, \mathbb{R}).$$

[1] Recall that an objects $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})})$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ belongs to $\mathbf{ind}\overline{\mathbf{Vect}}_X$ when the countably generated $\mathcal{O}_K$-module $\mathcal{F}(X)$

[2] For instance Theorem 8.2.3 (1), even restricted to suitable objects of $\mathbf{ind}\overline{\mathbf{Vect}}_X$, is out of reach by the methods of [**Bos20b**].

We establish various criteria that ensure that the measure $\beta_{\overline{F}^\vee}$ is actually supported by the Hilbert space $F_{\mathbb{R}}^{\vee\mathrm{Hilb}}$ of continuous $\mathbb{R}$-linear forms on $(F_{\mathbb{R}}, \|.\|)$. Our proofs rely on various classical facts concerning Borel probability measures on certain infinite dimensional locally convex spaces, due notably to Bochner, Prokhorov, Sazonov, and Minlos. These are discussed with some details, in a formulation appropriate to their use in this chapter, in Appendix C.

In Section 8.4, the results concerning the measures $\beta_{\overline{F}^\vee}$ established in Section 8.3 are used to investigate when condition (8.0.1) is satisfied. Here again the results of this chapter considerably extend the related results concerning objects in $\mathbf{ind\overline{Vect}}_X$ established in a dual form in [**Bos20b**, Chapter 7].[3] The central role of the property for the measure $\beta_{\overline{F}^\vee}$ to be supported by the Hilbert space $F_{\mathbb{R}}^{\vee\mathrm{Hilb}}$ in the characterization of the objects $\overline{F}$ of $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ admitting a well-defined and finite invariant h^1_θ is an essential feature of the more efficient approach in this chapter.[4]

Section 8.5 is devoted to a discussion of the θ^1-finite objects in $\mathbf{q\overline{Coh}}_X$. Basically it gathers various facts that have been previously established, and does not establish any actual new result. However the definition and the properties of the category θ^1_f-$\mathbf{q\overline{Coh}}_X$ of θ^1-finite objects in $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ will play an important role in applications to Diophantine geometry.

The final section 8.6 discusses an intriguing side issue: the existence of objects $\overline{\mathcal{F}}$ in $\mathbf{q\overline{Coh}}_X$ such that both $h^0_\theta(\overline{\mathcal{F}})$ and $h^1_\theta(\overline{\mathcal{F}})$ are finite, and the underlying $\mathcal{O}_X$-module $\mathcal{F}$ is *not* coherent.

8.1. The Invariant h^0_θ of Hermitian Quasi-coherent $\mathcal{O}_X$-Modules

8.1.1. Definitions of $h^0_\theta(\overline{\mathcal{F}})$. Let $\overline{\mathcal{F}} = (\mathcal{F}, (||.||_x)_{x\in X(\mathbb{C})})$ be an object in the category $\mathbf{q\overline{Coh}}_X$ of Hermitian quasi-coherent $\mathcal{O}_K$-module.

We shall denote by $\mathrm{coh}(\mathcal{F})$ the set of coherent $\mathcal{O}_X$-submodules $\mathcal{C}$ of $\mathcal{F}$, or equivalently, the set of finitely generated $\mathcal{O}_K$-submodules $\mathcal{C}(X)$ of $\mathcal{F}(X)$.

For any $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$, we may consider the semi-normed Hermitian coherent sheaf $\overline{\mathcal{C}}$, defined as $\mathcal{C}$ equipped with the restrictions of the Hermitian semi-norms $(||.||_x)_{x\in X(\mathbb{C})}$. As discussed in 7.3.2 above, it admits a well-defined theta invariant:

$$h^0_\theta(\overline{\mathcal{C}}) :== \log \sum_{s\in\mathcal{C}} e^{-\pi||s||^2_{\pi_*\overline{\mathcal{C}}(X)}} \quad \in [0, +\infty]. \tag{8.1.1}$$

The function

$$\mathrm{coh}(\mathcal{F}) \longrightarrow [0, +\infty], \quad \mathcal{C} \longmapsto h^0_\theta(\overline{\mathcal{C}})$$

is clearly nondecreasing over $\mathrm{coh}(\mathcal{F})$ ordered by inclusion, and we may define $h^0_\theta(\overline{\mathcal{F}})$ as its supremum:

$$h^0_\theta(\overline{\mathcal{F}}) = \sup_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} h^0_\theta(\overline{\mathcal{C}}) \in [0, +\infty]. \tag{8.1.2}$$

The partially ordered set $(\mathrm{coh}(\mathcal{F}), \subseteq)$ is directed, and this supremum may also be written as the limit:

$$h^0_\theta(\overline{\mathcal{F}}) = \lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} h^0_\theta(\overline{\mathcal{C}}) \in [0, +\infty]. \tag{8.1.3}$$

[3] For instance the existence and the properties of the measure $\beta_{\overline{\mathcal{F}}^\vee}$ — established for a general object in $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ by relying on the properties of the Banaszczyk functions $B_{\overline{E}}$ that constitute the main result in Chapter 7 — appear (implicitly) in *loc. cit.* only when $\overline{\mathcal{F}}$ belongs to $\mathbf{ind\overline{Vect}}_{\mathbb{Z}}$ and is θ^1-summable, and its construction *a priori* depends on the choice of some θ^1-summable filtration of F by saturated finitely generated submodules. Even when $\overline{F} := (F, \|.\|)$ admits a free underlying $\mathbb{Z}$-module F, the equivalence of (i) and (iv) in Theorem 8.4.7 is established in [**Bos20b**, Chapter 7] only when the seminorm $\|.\|$ is a norm. An extension of the reasoning in [**Bos20b**, Chapter 7], where the norms $\|.\|_i$ are replaced by definite quasi-norms, would be required to achieve this with the techniques of [**Bos20b**].

[4] Our new approach allows us also to avoid the recourse to the "convexity trick" that plays a key role in the proofs of [**Bos20b**, Chapter 7] (see notably [**Bos20b**, Corollary 7.3.3 and Section 7.5].

Observe that any finite subset of $\mathcal{F}(X)$ is contained in $\mathcal{C}(X)$ for some $\mathcal{C} \in \mathrm{coh}(\mathcal{F})$. This immediately implies that $h^0_\theta(\mathcal{F})$ may be defined as the logarithm of a theta series, defined by a straightforward extension of the formula (8.1.1) defining $h^0_\theta(\overline{\overline{\mathcal{C}}})$ when $\overline{\mathcal{C}}$ is coherent. Namely, we have:

$$h^0_\theta(\overline{\mathcal{F}}) = \log \sum_{s \in \mathcal{F}(X)} e^{-\pi \|s\|^2_{\pi_* \overline{\mathcal{F}}}}. \tag{8.1.4}$$

This definition of $h^0_\theta(\overline{\mathcal{F}})$ extends the one in [**Bos20b**, 6.1], where it is introduced when $\overline{\mathcal{F}}$ is an object of $\mathbf{ind}\overline{\mathbf{Vect}}_X$, namely when the countably generated $\mathcal{O}_K$-module $\mathcal{F}(X)$ is projective and the Hermitian semi-norms $(\|.\|_x)_{x \in X(\mathbb{C})}$ are Hermitian norms.

Finally observe that, for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, we may consider its direct image $\pi_* \overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and that the following equality holds in $[0, +\infty]$:

$$h^0_\theta(\pi_* \overline{\mathcal{F}}) = h^0_\theta(\overline{\mathcal{F}}).$$

8.1.2. The objects $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that $h^0_\theta(\overline{\mathcal{F}}) < +\infty$. The additional generality brought forth by the definition of $h^0_\theta(\overline{\mathcal{F}})$ for arbitrary objects $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ turns out to be limited when compared to its definition in [**Bos20b**, 6.1], as shown by the next proposition.

If $\overline{\mathcal{F}} = (\mathcal{F}, (\|.\|_x)_{x \in X(\mathbb{C})})$ is an object in $\mathbf{q}\overline{\mathbf{Coh}}_X$, we denote by $\overline{\mathcal{F}}_{/\mathrm{tor}}$the object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ defined by the $\mathcal{O}_K$-module

$$\mathcal{F}_{/\mathrm{tor}}(X) := \mathcal{F}(X)/\mathcal{F}(X)_{\mathrm{tor}}$$

and the Hermitian norms $(\|.\|_x)_{x \in X(\mathbb{C})}$.

PROPOSITION 8.1.1. *For every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following equality holds in* $[0, +\infty]$*:*

$$h^0_\theta(\overline{\mathcal{F}}) = \log |\mathcal{F}(X)_{\mathrm{tor}}| + h^0_\theta(\overline{\mathcal{F}}_{/\mathrm{tor}}). \tag{8.1.5}$$

Moreover if $\overline{\mathcal{F}}$ satisfies the condition:

$$h^0_\theta(\overline{\mathcal{F}}) < +\infty, \tag{8.1.6}$$

then the torsion $\mathcal{O}_K$-module $\mathcal{F}(X)_{\mathrm{tor}}$ is finite, the quotient $\mathcal{O}_K$-module $\mathcal{F}_{/\mathrm{tor}}(X) = \mathcal{F}(X)/\mathcal{F}(X)_{\mathrm{tor}}$ is projective, and the Hermitian semi-norms $(\|.\|_x)_{x \in X(\mathbb{C})}$ are Hermitian norms.

In other words, when $h^0_\theta(\overline{\mathcal{F}})$ is finite, $\overline{\mathcal{F}}$ is the direct sum of a torsion coherent $\mathcal{O}_X$-module and of an object in $\mathbf{ind}\overline{\mathbf{Vect}}_X$.

PROOF. The expression (8.1.4) for $h^0_\theta(\overline{\mathcal{F}})$ implies the validity of (8.1.5). In turn (8.1.5) implies that, when (8.1.6) holds, then $\mathcal{F}(X)_{\mathrm{tor}}$ is a finite $\mathcal{O}_K$-module and $h^0_\theta(\overline{\mathcal{F}/\mathcal{F}_{\mathrm{tor}}})$ is finite.

We are thus reduced to proving that, when $\mathcal{F}(X)_{\mathrm{tor}} = 0$ and $h^0_\theta(\overline{\mathcal{F}})$ is finite, then the following holds:

(a) the seminorms $(\|.\|_x)_{x \in X(\mathbb{C})}$ are norms;
(b) the countably generated $\mathcal{O}_K$-module $\mathcal{F}(X)$ is projective.

Observe that the finiteness of $h^0_\theta(\overline{\mathcal{F}})$ implies the finiteness of $h^0_\theta(\overline{\mathcal{C}})$ for any $\mathcal{C}$ in $\mathrm{coh}(\mathcal{F})$, and therefore, by Proposition 7.3.3, (i), that for any $x \in X(\mathbb{C})$), the seminorm $\|.\|_x$ restricted to $\mathcal{C}_x$ is a norm. Since

$$\mathcal{F}_x = \bigcup_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \mathcal{C}_x,$$

this implies (a).

Finally, formula (8.1.4) shows that, when $h^0_\theta(\overline{\mathcal{F}})$ is finite, the image of $\mathcal{F}(X)$ in $(\mathcal{F}_{\mathbb{R}}, \|.\|_{\pi_* \mathcal{F}})$ has finite intersection with any bounded subset of the pre-Hilbert space $(\mathcal{F}_{\mathbb{R}}, \|.\|_{\pi_* \mathcal{F}})$, and *a fortiori* is a discrete subgroup of this pre-Hilbert space. A fortiori, the image of $\mathcal{F}(X)$ in $\mathcal{F}_{\mathbb{R}}$ endowed with its inductive topology (see 1.4.1 above) is discrete, so that $\mathcal{F}(X)$ is a projective $\mathbb{Z}$-module by Corollary 1.4.3. (This is also a consequence of [**Bos20b**, Corollary 5.8.3].) According to Proposition 1.3.2, this implies that $\mathcal{F}(X)$ is a projective $\mathcal{O}_K$-module and completes the proof of (b). $\square$

The following proposition is a simple consequence of the expression (8.1.4) for h^0_θ on $\mathbf{q}\overline{\mathbf{Coh}}_X$, and the details of its proof will be left to the reader.

PROPOSITION 8.1.2. *Let $\overline{\mathcal{F}}$ be an element of $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that*

$$h^0_\theta(\overline{\mathcal{F}}) < +\infty.$$

For every $\varepsilon \in \mathbb{R}_-$, we have:

$$h^0_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)) < +\infty.$$

Moreover the function

$$\mathbb{R}_- \longrightarrow \mathbb{R}_+, \quad \varepsilon \longmapsto h^0_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon))$$

is continuous, increasing, and satisfies:

$$\lim_{\varepsilon \to -\infty} h^0_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)) = 0.$$

Finally the decreasing continuous function

$$[1, +\infty) \longrightarrow \mathbb{R}_+, \quad t \longmapsto h^0_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-(1/2)\log t))$$

is convex.

8.1.3. Monotonicity and Subadditivity Properties of h^0_θ on $\mathbf{q}\overline{\mathbf{Coh}}_X$. The basic property of the invariant

$$h^0_\theta : \overline{\mathbf{Coh}}_{\overline{X}} \longrightarrow \mathbb{R}_+$$

stated in Theorem 7.2.3 and its strong monotonicity property $\mathbf{StMon}^0$ established in Theorems 7.5.4 and 7.5.5 easily extends to the invariant:

$$h^0_\theta : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty].$$

In particular h^0_θ satisfies the following countable additivity property, as a straightforward consequence of the expression (8.1.4) for h^0_θ on $\mathbf{q}\overline{\mathbf{Coh}}_X$:

THEOREM 8.1.3. *For every countable family $(\overline{\mathcal{F}}_i)_{i\in I}$ of objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following equality holds in $[0, +\infty]$:*

$$h^0_\theta\Big(\bigoplus_{i\in I} \overline{\mathcal{F}}_i\Big) = \sum_{i\in I} h^0_\theta(\overline{\mathcal{F}}_i). \tag{8.1.7}$$

It also satisfies the following strong monotonicity properties.

THEOREM 8.1.4. *Let $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ be a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$.*

(1) If the morphism of $\mathcal{O}_X$-module $f : \mathcal{F} \to \mathcal{G}$ is injective, then:

$$h^0_\theta(\overline{\mathcal{F}}) \leq h^0_\theta(\overline{\mathcal{G}}). \tag{8.1.8}$$

(2) For every quasi-coherent $\mathcal{O}_C$-submodule $\mathcal{G}'$ of $\mathcal{G}$, of inverse image $\mathcal{F}' := f^{-1}(\mathcal{G}')$ in $\mathcal{F}$, the following inequality holds:

$$h^0_\theta(\mathcal{F}) + h^0_\theta(\mathcal{G}') \leq h^0_\theta(\mathcal{F}') + h^0_\theta(\mathcal{G}). \tag{8.1.9}$$

Indeed, starting from the validity of these properties when f is a morphism in $\overline{\mathbf{Coh}}_{\overline{X}}^{\leq 1}$, one may extend it first to morphisms in $\overline{\mathbf{Coh}}_X^{\leq 1}$, by writing the Hermitian seminorms defining objects in $\overline{\mathbf{Coh}}_X^{\leq 1}$ as limits of decreasing sequences of Hermitian norms, and then to arbitrary morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$, by using the definitions (8.1.2) and (8.1.3) of the invariant h^0_θ on objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

The easy details are left to the reader, as well as the formulation of the analogue of Corollary 7.5.6 in the present framework. Let us only indicate that as a special instance of Theorem 8.1.4 (2), one derives the subadditivity of h^0_θ on $\mathbf{q}\overline{\mathbf{Coh}}_X$. Namely for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ and every quasi-coherent $\mathcal{O}_C$-submodule $\mathcal{F}'$, the following inequality holds:

$$h^0_\theta(\overline{\mathcal{F}}) \leq h^0_\theta(\overline{\mathcal{F}}') + h^0_\theta(\overline{\mathcal{F}/\mathcal{F}'}). \tag{8.1.10}$$

8.1.4. θ^0-finite objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$.

Definition 8.1.5. We shall say that an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is θ^0-*finite* when the condition

$$h^0_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\delta)) < +\infty \tag{8.1.11}$$

is satisfied for every δ in $\mathbb{R}$, or equivalently by Proposition 8.1.2, when (8.1.11) holds for every δ in a subset D of $\mathbb{R}$ such that $\sup D = +\infty$.

The terminology "θ^0-finite" reflects that, as shown by the expression (8.1.4) for h^0_θ, an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is θ^0-finite precisely when the theta series:

$$\theta_{\pi_*\overline{\mathcal{F}}}(t) := \sum_{s \in \mathcal{F}(X)} e^{-\pi t ||s||^2_{\pi_*\overline{\mathcal{F}}}}$$

has a finite sum for every $t \in \mathbb{R}^*_+$.

An object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is θ^0-finite precisely when its directs image $\pi_*\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ is θ^0-finite. Proposition 8.1.2 shows that this holds if and only if it is the direct sum of a finite $\mathcal{O}_K$-module and of a θ^0-finite object of $\mathbf{ind}\overline{\mathbf{Vect}}_X$.

Moreover the estimates (8.1.8) and (8.1.10) show that the full subcategory $\theta^0_f\text{-}\mathbf{q}\overline{\mathbf{Coh}}^{\leq 1}_X$ of $\mathbf{q}\overline{\mathbf{Coh}}^{\leq 1}_X$ defined by the θ^0-finite objects is stable under subobjects and under extensions, in a sense we leave the reader to formulate precisely.

8.2. The Invariants $\underline{h}^1_\theta$ and $\overline{h}^1_\theta$, and the θ^1-Summable Quasi-coherent $\mathcal{O}_X$-Modules

In Chapter 7, we have shown that the invariant:

$$h^1_\theta : \overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

satisfies the strong monotonicity condition $\mathbf{StMon^1}$ (and notably conditions $\mathbf{Mon}^1$ adn $\mathbf{SubAdd}$), the downward continuity condition $\mathbf{Cont}^+$, and the additivity condition $\mathbf{Add}_\oplus$ introduced in Chapters 4 and 5. Moreover the invariant h^1_θ is small on Hermitian coherent sheaves generated by small sections, in the sense of Definition 4.2.11, and in particular, it satisfies condition $\mathbf{VT}$.

We may therefore apply to $\varphi := h^1_\theta$ the results of Chapters 4 and 5 as they are summarized in Section 5.7. In the Subsections 8.2.1–8.2.4, we recapitulate these results in a form that makes them accessible without familiarity with the content of these chapters; we refer the reader to Section 5.7 for precise references to their proofs in Chapters 4 and 5. Instead of h^1_θ-*summable*, we will use the terminology θ^1-*summable.*

In Subsection 8.2.5, we discuss the compatibility properties of the invariant h^1_θ and of the previous constructions with the operation of tensoring by a fixed object in $\overline{\mathbf{Coh}}_X$. This discussion would actually be valid in the more general context of Chapter 4, and has not been included there to limit formal developments.

Finally in Subsection 8.2.6, we spell out the compatibility properties presented in Section 4.6 of the invariants attached to $\varphi := h^1_\theta$ with the direct image functor:

$$\pi_* : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow \mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}.$$

8.2.1. The invariant $\overline{h}^1_\theta$. To the invariant h^1_θ on $\overline{\mathbf{Coh}}_X$, we may associate its upper extension,

$$\overline{h}^1_\theta : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

defined by:

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) := \liminf_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \varphi(\overline{\mathcal{C}})$$

for every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, where the inferior limit is taken over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$ of coherent $\mathcal{O}_X$-submodules of $\mathcal{F}$, or equivalently of finitely generated $\mathcal{O}_K$-submodules of $\mathcal{F}(X)$.

The invariant $\overline{h}^1_\theta$ extends h^1_θ — namely, for every $\overline{\mathcal{C}}$ in $\overline{\mathbf{Coh}}_X$, we have:

$$\overline{h}^1_\theta(\overline{\mathcal{C}}) = h^1_\theta(\overline{\mathcal{C}})$$

Moreover $\overline{h}^1_\theta$ is monotonic and subadditive on $\mathbf{q}\overline{\mathbf{Coh}}_X$:

PROPOSITION 8.2.1. *For every morphism $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ such that the morphism of K-vector spaces $f_K : \mathcal{F}_K \to \mathcal{G}_K$ is surjective, we have:*

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) \geq \overline{h}^1_\theta(\overline{\mathcal{G}}).$$

Moreover, for every admissible short exact sequence in $\overline{\mathbf{Coh}}_X$:

$$0 \longrightarrow \overline{\mathcal{E}} \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}} \longrightarrow 0,$$

the following inequality holds:

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) \leq \overline{h}^1_\theta(\overline{\mathcal{E}}) + \overline{h}^1_\theta(\overline{\mathcal{G}}).$$

Moreover $\overline{h}^1_\theta$ does not see the antiprojective part of Hermitian quasi-coherent sheaves. Namely, we have:

PROPOSITION 8.2.2. *For every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following equality holds:*

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}}^{\vee\vee}).$$

8.2.2. The category θ^1_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ of θ^1-summable Hermitian quasi-coherent sheaves. To any object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, we may also associate:

$$\mathrm{ev}\overline{h}^1_\theta(\overline{\mathcal{F}}) := \lim_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \overline{h}^1_\theta(\overline{\mathcal{F}/\mathcal{C}}) = \inf_{\mathcal{C} \in \mathrm{coh}(\mathcal{F})} \overline{h}^1_\theta(\overline{\mathcal{F}/\mathcal{C}}) \quad \in [0, +\infty],$$

where the limit and the infimum are taken over the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$.

If moreover $\mathcal{F}_\bullet := (\mathcal{F}_i)_{i \in \mathbb{N}}$ is a filtration of the $\mathcal{O}_K$-module $\mathcal{F}$ underlying $\overline{\mathcal{F}}$, we let:

$$\Sigma_{\overline{h}^1_\theta}(\overline{\mathcal{F}}, \mathcal{F}_\bullet) := \sum_{i=0}^{+\infty} \overline{h}^1_\theta(\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}) \quad (\in [0, +\infty]),$$

where by convention $\mathcal{F}_{-1} = 0$. In particular if $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i \in \mathbb{N}}$ is a filtration of $\mathcal{F}$ by submodules in $\mathrm{coh}(\mathcal{F})$, we have:

$$\Sigma_{\overline{h}^1_\theta}(\overline{\mathcal{F}}, \mathcal{C}_\bullet) = \Sigma_{h^1_\theta}(\overline{\mathcal{F}}, \mathcal{C}_\bullet) := \sum_{i=0}^{+\infty} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}).$$

DEFINITION AND THEOREM 8.2.1. *An object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is called* θ^1-summable *when there exists an exhaustive filtration $\mathcal{C}_\bullet$ of $\mathcal{F}$ by submodules in* $\mathrm{coh}(\mathcal{F})$ *such that*

$$\Sigma_{h^1_\theta}(\overline{\mathcal{F}}, \mathcal{C}_\bullet) < +\infty.$$

When this holds, we have:

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) = \lim_{k \to +\infty} h^1_\theta(\overline{\mathcal{C}}_k) \in \mathbb{R}_+. \tag{8.2.1}$$

Moreover, an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is θ^1-summable if and only if

$$\mathrm{ev}\overline{h}^1_\theta(\overline{\mathcal{F}}) = 0.$$

The objects of $\overline{\mathbf{Coh}}_X$ are θ^1-summable. Moreover the φ-summable objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ satisfy the following permanence properties:

THEOREM 8.2.3. (1) *Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$, and let $\mathcal{F}_\bullet := (\mathcal{F}_i)_{i\in\mathbb{N}}$ be an exhaustive filtration of the $\mathcal{O}_K$-module $\mathcal{F}$ underlying $\overline{\mathcal{F}}$. If the subquotients $\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}$ are θ^1-summable and if*

$$\Sigma_{\overline{h}^1_\theta}(\overline{\mathcal{F}}, \mathcal{F}_\bullet) < +\infty,$$

then $\overline{\mathcal{F}}$ is θ^1-summable and

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) = \lim_{i\to+\infty} \overline{h}^1_\theta(\overline{\mathcal{F}}_i)$$

(2) *Let $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ be a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$. If $\overline{\mathcal{F}}$ is θ^1-summable and if the K-linear map $f_K : \mathcal{F}_K \to \mathcal{G}_K$ is surjective, then $\overline{\mathcal{G}}$ is θ^1-summable.*

If we denote by θ^1_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ the subcategory of θ^1-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$, then the invariant

$$\overline{h}^1_\theta : \theta^1_\Sigma\text{-}\mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow \mathbb{R}_+$$

satisfies the conditions $\mathbf{Cont}^+$ (*cf.* Proposition 4.5.16) and a version of $\mathbf{StMon}^1$ (see Proposition 5.5.8 and Corollaries 5.5.9 to 5.5.12).

8.2.3. The invariant $\underline{h}^1_\theta$. To the invariant h^1_θ is also attached its lower extension

$$\underline{h}^1_\theta : \mathbf{q}\overline{\mathbf{Coh}}_X \longrightarrow [0, +\infty]$$

defined by:

$$\underline{h}^1_\theta(\overline{\mathcal{F}}) := \liminf_{\mathcal{F}'\in\mathrm{coft}(\mathcal{F})} h^1_\theta(\overline{\mathcal{F}/\mathcal{F}'}) = \sup_{\mathcal{F}'\in\mathrm{coft}(\mathcal{F})} h^1_\theta(\overline{\mathcal{F}/\mathcal{F}'}),$$

where the inferior limit is taken over the directed set $(\mathrm{coft}(\mathcal{F}), \supseteq)$, defined by the set $\mathrm{coft}(\mathcal{F})$ of $\mathcal{O}_K$-submodules $\mathcal{G}$ of $\mathcal{F}$ such that the quotient $\mathcal{O}_X$-module $\mathcal{F}/\mathcal{G}$ is coherent.

The lower extension $\underline{h}^1_\theta$ also extends h^1_θ, and for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, it satisfies:

$$\underline{h}^1_\theta(\overline{\mathcal{F}}) \leq \overline{h}^1_\theta(\overline{\mathcal{F}})$$

Moreover any object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that:

$$\underline{h}^1_\theta(\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}}) < +\infty$$

is θ^1-summable.

The invariant $\underline{\varphi}$ satisfies the conditions $\mathbf{Mon}^1$, $\mathbf{SubAdd}$ and $\mathbf{NSAp}$ on $\mathbf{q}\overline{\mathbf{Coh}}_X$ (*cf.* Propositions 4.3.6, 4.3.10, and 4.3.12). Moreover its restriction to the subcategory of $\mathbf{q}\overline{\mathbf{Coh}}_X$ where it takes finite values satisfies the conditions $\mathbf{Cont}^+$ and $\mathbf{StMon}^1$ (*cf.* Propositions 5.4.9 and 5.4.8).

8.2.4. Countable additivity. The invariant $\overline{h}^1_\theta$ (resp. $\underline{h}^1_\theta$) is countably additive on θ^1_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$ (resp. on $\mathbf{q}\overline{\mathbf{Coh}}_X$):

PROPOSITION 8.2.4. *For any countable family $(\overline{\mathcal{F}}_i)_{i\in I}$ of φ-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ such that*

$$\sum_{i\in I} \overline{h}^1_\theta(\overline{\mathcal{F}}_i) < +\infty,$$

the direct sum $\overline{\mathcal{F}} := \bigoplus_{i\in I} \overline{\mathcal{F}}_i$ is θ^1-summable and satisfies:

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) = \sum_{i\in I} \overline{h}^1_\theta(\overline{\mathcal{F}}_i).$$

Moreover, for any countable family $(\overline{\mathcal{F}}_i)_{i\in I}$ of objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ of direct sum $\mathcal{F}$ as above, we have:

$$\underline{h}^1_\theta(\mathcal{F}) = \sum_{i\in I} \underline{h}^1_\theta(\overline{\mathcal{F}}_i).$$

8.2.5. Tensor product with objects in $\overline{\mathbf{Coh}}_X$. The following easy proposition turns out to be useful in applications, and would actually be valid in the general context of Chapter 4.

PROPOSITION 8.2.5. *Let $\overline{\mathcal{C}}$ be an object of $\overline{\mathbf{Coh}}_X$.*

(1) Their exists $N \in \mathbb{N}$, $\eta \in \mathbb{R}$, and a morphism

$$p : \overline{\mathcal{O}}_X(-\eta)^{\oplus N} \longrightarrow \overline{\mathcal{C}}$$

in $\overline{\mathbf{Coh}}_X^{\leq 1}$ such the underlying morphism of $\mathcal{O}_X$-modules $p : \mathcal{O}_X^{\oplus N} \to \mathcal{C}$ is surjective.

(2) For N and η as in (1), and for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following estimates hold:

$$\underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{C}}) \leq N\, \underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)), \tag{8.2.2}$$

and:

$$\overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{C}}) \leq N\, \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)). \tag{8.2.3}$$

(3) If moreover $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)$ is θ^1-summable, then $\overline{\mathcal{F}} \otimes \overline{\mathcal{C}}$ is θ^1-summable.

PROOF. Let us choose a finite family $(s_i)_{1\leq i\leq N}$ of generators of the $\mathcal{O}_K$-module $\mathcal{F}(X)$. Then, if η is a large enough positive real number, condition (1) is satisfied by the morphism

$$p := (s_1, \ldots, s_N) : (f_i)_{1\leq i\leq N} \longmapsto \sum_{1\leq i\leq N} f_i s_i.$$

If N, η, and p satisfy (1), then for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, the map:

$$p \otimes \mathrm{Id}_{\mathcal{F}} : [\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)]^{\oplus N} \simeq \overline{\mathcal{F}} \otimes \overline{\mathcal{O}}_X(-\eta)^{\oplus N} \longrightarrow \overline{\mathcal{F}} \otimes \overline{\mathcal{C}}$$

is a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ such that the underlying morphism of $\mathcal{O}_X$-modules is surjective. Moreover, according to Proposition 8.2.4, we have:

$$\underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)^{\oplus N}) = N\, \underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)) \quad \text{and} \quad \underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)^{\oplus N}) = N\, \underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)).$$

The estimates (8.2.2) and (8.2.3) now follow from the monotonicity property of $\underline{h}^1_\theta$ and $\overline{h}^1_\theta$; see 8.2.3 and Proposition 8.2.1.

When $\overline{\mathcal{F}}(-\eta)$ is θ^1-summable, we may choose an exhaustive filtration $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by submodules in $\mathrm{coh}(\mathcal{F})$ such that:

$$\Sigma_{h^1_\theta}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta), \mathcal{C}_\bullet) := \sum_{i=0}^{+\infty} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{O}}(-\eta)) < +\infty.$$

If $\mathcal{C}'_i$ denotes the image of $\mathcal{C}_i \otimes \mathcal{C}$ in $\mathcal{F} \otimes \mathcal{C}$, then the subquotient $\mathcal{C}'_i/\mathcal{C}'_{i-1}$ may be identified to the quotient of $\mathcal{C}_i/\mathcal{C}_{i-1} \otimes \mathcal{C}'$ by a torsion module, and therefore we have:

$$h^1_\theta(\overline{\mathcal{C}'_i/\mathcal{C}'_{i-1}}) = h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{C}}).$$

Moreover the inequality (8.2.3), applied to $\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}$ instead of $\overline{\mathcal{F}}$, establishes the following estimate:

$$h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{C}}) \leq N\, h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{O}}(-\eta)).$$

This implies that the exhaustive filtration $\mathcal{C}'_\bullet := (\mathcal{C}'_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F} \otimes \mathcal{C})$ satisfies the summability condition:

$$\Sigma_{h^1_\theta}(\overline{\mathcal{F}} \otimes \overline{\mathcal{C}}, \mathcal{C}'_\bullet) := \sum_{i=0}^{+\infty} h^1_\theta(\overline{\mathcal{C}'_i/\mathcal{C}'_{i-1}}) < +\infty,$$

and completes the proof of the θ^1-summability of $\overline{\mathcal{F}} \otimes \overline{\mathcal{C}}$. □

8.2.6. Compatibility with direct images. The following proposition follows from Propositions 4.6.4 and 4.6.6 applied to $\varphi = h^1_\theta$.

PROPOSITION 8.2.6. *For every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following relations hold:*

$$\underline{h}^1_\theta(\overline{\mathcal{F}}) = \underline{h}^1_\theta(\pi_*\overline{\mathcal{F}}) \leq \overline{h}^1_\theta(\pi_*\overline{\mathcal{F}}) \leq \overline{h}^1_\theta(\overline{\mathcal{F}}). \tag{8.2.4}$$

If moreover $\overline{\mathcal{F}}$ is an object of θ^1_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_X$, then $\pi_*\overline{\mathcal{F}}$ is an object of θ^1_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and the following equality holds:

$$\overline{h}^1_\theta(\pi_*\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}}). \tag{8.2.5}$$

Let us emphasize that the equality (8.2.5) is not expected to hold for an arbitrary Hermitian quasi-coherent $\mathcal{O}_X$-module $\overline{\mathcal{F}}$. We refer to [**Bos20b**, Section 9.3.3] for a related counter-example in the geometric setting of quasi-coherent sheaves over projective curves considered in Chapter 3.

8.3. The Function $B_{\overline{F}}$ and the Measure $\beta_{\overline{F}^\vee}$ Attached to an Object $\overline{F}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$

In this section, we consider a Euclidean quasi-coherent sheaf $\overline{F} := (F, \|.\|)$. As before, we denote by $F_{/\mathrm{tor}}$ the quotient F/F_{tor}, and we identify it with the image of F in F_R.

We also consider:

$$F^\vee_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(F_{\mathbb{R}}, \mathbb{R}),$$

and we equip it with the topology of pointwise convergence (over F, or equivalently over $F_{\mathbb{R}}$). This topology makes $F^\vee_{\mathbb{R}}$ a Fréchet space. Indeed if $(f_a)_{a\in A}$ is a (countable) basis of the $\mathbb{R}$-vector space $F_{\mathbb{R}}$, the map

$$F^\vee_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^A, \quad \xi \longmapsto (\xi(f_a))_{a\in A}$$

is an isomorphism of topological vector spaces between $F^\vee_{\mathbb{R}}$ and $\mathbb{R}^A$ equipped with the product topology of the usual topology on each factor $\mathbb{R}$.

The $\mathbb{Z}$-module

$$F^\vee := \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(F_{/\mathrm{tor}}, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(F_{/\mathrm{ap}}, \mathbb{Z})$$

defines a closed subgroup of the Fréchet space $F^\vee_{\mathbb{R}}$. The $\mathbb{Z}$-module $F_{/\mathrm{ap}}$ is a free $\mathbb{Z}$-module with a countable basis $(g_b)_{b\in B}$, and the map

$$F^\vee \simeq \mathrm{Hom}_{\mathbb{Z}}(F_{/\mathrm{ap}}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}^B, \quad \xi \longmapsto (\xi(g_b))_{b\in B}$$

is an isomorphism of topological groups when $\mathbb{Z}^B$ is endowed with the product topology of the discrete topology on each factor $\mathbb{Z}$.

On $F^\vee_{\mathbb{R}}$ is defined the definite quasi-norm:

$$\|.\|_{\overline{F}^\vee} : F^\vee_{\mathbb{R}} \longrightarrow [0, +\infty]$$

defined by:

$$\|\xi\|_{\overline{F}^\vee} := \sup\left\{|\xi(x)|; x \in F_{\mathbb{R}} \text{ and } \|x\|_{\overline{F}} \leq 1\right\}.$$

As a function on the Fréchet space $F^\vee_{\mathbb{R}}$, it is lower semicontinuous. Moreover the $\mathbb{R}$-vector subspace

$$F^{\vee\mathrm{Hilb}}_{\mathbb{R}} := \left\{\xi \in F^\vee_{\mathbb{R}} \mid \|\xi\|_{\overline{F}^\vee} < +\infty\right\}$$

of $F^\vee_{\mathbb{R}}$, when equipped with $\|.\|_{\overline{F}^\vee}$, is a Hilbert space.

Observe that $F^{\vee\mathrm{Hilb}}_{\mathbb{R}}$ is a Borel subset of $F^\vee_{\mathbb{R}}$. It is actually a countable union of compact subsets since, for every $R \in \mathbb{R}_+$, the closed ball in $F^{\vee\mathrm{Hilb}}_{\mathbb{R}}$,

$$\overline{B}_{F^{\vee\mathrm{Hilb}}_{\mathbb{R}}}(R) := \left\{\xi \in F^\vee_{\mathbb{R}} \mid \|\xi\|_{\overline{F}^\vee} \leq R\right\}$$

is a compact subset of $F^\vee_{\mathbb{R}}$.

8.3.1. Constructions and basic properties of $B_{\overline{F}}$ and $\beta_{\overline{F}^\vee}$.

8.3.1.1. *The function $B_{\overline{F}}$.*

DEFINITION 8.3.1. For every Euclidean quasi-coherent sheaf $\overline{F} := (F, \|.\|)$ and every $x \in F_{\mathbb{R}}$, we define:

$$B_{\overline{F}}(x) := \sup \{B_{\overline{C}}(x); C \in \mathrm{coh}(F) \text{ and } x \in C_{\mathbb{R}}\}. \tag{8.3.1}$$

Observe that, for every $x \in F_{\mathbb{R}}$, the subset

$$\mathrm{coh}(F)_x := \{C \in \mathrm{coh}(F) \mid x \in C_{\mathbb{R}}\}$$

of $\mathrm{coh}(F)$ is cofinal in the directed set $(\mathrm{coh}(F), \subseteq)$. Moreover, according to the monotonicity property of the Banaszczyk functions attached to objects in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ stated in Proposition 7.7.2), for every two elements C_1 and C_2 of $\mathrm{coh}(F)_x$, the following implication holds:

$$C_1 \subseteq C_2 \Longrightarrow B_{\overline{C}_1}(x) \leq B_{\overline{C}_2}(x).$$

Consequently the right hand side of ((8.3.1)) is actually a limit over the directed set $(\mathrm{coh}(F), \subseteq)$:

$$B_{\overline{F}}(x) := \lim_{C \in \mathrm{coh}(F)} B_{\overline{C}}(x). \tag{8.3.2}$$

In particular, for every increasing filtration $(C_n)_{n\in\mathbb{N}}$ of F by finitely generated $\mathbb{Z}$-submodules and for every $x \in F_{\mathbb{R}}$, the value $B_{\overline{C}_n}(x)$ is defined when the integer is large enough, the sequence $(B_{\overline{C}_n}(x))$ is increasing and satisfies:

$$B_{\overline{F}}(x) := \lim_{n\to+\infty} B_{\overline{C}_n}(x). \tag{8.3.3}$$

Clearly the function $B_{\overline{F}}$ takes values in $(0, 1]$, and its construction "does not see torsion":

$$B_{\overline{F}} = B_{\overline{F}_{/\mathrm{tor}}}.$$

PROPOSITION 8.3.2. *For every object $\overline{F} := (F, \|.\|)$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the function $B_{\overline{F}}$ is a function of positive type on $F_{\mathbb{R}}$, is continuous on $(F_{\mathbb{R}}, \|.\|)$, and satisfies the following inequalities:*

$$e^{-\pi\|x\|^2} \leq B_{\overline{F}}(x) \leq 1 \quad \textit{for every } x \in F_{\mathbb{R}}. \tag{8.3.4}$$

Moreover, for every morphism $f : \overline{F} \to \overline{F}'$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ and every $x \in F_{\mathbb{R}}$, the following inequality holds:

$$B_{\overline{F}'}(f_{\mathbb{R}}(x)) \geq B_{\overline{F}}(x). \tag{8.3.5}$$

As for the Banaszczyk function associated to objects of $\overline{\mathbf{Vect}}_{\mathbb{Z}}$ and $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, we shall define the function $b_{\overline{F}}$ on $F_{\mathbb{R}}$, or on $F_{\mathbb{R}}/F_{/\mathrm{tor}}$, by the relation:

$$B_{\overline{F}}(x) = e^{-\pi b_{\overline{F}}(x)}.$$

It satisfies the estimates:

$$0 \leq b_{\overline{F}}(x) \leq \|x\|^2.$$

PROOF OF PROPOSITION 8.3.2. Being of positive type is clearly preserved by pointwise convergence. Since the functions $B_{\overline{C}}$, for C in $\mathrm{coh}(F)$, are of positive type as shown in Corollary 7.7.4, this establishes the first assertion. The estimates (8.3.4) also directly follow from the similar estimates (7.7.5) satisfied by the Banaszczyk functions of objects of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. Clearly (8.3.4) implies the continuity at the point 0 of the function $B_{\mathbb{E}}$ with respect to the seminorm $\|.\|$. Since $B_{\overline{E}}$ is a function of positive type, this implies its continuity on $(F_{\mathbb{R}}, \|.\|)$.

The inequality (8.3.5) follows its special case when $\overline{F}$ and $\overline{F}'$ are objects in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, stated in Proposition 7.7.2, and from the expression of $B_{\overline{F}}$ and $B_{\overline{F}'}$ as suprema of the functions $B_{\overline{C}}$ for C in $\mathrm{coh}(F)$ or $\mathrm{coh}(F')$, as in Definition 8.3.1. $\square$

8.3.1.2. *The measure* $\beta_{\overline{F}^\vee}$. With the notation of Proposition 8.3.2, the continuity of $B_{\overline{F}}$ on $(F_{\mathbb{R}}, \|.\|)$ trivially implies its continuity on $E_{\mathbb{R}}$ equipped with the inductive topology. Moreover, $B_{\overline{F}}(0) = 1$. According to the theorem of Bochner, the function $B_{\overline{F}}$ is the Fourier transform of a unique Borel probability measure on the Fréchet space $F_{\mathbb{R}}^\vee$.[5]

DEFINITION 8.3.3. For every Euclidean quasi-coherent sheaf $\overline{F}$, we denote by $\beta_{\overline{F}^\vee}$ the unique Borel probability measure on $F_{\mathbb{R}}^\vee$ such that:

$$B_{\overline{F}} = \mathcal{F}_{F_{\mathbb{R}}}^{-1} \beta_{\overline{F}^\vee}. \tag{8.3.6}$$

The relation (8.3.6) precisely means that, for every $x \in F_{\mathbb{R}}$, the following equality holds:

$$B_{\overline{F}}(x) = \int_{F_{\mathbb{R}}^\vee} e^{2\pi i \langle \xi, x\rangle}\, d\beta_{\overline{F}^\vee}(\xi). \tag{8.3.7}$$

Since the function $B_{\overline{F}}$ is real valued, the measure $\beta_{\overline{F}^\vee}$ on the Fréchet space $F_{\mathbb{R}}^\vee$ is symmetric. Namely, it satisfies:

$$[-1]_* \beta_{\overline{F}^\vee} = \beta_{\overline{F}^\vee}.$$

Observe also that the following generalized form of the identities (7.8.1) and (7.8.14) also holds, for every $x \in F_{\mathbb{R}}$:

$$1 + B_{\overline{E}}(x) = 2 \int_{E_{\mathbb{R}}} \cos^2(\pi \langle \xi, x\rangle)\, d\beta_{\overline{E}^\vee}(\xi) \tag{8.3.8}$$

8.3.2. The measure $\beta_{\overline{F}^\vee}$ as a limit.

8.3.2.1. *Positive Borel measures on* $F_{\mathbb{R}}^\vee$. Let us consider an exhaustive filtration $(C_n)_{n\in\mathbb{N}}$ of F by finitely generated $\mathbb{Z}$-submodules. For any two integers $0 \leq i \leq j$, let us denote by

$$\iota_{ji} : C_i \hookrightarrow C_j$$

the inclusion morphism and by

$$p_{ij} := \iota_{ji,\mathbb{R}}^\vee : C_{j,\mathbb{R}}^\vee \longrightarrow C_{i,\mathbb{R}}^\vee$$

the transpose of the associated injective $\mathbb{R}$-linear map

$$\iota_{ji,\mathbb{R}} : C_{i,\mathbb{R}} \hookrightarrow C_{j,\mathbb{R}}.$$

The maps $(p_{ji})_{0\leq i\leq j}$ define a projective systems of surjective $\mathbb{R}$-linear maps between the finite dimensional spaces $(C_{i,\mathbb{R}}^\vee)_{i\in\mathbb{N}}$.

For every $i \in \mathbb{N}$, the transpose of the inclusion map

$$C_{i,\mathbb{R}} \hookrightarrow F_{\mathbb{R}}$$

defines a surjective $\mathbb{R}$-linear map:

$$q_i : F_{\mathbb{R}}^\vee \longrightarrow C_{i,\mathbb{R}}^\vee.$$

The maps $(q_i)_{i\in\mathbb{N}}$ define an isomorphism

$$F_{\mathbb{R}}^\vee \xrightarrow{\sim} \lim_{i} C_{i,\mathbb{R}}^\vee$$

of locally convex spaces between the Fréchet space $F_{\mathbb{R}}^\vee$ and the projective limit of the $C_{i,\mathbb{R}}^\vee$, equipped with their usual (Hausdorff locally convex) topology.

We may also consider the maps between spaces of positive bounded measures induced by the maps p_{ji}:

$$p_{ij*} : \mathcal{M}_+^b(C_{j,\mathbb{R}}^\vee) \longrightarrow \mathcal{M}_+^b(C_{i,\mathbb{R}}^\vee). \tag{8.3.9}$$

[5] We refer the reader to Appendix C, section C.1, for an exposition of the theorems of Bochner and P. Lévy concerning probability measures on Fréchet spaces like $F_{\mathbb{R}}^\vee$ and their Fourier transforms used in this section.

and by the maps (q_i):

$$q_{i*} : \mathcal{M}^b_+(F^\vee_{\mathbb{R}}) \longrightarrow \mathcal{M}^b_+(C^\vee_{i,\mathbb{R}}).$$

They induce a map from $\mathcal{M}^b_+(F^\vee_{\mathbb{R}})$ to the limit (in the category of sets) of the projective system (8.3.9), which according to a classical theorem of Kolmogorov, is a bijection:

$$q_* : \mathcal{M}^b_+(F^\vee_{\mathbb{R}}) \xrightarrow{\sim} \lim_i \mathcal{M}^b_+(C^\vee_{i,\mathbb{R}}), \quad \mu \longmapsto (q_{i*}\mu)_{i\in\mathbb{N}}. \tag{8.3.10}$$

By restriction the bijection (8.3.10) defines a bijection:

$$q_* : \mathcal{M}^b_+(F^\vee) \xrightarrow{\sim} \lim_i \mathcal{M}^b_+(C^\vee_i). \tag{8.3.11}$$

Moreover it is compatible with the ordering of positive bounded measures. Namely, for any two measures μ and μ' in $\mathcal{M}^b_+(F^\vee_{\mathbb{R}})$, the following equivalence holds:

$$\mu' \geq \mu \iff \text{for every } i \in \mathbb{N},\ q_{i*}\mu' \geq q_{i*}\mu. \tag{8.3.12}$$

8.3.2.2. *Constructing $\beta_{\overline{F}^\vee}$ from the $\beta_{\overline{C}^\vee_n}$.* The following proposition exhibits a limiting procedure that allows one to construct the measure $\beta_{\overline{F}^\vee}$ on the Fréchet space $F^\vee_{\mathbb{R}}$ directly from the Banaszczyk measures $\beta_{\overline{C}^\vee_n}$ on the finite dimensional $\mathbb{R}$-vector spaces $\overline{C}^\vee_{n,\mathbb{R}}$.

PROPOSITION 8.3.4. *For every $i \in \mathbb{N}$, the sequence $(p_{ij*}\beta_{\overline{C}^\vee_j})_{j\in\mathbb{N}_{\geq i}}$ converges to some limit β_i in $\mathcal{M}^1_+(C^\vee_{i,\mathbb{R}})$ equipped with the topology of narrow convergence.*

The sequence of measures $(\beta_i)_{i\in\mathbb{N}}$ in $\prod_{i\in\mathbb{N}} \mathcal{M}^1_+(C^\vee_{i,\mathbb{R}})$ satisfies the relations:

$$p_{ij*}\beta_j = \beta_i \quad \textit{for every } 0 \leq i \leq j, \tag{8.3.13}$$

and therefore defines an element of $\lim_i \mathcal{M}^1_+(C^\vee_{i,\mathbb{R}})$. Finally,

$$q_*(\beta_{\overline{F}^\vee}) = (\beta_i)_{i\in\mathbb{N}}.$$

PROOF. For every $C \in \mathrm{coh}(F)$, we have:

$$B_{\overline{C}} = \mathcal{F}^{-1}_{C_{\mathbb{R}}}\beta_{\overline{C}^\vee}.$$

Consequently, for every $i \in \mathbb{N}$ and every $j \in \mathbb{N}_{\geq i}$, we have:

$$\mathcal{F}^{-1}(p_{ij*}\beta_{\overline{C}^\vee_j}) = p^{\vee *}_{ij}(\mathcal{F}^{-1}\beta_{\overline{C}^\vee_j}) = \iota^*_{ji,\mathbb{R}} B_{\overline{C}_j} = B_{\overline{C}_j|C_{i,\mathbb{R}}}.$$

Therefore the sequence of inverse Fourier transforms $(\mathcal{F}^{-1}(p_{ij*}\beta_{\overline{C}^\vee_j}))_{j\in\mathbb{N}_{\geq i}}$ converges pointwise to $B_{\overline{F}|C_{i,\mathbb{R}}}$. Moreover, according to the definition of $\beta_{\overline{F}^\vee}$, we have:

$$B_{\overline{F}|C_{i,\mathbb{R}}} = (\mathcal{F}^{-1}_{F_{\mathbb{R}}}\beta_{\mathcal{F}^\vee})_{|C_{i,\mathbb{R}}} = \mathcal{F}^{-1}_{C_{i,\mathbb{R}}} = \mathcal{F}^{-1}_{C_{i,\mathbb{R}}}(q_{i*}\beta_{\overline{F}^\vee}).$$

According to P. Lévy's continuity theorem, this establishes that, in the topology of narrow convergence, the sequence of measures $(p_{ij*}\beta_{\overline{C}^\vee_j})_{j\in\mathbb{N}_{\geq i}}$ converges to $\beta_i := q_{i*}\beta_{\overline{F}^\vee}$. □

Using the bijections (8.3.10) and (8.3.11), Proposition 8.3.4 immediately implies:

COROLLARY 8.3.5. *The measure $\beta_{\overline{F}^\vee}$ on the Fréchet space $F^\vee_{\mathbb{R}}$ is supported by the closed subgroup $F^\vee := \mathrm{Hom}_{\mathbb{Z}}(F, \mathbb{Z})$ of $F^\vee_{\mathbb{R}}$.*

Proposition 8.3.4 admits the following alternative formulation, which we leave as an exercise for the interested reader.

COROLLARY 8.3.6. *If $(\tilde{\beta}_i)_{i\in\mathbb{N}}$ is a sequence of Borel probability measures on $F^\vee_{\mathbb{R}}$ such that*

$$q_{i*}\tilde{\beta}_i = \beta_{\overline{C}^\vee_i} \quad \textit{for every } i \in \mathbb{N}, \tag{8.3.14}$$

then $(\tilde{\beta}_i)_{i\in\mathbb{N}}$ converges to $\beta_{\overline{F}^\vee}$ in the topology of narrow convergence.

Observe that, as the measures $\beta_{\overline{C}_i^\vee}$ are supported by the countable sets $C_i^\vee$, the existence of measures $\tilde{\beta}_i$ satisfying (8.3.14) is straightforward.

8.3.3. Inequalities between Banaszczyk measures and inverse monotonicity of Banaszczyk functions attached to Euclidean quasi-coherent sheaves. At this stage, the monotonicity properties concerning the functions $B_{\overline{E}}$ and the measures $\beta_{\overline{E}^\vee}$ attached to objects of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ established in Section 7.8 may be extended to objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

Namely we shall prove the following generalizations of Proposition 7.8.3 and Corollary 7.8.4, reformulated in terms θ^1-ranks as in Subsection 7.8.4, and of Corollary 7.8.5.

PROPOSITION 8.3.7. *Let $f : \overline{E} \to \overline{F}$ be a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$. The following inequality between positive measures on $E_{\mathbb{R}}^\vee$ is satisfied:*

$$e^{\overline{\mathrm{rk}}_\theta^1(f:E\to\overline{F})}\, f_{\mathbb{R}*}^\vee \beta_{\overline{F}^\vee} \leq e^{\overline{h}_\theta^1(\overline{E})}\, \beta_{\overline{E}^\vee}. \tag{8.3.15}$$

Moreover for every $x \in E_{\mathbb{R}}$, the following inequality holds:

$$e^{\overline{\mathrm{rk}}_\theta^1(f:E\to\overline{F})}\, (1 + B_{\overline{F}}(f_{\mathbb{R}}(x))) \leq e^{\overline{h}_\theta^1(\overline{E})}\, (1 + B_{\overline{E}}(x)). \tag{8.3.16}$$

PROPOSITION 8.3.8. *Let M be a countably generated $\mathbb{Z}$-module, $\overline{N}$ an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, and $f : M \to N$ a morphism of $\mathbb{Z}$-modules. For every $x \in M_{\mathbb{R}}$, the following inequality holds:*

$$B_{\overline{N}}(f_{\mathbb{R}}(x)) \geq 2e^{-\overline{\mathrm{rk}}_\theta^1(f:M\to\overline{N})} - 1. \tag{8.3.17}$$

Using the function κ introduced in 7.8.2, the inequality (8.3.17) may be rephrased as:

$$b_{\overline{N}}(f_{\mathbb{R}}(x)) \leq \kappa\big(\overline{\mathrm{rk}}_\theta^1(f : M \to \overline{N})\big). \tag{8.3.18}$$

PROPOSITION 8.3.9. *For every admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$:*

$$0 \longrightarrow \overline{G} \xrightarrow{\ i\ } \overline{E} \xrightarrow{\ p\ } \overline{F} \longrightarrow 0 \tag{8.3.19}$$

and for every $x \in E_{\mathbb{R}}$, the following inequality holds:

$$e^{-\overline{h}_\theta^1(\overline{G})}\, (1 + B_{\overline{F}}(p_{\mathbb{R}}(x))) \leq 1 + B_{\overline{E}}(x). \tag{8.3.20}$$

In the left-hand side of (8.3.21) and (8.3.20), the term $e^{-\overline{h}_\theta^1(\overline{E})}$ is by definition 0 when $\overline{h}_\theta^1(\overline{E})$ is $+\infty$.

With the notation of Proposition 8.3.9, it is actually possible to establish the following inequality between positive measures on $E_{\mathbb{R}}^\vee$:

$$e^{-\overline{h}_\theta^1(\overline{G})}\, p_{\mathbb{R}*}^\vee \beta_{\overline{F}^\vee} \leq \beta_{\overline{E}^\vee}. \tag{8.3.21}$$

This inequality is a refinement of the inequality (8.3.20) between Banaszczyk functions,[6] and generalizes the inequality (7.8.18) in Corollary 7.8.5. Starting from (7.8.18), the inequality (8.3.21) may be proved by arguments similar to the ones in the proofs of Propositions 8.3.7 and 8.3.9; we shall leave the details to the interested reader.

PROOF OF PROPOSITION 8.3.7. When $\overline{E}$ and $\overline{F}$ are objects in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the inequality (8.3.15) may be written:

$$e^{h_\theta^1(\overline{F})-h_\theta^1(\overline{F/f(E)})}\, f_{\mathbb{R}*}^\vee \beta_{\overline{F}^\vee} \leq e^{h_\theta^1(\overline{E})}\, \beta_{\overline{E}^\vee}. \tag{8.3.22}$$

and has been established in Proposition 7.8.3.

[6]Indeed (8.3.20) follows from (8.3.21) by integrating the non-negative function $(\xi \mapsto 2\cos^2(\pi\langle\xi, x\rangle))$, as in the proof of Corollary 7.8.5.

To establish the validity of (8.3.15) when $f : \overline{E} \to \overline{F}$ is an arbitrary morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$, let us first choose an exhaustive filtration $(E_i)_{i\in\mathbb{N}}$ of E by submodules in $\mathrm{coh}(E)$ such that:

$$\overline{h}^1_\theta(\overline{E}) = \lim_{i\to+\infty} h^1_\theta(\overline{E}_i). \tag{8.3.23}$$

Then we may also choose an exhaustive filtration $(F_i)_{i\in\mathbb{N}}$ of F by submodules in $\mathrm{coh}(F)$ such that:

$$f(E_i) = F_i \cap f(E) \quad \text{for every } i \in \mathbb{N}.$$

According to the definition of $\overline{\mathrm{rk}}^1_\theta$, the following estimate holds:

$$\overline{\mathrm{rk}}^1_\theta(f : E \to \overline{F}) \leq \liminf_{i\to+\infty} \mathrm{rk}^1_\theta(f_{|E_i} : E_i \to \overline{F}_i) := \liminf_{i\to+\infty} \left(h^1_\theta(\overline{F}_i) - h^1_\theta(\overline{F_i/f(E_i)})\right). \tag{8.3.24}$$

For every pair (i,j) of integers such that $0 \leq i \leq j$, let us introduce the $\mathbb{R}$-linear maps

$$p^E_{ij} : E^\vee_{j,\mathbb{R}} \longrightarrow E^\vee_{i,\mathbb{R}}$$

and

$$p^F_{ij} : F^\vee_{j,\mathbb{R}} \longrightarrow F^\vee_{i,\mathbb{R}}$$

defined as the transposes of the inclusion morphisms $E_{i,\mathbb{R}} \hookrightarrow E_{j,\mathbb{R}}$ and $F_{i,\mathbb{R}} \hookrightarrow F_{j,\mathbb{R}}$. For every $i \in \mathbb{N}$, we shall also consider the $\mathbb{R}$-linear maps

$$q^E_i : E^\vee_{\mathbb{R}} \longrightarrow E^\vee_{i,\mathbb{R}}$$

and

$$q^F_i : F^\vee_{\mathbb{R}} \longrightarrow F^\vee_{i,\mathbb{R}}$$

defined as the transposes of the inclusion morphisms $E_{i,\mathbb{R}} \hookrightarrow E_{\mathbb{R}}$ and $F_{i,\mathbb{R}} \hookrightarrow F_{\mathbb{R}}$.

The inequality (8.3.22) applied to

$$f_j := f_{|E_j} : E_j \longrightarrow \overline{F}_j,$$

shows that the following inequality between measures on $E^\vee_{j,\mathbb{R}}$ holds for every $j \in \mathbb{N}$:

$$e^{h^1_\theta(\overline{F}_j) - h^1_\theta(\overline{F_j/f(E_j)})}\, f^\vee_{j,\mathbb{R}*}\beta_{\overline{F}^\vee_j} \leq e^{h^1_\theta(\overline{E}_j)}\, \beta_{\overline{E}^\vee_j}.$$

Consequently, for every pair (i,j) of integers such that $0 \leq i \leq j$, the following inequality between measure on $E^\vee_{i,\mathbb{R}}$ holds:

$$e^{h^1_\theta(\overline{F}_j) - h^1_\theta(\overline{F_j/f(E_j)})}\, p^E_{ij*}(f^\vee_{j,\mathbb{R}*}\beta_{\overline{F}^\vee_j}) \leq e^{h^1_\theta(\overline{E}_j)}\, p^E_{ij*}\beta_{\overline{E}^\vee_j}. \tag{8.3.25}$$

Observe also that the diagrams

$$\begin{array}{ccc} E_{i,\mathbb{R}} & \xrightarrow{f_{i,\mathbb{R}}} & F_{i,\mathbb{R}} \\ \big\downarrow & & \big\downarrow \\ E_{j,\mathbb{R}} & \xrightarrow{f_{j,\mathbb{R}}} & F_{j,\mathbb{R}} \end{array}$$

and

$$\begin{array}{ccc} F^\vee_{j,\mathbb{R}} & \xrightarrow{f^\vee_{j,\mathbb{R}}} & E^\vee_{j,\mathbb{R}} \\ {\scriptstyle p^F_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle p^E_{ij}} \\ F^\vee_{i,\mathbb{R}} & \xrightarrow{f^\vee_{i,\mathbb{R}}} & E^\vee_{i,\mathbb{R}} \end{array}$$

are commutative. Therefore we have:

$$p^E_{ij*}(f^\vee_{j,\mathbb{R}*}\beta_{\overline{F}^\vee_j}) = f^\vee_{i,\mathbb{R}*}(p^F_{ij*}\beta_{\overline{F}^\vee_j}),$$

and the inequality between measures (8.3.25) may also be written:

$$e^{h^1_\theta(\overline{F}_j)-h^1_\theta(\overline{F_j/f(E_j)})}\, f^\vee_{i,\mathbb{R}*}(p^F_{ij*}\beta_{\overline{F}^\vee_j}) \leq e^{h^1_\theta(\overline{E}_j)}\, p^E_{ij*}\beta_{\overline{E}^\vee_j}. \tag{8.3.26}$$

For every $i \in \mathbb{N}$, when $j \in \mathbb{N}_{\geq i}$ goes to infinity, $p^F_{ij*}\beta_{\overline{F}^\vee_j}$ (resp. $p^E_{ij*}\beta_{\overline{E}^\vee_j}$) converges to $q^F_{i*}(\beta_{\overline{E}^\vee})$ (resp. to $q^E_{i*}(\beta_{\overline{E}^\vee})$ in the topology of narrow convergence. Taking (8.3.23) and (8.3.24) into account, the estimates (8.3.26) therefore implies, for every $i \in \mathbb{N}$, the validity of the following inequality of measures on $E^\vee_{i,\mathbb{R}}$:

$$e^{\overline{\mathrm{rk}}^1_\theta(f:E\to\overline{F})}\, f^\vee_{i,\mathbb{R}*}(q^F_{i*}(\beta_{\overline{E}^\vee})) \leq e^{\overline{h}^1_\theta(\overline{E})}\, q^E_{i*}(\beta_{\overline{E}^\vee}). \tag{8.3.27}$$

Observe that the diagrams

$$\begin{tikzcd} E_{i,\mathbb{R}} \arrow[r, "f_{i,\mathbb{R}}"] \arrow[d, hook] & F_{i,\mathbb{R}} \arrow[d, hook] \\ E_\mathbb{R} \arrow[r, "f_\mathbb{R}"] & F_\mathbb{R} \end{tikzcd}$$

and

$$\begin{tikzcd} F^\vee_{\mathbb{R}} \arrow[r, "f^\vee_\mathbb{R}"] \arrow[d, "q^F_i"'] & E^\vee_{\mathbb{R}} \arrow[d, "q^E_i"] \\ F^\vee_{i,\mathbb{R}} \arrow[r, "f^\vee_{i,\mathbb{R}}"] & E^\vee_{i,\mathbb{R}} \end{tikzcd}$$

also are commutative. Consequently we have:

$$f^\vee_{i,\mathbb{R}*}(q^F_{i*}(\beta_{\overline{E}^\vee})) = q^E_{i*}(f^\vee_{\mathbb{R}*}\beta_{\overline{F}^\vee}),$$

and the estimates (8.3.27) may be rephrased as:

$$e^{\overline{\mathrm{rk}}^1_\theta(f:E\to\overline{F})}\, q^E_{i*}(f^\vee_{\mathbb{R}*}\beta_{\overline{F}^\vee}) \leq e^{\overline{h}^1_\theta(\overline{E})}\, q^E_{i*}(\beta_{\overline{E}^\vee}). \tag{8.3.28}$$

According to the compatibility (8.3.12) of the Kolmogorov isomorphism (8.3.11) with the ordering of measures, the validity of (8.3.28) for every $i \in \mathbb{N}$ is equivalent to the inequality (8.3.15).

The inequality (8.3.16) follows from (8.3.15) by integration of the measures appearing in this inequality of measures against the non-negative function $(\xi \mapsto 2\cos^2(\pi\langle \xi, x\rangle))$, as in the second part of the proof of Proposition 7.8.3. It also follows from the estimate (7.8.10) in Proposition 7.8.3 applied to the morphisms $f_i : \overline{E}_i \to \overline{F}_i$ by letting i go to infinity. □

Proof of Proposition 8.3.8. When M and N are finitely generated $\mathbb{Z}$-modules, and therefore $\overline{N}$ is an object of $\overline{\mathbf{Coh}}_\mathbb{Z}$, Proposition 8.3.8 as already been established as Corollary 7.8.4; indeed the inequality 7.8.16, or equivalently 7.8.20, becomes (8.3.17) when $\overline{E} = \overline{N}$ and $E' = f(M)$.

To establish Proposition 8.3.8 in general, let us choose some exhaustive filtrations $(M_i)_{i\in\mathbb{N}}$ and $(N_i)_{i\in\mathbb{N}}$ by finitely generated submodules of M and N respectively such that the following conditions are satisfied:

- $M_{0,\mathbb{R}}$ contains x;
- for every $i \in \mathbb{N}$, $f(M_i) = f(M) \cap N_i$;
- $\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) = \lim_{i\to+\infty} \mathrm{rk}^1_\theta(f_i : M_i \to \overline{N}_i)$, where $f_i := f_{|M_i}$.

Then, as observed above, for every $i \in \mathbb{N}$, Proposition 8.3.8 holds for $f_i : M_i \to \overline{N}$, and therefore:

$$B_{\overline{N}_i}(f_\mathbb{R}(x)) \geq 2e^{-\mathrm{rk}^1_\theta(f_i:M_i\to\overline{N}_i)} - 1.$$

The inequality (8.3.17) follows by letting i go to infinity. □

Proof of Proposition 8.3.9. When $\overline{E}$, and therefore $\overline{F}$ and $\overline{G}$, are objects of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, Proposition 8.3.9 has been established as Corollary 7.8.5.

For every $x \in E_{\mathbb{R}}$ and every $\varepsilon \in \mathbb{R}_+^*$, we may choose D in $\mathrm{coh}(F)$ such $p_{\mathbb{R}}(x)$ is contained in $D_{\mathbb{R}}$ and the following estimate is satisfied:

$$B_{\overline{D}}(p_{\mathbb{R}}(x)) \geq B_{\overline{G}}(p_{\mathbb{R}}(x)) - \varepsilon.$$

Since p is surjective, we may find C in $\mathrm{coh}(E)$ such that:

$$p(C) = D.$$

Moreover there exists an exhaustive filtration $(B_n)_{n\in\mathbb{N}}$ of G by submodules in $\mathrm{coh}(G)$ such that:

$$\lim_{n\to+\infty} h^1_\theta(\overline{B}_n) = \overline{h}^1_\theta(\overline{G}).$$

For every $n \in \mathbb{N}$, let us consider the submodule

$$C_n := C + i(B_n)$$

of E. It is finitely generated, and if n is large enough — say $n \geq n_0$ — the kernel of $p_{|C}$ is contained in $i(C_n)$, and therefore the diagram

$$0 \longrightarrow B_n \xrightarrow{i_{|B_n}} C_n \xrightarrow{p_{|C_n}} D \longrightarrow 0$$

is an exact sequence of $\mathbb{Z}$-modules. Let us denote by

$$0 \longrightarrow \overline{B}_n \xrightarrow{i_{|B_n}} \overline{C}_n \xrightarrow{p_{|C_n}} \overline{D}_n \longrightarrow 0 \tag{8.3.29}$$

the associated admissible short exact sequence in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. By definition, the Euclidean seminorms defining $\overline{B}_n$ and $\overline{C}_n$ are the restrictions of the seminorms defining $\overline{G}$ and $\overline{E}$, and the seminorm $\|.\|_n$ on $D_{\mathbb{R}}$ defining $\overline{D}_n$ is the quotient seminorm deduced from the one on $\overline{C}_n$.

The sequence of seminorms $(\|.\|_{\overline{D}_n})_{n\geq n_0}$ is decreasing and converges to the seminorm $\|.\|_{\overline{D}}$ on $D_{\mathbb{R}}$ defined as the restriction of the seminorm of $\overline{F}$. As discussed in 7.7.1.1, this implies the equality:

$$B_{\overline{D}}(p_{\mathbb{R}}(x)) = \lim_{n\to+\infty} B_{\overline{D}_n}(p_{\mathbb{R}}(x)).$$

By the very definition of $B_{\overline{E}}$, we also have:

$$B_{\overline{E}}(x) = \lim_{n\to+\infty} B_{\overline{C}_n}(x).$$

Moreover Corollary 7.8.5 applied to the admissible short exact sequence (8.3.29) establishes the following estimates, for $n \geq n_0$:

$$1 + B_{\overline{C}_n}(x) \geq e^{-h^1_\theta(\overline{B}_n)}(1 + B_{\overline{D}_n}(p_{\mathbb{R}}(x))).$$

By taking the limit when n goes to infinity, we finally obtain:

$$1 + B_{\overline{F}}(x) \geq e^{-\overline{h}^1_\theta(\overline{E})}\,(1 + B_{\overline{D}}(p_{\mathbb{R}}(x)) \geq e^{-\overline{h}^1_\theta(\overline{E})}\,(1 + B_{\overline{G}}(p_{\mathbb{R}}(x)) - \varepsilon).$$

Since $\varepsilon \in \mathbb{R}_+^*$ is arbitrary, this establishes (8.3.20). □

For later reference, we spell out some simple consequences of Propositions 8.3.7, 8.3.8 and 8.3.9.

Recall that a Borel probability measure μ on the Polish space $F_{\mathbb{R}}^\vee$ is said to be *discrete* when there exists a countable subset C of $F_{\mathbb{R}}^\vee$ such that $\mu(F_{\mathbb{R}}^\vee \setminus C) = 0$, or equivalently such that there exists $(\lambda_c)_{c\in C}$ in $[0,1]^C$ satisfying the conditions:

$$\sum_{c\in C} \lambda_c = 1 \quad \text{and} \quad \mu = \sum_{c\in C} \lambda_c \delta_c.$$

Corollary 8.3.10. *Let $f : \overline{E} \to \overline{F}$ be a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$ such that $f_{\mathbb{Q}}$ is surjective. If the measure $\beta_{\overline{E}^\vee}$ is discrete and if $\overline{h}^1_\theta(\overline{E})$ is finite, then the measure $\beta_{\overline{F}^\vee}$ is discrete.*

PROOF. The finiteness of $\overline{h}^1_\theta(\overline{E})$ implies the finiteness of $\overline{\mathrm{rk}}^1_\theta(f : E \to \overline{F})$. Therefore Proposition 8.3.7 shows that, when $\overline{h}^1_\theta(\overline{E})$ is finite, the measure $f^\vee_{\mathbb{R}*}\beta_{\overline{F}^\vee}$ is bounded from above by some positive multiple of $\beta_{\overline{E}^\vee}$, and therefore is discrete if $\beta_{\overline{E}^\vee}$ is. When moreover $f_\mathbb{Q}$ — or equivalently $f_\mathbb{R}$ — is surjective, then $f^\vee_{\mathbb{R}*} : F^\vee_\mathbb{R} \to E^\vee_\mathbb{R}$ is injective, has a closed image, and establishes a homeomorphism from $F^\vee_\mathbb{R}$ onto this image. Therefore $\beta_{\overline{F}^\vee}$ also is a discrete measure. □

Applied to $M = N$ and $f = \mathrm{Id}_M$, Proposition 8.3.8 becomes the following generalization of the lower bound (7.8.2) in Proposition 7.8.1.

COROLLARY 8.3.11. *For every object $\overline{N}$ of $\mathbf{q}\overline{\mathbf{Coh}}_\mathbb{Z}$ and every $x \in N_\mathbb{R}$, the following inequality holds:*

$$B_{\overline{N}}(x) \geq 2e^{-\overline{h}^1_\theta(\overline{N})} - 1. \tag{8.3.30}$$

Equivalently, we have:

$$b_{\overline{N}}(x) \leq \kappa(\overline{h}^1_\theta(\overline{N})). \tag{8.3.31}$$

From Proposition 8.3.9, we deduce :

COROLLARY 8.3.12. *For every admissible short exact sequence in $\mathbf{q}\overline{\mathbf{Coh}}_Z$:*

$$0 \longrightarrow \overline{G} \xrightarrow{\ i\ } \overline{E} \xrightarrow{\ p\ } \overline{F} \longrightarrow 0, \tag{8.3.32}$$

such that:

$$\overline{h}^1_\theta(\overline{G}) = 0,$$

the following equality of positive measures on $E^\vee_\mathbb{R}$ is satisfied:

$$\beta_{\overline{E}^\vee} = p^\vee_{\mathbb{R}*}\beta_{\overline{F}^\vee}, \tag{8.3.33}$$

and for every $x \in E_\mathbb{R}$, we have:

$$B_{\overline{E}}(x) = B_{\overline{F}}(p_\mathbb{R}(x))). \tag{8.3.34}$$

PROOF. This is a straightforward consequence of the inequality (8.3.21) mentioned after the statement of Proposition 8.3.9. Indeed, when $\overline{h}^1_\theta(\overline{G})$ vanishes, the inequality (8.3.21) becomes the following inequality of measures:

$$p^\vee_{\mathbb{R}*}\beta_{\overline{F}^\vee} \leq \beta_{\overline{E}^\vee}.$$

Since both $p^\vee_{\mathbb{R}*}\beta_{\overline{F}^\vee}$ and $\beta_{\overline{E}^\vee}$ are probability measures, this inequality is actually an equality. This proves (8.3.33). Finally the equality (8.3.34) follows from (8.3.33) by Fourier transform.

The equality (8.3.34) is also a consequence of the monotonicity of Banaszczyk functions applied to the morphism p — which implies the estimate $B_{\overline{E}}(x) \leq B_{\overline{F}}(p_\mathbb{R}(x)))$ — and of the estimate (8.3.20) established in Proposition 8.3.9. Indeed the estimate (8.3.20) becomes the estimate $B_{\overline{E}}(x) \geq B_{\overline{F}}(p_\mathbb{R}(x))$ when $\overline{h}^1_\theta(\overline{G})$ vanishes. Moreover the equality (8.3.33) follows from (8.3.34) by the injectivity of the Fourier transform. □

Corollary 8.3.12 applies notably to the canonical dévissage of an arbitrary object $\overline{E}$ of $\mathbf{q}\overline{\mathbf{Coh}}_Z$:

$$0 \longrightarrow \overline{E}_{\mathrm{ap}} \longrightarrow \overline{E} \xrightarrow{\delta_E} \overline{E}_{/\mathrm{ap}} := \overline{E}^{\vee\vee} \longrightarrow 0. \tag{8.3.35}$$

Thus we obtain:

COROLLARY 8.3.13. *For every object $\overline{E}$ of $\mathbf{q}\overline{\mathbf{Coh}}_\mathbb{Z}$, the isomorphism of topological groups*

$$\delta^\vee_E : E^\vee_{/\mathrm{ap}} \xrightarrow{\sim} E^\vee$$

maps the measure $\beta_{\overline{E}^\vee_{/\mathrm{ap}}}$ to the measure $\beta_{\overline{E}^\vee}$; namely the following equality holds:

$$\delta^\vee_{E*}\beta_{\overline{E}^\vee_{/\mathrm{ap}}} = \beta_{\overline{E}^\vee}. \tag{8.3.36}$$

Moreover the functions $B_{\overline{E}}$ and $B_{\overline{E}_{/\mathrm{ap}}}$ associated to $\overline{E}$ and $\overline{E}_{/\mathrm{ap}}$ satisfy:

$$B_{\overline{E}}(x) = B_{\overline{E}_{/\mathrm{ap}}}(\delta_{E,\mathbb{R}}(x)) \quad \text{for every } x \in E_{\mathbb{R}}. \tag{8.3.37}$$

8.3.4. Criteria for $\beta_{\overline{F}^\vee}$ to be supported by $F_{\mathbb{R}}^{\vee\mathrm{Hilb}}$. In the next section, we will show that the invariants $\underline{h}^1_\theta(\overline{F})$ and $\overline{h}^1_\theta(\overline{F})$ attached to a θ^1-summable object in $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ coincide if and only if the measure $\beta_{\overline{F}^\vee}$ is supported by $F_{\mathbb{R}}^{\mathrm{Hilb}}$. In this paragraph, we establish diverse criteria ensuring that the measure $\beta_{\overline{F}^\vee}$ associated to an object $\overline{F}$ of $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ satisfies this property.

8.3.4.1. The following proposition is basically a consequence of the theorem of Prokhorov-Sazonov characterizing the Borel measures on a Hilbert space in terms of their Fourier transform and of their continuity for the Sazonov topology. We refer the reader to Appendix C, section C.2, for an exposition of this theorem fitted to its present application — notably for a definition of the Sazonov topology — and for further references.

PROPOSITION 8.3.14. *The following conditions are equivalent:*

(i) The measure $\beta_{\overline{F}^\vee}$ is supported by $F_{\mathbb{R}}^{\vee\mathrm{Hilb}}$.
(ii) The function $B_{\overline{F}}$ is continuous with respect to the Sazonov topology on $(F_{\mathbb{R}}, \|.\|)$.
(iii) The function $B_{\overline{F}}$ is continuous at 0 with respect to the Sazonov topology on $(F_{\mathbb{R}}, \|.\|)$. Equivalently, for every $\varepsilon \in \mathbb{R}_+^*$ there exists a Euclidean seminorm $\|.\|'$ on $E_{\mathbb{R}}$ such that:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty,$$

and such that the following implication holds, for every $x \in F_{\mathbb{R}}$:

$$\|x\|' < 1 \Longrightarrow b_{\overline{F}}(x) < \varepsilon.$$

By definition, condition (i) is equivalent to the equality:

$$\beta_{\overline{F}^\vee}(F_{\mathbb{R}}^\vee \setminus F_{\mathbb{R}}^{\vee\mathrm{Hilb}}) = 0, \tag{8.3.38}$$

or alternatively to the equality:

$$\beta_{\overline{F}^\vee}(F_{\mathbb{R}}^{\vee\mathrm{Hilb}}) = 1. \tag{8.3.39}$$

PROOF OF PROPOSITION 8.3.14. This follows from Theorem C.2.3 applied to $V := F_{\mathbb{R}}$ and $\mu := \beta_{\overline{F}^\vee}$. Indeed, then we have:

$$\Phi := \mathcal{F}\beta_{\overline{F}^\vee} = B_{\overline{F}} = 1 - e^{-\pi b_{\overline{F}}},$$

and condition (i) (resp. (iii), resp. (iv)) in Theorem C.2.3 becomes condition (i) (resp. (ii), resp. (iii)) in Proposition 8.3.14. □

8.3.4.2. The following proposition is a consequence of the construction of the measure $\beta_{\overline{F}^\vee}$ by means of the limiting procedure in Proposition 8.3.4. It leads to simple sufficient conditions for $\beta_{\overline{F}^\vee}$ to be supported by $F_{\mathbb{R}}^{\vee\mathrm{Hilb}}$.

PROPOSITION 8.3.15. *Let $C_\bullet := (C_n)_{n\in\mathbb{N}}$ be an exhaustive filtration of F by finitely generated $\mathbb{Z}$-submodules, and let ε be an element of $\mathbb{R}_+$ and $\eta := e^{2\varepsilon} - 1$.*

Assume that the following two limits exist in $\mathbb{R}_+$:

$$h^1_\theta(\overline{C}_\bullet) := \lim_{n\to+\infty} h^1_\theta(\overline{C}_n) \quad \text{and} \quad h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(\varepsilon)) := \lim_{n\to+\infty} h^1_\theta(\overline{C}_n \otimes \overline{\mathcal{O}}(\varepsilon)).$$

Then the following inequality between finite positive Borel measures on $F_{\mathbb{R}}^\vee$ is satisfied:

$$e^{h^1_\theta(\overline{C}_\bullet)}\, e^{-\pi\eta\|.\|^2_{\overline{F}^\vee}}\, \beta_{\overline{F}^\vee} \geq e^{h^1_\theta(\overline{C}_\bullet\otimes\overline{\mathcal{O}}(\varepsilon))}\, \beta_{\overline{F}^\vee\otimes\overline{\mathcal{O}}(-\varepsilon)}. \tag{8.3.40}$$

In the left hand side of (8.3.40), the function $e^{-\pi\eta\|.\|^2_{\overline{F}^\vee}}$ is a Borel function[7] from $F^\vee_{\mathbb{R}}$ to $[0,1]$, which vanishes precisely on $F^\vee_{\mathbb{R}} \setminus F^{\vee \mathrm{Hilb}}_{\mathbb{R}}$ when $\varepsilon > 0$, and its product with the probability measure $\beta_{\overline{F}^\vee}$ is a well-defined finite positive measure on $F^\vee_{\mathbb{R}}$.

PROOF. We shall use the description of the measures $\beta_{\overline{F}^\vee}$ and $\beta_{\overline{F}^\vee \otimes \overline{\mathcal{O}}(-\varepsilon)}$ provided by Proposition 8.3.4, and will accordingly use the notation introduced in 8.3.2.1.

For every $n \in \mathbb{N}$, consider the following measures on $C^\vee_{n,\mathbb{R}}$, supported by $C^\vee_n$:

$$\gamma_{\overline{C}^\vee_n} := e^{h^1_\theta(\overline{C}_n)} \beta_{\overline{C}^\vee_n} = e^{-\pi\|.\|^2_{\overline{C}^\vee_n}} \sum_{\xi \in C^\vee_n} \delta_\xi$$

and:

$$\gamma_{\overline{C}^\vee_n \otimes \overline{\mathcal{O}}(-\varepsilon)} := e^{h^1_\theta(\overline{C}_n \otimes \overline{\mathcal{O}}(\varepsilon))} \beta_{\overline{C}^\vee_n \otimes \overline{\mathcal{O}}(-\varepsilon)} = e^{-\pi\|.\|^2_{\overline{C}^\vee_n \otimes \overline{\mathcal{O}}(-\varepsilon)}} \sum_{\xi \in C^\vee_n} \delta_\xi.$$

By the very definition of $\overline{\mathcal{O}}(-\varepsilon)$, we have:

$$\|.\|^2_{\overline{C}^\vee_n \otimes \overline{\mathcal{O}}(-\varepsilon)} = e^{2\varepsilon} \|.\|^2_{\overline{C}^\vee_n}.$$

This implies the following equality of measures on $C^\vee_{n,\mathbb{R}}$:

$$\gamma_{\overline{C}^\vee_n \otimes \overline{\mathcal{O}}(-\varepsilon)} = e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_n}} \gamma_{\overline{C}^\vee_n},$$

or equivalently:

$$e^{h^1_\theta(\overline{C}_n \otimes \overline{\mathcal{O}}(\varepsilon))} \beta_{\overline{C}^\vee_n \otimes \overline{\mathcal{O}}(-\varepsilon)} = e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_n}} e^{h^1_\theta(\overline{C}_n)} \beta_{\overline{C}^\vee_n}.$$

Consequently, if two integers i and j satisfy $0 \leq i \leq j$, the following equality holds between measures over $C^\vee_i$:

$$e^{h^1_\theta(\overline{C}_j \otimes \overline{\mathcal{O}}(\varepsilon))} p_{ij*} \beta_{\overline{C}^\vee_j \otimes \overline{\mathcal{O}}(-\varepsilon)} = e^{h^1_\theta(\overline{C}_j)} p_{ij*} \left(e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_j}} \beta_{\overline{C}^\vee_j} \right). \tag{8.3.41}$$

Moreover, since $p_{ij} : \overline{C}^\vee_j \to \overline{C}^\vee_i$ is a morphism if $\overline{\mathbf{Vect}}^{[\infty]\leq 1}_{\mathbb{Z}}$, the following inequality holds between functions over $C^\vee_{j,\mathbb{R}}$:

$$p^*_{ij}\|.\|_{\overline{C}^\vee_i} \leq \|.\|_{\overline{C}^\vee_j},$$

and consequently:

$$p^*_{ij} e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_i}} \geq e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_j}}.$$

This implies the following inequality between measures over $C^\vee_i$:

$$p_{ij*} \left(e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_j}} \beta_{\overline{C}^\vee_j} \right) \leq p_{ij*} \left(p^*_{ij} e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_i}} \beta_{\overline{C}^\vee_j} \right) = e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_i}} p_{ij*} \beta_{\overline{C}^\vee_j}. \tag{8.3.42}$$

From (8.3.41) and (8.3.42), we get the inequality of measures:

$$e^{h^1_\theta(\overline{C}_j \otimes \overline{\mathcal{O}}(\varepsilon))} p_{ij*} \beta_{\overline{C}^\vee_j \otimes \overline{\mathcal{O}}(-\varepsilon)} \leq e^{h^1_\theta(\overline{C}_j)} e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_i}} p_{ij*} \beta_{\overline{C}^\vee_j}.$$

Using Proposition 8.3.4, applied to $\overline{F}$ and $\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon)$, and letting j go to infinity, this establishes the inequality:

$$e^{h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(\varepsilon))} \beta_i(\varepsilon) \leq e^{h^1_\theta(\overline{C}_\bullet)} e^{-\pi\eta\|.\|^2_{\overline{C}^\vee_i}} \beta_i, \tag{8.3.43}$$

where

$$\beta_i := q_{i*}\beta_{\overline{F}^\vee} \quad \text{and} \quad \beta_i(\varepsilon) := q_{i*}\beta_{\overline{F}^\vee \otimes \overline{\mathcal{O}}(-\varepsilon)}.$$

[7] It is actually upper semi-continous.

By applying $p_{i'i*}$ to (8.3.43), we obtain that, for any two integers i' and i such that $0 \leq i' \leq i$, the following inequality holds:

$$e^{h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(\varepsilon))}\, q_{i'*}\beta_{\overline{F}^\vee \otimes \overline{\mathcal{O}}(-\varepsilon)} \leq e^{h^1_\theta(\overline{C}_\bullet)}\, q_{i'*}\left(e^{-\pi\eta \|q_i\|^2_{\overline{C}_i^\vee}}\, \beta_{\overline{F}^\vee}\right).$$

When i goes to infinity, the sequence of functions $\left(e^{-\pi\eta\|q_i\|^2_{\overline{C}_i^\vee}}\right)_i$ on $F^\vee_{\mathbb{R}}$ is decreasing and converges pointwise to the function $e^{-\pi\eta\|.\|^2_{\overline{F}^\vee}}$. Therefore, by dominated convergence, the measure on $F^\vee$:

$$e^{-\pi\eta\|q_i\|^2_{\overline{C}_i^\vee}}\, \beta_{\overline{F}^\vee}$$

converges in the topology of narrow convergence to the measure:

$$e^{-\pi\eta\|.\|^2_{\overline{F}^\vee}}\, \beta_{\overline{F}^\vee}.$$

Therefore, for every $i' \in \mathbb{N}$, the following inequality holds between measures on $C_i^\vee$:

$$e^{h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(\varepsilon))}\, q_{i'*}\beta_{\overline{F}^\vee \otimes \overline{\mathcal{O}}(-\varepsilon)} \leq e^{h^1_\theta(\overline{C}_\bullet)}\, q_{i'*}\left(e^{-\pi\eta\|.\|^2_{\overline{F}^\vee}}\, \beta_{\overline{F}}\right).$$

According to the equivalence (8.3.12), this sequence of inequalities is equivalent to the inequality (8.3.40). □

Corollary 8.3.16. *If there exists an exhaustive filtration $(C_n)_{n\in\mathbb{N}}$ of F by finitely generated $\mathbb{Z}$-submodules and ε_0 in $\mathbb{R}^*_+$ such that, for every $\varepsilon \in [0, \varepsilon_0]$, the limit*

$$h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(\varepsilon)) := \lim_{n\to+\infty} h^1_\theta(\overline{C}_n \otimes \overline{\mathcal{O}}(\varepsilon))$$

exists in $\mathbb{R}_+$, and if

$$\lim_{\varepsilon\to 0_+} h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(\varepsilon)) = h^1_\theta(\overline{C}_\bullet), \tag{8.3.44}$$

then $\beta_{\overline{F}^\vee}$ is supported by $F^{\vee\mathrm{Hilb}}_{\mathbb{R}}$.

Proof. When $\varepsilon \in \mathbb{R}^*_+$ decreases to 0, the function on $F^\vee_{\mathbb{R}}$:

$$\exp\left(-\pi(e^{2\varepsilon}-1)\|.\|^2_{\overline{F}^\vee}\right)$$

increases and converges pointwise to the characteristic function of $F^{\vee\mathrm{Hilb}}_{\mathbb{R}}$. Therefore, by dominated convergence, the following equality holds:

$$\beta_{\overline{F}^\vee}(F^{\vee\mathrm{Hilb}}_{\mathbb{R}}) = \lim_{\varepsilon\to 0_+} \int_{F^\vee_{\mathbb{R}}} \exp\left(-\pi(e^{2\varepsilon}-1)\|\xi\|^2_{\overline{F}^\vee}\right)\, d\beta_{\overline{F}^\vee}(\xi). \tag{8.3.45}$$

Under the hypotheses of Corollary 8.3.16, we may apply Proposition 8.3.15 to any ε in $(0, \varepsilon_0]$. Thus we have:

$$\int_{F^\vee_{\mathbb{R}}} \exp\left(-\pi(e^{2\varepsilon}-1)\|\xi\|^2_{\overline{F}^\vee}\right)\, d\beta_{\overline{F}^\vee}(\xi) \geq e^{h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(\varepsilon)) - h^1_\theta(\overline{C}_\bullet)} \quad \text{for every } \varepsilon \in (0, \varepsilon_0]. \tag{8.3.46}$$

From (8.3.45), (8.3.46), and (8.3.44), we obtain:

$$\beta_{\overline{F}^\vee}(F^{\vee\mathrm{Hilb}}_{\mathbb{R}}) \geq 1.$$

Since $\beta_{\overline{F}^\vee}$ is a probability measure, this establishes that it is supported by $F^{\vee\mathrm{Hilb}}_{\mathbb{R}}$. □

Corollary 8.3.17. *If there exists an exhaustive filtration $(C_n)_{n\in\mathbb{N}}$ of F by finitely generated $\mathbb{Z}$-submodules and ε in $\mathbb{R}^*_+$ such that the limits*

$$h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(-\varepsilon)) := \lim_{n\to+\infty} h^1_\theta(\overline{C}_n \otimes \overline{\mathcal{O}}(-\varepsilon)) \quad \textit{and} \quad h^1_\theta(\overline{C}_\bullet) := \lim_{n\to+\infty} h^1_\theta(\overline{C}_n)$$

exist in $\mathbb{R}_+$, then $\beta_{\overline{F}^\vee}$ is supported by $F^{\vee\mathrm{Hilb}}_{\mathbb{R}}$.

PROOF. Under the hypothesis of Corollary 8.3.17, we may apply Proposition 8.3.15 with $\overline{F}$ replaced by $\overline{F} \otimes \overline{\mathcal{O}}(-\varepsilon)$. Then the estimate (8.3.40) reads:

$$\beta_{\overline{F}^\vee} \leq e^{h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(-\varepsilon)) - h^1_\theta(\overline{C}_\bullet)}\, e^{-\pi\eta \|.\|^2_{\overline{F}^\vee}}\, \beta_{\overline{F}^\vee \otimes \overline{\mathcal{O}}(\varepsilon)}.$$

Since $\eta := e^{2\varepsilon} - 1$ is strictly positive, the function $e^{-\pi\eta\|.\|^2_{\overline{F}^\vee}}$ vanishes on $F^\vee_\mathbb{R} \setminus F^{\vee \mathrm{Hilb}}_\mathbb{R}$, and (8.3.38) follows. □

REMARK 8.3.18. For every $\varepsilon \in \mathbb{R}^*_+$, there exists an exhaustive filtration $(C_n)_{n \in \mathbb{N}}$ of F such that:

$$h^1_\theta(\overline{C}_\bullet) = h^1_\theta(\overline{F}) \quad (\text{resp. } h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(-\varepsilon)) = h^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(-\varepsilon)))$$

and such that the limit defining $h^1_\theta(\overline{C}_\bullet \otimes \overline{\mathcal{O}}(\varepsilon))$ (resp. $h^1_\theta(\overline{C}_\bullet)$) exists in $[0, +\infty]$. This limit is actually bounded from above by $h^1_\theta(\overline{F})$ (resp. by $h^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(-\varepsilon))$).

Consequently Proposition 8.3.15 implies that, *when $h^1_\theta(\overline{F}) < +\infty$, the following inequality holds, where c_ε denotes some positive real number and $\eta := e^{2\varepsilon} - 1$:*

$$\beta_{\overline{F}^\vee \otimes \overline{\mathcal{O}}(-\varepsilon)} \leq e^{c_\varepsilon}\, e^{-\pi\eta\|.\|^2_{\overline{F}^\vee}}\, \beta_{\overline{F}^\vee},$$

and Corollary 8.3.17 shows that, *if $\overline{h}^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(-\varepsilon)) < +\infty$ for some $\varepsilon \in \mathbb{R}^*_+$, then $\beta_{\overline{F}^\vee}$ is supported by $F^{\mathrm{Hilb}}_\mathbb{R}$.*

8.4. Coincidence of $\underline{h}^1_\theta$ and $\overline{h}^1_\theta$, θ^1-summability, and Banaszczyk Measures

8.4.1. The invariant $\underline{h}^1_\theta(\overline{F})$ as the logarithm of a θ-series.

PROPOSITION 8.4.1. *For every object $\overline{F}$ in $\mathbf{q}\overline{\mathbf{Coh}}_\mathbb{Z}$, the following equality holds:*

$$\underline{h}^1_\theta(\overline{F}) = \log \sum_{\xi \in F^\vee \cap F^{\vee \mathrm{Hilb}}_\mathbb{R}} e^{-\pi\|\xi\|^2_{\overline{F}^\vee}}. \tag{8.4.1}$$

The sum in the right hand-side of (8.4.1) is the supremum in $[0, +\infty]$ of the sums $\sum_{\xi \in A} e^{-\pi\|\xi\|^2_{\overline{F}^\vee}}$ when A runs over the finite subsets of the set $F^\vee \cap F^{\vee \mathrm{Hilb}}_\mathbb{R}$, which is possibly non-countable. We also use the convention: $\log(+\infty) := +\infty$.

Since the quasi-norm $\|.\|_{\overline{F}^\vee}$ takes the values $+\infty$ and consequently $e^{-\pi\|.\|^2_{\overline{F}^\vee}}$ vanishes on $F^\vee_\mathbb{R} \setminus F^{\vee \mathrm{Hilb}}_\mathbb{R}$, the equality (8.4.1) may equivalently be written as:

$$\underline{h}^1_\theta(\overline{F}) = \log \sum_{\xi \in F^\vee} e^{-\pi\|\xi\|^2_{\overline{F}^\vee}}. \tag{8.4.2}$$

PROOF. Consider $\overline{F} := (F, \|.\|)$ an object of $\mathbf{q}\overline{\mathbf{Coh}}_\mathbb{Z}$. Recall that the invariant $\underline{h}^1_\theta(\overline{F})$ is defined as:

$$\underline{h}^1_\theta(\overline{F}) := \sup_{C \in \mathrm{coft}(F)} h^1_\theta(\overline{F/C}). \tag{8.4.3}$$

To prove that the right-hand sides of (8.4.1) and (8.4.3) coincide, we proceed in three steps.

(1) For every C in $\mathrm{coft}(F)$, we may consider the following admissible surjective morphisms in $\mathbf{q}\overline{\mathbf{Coh}}_\mathbb{Z}$: the quotient map

$$p_C : \overline{F} \longrightarrow \overline{F/C},$$

the vectorization of $\overline{F/C}$

$$\nu_{\overline{F/C}} : \overline{F/C} \longrightarrow \overline{F/C}^{\mathrm{vect}},$$

and their composition

$$\nu_{\overline{F/C}} \circ p_C : \overline{F} \longrightarrow \overline{F/C}^{\mathrm{vect}}.$$

The kernel of this composition:

$$C^{\mathrm{vect}} := \ker \nu_{\overline{F/C}} \circ p_C$$

is a submodule of F in

$$\mathrm{covect}(\overline{F}) := \big\{ C \in \mathrm{scoft}(F) \mid C_{\mathbb{R}} \text{ closed in } (F_{\mathbb{R}}, \|.\|) \big\}.$$

Moreover, according to Scholium 7.3.1 applied to $\overline{F/C}$, the following equalities hold:

$$h^1_\theta(\overline{F/C^{\mathrm{vect}}}) = h^1_\theta(\overline{F/C}^{\mathrm{vect}}) = h^1_\theta(\overline{F/C}).$$

This establishes the following alternative expression for $\underline{h}^1_\theta(\overline{F})$:

$$\underline{h}^1_\theta(\overline{F}) := \sup_{C \in \mathrm{covect}(\overline{F})} h^1_\theta(\overline{F/C}). \tag{8.4.4}$$

(2) Let us consider the set $\mathrm{cohsat}(F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}})$ of finitely generated saturated $\mathbb{Z}$-submodules of $F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}}$. There is a canonical bijection:

$$\mathrm{covect}(\overline{F}) \xrightarrow{\sim} \mathrm{cohsat}(F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}}), \quad C \longmapsto C^\perp := \{\xi \in F^\vee \mid \xi_{|C} = 0\}.$$

Moreover the Euclidean lattice $\overline{F/C}^\vee$ is canonically isomorphic to $C^\perp$ equipped with the restriction $\|.\|_{\overline{F}^\vee |(C^\perp)_{\mathbb{R}}}$ of the quasi-norm $\|.\|_{\overline{F}^\vee}$. Notably we have the following equality of θ-invariants:

$$h^1_\theta(\overline{F/C}) = h^0_\theta(\overline{C^\perp}). \tag{8.4.5}$$

This easily follows from the definitions, and we leave the details to the reader.

(3) If A is a finite subset of $F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}}$, then the saturation $\mathbb{Z}\langle A \rangle^{\mathrm{sat}}$ in $F^\vee$ of the $\mathbb{Z}$-submodule $\mathbb{Z}\langle A\rangle$ generated by A is finitely generated[8] and is clearly contained in $F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}}$. Therefore $\mathbb{Z}\langle A \rangle^{\mathrm{sat}}$ belongs to $\mathrm{cohsat}(F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}})$.

This establishes the equality:

$$\begin{aligned} \log \sum_{\xi \in F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}}} e^{-\pi \|\xi\|^2_{\overline{F}^\vee}} &= \sup_{D \in \mathrm{cohsat}(F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}})} \log \sum_{\xi \in D} e^{-\pi \|\xi\|^2_{\overline{F}^\vee}} \\ &= \sup_{D \in \mathrm{cohsat}(F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}})} h^0_\theta(D, \|.\|_{\overline{F}^\vee |D_{\mathbb{R}}}). \end{aligned} \tag{8.4.6}$$

Finally the equality (8.4.1) follows from (8.4.4), (8.4.5), and (8.4.6). □

For later reference, observe that Proposition 8.4.1 admits the following straigthtforward corollary.

COROLLARY 8.4.2. *For every object $\overline{F}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following implication holds:*

$$\underline{h}^1_\theta(\overline{F}) < +\infty \Longrightarrow F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}} \text{ is countable.}$$

Proposition 8.4.1 also implies the following characterization of the objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ with non-vanishing invariant $\underline{h}^1_\theta$, to be compared with Proposition 6.3.12.

COROLLARY 8.4.3. *For every object $\overline{F}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following conditions are equivalent:*

(i) $\underline{h}^1_\theta(\overline{M}) > 0$;
(ii) $F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}} \neq \{0\}$;
(iii) there exists a non-zero morphism $\xi : \overline{M} \longrightarrow \overline{\mathcal{O}}_{\mathbb{Z}} := (\mathbb{Z}, |.|)$ *in* $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

[8] This is a special case of [**Bos20b**, Corollary 4.4.2].

EXAMPLE 8.4.4. Using Corollary 8.4.3, we may show as in [**Bos20b**, Section 6.4.4] that the Euclidean quasi-coherent sheaf $\overline{M}$ introduced in Example 6.3.3.4 satisfies the conditions:

$$\underline{h}^1_\theta(\overline{M}) = 0 \quad \text{and} \quad \overline{h}^1_\theta(\overline{M}) > 0.$$

Indeed as stated as assertion (i) in Example 6.3.3.4, the set $M^\vee \cap M_{\mathbb{R}}^{\vee \mathrm{Hilb}}$ of continuous linear forms ξ on $\overline{M}_{\mathbb{R}} := (\mathbb{R}^{\mathbb{N}}, \|.\|)$ such that $\xi(M) \subseteq \mathbb{Z}$ is reduced to $\{0\}$. The vanishing of $\underline{h}^1_\theta(\overline{M})$ therefore follows from Corollary 8.4.3.

Moreover, according to the assertion (iii) in Example 6.3.3.4, for every non-zero finitely generated submodule C of M, there exists $n \in \mathbb{N}$ such that the linear form ξ_n defines an non-zero morphism in $\overline{\mathbf{Vect}}^{\leq 1}_{\mathbb{R}}$:

$$\xi_n : \overline{C} \longrightarrow \overline{\mathcal{O}}_{\mathbb{Z}} := (\mathbb{Z}, |.|).$$

Consequently, the following inequality holds:

$$h^1_\theta(\overline{C}) \geq h^1_\theta(\overline{\mathcal{O}}_{\mathbb{Z}}) =: \eta.$$

This establishes the lower bound:

$$\overline{h}^1_\theta(\overline{M}) \geq \eta > 0.$$

8.4.2. The measure $\beta_{\overline{F}^\vee}$ attached to an object $\overline{F}$ in θ^1_Σ-$\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

PROPOSITION 8.4.5. *Let $\overline{F}$ be an object in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.*

(1) For every ξ in $F^\vee$, the following inequality holds:

$$\beta_{\overline{F}^\vee}(\{\xi\}) \geq e^{-\pi\|\xi\|^2_{\overline{F}^\vee}} e^{-\overline{h}^1_\theta(\overline{F})}. \tag{8.4.7}$$

(2) If moreover $\overline{F}$ is θ^1-summable, then:

$$\beta_{\overline{F}^\vee}(\{\xi\}) = e^{-\pi\|\xi\|^2_{\overline{F}^\vee}} e^{-\overline{h}^1_\theta(\overline{F})}. \tag{8.4.8}$$

PROOF. Let $(C_\bullet) := (C_n)_{n\in\mathbb{N}}$ be an exhaustive filtration of F by finitely generated submodules such that the limit

$$h^1_\theta(\overline{C}_\bullet) := \lim_{n\to+\infty} h^1_\theta(\overline{C}_n)$$

exists in $[0, +\infty]$, and let ξ be an element of $F^\vee$. To establish the proposition, we have to establish the lower bound:

$$\beta_{\overline{F}^\vee}(\{\xi\}) \geq e^{-\pi\|\xi\|^2_{\overline{F}^\vee}} e^{-\overline{h}^1_\theta(\overline{C}_\bullet)}, \tag{8.4.9}$$

and to prove that (8.4.9) is actually an equality when the filtration $\overline{C}_\bullet$ satisfies the condition:

$$\Sigma_{h^1_\theta}(\overline{F}, C_\bullet) < +\infty. \tag{8.4.10}$$

As in 8.3.2.1, let us denote by

$$q_i : F^\vee_{\mathbb{R}} \longrightarrow C^\vee_{i,\mathbb{R}} \quad (\text{resp. } p_{ij} : C^\vee_{j,\mathbb{R}} \longrightarrow C^\vee_{i,\mathbb{R}})$$

the transpose of the inclusion map $C_{i,\mathbb{R}} \hookrightarrow F_{\mathbb{R}}$ (resp. $C_{i,\mathbb{R}} \hookrightarrow C_{j,\mathbb{R}}$). Let us also use the notation

$$\beta_i := q_{i*}\beta_{\overline{F}^\vee}, \quad \text{for } i \in \mathbb{N},$$

introduced in Proposition 8.3.4.

If we define, for $i \in \mathbb{N}$:

$$\xi_i := q_i(\xi) \in C_i^\vee,$$

then $(q_i^{-1}(\xi_i))_{i\in\mathbb{N}}$ is a decreasing sequence of closed subsets in $F^\vee_{\mathbb{R}}$, and we have:

$$\bigcap_{i\in\mathbb{N}} q_i^{-1}(\xi_i) = \{\xi\}.$$

This implies the equalities:

$$\beta_{\overline{F}^\vee}(\{\xi\}) = \lim_{i\to+\infty} \beta_{\overline{F}^\vee}(q_i^{-1}(\xi_i)) = \lim_{i\to+\infty} \beta_i(\{\xi_i\}). \tag{8.4.11}$$

Moreover the sequence $(\|\xi_i\|_{\overline{C}_i^\vee})_{i\in\mathbb{N}}$ of elements of $[0,+\infty]$ is increasing and satisfies:

$$\lim_{i\to+\infty} \|\xi_i\|_{\overline{C}_i^\vee} = \|\xi\|_{\overline{F}^\vee}. \tag{8.4.12}$$

According to Proposition 8.3.4, for every $i\in\mathbb{N}$, we have:

$$\beta_i(\{\xi_i\}) = \lim_{j\to+\infty} \beta_{\overline{C}_j^\vee}(p_{ij}^{-1}(\xi_i)). \tag{8.4.13}$$

Moreover $(p_{ij}^{-1}(\xi_i)$ contains ξ_j, and therefore:

$$\beta_{\overline{C}_j^\vee}(p_{ij}^{-1}(\xi_i)) \geq \beta_{\overline{C}_j^\vee}(\{\xi_j\}) = e^{-h^1_\theta(\overline{C}_j)}\, e^{-\pi\|\xi_j\|^2_{\overline{C}_j^\vee}}. \tag{8.4.14}$$

Using (8.4.13), (8.4.14), and (8.4.12), we obtain:

$$\beta_i(\{\xi_i\}) \geq \lim_{j\to+\infty} e^{-h^1_\theta(\overline{C}_j)}\, e^{-\pi\|\xi_j\|^2_{\overline{C}_j^\vee}} = e^{-h^1_\theta(\overline{C}_\bullet)}\, e^{-\pi\|\xi\|^2_{\overline{F}^\vee}}.$$

Together with (8.4.11), this establishes the estimate (8.4.9).

From now on, let us assume that the filtration $C_\bullet$ satisfies the condition (8.4.10) of θ^1-summability.

For any pair (i,j) of integers such that $0\leq i\leq j$, we may apply Proposition 7.8.3 to the admissible short exact sequence:

$$0 \longrightarrow \overline{C}_i \hookrightarrow \overline{C}_j \longrightarrow \overline{C_j/C_i} \longrightarrow 0.$$

This establishes the following inequality between positive measure of finite total mass on $C^\vee_{i,\mathbb{R}}$:

$$p_{ij*}\beta_{\overline{C}_j^*} \leq e^{h^1_\theta(\overline{C}_i)+h^1_\theta(\overline{C_j/C_i})-h^1_\theta(\overline{C}_j)}\, \beta_{\overline{C}_i^\vee}. \tag{8.4.15}$$

When j goes to infinity, the left-hand side of (8.4.15) converges to β_i by Proposition 8.3.4. Moreover, since $C_\bullet$ satisfies the condition (8.4.10), we have:

$$\overline{h}^1_\theta(\overline{F}) = \lim_{j\to+\infty} h^1_\theta(\overline{C}_j).$$

Moreover the filtration $C_\bullet/C_i := (C_j/C_i)_{j\in\mathbb{N}_{\geq i}}$ of F/C_i satisfies the condition

$$\Sigma_{h^1_\theta}(\overline{F/C_i}, C_\bullet/C_i) < +\infty,$$

and therefore:

$$\overline{h}^1_\theta(\overline{F/C_i}) = \lim_{j\to+\infty} h^1_\theta(\overline{C_j/C_i}) < +\infty.$$

Together with (8.4.15), this establishes the inequality of measures on $C^\vee_{i,\mathbb{R}}$:

$$\beta_i \leq e^{h^1_\theta(\overline{C}_i)+\overline{h}^1_\theta(\overline{F/C_i})-\overline{h}^1_\theta(\overline{F})}\, \beta_{\overline{C}_i^\vee}. \tag{8.4.16}$$

Consequently, for every $i\in\mathbb{N}$, we have:

$$\beta_i(\{\xi_i\}) \leq e^{h^1_\theta(\overline{C}_i)+\overline{h}^1_\theta(\overline{F/C_i})-\overline{h}^1_\theta(\overline{F})}\, \beta_{\overline{C}_i^\vee}(\{\xi_i\}) = e^{\overline{h}^1_\theta(\overline{F/C_i})-\overline{h}^1_\theta(\overline{F})}\, e^{-\pi\|\xi_i\|^2_{\overline{C}_i^\vee}}. \tag{8.4.17}$$

Moreover ev$\overline{h}^1_\theta(\overline{F})$ vanishes, and therefore:

$$\lim_{i\to\infty} \overline{h}^1_\theta(\overline{F/C_i}) = 0.$$

Together with (8.4.11), (8.4.17), and (8.4.12), this establishes the estimate:

$$\beta_{\overline{F}^\vee}(\{\xi\}) \leq e^{-\pi\|\xi\|^2_{\overline{F}^\vee}}\, e^{-\overline{h}^1_\theta(\overline{F})},$$

and completes the proof of (8.4.8). □

The following scholium gathers various facts established so far about the measure $\beta_{\overline{F}^\vee}$ associated to some θ^1-summable object $\overline{F}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

SCHOLIUM 8.4.6. *Let $\overline{F}$ be a θ^1-summable object in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$. The set $F^\vee \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}$ is countable. The discrete part*[9] *of the measure $\beta_{\overline{F}^\vee}$ coincides with the measure $\mathbf{1}_{F_{\mathbb{R}}^{\vee \mathrm{Hilb}}} \beta_{\overline{F}^\vee}$ defined by:*

$$\mathbf{1}_{F_{\mathbb{R}}^{\vee \mathrm{Hilb}}} \beta_{\overline{F}^\vee}(A) := \beta_{\overline{F}^\vee}(A \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}),$$

for every Borel subset A of $F_{\mathbb{R}}^\vee$, and satisfies:

$$\mathbf{1}_{F_{\mathbb{R}}^{\vee \mathrm{Hilb}}} \beta_{\overline{F}^\vee} = e^{-\overline{h}^1_\theta(\overline{F})} \sum_{\xi \in F^\vee \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}} e^{-\pi \|\xi\|^2_{\overline{F}^\vee}} \, \delta_\xi. \tag{8.4.18}$$

Finally we have:

$$\beta_{\overline{F}^\vee}(F_{\mathbb{R}}^{\vee \mathrm{Hilb}}) = \beta_{\overline{F}^\vee}(F^\vee \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}) = e^{\underline{h}^1_\theta(\overline{F}) - \overline{h}^1_\theta(\overline{F})}. \tag{8.4.19}$$

PROOF. According to Theorem 8.2.1, the invariant $\overline{h}^1_\theta(\overline{\mathcal{F}})$ is finite. A fortiori $\underline{h}^1_\theta(\overline{\mathcal{F}})$ is finite, and therefore $F^\vee \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}$ is countable by Corollary 8.4.2. Moreover Proposition 8.4.5, (2), shows that , for every $\xi \in F_{\mathbb{R}}^\vee$, the following equivalence holds:

$$\beta_{\overline{F}^\vee}(\{\xi\}) > 0 \Longleftrightarrow \xi \in F^\vee \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}.$$

This implies the first two assertions of the scholium.

Since the measure $\beta_{\overline{F}^\vee}$ is supported by $F^\vee$ and satisfies (8.4.18), we have:

$$\beta_{\overline{F}^\vee}(F_{\mathbb{R}}^{\vee \mathrm{Hilb}}) = \beta_{\overline{F}^\vee}(F^\vee \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}) = e^{-\overline{h}^1_\theta(\overline{F})} \sum_{\xi \in F^\vee \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}} e^{-\pi \|\xi\|^2_{\overline{F}^\vee}}.$$

Together with Proposition 8.4.1, this establishes the equality (8.4.19). □

8.4.3. The objects $\overline{F}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ with a well-defined and finite θ-invariant $h^1_\theta(\overline{F})$. In Proposition 5.6.7, we have shown that, if

$$\varphi : \overline{\mathbf{Coh}}_{\mathbb{Z}} \longrightarrow \mathbb{R}_+$$

is a strongly monotonic invariant, then the objects $\overline{F}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ such that the invariants $\underline{\varphi}(\overline{F})$ and $\overline{\varphi}(\overline{F})$ are finite and coincide are always φ-summable.

The following theorem, which constitutes the main result of this section, establishes a suitable converse to this statement when φ is the theta invariant h^1_θ.

THEOREM 8.4.7. *For every object $\overline{F}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following conditions are equivalent:*

(i) $\underline{h}^1_\theta(\overline{F}) = \overline{h}^1_\theta(\overline{F}) < +\infty$.
(ii) *$\overline{F}$ is θ^1-summable and the measure $\beta_{\overline{F}^\vee}$ is supported by $F^\vee \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}$.*
(iii) *$\overline{F}$ is θ^1-summable and the measure $\beta_{\overline{F}^\vee}$ is a discrete measure.*
(iv) *$\overline{F}$ is θ^1-summable and the following condition is satisfied:*

$$\lim_{\varepsilon \to 0_+} \overline{h}^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon)) = \overline{h}^1_\theta(\overline{F}). \tag{8.4.20}$$

[9] The discrete part of a finite positive Borel measure μ on some Polish space X is the measure defined as the countable sum: $\sum_{x \in X, \mu(\{x\}) \neq 0} \mu(\{x\}) \delta_x$.

Let us formulate a few observations concerning these conditions.

Concerning (ii), recall that the measure $\beta_{\overline{F}^\vee}$ is always supported by $F^\vee$. Consequently it is supported by $F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}}$ if and only if the equivalent conditions in Proposition 8.3.14 are satisfied, notably if and only if $B_{\overline{F}}$ is continuous with respect to the Sazonov topology on $(F_{\mathbb{R}}, \|.\|)$.

Concerning (iii), the definition of a discrete measure has been recalled in Subsection 8.3.3 above, before Corollary 8.3.10.

Concerning (iv), observe that if $\overline{F}$ is θ^1-summable, then $\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon)$ also is θ^1-summable for every $\varepsilon \in \mathbb{R}_+$ and the θ-invariant $\overline{h}^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon))$ defines a decreasing function of $\varepsilon \in \mathbb{R}_+$ with values in $\mathbb{R}_+$. In particular the limit in the left-hand side of (8.4.20) exists and satisfies:

$$\lim_{\varepsilon \to 0_+} \overline{h}^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon)) \leq \overline{h}^1_\theta(\overline{F}).$$

Condition (8.4.20) asserts that this inequality is indeed an equality.

Proof. When (i) holds, then $\overline{F}$ is θ^1-summable by Proposition 5.6.7. Moreover the last equation (8.4.19) in Scholium 8.4.6 shows that $\beta_{\overline{F}^\vee}$ is supported by $F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}}$, and therefore that (ii) holds. It also establishes the converse implication $(ii) \Rightarrow (i)$.

The equivalence $(ii) \Leftrightarrow (iii)$ follows from the first assertion in Scholium 8.4.6.

Let us now assume that $\overline{F}$ is θ^1-summable and let us choose an exhaustive filtration $C_\bullet := (C_n)_{n \in \mathbb{N}}$ of F by finitely generated submodules such that the sum:

$$\Sigma_{h^1_\theta}\left(\overline{F}, C_\bullet\right) := \sum_{i=0}^{+\infty} h^1_\theta(\overline{C_i / C_{i-1}})$$

is finite. Then for every $\varepsilon \in \mathbb{R}_+$, the sum:

$$\Sigma_{h^1_\theta}\left(\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon), C_\bullet\right) := \sum_{i=0}^{+\infty} h^1_\theta(\overline{C_i / C_{i-1}} \otimes \overline{\mathcal{O}}(\varepsilon))$$

is also finite, and, according to Theorem 8.2.1, the following equality holds:

$$\overline{h}^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon)) = \lim_{n \to +\infty} h^1_\theta(\overline{C}_n \otimes \overline{\mathcal{O}}(\varepsilon)).$$

If the equality (8.4.20) holds, then Corollary 8.3.16 shows that the measure $\beta_{\overline{F}^\vee}$ is supported by $F^{\vee \mathrm{Hilb}}_{\mathbb{R}}$. This establishes the implication $(iv) \Rightarrow (ii)$.

Corollary 8.3.17 applied to $\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon)$ shows that, for every $\varepsilon \in \mathbb{R}^*_+$, the measure $\beta_{\overline{\mathcal{F}}^\vee \otimes \overline{\mathcal{O}}(-\varepsilon)}$ is supported by $F^{\vee \mathrm{Hilb}}_{\mathbb{R}}$. Therefore, according to the implication $(ii) \Rightarrow (i)$, the following equality holds, for every $\varepsilon \in \mathbb{R}^*_+$:

$$\underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)) = \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)). \tag{8.4.21}$$

Moreover, according to Proposition 8.4.1, the following equality holds for every $\varepsilon \in \mathbb{R}$:

$$\underline{h}^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon)) = \log \sum_{\xi \in F^\vee \cap F^{\vee \mathrm{Hilb}}_{\mathbb{R}}} e^{-\pi e^{2\varepsilon} \|\xi\|^2_{\overline{F}^\vee}}.$$

By "monotone convergence," this implies:

$$\lim_{\varepsilon \to 0_+} \underline{h}^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon)) = \underline{h}^1_\theta(\overline{F}). \tag{8.4.22}$$

The relations (8.4.21) and (8.4.22) show that, if $\underline{h}^1_\theta(\overline{F})$ and $\overline{h}^1_\theta(\overline{F})$ coincide then the equality (8.4.20) is satisfied. This establishes the implication $(i) \Rightarrow (iv)$ and completes the proof. ☐

DEFINITION 8.4.8. If $\overline{F}$ is an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, we shall say that *the θ-invariant $h^1_\theta(\overline{F})$ is well-defined and finite* when the equivalent conditions in Theorem 8.4.7 are satisfied. Then we shall use the following notation:

$$h^1_\theta(\overline{F}) := \underline{h}^1_\theta(\overline{F}) = \overline{h}^1_\theta(\overline{F}).$$

The following scholium is a straighforward consequence of Theorem 8.4.7 and of Scholium 8.4.6.

SCHOLIUM 8.4.9. *If $\overline{F}$ is an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ such that $h^1_\theta(\overline{F})$ is well-defined and finite, then $F^\vee \cap F^{\vee\mathrm{Hilb}}_{\mathbb{R}}$ is countable, and we have:*

$$\sum_{\xi \in F^\vee \cap F^{\vee\mathrm{Hilb}}_{\mathbb{R}}} e^{-\pi\|\xi\|^2_{\overline{F}^\vee}} < +\infty, \tag{8.4.23}$$

$$h^1_\theta(\overline{F}) := \underline{h}^1_\theta(\overline{F}) = \overline{h}^1_\theta(\overline{F}) = \log \sum_{\xi \in F^\vee \cap F^{\vee\mathrm{Hilb}}_{\mathbb{R}}} e^{-\pi\|\xi\|^2_{\overline{F}^\vee}}, \tag{8.4.24}$$

$$\beta_{\overline{F}^\vee} = e^{-h^1_\theta(\overline{F})} \sum_{\xi \in F^\vee \cap F^{\vee\mathrm{Hilb}}_{\mathbb{R}}} e^{-\pi\|\xi\|^2_{\overline{F}^\vee}} \delta_\xi, \tag{8.4.25}$$

and, for every $x \in F_{\mathbb{R}}$:

$$B_{\overline{F}}(x) = e^{-h^1_\theta(\overline{F})} \sum_{\xi \in F^\vee \cap F^{\vee\mathrm{Hilb}}_{\mathbb{R}}} e^{-\pi\|\xi\|^2_{\overline{F}^\vee}} \cos(2\pi\langle\xi, x\rangle). \tag{8.4.26}$$

In spite of its elementary formulation, which involves only the definitions of the lower and upper extensions $\underline{h}^1_\theta$ and $\overline{h}^1_\theta$, the proof of the following proposition relies on the full strength of the formalism behind Theorem 8.4.7.

PROPOSITION 8.4.10. *Let $f : \overline{E} \to \overline{F}$ be a morphism in $\mathbf{q}\overline{\mathbf{Coh}}^{\leq 1}_{\mathbb{Z}}$. If $f_{\mathbb{Q}}$ is surjective and if the θ-invariant $h^1_\theta(\overline{E})$ is well-defined and finite, then the θ-invariant $h^1_\theta(\overline{F})$ is well-defined and finite.*

PROOF. Let us assume that $f_{\mathbb{Q}}$ is surjective. Then if $\overline{E}$ is θ^1-summable, then $\overline{F}$ is θ^1-summable according to Theorem 8.2.3, (2). Moreover, if the measure $\beta_{\overline{E}^\vee}$ is discrete and if $\overline{h}^1_\theta(\overline{E})$ is finite, then the measure $\beta_{\overline{F}^\vee}$ is discrete according to Corollary 8.3.10. The proposition therefore follows from the characterization of the objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ with well-defined and finite invariant h^1_θ by condition (iii) in Theorem 8.4.7. $\square$

In practice one needs simple criteria to check that some object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ satisfis the equivalent conditions in Theorem 8.4.7. The second part of the following proposition provides such a criterion.

PROPOSITION 8.4.11. *Let $\overline{F} := (F, \|.\|)$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.*

(i) If $h^1_\theta(\overline{F})$ is well-defined and finite, then for every Euclidean seminorm $\|.\|'$ over $F_{\mathbb{R}}$ such that:

$$\|.\|' \leq \|.\|, \tag{8.4.27}$$

the θ-invariant $h^1_\theta(\overline{F}')$ of $\overline{F}' := (F, \|.\|')$ is well-defined and finite. In particular, for every $\varepsilon \in \mathbb{R}_+$, the θ-invariant $h^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon))$ of $\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon) := (F, e^{-\varepsilon}\|.\|)$ is well-defined and finite.

(ii) Conversely, if there exists $\varepsilon \in \mathbb{R}^*_+$ such that $\overline{F} \otimes \overline{\mathcal{O}}(-\varepsilon) := (F, e^{\varepsilon}\|.\|)$ is θ^1-summable, then $h^1_\theta(\overline{F})$ is well-defined and finite.

Observe that, according to the monotonicity properties of h^1_θ and its lower and upper extensions, under the assumption of (i) we also have:

$$h^1_\theta(\overline{F}') \leq h^1_\theta(\overline{F}).$$

Proof. Assertion (i) is a special case of Proposition 8.4.10.

Conversely observe that when, for some positive ε, $\overline{F} \otimes \overline{\mathcal{O}}(-\varepsilon)$ is θ^1-summable, then $\overline{F}$ also is and the measure $\beta_{\overline{F}^\vee}$ is supported by $F_{\mathbb{R}}^{\vee \mathrm{Hilb}}$ according to Corollary 8.3.17. Assertion (ii) follows therefore from the characterization of the objects $\overline{F}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ such that $h^1_\theta(\overline{F})$ is well-defined and finite by condition (ii) in Theorem 8.4.7. □

8.4.4. The objects $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$ with a well-defined and finite θ-invariant $h^1_\theta(\overline{\mathcal{F}})$. At this stage, the characterization in Theorem 8.4.7 of objects in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ with well-defined and finite invariant h^1_θ easily extends to the situation where an arbitrary arithmetic curve X replaces $\operatorname{Spec}\mathbb{Z}$.

Theorem 8.4.12. *Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, and let $\overline{F} := \pi_*\overline{\mathcal{F}}$ be its direct image in the following conditions are equivalent:*

(i) $\underline{h}^1_\theta(\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}}) < +\infty$.
(ii) $\overline{\mathcal{F}}$ *is θ^1-summable and* $\underline{h}^1_\theta(\overline{F}) = \overline{h}^1_\theta(\overline{F}) < +\infty$.
(iii) $\overline{\mathcal{F}}$ *is θ^1-summable and the measure $\beta_{\overline{F}^\vee}$ is supported by $F^\vee \cap F_{\mathbb{R}}^{\vee \mathrm{Hilb}}$.*
(iv) $\overline{\mathcal{F}}$ *is θ^1-summable and the measure $\beta_{\overline{F}^\vee}$ is a discrete measure.*
(v) $\overline{\mathcal{F}}$ *is θ^1-summable and the following condition is satisfied:*

$$\lim_{\varepsilon \to 0_+} \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)) = \overline{h}^1_\theta(\overline{\mathcal{F}}). \tag{8.4.28}$$

Proof. When $\overline{\mathcal{F}}$ is θ^1-summable, then according to Proposition 8.2.6, the following equalities hold:

$$\underline{h}^1_\theta(\overline{F}) = \underline{h}^1_\theta(\overline{\mathcal{F}}) \quad \text{and} \quad \overline{h}^1_\theta(\overline{F}) = \overline{h}^1_\theta(\overline{\mathcal{F}}).$$

Moreover, for every $\varepsilon \in \mathbb{R}_+$, $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)$ also is θ^1-summable, and therefore:

$$\overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)) = \overline{h}^1_\theta(\pi_*(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)) = \overline{h}^1_\theta(\overline{F} \otimes \overline{\mathcal{O}}(\varepsilon)).$$

Besides, when (i) holds, then $\overline{\mathcal{F}}$ is θ^1-summable as recalled in Subsection 8.2.3.

Together with Theorem 8.4.7, these observations imply the equivalence of conditions (i)–(v). □

We may generalize the definition 8.4.8 by declaring an object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ to have a *θ-invariant $h^1_\theta(\overline{\mathcal{F}})$ well-defined and finite* when the equivalent conditions in Theorem 8.4.12 are satisfied. This holds precisely when $\overline{\mathcal{F}}$ is θ^1-summable and when the θ-invariant $h^1_\theta(\pi_*\overline{\mathcal{F}})$ is well-defined and finite, and then we let:

$$h^1_\theta(\overline{\mathcal{F}}) := \underline{h}^1_\theta(\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}}) = h^1_\theta(\pi_*\overline{\mathcal{F}}).$$

Recall also that, for every object $\overline{\mathcal{F}}$ in $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following equalities hold:

$$\underline{h}^1_\theta(\overline{\mathcal{F}}) = \underline{h}^1_\theta(\overline{\mathcal{F}}^{\vee\vee}) \quad \text{and} \quad \overline{h}^1_\theta(\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}}^{\vee\vee});$$

see Proposition 8.2.2 and 8.2.3. Therefore the θ-invariant $h^1_\theta(\overline{\mathcal{F}})$ well-defined and finite if and only if $h^1_\theta(\overline{\mathcal{F}}^{\vee\vee})$ is well-defined and finite.

Propositions 8.4.10 and 8.4.11 also immediately extends to objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$. For future reference, we spell out the following generalization of Proposition 8.4.11 (ii), which immediately follows from its original version combined with the previous observations.

Proposition 8.4.13. *Let $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x \in X(\mathbb{C})})$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$. If there exists $\varepsilon \in \mathbb{R}^*_+$ such that*

$$\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\varepsilon) := (\mathcal{F}, (e^\varepsilon \|.\|_x)_{x \in X(\mathbb{C})})$$

is θ^1-summable, then $h^1_\theta(\overline{\mathcal{F}})$ is well-defined and finite.

8.4.5. Two questions. One may wonder whether, in conditions (ii)-(iv) in Theorem 8.4.7, the conditions on the measure $\beta_{\overline{F}^\vee}$, or the condition (8.4.28), are actually necessary, or equivalently whether there exists a θ^1-summable Euclidean quasi-coherent sheaf $\overline{F}$ such that $\underline{h}^1_\theta(\overline{\mathcal{F}}) < \overline{h}^1_\theta(\overline{\mathcal{F}})$. This seems quite likely, although we do not know any example yet.

Another question, suggested by Theorem 3.2.7 (3), is whether, for an object $\overline{F}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following implication holds:

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) < +\infty \overset{?}{\Longrightarrow} \forall \varepsilon > 0,\ \overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)\ \theta^1\text{-summable}.$$

According to Proposition 8.4.11 (ii), this would imply:

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) < +\infty \overset{?}{\Longrightarrow} \forall \varepsilon > 0,\ \underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)) = \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)) < +\infty.$$

8.5. θ^1-finite Hermitian Quasi-coherent Sheaves

8.5.1. Defining θ^1-finite Hermitian quasi-coherent sheaves. The following proposition is a straightforward consequences of the results on the theta invariants $\overline{h}^1_\theta$ and $\underline{h}^1_\theta$ established so far.

PROPOSITION 8.5.1. *For every object $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})})$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following conditions are equivalent:*

(i) For every δ in $\mathbb{R}$, $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ is θ^1-summable.
(ii) For every δ in $\mathbb{R}$, $\underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)) = \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)) < +\infty$.
(iii) There exists a family $(\delta_i)_{i\in I}$ of real numbers such that $\sup_{i\in I} \delta_i = +\infty$ and $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_i)$ is θ^1-summable for every $i \in I$.
(iv) There exists a family $(\delta_i)_{i\in I}$ of real numbers such that $\sup_{i\in I} \delta_i = +\infty$ and

$$\underline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_i)) = \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta_i)) < +\infty \quad \text{for every } i \in I.$$

(v) For every family $(\|.\|'_x)_{x\in X(\mathbb{C})}$ invariant under complex conjugation of Hermitian semi-norms on the $\mathbb{C}$-vector spaces $(\mathcal{F}_x)_{x\in X(\mathbb{C})}$ such that $\|.\|'_x/\|.\|_x$ is bounded[10] *for each x in $X(\mathbb{C})$, the Hermitian quasi-coherent sheaf $\overline{\mathcal{F}}' := (\mathcal{F}, (\|.\|'_x)_{x\in X(\mathbb{C})})$ is θ^1-summable.*
(vi) For every family $(\|.\|'_x)_{x\in X(\mathbb{C})}$ invariant under complex conjugation of Hermitian semi-norms on the $\mathbb{C}$-vector spaces $(\mathcal{F}_x)_{x\in X(\mathbb{C})}$ such that $\|.\|'_x/\|.\|_x$ is bounded for each x in $X(\mathbb{C})$, the Hermitian quasi-coherent sheaf $\overline{\mathcal{F}}' := (\mathcal{F}, (\|.\|'_x)_{x\in X(\mathbb{C})})$ satisfies:

$$\underline{h}^1_\theta(\overline{\mathcal{F}}') = \overline{h}^1_\theta(\overline{\mathcal{F}}') < +\infty.$$

(vii) For every δ in $\mathbb{R}$, there exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ such that the following condition holds:

$$\sum_{i\in\mathbb{N}} h^1_\theta(\overline{\mathcal{C}_{i+1}/\mathcal{C}_i} \otimes \overline{\mathcal{O}}(-\delta)) < +\infty. \tag{8.5.1}$$

(viii) There exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by elements of $\mathrm{coh}(\mathcal{F})$ such that (8.5.1) holds for every δ in $\mathbb{R}$.
(ix) For every δ in $\mathbb{R}$, $\mathrm{ev}\overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)) = 0$.

PROOF. The equivalence of condition (i) to (vi) follows from the basic properties of θ^1-summable objects in $\mathbf{q}\overline{\mathbf{Coh}}_X$ stated in Subsections 8.2.2 and 8.2.3 and from Proposition 8.4.13.

Conditions (ii) and (vii) are equivalent by the definition of θ^1-summability. The implication $(viii) \Rightarrow (vii)$ is clear, and the converse implication $(vii) \Rightarrow (viii)$ and the equivalence $(vii) \Leftrightarrow (ix)$ follow from Proposition 5.6.4 applied with $\varphi = h^1_\theta$. □

[10] Recall that this means that there exists C_x in $\mathbb{R}_+$ such that $\|.\|'_x \leq C_x \|.\|_x$.

DEFINITION 8.5.2. An object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is called *θ^1-finite* when it satisfies the equivalent conditions in Proposition 8.5.1.

The full subcategories of the categories $\mathbf{q}\overline{\mathbf{Coh}}_X$ and $\mathbf{q}\overline{\mathbf{Coh}}^{\leq 1}$ defined by the θ^1-finite objects will be denoted by θ^1_f-$\mathbf{q}\overline{\mathbf{Coh}}_X$ and θ^1_f-$\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$.

These categories are the special instances of the categories φ_f-$\mathbf{q}\overline{\mathbf{Coh}}_X$ and φ_f-$\mathbf{q}\overline{\mathbf{Coh}}_X^{\leq 1}$ introduced in Subsection 5.6.2 when $\varphi = h^1_\theta$. Observe that the general formalism in *loc. cit.* covers the equivalence of of conditions (i), (iii), (v), (vii), and (ix) in Proposition 8.5.1. The equivalence between these conditions and conditions (ii), (iv), and (vi) relies on the more delicate results concerning Banaszczyk measures established in the previous sections.

The θ^1-finite objects of $\mathbf{q}\overline{\mathbf{Coh}}_X$ are precisely those objects $\overline{\mathcal{F}}$ such that, for every $\delta \in \mathbb{R}$, the θ-invariant $h^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta))$ is well-defined and finite. The terminology is parallel to the one in 8.1.4, where we introduced the notion of θ^0-finiteness. Let us emphasize that, while the definition of θ^0-finiteness is rather straighforward, the proof of the equivalence between the various conditions defining θ^1-finiteness uses virtually all the results established in Chapters 4, 5, and 7.

8.5.2. Permanence properties of θ^1-finiteness. As a consequence of the permanence properties of θ^1-summability stated in Section 8.2, θ^1-finiteness satisfies various permanence properties. It is notably preserved under general constructions of quotients and extensions.

PROPOSITION 8.5.3. *Let $f : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ be a morphism in $\mathbf{q}\overline{\mathbf{Coh}}_X$. If $f_K : \mathcal{F}_K \to \mathcal{G}_K$ is surjective and if $\overline{\mathcal{F}}$ is θ^1-finite, then $\overline{\mathcal{G}}$ is θ^1-finite.*

PROOF. This follows from Theorem 8.2.3 (2). □

When $\mathcal{F} = \mathcal{G}$ and f is the identity morphism, Proposition 8.5.3 becomes the following statement:

COROLLARY 8.5.4. *Let $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x \in X(\mathbb{C})})$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$, and let $(\|.\|'_x)_{x \in X(\mathbb{C})}$ be a family, invariant under complex conjugation, of Hermitian seminorms of the $\mathbb{C}$-vector spaces $(\mathcal{F}_x)_{x \in X(\mathbb{C})}$.*

If $\overline{\mathcal{F}}$ is θ^1-finite and if, for every $x \in X(\mathbb{C})$, $\|.\|'_x$ is bounded with respect to $\|.\|_x$,[11] then $\overline{\mathcal{F}}' := (\mathcal{F}, (\|.\|'_x)_{x \in X(\mathbb{C})})$ is θ^1-finite.

In particular, the θ^1-finiteness of an object $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x \in X(\mathbb{C})})$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ depends only on the topology defined by the seminorms $(\|.\|_x)_{x \in X(\mathbb{C})}$ on the $\mathbb{C}$-vector spaces $(\mathcal{F}_x)_{x \in X(\mathbb{C})}$.

PROPOSITION 8.5.5. *Let $\overline{\mathcal{F}}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and let D be a subset of $\mathbb{R}$ such that $\sup D = +\infty$. If $\mathcal{F}_\bullet := (\mathcal{F}_i)_{i \in \mathbb{N}}$ is an exhaustive filtration of $\mathcal{F}$ be quasi-coherent $\mathcal{O}_X$-submodules such that the following conditions are satisfied:*

(i) for every $i \in \mathbb{N}$, the quotient[12] $\overline{\mathcal{F}_i/\mathcal{F}_{i-1}}$ is θ^1-finite, and
(ii) for every $\delta \in D$, $\sum_{i \in \mathbb{N}} h^1_\theta(\overline{\mathcal{F}_i/\mathcal{F}_{i-1}} \otimes \overline{\mathcal{O}}(-\delta)) < +\infty$,

then $\overline{\mathcal{F}}$ is θ^1-finite. Moreover, for every $\delta \in \mathbb{R}$,

$$h^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)) = \lim_{i \to +\infty} h^1_\theta(\overline{\mathcal{F}}_i \otimes \overline{\mathcal{O}}(-\delta)).$$

PROOF. Let us assume that the exhaustive filtration $\mathcal{F}_\bullet$ satisfies condition (i) and (ii). For every δ_D, Theorem 8.2.3 applied to $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ and to the filtration $\mathcal{F}_\bullet$ shows that $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ is θ^1-summable and that the following relations hold:

$$\overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)) = \lim_{i \to +\infty} h^1_\theta(\overline{\mathcal{F}}_i \otimes \overline{\mathcal{O}}(-\delta)) < +\infty.$$

[11] Namely, if there exists $C_x \in \mathbb{R}_+$ such that $\|.\|'_x \leq C_x \|.\|_x$.

[12] As usual, we let $\mathcal{F}_{-1} = 0$.

In particular $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ satisfies condition (iii) in Proposition 8.5.1. This establishes that $\overline{\mathcal{F}}$ is θ^1-finite and completes the proof. □

REMARK 8.5.6. As a special case of Proposition 8.5.5, we see that the property of being θ^1-finite is preserved by admissible extensions. Namely, *for every admissible short exact sequence in* $\mathbf{q}\overline{\mathbf{Coh}}_X$*:*

$$0 \longrightarrow \overline{\mathcal{F}}' \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}'' \longrightarrow 0, \tag{8.5.2}$$

if $\overline{\mathcal{F}}'$ *and* $\overline{\mathcal{F}}''$ *are* θ^1*-finite, then* $\overline{\mathcal{F}}$ *also is* θ^1*-finite.*

In particular, if $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x\in X(\mathbb{C})})$ is an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$ and *if* $\mathcal{F}'$ *is a* $\mathcal{O}_X$*-submodule of* $\mathcal{F}$ *such that, for every* $x \in X(\mathbb{C})$, $\mathcal{F}'_x$ *is dense in* $(\mathcal{F}_x, \|.\|_x)$ *and if* $\overline{\mathcal{F}}' := (\mathcal{F}', (\|.\|_{x|\mathcal{F}'_x})_{x\in X(\mathbb{C})})$ *is* θ^1*-finite, then* $\overline{\mathcal{F}}$ *is* θ^1*-finite.*

Indeed $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}'$ fits into an admissible short exact sequence (8.5.2) in $\mathbf{q}\overline{\mathbf{Coh}}_X$ where the Hermitian norms defining $\overline{\mathcal{F}}''$ are all zero, and therefore $\overline{\mathcal{F}}''$ is trivially θ^1-finite.

PROPOSITION 8.5.7. *If* $\overline{\mathcal{F}}$ *is a* θ^1*-finite object of* θ^1_f-$\mathbf{q}\overline{\mathbf{Coh}}_X$ *and if* $\overline{\mathcal{C}}$ *is an object of* $\overline{\mathbf{Coh}}_X$*, then their tensor product* $\overline{\mathcal{F}} \otimes \overline{\mathcal{C}}$ *in* $\mathbf{q}\overline{\mathbf{Coh}}_X$ *is* θ^1*-finite.*

PROOF. This follows from Proposition 8.2.5, (3), applied to $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ with $\delta \in \mathbb{R}_+$ arbitrarily large. □

8.5.3. Compatibility of θ^1-finiteness with inverse and direct images.

8.5.3.1. In this paragraph, we consider a finite extension K' of the number field K. We denote by $X' := \operatorname{Spec} \mathcal{O}_{K'}$ the arithmetic curve defined by its ring of integers $\mathcal{O}_{K'}$ and by

$$f : X' \longrightarrow X$$

the finite and flat morphism of schemes defined by the inclusion morphism $\mathcal{O}_K \hookrightarrow \mathcal{O}_{K'}$.

A finite family $(\alpha_i)_{1\leq i\leq N}$ of generators of the $\mathcal{O}_K$-module $\mathcal{O}_{K'}$ defines a surjective morphism of $\mathcal{O}_X$-modules:

$$p := (\alpha_1, \dots, \alpha_N) : \mathcal{O}_X^{\oplus N} \longrightarrow f_*\mathcal{O}_{X'}.$$

If $\eta \in \mathbb{R}_+$ is large enough, then p defines a morphism

$$p : \overline{\mathcal{O}}_X(-\eta)^{\oplus N} \longrightarrow f_*\overline{\mathcal{O}}_{X'}$$

in $\overline{\mathbf{Coh}}_X^{\leq 1}$. From now one, we assume $(\alpha_i)_{1\leq i\leq N}$ and η to have been chosen so that these conditions are satisfied. %jbgOn pourrait demander seulement la surjectivité au point générique. La suite marcherait encore. Mais l'exposé devient ainsi un peu pédant... Une footnote ?

We do not expect any simple compatibility between the invariants h^0_θ and h^1_θ and the pull-back functors:

$$f^* : \overline{\mathbf{Coh}}_X \longrightarrow \overline{\mathbf{Coh}}_{X'} \quad \text{and} \quad f^* : \overline{\mathbf{Coh}}_X \longrightarrow \overline{\mathbf{Coh}}_{X'}.$$

It is however possible to establish the following simple results.

PROPOSITION 8.5.8. *(1) For every* $\overline{\mathcal{F}}$ *in* $\overline{\mathbf{Coh}}_X$*, the following relations hold:*

$$h^1_\theta(f^*\overline{\mathcal{F}}) = h^1_\theta(\overline{\mathcal{F}} \otimes f_*\overline{\mathcal{O}}_{X'}) \leq N\, h^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}_X(-\eta)). \tag{8.5.3}$$

(2) Let $\overline{\mathcal{F}}$ *be an object of* $\mathbf{q}\overline{\mathbf{Coh}}_X$*. If* $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)$ *is* θ^1*-summable, then* $f^*\overline{\mathcal{F}}$ *is* θ^1*-summable, and the following relations hold:*

$$\overline{h}^1_\theta(f^*\overline{\mathcal{F}}) = \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes f_*\overline{\mathcal{O}}_{X'}) \leq N\, \overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)). \tag{8.5.4}$$

(3) If $\overline{\mathcal{F}}$ *is* θ^1*-finite, then* $f^*\overline{\mathcal{F}}$ *is* θ^1*-finite.*

PROOF. The definition of h^1_θ shows that:

$$h^1_\theta(f^*\overline{\mathcal{F}}) = h^1_\theta(f_*f^*\overline{\mathcal{F}}).$$

Moreover $f_*f^*\overline{\mathcal{F}}$ is canonically isomorphic to $\overline{\mathcal{F}} \otimes f_*\overline{\mathcal{O}}_{X'}$. This proves the first equality in (8.5.3).

Like the map p, the tensor product

$$\mathrm{Id}_{\mathcal{F}} \otimes p : \overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)^{\oplus N} \longrightarrow \overline{\mathcal{F}} \otimes f_*\overline{\mathcal{O}}_X$$

is a morphism in $\overline{\mathbf{Coh}}_X^{\leq 1}$ with surjective underlying morphism of $\mathcal{O}_X$-modules. The monotonicity and the additivity of h^1_θ implies therefore:

$$h^1_\theta(\overline{\mathcal{F}} \otimes f_*\overline{\mathcal{O}}_X) \leq h^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)^{\oplus N}) = N\, h^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)).$$

This completes the proof of (1).

To prove (2), let us assume that $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)$ is θ^1-summable, and let us choose an exhaustive filtration $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by $\mathcal{O}_X$-submodules in $\mathrm{coh}(\mathcal{F})$ such that the following summability condition is satisfied:

$$\Sigma_{h^1_\theta}(\mathcal{C}_\bullet, \overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)) := \sum_{i=0}^{+\infty} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{O}}(-\eta)) < +\infty.$$

The estimates (8.5.3), applied to $\overline{\mathcal{C}}_i$ and to $\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}$, imply the following inequalities, valid for every $i \in \mathbb{N}$:

$$h^1_\theta(f^*\overline{\mathcal{C}}_i) \leq N\, h^1_\theta(\overline{\mathcal{C}}_i \otimes \overline{\mathcal{O}}(-\eta)), \tag{8.5.5}$$

and:

$$h^1_\theta(f^*\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) \leq N\, h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{O}}(-\eta)).$$

Therefore we have:

$$\Sigma_{h^1_\theta}(f^*\mathcal{C}_\bullet, f^*\overline{\mathcal{F}}) := \sum_{i=0}^{+\infty} h^1_\theta(f^*\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) \leq N \sum_{i+0}^{+\infty} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{O}}(-\eta)) = N\, \Sigma_{h^1_\theta}(\mathcal{C}_\bullet, \overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)) < +\infty.$$

Since $f^*\mathcal{C}_\bullet := (f^*\mathcal{C}_i)_{i\in\mathbb{N}}$ is an exhaustive filtration of $f^*\mathcal{F}$ by elements of $\mathrm{coh}(f^*\mathcal{F})$. This shows that $f^*\overline{\mathcal{F}}$, like $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)$, is θ^1-summable.

Moreover, according to Theorem 8.2.1, we have:

$$\overline{h}^1_\theta(f^*\overline{\mathcal{F}}) = \lim_{i\to+\infty} h^1_\theta(f^*\overline{\mathcal{C}}_i)$$

and:

$$\overline{h}^1_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)) = \lim_{i\to+\infty} h^1_\theta(\overline{\mathcal{C}}_i \otimes \overline{\mathcal{O}}(-\eta)).$$

The inequality (8.5.4) follows therefore from (8.5.5).

Finally (3) follows from (2) applied to $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ with $\delta \in \mathbb{R}_+$ arbitrary large. □

8.5.3.2. Observe that the compatibility of θ^1-summability and direct images stated in Proposition 8.2.6 shows that, for every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following implication holds:

$$\overline{\mathcal{F}}\ \theta^1\text{-finite} \Longrightarrow \pi_*\overline{\mathcal{F}}\ \theta^1\text{-finite}. \tag{8.5.6}$$

In this paragraph, we shall show that this implication is actually an equivalence.

As above, with now π instead of f, let us choose a positive integer N, η in $\mathbb{R}_+$, and a morphism

$$p : \overline{\mathcal{O}}(-\eta)^{\oplus N} \longrightarrow \pi_*\overline{\mathcal{O}}_X$$

in $\overline{\mathbf{Coh}}_X^{\leq 1}$ such the underlying morphism of $\mathcal{O}_X$-modules

$$p : \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}^{\oplus N} \to \pi_*\mathcal{O}_X,$$

or equivalently the morphism of $\mathbb{Z}$-modules:

$$p : \mathbb{Z}^{\oplus N} \longrightarrow \mathcal{O}_K,$$

is surjective. (The integer $N := [K : \mathbb{Q}]$ will do.)

Let moreover $\overline{\mathcal{F}} := (\mathcal{F}, (\|.\|_x)_{x \in X(\mathbb{C})})$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_X$.

The morphism of $\mathbb{Z}$-modules:

$$\mathcal{F}(X) \otimes_{\mathbb{Z}} \mathcal{O}_K \longrightarrow \mathcal{F}(X) \tag{8.5.7}$$

defined by the structure of $\mathcal{O}_K$-modules of $\mathcal{F}(X)$ defines a morphism of $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}}$-modules:

$$\varpi : \pi_*\mathcal{F} \otimes \pi_*\mathcal{O}_X \longrightarrow \pi_*\mathcal{F}. \tag{8.5.8}$$

PROPOSITION 8.5.9. *The morphism ϖ defines a surjective admissible morphism in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$:*

$$\varpi : \pi_*\overline{\mathcal{F}} \otimes \pi_*\overline{\mathcal{O}}_X \longrightarrow \pi_*\overline{\mathcal{F}}.$$

PROOF. We have to show that the Hermitian norm of $\pi_*\overline{\mathcal{F}}_{\mathbb{C}}$ is the quotient norm deduced from the Hermitian norm on $(\pi_*\overline{\mathcal{F}} \otimes \pi_*\overline{\mathcal{O}}_X)_{\mathbb{C}}$ by means of the $\mathbb{C}$-linear map $\varpi_{\mathbb{C}}$. This is a consequence of the following description of these objects in terms of the set $X(\mathbb{C})$ of field embeddings of K in $\mathbb{C}$ and of the Hermitian (seminormed) complex vector spaces $(\mathcal{F}_x, \|.\|_x)$, $x \in X(\mathbb{C})$.

In terms of the isomorphism of $\mathbb{C}$-vector space:

$$\iota = (\iota_x)_{x\in X(\mathbb{C})} : (\pi_*\mathcal{O}_X)_{\mathbb{C}} := \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}^{X(\mathbb{C})}, \quad \alpha\otimes\lambda \longmapsto (\iota_x(\alpha\otimes\lambda)_{x\in X(\mathbb{C})} := (x(\alpha)\lambda))_{x\in X(\mathbb{C})},$$

the Hermitian norm $\|.\|_{\pi_*\overline{\mathcal{O}}_X}$ admits the following expression:

$$\|a\|^2_{\pi_*\overline{\mathcal{O}}_X} := \sum_{x\in X(\mathbb{C})} |\iota_x(a)|^2.$$

The isomorphism ι determines an isomorphism of $\mathbb{C}$-vector spaces:

$$\iota_{\mathcal{F}} := \mathrm{Id}_{\mathcal{F}} \otimes \iota : (\pi_*\mathcal{F})_{\mathbb{C}} := \mathcal{F}(X)\otimes_{\mathbb{Z}}\mathbb{C} \xrightarrow{\sim} \mathcal{F}(X)\otimes_{\mathcal{O}_K}(\mathcal{O}_K\otimes_{\mathbb{Z}}\mathbb{C}) \xrightarrow{\sim} \bigoplus_{x\in X(\mathbb{C})} \mathcal{F}(X)\otimes_x \mathbb{C} =: \bigoplus_{x\in X(\mathbb{C})} \mathcal{F}_x.$$

It maps an element $\varphi \otimes_{\mathbb{Z}} \lambda$ of $\mathcal{F}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ to

$$\iota_{\mathcal{F}}(\varphi\otimes\lambda) = (\iota_{\mathcal{F},x}(\varphi\otimes\lambda))_{x\in X(\mathbb{C})} := (\varphi\otimes_x\lambda)_{x\in X(\mathbb{C})}.$$

In terms of $\iota_{\mathcal{F}}$, the Hermitian seminorm $\|.\|_{\pi_*\overline{\mathcal{F}}}$ admits the following expression:

$$\|f\|^2_{\pi_a st\overline{\mathcal{F}}} = \sum_{x\in X(\mathbb{C})} \|\iota_{\mathcal{F},x}(f)\|^2_x.$$

Using these isomorphisms, the Hermitian $\mathbb{C}$-vector space $(\pi_*\overline{\mathcal{F}} \otimes \pi_*\overline{\mathcal{O}}_X)_{\mathbb{C}}$ may be identified with the direct sum:

$$\bigoplus_{x\in X(\mathbb{C})} \mathcal{F}_x^{X(\mathbb{C})}$$

equipped with the Hermitian norm $\|.\|$ defined by:

$$\|(g_x)_{x\in X(\mathbb{C})}\|^2 := \sum_{(x,x')\in X(\mathbb{C})^2} \|g_x(x')\|^2_x,$$

where $(g_x)_{x\in X(\mathbb{C})}$ denotes a family of maps $g_x : X(\mathbb{C}) \to \mathcal{F}_x$, and the map $\varpi_{\mathbb{C}}$ becomes the "codiagonal morphism":

$$\bigoplus_{x\in X(\mathbb{C})} \mathcal{F}_x^{X(\mathbb{C})} \longrightarrow \bigoplus_{x\in X(\mathbb{C})} \mathcal{F}_x, \quad (g_x)_{x\in X(\mathbb{C})} \longmapsto (g_x(x))_{x\in X(\mathbb{C})}.$$

□

If C is a $\mathbb{Z}$-submodule of $\mathcal{F}(X)$, the image of $C \otimes_{\mathbb{Z}} \mathcal{O}_K$ by the map (8.5.7) is the $\mathcal{O}_K$-submodule $\tilde{C}$ of $\mathcal{F}(X)$ generated by C. If we denote by $\mathcal{C}$ (resp. C') the quasi-coherent $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}}$-submodule (resp. $\mathcal{O}_X$-submodule) of $\pi_*\mathcal{F}$ (resp. of $\mathcal{F}$) defined by C (resp. C'), then ϖ defines by restriction a surjective morphism of $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}}$-modules:

$$\varpi_{\mathcal{C}} : \mathcal{C} \otimes \pi_*\mathcal{O}_X \longrightarrow \pi_*\mathcal{C}'. \tag{8.5.9}$$

Lemma 8.5.10. *With the above notation, if $\mathcal{C}$ belongs to $\operatorname{coh}(\pi_*\mathcal{F})$, then $\mathcal{C}'$ belongs to $\operatorname{coh}(\mathcal{F})$, and the map $\varpi_{\mathcal{C}}$ defines a morphism in $\overline{\mathbf{Coh}}_{\mathbb{Z}}^{\leq 1}$:*

$$\varpi_{\mathcal{C}} : \overline{\mathcal{C}} \otimes \pi_*\overline{\mathcal{O}}_X \longrightarrow \pi_*\overline{\mathcal{C}}', \tag{8.5.10}$$

with surjective underlying morphism of $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}}$-modules. Moreover the following relations hold:

$$h^1_\theta(\overline{\mathcal{C}}') = h^1_\theta(\pi_*\overline{\mathcal{C}}') \leq h^1_\theta(\overline{\mathcal{C}} \otimes \pi_*\overline{\mathcal{O}}_X) \leq N\, h^1_\theta(\overline{\mathcal{C}} \otimes \overline{\mathcal{O}}(-\eta)) \tag{8.5.11}$$

Proof. The first assertion is an immediate consequence of the above observation and from the fact, established in Proposition 8.5.9, that ϖ has an operator norm at most 1. The first inequality in (8.5.11) follows from the monotonicity of h^1_θ on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, and the second one from Proposition 8.2.5 applied to $(\operatorname{Spec}\mathbb{Z}, \pi_*\overline{\mathcal{O}}_X, \overline{\mathcal{C}})$ instead of $(X, \overline{\mathcal{C}}, \overline{\mathcal{F}})$. □

Proposition 8.5.11. *For every object $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$, the following estimates hold:*

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) \leq N\, \overline{h}^1_\theta(\pi_*\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)), \tag{8.5.12}$$

and:

$$\operatorname{ev}\overline{h}^1_\theta(\overline{\mathcal{F}}) \leq N\, \operatorname{ev}\overline{h}^1_\theta(\pi_*\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)) \tag{8.5.13}$$

Proof. As observed in Proposition 4.3.3, we may choose an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\pi_*\mathcal{F}$ by $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}}$-submodules in $\operatorname{coh}(\pi_*\mathcal{F})$ such that:

$$\overline{h}^1_\theta(\pi_*\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)) = \lim_{i\to+\infty} h^1_\theta(\overline{\mathcal{C}}_i \otimes \overline{\mathcal{O}}(-\eta)). \tag{8.5.14}$$

Lemma 8.5.10 shows that the invariant h^1_θ of the coherent $\mathcal{O}_X$-submodules $\mathcal{C}'_i$ of $\mathcal{F}$ generated by the $\mathcal{C}_i$ satisfy the following estimates:

$$h^1_\theta(\overline{\mathcal{C}}'_i) \leq N\, h^1_\theta(\overline{\mathcal{C}}_i \otimes \overline{\mathcal{O}}(-\eta)). \tag{8.5.15}$$

Moreover $(\mathcal{C}'_i)$ is clearly an exhaustive filtration of $\mathcal{F}$ by $\mathcal{O}_X$-submodules in $\operatorname{coh}(\mathcal{F})$, and consequently:

$$\overline{h}^1_\theta(\overline{\mathcal{F}}) \leq \liminf_{i\to+\infty} h^1_\theta(\overline{\mathcal{C}}'_i). \tag{8.5.16}$$

The inequality (8.5.12) follows from (8.5.14), (8.5.15), and (8.5.16).

Observe that the direct image define an increasing map

$$\pi_* : \operatorname{coh}(\mathcal{F}) \longrightarrow \operatorname{coh}(\pi_*\mathcal{F}), \quad \mathcal{C} \longmapsto \pi_*\mathcal{C},$$

the image of which is cofinal in the directed set $(\operatorname{coh}(\pi_*\mathcal{F}), \subseteq)$. This establishes the following expression for $\operatorname{ev}\overline{h}^1_\theta(\pi_*\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta))$:

$$\begin{aligned}\operatorname{ev}\overline{h}^1_\theta(\pi_*\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)) &:= \inf_{\mathcal{C}'\in\operatorname{coh}(\pi_*\mathcal{F})} \overline{h}^1_\theta(\overline{\pi_*\mathcal{F}/\mathcal{C}'} \otimes \overline{\mathcal{O}}(-\eta)) \\ &= \inf_{\mathcal{C}\in\operatorname{coh}(\mathcal{F})} \overline{h}^1_\theta(\overline{\pi_*\mathcal{F}/\pi_*\mathcal{C}} \otimes \overline{\mathcal{O}}(-\eta)) = \inf_{\mathcal{C}\in\operatorname{coh}(\mathcal{F})} \overline{h}^1_\theta(\pi_*\overline{\mathcal{F}/\mathcal{C}} \otimes \overline{\mathcal{O}}(-\eta)).\end{aligned} \tag{8.5.17}$$

Moreover the estimate (8.5.12) shows that, for every $\mathcal{C}$ in $\operatorname{coh}(\mathcal{F})$, the following inequality holds:

$$h^1_\theta(\overline{\mathcal{F}/\mathcal{C}}) \leq N\, \overline{h}^1_\theta(\pi_*\overline{\mathcal{F}/\mathcal{C}} \otimes \overline{\mathcal{O}}(-\eta)). \tag{8.5.18}$$

The inequality (8.5.13) follows from (8.5.17) and (8.5.18). □

COROLLARY 8.5.12. *Let* $\overline{\mathcal{F}}$ *be an object of* $\mathbf{q}\overline{\mathbf{Coh}}_X$.
(1) If $\pi_*\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\eta)$ *is* θ^1*-summable, then* $\overline{\mathcal{F}}$ *is* θ^1*-summable.*
(2) If $\pi_*\overline{\mathcal{F}}$ *is* θ^1*-finite, then* $\overline{\mathcal{F}}$ *is* θ^1*-finite.*

PROOF. Assertion (1) follows from the inequality (8.5.13) and from the characterization of the θ^1-summability by the vanishing of $\mathrm{ev}h^1_\theta$ stated in Theorem 8.2.1. Assertion (2) follows from assertion (1) applied to $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\delta)$ with $\delta \in \mathbb{R}_+$ arbitrarily large. □

8.5.3.3. The results of the last paragraphs may be extended to the situation where, instead of the direct images by the morphism
$$\pi : X \longrightarrow \operatorname{Spec} \mathbb{Z},$$
one considers direct images under an arbitrary morphism of arithmetic curves
$$f : X' \longrightarrow X$$
as in 8.5.3.1. We shall leave the details to the interested reader, and content ourselves with the following simple consequence of our previous results.

PROPOSITION 8.5.13. *With the notation of 8.5.3.1, an object* $\overline{\mathcal{F}}$ *of* $\mathbf{q}\overline{\mathbf{Coh}}_{X'}$ *is* θ^1*-finite if and only if the object* $f_*\overline{\mathcal{F}}$ *of* $\mathbf{q}\overline{\mathbf{Coh}}_X$ *is* θ^1*-finite.*

PROOF. Recall that, together with the implication 8.5.6, Corollary 8.5.12 (2) show that an object $\overline{\mathcal{G}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ is θ^1-finite if and only if $\pi_*\overline{\mathcal{G}}$ is θ^1-finite. Applied to the morphism:
$$\pi' := \pi \circ f : X' \longrightarrow \operatorname{Spec} \mathbb{Z}$$
instead of π, this shows that $\overline{\mathcal{F}}$ is θ^1-finite if and only if $\pi'_*\overline{\mathcal{F}}$ is θ^1-finite.

Moreover there exists a canonical isomorphism:
$$\pi'_*\overline{\mathcal{F}} \xrightarrow{\sim} \pi_*(f_*\overline{\mathcal{F}}).$$

This implies the successive equivalence of the following conditions:

(i) $\overline{\mathcal{F}}$ is θ^1-finite;
(ii) $\pi'_*\overline{\mathcal{F}}$ is θ^1-finite;
(iii) $\pi_*(f_*\overline{\mathcal{F}})$ is θ^1-finite;
(iv) $f_*\overline{\mathcal{F}}$ is θ^1-finite. □

8.6. Conjoint θ^0- and θ^1-Finiteness

In this final section, we discuss the possible existence of objects $\overline{\mathcal{F}}$ of $\mathbf{q}\overline{\mathbf{Coh}}_X$ whose both theta invariants $h^0_\theta(\overline{\mathcal{F}})$ and $h^1_\theta(\overline{\mathcal{F}})$ are finite, or such that $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(\varepsilon)$ satisfy these finiteness property for some nonzero ε. We begin by some observations on the geometric analogue of these questions, concerning quasi-coherent sheaves over a projective curve.

8.6.1. Preliminary: the geometric case. Let us return to the framework of Chapter 3. Namely we consider a smooth, projective, geometrically connected curve C of genus g over some base field k, and we study quasi-coherent $\mathcal{O}_C$-modules $\mathcal{F}$ over C, and their invariant $h^1(C, \mathcal{F})$ and $\overline{h}^1(C, \mathcal{F})$.

Beside these invariants defined in terms of the first cohomology groups, we may also consider the dimension of the space of sections:
$$h^0(C, \mathcal{F}) := \dim_k \Gamma(C, \mathcal{F})).$$

The function
$$h^0(C,.) : \mathrm{coh}(\mathcal{F}) \longrightarrow \mathbb{N}$$
is increasing on the directed set $(\mathrm{coh}(\mathcal{F}), \subseteq)$ of coherent subsheaves of $\mathcal{F}$, and clearly we have:
$$h^0(C,\mathcal{F}) = \sup_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} h^0(C,\mathcal{C}) = \lim_{\mathcal{C}\in\mathrm{coh}(\mathcal{F})} h^0(C,\mathcal{C}).$$

8.6.1.1. *Quasi-coherent $\mathcal{O}_C$-modules $\mathcal{F}$ such that $h^0(C,\mathcal{F}) < +\infty$.* The following proposition may be seen as an arithmetic counterpart of Proposition 8.1.1. Indeed it implies that, if a quasi-coherent $\mathcal{O}_C$-module of countable type $\mathcal{F}$ satisfies the condition: $h^0(C,\mathcal{F}) < +\infty$, then $\mathcal{F}_{\mathrm{tor}}$ is coherent and $\mathcal{F}_{/\mathrm{tor}}$ is locally free.

PROPOSITION 8.6.1. *Let $\mathcal{F}$ be a quasi-coherent sheaf over C.*

(1) If the $k(C)$-vector space $\mathcal{F}_\eta$ is finite dimensional, then the following two conditions are equivalent:

(i) $h^0(C,\mathcal{F}) < +\infty$;
(ii) $\mathcal{F}$ is a coherent $\mathcal{O}_C$-module.

(2) If the $k(C)$-vector space $\mathcal{F}_\eta$ is has infinite countable dimension and if $h^0(C,\mathcal{F})$ is finite, then the torsion subsheaf $\mathcal{F}_{\mathrm{tor}}$ of $\mathcal{F}$ is coherent, and the quotient $\mathcal{F}_{/\mathrm{tor}} := \mathcal{F}/\mathcal{F}_{\mathrm{tor}}$ admits admits an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ by coherent subsheaves such that each $\mathcal{C}_i$ is locally free, of rank i, and saturated in $\mathcal{F}_{/\mathrm{tor}}$.

Observe that when (2) holds, then for any open affine subscheme U in C, the $\mathcal{O}_C(U)$-module $\mathcal{F}(U)$ is projective, and therefore free (see for instance [**Bos20b**, 4.1]).

We leave the proof of Proposition 8.6.1 as an exercise for the reader.

8.6.1.2. *Quasi-coherent $\mathcal{O}_C$-modules $\mathcal{F}$ such that $h^0(C,\mathcal{F}) < +\infty$ and $\overline{h}^1(C,\mathcal{F}) < +\infty$.* Observe that, under some suitable technical assumptions on the curve C, it is possible to construct quasi-coherent $\mathcal{O}_C$-modules $\mathcal{F}$ of infinite rank[13] such that both conditions:
$$h^0(C,\mathcal{F}) < +\infty \quad \text{and} \quad h^1(C,\mathcal{F}) < +\infty,$$
or even the stronger conditions:
$$h^0(C,\mathcal{F}) < +\infty \quad \text{and} \quad \overline{h}^1(C,\mathcal{F}) < +\infty,$$
are satisfied.

This is demonstrated by the construction in Section 3.4, when $g > 0$. This also follows from the following simple construction.

EXAMPLE 8.6.2. Assume that there exists a line bundle L over C that satisfies the condition:
$$h^0(C,L) = h^1(C,L) = 0. \tag{8.6.1}$$
A line bundle L satisfying (8.6.1) has degree $g-1$, and there exists such a line bundle as soon as both C and $\mathrm{Pic}^{g-1}_{C/k} \setminus \Theta$ — where Θ denotes the theta divisor[14] in $\mathrm{Pic}^{g-1}_{C/k}$ — have a k-rational point. These conditions are satisfied for instance when k is algebraically closed, or when the field k is finite of cardinality larger than some function of g.

Then the quasi-coherent $\mathcal{O}_C$-module $\mathcal{F} := L^{(\mathbb{N})}$, defined as the direct sum of an infinite countable family of copies of L, is clearly locally free of infinite rank and satisfies:
$$h^0(C,\mathcal{F}) = h^1(C,\mathcal{F}) = \overline{h}^1(C,\mathcal{F}).$$

---

[13]Observe that according to Proposition 8.6.1, if $h^0(C,\mathcal{F})$ is finite, $\mathcal{F}$ has infinite rank if and only if it is not coherent.

[14]Recall that $\Theta(\overline{k})$ parametrizes the line bundles M of degree $g-1$ over $C_{\overline{k}}$ such that $h^0(C_{\overline{k}},M) = h^1(C_{\overline{k}},M)$ is positive.

8.6.1.3. In this paragraph, we want to prove that, if a quasi-coherent $\mathcal{O}_C$-module of countable type satisfies $\mathcal{F}$ the conditions:

$$h^0(C,\mathcal{F}) < +\infty \quad \text{and} \quad \overline{h}^1(C,\mathcal{F}) < +\infty,$$

and if $\mathcal{F}$ twisted by a line bundle of sufficiently large degree still satisfies these, then $\mathcal{F}$ is necessarily coherent.

Our proof will rely on the following observation.

LEMMA 8.6.3. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_C$-module and let $\mathcal{C}_\bullet := (\mathcal{C}_i)$ be an exhaustive filtration of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules. If the following two conditions are satisfied:*

$$h^0(C,\mathcal{F}) < +\infty \tag{8.6.2}$$

and

$$h^1(C,\mathcal{C}_{i+1}/\mathcal{C}_i) = 0 \quad \textit{for } i \textit{ large enough,} \tag{8.6.3}$$

then we also have:

$$h^0(C,\mathcal{C}_{i+1}/\mathcal{C}_i) = 0 \quad \textit{for } i \textit{ large enough.} \tag{8.6.4}$$

PROOF. For every $i \in \mathbb{N}$, we may consider the short exact sequence of coherent $\mathcal{O}_C$-modules:

$$0 \longrightarrow \mathcal{C}_i \xrightarrow{\iota_i} \mathcal{C}_{i+1} \xrightarrow{p_i} \mathcal{C}_{i+1}/\mathcal{C}_i \longrightarrow 0,$$

and associated long exact sequences of finite dimensional cohomology groups:

$$\begin{aligned} 0 \longrightarrow H^0(C,\mathcal{C}_i) \xrightarrow{\iota_i^0} H^0(C,\mathcal{C}_{i+1}) \xrightarrow{p_i^0} H^0(C,\mathcal{C}_{i+1}/\mathcal{C}_i) \\ \xrightarrow{\delta_i} H^1(C,\mathcal{C}_i) \xrightarrow{\iota_i^1} H^1(C,\mathcal{C}_{i+1}) \xrightarrow{p_i^1} H^1(C,\mathcal{C}_{i+1}/\mathcal{C}_i) \longrightarrow 0. \end{aligned}$$

According to (8.6.3)), for i large enough the map p_i^1 vanishes and therefore ι_i^1 is surjective. Consequently the sequence $(h^1(C,\mathcal{C}_i))_{i\in\mathbb{N}}$ is eventually decreasing, and therefore eventually constant. This shows that, for i large enough, ι_i^1 is an isomorphism and consequently δ_i vanishes.

According to (8.6.2), the increasing sequence $(h^0(C,\mathcal{C}_i))_{i\in\mathbb{N}}$ is eventually constant and equal to $h^0(C,\mathcal{F})$. This shows that, for i large enough, ι_i^0 is an isomorphism and consequently p_i^0 vanishes.

This establishes that both p_i^0 and δ_i, and therefore $H^0(C,\mathcal{C}_{i+1}/\mathcal{C}_i)$, vanish for i large enough. □

PROPOSITION 8.6.4. *Let L and L' be two line bundles over C such that $\deg_C L > g-1$ and such that L' is generated by its global sections and $\deg_C L' > 0$.*[15] *For every quasi-coherent $\mathcal{O}_C$-module $\mathcal{F}$ of countable type, the following two conditions are equivalent:*

(i) *$\mathcal{F}$ is a coherent $\mathcal{O}_C$-module;*
(ii) *$h^0(C,\mathcal{F}) < +\infty$ and $\overline{h}^1(C,\mathcal{F}\otimes L^\vee \otimes L'^\vee) < \infty$.*

PROOF. According to Theorem 3.2.7 (3) and (2) applied to $\mathcal{F}\otimes L^\vee\otimes L'^\vee$ and to $\mathcal{F}\otimes L'^\vee$, there exists an exhaustive filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of $\mathcal{F}$ by coherent $\mathcal{O}_C$-submodules such that:

$$h^1(C,(\mathcal{C}_{i+1}/\mathcal{C}_i)\otimes L'^\vee) = 0 \quad \text{for } i \text{ large enough.} \tag{8.6.5}$$

Since L' is generated by its global sections, this implies that the coherent $\mathcal{O}_C$-modules

$$\mathcal{C}_{i+1}/\mathcal{C}_i \simeq (\mathcal{C}_{i+1}/\mathcal{C}_i)\otimes L'^\vee)\otimes L'$$

also satisfy:

$$h^1(C,\mathcal{C}_{i+1}/\mathcal{C}_i) = 0 \quad \text{for } i \text{ large enough.} \tag{8.6.6}$$

[15] The line bundle L' satisfies these conditions if $\deg_C L' \geq \max(1,2g)$.

Moreover, since the $\mathcal{O}_C$-module $L'^\vee$ injects into $\mathcal{O}_C$, the finiteness of $h^0(C,\mathcal{F})$ implies the one of $h^0(C,\mathcal{F}\otimes L'^\vee)$.

We may therefore apply Lemma 8.6.3 both to $\mathcal{F}$, equipped with the filtration $(\mathcal{C}_i)_{i\in\mathbb{N}}$, and to $\mathcal{F}\otimes L'^\vee$ equipped with the filtration $(\mathcal{C}_i\otimes L'^\vee)_{i\in\mathbb{N}}$. This implies that

$$\chi(C,\mathcal{C}_{i+1}/\mathcal{C}_i) := h^0(C,\mathcal{C}_{i+1}/\mathcal{C}_i) - h^1(C,\mathcal{C}_{i+1}/\mathcal{C}_i) \tag{8.6.7}$$

vanishes for i large enough, and $\chi(C,(\mathcal{C}_{i+1}/\mathcal{C}_i)\otimes L'^\vee)$ as well.

Moreover, according to the Riemann-Roch formula, the following equality holds:

$$\chi(C,\mathcal{C}_{i+1}/\mathcal{C}_i) - \chi(C,(\mathcal{C}_{i+1}/\mathcal{C}_i)\otimes L'^\vee) = \operatorname{rk}\mathcal{C}_{i+1}/\mathcal{C}_i \,\deg_C L.$$

This shows that, for i large enough, the rank of the coherent sheaf $\mathcal{C}_{i+1}/\mathcal{C}_i$ vanishes, or equivalently that $\mathcal{C}_{i+1}/\mathcal{C}_i$ is a coherent sheaf with finite support.

Since $\mathcal{F}_{\mathrm{tor}}$ is coherent and $\mathcal{F}_{/\mathrm{tor}}$ is locally free according to Proposition 8.6.1, this implies that $\mathcal{F}$ is coherent. □

8.6.2. The objects $\overline{\mathcal{F}}$ in θ^1_Σ-q$\overline{\mathbf{Coh}}_X$ such that $h^0_\theta(\overline{\mathcal{F}})<+\infty$. In this subsection, we establish an arithmetic analogue of Proposition 8.6.4, namely Proposition 8.6.6 below. Its derivation will rely on the following arithmetic counterpart of Lemma 8.6.3, that will also be of crucial when investigating the analogue of Example 8.6.2 in the next subsection.

Proposition 8.6.5. *Let $\overline{\mathcal{F}} := (\mathcal{F},(\|.\|_x)_{x\in X(\mathbb{C})})$ be an object in θ^1_Σ-q$\overline{\mathbf{Coh}}_X$ and let $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ be an exhaustive filtration of $\mathcal{F}$ by objects in* $\mathrm{coh}(\mathcal{F})$ *that satisfies the following θ^1-summability condition:*[16]

$$\Sigma_{h^1_\theta}(\overline{\mathcal{F}},\mathcal{C}_\bullet) := \sum_{i=0}^{+\infty} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty. \tag{8.6.8}$$

If $h^0_\theta(\overline{\mathcal{F}})<+\infty$, then, when i goes to $+\infty$,

$$\widehat{\deg}\,\pi_*\overline{\mathcal{C}}_i = \widehat{\deg}\,\overline{\mathcal{C}}_i - (1/2)\log|\Delta_K|\mathrm{rk}\mathcal{C}_i$$

admits in limit in $\mathbb{R}$. Moreover:

$$\sum_{i=0}^{+\infty} h^0_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty. \tag{8.6.9}$$

Recall that, according to Proposition 8.1.1, the finiteness of $h^0_\theta(\overline{\mathcal{F}})$ implies that the Hermitian seminorms $\|.\|_x$ defining $\overline{\mathcal{F}}$ are actually norms. Therefore the $\overline{\mathcal{C}}_i$ are objects in $\overline{\mathbf{Coh}}_{\overline{X}}$, and have well-defined invariants $h^0_\theta(\overline{\mathcal{C}}_i)$ and $h^1_\theta(\overline{\mathcal{C}}_i)$ in $\mathbb{R}_+$, which satisfy the Poisson-Riemann-Roch formula:

$$h^0_\theta(\mathcal{C}_i) - h^1_\theta(\mathcal{C}_i) = \widehat{\deg}\,\pi_*\overline{\mathcal{C}}_i. \tag{8.6.10}$$

Proof. By the very definition of $h^0_\theta(\overline{\mathcal{F}})$, the increasing sequence $\big(h^0_\theta(\overline{\mathcal{C}}_i)\big)_{i\in\mathbb{N}}$ converges in $\mathbb{R}_+$ to $h^0_\theta(\overline{\mathcal{F}})$. According to Theorem 8.2.1, $\big(h^1_\theta(\overline{\mathcal{C}}_i)\big)_{i\in\mathbb{N}}$ converges in $\mathbb{R}_+$ to $h^1_\theta(\overline{\mathcal{F}})$. Using (8.6.10), this implies that the sequence $\big(\widehat{\deg}\,\pi_*\overline{\mathcal{C}}_i\big)_{i\in\mathbb{N}}$ converges in $\mathbb{R}$ to $h^0_\theta(\overline{\mathcal{F}}) - \overline{h}^1_\theta(\overline{\mathcal{F}})$.

Using the Poisson-Riemann-Roch formula again and the additivity of the Arakelov degree, we obtain, for every $N\in\mathbb{N}$:

$$\sum_{i=0}^{N} h^0_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) = \sum_{i=0}^{N} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) + \sum_{i=0}^{N}\widehat{\deg}\,\pi_*\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} = \sum_{i=0}^{N} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) + \widehat{\deg}\,\pi_*\overline{\mathcal{C}}_N.$$

[16] As usual, we let: $\mathcal{C}_{-1}=0$.

When N goes to infinity, this admits a limit in $\mathbb{R}$, namely:

$$\Sigma_{h^1_\theta}(\overline{\mathcal{F}}, \mathcal{C}_\bullet) + h^0_\theta(\overline{\mathcal{F}}) - \overline{h}^1_\theta(\overline{\mathcal{F}}). \qquad \square$$

PROPOSITION 8.6.6. *For every object $\overline{\mathcal{F}}$ in $\mathbf{q\overline{Coh}}_X$ and every $\varepsilon \in \mathbb{R}^*_+$, the following conditions are equivalent:*

(i) $\overline{\mathcal{F}}$ is an object of $\overline{\mathbf{Coh}}_{\overline{X}}$;
(ii) $h^0_\theta(\overline{\mathcal{F}}) < +\infty$ and $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\varepsilon)$ is θ^1-summable.

PROOF. The implication $(i) \Rightarrow (ii)$ is clear.

Conversely when (ii) is satisfied, since $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\varepsilon)$ is θ^1-summable, there exists an exhaustive filtration $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i \in \mathbb{N}}$ of $\mathcal{F}$ by objects in $\mathrm{coh}(\mathcal{F})$ such that:

$$\Sigma_{h^1_\theta}(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\varepsilon), \mathcal{C}_\bullet) := \sum_{i=0}^{+\infty} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}} \otimes \overline{\mathcal{O}}(-\varepsilon)) < +\infty.$$

According to the monotonicity of h^1_θ, it also satisfies:

$$\Sigma_{h^1_\theta}(\overline{\mathcal{F}}, \mathcal{C}_\bullet) := \sum_{i=0}^{+\infty} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty.$$

Moreover the montonicity of h^0_θ and the finiteness of $h^0_\theta(\overline{\mathcal{F}})$ implies the finiteness of $h^0_\theta(\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\varepsilon))$.

Consequently we may apply Proposition 8.6.5 both to $\overline{\mathcal{F}}$ and to $\overline{\mathcal{F}} \otimes \overline{\mathcal{O}}(-\varepsilon)$, equipped with the filtration $\mathcal{C}_\bullet$. This establishes that, when i goes to infinity, both $\widehat{\deg}\, \pi_* \overline{\mathcal{C}}_i$ and

$$\widehat{\deg}\, \pi_*(\overline{\mathcal{C}}_i \otimes \overline{\mathcal{O}}(-\varepsilon)) = \widehat{\deg}\, \pi_* \overline{\mathcal{C}}_i - \varepsilon [K:\mathbb{Q}] \mathrm{rk}_K \mathcal{C}_{i,K}$$

admit a limit in $\mathbb{R}$.

This implies that $\mathrm{rk}_K \mathcal{C}_{i,K}$ vanishes when i is large enough. Since, according to Proposition 8.1.1, the finiteness of $h^0_\theta(\overline{\mathcal{F}})$ implies that $\mathcal{F}_{\mathrm{tor}}$ is finite and $\mathcal{F}_{/\mathrm{tor}} := \mathcal{F}/\mathcal{F}_{\mathrm{tor}}$ is projective, this implies that $\mathcal{F}$ is coherent. Proposition 8.1.1 also shows that the Hermitian seminorms defining $\overline{\mathcal{F}}$ are actually norms, and this completes the proof of (i). $\square$

Proposition 8.6.6 admits the following straightforward consequence:

COROLLARY 8.6.7. *For every objects $\overline{\mathcal{F}}$ in $\mathbf{q\overline{Coh}}_X$, the following conditions are equivalent:*

(i) $\overline{\mathcal{F}}$ is an object of $\overline{\mathbf{Coh}}_{\overline{X}}$;
(ii) $\overline{\mathcal{F}}$ is θ^0-finite and θ^1-finite.

8.6.3. An intriguing question.

8.6.3.1. For N a positive integer and $\delta \in \mathbb{R}$, we may consider the non-negative real number:

$$h^0_{\theta,\min}(N, \delta) := \inf \left\{ h^0_\theta(\overline{E}); \overline{E} \in \overline{\mathbf{Vect}}_{\mathbb{Z}}, \mathrm{rk}\overline{E} = N, \text{ and } \widehat{\deg}\, \overline{E} = \delta \right\}.$$

Recall that the theta-invariant h^0_θ defines a continuous function on the orbifold $\mathcal{L}(N, \delta)$ of isomorphism classes of Euclidean lattices of rank N and Arakelov degree δ. This function is actually an exhaustion function;[17] this follows from Mahler's compactness criterion — which asserts that the inverse $\lambda_1(\overline{E})^{-1}$ of the first of the successive minima defines an exhaustion function on $\mathcal{L}(N, \delta)$ — combined with the lower-bound:

$$h^0_\theta(\overline{E}) \geq h^0_\theta(\lambda_1(\overline{E})\mathbb{Z}, |.|) \geq \widehat{\deg}\,(\lambda_1(\overline{E})\mathbb{Z}, |.|) = \log \lambda_1(\overline{E})^{-1}.$$

[17] In other words the map $h^0_\theta : \mathcal{L}(N, \delta) \longrightarrow \mathbb{R}_+$ is proper.

Consequently the infimum defining $h^0_{\theta,\min}(N,\delta)$ is attained at some point of $\mathcal{L}(N,\delta)$. In particular, we have:

$$h^0_{\theta,\min}(N,\delta) > 0.$$

Moreover the Poisson-Riemann-Roch formula implies:

$$h^0_{\theta,\min}(N,\delta) > \delta, \tag{8.6.11}$$

and finally we obtain the following lower bound:

$$h^0_{\theta,\min}(N,\delta) > \delta^+ := \max(0,\delta) \tag{8.6.12}$$

Moreover an application of Siegel's Mean Value Theorem allows one to establish the following upper bound:

$$h^0_{\theta,\min}(N,\delta) < \log(1+e^\delta). \tag{8.6.13}$$

We refer to [**Bos20b**, 3.5] for more details on the proof of (8.6.13), and simply emphasize that it is an illustration of the so-called "probabilistic method" applied to spaces of isomorphism classes of Euclidean lattices — which goes back to Minkowski, Hlawka, and Siegel — and accordingly is non-constructive.

Similarly we define:

$$h^1_{\theta,\min}(N,\delta) := \inf\left\{h^1_\theta(\overline{E}); \overline{E} \in \overline{\mathbf{Vect}}_{\mathbb{Z}}, \mathrm{rk}\overline{E} = N, \text{ and } \widehat{\deg}\,\overline{E} = \delta\right\}.$$

and

$$h^{0+1}_{\theta,\min}(N,\delta) := \inf\left\{h^0_\theta(\overline{E}) + h^1_\theta(\overline{E}); \overline{E} \in \overline{\mathbf{Vect}}_{\mathbb{Z}}, \mathrm{rk}\overline{E} = N, \text{ and } \widehat{\deg}\,\overline{E} = \delta\right\}.$$

Here again these number are the infima of positive continuous exhaustion functions, and are therefore attained and positive.

Recall that the Arakelov degree and the theta invariants associated to some Euclidean lattice $\overline{E}$ and of its dual $\overline{E}^\vee$ satisfy the relations:

$$\widehat{\deg}\,\overline{E}^\vee = -\widehat{\deg}\,\overline{E}, \quad \widehat{\deg}\,\overline{E}^{\oplus 2} = 2\,\widehat{\deg}\,\overline{E}, \quad \text{and} \quad \widehat{\deg}\,(\overline{E} \oplus \overline{E}^\vee) = 0,$$

and:

$$h^1_\theta(\overline{E}) = h^0_\theta(\overline{E}^\vee), \quad h^0_\theta(\overline{E}^{\oplus 2}) = 2h^0_\theta(\overline{E}), \quad \text{and} \quad h^0_\theta(\overline{E} \oplus \overline{E}^\vee) = h^0_\theta(\overline{E}) + h^1_\theta(\overline{E}),$$

and the Poisson-Riemann-Roch formula:

$$h^0_\theta(\overline{E}) - h^1_\theta(\overline{E}) = \widehat{\deg}\,\overline{E}.$$

This immediately implies the following proposition:

PROPOSITION 8.6.8. *For every positive integer N and every $\delta \in \mathbb{R}$, the following relations hold:*

$$h^{0+1}_{\theta,\min}(N,0) = 2h^0_{\theta,\min}(N,0),$$

$$h^1_{\theta,\min}(N,\delta) = h^0_{\theta,\min}(N,-\delta),$$

$$2\,h^0_{\theta,\min}(N,\delta) \geq h^0_{\theta,\min}(2N,2\delta),$$

and:

$$h^{0+1}_{\theta,\min}(N,\delta) \geq h^0_{\theta,\min}(2N,0).$$

In turn, Proposition 8.6.8 admits the following straightforward consequence:

COROLLARY 8.6.9. *For every positive integer N, the following estimates hold:*

$$2h^0_{\theta,\min}(N,0) \geq \inf_{\delta\in\mathbb{R}} h^{0+1}_{\theta,\min}(N,\delta) \geq h^0_{\theta,\min}(2N,0).$$

Consequently, we have:

$$2\inf_{N>0} h^0_{\theta,\min}(N,0) \geq \inf_{N>0,\delta\in\mathbb{R}} h^{0+1}_{\theta,\min}(N,\delta) \geq \inf_{N>0} h^0_{\theta,\min}(2N,0) \geq \inf_{N>0} h^0_{\theta,\min}(N,0). \tag{8.6.14}$$

8.6.3.2. The following proposition relates the vanishing of the infima occurring in the estimates (8.6.14) with the existence of some arithmetic counterpart to the constructions of quasi-coherent $\mathcal{O}_C$-modules discussed in paragraph 8.6.1.2 and Example 8.6.2.

PROPOSITION 8.6.10. *The following three conditions are equivalent:*

(i) there exists $\overline{F} := (F, \|.\|)$ *in* $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ *that is* θ^1*-summable, satisfies* $h^0_\theta(\overline{F}) < +\infty$*, and such that* F *has infinite rank.*

(ii) $\inf \left\{ h^0_\theta(\overline{E}) + h^1_\theta(\overline{E}); \overline{E} \in \overline{\mathbf{Vect}}_{\mathbb{Z}}, \overline{E} \neq \{0\} \right\} = 0;$

(iii) $\inf \left\{ h^0_\theta(\overline{E}); \overline{E} \in \overline{\mathbf{Vect}}_{\mathbb{Z}}, \overline{E} \neq \{0\} \text{ and } \widehat{\deg}\, \overline{E} = 0 \right\} = 0.$

Concerning condition (i), recall that, according to Proposition 8.1.1, the finiteness of $h^0_\theta(\overline{F})$ implies the finiteness of $F_{\rm tor}$ and the freeness of $F_{/\rm tor} := F/F_{\rm tor}$, and that the Euclidean seminorm $\|.\|$ is a norm. In particular, the rank of F is the rank of the free $\mathbb{Z}$-module $F_{/\rm tor}$, and is infinite if and only if F is not finitely generated.

PROOF. Let $\overline{F}$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ satisfying the conditions in (i). Since $\overline{F}$ is θ^1-summable, we may find an exhaustive filtration $\mathcal{C}_\bullet := (\mathcal{C}_i)_{i\in\mathbb{N}}$ of F that satisfies the θ^1-summability condition:

$$\sum_{i=0}^{+\infty} h^1_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty. \tag{8.6.15}$$

Proposition 8.6.5, applied with $X = \operatorname{Spec}\mathbb{Z}$, establishes the following θ^0-summability:

$$\sum_{i=0}^{+\infty} h^0_\theta(\overline{\mathcal{C}_i/\mathcal{C}_{i-1}}) < +\infty. \tag{8.6.16}$$

From ((8.6.15)) and ((8.6.16)), we get:

$$\lim_{i\to+\infty} (h^0_\theta(\overline{\mathcal{C}_{i+1}/\mathcal{C}_i}) + h^0_\theta(\overline{\mathcal{C}_{i+1}/\mathcal{C}_i})) = 0,$$

and therefore:

$$\lim_{i\to+\infty} (h^0_\theta(\overline{(\mathcal{C}_{i+1}/\mathcal{C}_i)}_{/\rm tor}) + h^0_\theta(\overline{(\mathcal{C}_{i+1}/\mathcal{C}_i)}_{/\rm tor})) = 0. \tag{8.6.17}$$

Since F has infinite rank, there exists and infinite subset I of $\mathbb{N}$ such that, for every $i \in I$, the Euclidean lattice $\overline{(\mathcal{C}_{i+1}/\mathcal{C}_i)}_{/\rm tor}$ has positive rank. Since ((8.6.17)) implies:

$$\inf_{i\in I} (h^0_\theta(\overline{(\mathcal{C}_{i+1}/\mathcal{C}_i)}_{/\rm tor}) + h^0_\theta(\overline{(\mathcal{C}_{i+1}/\mathcal{C}_i)}_{/\rm tor})) = 0,$$

this completes the proof of (ii).

Conversely if condition (ii) holds, we may find a sequence $(\overline{E}_i)_{i\in\mathbb{N}}$ of Euclidean lattices of positive rank such that, for every $i \in \mathbb{N}$:

$$h^0_\theta(\overline{E}_i) + h^1_\theta(\overline{E}_i) \leq 2^{-i}.$$

Then the Euclidean quasi-coherent sheaf of infinite rank $\overline{F} := \bigoplus_{i\in\mathbb{N}} \overline{E}_i$ satisfies:

$$h^0_\theta(\overline{F}) = \sum_{i\in\mathbb{N}} h^0_\theta(\overline{E}_i) < +\infty.$$

Moreover $\overline{F}$ is clearly θ^1-summable, and according to Proposition 8.2.4, it satisfies:

$$\underline{h}^1_\theta(\overline{F}) = \overline{h}^1_\theta(\overline{F}) = \sum_{i\in\mathbb{N}} h^1_\theta(\overline{E}_i) < +\infty.$$

This completes the proof of (i).

The infimum in condition (ii) (resp. in condition (iii)) is precisely:

$$\inf_{N>0,\delta\in\mathbb{R}} h^{0+1}_{\theta,\min}(N,\delta) \quad (\text{resp. } \inf_{N>0} h^0_{\theta,\min}(N,0)).$$

According to (8.6.14), these two numbers vanish simultaneously. This establishes the equivalence of (*ii*) and (*iii*). □

According to Proposition 8.6.10, we face the following alternative: either (a) there exists some $\overline{F}$ of infinite rank as in (*i*); or (b) the following holds:

$$\inf_{N>0} h^0_{\theta,\min}(N,0) > 0, \tag{8.6.18}$$

or equivalently: there exists a universal positive lower bound on the invariant $h^0_\theta(\overline{E})$ attached to a Euclidean lattice $\overline{E}$ of positive rank and covolume 1.

Observe that for every positive integer N, the estimates (8.6.12) and (8.6.13) take the following form when $\delta = 0$:

$$0 < h^0_{\theta,\min}(N,0) < \log 2. \tag{8.6.19}$$

At this stage, this appears to be the strongest information available concerning the behavior of $h^0_{\theta,\min}(N,0)$ for large N. As recalled above, the upper bound in (8.6.19) is established by some "probabilistic arguments" *à la* Minkowski-Hlawka-Siegel.

The validity of (8.6.18) would be rather surprising. At the same time, contradicting (8.6.18) would constitute a significant improvement on these probabilistic arguments, and seems to require some original idea.[18]

[18]Recall that, concerning the similar question of constructing Euclidean lattices of rank N of "large density", or equivalently with "large first minimum," the original non-constructive lower bounds of Minkowski have never been improved for large N; see for instance [**CS99**, Chapter 1, 1.5].

CHAPTER 9

Theta Invariants and Infinite-Dimensional Geometry of Numbers

This final chapter is devoted to the relations between the theta invariants attached to some Euclidean quasi-coherent sheaf $\overline{M}$, notably $\overline{h}_\theta^1(\overline{M})$, and the more naive invariants introduced in Chapter 6, especially its covering radius $\rho(\overline{M})$.

In the application of theta invariants to Diophantine geometry in the sequel of this monograph, the formalism developed in the previous chapters will be used to transpose in Diophantine geometry various cohomological techniques familiar in classical algebraic geometry. It will lead to results concerning the smallness or the vanishing of the invariants $\overline{h}_\theta^1(\overline{N})$ associated to Euclidean quasi-coherent sheaves $\overline{N}$ naturally associated to the Diophantine problem under study.

To derive "concrete" consequences — for instance density results — from these results about the theta invariants $\overline{h}_\theta^1(\overline{N})$, it will be crucial to know that they imply similar smallness properties of the covering radii $\rho(\overline{N})$ of these Euclidean quasi-coherent sheaves. The main result of this chapter, Theorem 9.3.3 and its Corollary 9.3.4, will provide the needed control of covering radii in terms of theta invariants.

This chapter also discusses further results comparing theta invariants and "naive invariants" of Euclidean lattices and quasi-coherent sheaves that put into perspective this main result, or are established by similar techniques.

9.0.1. Before investigating upper bounds on covering radii in terms of theta invariants, in Section 9.1 we establish estimates in the reverse direction. In their simplest form, concerning a Euclidean lattice $\overline{E}$, these estimates read as follows:

$$h_\theta^1(\overline{E}) \leq \pi\rho(\overline{E})^2. \tag{9.0.1}$$

At this stage, the inequality (9.0.1) is a straightforward consequences of the basic properties of Banaszczyk functions established in Chapter 7; see equation (7.1.23) and Corollary 7.4.4. Actually the derivation of (9.0.1) highlights the significance of a noteworthy invariant of Euclidean lattices, which may be defined as follows in terms of integrals of Gaussian functions over their Voronoi domains.

Recall that the Voronoi cell $\mathcal{V}(\overline{E})$ of a Euclidean lattice $\overline{E} := (E, \|.\|)$ is the symmetric polytope in $E_\mathbb{R}$ defined as:

$$\begin{aligned}\mathcal{V}(\overline{E}) : &= \big\{x \in E_\mathbb{R} \mid \forall m \in E, \|x\| \leq \|x-m\|\big\}\\ &= \bigcap_{m\in E\setminus\{0\}} \big\{x \in E_\mathbb{R} \mid 2\langle m, x\rangle \leq \langle m, m\rangle\big\};\end{aligned} \tag{9.0.2}$$

see Section 6.2. The average value of the Gaussian function $e^{-\pi\|x\|^2}$ of $x \in E_{\mathbb{R}}$ over $\mathcal{V}(\overline{E})$, namely[1]:

$$\operatorname{covol}(\overline{E})^{-1} \int_{\mathcal{V}(\overline{E})} e^{-\pi\|x\|^2}\, d\lambda_{\overline{E}_{\mathbb{R}}}(x), \tag{9.0.3}$$

belongs to the interval $(0,1]$, and we may define[2]:

$$\operatorname{gv}(\overline{E}) := -\log\Big(\operatorname{covol}(\overline{E})^{-1} \int_{\mathcal{V}(\overline{E})} e^{-\pi\|x\|^2}\, d\lambda_{\overline{E}_{\mathbb{R}}}(x)\Big) \in \mathbb{R}_+. \tag{9.0.4}$$

The average value (9.0.3) of the Gaussian function over the Voronoi cell of some Euclidean lattice has been considered for decades in the literature devoted to the design of signals for data transmission based on Euclidean lattices — it represents the probability that a "quantizer" designed from some Euclidean lattice provides a correct answer when disrupted by some Gaussian noise; see for instance [**CS99**, Section 3.1]. The recent investigations of Euclidean lattices by computer scientists working on lattice based cryptography, focusing on involving theta invariants and on Banaszczyk functions and measures — notably in [**CDLP13**] and [**RSD17b**] — have demonstrated the significance of the average value (9.0.3) for the understanding of Euclidean lattices and of their invariants, from a purely mathematical perspective.

The estimate (9.0.1) is indeed a consequence of the successive estimates:

$$h^1_\theta(\overline{E}) \leq \operatorname{gv}(\overline{E}) \leq \pi\rho(\overline{E})^2. \tag{9.0.5}$$

The first inequality in (9.0.5) appears in substance in the work [**CDLP13**] by Chung, Dadush, Liu, and Peikert. The second inequality is a straightforward consequence of the definitions.

From the perspective of this monograph — where we study invariants attached not only to objects of categories of Euclidean lattices and of their possibly infinite dimensional generalizations, but also to morphisms in these categories — it is remarkable that the estimates (9.0.5) admit a more general version concerning morphisms in $\ddot{\mathbf{Coh}}_{\mathbb{Z}}$; see Theorem 9.1.3. It involves the theta rank $\operatorname{rk}^1_\theta$, the relative covering radius, and a suitable general generalization of $\operatorname{gv}(\overline{E})$, instead of $h^1_\theta(\overline{E})$, $\rho(\overline{E})$, and $\operatorname{gv}(\overline{E})$.

In Subsection 9.1.3, we indicate a few additional properties of the invariant $\operatorname{gv}(\overline{E})$ defined by (9.0.4), besides the estimates (9.0.5) and their generalizations to morphisms. It turns out, that after the squared covering radius ρ^2 and the theta invariant h^1_θ, the invariant gv provides another instance of a positive invariant on $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ to which may be applied the formalism of Chapter 4 concerning monotonic and subadditive invariants and their infinite dimensional extensions. But we do not pursue here this line of investigation, how interesting it promises to be.

9.0.2. Concerning upper bounds on covering radii in terms of theta invariants and a possible converse to the estimates (9.0.5), it should be emphasized that the covering radius $\rho(\overline{E})$ of a Euclidean lattice — although it can be bounded in terms of $h^1_\theta(\overline{E})$ and of the rank of $\overline{E}$ — *cannot* be bounded in terms of $h^1_\theta(\overline{E})$ alone.

Indeed one easily checks that the sequence of real numbers $(\lambda_n)_{n\geq 1}$ defined by the conditions:

$$h^1_\theta(\overline{\mathcal{O}}(\lambda_n)^{\oplus n}) = 1/2$$

satisfies:

$$\lambda_n = \log\log n + O(1) \quad \text{when } n \to +\infty,$$

and accordingly:

$$\lim_{n\to+\infty} \rho(\overline{\mathcal{O}}(\lambda_n)^{\oplus n}) = \lim_{n\to+\infty} e^{-\lambda_n}\sqrt{n}/2 = +\infty;$$

[1]Recall that $\lambda_{\overline{E}_{\mathbb{R}}}$ denotes the Lebesgue measure on the finite dimensional Euclidean vector space $\overline{E}_{\mathbb{R}} := (E_{\mathbb{R}}, \|.\|)$. The measure $\lambda_{\overline{E}_{\mathbb{R}}}(\mathcal{V}(\overline{E}))$ of the Voronoi cell of the Euclidean lattice $\overline{E}$ coincides with its covolume $\operatorname{covol}(\overline{E})$.

[2]The notation gv stands for **G**auss and **V**oronoi. The use of lower-case letters is intended to reflect the logarithmic character of the invariant gv.

see Example 9.4.4 below.

When looking for bounds on the covering radius $\rho(\overline{E})$ of a Euclidean lattice $\overline{E}$ in terms of $h^1_\theta(\overline{E})$ that will still make sense in the infinite dimensional setting, one is therefore led to consider, not simply a Euclidean lattice $\overline{E} := (E, \|.\|)$, but a *Euclidean lattice* $\overline{E} := (E, \|.\|)$ *together with a second Euclidean norm* $\|.\|'$ *on* $E_{\mathbb{R}}$, and to look for some control of the covering radius $\rho(\overline{E}')$ of the Euclidean lattice $\overline{E}' := (E, \|.\|')$ in terms of $h^1_\theta(\overline{E})$ and of some relative trace attached to the norms $\|.\|'$ and $\|.\|$.

We actually followed a such an approach in Chapter 6, when establishing upper bounds on the invariants $\rho(\overline{E}')$ and $\gamma(\overline{E}')$ in terms of $\lambda^{[0]}(\overline{E})$, where $\overline{E} := (E, \|.\|)$ is an arbitrary Euclidean quasi-coherent sheaf, and where $\overline{E}' := (E, \|.\|')$ is defined by some Euclidean seminorm $\|.\|'$ satisfying a trace class condition with respect to $\|.\|$, expressed by the finiteness of the relative trace $\mathrm{Tr}(\|.\|'/\|.\|)$ or $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$; see Proposition 6.4.13 and Corollary 6.4.15.

When E is a $\mathbb{Z}$-module of finite rank, we could apply these upper bounds with $\|.\|$ a Euclidean norm and with $\|.\|' = \|.\|$. In this case, the relative traces $\mathrm{Tr}(\|.\|'/\|.\|)$ and $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$ are equal to the rank n of E, and our upper bounds specialized to "classical" estimates relating invariants of a Euclidean lattice and involving its rank. In the infinite rank case, the finiteness of these relative traces means heuristically that the seminorm $\|.\|'$ is much smaller than $\|.\|$ when restricted to suitable subspaces of $E_{\mathbb{R}}$ of large codimension.

Our main result, established in Sections 9.2 and 9.3, is an upper bound of this kind. Namely, we show that, with the above notation, when $\|.\|' \leq \|.\|$, the following implication holds for suitable universal positive constant ε, a, and b:

$$h^1_\theta(\overline{E}) < \varepsilon \Longrightarrow \rho(\overline{E}')^2 \leq a\, h^1_\theta(\overline{E}) + b\,\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.0.6}$$

Actually (9.0.6) holds with $\varepsilon = 1/2$, $a = 2$, and $b = 1/4$.

At this stage, a formal argument allows one to extend (9.0.6) to the infinite dimensional setting and to prove that, for every object $\overline{N} := (N, \|.\|)$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and every Euclidean seminorm $\|.\| \leq \|.\|'$ on $N_{\mathbb{R}}$, the following implication holds:

$$\overline{h}^1_\theta(\overline{N}) < \varepsilon \Longrightarrow \overline{\rho}(\overline{N}')^2 \leq a\, \overline{h}^1_\theta(\overline{N}) + b\,\mathrm{Tr}(\|.\|'^2/\|.\|^2), \tag{9.0.7}$$

where $\overline{N}' := (N, \|.\|')$.

Here again, we are actually able to establish a version of the estimates in (9.0.6) and (9.0.7) that covers "relative invariants" associated to a morphism $f : M \to \overline{N}$ in $\mathbf{q}\dddot{\overline{\mathbf{Coh}}}_{\mathbb{Z}}$. Namely we establish an estimate of the form:

$$\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) < \varepsilon \Longrightarrow \overline{\rho}(f : M \to \overline{N}')^2 \leq a\, \overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) + b\,\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.0.8}$$

Having at our disposal such relative versions of the upper bounds on covering radii will be crucial in the Diophantine applications of our formalism.

Our derivation of the estimate (9.0.6) and of its generalizations (9.0.7) and (9.0.8) in Section 9.3 will rely on our previous results on Banaszczyk functions in Chapter 8, combined with the following estimates. Consider a free $\mathbb{Z}$-module N of finite rank n, and let $\|.\|$ (resp. $\|.\|'$) be a Euclidean norm (resp. a Euclidean seminorm) on $N_{\mathbb{R}}$ such that $\|.\|' \leq \|.\|$. *For every* $\eta \in [0, 1)$, *if we denote by* $\|.\|_\eta$ *the Euclidean norm on* $N_{\mathbb{R}}$ *defined by the equality:*

$$\|x\|^2_\eta := \|x\|^2 - \eta \|x\|'^2,$$

then the following inequality holds:

$$0 \leq h^0_\theta(N, \|.\|_\eta) - h^0_\theta(N, \|.\|) \leq \frac{1}{2} \log(1-\eta)^{-1}\ \mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.0.9}$$

The possibility of establishing upper bounds on covering radii in terms of theta invariants of the form (9.0.6) — where the "error term" is a relative trace $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$ — crucially relies on the inequality (9.0.9), which we shall establish at the beginning of Section 9.2.

Besides the derivation of (9.0.6) and its generalizations, the inequality (9.0.9) actually admits other significant applications, concerning bounds on small vectors in Euclidean quasi-coherent sheaves in terms of their invariants h^0_θ. These bounds are presented in Subsections 9.2.2 and 9.2.3, and constitutes infinite dimensional extensions of results on Euclidean lattices established by means of Banaszczyk method in [**Bos20b**].

For instance, we show that if $\overline{E} := (E, \|.\|)$ is a Euclidean quasi-coherent sheaf, and if $\|.\|'$ is a Euclidean seminorm on $E_\mathbb{R}$ that satisfies the conditions:

$$\|.\|' \leq \|.\| \quad \text{and} \quad T := \mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty,$$

then the first minimum of the quasi-coherent sheaf $(E, \|.\|')$, previously introduced in Definition 6.6.1:

$$\lambda_1(E, \|.\|') := \inf_{v \in E} \|v\|'$$

satisfies the following upper bound:

$$h^0_\theta(E, T^{1/2}\|.\|) \geq \log 2 \Longrightarrow \lambda_1(E, \|.\|') \leq 1; \tag{9.0.10}$$

see Theorem 9.2.4 and Corollary 9.2.5 applied with $e^{2\delta} = T$. This constitutes an infinite dimensional version of Minkowski's first theorem, which one recovers when $\overline{E}$ is a Euclidean lattice and $\|.\|' = \|.\|$.[3]

In Subsection 9.2.3, we actually show how a variation on the arguments leading to the estimate (9.0.8) allows one to establish similar upper bounds on the first minimum of the transpose morphism $f^\vee$. In Section 9.5, these will be used to establish generalizations of the famous transference estimates proved by Banaszczyk in his seminal paper [**Ban93**].

Section 9.4 is devoted to diverse consequences of the comparison estimates established in Sections 9.1 and 9.3. Firstly, in Subsection 9.4.1, we discuss their consequences concerning (finite rank) Euclidean lattices. Then, in Subsection 9.4.2, we apply them to relate the properties of θ^1-finiteness and ρ^2-summability of Euclidean quasi-coherent sheaves. Finally, in Subsection 9.4.3, we establish the density theorems suited to Diophantine applications that have been alluded to at the beginning of this introduction.

The estimate (9.0.8) established in Section 9.3 and its consequences in Section 9.4 constitute the main results of this chapter. Its final sections are devoted to additional results concerning naive and theta invariants of Euclidean quasi-coherent sheaves, which put into perspective the estimate (9.0.8) and its proof.

9.0.3. Statements relating the invariants attached to some Euclidean lattice $\overline{E}$ to those of its dual $\overline{E}^\vee$ are classically known as *transference theorems*,[4] and a major breakthrough in geometry of numbers had been achieved by Banaszczyk when — in his seminal paper [**Ban93**] which inaugurates the use of theta series and harmonic analysis in the study of general Euclidean lattices[5] — he established transference inequalities in which the constants, depending on the rank n of the Euclidean lattice $\overline{E}$ under study, are basically optimal.

---

[3]Indeed, when this holds, then T is the rank n of E, and the lower bound $h^0_\theta(E, n^{1/2}\|.\|) \geq \widehat{\deg}\,(E, n^{1/2}\|.\|) := -\log(n^{n/2}\mathrm{covol}\,\overline{E})$ together with (9.0.10) establish the implication: $(\mathrm{covol}\,\overline{E})^{1/n} \leq 2^{-1/n}\, n^{-1/2} \Longrightarrow \lambda_1(\overline{E}) \leq 1$. By a straightforward scaling argument, this may be reformulated as the estimate: $\lambda_1(\overline{E}) \leq 2^{1/n}\, n^{1/2}(\mathrm{covol}\,\overline{E})^{1/n}$.

[4]Originally, *Übertragungssätze*; see for instance [**Cas71**, Chapter XI].

[5]Since Jacobi's *Fundamenta Nova* [**Jac29**], the theta series $\theta_{\overline{E}}$ associated to *integral* Euclidean lattices — namely Euclidean lattices $\overline{E}$ defined by a Euclidean scalar product that is $\mathbb{Z}$ valued on $E \times E$ — are known to define modular forms, and through this construction the theory of modular forms classically plays a central role in the study and in the classification of integral lattices; see for instance [**Ebe13**] for a modern introduction and references. The techniques introduced in [**Ban93**] demonstrated the significance of theta series when investigating the fine properties of *general* Euclidean lattices.

Banaszczyk notably established in [**Ban93**] the following inequality relating the covering radius $\rho(\overline{E})$ of some Euclidean lattice $\overline{E} := (E, \|.\|)$ of positive rank n and the first of the successive minima $\lambda_1(\overline{E}^\vee)$ of its dual $\overline{E}^\vee := (E^\vee, \|.\|^\vee)$:

$$\rho(\overline{E})\,\lambda_1(\overline{E}^\vee) \leq n/2. \tag{9.0.11}$$

Recall that, in the left-hand side of (9.0.11), $\rho(\overline{E})$ is defined by the equality (6.0.1), which in the special case of Euclidean lattice takes the classical form:

$$\rho(\overline{E}) := \max_{x \in E_\mathbb{R}} \min_{e \in E} \|x - e\|,$$

and that $\lambda_1(\overline{E}^\vee)$ is defined as the length of the smallest non-zero vector in $E^\vee$:

$$\lambda_1(\overline{E}^\vee) := \min_{\xi \in E^\vee} \|\xi\|^\vee.$$

Moreover, as observed by Banaszczyk, the transference inequalities (9.0.11) are optimal, up to some multiplicative error term, uniformly bounded when n varies.[6]

Section 9.5 is devoted to the proof of transference inequalities which extend Banaszczyk's transference inequality (9.0.11) and its variants, concerning Euclidean lattices, to the framework of this monograph, where relative versions of classical invariants are considered, and where one deals with general Euclidean quasi-coherent sheaves.

Here again, in our possibly infinite dimensional framework, one deals with a countably generated $\mathbb{Z}$-module M whose underlying the $\mathbb{R}$-vector space $M_\mathbb{R}$ is equipped with two Euclidean seminorms $\|.\|$ and $\|.\|'$, and our general transference estimate involve the relative trace $\mathrm{Tr}\,(\|.\|'/\|.\|)$ of these seminorms.

Recall, that to a morphism in $\ddot{\overline{\mathbf{qCoh}}}_\mathbb{Z}$:

$$f : M \longrightarrow \overline{N} := (N, \|.\|), \tag{9.0.12}$$

defined by a morphism of countably generated $\mathbb{Z}$-modules:

$$f : M \longrightarrow N$$

and a Euclidean seminorm $\|.\|$ on the real vector space $N_\mathbb{R}$, we have attached its relative covering radius:

$$\rho(f : M \to \overline{N}) := \sup_{x \in M_\mathbb{R}} \inf_{n \in N_{/\mathrm{tor}}} \|f_\mathbb{R}(x) - n\| \in [0, +\infty]$$

and its lower variant:

$$\underline{\rho}(f : M \to \overline{N}) := \sup_{N' \in \mathrm{coft}(N)} \rho(f_{N'} : M \to \overline{N/N'}) \in [0, +\infty],$$

where $f_{N'}$ denotes the composition of f with the quotient morphism $N \to N/N'$; see Definition 6.7.1 and Subsection 6.7.3.

If moreover $\overline{\widehat{E}} := (\widehat{E}, \|.\|)$ is a generalized pro-Euclidean vector bundle, namely an object of $\mathbf{pro}\overline{\mathbf{Vect}}_\mathbb{Z}^{[\infty]}$, defined by an object $\widehat{E}$ of $\mathbf{pro}\overline{\mathbf{Vect}}_\mathbb{Z}$ and a lower continuous definite Euclidean quasi-norm on $\widehat{E}_\mathbb{R}$, we may consider its "first minimum":

$$\lambda_1(\overline{\widehat{E}}) := \inf_{\xi \in \widehat{E} \setminus \{0\}} \|\xi\|. \tag{9.0.13}$$

More generally, if we are also given a morphism

$$\psi : \widehat{E} \longrightarrow \widehat{F}$$

[6] This follows from the existence, established by Conway and Thompson, of a sequence of Euclidean lattices $\overline{\mathrm{CT}}_n$ such that $\mathrm{rk}\overline{\mathrm{CT}}_n = n$, $\overline{\mathrm{CT}}_n^\vee \simeq \overline{\mathrm{CT}}_n$, and $\lambda_1(\overline{\mathrm{CT}}_n) \geq \sqrt{n/2\pi e}(1 + o(n))$ when $n \to +\infty$. The Euclidean lattices $\overline{\mathrm{CT}}_n$ are actually integral unimodular lattices, and their existence follows from the mass formula of Smith-Minkowski-Siegel; see [**MH73**, Chapter II, Theorem 9.5].

in $\mathbf{CTC}_{\mathbb{Z}}$ of domain the topological module $\widehat{E}$ underlying $\widehat{\overline{E}}$, we may consider the following relative version of the first minimum (9.0.13):

$$\lambda_1(\psi : \widehat{\overline{E}} \to \widehat{F}) := \inf_{\xi \in \widehat{E} \setminus \ker \psi} \|\xi\|. \tag{9.0.14}$$

When $\widehat{F}$ is $\widehat{E}$ and ψ is $\mathrm{Id}_{\widehat{E}}$, this coincides with $\lambda_1(\widehat{\overline{E}})$.

To the data of the morphism (9.0.12) in $\mathbf{q}\dddot{\mathbf{Coh}}_{\mathbb{Z}}$ is attached by duality an object $\overline{N}^{\vee} := (N^{\vee}, \|.\|^{\vee})$ of $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[\infty]}$, and a morphism in $\mathbf{CTC}_{\mathbb{Z}}$:

$$f^{\vee} := \cdot \circ f : N^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \longrightarrow M^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z});$$

see Subsection 2.5.2. Consequently it makes sense to consider the associated relative first minimum, which by its very definition satisfies:

$$\lambda_1(f^{\vee} : \overline{N}^{\vee} \to M^{\vee}) := \inf \left\{ \|\xi\|^{\vee}, \xi \in N^{\vee} \text{ and } \xi \circ f \neq 0 \right\},$$

where:

$$\|\xi\|^{\vee} := \sup \left\{ |\xi(x)|, x \in N_{\mathbb{R}} \text{ and } \|x\| \leq 1 \right\}.$$

With this notation, our generalization of Banaszczyk transference estimate (9.0.11) to the relative and infinite dimensional setting asserts that, *for every Euclidean seminorm* $\|.\|'$ *on* $N_{\mathbb{R}}$*, if we let* $\overline{N}' := (N, \|.\|')$*, then the following inequality holds:*

$$\underline{\rho}(f : M \to \overline{N}')\, \lambda_1(f^{\vee} : \overline{N}^{\vee} \to M^{\vee}) \leq \mathrm{Tr}(\|.\|'/\|.\|); \tag{9.0.15}$$

see Theorem 9.5.1.

Even if one restricts to the situation where the $\mathbb{Z}$-module M has finite rank, the extra flexibility added by this relative version involving pair of seminorms allows one to recover from (9.0.15) the original transference estimates in Banasczyk [**Ban93**] concerning successive minima of higher order, and also their refinements involving successive covering radii, as in [**Cai03**], and concerning Euclidean lattices with "gaps" in the sequence of their successive minima.

As already mentioned, the proof of the transference estimate (9.0.15) crucially relies on the preliminary inequalities established in Subsections 9.2.2 and 9.2.3, by a variation on the arguments leading to the estimates (9.0.9)

9.0.4. In Section 9.6, we complete our investigation of the relations between the θ^1-finiteness of Euclidean quasi-coherent sheaves and the finiteness of their naive invariants ρ, γ, and $\lambda^{[0]}$ introduced in Chapter 6. Namely we show that, if a Euclidean quasi-coherent sheaf $\overline{M} = (M, \|.\|)$ satisfies the condition: $\lambda^{[0]}(\overline{M}) < +\infty$, then, after replacing the Euclidean seminorm $\|.\|$ by a seminorm $\|.\|'$ that satisfies a very weak quantitative strengthening of the compactness of $\|.\|'$ with respect to $\|.\|$, we obtain a Euclidean quasi-coherent sheave $\overline{M}' := (M, \|.\|')$ that is θ^1-finite.

This result result will be a consequence of the criteria for θ^1-finiteness established in Section 8.5 and of the Peierls-Bogoliubov inequality, concerning the trace of convex functions of compact positive operators in Hilbert-spaces, which is recalled in Appendix A .

Finally, in Section 9.7, we summarize the diverse relations between the finiteness properties of the theta invariants and of the elementary invariants ρ, γ, and $\lambda^{[0]}$ established so far. We finallly discuss how these implications apply to families $(M, \|.\|_{\alpha})_{\alpha \in A}$ of Euclidean quasi-coherent sheaves defined by a fixed $\mathbb{Z}$-module M equipped with a family $(\|.\|_{\alpha})_{\alpha \in A}$ of Euclidean seminorms satisfying a suitable nuclearity property, and how such families naturally arise in Diophantine geometry.

Although they are devoted to a common theme, the contents of the successive sections of this chapter are to a large extent independent.

Sections 9.1 and 9.7 require only a basic knowledge of diverse invariants of Euclidean lattices and Euclidean quasi-coherent sheaves, and their results are not used elsewhere in this chapter, besides the final observations in Section 9.7.

A reader interested in upper bounds on covering radii in terms of theta invariants (resp. in transference inequalities) only might focus on Subsection 9.2.1 and Sections 9.3 and 9.4 (resp. on Subsections 9.2.1 and 9.2.3 and Section 9.5) and skip the remainder of this chapter.

9.1. The Estimates $h^1_\theta(\overline{E}) \leq \pi\rho(\overline{E})^2$ and $\mathrm{rk}^1_\theta(f) \leq \pi\rho(f)^2$ and the Invariants $\mathrm{gv}(\overline{E})$ and $\mathrm{gv}(f)$

9.1.1. Bounding theta invariants in terms of covering radii: statement of results. In this section, we shall prove the following upper on covering radii in terms of theta invariants.

THEOREM 9.1.1. *For every object $\overline{N} := (N, \|.\|)$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the following estimates hold:*

$$\underline{h}^1_\theta(\overline{N}) \leq \pi\underline{\rho}(\overline{N})^2 \quad \text{and} \quad \overline{h}^1_\theta(\overline{N}) \leq \pi\overline{\rho}(\overline{N})^2. \tag{9.1.1}$$

More generally, for every countably generated $\mathbb{Z}$-module M and every morphism of $\mathbb{Z}$-modules $f : M \to N$, the following estimates hold:

$$\underline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) \leq \pi\underline{\rho}(f : M \to \overline{N})^2 \quad \text{and} \quad \overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) \leq \pi\overline{\rho}(f : M \to \overline{N})^2. \tag{9.1.2}$$

When $\overline{N}$ is an object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the estimates (9.1.1) reduce to a single estimate:

$$h^1_\theta(\overline{E}) \leq \pi\rho(\overline{N})^2. \tag{9.1.3}$$

More generally, if $f : F \to E$ is a morphism of finitely generated $\mathbb{Z}$-modules, the estimates (9.1.2) reduce to:

$$\mathrm{rk}^1_\theta(f : F \to \overline{E}) \leq \pi\rho(f : F \to \overline{E})^2. \tag{9.1.4}$$

Conversely, from the validity of (9.1.3) (resp. of (9.1.4)) for an arbitrary object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ (resp. for an arbitrary morphism $f : F \to \overline{E}$ in $\dddot{\mathbf{Coh}}_{\mathbb{Z}}$), one derives the validity of (9.1.1) (resp. of (9.1.2)) simply by spelling out the definitions of the lower and upper theta invariants and covering radii (resp. of the lower and upper theta ranks and relative covering radii).

Observe also that, since h^1_θ, rk^1_θ, and ρ are downward continuous and "do not see torsion," to prove (9.1.3) or (9.1.4), one may assume that $\overline{E}$ is actually a Euclidean lattice.

We shall establish (9.1.3) or (9.1.4) in a more precise form, that will involve the invariants attached to objects of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and to morphisms in $\dddot{\mathbf{Coh}}_{\mathbb{Z}}$ defined as follows:

DEFINITION 9.1.2. For every object $\overline{E} := (E, \|.\|)$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, we let:

$$\mathrm{gv}(\overline{E}) := -\log \int_{E_{\mathbb{R}}/E_{/\mathrm{tor}}} e^{-\pi d_{\overline{E}_{\mathbb{R}}}(x,E)^2}\, d\mu_{E_{\mathbb{R}/\mathbb{Z}}}([x]). \tag{9.1.5}$$

If moreover F is a finitely generated $\mathbb{Z}$-modules and $f : F \to E$ a morphism of $\mathbb{Z}$-modules, we let:

$$\mathrm{gv}(f : F \to \overline{E}) := -\log \int_{F_{\mathbb{R}}/F_{/\mathrm{tor}}} e^{-\pi d_{\overline{E}_{\mathbb{R}}}(f_{\mathbb{R}}(y),E)^2}\, d\mu_{F_{\mathbb{R}/\mathbb{Z}}}([y]). \tag{9.1.6}$$

In the right-hand side of (9.1.5), we denote by $d_{\overline{E}_{\mathbb{R}}}(x, E)$ the "distance" of $x \in E_{\mathbb{R}}$ to (the image $E_{/\mathrm{tor}}$ in $E_{\mathbb{R}}$) of E in the seminormed vector space $\overline{E}_{\mathbb{R}} := (E_{\mathbb{R}}, \|.\|)$:

$$d_{\overline{E}_{\mathbb{R}}}(x, E) := \inf_{v \in E_{/\mathrm{tor}}} \|x - v\|.$$

It depends only on the class $[x]$ of x in the compact torus $E_{\mathbb{R}}/E_{/\mathrm{tor}}$. Moreover $\mu_{E_{\mathbb{R}/\mathbb{Z}}}$ denotes the Haar probability measure on this torus.[7] A similar notation, with E replaced by F, is used in the right-hand side of (9.1.6).

Observe that the invariant (9.1.6) of the identity morphism boils down to the invariant (9.1.5):

$$\mathrm{gv}(\mathrm{Id}_E : E \to \overline{E}) = \mathrm{gv}(\overline{E}).$$

Moreover one easily checks that the invariant $\mathrm{gv}(f : F \to \overline{E})$ attached to the morphism $f : F \to \overline{E}$ in $\dddot{\mathbf{Coh}}_{\mathbb{Z}}$ depends only on $\overline{E}$ and on the image $f_{\mathbb{R}}(F_{\mathbb{R}})$ of $f_{\mathbb{R}}$ in $E_{\mathbb{R}}$.

We may now state the more precise version of the estimates (9.1.3) and (9.1.4) alluded to above:

THEOREM 9.1.3. *For every Euclidean lattice $\overline{E}$, the following inequalities hold:*

$$h^1_\theta(\overline{E}) \leq \mathrm{gv}(\overline{E}) \leq \pi\rho(\overline{E})^2. \tag{9.1.7}$$

If moreover F is a finitely generated $\mathbb{Z}$-modules and $f : F \to E$ a morphism of $\mathbb{Z}$-modules, we have:

$$\mathrm{rk}^1_\theta(f : F \to \overline{E}) \leq \mathrm{gv}(f : F \to \overline{E}) \leq \pi\rho(f : F \to \overline{E})^2. \tag{9.1.8}$$

9.1.2. Proof of Theorem 9.1.3.

9.1.2.1. The second inequality in (9.1.7) (resp. in (9.1.8)) is a straightforward consequence of the definition (9.1.5) of $\mathrm{gv}(\overline{E})$ (resp. of the definition (9.1.6) of $\mathrm{gv}(f : F \to \overline{E})$) and of the definition of $\rho(\overline{E})$ (resp. of $\rho(f : F \to \overline{E})$):

$$\rho(\overline{E}) := \max_{x \in E_{\mathbb{R}}} d_{\overline{E}_{\mathbb{R}}}(x, E) \quad (\text{ resp. } \rho(f : F \to \overline{E}) := \max_{y \in F_{\mathbb{R}}} d_{\overline{E}_{\mathbb{R}}}(f_{\mathbb{R}}(y), E)\ ). \tag{9.1.9}$$

.

9.1.2.2. The first inequality in (9.1.7) already appears in [**CDLP13**] and follows from the expression (7.7.6) for $h^1_\theta(\overline{E})$ as an integral:

$$h^1_\theta(\overline{E}) = -\log \int_{E_{\mathbb{R}}/E_{/\mathrm{tor}}} e^{-\pi b_{\overline{E}}(x)}\, d\lambda_{E_{\mathbb{R}/\mathbb{Z}}}(x),$$

and of the estimate:

$$b_{\overline{E}}(x) \leq d_{\overline{E}}(x, E)^2,$$

which follows from (7.7.5) and from the invariance of $b_{\overline{E}}$ under translation by $E_{/\mathrm{tor}}$, as in Corollary 7.4.4.

9.1.2.3. Let us finally prove the inequality[8]:

$$\mathrm{rk}^1_\theta(f : F \to \overline{E}) \leq \mathrm{gv}(f : F \to \overline{E}). \tag{9.1.10}$$

As observed above, to achieve this, we may assume that $\overline{E} := (E, \|.\|)$ is a Euclidean lattice. By replacing F by the saturation of $f(F)$ in E, we may also assume that F is a saturated $\mathbb{Z}$-submodule of E and that f is the inclusion morphism. Then we have:

$$\mathrm{rk}^1_\theta(f : F \to \overline{E}) = h^1_\theta(\overline{E}) - h^1_\theta(\overline{E/F}).$$

Consider the closed subgroup $F_{\mathbb{R}} + E$ of $E_{\mathbb{R}}$. Its connected component is $F_{\mathbb{R}}$, and we shall denote by ν the Haar measure on $F_{\mathbb{R}} + E$ the restriction of which to $F_{\mathbb{R}}$ coincides with the Lebesgue measure $\lambda_{\overline{F}_{\mathbb{R}}}$ on the Euclidean vector space $\overline{F}_{\mathbb{R}} := (F_{\mathbb{R}}, \|.\|_{|F_{\mathbb{R}}})$.

Since F is saturated in E, the image of $F_{\mathbb{R}} + E$ in $E_{\mathbb{R}}/E$ may be identified with $F_{\mathbb{R}}/F$, and we shall denote by:

$$p : F_{\mathbb{R}} + E \longrightarrow (F_{\mathbb{R}} + E)/E \simeq F_{\mathbb{R}}/F$$

the quotient map.

---

[7] The notation $\mu_{E_{\mathbb{R}/\mathbb{Z}}}$ is motivated by the identification of $\mathbb{Z}$-modules: $E \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \simeq E_{\mathbb{R}}/E_{/\mathrm{tor}}$.

[8] The proof below, applied to $f = \mathrm{Id}_E$, provides a proof of the first inequality in (9.1.7) logically independent of the one in 9.1.2.2, but that is in substance the same proof.

To a Borel function on $F_{\mathbb{R}} + E$:

$$\varphi : F_{\mathbb{R}} + E \longrightarrow [0, +\infty],$$

we may attach the Borel function on $F_{\mathbb{R}}/F$:

$$p_*\varphi : F_{\mathbb{R}}/F \longrightarrow [0, +\infty]$$

defined by the following equality, for every $y \in F_{\mathbb{R}}$, of class $[y]$ in $F_{\mathbb{R}}/F$:

$$\pi_*\varphi([y]) := \sum_{v \in E} \varphi(y + v).$$

Then the following equality holds in $[0, +\infty]$:

$$\int_{F_{\mathbb{R}}+E} \varphi(y)\, d\nu(y) = \mathrm{covol}\,(\overline{F}) \int_{F_{\mathbb{R}}/F} p_*\varphi([y])\, d\mu_{F_{\mathbb{R}/\mathbb{Z}}}([y]). \tag{9.1.11}$$

LEMMA 9.1.4. *Let φ be the Gaussian function on $F_{\mathbb{R}} + E$, defined by the following equality, for every $y \in F_{\mathbb{R}} + E$:*

$$\varphi(y) := e^{-\pi\|y\|^2}.$$

(1) The following equality holds:

$$\int_{F_{\mathbb{R}}+E} e^{-\pi\|y\|^2}\, d\nu(y) = \sum_{w \in E/F} e^{-\pi\|w\|^2_{\overline{E/F}}}. \tag{9.1.12}$$

(2) For every $y \in F_{\mathbb{R}} + E$, we have:

$$p_*\varphi([y]) \geq e^{-\pi d_{\overline{E}_{\mathbb{R}}}(y,E)^2} \sum_{v \in E} e^{-\pi\|v\|^2}. \tag{9.1.13}$$

PROOF. (1) For every $w \in E/F$, let us denote by $w^\perp$ the unique element of the orthogonal $F_{\mathbb{R}}^\perp$ of $F_{\mathbb{R}}$ in the Euclidean vector space $\overline{E}_{\mathbb{R}}$ whose image in $E_{\mathbb{R}}/F_{\mathbb{R}} \simeq (E/F)_{\mathbb{R}}$ is w. The map:

$$E/F \oplus F_{\mathbb{R}} \longrightarrow E + F_{\mathbb{R}}, \quad (w, t) \longmapsto w^\perp + t$$

is an isomorphism, satisfies:

$$\|w\|^2_{\overline{E/F}} + \|t\|^2 = \|w^\perp + t\|^2,$$

and maps the product of the counting measure on E/F and of the Lebesgue measure $\lambda_{\overline{F}_{\mathbb{R}}}$ on $F_{\mathbb{R}}$ to the measure ν on $F_{\mathbb{R}} + E$.

Consequently we have:

$$\begin{aligned}
\int_{F_{\mathbb{R}}+E} e^{-\pi\|y\|^2}\, d\nu(y) &= \sum_{w \in E/F} \int_{F_{\mathbb{R}}} e^{-\pi\|w^\perp + t\|^2}\, d\lambda_{\overline{F}_{\mathbb{R}}}(t) \\
&= \sum_{w \in E/F} e^{-\pi\|w\|^2_{\overline{E/F}}} \int_{F_{\mathbb{R}}} e^{-\pi\|t\|^2}\, d\lambda_{\overline{F}_{\mathbb{R}}}(t) \\
&= \sum_{w \in E/F} e^{-\pi\|w\|^2_{\overline{E/F}}}.
\end{aligned}$$

(2) For every $y \in F_{\mathbb{R}} + E$, we have:

$$\begin{aligned}
p_*\varphi([y]) &= \sum_{v \in E} e^{-\pi\|y+v\|^2} = \sum_{v \in E} e^{-\pi\|y-v\|^2} \\
&= \sum_{v \in E} \left(e^{-\pi\|y+v\|^2} + e^{-\pi\|y-v\|^2}\right)/2 \\
&\geq e^{-\pi\|y\|^2} \sum_{v \in E} e^{-\pi\|v\|^2}.
\end{aligned}$$

Therefore, for every $w \in E$:

$$p_*\varphi([y]) \geq e^{-\pi\|y-w\|^2} \sum_{v\in E} e^{-\pi\|v\|^2}.$$

By taking the supremum over $w \in E$, (9.1.13) follows. □

From (9.1.11), (9.1.12), and (9.1.13), we deduce:

$$\sum_{w\in E/F} e^{-\pi\|w\|^2_{\overline{E/F}}} \geq \operatorname{covol}(\overline{F}) \sum_{v\in E} e^{-\pi\|v\|^2} \int_{F_{\mathbb{R}}/F} e^{-\pi d_{\overline{E}_{\mathbb{R}}}(y,E)^2}\, d\mu_{F_{\mathbb{R}/\mathbb{Z}}}([y]),$$

and therefore, by taking logarithms:

$$\operatorname{gv}(f : F \to \overline{E}) := -\log \int_{F_{\mathbb{R}}/F} e^{-\pi d_{\overline{E}_{\mathbb{R}}}(y,E)^2}\, d\mu_{F_{\mathbb{R}/\mathbb{Z}}}([y]) \geq -h^0_\theta(\overline{E/F}) + h^0_\theta(\overline{E}) - \widehat{\deg}\,\overline{F}.$$

Moreover, according to the Poisson-Riemann-Roch formula applied to $\overline{E/F}$ and $\overline{E}$ and to the additivity of the Arakelov degree, we have:

$$\begin{aligned} -h^0_\theta(\overline{E/F}) + h^0_\theta(\overline{E}) - \widehat{\deg}\,\overline{F} &= -h^1_\theta(\overline{E/F}) - \widehat{\deg}\,\overline{E/F} + h^1_\theta(\overline{E}) + \widehat{\deg}\,\overline{E} - \widehat{\deg}\,\overline{F} \\ &= -h^1_\theta(\overline{E/F}) + h^1_\theta(\overline{E}) = \operatorname{rk}^1_\theta(f : F \to \overline{E}). \end{aligned}$$

This completes the proof of (9.1.10).

9.1.3. The invariant $\operatorname{gv} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \to \mathbb{R}_+$.

9.1.3.1. If $\overline{E} := (E, \|.\|)$ is a Euclidean lattice, its Voronoi cell $\mathcal{V}(\overline{E})$, defined by (9.0.2), constitutes — up to some Lebesgue negligeable subset — a fundamental domain for the action by translation of E on $E_{\mathbb{R}}$, and for every $x \in \mathcal{V}(\overline{E})$, we have:

$$d_{\overline{E}_{\mathbb{R}}}(x, E) = \|x\|.$$

Consequently the average of the function $e^{-\pi d_{\overline{E}_{\mathbb{R}}}(.,E)^2}$ over $E_{\mathbb{E}}/E$ which occurs in the definition (9.1.5) of $\operatorname{gv}(\overline{E})$ coincide with the average of the Gaussian function $e^{-\pi\|x\|^2}$ of $x \in E_{\mathbb{R}}$ over $\mathcal{V}(\overline{E})$:

$$\int_{E_{\mathbb{R}}/E_{/\mathrm{tor}}} e^{-\pi d_{\overline{E}_{\mathbb{R}}}(x,E)^2}\, d\mu_{E_{\mathbb{R}/\mathbb{Z}}}([x]) = \operatorname{covol}(\overline{E})^{-1} \int_{\mathcal{V}(\overline{E})} e^{-\pi\|x\|^2}\, d\lambda_{\overline{E}_{\mathbb{R}}}(x), \tag{9.1.14}$$

and therefore the invariant $\operatorname{gv}(\overline{E})$ admits the following expression:

$$\operatorname{gv}(\overline{E}) := -\log\Big(\operatorname{covol}(\overline{E})^{-1} \int_{\mathcal{V}(\overline{E})} e^{-\pi\|x\|^2} d\lambda_{\overline{E}_{\mathbb{R}}}(x)\Big). \tag{9.1.15}$$

As mentioned in the introduction to this chapter, the average value (9.1.14) of the Gaussian function over the Voronoi cell of some Euclidean lattice has been considered in the literature devoted to the design of signals for data transmission based on Euclidean lattice, since the beginning of the 1970's. The problem of finding Euclidean lattices of a given rank and covolume for which (9.1.14) is maximal is known as the *lattice version of the Gaussian channel coding problem.* We refer the reader to [**CS99**, Section 3.1] for a discussion and references on this circle of questions.

It turns out that the quantity (9.1.14), and the Euclidean lattices for which it achieves a maximum, play a key role in the work in the proof [**RSD17b**] by Regev and Stephens-Davidowitz of a conjecture of Dadush, which in turns implies the ℓ_2 case of the Kannan-Lovász conjecture [**KL88**]; see [**DR16**] and [**RSD17b**].[9]

[9]See also Theorem 9.4.1 and ... below for a statement of the main results in [**DR16**] and [**RSD17b**], and [**Bos20a**] for a survey of their proofs.

Actually the proof of Regev and Stephens-Davidowitz rely crucially on diverse properties of the quantity (9.1.14), which may be expressed as the inequality:

$$h^1_\theta(\overline{E}) \leq \mathrm{gv}(\overline{E}),$$

established in substance in [**CDLP13**], and as the fact that $\mathrm{gv}(\overline{E})$ is a subadditive invariant of the Euclidean lattice $\overline{E}$.

9.1.3.2. Actually the invariant $\mathrm{gv}(\overline{E})$, defined by (9.1.5) for every object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, turns out to provide interesting instances of the properties of invariants investigated in Chapter 4.

PROPOSITION 9.1.5. *The invariant* $\mathrm{gv} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \to \mathbb{R}_+$ *defined by* (9.1.5) *satisfies the monotonicity and the subadditivity conditions* $\mathbf{Mon}^1_{\mathbb{Q}}$ *and* **SubAdd***, and the downward continuity condition* $\mathbf{Cont}^+$*.*

It also satisfies the additivity condition $\mathbf{Add}_\oplus$*, and is small on Euclidean coherent sheaves generated by small sections, and compatible with vectorization.*[10]

We leave the proof of this proposition as an exercise for the interested reader.[11]

We ignore whether the invariant $\mathrm{gv} : \overline{\mathbf{Coh}}_{\mathbb{Z}} \to \mathbb{R}_+$ satisfies the strong monotonicity condition introduced in Chapter 5.

9.2. Relative Traces, Theta Invariants, and First Minima

9.2.1. A key computation: variation of arithmetic degrees in terms of singular values. As mentioned in 9.0.2 above, the following lemma will play a crucial role for establishing estimates relating invariants attached to pairs of Euclidean quasi-coherent sheaves $\overline{N} := (N, \|.\|)$ and $\overline{N}' := (N, \|.\|')$ with the same underlying $\mathbb{Z}$-module N, when the relative trace $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$ is finite.

LEMMA 9.2.1. *Let N be a free $\mathbb{Z}$-module of finite rank n, and let $\|.\|$ (resp. $\|.\|'$) be a Euclidean norm (resp. a Euclidean seminorm) on $N_{\mathbb{R}}$ such that:*

$$\|.\|' \leq \|.\|.$$

Let us consider the singular values[12]*:*

$$0 \leq \lambda_n \leq \cdots \leq \lambda_1 \leq 1$$

of $\|.\|'$ with respect to $\|.\|$, and, for $\eta \in [0,1)$, let us denote by $\|.\|_\eta$ the Euclidean norm on $N_{\mathbb{R}}$ defined by the equality:

$$\|x\|^2_\eta := \|x\|^2 - \eta\|x\|'^2, \tag{9.2.1}$$

for every $x \in N_{\mathbb{R}}$.

Then, for every $\eta \in [0,1)$, the following relations hold:

$$\widehat{\deg}\,(N, \|.\|_\eta) - \widehat{\deg}\,(N, \|.\|) = -\frac{1}{2}\sum_{1\leq 1\leq n} \log(1-\eta\lambda_i^2)^{1/2}, \tag{9.2.2}$$

and:

$$0 \leq h^0_\theta(N, \|.\|_\eta) - h^0_\theta(N, \|.\|) \leq \widehat{\deg}\,(N, \|.\|_\eta) - \widehat{\deg}\,(N, \|.\|) \leq \frac{1}{2}\log(1-\eta)^{-1}\,\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.2.3}$$

[10]Namely, for every object $\overline{E}$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, $\mathrm{gv}(\overline{E}) = \mathrm{gv}(\overline{E}^{\mathrm{vect}})$.

[11]All its assertions are rather straightforward consequences of the definitions, with the possible exception of the subadditivity, the proof of which appears in [**RSD17b**, Proof of Proposition 4.14] and in [**Bos20a**, Proposition 6.3].

[12]See Appendix A.

In particular the following estimates are satisfied:

$$0 \leq h^0_\theta(N, \|.\|_\eta) - h^0_\theta(N, \|.\|) \leq \frac{1}{2}\log(1-\eta)^{-1}\ \mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.2.4}$$

A key feature of the estimates (9.2.4) is that they do not explicitly involve the rank n of the $\mathbb{Z}$-module N, but only the relative trace $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$. This will be crucial when extending consequences of these estimates to general quasi-coherent Euclidean sheaves.

In the same vein, observe that, when written in the equivalent form:

$$h^0_\theta(N, \|.\|) \leq h^0_\theta(N, \|.\|_\eta) \leq h^0_\theta(N, \|.\|) + \frac{1}{2}\log(1-\eta)^{-1}\ \mathrm{Tr}(\|.\|'^2/\|.\|^2), \tag{9.2.5}$$

the estimates (9.2.4) extends to quasi-coherent Euclidean[13] sheaves by an easy approximation argument. We leave this to the interested reader, as well as extensions of (9.2.5) to pro-Hermitian vector bundles.

Proof. By introducing a basis of $N_\mathbb{R}$ that is orthogonal both for the Euclidean (semi)norms $\|.\|$ and $\|.\|'$, one sees that the ratio of the Lebesgue measures attached to the Euclidean vector spaces $(N_\mathbb{R}, \|.\|_\eta)$ and $(N_\mathbb{R}, \|.\|)$ is:

$$\prod_{1\leq i\leq n}(1-\eta\lambda_i^2)^{1/2}.$$

This implies the equality:

$$\frac{\mathrm{covol}(N, \|.\|_\eta)}{\mathrm{covol}(N, \|.\|)} = \prod_{1\leq i\leq n}(1-\eta\lambda_i^2)^{1/2},$$

and (9.2.2) follows.

The estimates (9.2.3) follow from Corollary 7.2.5 applied to the morphism Id_N from $(N, \|.\|)$ to $(N, \|.\|_\eta)$, and from (9.2.2). Indeed we have:

$$-\frac{1}{2}\sum_{1\leq 1\leq n}\log(1-\eta\lambda_i^2)^{1/2} \leq \frac{1}{2}\log(1-\eta)^{-1}\sum_{1\leq i\leq n}\lambda_i^2 \leq \frac{1}{2}\log(1-\eta)^{-1}\ \mathrm{Tr}(\|.\|'^2/\|.\|^2),$$

since $-x^{-1}\log(1-x) = \sum_{k=0}^\infty x^k/(k+1)$ is an increasing function of $x \in [0,1)$, and therefore:

$$-\log(1-\eta\lambda_i^2) \leq -\log(1-\eta)\,\lambda_i^2$$

for every $i \in \{1, \cdots, n\}$. □

9.2.2. Rank-free comparisons between h^0_θ and h^0_{Ar}. This subsection parallels the discussion of [**Bos20b**, 3.1]. We show how Lemma 9.2.1 above allows to remove the dependance in the ranks of the Euclidean lattice considered, and as a consequence to obtain statements for arbitrary quasi-coherent Euclidean sheaves.

We will be concerned with the following invariants of a Euclidean quasi-coherent sheaf $\overline{N} = (N, \|.\|)$, see [**Bos20b**, 3.1.1] for the Euclidean case. We define:

$$h^0_{\mathrm{Ar}}(\overline{N}) = \log\left|\{v \in N \mid \|v\| \leq 1\}\right|;$$

$$h^0_{\mathrm{Ar}-}(\overline{N}) = \log\left|\{v \in N \mid \|v\| < 1\}\right|;$$

$$h^0_{\mathrm{Bl}}(\overline{N}) = \log\max_{x\in N_\mathbb{R}}\left|\{v \in N \mid \|v-x\| \leq 1\}\right|.$$

Clearly, we have:

$$h^0_{\mathrm{Ar}-}(\overline{N}) \leq h^0_{\mathrm{Ar}}(\overline{N}) \leq h^0_{\mathrm{Bl}}(\overline{N}).$$

[13]Namely to the situation where $\overline{N} = (N, \|.\|)$ is an arbitrary object of $\mathbf{q}\overline{\mathbf{Coh}}_\mathbb{Z}$, where $\|.\|'$ is an arbitrary Euclidean seminorm over $N_\mathbb{R}$ satisfying $\|.\|' \leq \|.\|$, and where $\|.\|_\eta$ is defined by (9.2.1).

9.2.2.1. One of the general guiding principles of this chapter is the idea that results in the geometry of numbers for Euclidean lattices should generalize to arbitrary Euclidean quasi-coherent sheaves as long as they don't refer to the ranks of the lattices involved. As a first illustration, we obtain the following generalization of [**Bos20b**, Proposition 3.1.3]:

PROPOSITION 9.2.2. *Let $\overline{N}$ be a Euclidean quasi-coherent sheaf. We have:*

$$h^0_{\mathrm{Bl}}(\overline{N}) \leq h^0_\theta(\overline{N}) + \pi. \tag{9.2.6}$$

PROOF. We could adapt the proof of [**Bos20b**, Proposition 3.1.3] and give a direct proof of (9.2.6). However, we illustrate our general method of extending estimates on Euclidean lattices to estimates on quasi-coherent Euclidean shaves.

If $\overline{N}$ is a Euclidean lattice, then the statement is [**Bos20b**, Proposition 3.1.3]. Let us assume that $\overline{N}$ is an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. We have:

$$h^0_{\mathrm{Bl}}(\overline{N}) = h^0_{\mathrm{Bl}}(N_{/\mathrm{tor}}, \|.\|) + \log \left|N_{\mathrm{tor}}\right|$$

and

$$h^0_\theta(\overline{N}) = h^0_\theta(N_{/\mathrm{tor}}, \|.\|) + \log \left|N_{\mathrm{tor}}\right|.$$

As a consequence, after replacing N with $N_{/\mathrm{tor}}$, we may assume that N is torsion free. We may choose a decreasing sequence $(\|.\|_n)_{n\geq 0}$ of Euclidean norms on $N_{\mathbb{R}}$ that converges to $\|.\|$. By the previous paragraph, the estimate (9.2.6) is valid when $\|.\|$ is replaced with $\|.\|_n$. Letting n tend to infinity finishes proves that (9.2.6) holds for $\overline{N}$.

Finally, assume that $\overline{N}$ is an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$. First assume that $h^0_{\mathrm{Bl}}(\overline{N})$ is finite and equals e^n for some positive integer n. Then there exists $x \in N_{\mathbb{R}}$ and elements $v_1, \ldots, v_n$ of N such that, for every integer i between 1 and n, $\|v_i - x\| \leq 1$.

Let C be the $\mathbb{Z}$-submodule of N generated by $v_1, \ldots, v_n$. After replacing x by its projection onto $C_{\mathbb{R}}$, we may assume that x belongs to $C_{\mathbb{R}}$ and we get the equality:

$$h^0_{\mathrm{Bl}}(C, \|.\|) = h^0_{\mathrm{Bl}}(\overline{N}).$$

Using the coherent case treated above, we may write:

$$h^0_{\mathrm{Bl}}(\overline{N}) = h^0_{\mathrm{Bl}}(C, \|.\|) \leq h^0_\theta(C, \|.\|) \leq h^0_\theta(\overline{N}).$$

We leave it to the interested reader to adapt the argument above to the case where $h^0_{\mathrm{Bl}}(\overline{N})$ is infinite. □

9.2.2.2. The following statement is proved exactly as [**Bos20b**, Lemma 3.1.5], once Lemma 9.2.1 is taken into account. We use its notation: N is a free $\mathbb{Z}$-module of finite rank n, $\|.\|$ (resp. $\|.\|'$) is a Euclidean norm (resp. a Euclidean seminorm) on $N_{\mathbb{R}}$ such that:

$$\|.\|' \leq \|.\|.$$

We let $\lambda_n \leq \ldots \leq \lambda_1$ be the singular values of $\|.\|'$ with respect to $\|.\|$ and, for $\eta \in [0, 1)$, we consider the Euclidean norm $\|.\|_\eta$ as above.

LEMMA 9.2.3. *(1) The expression*

$$h^0_\theta(N, \|.\|_\eta) + \frac{1}{2} \sum_{1 \leq 1 \leq n} \log(1 - \eta\lambda_i^2)$$

defines a decreasing function of $\eta \in [0, 1)$.

(2) For any $\eta \in [0, 1)$, we have:

$$\sum_{v \in N} \|v\|'^2 e^{-\pi\|v\|^2_\eta} \leq \frac{1}{2}\Big(\sum_{i=1}^n \frac{\lambda_i^2}{1 - \eta\lambda_i^2}\Big)\Big(\sum_{v \in N} e^{-\pi\|v\|^2_\eta}\Big) \tag{9.2.7}$$

(3) For any $r \in \mathbb{R}_+^*$ and any $\eta \in [0,1)$, we have:

$$\sum_{v \in N, \|v\|' < r} e^{-\pi \|v\|_\eta^2} \geq \Big(1 - \frac{1}{2r^2} \sum_{i=1}^n \frac{\lambda_i^2}{1 - \eta \lambda_i^2}\Big) \sum_{v \in N} e^{-\pi \|v\|_\eta^2}. \tag{9.2.8}$$

Proof. Lemma 9.2.1 shows that the expression $h^0_\theta(N, \|.\|_\eta) + \frac{1}{2} \sum_{1 \leq 1 \leq n} \log(1 - \eta\lambda_i^2)$ is equal to $h^1_\theta(N, \|.\|_\eta) - \widehat{\deg}(N, \|.\|)$, which is indeed a decreasing function of η. This proves (1).

To prove (2), we simply take the derivative of $h^0_\theta(N, \|.\|_\eta) + \frac{1}{2} \sum_{1 \leq 1 \leq n} \log(1 - \eta\lambda_i^2)$ as a function of η.

Finally, to prove (3), we combine (9.2.7) with the straightforward estimate:

$$\sum_{v \in N, \|v\|' \geq r} e^{-\pi \|v\|_\eta^2} \leq \frac{1}{r^2} \sum_{v \in N} \|v\|'^2 e^{-\pi \|v\|_\eta^2}.$$

This yields:

$$\sum_{v \in N, \|v\|' \geq r} e^{-\pi \|v\|_\eta^2} \leq \frac{1}{2r^2} \Big(\sum_{i=1}^n \frac{\lambda_i^2}{1 - \eta\lambda_i^2}\Big) \Big(\sum_{v \in N} e^{-\pi \|v\|_\eta^2}\Big)$$

or, equivalently:

$$\sum_{v \in N, \|v\|' < r} e^{-\pi \|v\|_\eta^2} \geq \Big(1 - \frac{1}{2r^2} \sum_{i=1}^n \frac{\lambda_i^2}{1 - \eta \lambda_i^2}\Big) \sum_{v \in N} e^{-\pi \|v\|_\eta^2}.$$

□

9.2.2.3. Theorem 9.2.4 below is a first instance concerning theta invariants of the general principle, which plays a key role in this monograph and its applications, that estimates of geometry of numbers which involve the rank of Euclidean lattices still hold in the infinite setting, provided ranks are suitably replaced by relative traces.

Theorem 9.2.4. *Let $\overline{N} = (N, \|.\|)$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ and let $\|.\|'$ be a Euclidean seminorm on $N_{\mathbb{R}}$ such that $\|.\|' \leq \|.\|$. Let δ be a real number such that:*

$$\mathrm{Tr}(\|.\|'^2 / \|.\|^2) < 2e^{2\delta}.$$

Then the following inequalities holds:

$$h^0_{\mathrm{Ar}-}(N, e^{-\delta}\|.\|') \geq h^0_\theta(\overline{N}) + \log\big(1 - \frac{1}{2} e^{-2\delta} \mathrm{Tr}(\|.\|'^2/\|.\|^2)\big) \tag{9.2.9}$$

and:

$$h^0_{\mathrm{Ar}-}(N, \|.\|') \geq h^0_\theta(N, e^{\delta}\|.\|) + \log\big(1 - \frac{1}{2} e^{-2\delta} \mathrm{Tr}(\|.\|'^2/\|.\|^2)\big) \tag{9.2.10}$$

Proof. We first note that (9.2.10) follows from (9.2.9) applied after replacing $\|.\|$ with $e^{-\delta}\|.\|$ and $\|.\|'$ with $e^{-\delta}\|.\|'$. We will prove the estimate (9.2.9).

First consider the case where N is a free $\mathbb{Z}$-module of rank n and $\|.\|$ is a Euclidean norm. Applying (9.2.8) with $\eta = 0$ and $r = e^\delta$, we get:

$$\sum_{v \in N, \|v\|' < e^\delta} e^{-\pi\|v\|^2} \geq \Big(1 - \frac{1}{2} e^{-2\delta} \mathrm{Tr}(\|.\|'^2/\|.\|^2)\Big) \sum_{v \in N} e^{-\pi\|v\|^2}.$$

The left-hand side of this inequality is bounded above by $\exp(h^0_{\mathrm{Ar}-}(N, e^{-\delta}\|.\|')$. Taking logarithms, we obtain (9.2.9).

Once the estimate (9.2.9) has been shown to hold for Euclidean lattices, its derivation in the general case follows the exact same steps as in Proposition 9.2.2. □

As a direct consequence of Theorem 9.2.4, we obtain the following analogue of [**Bos20b**, Corollary 3.1.2], which may be considered as a variant of Minkowski's first theorem in infinite rank:

COROLLARY 9.2.5. *With the notation of Theorem 9.2.4, we have:*

$$\lambda_1(N, \|.\|') \geq e^{\delta} \implies h^0_\theta(\overline{N}) \leq \log\Big(1 - \frac{1}{2}e^{-2\delta}\mathrm{Tr}(\|.\|'^2/\|.\|^2)\Big)^{-1}$$

and:

$$\lambda_1(N, \|.\|') \geq 1 \implies h^0_\theta(N, e^{\delta}\|.\|) \leq \log\Big(1 - \frac{1}{2}e^{-2\delta}\mathrm{Tr}(\|.\|'^2/\|.\|^2)\Big)^{-1}.$$

We leave it to the reader to state and prove an analogue of Corollary 9.2.5 for the relative version of first minima.

9.2.3. Bounding first minima in terms of theta invariants. We now turn our attention to the first minimum and its relative version, and will derive below estimates that are analogue to [**Bos20b**, 3.2.2]. Aside from their own intrinsic interest, and from the illustration they provide of our general method of replacing ranks by relative traces to prove estimates in infinite rank, those estimates will be essential to our proof of general transference inequalities in Section 9.5.

9.2.3.1. Let N be a free $\mathbb{Z}$-module of finite rank, and let $\|.\|$ (resp. $\|.\|'$) be a Euclidean norm (resp. a Euclidean seminorm) on $N_\mathbb{R}$ such that:

$$\|.\|' \leq \|.\|.$$

For $\eta \in [0,1)$, let us denote by $\|.\|_\eta$ the Euclidean norm on $N_\mathbb{R}$ defined by the equality:

$$\|x\|^2_\eta := \|x\|^2 - \eta\|x\|'^2, \tag{9.2.11}$$

for every $x \in N_\mathbb{R}$.

If K is a sub-$\mathbb{Z}$-module of N, let $\lambda_1(N, K, \|.\|')$ denote the quantity

$$\lambda_1(N, K, \|.\|') := \min_{v \in N \setminus K} \|v\|'.$$

If $K = 0$, this is simply the first minimum of $(N, \|.\|')$. Write $\overline{N} = (N, \|.\|)$ and $\overline{K} = (K, \|.\|)$.

PROPOSITION 9.2.6. *For any $\eta \in [0,1)$, the following inequality holds:*

$$e^{h^0_\theta(\overline{K}) - h^0_\theta(\overline{N})} + e^{-\pi\eta\lambda_1(N,K,\|.\|')^2 + 1/2\log(1-\eta)^{-1}\,\mathrm{Tr}(\|.\|'^2/\|.\|^2)} \geq 1. \tag{9.2.12}$$

PROOF. We have:

$$\begin{aligned} e^{h^0_\theta(\overline{N})} &= \sum_{c \in N/K}\sum_{v \in c+K} e^{-\pi\|v\|^2} \\ &= \sum_{v \in K} e^{-\pi\|v\|^2} + \sum_{c \in (N/K)\setminus\{0\}}\sum_{v \in c+K} e^{-\pi\|v\|^2} \\ &= e^{h^0_\theta(\overline{K})} + \sum_{c \in (N/K)\setminus\{0\}}\sum_{v \in c+K} e^{-\pi\|v\|^2_\eta - \pi\eta\|v\|'^2} \qquad (9.2.13) \\ &\leq e^{h^0_\theta(\overline{K})} + e^{-\pi\lambda_1(N,K,\|.\|')^2}\sum_{c \in (N/K)\setminus\{0\}}\sum_{v \in c+K} e^{-\pi\|v\|^2_\eta} \\ &\leq e^{h^0_\theta(\overline{K})} + e^{-\pi\lambda_1(N,K,\|.\|')^2 + h^0_\theta(N, \|.\|_\eta)}. \end{aligned}$$

Applying (9.2.4), this proves (9.2.13). ∎

COROLLARY 9.2.7. *Write $T = \mathrm{Tr}(\|.\|'^2/\|.\|^2)$. Assume that we have:*

$$\lambda_1(N, K, \|.\|') > \sqrt{\frac{T}{2\pi}}$$

and define:

$$\widetilde{\lambda} := \sqrt{\frac{2\pi}{T}}\lambda_1\,(N, K, \|.\|') \in (1, +\infty).$$

Then:

$$e^{-(h^0_\theta(\overline{N})-h^0_\theta(\overline{K}))} + e^{T/2(1-\widetilde{\lambda}^2+2\log\widetilde{\lambda})} \geq 1. \tag{9.2.14}$$

Equivalently:

$$h^0_\theta(\overline{N}) - h^0_\theta(\overline{K}) \leq \log\left(1 - \left(\widetilde{\lambda}e^{-1/2(\widetilde{\lambda}^2-1)}\right)^T\right)^{-1}. \tag{9.2.15}$$

Proof. We apply the estimate (9.2.12) with $\eta = 1 - \widetilde{\lambda}^2$; the reader might check that this is the value of η that gives the optimal inequality. Then

$$-\pi\eta\,\lambda_1(N, K, \|.\|')^2 + (1/2)\log(1-\eta)^{-1}\,T = \frac{T}{2}\big(1 - \widetilde{\lambda}^2 + 2\log\widetilde{\lambda}\big)$$

and (9.2.12) is equivalent to (9.2.14). □

Note that when $K = 0$ and $\|.\| = \|.\|'$, then $h^0_\theta(\overline{K}) = 0$, T is the rank of N and Corollary 9.2.7 is equivalent to [**Bos20b**, Corollary 3.2.3].

9.2.3.2. We will use the general invariant $\lambda_1(N, K, \|.\|')$ below when proving general transference inequalities. Here we simply focus on the case where $K = 0$, so that we are simply considering the first minimum. We obtain the following general statement:

Theorem 9.2.8. *Let $\overline{N} = (N, \|.\|)$ be an object of $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ and let $\|.\|'$ be a Euclidean seminorm on $N_{\mathbb{R}}$ such that $\|.\|' \leq \|.\|$. Assume that the relative trace*

$$T := \mathrm{Tr}(\|.\|'^2/\|.\|^2)$$

is finite, and that we have:

$$\lambda_1(N, \|.\|') > \sqrt{\frac{T}{2\pi}}.$$

Define:

$$\widetilde{\lambda} := \sqrt{\frac{2\pi}{T}}\,\lambda_1(N, \|.\|') \in (1, +\infty).$$

Then:

$$h^0_\theta(\overline{N}) \leq \log\left(1 - \left(\widetilde{\lambda}e^{-1/2(\widetilde{\lambda}^2-1)}\right)^T\right)^{-1}. \tag{9.2.16}$$

Proof. When $\overline{N}$ is a Euclidean lattice, (9.2.16) is simply (9.2.14) in Corollary 9.2.7. The derivation of (9.2.16) in the general case from the cases of Euclidean lattices follows the exact same steps as in Proposition 9.2.2, once taken into account the fact that

$$\widetilde{\lambda} \mapsto \log\left(1 - \left(\widetilde{\lambda}e^{-1/2(\widetilde{\lambda}^2-1)}\right)^T\right)^{-1}$$

is an decreasing function of $\widetilde{\lambda} \in (1, +\infty)$. □

We leave it to the reader to state and prove analogues in infinite rank of Corollary 9.2.7.

9.3. Bounding Covering Radii in Terms of Theta Invariants

9.3.1. Estimates on Banaszczyk functions.

9.3.1.1. The bounds on covering radii in terms of theta invariants that we shall establish in this section will be consequences of Lemma 9.2.1, the upper bounds on the Banaszczyk function $b_{\overline{N}}$ attached to an object $\overline{N}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ in terms of $\overline{h}^1_\theta(\overline{N})$ established in Subsection 8.3.3, combined with the next lemma.

LEMMA 9.3.1. *Let N be a finitely generated free $\mathbb{Z}$-module, and $\|.\|'$ (resp. $\|.\|''$) a Euclidean seminorm (resp. a Euclidean norm) on the $\mathbb{R}$-vector space $N_{\mathbb{R}}$.*

We define a Euclidean norm $\|.\|$ on $N_{\mathbb{R}}$ by the equality:

$$\|.\|^2 = \|.\|'^2 + \|.\|''^2.$$

and we denote by $\overline{N}$ (resp. by $\overline{N}''$) the Euclidean lattice $(N, \|.\|)$ (resp. $(N, \|.\|'')$). Moreover for every $x \in N_{\mathbb{R}}$, we let:

$$d'(x, N) := d_{\|.\|'}(x, N) = \inf_{n \in N} \|x - n\|'.$$

Then the following inequality holds for every $x \in N_{\mathbb{R}}$:

$$b_{\overline{N}''}(x) + d'(x, N)^2 \leq b_{\overline{N}}(x) + \pi^{-1}\big(h^0_\theta(\overline{N}'') - h^0_\theta(\overline{N})\big). \tag{9.3.1}$$

In (9.3.1), $b_{\overline{N}''}$ and $b_{\overline{N}}$ denote the Banaszczyk functions on $N_{\mathbb{R}}$ attached to the Euclidean lattices $\overline{N}''$ and $\overline{N}$, as defined in 7.1.2.1 above.

PROOF. Let x be an element of $N_{\mathbb{R}}$. For any $v \in N$, we have:

$$\|v - x\|'^2 \geq d'(x, N)^2.$$

Moreover, according to the definition of the Banaszczyk function $B_{\overline{N}}$ and of the Euclidean lattice $\overline{N}$, we have:

$$B_{\overline{N}}(x) = e^{-h^0_\theta(\overline{N})} \sum_{v \in N} e^{-\pi \|v - x\|^2} = e^{-h^0_\theta(\overline{N})} \sum_{v \in N} e^{-\pi \|v - x\|'^2 - \pi \|v - x\|''^2}.$$

Consequently the following inequality holds:

$$B_{\overline{N}}(x) \leq e^{-h^0_\theta(\overline{N}) - \pi d'(x,N)^2} \sum_{v \in N} e^{-\pi \|v - x\|''^2} = e^{-h^0_\theta(\overline{N}) - \pi d'(x,N)^2 + h^0_\theta(\overline{N}'')} B_{\overline{N}''}(x),$$

and (9.3.1) follows by taking logarithms. □

9.3.1.2. The following proposition is technically the central result of this section. At this stage, it is a rather formal consequence of Lemmas 9.2.1 and 9.3.1 established in the previous subsection.

PROPOSITION 9.3.2. *Let $\overline{N} = (N, \|.\|)$ be an object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$, let $\|.\|'$ be a Euclidean seminorm on $N_{\mathbb{R}}$ such that*

$$\|.\|' \leq \|.\|$$

on $N_{\mathbb{R}}$, and let η be an element of the interval $[0, 1)$.

We define a Euclidean seminorm $\|.\|_\eta$ on $M_{\mathbb{R}}$ by the equality:

$$\|x\|^2_\eta := \|x\|^2 - \eta \|x\|'^2, \tag{9.3.2}$$

for every $x \in N_{\mathbb{R}}$, and we denote by $\overline{N}_\eta$ the object $(N, \|.\|_\eta)$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$. Moreover, for every $x \in N_{\mathbb{R}}$, we let:

$$d'(x, N) := d_{\|.\|'}(x, N_{/\mathrm{tor}}) = \inf_{n \in N_{/\mathrm{tor}}} \|x - n\|'.$$

Then the following inequality holds, for every $x \in N_{\mathbb{R}}$:

$$b_{\overline{N}_\eta}(x) + \eta\, d'(x, N)^2 \leq b_{\overline{N}}(x) + (2\pi)^{-1} \log(1 - \eta)^{-1} \operatorname{Tr}(\|.\|'^2 / \|.\|^2). \tag{9.3.3}$$

In particular, we have:

$$\eta\, d'(x, N)^2 \leq b_{\overline{N}}(x) + (2\pi)^{-1} \log(1 - \eta)^{-1} \operatorname{Tr}(\|.\|'^2 / \|.\|^2). \tag{9.3.4}$$

PROOF. We shall divide the proof of the estimate (9.3.3) in three successive steps: firstly when $\overline{N}$ is a Euclidean lattice; secondly when $\overline{N}$ is an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$; finally when $\overline{N}$ is an arbitrary object of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$. The estimate (9.3.4) will follow from (9.3.3) and the non-negativity of the function $b_{\overline{N}_\eta}$.

(1) When $\overline{N}$ is a Euclidean lattice, the inequality (9.3.3) is a straightforward consequence of Lemma 9.3.1 applied with $\|.\|'$ replaced by $\eta^{1/2}\|.\|'$ and $\|''$ by $\|.\|_\eta$, and of Lemma 9.2.1. Indeed we have, for every $x \in N_R$:

$$\eta\, d_{\eta^{1/2}\|.\|'}(x, N)^2 = \inf_{n \in N} \eta \|x - n\|'^2 = \eta\, d'(x, N)^2.$$

(2) Let us assume that $\overline{N}$ is an object of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

The various terms of (9.3.3) are unchanged when $\overline{N} = (N, \|.\|)$ is replaced by:

$$\overline{N}_{/\mathrm{tor}} := (N_{/\mathrm{tor}}, \|.\|).$$

Consequently, to establish (9.3.3), we may also assume that N is torsion free.

Then we may choose a decreasing sequence $(\|.\|_n)_{n\in\mathbb{N}}$ of Euclidean norms on $N_{\mathbb{R}}$ that converges to $\|.\|$. According to part (1) of this proof, for every $n \in \mathbb{N}$, the estimate (9.3.3) is valid when $\|.\|$ is replaced by $\|.\|_n$ and $\|.\|_\eta$ is replaced by the Euclidean norm $\|.\|_{n,\eta}$ defined by the equality:

$$\|x\|^2_{n,\eta} := \|x\|^2_n - \eta \|x\|'^2.$$

The decreasing sequence of Euclidean norms $(\|.\|_{n,\eta})_{n\in\mathbb{N}}$ converges to $\|.\|_\eta$, and the validity of (9.3.3) follows by taking the limit when n goes to infinity, according to the downward continuity of Banaszczyk functions as functions of the seminorms observed in 7.7.1.1 above; see notably (7.7.4).

(3) Let us finally consider an arbitrary object $\overline{N}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$. According to part (2) of this proof, for every $C \in \mathrm{coh}(N)$ — that is for every finitely generated $\mathbb{Z}$-submodule C of N — and every x in $C_{\mathbb{R}}$, the following inequality holds:

$$b_{(C,\|.\|_{\eta|C_{\mathbb{R}}})}(x) + \eta\, d'(x, C)^2 \leq b_{(C,\|.\|_{|C_{\mathbb{R}}})}(x) + (2\pi)^{-1}\log(1-\eta)^{-1}\,\mathrm{Tr}(\|.\|'^2_{|C_{\mathbb{R}}}/\|.\|^2_{|C_{\mathbb{R}}}). \tag{9.3.5}$$

Observe that, for every $C \in \mathrm{coh}(N)$, the following inequality holds:

$$\mathrm{Tr}(\|.\|'^2_{|C_{\mathbb{R}}}/\|.\|^2_{|C_{\mathbb{R}}}) \leq \mathrm{Tr}(\|.\|'^2/\|.\|^2).$$

This follows from the very definition of the relative trace of the square of two Euclidean semi norms; see Appendix B. Consequently, from (9.3.5), we derive the estimate:

$$b_{(C,\|.\|_{\eta|C_{\mathbb{R}}})}(x) + \eta\, d'(x, C)^2 \leq b_{(C,\|.\|_{|C_{\mathbb{R}}})}(x) + (2\pi)^{-1}\log(1-\eta)^{-1}\,\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.3.6}$$

By the very construction of Banaszczyk functions attached to objects of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ in 8.3.1.1 above, the Banaszczyk functions $b_{(C,\|.\|_{\eta|C_{\mathbb{R}}})}(x)$ and $b_{(C,\|.\|_{|C_{\mathbb{R}}})}(x)$, as functions on the directed set

$$\big(\{C \in \mathrm{coh}(N) \mid x \in C_{\mathbb{R}}\}, \subseteq\big) \tag{9.3.7}$$

are decreasing, and satisfy:

$$\lim_{C\in\mathrm{coh}(N)} b_{(C,\|.\|_{\eta|C_{\mathbb{R}}})}(x) = \inf_{\substack{C\in\mathrm{coh}(N)\\ x\in C_{\mathbb{R}}}} b_{(C,\|.\|_{\eta|C_{\mathbb{R}}})}(x) = b_{\overline{N}_\eta}(x) \tag{9.3.8}$$

and

$$\lim_{C\in\mathrm{coh}(N)} b_{(C,\|.\|_{|C_{\mathbb{R}}})}(x) = \inf_{\substack{C\in\mathrm{coh}(N)\\ x\in C_{\mathbb{R}}}} b_{(C,\|.\|_{|C_{\mathbb{R}}})}(x) = b_{\overline{N}}(x). \tag{9.3.9}$$

The "distance to x" $d'(x, C)$ also is a decreasing function of C in the directed set (9.3.7), and it is straightforward that we have:

$$\lim_{C\in\mathrm{coh}(N)} d'(x, C) = \inf_{\substack{C\in\mathrm{coh}(N)\\ x\in C_{\mathbb{R}}}} d'(x, C) = d'(x, N). \tag{9.3.10}$$

Using (9.3.8), (9.3.9), and (9.3.10), the estimate (9.3.3) follows from (9.3.6) by taking the infimum, or indeed the limit, over C in the directed set (9.3.7). □

9.3.2. The main inequality on covering radii.

9.3.2.1. Recall that in Section 7.8.2, we have introduced the function:

$$\kappa : \mathbb{R}_+ \longrightarrow [0, +\infty]$$

defined by:

$$\kappa(x) := \begin{cases} \pi^{-1}\log(2e^{-x}-1)^{-1} & \text{if } 0 \leq x < \log 2, \\ +\infty & \text{if } x \geq \log 2. \end{cases}$$

In Subsection 8.3.3, we have shown that, for every object $\overline{N}$ of $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ and every $x \in N_{\mathbb{R}}$, the following inequality holds:

$$b_{\overline{N}}(x) \leq \kappa(\overline{h}^1_\theta(\overline{N})); \tag{9.3.11}$$

see Corollary 8.3.11 and (8.3.31).

More generally, if M is a countably generated $\mathbb{Z}$-module, $\overline{N}$ an object of $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$, and $f : M \to N$ a morphism of $\mathbb{Z}$-modules, and $f : M \to N$ a morphism of $\mathbb{Z}$-modules, then for every $x \in M_{\mathbb{R}}$, the following inequality holds:

$$b_{\overline{N}}(f_{\mathbb{R}}(x)) \leq \kappa\big(\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N})\big); \tag{9.3.12}$$

see Proposition 8.3.8 and (8.3.18).

By combining the upper-bounds (9.3.11) and (9.3.12) on the Banaszczyk function $b_{\overline{N}}$ with the last inequality (9.3.4) in Proposition 9.3.2, we may finally establish the following theorem.

THEOREM 9.3.3. *Let $\overline{N} := (N, \|.\|)$ be an object of $\mathbf{q\overline{Coh}}_{\mathbb{Z}}$ and $\|.\|'$ a Euclidean seminorm on $N_{\mathbb{R}}$ such that $\|.\|' \leq \|.\|$. Then, if we let $\overline{N}' := (N, \|.\|')$, the following inequality holds for every $\eta \in [0,1)$:*

$$\eta\, \rho(\overline{N}')^2 \leq \kappa\big(\overline{h}^1_\theta(\overline{N})\big) + (2\pi)^{-1}\log(1-\eta)^{-1}\,\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.3.13}$$

More generally, for every countably generated $\mathbb{Z}$-module M and every morphism of $\mathbb{Z}$-modules $f : M \to N$, the following equality holds:

$$\eta\, \rho(f : M \to \overline{N}')^2 \leq \kappa\big(\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N})\big) + (2\pi)^{-1}\log(1-\eta)^{-1}\,\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.3.14}$$

Due to its importance in the applications of the formalism developed in this monograph, we will refer to the estimate (9.3.13) or to its more general version (9.3.14) as the *main inequality on covering radii*.

PROOF. The estimate (9.3.13) (resp. (9.3.14)) follows from the estimate (9.3.4) by taking the supremum over x in $N_{\mathbb{R}}$ (resp. in $f_{\mathbb{R}}(M_{\mathbb{R}})$) and by using (9.3.11) (resp. (9.3.12)). Indeed by the very definition of the (relative) covering radius $\rho(\overline{N}')$ (resp. $\rho(f : M \to \overline{N}')$), we have:

$$\rho(\overline{N}')^2 = \sup_{x \in N_{\mathbb{R}}} d'(x, N)^2 \quad (\text{resp. } \rho(f : M \to \overline{N}')^2 = \sup\nolimits_{x \in M_{\mathbb{R}}} d'(f_{\mathbb{R}}(x), N)^2). \qquad \square$$

9.3.2.2. Observe that, since $\kappa(\delta) = +\infty$ if $\delta \geq \log 2$, the estimate (9.3.13) (resp. (9.3.14)) has a non-trivial content only when $\overline{h}^1_\theta(\overline{N})$ (resp. $\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N})$) is smaller than $\log 2$.

In the sequel, rather than using the full strength of these estimates, we will only use the following straightforward consequence: $\rho(\overline{N}')^2$ *(resp. $\rho(f : M \to \overline{N}')^2$)) is bounded from above by some positive multiple of*

$$\overline{h}^1_\theta(\overline{N}) + +\mathrm{Tr}(\|.\|'^2/\|.\|^2) \quad \textit{(resp. of } \overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) + +\mathrm{Tr}(\|.\|'^2/\|.\|^2))$$

provided $\overline{h}^1_\theta(\overline{N})$ *(resp.* $\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N})$*) is small enough.* This indeed follows from (9.3.13) (resp. (9.3.14)) for some fixed value of η in $(0,1)$, and from the observation that:

$$\kappa(\delta) = O(\delta) \quad \text{when } \delta \to 0.$$

To derive an explicit version of such a bound, we may choose $\eta = 1/2$. Then:

$$\frac{1}{2\pi\eta}\log(1-\eta)^{-1} = \frac{\log 2}{\pi} = 0.22\ldots < \frac{1}{4}. \tag{9.3.15}$$

It was shown in (7.8.5) that, for every δ in $[0, 1/2]$, we have:

$$\kappa(\delta) \leq \delta. \tag{9.3.16}$$

Combined with (9.3.15) and (9.3.16), the special case $\eta = 1/2$ of Theorem 9.3.3 implies:

COROLLARY 9.3.4. *Let* $\overline{N} := (N, \|.\|)$ *be an object of* $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ *and* $\|.\|'$ *a Euclidean seminorm on* $N_{\mathbb{R}}$ *such that* $\|.\|' \leq \|.\|$*. Then, if we let* $\overline{N}' := (N, \|.\|')$*, the following inequality holds when* $\overline{h}^1_\theta(\overline{N}) \leq 1/2$*:*

$$\rho(\overline{N}')^2 \leq 2\,\overline{h}^1_\theta(\overline{N}) + \frac{1}{4}\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.3.17}$$

More generally, for every countably generated $\mathbb{Z}$*-module* M *and every morphism of* $\mathbb{Z}$*-modules* $f : M \to N$*, the following equality holds when* $\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) \leq 1/2$ *:*

$$\rho(f : M \to \overline{N}')^2 \leq 2\,\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) + \frac{1}{4}\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.3.18}$$

As in Corollary 9.2.7, we may optimize our main estimate with regards to η as long as ρ is large enough.

COROLLARY 9.3.5. *Write* $T = \mathrm{Tr}(\|.\|'^2/\|.\|^2)$*. Assume that we have:*

$$\rho(f : M \to \overline{N}') > \sqrt{\frac{T}{2\pi}}$$

and define:

$$\widetilde{\rho} := \sqrt{\frac{2\pi}{T}}\rho(f : M \to \overline{N}') \in (1, +\infty).$$

Then:

$$\kappa\big(\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N})\big) \geq -\frac{T}{2\pi}\big(1 - \widetilde{\rho}^2 + 2\log\widetilde{\rho}\big). \tag{9.3.19}$$

Equivalently:

$$2e^{-\overline{\mathrm{rk}}^1_\theta(f:M\to\overline{N})} \leq 1 + e^{\frac{T}{2}\left(1-\widetilde{\rho}^2+2\log\widetilde{\rho}\right)}. \tag{9.3.20}$$

PROOF. We apply the estimate (9.3.14) with $\eta = 1 - \widetilde{\rho}^2$; here again, this is the value of η that gives the optimal inequality. Then

$$-\eta\rho(f : M \to \overline{N}')^2 + (2\pi)^{-1}\log(1-\eta)^{-1}T = \frac{T}{2\pi}\big(1 - \widetilde{\rho}^2 + 2\log\widetilde{\rho}\big)$$

and (9.3.14) is equivalent to (9.3.19).

Clearly, (9.3.19) is equivalent to (9.3.20) when $\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) < \log 2$. When $\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) > \log 2$, then the left-hand side of (9.3.20) is bounded above by 1, so that (9.3.20) holds. $\square$

9.3.2.3. The upper bounds on covering radii established in Theorem 9.3.3 and Corollary 9.3.4 admit variants concerning the upper and lower covering radii introduced in Subsections 6.3.3 and 6.7.3.

Consider for instance the inequality (9.3.18) in Corollary 9.3.4. For every submodule $\widetilde{N}$ of N in $\mathrm{coft}(N)$, we may consider the morphism:

$$f_{\widetilde{N}} : f(M) \longrightarrow N/\widetilde{N},$$

defined as the composition of the inclusion $f(M) \hookrightarrow N$ with the quotient morphism $N \to N/\widetilde{N}$, and the quotient Euclidean coherent sheaves $\overline{N/\widetilde{N}}$ and $\overline{N/\widetilde{N}}'$. These are defined by the Euclidean seminorms $\|.\|_{\overline{N/\widetilde{N}}}$ and $\|.\|'_{\overline{N/\widetilde{N}}}$ on $(N/\widetilde{N})_{\mathbb{R}}$ quotients of the seminorms $\|.\|$ and $\|.\|'$ on $N_{\mathbb{R}}$.

Clearly, we have:

$$\|.\|_{\overline{N/\widetilde{N}}'} \leq \|.\|_{\overline{N/\widetilde{N}}},$$

and applied to the morphisms $f_{\widetilde{N}} : f(M) \to \overline{N/\widetilde{N}}$ and $f_{\widetilde{N}} : f(M) \to \overline{N/\widetilde{N}}'$ in $\overline{\mathbf{Coh}}_{\mathbb{Z}}$, the estimate (9.3.18) shows that:

$$\rho\Big(f_{\widetilde{N}} : f(M) \to \overline{N/\widetilde{N}}'\Big)^2 \leq 2\,\mathrm{r}k^1_\theta\Big(f_{\widetilde{N}} : f(M) \to \overline{N/\widetilde{N}}\Big) + \frac{1}{4}\mathrm{Tr}\Big(\|.\|^2_{\overline{N/\widetilde{N}}'}/\|.\|^2_{\overline{N/\widetilde{N}}}\Big), \tag{9.3.21}$$

provided $\mathrm{r}k^1_\theta\big(f_{\widetilde{N}} : f(M) \to \overline{N/\widetilde{N}}\big) \leq 1/2$.

Moreover, the following inequality holds:

$$\mathrm{Tr}\Big(\|.\|^2_{\overline{N/\widetilde{N}}'}/\|.\|^2_{\overline{N/\widetilde{N}}}\Big) \leq \mathrm{Tr}\big(\|.\|'^2/\|.\|^2\big).$$

Therefore, by taking the limit over $\widetilde{N}$ in the directed set $(\mathrm{coft}(N), \supseteq)$ and using the definitions (6.7.17) and (5.4.2) of $\underline{\rho}(f : M \to \overline{N}')$ and $\underline{\mathrm{r}k}^1_\theta(f : M \to \overline{N})$, we obtain that, *when* $\underline{\mathrm{r}k}^1_\theta(f : M \to \overline{N}) \leq 1/2$, *the following "lower variant" of the inequality* (9.3.26) *holds*:

$$\underline{\rho}(f : M \to \overline{N}')^2 \leq 2\,\underline{\mathrm{r}k}^1_\theta(f : M \to \overline{N}) + \frac{1}{4}\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.3.22}$$

In turn, specialized to $M := N$ and $f := \mathrm{Id}_N$, (9.3.22) shows that, *when* $\underline{h}^1_\theta(\overline{N}) \leq 1/2$, *the following variant of* (9.3.17) *holds*:

$$\underline{\rho}(\overline{N}')^2 \leq 2\,\underline{h}^1_\theta(\overline{N}) + \frac{1}{4}\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.3.23}$$

The same argument, starting this time from the estimate (9.3.20) in Corollary 9.3.5, proves that, writing T for the relative trace $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$, *when* $\underline{\rho}(f : M \to \overline{N}') > \sqrt{\frac{T}{2\pi}}$, *the following "lower variant" of the inequality* (9.3.19) *holds*:

$$2e^{-\underline{\mathrm{r}k}^1_\theta(f:M\to\overline{N})} \leq 1 + e^{\frac{T}{2}\left(1-\underline{\widetilde{\rho}}^2+2\log\underline{\widetilde{\rho}}\right)}. \tag{9.3.24}$$

Here we wrote $\underline{\widetilde{\rho}} = \sqrt{T/2\pi}\,\underline{\rho}(f : M \to \overline{N}')$.

9.3.2.4. Similarly, for every submodule C of N in $\mathrm{coh}(N)$, we may apply (9.3.18) to the inclusion morphism $f^C : C \cap \mathrm{im}\, f \to C$ and to the Euclidean coherent sheaves $\overline{C} := (C, \|.\|_{|C_{\mathbb{R}}})$ and $\overline{C}' := (C, \|.\|_{|C_{\mathbb{R}}})$, and we obtain:

$$\rho\big(f^C : C \cap \mathrm{im}\, f \to \overline{C}'\big)^2 \leq 2\,\mathrm{rk}^1_\theta\big(f^C : C \cap \mathrm{im}\, f \to \overline{C}\big) + \frac{1}{4}\mathrm{Tr}\big(\|.\|'^2_{|C_{\mathbb{R}}}/\|.\|^2_{|C_{\mathbb{R}}}\big). \tag{9.3.25}$$

Indeed we have: $\|.\|'_{|C_{\mathbb{R}}} \leq \|.\|_{|C_{\mathbb{R}}}$.

By using the inequality:

$$\mathrm{Tr}\big(\|.\|'^2_{|C_{\mathbb{R}}}/\|.\|^2_{|C_{\mathbb{R}}}\big) \leq \mathrm{Tr}\big(\|.\|'^2/\|.\|^2\big)$$

and by taking the lower limit over C in the directed set $(\mathrm{coh}(N), \subseteq)$ and using the definitions (6.7.18) and (5.5.1) of $\overline{\rho}(f : M \to \overline{N}')$ and $\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N})$, we deduce from (9.3.25) that, *when* $\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) < 1/2$, *the following inequality holds:*

$$\overline{\rho}(f : M \to \overline{N}')^2 \leq 2\, \overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) + \frac{1}{4}\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.3.26}$$

In turn, this implies that, *when* $h^1_\theta(\overline{N}) < 1/2$, *the following inequality holds:*

$$\overline{\rho}(\overline{N}')^2 \leq 2\, \overline{h}^1_\theta(\overline{N}) + \frac{1}{4}\mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{9.3.27}$$

We leave it to the reader to establish by similar arguments the validity of lower and upper variants of the estimates (9.3.13) and (9.3.14) in Theorem 9.3.3, and to deduce the validity of (9.3.26) (resp. (9.3.27)) when $\overline{\mathrm{rk}}^1_\theta(f : M \to \overline{N}) \leq 1/2$ (resp. when $h^1_\theta(\overline{N}) \leq 1/2$).

Recall finally that, according to Proposition 6.7.16 (resp. to Proposition 6.3.10), we have:

$$\rho(f : M \to \overline{N}') \leq \overline{\rho}(f : M \to \overline{N}') \quad (\text{resp. } \rho(\overline{N}') \leq \overline{\rho}(\overline{N}')).$$

Consequently the estimate (9.3.26) (resp. (9.3.27)) is a strengthening of (9.3.18) (resp. of (9.3.17))

9.4. Applications of the Main Inequality on Covering Radii

In this section, we spell out various consequences of the estimates relating covering radii and theta invariants established in the previous two sections, with a special emphasis on the consequences of the main inequality on covering radii established in Theorem 9.3.3.

9.4.1. Application to Euclidean lattices: comparing $\log \rho(\overline{E})$ and $s^1_{\theta,\varepsilon}(\overline{E})$. Let $\overline{E} := (E, \|.\|)$ be a Euclidean lattice of rank n.

According to Proposition 9.1.3, we have:

$$h^1_\theta(\overline{E}) \leq \pi \rho(\overline{E})^2. \tag{9.4.1}$$

Moreover Corollary 9.3.4 applied to $\overline{N} = \overline{E}$ and to $\|.\|' = \|.\|$ implies:

$$h^1_\theta(\overline{E}) \leq 1/2 \Longrightarrow \rho(\overline{E})^2 \leq 1 + n/4. \tag{9.4.2}$$

In this subsection, we show that, by combining the estimates (9.4.1) and (9.4.2) and the 1-homogeneity of the covering radius $\rho(\overline{E})$ as a function of the Euclidean norm $\|.\|$ that defines $\overline{E}$, we may establish comparison estimates relating the covering radius $\rho(\overline{E})$ and the smoothing parameter $\eta_\varepsilon(\overline{E})$ of $\overline{E}$, as defined by Micciancio and Regev [**MR07**].

From the perspective of the analogy between number fields and function fields, it is actually more natural to consider the logarithm of the covering radius — in terms of which the 1-homogeneity of the covering radius may be expressed as the equality:

$$\log \rho(\overline{E} \otimes \overline{\mathcal{O}}(\delta)) = \log \rho(\overline{E}) - \delta, \tag{9.4.3}$$

valid for every $\delta \in \mathbb{R}$ — and the logarithmic variant $s^1_{\theta,\varepsilon}(\overline{E})$ of the smoothing parameter $\eta_\varepsilon(\overline{E})$ considered in [**Bos20a**].

9.4.1.1. *The invariants* $\eta_\varepsilon(\overline{E})$ *and* $s^1_{\theta,\overline{E}}(\overline{E})$*.* In this paragraph, we briefly recall the definitions and the basic properties of the invariants $\eta_\varepsilon(\overline{E})$ and $s^1_{\theta,\overline{E}}(\overline{E})$. We refer the reader to [**Bos20a**] for more details and references, and notably for a discussion of the interpretation of $s^1_{\theta,\overline{E}}(\overline{E})$ as a "threshold for the vanishing of cohomology."

For every $\varepsilon \in \mathbb{R}_+^*$, following Micciancio and Regev [**MR07**], we attach to a Euclidean lattice $\overline{E}$ of positive rank its *smoothing parameter* $\eta_\varepsilon(\overline{E})$. This is the positive real number defined by the equality:

$$\theta_{\overline{E}^\vee}(\eta_\varepsilon(\overline{E})^2) = 1 + \varepsilon, \tag{9.4.4}$$

where $\theta_{\overline{E}^\vee}$ denotes[14] the theta function attached to the dual Euclidean lattice $\overline{E}^\vee$, namely the function defined by:

$$\theta_{\overline{E}^\vee}(t) := \sum_{\xi \in E^\vee} e^{-\pi t \|\xi\|^2_{\overline{E}^\vee}}$$

for every $t \in \mathbb{R}_+^*$, which establishes a decreasing real analytic diffeomorphism:

$$\theta_{\overline{E}^\vee} : \mathbb{R}_+^* \xrightarrow{\sim} \mathbb{R}_+^*.$$

As discussed in [**Bos20a**], when one is concerned by the analogy between number fields and function fields, it is arguably more natural to consider the logarithm of the smoothing parameter:

$$s^1_{\theta,\varepsilon}(\overline{E}) := \log \eta_\varepsilon(\overline{E}). \tag{9.4.5}$$

This is a real number, characterized by the following equivalence, valid for every $\delta \in \mathbb{R}$:

$$\delta \geq s^1_{\theta,\varepsilon}(\overline{E}) \Longleftrightarrow h^1_\theta(\overline{E} \otimes \overline{\mathcal{O}}(\delta)) \leq \log(1+\varepsilon). \tag{9.4.6}$$

By construction, $s^1_{\theta,\varepsilon}(\overline{E})$ has the same behavior under scaling as $\log \rho(\overline{E})$, as formulated in (9.4.3) above. Namely, for every $\delta \in \mathbb{R}$, we have:

$$s^1_{\theta,\varepsilon}(\overline{E} \otimes \overline{\mathcal{O}}(\delta)) = s^1_{\theta,\varepsilon}\rho(\overline{E}) - \delta. \tag{9.4.7}$$

The definitions of $\eta_\varepsilon(\overline{E})$ and $s^1_{\theta,\varepsilon}$ depend on the choice of the "threshold" ε. However up to some universal multiplicative or additive constant, this choice is irrelevant. Indeed, as pointed out in [**CDLP13**, Section 2], if $0 < \varepsilon' \leq \varepsilon < 1$, we have:

$$0 \leq s^1_{\theta,\varepsilon'}(\overline{E}) - s^1_{\theta,\varepsilon}(\overline{E}) \leq (1/2)\log\log \varepsilon'^{-1} - (1/2)\log\log\varepsilon^{-1}. \tag{9.4.8}$$

Indeed the fact that the function $\theta_{\overline{E}^\vee}$ is decreasing implies the first inequality in (9.4.8). The second inequality asserts that $s^1_{\theta,\varepsilon}(\overline{E}) - (1/2)\log\log\varepsilon^{-1}$ is an increasing function of $\varepsilon \in (0,1)$, and follows from the following estimate, valid for $x \in \mathbb{R}_+^*$ and $t \in [1, +\infty)$:

$$\theta_{\overline{E}^\vee}(tx) - 1 \leq (\theta_{\overline{E}^\vee}(x) - 1)^t.$$

The estimates (9.4.8) show that comparison estimates involving $s^1_{\theta,\varepsilon}$ are basically independent of the choice of $\varepsilon \in (0,1)$. As in various classical papers concerning the smoothing parameter, notably [**RSD17b**], we will choose $\varepsilon = 1/2$.

Let us finally indicate that Regev and Stephens-Davidowitz have proved that the invariant $s^1_{\theta,1/2}(\overline{E})$ coincides with the opposite of the minimal slope $-\widehat{\mu}_{\min}(\overline{E})$ up to some additive error term which grows very slowly with the rank n of $\overline{E}$, namely which is $O(\log\log n)$:

THEOREM 9.4.1. *For every Euclidean lattice $\overline{E}$ of positive rank n, the following estimates are satisfied:*

$$-\widehat{\mu}_{\min}(\overline{E}) - \log(3/2) \leq s^1_{\theta,1/2}(\overline{E}) \leq -\widehat{\mu}_{\min}(\overline{E}) + t(n), \tag{9.4.9}$$

where $t(n) := \log[10(\log n + 2)]$.

[14] This notation is not compatible with the one used in Chapter 7, where $\theta_{\overline{E}^\vee}$ denoted a function on $E^\vee_\mathbb{R}$, but coincides with the one used in [**Bos20a**].

The first inequality in (9.4.9) is actually a direct consequence of the Poisson-Riemann-Roch formula, which indeed implies the inequality:

$$-\widehat{\mu}_{\min}(\overline{E}) - \log(1+\varepsilon) \leq s^1_{\theta,\varepsilon}(\overline{E}),$$

for every $\varepsilon \in \mathbb{R}^*_+$.

The second inequality, initially conjectured by Dadush, is an outstanding result, established in [**RSD17b**]. Its proof involves in substance the invariant $\mathrm{gv}(\overline{E})$ considered in Section 9.1, and diverse deep results concerning Gaussian measures and related isoperimetric inequalities.[15]

9.4.1.2. From the estimate (9.4.1), we immediately derive the implication:

$$\rho(\overline{E}) \leq \left(\pi^{-1}\log(1+\varepsilon)\right)^{1/2} \Longrightarrow h^1_\theta(\overline{E}) \leq \log(1+\varepsilon),$$

which may also be expressed as follows:

$$\log\rho(\overline{E}) \leq (1/2)\log\left(\pi^{-1}\log(1+\varepsilon)\right) \Longrightarrow s^1_{\theta,\varepsilon}(\overline{E}) \leq 0. \tag{9.4.10}$$

According to (9.4.3) and (9.4.7), the invariants $\log\rho(\overline{E})$ and $s^1_{\theta,\varepsilon}(\overline{E})$ have the same behavior under scaling. Therefore, by applying (9.4.10) to $\overline{E}\otimes\overline{\mathcal{O}}(\delta)$ with

$$\delta := \log\rho(\overline{E}) - (1/2)\log\left(\pi^{-1}\log(1+\varepsilon)\right),$$

we obtain:

Proposition 9.4.2. *For every Euclidean lattice $\overline{E}$ of positive rank and every $\varepsilon \in \mathbb{R}^*_+$, the following inequality holds:*

$$s^1_{\theta,\varepsilon}(\overline{E}) \leq \log\rho(\overline{E}) - (1/2)\log\left(\pi^{-1}\log(1+\varepsilon)\right). \tag{9.4.11}$$

We may specialize the estimate (9.4.11) to the case $\varepsilon = 1/2$. Since we have:

$$-(1/2)\log\left(\pi^{-1}\log(3/2)\right) = 1.02372...$$

we obtain the inequality:

$$s^1_{\theta,1/2}(\overline{E}) \leq \log\rho(\overline{E}) + 1.1 \quad . \tag{9.4.12}$$

Similarly, starting from (9.4.2), we may derive an estimate in the opposite direction:

Proposition 9.4.3. *For every Euclidean lattice $\overline{E}$ of positive rank n, the following inequality holds:*

$$\log\rho(\overline{E}) \leq s^1_{\theta,1/2}(\overline{E}) + (1/2)\log(1+n/4). \tag{9.4.13}$$

Proof. Observe that, by using (9.4.6) with $\delta = 0$ and (9.4.2), we get:

$$\begin{aligned} s^1_{\theta,1/2}(\overline{E}) \leq 0 \quad &\Longleftrightarrow \quad h^1_\theta(\overline{E}) \leq \log(1+1/2) \\ &\Longrightarrow \quad h^1_\theta(\overline{E}) \leq 1/2 \\ &\Longrightarrow \quad \rho(\overline{E})^2 \leq 1+n/4 \\ &\Longrightarrow \quad \log\rho(\overline{E}) \leq (1/2)\log(1+n/4). \end{aligned}$$

In turn, applied to $\overline{E}\otimes\overline{\mathcal{O}}(\delta)$ with $\delta := s^1_{\theta,\varepsilon}(\overline{E})$, this implication yields (9.4.13), thanks to the relations (9.4.3) and (9.4.7). □

[15]See [**Bos20a**] for a survey of this proof and related results, written for an audience of arithmetic geometers.

9.4.1.3. Propositions 9.4.2 and 9.4.3 show that, for some suitable constants $a(n)$ and $b(n)$, depending only on the rank n of E, the following comparison estimates hold:

$$\log \rho(\overline{E}) - a(n) \leq s^1_{\theta,1/2}(\overline{E}) \leq \log \rho(\overline{E}) + b(n). \tag{9.4.14}$$

One may wonder about the best possible constant $a(n)$ and $b(n)$ in (9.4.14), namely about the functions of the positive integer n defined by:

$$a(n) := \sup_{\|.\|} \Big(\log \rho(\mathbb{Z}^n, \|.\|) - s^1_{\theta,1/2}(\mathbb{Z}^n, \|.\|) \Big) \tag{9.4.15}$$

and:

$$b(n) := \sup_{\|.\|} \Big(s^1_{\theta,1/2}(\mathbb{Z}^n, \|.\|) - \log \rho(\mathbb{Z}^n, \|.\|) \Big), \tag{9.4.16}$$

where the supremum in the right-hand side of (9.4.15) and (9.4.16) is taken over $\|.\|$ in the cone $\overset{\circ}{\mathcal{Q}}(\mathbb{R}^n)$ of Euclidean norms on $\mathbb{R}^n$.

According to (9.4.13) and (9.4.12), we have:

$$a(n) \leq (1/2)\log(1 + n/4) = (1/2)\log n + O(1). \tag{9.4.17}$$

and

$$b(n) \leq 1.1 \quad . \tag{9.4.18}$$

Together with these upper bounds, the following examples show that, when n goes to infinity:

$$a(n) \sim (1/2)\log n,$$

and that $b(n)$ is a bounded function of n. This demonstrates that the upper bounds (9.4.17) and (9.4.18) on $a(n)$ and $b(n)$ are basically optimal.

Example 9.4.4. For every positive integer n, consider the direct sum $\overline{\mathcal{O}}^{\oplus n}$ of n copies of the trivial Euclidean lattice of rank one $\overline{\mathcal{O}} := (\mathbb{Z}, |\cdot|)$.

Then we have:

$$\rho(\overline{\mathcal{O}}^{\oplus n}) = (1/2)n^{1/2}$$

and therefore:

$$\log \rho(\overline{\mathcal{O}}^{\oplus n}) = (1/2)\log n - \log 2.$$

Moreover, for every $\varepsilon \in \mathbb{R}^*_+$, we have:

$$\theta_{\overline{\mathcal{O}}}(\eta_\varepsilon(\overline{\mathcal{O}}^{\oplus n})^2)^n = \theta_{\overline{\mathcal{O}}^{\oplus n}}(\eta_\varepsilon(\overline{\mathcal{O}})^2) = 1 + \varepsilon,$$

and therefore:

$$\log \theta_{\overline{\mathcal{O}}}(\eta_\varepsilon(\overline{\mathcal{O}}^{\oplus n})^2) = n^{-1}\log(1+\varepsilon).$$

This implies that, when n goes to infinity, the following relations hold:

$$2e^{-\pi\eta_\varepsilon(\overline{\mathcal{O}}^{\oplus n})^2)} \sim n^{-1}\log(1+\varepsilon),$$

$$\pi\eta_\varepsilon(\overline{\mathcal{O}}^{\oplus n})^2 = \log n + O(1),$$

and finally:

$$s^1_{\theta,\varepsilon}(\overline{\mathcal{O}}^{\oplus n}) = (1/2)\log\log n + O(1).$$

Consequently,

$$a(n) \geq \log \rho(\overline{\mathcal{O}}^{\oplus n}) - s^1_{\theta,\varepsilon}(\overline{\mathcal{O}}^{\oplus n}) = (1/2)\log n - (1/2)\log\log n - O(1).$$

EXAMPLE 9.4.5. For every positive integer n and every $\lambda \in \mathbb{R}$, consider the Euclidean lattice:

$$\overline{E}_{n,\lambda} := \overline{\mathcal{O}}^{\oplus n-1} \oplus \overline{\mathcal{O}}(-\lambda).$$

Then we have:

$$\rho(\overline{E}_{n,\lambda})^2 = (n-1+e^{2\lambda})/4,$$

and therefore:

$$\lambda \geq (1/2)\log n \Longrightarrow \log \rho(\overline{E}_{n,\lambda}) = \lambda + (1/2)\log[(1/4)(1+(n-1)e^{-2\lambda})] \leq \lambda.$$

Moreover:

$$s^1_{\theta,\varepsilon}(\overline{E}_{n,\lambda}) \geq s^1_{\theta,\varepsilon}(\overline{\mathcal{O}}(-\lambda)) = \lambda + s^1_{\theta,\varepsilon}(\overline{\mathcal{O}}).$$

Consequently, choosing λ in $[(1/2)\log n, +\infty)$, we obtain:

$$b(n) \geq s^1_{\theta,\varepsilon}(\overline{E}_{n,\lambda}) - \log \rho(\overline{E}_{n,\lambda}) \geq s^1_{\theta,\varepsilon}(\overline{\mathcal{O}}).$$

9.4.2. ρ^2-summability and θ^1-finiteness. Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf. A straightforward consequence of Theorem 9.1.1 is the following implication:

$$\overline{\rho}(\overline{M}) < +\infty \Longrightarrow \overline{h}^1_\theta(\overline{M}) < +\infty.$$

Moreover, if $\|.\|'$ denotes a Euclidean semi-norm on $M_\mathbb{R}$ such that $\|.\|' \leq \|.\|$ and $\mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty$, and if $\overline{M}'$ denotes the Euclidean quasi-coherent sheaf $(M, \|.\|')$, the variant concerning upper covering radii discussed in 9.3.2.4 of the main inequality on covering radii (9.3.13) establishes the following implication:

$$\overline{h}^1_\theta(\overline{M}) < \log 2 \Longrightarrow \overline{\rho}(\overline{M}') < +\infty.$$

It is possible to establish similar implications concerning the θ^1-finiteness of $\overline{M}$ and the ρ^2-summability[16] of $\overline{M}$ and $\overline{M}'$.

PROPOSITION 9.4.6. *Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf.*

If $\overline{M}$ is ρ^2-summable, then $\overline{M}$ is θ^1-finite, and for every $\delta \in \mathbb{R}$, the following inequality holds:

$$h^1_\theta\big(\overline{M} \otimes \overline{\mathcal{O}}(-\delta)\big) \leq \pi e^{2\delta}\, \overline{\rho}(\overline{E})^2. \tag{9.4.19}$$

Let moreover $\|.\|'$ be a Euclidean seminorm on $M_\mathbb{R}$, and let $\overline{M}' := (M, \|.\|')$. If $\overline{M}$ is θ^1-summable and if $\|.\|'$ is Hilbert-Schmidt[17] *with respect to $\|.\|$, then $\overline{M}'$ is ρ^2-summable.*

In the second part of Proposition 9.4.6, instead of asking $\overline{M}$ to be θ^1-summable, one might more generally ask for the existence of $\delta \in \mathbb{R}$ such that $\overline{M} \otimes \overline{\mathcal{O}}(\delta)$ is θ^1-summable, and the conclusion would still hold. This follows from Proposition 9.4.6 as stated above applied to the seminorms $e^{-\delta}\|.\|$ and $e^{-\delta}\|.\|'$ in place of $\|.\|$ and $\|.\|'$.

Recall also that the ρ^2-summability of $\overline{M}'$ implies the vanishing of $\mathrm{ev}\overline{\rho}(\overline{M}')$, and a fortiori of $\mathrm{ev}\rho(\overline{M}')$. In other words, we have:

COROLLARY 9.4.7. *If $\overline{M}$ is θ^1-summable and if $\|.\|'$ is Hilbert-Schmidt with respect to $\|.\|$, then $\overline{M}'$ has eventually vanishing covering radius.*

PROOF OF PROPOSITION 9.4.6. Let us assume that $\overline{M}$ is ρ^2-summable, and consider an exhaustive filtration $(C_i)_{i\in\mathbb{N}}$ of M by submodules in $\mathrm{coh}(M)$ such that the following summability condition holds:

$$\sum_{i\in\mathbb{N}} \rho(\overline{C_i/C_{i-1}})^2 < +\infty.$$

[16] defined in Subsection 6.3.3.

[17] that is, if $\mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty$.

Then, for every $\delta \in \mathbb{R}$,

$$\sum_{i\in\mathbb{N}} \rho\big(\overline{C_i/C_{i-1}} \otimes \overline{\mathcal{O}}(-\delta)\big)^2 = \sum_{i\in\mathbb{N}} e^{2\delta} \rho\big(\overline{C_i/C_{i-1}}\big)^2 < +\infty,$$

where, as usual, we let $C_{-1} := 0$. Since, according to the estimate (9.1.3), the following inequality holds for every $i \in \mathbb{N}$:

$$h^1_\theta\big(\overline{C_i/C_{i-1}} \otimes \overline{\mathcal{O}}(-\delta)\big)^2 \leq \pi\rho\big(\overline{C_i/C_{i-1}} \otimes \overline{\mathcal{O}}(-\delta)\big)^2,$$

this shows that, for every $\delta \in \mathbb{R}$,

$$\sum_{i\in\mathbb{N}} h^1_\theta\big(\overline{C_i/C_{i-1}} \otimes \overline{\mathcal{O}}(-\delta)\big)^2 < +\infty,$$

and establishes the θ^1-finiteness of $\overline{M}$.

Moreover, as a consequence of the general formalism of φ-summability developed in Section 4.5, we have:

$$\overline{\rho}(\overline{E}) = \lim_{i\to+\infty} \rho(\overline{C}_i)^2$$

and

$$\overline{h}^1_\theta\big(\overline{E} \otimes \overline{\mathcal{O}}(-\delta)\big) = \lim_{i\to+\infty} h^1_\theta\big(\overline{C}_i \otimes \overline{\mathcal{O}}(-\delta)\big);$$

see (4.5.3), and also (6.3.18) and (8.2.1). Moreover, applied to $\overline{C}_i \otimes \overline{\mathcal{O}}(-\delta)$, the estimate (9.1.3) implies the inequality:

$$\overline{h}^1_\theta\big(\overline{E} \otimes \overline{\mathcal{O}}(-\delta)\big) \leq \pi\overline{\rho}\big((\overline{E} \otimes \overline{\mathcal{O}}(-\delta)\big)^2 = \pi e^{2\delta}\, \overline{\rho}(\overline{E})^2,$$

and (9.4.19) follows by letting i go to infinity.

Conversely let us assume that $\overline{M}$ is θ^1-summable and that:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty, \tag{9.4.20}$$

and let us choose an exhaustive filtration $(C_i)_{i\in\mathbb{N}}$ of M by submodules in $\mathrm{coh}(M)$ such that the following summability condition holds:

$$\sum_{i\in\mathbb{N}} h^1_\theta(\overline{C_i/C_{i-1}}) < +\infty. \tag{9.4.21}$$

Then the sequence $(h^1_\theta(\overline{C_i/C_{i-1}}))_{i\in\mathbb{N}}$ converges to 0, and the seminorm $\|.\|'$ is compact with respect to $\|.\|$. Consequently, we may choose $i_0 \in \mathbb{N}$ such that, for every integer $i > i_0$:

$$h^1_\theta(\overline{C_i/C_{i-1}}) \leq 1/2,$$

and:

$$\|.\|'_{M/C_{i-1}} \leq \|.\|_{M/C_{i-1}},$$

where $\|.\|_{M/C_{i-1}}$ (resp. $\|.\|'_{M/C_{i-1}}$) denotes the Euclidean seminorm on the quotient $(M/C_{i-1})_\mathbb{R}$ deduced from the seminorm $\|.\|$ (resp. $\|.\|'$) on $M_\mathbb{R}$.

Then, for every $i > i_0$, we also have:

$$\|.\|'_{C_i/C_{i-1}} \leq \|.\|_{C_i/C_{i-1}},$$

where $\|.\|_{C_i/C_{i-1}}$ (resp. $\|.\|'_{C_i/C_{i-1}}$) denotes the Euclidean seminorm on the subquotient $(C_i/C_{i-1})_\mathbb{R}$ deduced from the seminorm $\|.\|$ (resp. $\|.\|'$) on $M_\mathbb{R}$, and according to Corollary 9.3.4, the following inequality holds:

$$\rho\big(\overline{C_i/C_{i-1}}'\big)^2 \leq 2h^1_\theta(\overline{C_i/C_{i-1}}) + \frac{1}{4}\mathrm{Tr}\big(\|.\|'^2_{C_i/C_{i-1}}/\|.\|^2_{C_i/C_{i-1}}\big),$$

where $\overline{C_i/C_{i-1}}'$ denotes the object $(C_i/C_{i-1}, \|.\|'_{C_i/C_{i-1}})$ of $\overline{\mathbf{Coh}}_\mathbb{Z}$.

This implies the inequality:

$$\sum_{i>i_0} \rho\big(\overline{C_i/C_{i-1}}'\big)^2 \leq 2 \sum_{i>i_0} h^1_\theta(\overline{C_i/C_{i-1}}) + \frac{1}{4} \sum_{i>i_0} \mathrm{Tr}\big(\|.\|'^2_{C_i/C_{i-1}}/\|.\|^2_{C_i/C_{i-1}}\big) \tag{9.4.22}$$

Moreover, we have:

$$\sum_{i>i_0} \mathrm{Tr}\big(\|.\|'^2_{C_i/C_{i-1}}/\|.\|^2_{C_i/C_{i-1}}\big) \leq \mathrm{Tr}\big(\|.\|'^2_{M/C_{i_0}}/\|.\|^2_{M/C_{i_0}}\big) \leq \mathrm{Tr}(\|.\|'^2/\|.\|^2), \tag{9.4.23}$$

where $\|.\|_{M/C_{i_0}}$ (resp. $\|.\|'_{M/C_{i_0}}$) denotes the Euclidean seminorm on the quotient $(M/C_{i_0})_\mathbb{R}$ deduced from the seminorm $\|.\|$ (resp. $\|.\|'$) on $M_\mathbb{R}$.

Together with (9.4.20), (9.4.21), and (9.4.23), the inequality (9.4.22) implies:

$$\sum_{i>i_0} \rho\big(\overline{C_i/C_{i-1}}'\big)^2 < +\infty.$$

This show that the summability condition:

$$\sum_{i\in\mathbb{N}} \rho\big(\overline{C_i/C_{i-1}}'\big)^2 < +\infty$$

is satisfied, and therefore that $\overline{M}'$ is ρ^2-summable. □

9.4.3. Density theorems. In the sequel, the main inequality on covering radii in Theorem 9.3.3 will play a crucial role in the derivation of density results concerning sections of coherent sheaves on arithmetic schemes. Indeed, the main inequality may be used to derive density properties from vanishing properties of suitable theta invariants.

9.4.3.1. Let us start by a simple instance of such density results, which directly follows from Theorem 9.3.3.

PROPOSITION 9.4.8. *Let $\overline{M} = (M, \|.\|)$ be a Euclidean quasi-coherent sheaf such that, for every $\delta \in \mathbb{R}$,*

$$\overline{h}^1_\theta(\overline{M} \otimes \overline{\mathcal{O}}(-\delta)) = 0. \tag{9.4.24}$$

If $\|.\|'$ is a Euclidean seminorm on $M_\mathbb{R}$ that is Hilbert-Schmidt with respect to $\|.\|$, then the Euclidean quasi-coherent sheaf $\overline{M}' := (M, \|.\|')$ satisfies:

$$\overline{\rho}(\overline{M}') = 0, \tag{9.4.25}$$

and the image $M_{/\mathrm{tor}}$ of M in $M_\mathbb{R}$ is dense in the seminormed space $(M_\mathbb{R}, \|.\|')$.

Concerning condition (9.4.24), observe that a Euclidean quasi-coherent sheaf $\overline{M}$ satisfies the condition

$$\overline{h}^1_\theta(\overline{M}) = 0$$

if and only $\overline{M}$ is θ^1-summable and the Banaszczyk measure $\beta_{\overline{M}^\vee}$ is the Dirac measure δ_0; this last condition is equivalent to the vanishing of the Banaszczyk function $b_{\overline{M}}$. Indeed, this follows from the results in Section 8.4; see notably Theorem 8.4.7 and Proposition 8.4.5.

PROOF. For every large enough positive real number δ, we may apply the inequality (9.3.27) to the Euclidean quasi-coherent sheaf

$$\overline{N} := \overline{M} \otimes \overline{\mathcal{O}}(-\delta) = (M, e^\delta \|.\|).$$

Indeed, the seminorm $\|.\|'$ is continuous with respect to $\|.\|$, and therefore satisfies $\|.\|' \leq e^\delta \|.\|$ for δ large enough. This leads to the upper bound:

$$\rho(M, \|.\|')^2 \leq \frac{e^{-2\delta}}{4} \mathrm{Tr}(\|.\|'^2/\|.\|^2),$$

and the vanishing of $\rho(M, \|.\|')$ follows by letting δ go to infinity.

The density of $M_{/\mathrm{tor}}$ in $(M_{\mathbb{R}}, \|.\|')$ is equivalent to the vanishing of the covering radius $\rho(M, \|.\|')$, and follows from the vanishing of $\overline{\rho}(\overline{M}')$. $\square$

9.4.3.2. *A density theorem for maps from Euclidean quasi-coherent sheaves to projective systems of Euclidean lattices.* We shall now formulate and establish a density theorem suited for the Diophantine applications alluded to above.

The data for this theorem are *a Euclidean quasi-coherent sheaf* $\overline{M} := (M, \|.\|)$ *and a decreasing sequence* $(M_i)_{i\in\mathbb{N}}$ *of* $\mathbb{Z}$*-submodules of* M.

For any nonnegative integer i, we will denote by :

$$f_i : M_i \longrightarrow M$$

the inclusion map, by $N_i := M/M_i$ the quotient $\mathbb{Z}$-module, and by:

$$p_i : M \longrightarrow N_i$$

the quotient map.

Since the M_i are decreasing, the N_i fit into a projective system of surjective morphisms of $\mathbb{Z}$-modules:

$$N_0 := M/M_0 \longleftarrow N_1 := M/M_1 \longleftarrow \cdots \longleftarrow N_i := M/M_i \longleftarrow N_{i+1} := M/M_{i+1} \longleftarrow \cdots,$$

We may introduce its limit:

$$\widehat{N} := \lim_{\substack{\longleftarrow \\ i}} N_i,$$

and denote by:

$$p : M \longrightarrow \widehat{N}$$

be the morphism of $\mathbb{Z}$-modules induced by the maps p_i.

The $\mathbb{R}$-linear map

$$p_{\mathbb{R}} : M_{\mathbb{R}} \longrightarrow \widehat{N}_{\mathbb{R}} := \widehat{N} \mathbin{\widehat{\otimes}}_{\mathbb{Z}} \mathbb{R} \simeq \lim_{\substack{\longleftarrow \\ i}} N_{i,\mathbb{R}},$$

defined by the compatible system of $\mathbb{R}$-linear maps $(p_{i,\mathbb{R}} : M_{\mathbb{R}} \to N_{i,\mathbb{R}})_{i\in\mathbb{N}}$, sends the image $M_{/\mathrm{tor}}$ of M in $M_{\mathbb{R}}$ to the $\mathbb{Z}$-submodule

$$\widehat{N}_{/\mathrm{tor}} := \lim_{\substack{\longleftarrow \\ i}} N_{i/\mathrm{tor}}$$

of $\widehat{N}_{\mathbb{R}}$, image of $\widehat{N}$ in $\widehat{N}_{\mathbb{R}}$. In other words, the $\mathbb{Z}$-submodule $M_{/\mathrm{tor}}$ of $M_{\mathbb{R}}$ is contained in $p_{\mathbb{R}}^{-1}(\widehat{N}_{/\mathrm{tor}})$.

LEMMA 9.4.9. *If the relative covering radii* $\rho(f_i : M_i \to \overline{M})$ *satisfy the condition:*

$$\lim_{i\to+\infty} \rho(f_i : M_i \to \overline{M}) = 0, \tag{9.4.26}$$

then $M_{/\mathrm{tor}}$ *is dense in* $p_{\mathbb{R}}^{-1}(\widehat{N}_{/\mathrm{tor}})$ *equipped with the distance defined by the seminorm* $\|.\|$.

PROOF. According to (9.4.26), there exists a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ of positive real numbers such that

$$\lim_{i\to+\infty} \varepsilon_i = 0$$

and

$$\rho(f_i : M_i \to \overline{M}) < \varepsilon_i$$

for every $i \in \mathbb{N}$.

Then the following inclusion holds, for every $i \in \mathbb{N}$:

$$p_{i,\mathbb{R}}^{-1}(N_{i/\mathrm{tor}}) = f_{i,\mathbb{R}}(M_{i,\mathbb{R}}) \subseteq M_{/\mathrm{tor}} + B(\overline{M}_{\mathbb{R}}; \varepsilon_i), \tag{9.4.27}$$

where:

$$B(\overline{M}_{\mathbb{R}}; \varepsilon_i) := \big\{ v \in M_{\mathbb{R}} \mid \|v\| < \varepsilon_i \big\}.$$

Let m be an element in $p_{\mathbb{R}}^{-1}(\widehat{N}_{/\mathrm{tor}})$, that is, an element of $M_{\mathbb{R}}$ such that, for every $i \in \mathbb{N}$, $p_i(m)$ lies in $N_{i/\mathrm{tor}}$. According to (9.4.27), we may find a sequence (m_i) of elements of $M_{/\mathrm{tor}}$ such that

$$\|m - m_i\| \leq \varepsilon_i.$$

This shows that m belongs to the closure of $M_{/\mathrm{tor}}$ in $M_{\mathbb{R}}$ equipped with the distance defined by the seminorm $\|.\|$. □

THEOREM 9.4.10. *With the above notation, assume that, for any $\delta \in \mathbb{R}_+$, the theta ranks $\overline{\mathrm{rk}}_\theta^1(f_i : M_i \to \overline{M} \otimes \overline{\mathcal{O}}(-\delta))$ satisfy the following condition:*

$$\lim_{i \to +\infty} \overline{\mathrm{rk}}_\theta^1(f_i : M_i \to \overline{M} \otimes \overline{\mathcal{O}}(-\delta)) = 0. \tag{9.4.28}$$

Then, for every Euclidean seminorm $\|.\|'$ on $M_{\mathbb{R}}$ that is Hilbert-Schmidt relatively to $\|.\|$, we have:

$$\lim_{i \to +\infty} \overline{\rho}(f_i : M_i \to \overline{M}') = 0, \tag{9.4.29}$$

where $\overline{M}' := (M, \|.\|')$.

Consequently the $\mathbb{Z}$-submodule $M_{/\mathrm{tor}}$ of $M_{\mathbb{R}}$ is dense in $p_{\mathbb{R}}^{-1}(\widehat{N}_{/\mathrm{tor}})$ equipped with the distance defined by the seminorm $\|.\|'$.

The strong monotonicity[18] of h_θ^1 shows that condition (9.4.28) is satisfied for every $\delta \in \mathbb{R}$ provided it holds for δ in some unbounded subset of $\mathbb{R}_+$. The proof of Theorem 9.4.10 will actually establish its validity under this formally weaker assumption.

Observe also that, when we specialize Theorem 9.4.10 to the situation where all the M_i are equal to M, we recover Proposition 9.4.8 above.[19] Actually, with Lemma 9.4.9 at hand, the proof of Theorem 9.4.10 appears as a variation on the proof of Proposition 9.4.8, where now the full strength of the main equality on covering radii, concerning *relative* covering radii, plays a crucial role.

PROOF. Let $\|.\|'$ be a Euclidean seminorm on $M_{\mathbb{R}}$ that is Hilbert-Schmidt relatively to $\|.\|$, and choose $\delta_0 \in \mathbb{R}$ such that:

$$\|.\|' \leq e^{\delta_0} \|.\|.$$

For every $\delta \in \mathbb{R}_+$, the validity of (9.4.28) implies the existence of $n(\delta) \in \mathbb{N}$ such that, for every integer $n \geq n(\delta)$,

$$\overline{\mathrm{rk}}_\theta^1(f_i : M_i \to \overline{M} \otimes \overline{\mathcal{O}}(-\delta)) \leq 1/2.$$

Therefore, for every $\delta \in [\delta_0, +\infty)$ and very $i \geq n(\delta)$, we may apply the inequality (9.3.26) to $f_i : M_i \to \overline{M} \otimes \overline{\mathcal{O}}(-\delta)$ and to the Euclidean seminorm $\|.\|'$ on $M_{\mathbb{R}}$. It reads as follows:

$$\overline{\rho}(f_i : M_i \to \overline{M}')^2 \leq 2\, \overline{\mathrm{rk}}_\theta^1(f_i : M_i \to \overline{M} \otimes \overline{\mathcal{O}}(-\delta)) + \frac{e^{-2\delta}}{4} \operatorname{Tr}(\|.\|'^2/\|.\|^2).$$

Together with (9.4.28), this implies:

$$\limsup_{i \to +\infty} \overline{\rho}(f_i : M_i \to \overline{M}')^2 \leq \frac{e^{-2\delta}}{4} \operatorname{Tr}(\|.\|'^2/\|.\|^2),$$

and therefore, since δ may be chosen arbitrary large, proves (9.4.29).

The final assertion follows from the estimates:

$$\rho(f_i : M_i \to \overline{M}') \leq \overline{\rho}(f_i : M_i \to \overline{M}')$$

and from Lemma 9.4.9 applied with $\|.\|$ replaced by $\|.\|'$. □

[18] More specifically, the fact that h_θ^1 satisfies the metric monotonicity condition $\mathbf{StMon_4^1}$.

[19] However, in Diophantine applications, we will be mainly interested in situations where the map p from M to its "completion" $\widehat{N}$ is injective, or equivalently where $\bigcap_{i \in \mathbb{N}} M_i = \{0\}$.

9.5. A Relative Infinite Dimensional Transference Inequality

In the first part of this section, we formulate and we establish a generalization of Banaszczyk's classical transference inequalities (9.0.11) to the framework of this monograph, where one deals not only with Euclidean lattices, but with general Euclidean quasi-coherent sheaves and with relative versions of classical invariants.

Then in Subsections 9.5.4 and 9.5.5, we show that, even when one considers only the finite rank situation, the extra flexibility added to the basic transference inequalities (9.0.11) by working with relative invariants and pairs of seminorms allows one to recover easily the more refined transference estimates relating successive minima of higher order and successive covering radii attached to Euclidean lattices and their duals, and also their strengthenings concerning Euclidean lattices with "gaps" in the sequence of their successive minima — a class of Euclidean lattices which naturally arise in lattice-based cryptography.

9.5.1. Statement of the generalized transference inequality.

9.5.1.1. To formulate our generalized transference inequality, we need to introduce a generalized version of the first minima of a Euclidean lattice that make sense in the context of arbitrary $\mathbb{Z}$-modules whose associated $\mathbb{R}$-vector space is equipped with a quasinorm,[20] which possibly is not a seminorm. generalized pro-Euclidean $\mathbb{Z}$-modules.

Let M be a $\mathbb{Z}$-module, let $\|.\|$ be a quasinorm on the real vector space $M_{\mathbb{R}}$, and let K be a $\mathbb{Z}$-submodule of M. We define the *first minimum of* $(M, \|.\|)$ *relative to* K by the equality:

$$\lambda_1(M, K, \|.\|) := \inf \left\{ \|m\| \, ; \, m \in M \setminus K \right\} \quad (\in [0, +\infty]).$$

If $\overline{N} := (N, \|.\|)$ is a Euclidean quasi-coherent sheaf, we may consider its dual $\overline{N}^{\vee} := (N^{\vee}, \|.\|^{\vee})$ in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[\infty]}$ as defined in Subsection 2.5.2.1. Recall that the dual quasinorm $\|.\|^{\vee}$ is defined on $N^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, identified to a $\mathbb{Z}$-submodule of $N_{\mathbb{R}}^{\vee} := \mathrm{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R})$, by the following equality for every $\xi \in N^{\vee}$:

$$\|\xi\|^{\vee} := \sup \left\{ \|\xi(v)\| \mid v \in N_{\mathbb{R}} \text{ and } \|v\| \leq 1 \right\} \quad (\in [0, +\infty]).$$

The version of the first minimum which shall enters in our generalized transference inequality is defined as follows.

Consider a morphism:

$$f : M \longrightarrow N$$

of countably generated $\mathbb{Z}$-modules, and $\|.\|$ a Euclidean seminorm on $N_{\mathbb{R}}$.

To these data are associated the transpose of f:

$$f^{\vee} : N^{\vee} \longrightarrow M^{\vee}, \quad \xi \longmapsto \xi \circ f,$$

which is a morphism in $\mathbf{CTC}_{\mathbb{Z}}$, the Euclidean quasi-coherent sheaf $\overline{N} := (N, \|.\|)$, and its dual $\overline{N}^{\vee} := (N^{\vee}, \|.\|^{\vee})$ in $\mathbf{pro}\overline{\mathbf{Vect}}_{\mathbb{Z}}^{[\infty]}$. Observe that we have:

$$\ker f^{\vee} = \left\{ \xi \in N^{\vee} \mid \xi_{|f(M)} = 0 \right\} =: f(M)^{\perp}.$$

Then we define the *relative first minimum* associated to the data above as:

$$\lambda_1(f^{\vee} : \overline{N}^{\vee} \to M^{\vee}) := \lambda_1(N^{\vee}, f(M)^{\perp}, \|.\|^{\vee}) = \inf \left\{ \|\xi\|^{\vee} \mid \xi \in N^{\vee}, \, \xi \circ f \neq 0 \right\}.$$

---

[20] See Definition 2.4.1.

When M and N coincide and f is the identity morphism Id_N, the relative first minimum specializes to the *first minimum of* $\overline{N}^\vee$, defined as:

$$\begin{aligned}\lambda_1(\overline{N}^\vee) &:= \lambda_1(\mathrm{Id}_{N^\vee} : \overline{N}^\vee \to N^\vee) \\ &= \lambda_1(N^\vee, \{0\}, \|.\|^\vee) \\ &= \inf\left\{\|\xi\|^\vee; \xi \in N^\vee \setminus \{0\}\right\} \quad (\in [0, +\infty]).\end{aligned}$$

9.5.1.2. The main result of this section is the following transference theorem:

THEOREM 9.5.1. *Consider a morphism of countably generated $\mathbb{Z}$-modules:*

$$f : M \longrightarrow N,$$

and $\|.\|_1$ and $\|.\|_2$ two Euclidean seminorms on $N_\mathbb{R}$.

If $\overline{N}_1 := (N, \|.\|_1)$ and $\overline{N}_2 := (N, \|.\|_2)$ denote the associated Euclidean quasi-coherent sheaves, then the following inequality is satisfied:

$$\underline{\rho}(f : M \to \overline{N}_1)\, \lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee) \leq \mathrm{Tr}(\|.\|_1/\|.\|_2). \tag{9.5.1}$$

If moreover the following conditions hold:

$$\|.\|_1 \leq \|.\|_2 \quad \textit{and} \quad \mathrm{Tr}(\|.\|_1/\|.\|_2) \geq 1, \tag{9.5.2}$$

then we have:

$$\underline{\rho}(f : M \to \overline{N}_1)\, \lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee) \leq \frac{1}{2\pi}\mathrm{Tr}(\|.\|_1/\|.\|_2) + \frac{2}{\pi}\sqrt{\mathrm{Tr}(\|.\|_1/\|.\|_2)}. \tag{9.5.3}$$

The diverse terms in (9.5.1) and (9.5.3) belong to $[0, +\infty]$, and we adopt the usual convention:

$$0.(+\infty) = 0.$$

Applied to $M = N$ and $f = \mathrm{Id}_N$, the estimates (9.5.1) and (9.5.3) become:

$$\underline{\rho}(\overline{N}_1)\, \lambda_1(\overline{N}_2^\vee) \leq \mathrm{Tr}(\|.\|_1/\|.\|_2). \tag{9.5.4}$$

and:

$$\underline{\rho}(\overline{N}_1)\, \lambda_1(\overline{N}_2^\vee) \leq \frac{1}{2\pi}\mathrm{Tr}(\|.\|_1/\|.\|_2) + \frac{2}{\pi}\sqrt{\mathrm{Tr}(\|.\|_1/\|.\|_2)}. \tag{9.5.5}$$

We have not tried to optimize the constants occuring in the right-hand side of (9.5.1) and (9.5.3). Actually the proof below will establish a slightly stronger version of (9.5.1) and (9.5.4), namely:

$$\underline{\rho}(f : M \to \overline{N}_1)\, \lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee) \leq \frac{5}{2\pi}\mathrm{Tr}(\|.\|_1/\|.\|_2), \tag{9.5.6}$$

and

$$\underline{\rho}(\overline{N}_1)\, \lambda_1(\overline{N}_2^\vee) \leq \frac{5}{2\pi}\mathrm{Tr}(\|.\|_1/\|.\|_2). \tag{9.5.7}$$

The proof will also make clear that the factor $2/\pi$ in the right-hand side of (9.5.3) and (9.5.5) may equally be slightly improved.

We do not know whether the inequalities (9.5.3)-(9.5.7) still hold when the lower covering radii $\underline{\rho}(f : M \to \overline{N}_1)$ and $\underline{\rho}(\overline{N}_1)$ are replaced by the "naive" covering radii $\rho(f : M \to \overline{N}_1)$ and $\rho(\overline{N}_1)$.[21]

9.5.2. Preliminary estimates. In this subsection, we gather various technical statements on which we will rely in the proof of Theorem 9.5.1.

[21]Note that the construction in paragraph 6.3.3.4 establishes the existence of some Euclidean quasi-coherent sheaf $\overline{M}$ such that $\underline{\rho}(\overline{M}) = 0$ and $\rho(\overline{M}) > 0$. Since the vanishing of $\underline{\rho}(\overline{M})$ implies that there does not exist any nonzero continuous linear form $M \to \mathbb{Z}$, it also satisfies: $\lambda_1(\overline{M}^\vee) = +\infty$, and consequently: $\rho(\overline{M})\lambda_1(\overline{M}^\vee) = +\infty$.

9.5.2.1. Our first preliminary result deals with traces.

PROPOSITION 9.5.2. *Let V be a real vector space. Let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on V such that:*

$$\|.\|' \leq \|.\|.$$

For every Euclidean seminorm $\|.\|''$ on V that satisfies:

$$\|.\|' \leq \|.\|'' \leq \|.\|, \tag{9.5.8}$$

the following inequality holds:

$$\mathrm{Tr}(\|.\|'/\|.\|)^2 \leq \mathrm{Tr}(\|.\|'^2/\|.\|''^2)\,\mathrm{Tr}(\|.\|''^2/\|.\|^2). \tag{9.5.9}$$

Moreover there exists a Euclidean seminorm $\|.\|''$ on V such that (9.5.8) *and following equalities hold:*

$$\mathrm{Tr}(\|.\|'^2/\|.\|''^2) = \mathrm{Tr}(\|.\|''^2/\|.\|^2) = \mathrm{Tr}(\|.\|'/\|.\|).$$

PROOF. To prove (9.5.9), we may assume that V is finite-dimensional after replacing V by an arbitrarily large subspace. Modding out by the kernel of $\|.\|$, we assume that $\|.\|$ is definite positive. Write $(.,.)$ (resp. $(.,.)'$, resp. $(.,.)''$) for the bilinear form defined by $\|.\|$ (resp. $\|.\|'$, resp. $\|.\|''$).

Let f be the endomorphism of V such that, for all v, w in V:

$$(v,w)' = (f(v), w).$$

The inequality:

$$\|.\|' \leq \|.\|''$$

implies that we may find an endomorphism g of V such that, for all v, w in V:

$$(v,w)'' = (g(v), w)'.$$

In particular:

$$(v,w)'' = ((g \circ f)(v), w)$$

and the inequality (9.5.9) becomes

$$\mathrm{Tr}(g \circ f)^2 \leq \mathrm{Tr}(f)^2 \mathrm{Tr}(g)^2. \tag{9.5.10}$$

The inequality (9.5.10) is the Cauchy-Schwartz inequality for the positive semi-definite bilinear form

$$(g, f) \mapsto \mathrm{Tr}(g \circ f)$$

on the space of those endomorphisms of V that that vanish on the kernel of $\|.\|$ and are symmetric and positive semi-definite with respect to $\|.\|$.

To prove the second statement, let $\varphi : V \to H$ be the natural linear map from V to the Hilbert space H obtained as the separated completion of V with respect to $\|.\|$. Let $A : H \to H$ be the positive linear map such that, for all $v \in V$,

$$\|v\|' = \|A(\varphi(v))\|.$$

Let $(e_i)_{i \in I}$ be a Hilbert space basis of H consisting of eigenvectors of A, and let λ_i be the eighenvalue corresponding to e_i for any $i \in I$. The inequality

$$\|.\|' \leq \|.\|$$

guarantees that, for all $i \in I$, we have $0 \leq \lambda_i \leq 1$.

Let B be the continuous endomorphism of H such that, for any $i \in I$,

$$B(e_i) = \sqrt{\lambda_i},$$

and let $\|.\|''$ be the Euclidean seminorm on V such that, for all $v \in V$,

$$\|v\|'' = \|B(\varphi(v))\|.$$

Since, for all $i \in I$, we have:

$$\sqrt{\lambda_i} \leq \lambda_i \leq 1,$$

we also have:

$$\|.\|' \leq \|.\|'' \leq \|.\|,$$

and the following equalities are readily checked:

$$\mathrm{Tr}(\|.\|'^2/\|.\|''^2) = \sum_{i \in I} \lambda_i^2 = \mathrm{Tr}(\|.\|''^2/\|.\|^2) = \mathrm{Tr}(\|.\|'/\|.\|). \qquad \square$$

9.5.2.2. We record an elementary calculus computation.

PROPOSITION 9.5.3. *For any $T \geq 1$, we have:*

$$\frac{4}{\sqrt{T}} - \log\Big(1 + \frac{4}{\sqrt{T}}\Big) \geq \frac{2}{T}\log 3.$$

PROOF. After multiplication by T, the inequality we need to prove is equivalent to:

$$4\sqrt{T} - T\log\Big(1 + \frac{4}{\sqrt{T}}\Big) \geq 2\log 3. \tag{9.5.11}$$

The function

$$f : \mathbb{R}_+^* \longrightarrow \mathbb{R}, \quad x \longmapsto 4x - x^2\log(1 + 4/x)$$

is increasing. Indeed, for every $x \in \mathbb{R}_+^*$, we have:

$$\begin{aligned} f'(x) &= 4 - 2x\log(1+4/x) + \frac{4}{1+4/x} \\ &= x[(1+4/x) - (1+4/x)^{-1} - 2\log(1+4/x)] \\ &= x\int_1^{1+4/x} t^{-2}(t-1)^2\, dt > 0. \end{aligned}$$

In particular, for any $T \geq 1$, we have $f(\sqrt{T}) \geq f(1)$, and proving (9.5.11) amounts to proving the estimate:

$$4 - \log 5 \geq 2\log 3.$$

It holds, since $e^2 \geq 7$ and therefore $e^4 \geq 45$. $\square$

9.5.2.3. Finally, we record the following consequence of Corollary 9.2.7. Our notation is in line with its application to the proof Theorem 9.5.1 in the next subsection.

PROPOSITION 9.5.4. *Let $f : M \to N$ be a morphism of countably generated $\mathbb{Z}$-modules, and let $\|.\|_2$ and $\|.\|$ be two Euclidean seminorms on $N_{\mathbb{R}}$, and $\overline{N} := (N, \|.\|)$ and $\overline{N}_2 := (N, \|.\|_2)$ the Euclidean quasi-coherent sheaves they define.*

Assume that these Euclidean seminorms satisfy:

$$\|.\| \leq \|.\|_2 \quad \text{and} \quad T := \mathrm{Tr}\big(\|.\|^2/\|.\|_2^2\big) < +\infty,$$

and that the following inequality holds:

$$\lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee) > \sqrt{\frac{T}{2\pi}}.$$

Then, if we define:

$$\widetilde{\lambda} := \sqrt{\frac{2\pi}{T}}\,\lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee) \quad (\in (1, +\infty]),$$

we have:

$$e^{-\underline{\mathrm{rk}}_\theta^1(f : M \to \overline{N})} + e^{T(1 - \widetilde{\lambda}^2 + 2\log\widetilde{\lambda})/2} \geq 1. \tag{9.5.12}$$

In the left-hand side of (9.5.12), as usual we extend the exponential map to $[-\infty, +\infty)$ by letting $e^{-\infty} := 0$. Moreover we define $1 - \widetilde{\lambda}^2 + 2 \log \widetilde{\lambda}$ to be $-\infty$ when $\widetilde{\lambda} = +\infty$.

PROOF. (1) First assume that M and N the $\mathbb{Z}$-modules are free of finite rank, and that $\|.\|$ and $\|.\|_2$ are both Euclidean norms on $N_{\mathbb{R}}$. Then, with the notation of 9.2.3.1:

$$\lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee) = \lambda_1(N^\vee, f(M)^\perp, \|.\|_2^\vee).$$

Furthermore, an immediate computation shows that the dual Euclidean norms $\|.\|^\vee$ and $\|.\|_2^\vee$ on $N_{\mathbb{R}}^\vee$ satisfy the equality:

$$T = \mathrm{Tr}\big(\|.\|_2^{\vee\, 2}/\|.\|^{\vee\, 2}\big).$$

As a consequence, Corollary 9.2.7 establishes the estimate:

$$h^0_\theta(N^\vee, \|.\|^\vee) - h^0_\theta(f(M)^\perp, \|.\|^\vee) \leq \log\Big(1 - \big(\widetilde{\lambda} e^{-1/2(\widetilde{\lambda}^2 - 1)}\big)^T\Big)^{-1}. \tag{9.5.13}$$

By the very definition of theta-invariants, we have:

$$h^0_\theta(N^\vee, \|.\|^\vee) = h^1_\theta(\overline{N}).$$

Moreover we may identify the $\mathbb{Z}$-module $f(M)^\perp \subset N^\vee$ with the dual of $N/f(M)$ in such a way that the restriction of $\|.\|^\vee$ to $f(M)^\perp$ is identified with the dual of the quotient norm $\|.\|$ on $N/f(M)$. As a consequence, we have:

$$h^0_\theta(f(M)^\perp, \|.\|^\vee) = h^1_\theta(N/f(M), \|.\|)$$

and:

$$h^0_\theta(N^\vee, \|.\|^\vee) - h^0_\theta(f(M)^\perp, \|.\|^\vee) = \mathrm{rk}^1_\theta(f : M \to \overline{N}),$$

so that (9.5.12) is equivalent to (9.5.13).

(2) Now assume that M and N are arbitrary $\mathbb{Z}$-modules of finite type, and that $\|.\|$ and $\|.\|_2$ are arbitrary Euclidean seminorms on $N_{\mathbb{R}}$. Both the left-hand side and the right-hand side of (9.5.12) are left unchanged when N is replaced by its maximal torsion-free quotient $N_{/\mathrm{tor}} := N/N_{\mathrm{tor}}$. As a consequence, we may assume that N is free.

We may choose decreasing sequences $(\|.\|_n)_{n \geq 0}$ and $(\|.\|_{2,n})_{n \geq 0}$ of Euclidean norms on $N_{\mathbb{R}}$ that converge to $\|.\|$ and $\|.\|_2$ respectively, such that, for any $n \geq 0$, $\|.\| \leq \|.\|_n$. By Part (1) of this proof, the estimate (9.5.12) is valid when $\|.\|$ and $\|.\|_2$ are replaced with $\|.\|_n$ and $\|.\|_{2,n}$ respectively. Letting n go to infinity and using the downward continuity of θ-ranks as functions of the seminorms — ultimately a consequence of the similar downward continuity of Banaszczyk functions, see (7.7.4) —finishes the proof of (9.5.12) in this case.

(3) To prove the estimate (9.5.12) in the general case, we proceed by using finite dimensional approximations, as in Part (3) of the proof of Proposition 9.3.2.

By definition, the invariant $\underline{\mathrm{rk}}^1_\theta(f, \|.\|)$ is the limit of the expressions

$$\mathrm{rk}^1_\theta\big(f_{N'} : M/f^{-1}(N') \longrightarrow (N/N', \|.\|)\big)$$

as N' ranges through the directed set $(\mathrm{coft}(N), \supseteq)$.

Moreover we have:

$$\lambda_1\big(f_{N'}^\vee : (N/N', \|.\|_2)^\vee \to (M/f^{-1}(N')^\vee\big) = \inf\big\{\|\xi\|_2^\vee \mid \xi \in N^\vee,\, \xi \circ f \neq 0,\, \xi_{|N'} = 0\big\},$$

and consequently:

$$\lambda_1\big(f_{N'}^\vee : (N/N', \|.\|_2)^\vee \to (M/f^{-1}(N')^\vee\big) \geq \lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee).$$

Consequently, going to the limit over N' in the estimate (9.5.12) applied to $f_{N'}$ finishes the proof. $\square$

9.5.3. Proof of Theorem 9.5.1. When the seminorm $\|.\|_1$ vanishes, or when the relative trace $\mathrm{Tr}(\|.\|_1/\|.\|_2)$ is infinite, the estimates (9.5.1) and (9.5.3) are obviously satisfied. We may therefore assume that $\|.\|_1$ is not identically zero and that the relative trace $\mathrm{Tr}(\|.\|_1/\|.\|_2)$ is finite, and therefore that the seminorm $\|.\|_1$ is compact with respect to $\|.\|_2$.

Observe that, for every $\lambda \in \mathbb{R}_+^*$, when replacing $\|.\|_1$ by $\lambda\|.\|_1$ both the left- and right-hand term in (9.5.1) are multiplied by λ. Consequently, to prove (9.5.1), we may also assume that the first characteristic value of $\|.\|_1$ with respect to $\|.\|_2$ is 1, and therefore that the following conditions are satisfied:

$$\|.\|_1 \leq \|.\|_2 \quad \text{and} \quad T := \mathrm{Tr}(\|.\|_1/\|.\|_2) \geq 1. \tag{9.5.14}$$

When this last condition holds, the right-hand side of (9.5.3) is bounded from above by:

$$(5/2\pi)\mathrm{Tr}(\|.\|_1/\|.\|_2),$$

and therefore the validity of (9.5.3) implies the one (9.5.6), and consequently of (9.5.1).

To prove Theorem 9.5.1, we are therefore left to establish (9.5.3) when conditions (9.5.2), or equivalently (9.5.14), are satisfied. From now on we shall assume that is the case. In order to prove (9.5.3), we may also assume:

$$\underline{\rho}(f : M \to \overline{N}_1)\, \lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee) > \frac{T}{2\pi}.$$

According to Proposition 9.5.2, we may find a Euclidean seminorm $\|.\|$ on $N_{\mathbb{R}}$ which satisfies the following conditions:

$$\|.\|_1 \leq \|.\| \leq \|.\|_2 \quad \text{and} \quad \mathrm{Tr}(\|.\|_1^2/\|.\|^2) = \mathrm{Tr}(\|.\|^2/\|.\|_2^2) = T.$$

For any real number δ, replacing the seminorms $\|.\|_1$, $\|.\|$ and $\|.\|_2$ by $e^{-\delta}\|.\|_1$, $e^{-\delta}\|.\|$ and $e^{-\delta}\|.\|_1$ does not change the relative traces. It replaces $\underline{\rho}(f : M \to \overline{N}_1))$ with $e^{-\delta}\underline{\rho}(f : M \to \overline{N}_1)$ and $\lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee)$ with $e^{\delta}\lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee)$. In particular, after choosing δ suitably, we may assume that the following inequalities hold:

$$\underline{\rho}(f : M \to \overline{N}_1) > \sqrt{\frac{T}{2\pi}} \quad \text{and} \quad \lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee) > \sqrt{\frac{T}{2\pi}}.$$

We may write:

$$\widetilde{\underline{\rho}} := \sqrt{2\pi/T}\,\underline{\rho}(f : M \to \overline{N}_1) \quad \text{and} \quad \widetilde{\lambda} := \sqrt{2\pi/T}\,\lambda_1(f^\vee : \overline{N}_2^\vee \to M^\vee),$$

so that we have:

$$\widetilde{\underline{\rho}} > 1 \quad \text{and} \quad \widetilde{\lambda} > 1.$$

Corollary 9.3.5, in its variant (9.3.24) for the lower covering radius, shows:

$$2e^{-\underline{\mathrm{rk}}^1_\theta(f:M\to\overline{N})} \leq 1 + e^{\frac{T}{2}\left(1-\widetilde{\underline{\rho}}^2+2\log\widetilde{\underline{\rho}}\right)}. \tag{9.5.15}$$

Proposition 9.5.4 shows:

$$e^{-\underline{\mathrm{rk}}^1_\theta(f,\|.\|)} + e^{T/2(1-\widetilde{\lambda}^2+2\log\widetilde{\lambda})} \geq 1. \tag{9.5.16}$$

As a consequence of (9.5.15) and (9.5.16), we obtain:

$$e^{\frac{T}{2}\left(1-\widetilde{\underline{\rho}}^2+2\log\widetilde{\underline{\rho}}\right)} + 2e^{T/2(1-\widetilde{\lambda}^2+2\log\widetilde{\lambda})} \geq 1. \tag{9.5.17}$$

If $\widetilde{\underline{\rho}} = +\infty$, then (9.5.17) becomes:

$$e^{T/2(1-\widetilde{\lambda}^2+2\log\widetilde{\lambda})} \geq 1. \tag{9.5.18}$$

If $\widetilde{\lambda} \neq 0$, after again replacing the seminorms $\|.\|_1$, $\|.\|$ and $\|.\|_2$ by $e^{-\delta}\|.\|_1$, $e^{-\delta}\|.\|$ and $e^{-\delta}\|.\|_1$, we may assume that $\widetilde{\lambda}$ is arbitrarily large, which is a contradiction, so that $\widetilde{\lambda} = 0$. Similarly, if $\widetilde{\lambda} = +\infty$, then $\widetilde{\underline{\rho}} = 0$.

Assume that both $\widetilde{\lambda}$ and $\widetilde{\underline{\rho}}$ are finite. Again, after replacing the seminorms $\|.\|_1$, $\|.\|$ and $\|.\|_2$ by $e^{-\delta}\|.\|_1$, $e^{-\delta}\|.\|$ and $e^{-\delta}\|.\|_1$ for a suitable δ, we may assume the equality:

$$\widetilde{\lambda} = \widetilde{\underline{\rho}}.$$

Our goal, i.e., proving (9.5.3), becomes the estimate:

$$\widetilde{\lambda}^2 \leq 1 + \frac{4}{\sqrt{T}}, \tag{9.5.19}$$

and (9.5.17) becomes:

$$3e^{T/2(\log(\widetilde{\lambda}^2) - \widetilde{\lambda}^2 + 1)} \geq 1. \tag{9.5.20}$$

If we write $\widetilde{\lambda}^2 = 1 + x$, the inequality (9.5.19) we need to prove becomes:

$$x \leq \frac{4}{\sqrt{T}}, \tag{9.5.21}$$

Then from (9.5.20), we find:

$$x - \log(1+x) \leq \frac{2}{T}\log 3.$$

The function $(x \mapsto x - \log(1+x))$ is increasing for $x \geq 1$. As a consequence, to prove (9.5.21), it suffices to prove the inequality:

$$\frac{4}{\sqrt{T}} - \log\left(1 + \frac{4}{\sqrt{T}}\right) \geq \frac{2}{T}\log 3 \tag{9.5.22}$$

for any $T \geq 1$. This is the content of Proposition 9.5.3, which completes the proof of Theorem 9.5.1.

9.5.4. Application to successive minima and covering radii. As alluded above, Theorem 9.5.1 may be used to recover transference inequalities on successive minima.[22]

THEOREM 9.5.5. *Let M be a countably generated $\mathbb{Z}$-module abelian group and let $\|.\|_1$ and $\|.\|_2$ be two Euclidean seminorms on $M_{\mathbb{R}}$.*

If $\overline{M}_1 := (M, \|.\|_1)$ and $\overline{M}_2 := (M, \|.\|_2)$ denote the associated Euclidean quasi-coherent sheaves, then, for any integer i such that:

$$0 \leq i < \dim_{\mathbb{R}} M_{\mathbb{R}},$$

the following inequality is satisfied:

$$\underline{\rho}^{[i]}(\overline{M}_1)\,\lambda_{i+1}(\overline{M}_2^{\vee}) \leq \mathrm{Tr}(\|.\|_1/\|.\|_2). \tag{9.5.23}$$

When moreover the following conditions hold:

$$\|.\|_1 \leq \|.\|_2 \quad \textit{and} \quad \mathrm{Tr}(\|.\|_1/\|.\|_2) \geq 1,$$

then the inequality (9.5.23) *admits the following stronger variant:*

$$\underline{\rho}^{[i]}(\overline{M}_1)\,\lambda_{i+1}(\overline{M}_2^{\vee}) \leq \frac{1}{2\pi}\mathrm{Tr}(\|.\|_1/\|.\|_2) + \frac{2}{\pi}\sqrt{\mathrm{Tr}(\|.\|_1/\|.\|_2)}. \tag{9.5.24}$$

[22]The successive lower covering radii $\underline{\rho}^{[i]}(\overline{N})$ attached to an object $\overline{N}$ of $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$ have been defined in Subsection 6.7.3.

Proof. Let V be the vector subspace of $M^\vee_{\mathbb{R}}$ generated by those $\varphi \in M^\vee$ with

$$\|\varphi\|_2^\vee < \lambda_{i+1}(\overline{M}_2^\vee).$$

Then V has dimension at most i. Let j be the dimension of V, and consider $\varphi_1, \ldots, \varphi_j \in M^\vee$ such that the φ_k are linearly independent in $M^\vee_{\mathbb{R}}$ and, for all k between 1 and j,

$$\|\varphi_k\|_2^\vee \leq \lambda_{i+1}(\overline{M}_2^\vee).$$

Let N be the intersection of the j subspaces $\operatorname{Ker}\varphi_1, \ldots, \operatorname{Ker}\varphi_j$ of M, and let $f : N \to M$ be the inclusion. Then $N_{\mathbb{R}}$ has codimension $j \leq i$ in $M_{\mathbb{R}}$. In particular, we have:

$$\underline{\rho}(f : N \to \overline{M}_1) \geq \underline{\rho}^{[j]}(\overline{M}_1) \geq \underline{\rho}^{[i]}(\overline{M}_1). \tag{9.5.25}$$

The morphism $f^\vee : M^\vee \to N^\vee$ sends an element $\varphi \in M^\vee$ to its restriction to N. Let φ be an element of $M^\vee$ such that $f^\vee(\varphi)$ does not vanish. Then φ does not not belong to the subspace V of $M^\vee_{\mathbb{R}}$ generated by $\varphi_1, \ldots, \varphi_j$, so that:

$$\|\varphi\|_2^\vee \geq \lambda_{i+1}(\overline{M}_2^\vee).$$

In particular, we obtain:

$$\lambda_1(f^\vee : \overline{M}_2^\vee \to N^\vee) \geq \lambda_{i+1}(\overline{M}_2^\vee). \tag{9.5.26}$$

The second inequality in Theorem 9.5.1 proves the inequality:

$$\underline{\rho}(f : N \to \overline{M}_1)\, \lambda_1(f^\vee : \overline{M}_2^\vee \to N^\vee) \leq \operatorname{Tr}(\|.\|_1/\|.\|_2).$$

Together with (9.5.25) and (9.5.26), this implies:

$$\underline{\rho}^{[i]}(\overline{M}_1)\, \lambda_{i+1}(\overline{M}_2^\vee) \leq \operatorname{Tr}(\|.\|_1/\|.\|_2).$$

This proves (9.5.23).

Similarly the estimate (9.5.24) follows from the second estimate (9.5.3) in Theorem 9.5.1. □

We may replace covering radii with successive minima and the invariants γ^i introduced in 6.6.2. Recall that $\gamma^i(M, \|.\|)$ is the infimum of those positive real numbers R such that there exists a subroup N of $M_{/\mathrm{tor}}$ generated by elements v with $\|v\| \leq R$ such that M_{tor}/N is free of rank i.

As the invariants $\gamma^{[i]}$ and $\lambda^{[i]}$ are monotonous, we may consider their lower extensions $\underline{\gamma}^{[i]}$ and $\underline{\lambda}^{[i]}$ as defined in Section 4.3. As $\gamma^{[i]}$ and $\lambda^{[i]}$ are bounded above by $\rho^{[i]}$, $\underline{\gamma}^{[i]}$ and $\underline{\lambda}^{[i]}$ are bounded above by $\underline{\rho}^{[i]}$ and we obtain the following consequence of Theorem 9.5.5.

Corollary 9.5.6. *Let M be a countably generated $\mathbb{Z}$-module and let $\|.\|_1$ and $\|.\|_2$ be two Euclidean seminorms on $M_{\mathbb{R}}$.*

If $\overline{M}_1 := (M, \|.\|_1)$ and $\overline{M}_2 := (M, \|.\|_2)$ denote the associated Euclidean quasi-coherent sheaves, then, for any integer i such that:

$$0 \leq i < \dim_{\mathbb{R}} M_{\mathbb{R}},$$

the following inequality is satisfied:

$$\underline{\gamma}^{[i]}(\overline{M}_1)\, \underline{\lambda}_{i+1}(\overline{M}_2^\vee) \leq 2\operatorname{Tr}(\|.\|_1/\|.\|_2). \tag{9.5.27}$$

In particular:

$$\underline{\lambda}^{[i]}(\overline{M}_1)\, \underline{\lambda}_{i+1}(\overline{M}_2^\vee) \leq 2\operatorname{Tr}(\|.\|_1/\|.\|_2). \tag{9.5.28}$$

When moreover the following conditions hold:

$$\|.\|_1 \leq \|.\|_2 \quad \textit{and} \quad \operatorname{Tr}(\|.\|_1/\|.\|_2) \geq 1,$$

then inequality (9.5.23) *may be strengthened in:*

$$\lambda^{[i]}(\overline{M}_1)\,\lambda_{i+1}(\overline{M}_2^\vee) \leq \frac{1}{\pi}\mathrm{Tr}(\|.\|_1/\|.\|_2) + \frac{4}{\pi}\sqrt{\mathrm{Tr}(\|.\|_1/\|.\|_2)}.$$

Proof. This is a straightforward consequence of Theorem 9.5.5 and Proposition 6.6.10. □

9.5.5. The case of a single seminorm. Application to lattices with gaps. The transference estimates established in Theorem 9.5.5 and Corollary 9.5.6 may be applied to some Euclidean quasi-coherent sheaves $\overline{M} = (M, \|.\|)$ without explicitely referring to a second seminorm on $M_\mathbb{R}$ when this $\mathbb{R}$-vector space is finite dimensional.

In this section, we assume that $\overline{M} = (M, \|.\|)$ is a Euclidean quasi-coherent sheaf such that $M_\mathbb{R}$ is finite dimensional — this is for instance the case if $\overline{M}$ is a Euclidean lattice — and we let:

$$n := \dim_\mathbb{R} M_\mathbb{R}.$$

First, we recover generalizations of the classical transference inequalities of [**Ban93**] as follows.

Theorem 9.5.7. *For any integer i such that $0 \leq i < n$, the following estimates hold:*

$$\rho^i(\overline{M})\,\lambda_{i+1}(\overline{M}^\vee) \leq \frac{n}{2\pi} + \frac{2\sqrt{n}}{\pi} \tag{9.5.29}$$

and:

$$\gamma_{n-i}(\overline{M})\,\lambda_{i+1}(\overline{M}^\vee) \leq \frac{n}{\pi} + \frac{4\sqrt{n}}{\pi}. \tag{9.5.30}$$

In particular, we have:

$$\lambda_{n-i}(\overline{M})\,\lambda_{i+1}(\overline{M}^\vee) \leq \frac{n}{\pi} + \frac{4\sqrt{n}}{\pi}. \tag{9.5.31}$$

Proof. This is a consequence of Theorem 9.5.1, Theorem 9.5.5 and Corollary 9.5.6 in the case $\|.\|_1 = \|.\|_2 = \|.\|$. □

Remark 9.5.8. The numerical constants in Theorem 9.5.7 are slightly better than those appearing for instance in [**MSD19**, (1.9)]. The inequality (9.5.30) is a more precise version of [**Cai03**, Theorem 4.1]. One may similarly improve on the transference inequalities of [**WL16**].

We may improve the inequalities of Theorem 9.5.7 by choosing carefully the two seminorms appearing in Theorem 9.5.1. The next proposition contains the relevant construction.

For any integer $i \in \{1, \ldots, n\}$, we let:

$$\lambda_i := \lambda_i(\overline{M}).$$

Proposition 9.5.9. *For every integer $i \in \{1, \ldots, n\}$, there exists a Euclidean seminorm $\|.\|_i$ on M that satisfies the following properties:*

(i) $\|.\|_i \leq \|.\|$;
(ii) $\lambda_i(M, \|.\|_i) = \lambda_i$;
(iii) $\mathrm{Tr}(\|.\|_i/\|.\|) \leq i + \sum_{j=i+1}^n \frac{\lambda_i}{\lambda_j}$.

In (iii) above, we use the convention $\lambda_i/\lambda_j = 0$ if $\lambda_i = \lambda_j = 0$. The proof will show that, when $\lambda_i \neq 0$, we may require the inequality in (iii) to be an equality.

Proof. For any nonnegative real number R, let V_R denote the vector subspace of $M_\mathbb{R}$ generated by those v in $M_{/\mathrm{tor}}$ with $\|v\| \leq R$. The vector spaces V_R form an increasing family of vector subspaces of $M_\mathbb{R}$, and their reunion is $M_\mathbb{R}$ itself. In particular, there are only a finite numbers of subspaces of $M_\mathbb{R}$ that are of the form V_R for some real number R and, for any integer j between 1 and n, the space $V_{\lambda_j+\varepsilon}$ does not depend on $\varepsilon > 0$ if ε is chosen to be small enough.

Denote by V_j the vector space $V_{\lambda_j+\varepsilon}$ for small enough ε, and set $V_0 = 0$. By definition of successive minima, if v is an element of $M_{/\mathrm{tor}}$, then:

$$v \notin V_j \implies \|v\| \geq \lambda_{j+1} \tag{9.5.32}$$

and, for any $\varepsilon > 0$, V_{j+1} is generated by those elements of $V_j \cap M_{/\mathrm{tor}}$ of norm at most $\lambda_{j+1} + \varepsilon$.

For any j between 2 and n, let H_j be an orhogonal complement of V_{j-1} in V_j. We set $H_1 = V_1$, so that $M_{\mathbb{R}}$ is the orthogonal direct sum of the H_j and, for any j between 1 and n, V_j is the orthogonal direct sum of the H_k, $k \leq j$.

We define a seminorm $\|.\|_i$ of $M_{\mathbb{R}}$ by requiring the decomposition

$$M_{\mathbb{R}} = V_i \oplus H_{i+1} \oplus \ldots \oplus H_n$$

to be orthogonal, and by requiring following equalities:

(i) $(\|.\|_i)_{|V_i} = \|.\|_{|V_i}$;
(ii) for any $j \geq i+1$, $(\|.\|_i)_{|H_j} = \frac{\lambda_i}{\lambda_j}\|.\|_{|H_j}$.

By construction, we have $\|.\|_i \leq \|.\|$. If $\lambda_i = 0$, then $\|.\|_i = 0$ and conditions $(i), (ii), (iii)$ of the Proposition are satisfied. Assume that λ_i is nonzero. By construction, we have $\|.\|_i \leq \|.\|$, namely, condition (ii) holds.

We prove condition (ii), namely, the equality:

$$\lambda_i(M, \|.\|_i) = \lambda_i.$$

Since $\|.\|_i$ is bounded above by $\|.\|$, we know:

$$\lambda_i(M, \|.\|_i) \leq \lambda_i.$$

Given $\varepsilon > 0$, let W_ε be the vector subspace of $M_{\mathbb{R}}$ generated by those elements v of $M_{/\mathrm{tor}}$ with $\|v\|_i \leq \lambda_i - \varepsilon$. It remains to prove that W_ε has dimension at most $i-1$.

Let j be an integer between $i+1$ and n. By construction, for any $v \in V_j$, we have:

$$\|v\|_i \geq \frac{\lambda_i}{\lambda_j}\|v\|. \tag{9.5.33}$$

Together with (9.5.32), (9.5.33) implies, for any nonzero v in $M_{/\mathrm{tor}}$ and any j between i and $n-1$:

$$v \in V_{j+1} \setminus V_j \implies \|v\|_i \geq \frac{\lambda_i}{\lambda_{j+1}}\lambda_{j+1} = \lambda_i.$$

As a consequence, for any nonzero v in $M_{/\mathrm{tor}}$:

$$v \notin V_i \implies \|v\|_i \geq \lambda_i.$$

In other words, W_ε is contained in V_i. As a consequence, to prove that the dimension of W_ε is at most $i-1$, we may replace $\overline{M}$ with $(M_{/\mathrm{tor}} \cap V_i, \|.\|)$. By definition, $\|.\|_i$ coincides with $\|.\|$ on V_i, so that

$$\lambda(M_{/\mathrm{tor}} \cap V_i, \|.\|_i) = \lambda_i,$$

which finishes the proof of condition (ii).

We now prove that condition (iii) in the proposition holds, i.e., we compute $\mathrm{Tr}(\|.\|_i/\|.\|)$. Consider the function

$$f : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$$

defined by letting $f(j)$ be the largest integer between 1 and n such that $\lambda_j = \lambda_{f(j)}$. By definition, for any $R < \lambda_{f(j)+1}$, the space V_R has dimension at most $f(j)$ and, for any $R > \lambda_j = \lambda_{f(j)}$, the space V_R has dimension at least $f(j)$. As a consequence, we obtain, for any j between 1 and n:

$$\dim V_j = f(j). \tag{9.5.34}$$

As a consequence, we have:

$$\dim H_j = f(j) - f(j-1)$$

for any $j \geq i+1$.

By definition of the function f, we have $f(j) = f(j-1)$ if and only if $\lambda_{j-1} = \lambda_j$. If $\lambda_j > \lambda_{j-1}$, then $f(j-1) = j-1$ and $f(j) - f(j-1)$ is the number of integers k with $\lambda_k = \lambda_j$. In other words:

$$\dim H_j = \delta^{\lambda_j}_{\lambda_{j-1}} |\{k \in \{i+1, \ldots, n\}, \lambda_k = \lambda_j\}|, \tag{9.5.35}$$

where δ^a_b is 0 when $a \neq b$ and 1 when $a = b$, and $|X|$ denotes the cardinality of the finite set X.

Finally, we have:

$$\operatorname{Tr}(\|.\|_i/\|.\|) = \dim V_i + \sum_{j=i+1}^{n} \dim H_j \frac{\lambda_i}{\lambda_j}. \tag{9.5.36}$$

By (9.5.34), we may write:

$$\dim V_i = f(i) = i + \sum_{j=i+1}^{f(i)} 1 = f(i) = i + \sum_{j=i+1}^{f(i)} \frac{\lambda_i}{\lambda_j}. \tag{9.5.37}$$

Similarly, by (9.5.35), for any $j \geq i+1$, we may write:

$$\sum_{k\in\{i+1,\ldots,n\},\lambda_k=\lambda_j} \dim H_k \frac{\lambda_i}{\lambda_k} = |\{k \in \{i+1, \ldots, n\}, \lambda_k = \lambda_j\}| \frac{\lambda_i}{\lambda_j} = \sum_{k\in\{i+1,\ldots,n\},\lambda_k=\lambda_j} \frac{\lambda_i}{\lambda_k}. \tag{9.5.38}$$

Plugging (9.5.37) and (9.5.38) in (9.5.36) finishes the proof. □

As a consequence of the construction of Proposition 9.5.9, we obtain the following improvement on Banaszczyk's transference inequalities.

THEOREM 9.5.10. *Let $\overline{M} = (M, \|.\|)$ be a Euclidean lattice of rank n with successive minima:*

$$\lambda_1 \leq \ldots \leq \lambda_n.$$

For any integer $i \in \{1, \ldots, n\}$, we let:

$$T_i := i + \sum_{j=i+1}^{n} \frac{\lambda_i}{\lambda_j}.$$

Then we have:

$$\lambda_i\, \rho^{[i-1]}(\overline{M}^\vee) \leq \frac{T_i}{2\pi} + \frac{2\sqrt{T_i}}{\pi} \tag{9.5.39}$$

and

$$\lambda_i\, \gamma_{n-i+1}(\overline{M}^\vee) \leq \frac{T_i}{\pi} + \frac{4\sqrt{T_i}}{\pi}. \tag{9.5.40}$$

In particular:

$$\lambda_i\, \lambda_{n-i+1}(\overline{M}^\vee) \leq \frac{T_i}{\pi} + \frac{4\sqrt{T_i}}{\pi}. \tag{9.5.41}$$

PROOF. The inequalities (9.5.40) and (9.5.41) are consequences of (9.5.39) and Proposition 6.6.10. To prove (9.5.39), apply Theorem 9.5.5 to $\overline{M}^\vee$ equipped with the norm $\|.\|^\vee$ and the norm $\|.\|_i^\vee$ dual to that constructed in Proposition 9.5.9. After noting:

$$\operatorname{Tr}(\|.\|^\vee/\|.\|_i^\vee) = \operatorname{Tr}(\|.\|_i/\|.\|) = T_i \geq 1,$$

we find:

$$\lambda_i(M, \|.\|_i)\, \rho^{[i-1]}(\overline{M}^\vee) \leq \frac{T_i}{2\pi} + \frac{2\sqrt{T_i}}{\pi}.$$

Since $\lambda_i(M, \|.\|_i) = \lambda_i$, this proves (9.5.39). □

REMARK 9.5.11. It would be possible to improve (9.5.41) in Theorem 9.5.10 mildly by applying Proposition 9.5.9 to $\lambda_{n-i+1}(\overline{M})$ as well.

Note that we have obvious inequalities:

$$T_i \leq i + (n-i)\frac{\lambda_i}{\lambda_{i+1}} \leq n.$$

In particular, we have:

$$T_1 \leq 1 + (n-1)\frac{\lambda_1}{\lambda_2},$$

so that (9.5.41) provides an upper bound for the quantity $\lambda_1(\overline{M})\,\lambda_n(\overline{M}^\vee)$ in terms of the gap λ_1/λ_2 between the first two minima of $\overline{M}$.

Theorem 9.5.10 shows that transference inequalities may be improved when applied to Euclidean lattices $\overline{M}$ for which the sequence $\lambda_1 \leq \lambda_2 \leq \dots$ of successive minima contains large gaps.

It turns out that the main cryptosystems that appear in lattice-based cryptography do indeed use lattices with gaps, starting with the classical cryptosystem of Ajtai-Dwork [**AD99**]. There is a gap between λ_1 and λ_2 for cryptosystems based on the LWE problem of [**Reg05**], and it is noticed in [**HPS98**] that the lattices of rank $2n$ appearing in the widely used NTRU cryptosystem admit a gap between λ_n and λ_{n+1}.

Motivated by the special role of lattices with gaps, transference inequalities for these lattices have been investigated in the computer science literature; see for instance [**WTW14, WLW15**] whose results may be considered as special cases of our Theorem 9.5.10.

9.6. Finiteness of $\lambda^{[0]}$ and θ^1-Finiteness

At this stage, by combining results proved so far, one sees that, *if $\overline{M} := (M, \|.\|)$ is a Euclidean quasi-coherent sheaf that satisfies:*

$$\lambda^{[0]}(\overline{M}) < +\infty,$$

and if $\|.\|'$ is a Euclidean seminorm on $M_{\mathbb{R}}$ that is Hilbert-Schmidt with respect to $\|.\|$, then the Euclidean quasi-coherent sheaf $\overline{M}' := (M, \|.\|')$ is θ^1-finite.

Indeed, according to Remark 6.4.14, there exists a $\mathbb{Z}$-submodule N of M such that $N_{\mathbb{R}}$ is dense in $(M_{\mathbb{R}}, \|.\|)$ (and a fortiori in $(M_{\mathbb{R}}, \|.\|')$) and the Euclidean quasi-coherent sheaf $\overline{N}' := (N, \|.\|'_{|N_{\mathbb{R}}})$ is ρ^2-summable. According to Proposition 9.4.6, $\overline{N}'$ is θ^1-finite, and therefore, by Remark 8.5.6, $\overline{M}'$ also is θ^1-finite.

In this section, we shall show that the above implication still holds with the assumption on $\|.\|'$ to be Hilbert-Schmidt with respect to $\|.\|$ replaced by a much weaker assumption. The proof will not use previous results in this chapter, but will be an application of the criteria of θ^1-finiteness established in Section 8.5, combined with the consequence of the Peierls-Bogoliubov inequality stated in Corollary A.3.2.

9.6.1. The condition SE on pairs of Euclidean seminorms. Let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on some $\mathbb{R}$-vector space V.

Let us assume that $\|.\|'$ is compact with respect to $\|.\|$, and let

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

be the sequence of singular values of $\|.\|'$ with respect to $\|.\|$. For every $T \in \mathbb{R}_+^*$, let us define:

$$N(T) := |\{i \in \mathbb{N}_{\geq 1} \mid \lambda_i^{-1} \leq T\}|.$$

The following two conditions are equivalent:

SE$_1$: *for every* $\varepsilon \in \mathbb{R}_+^*$,

$$N(T) = O(e^{\varepsilon T}) \quad \text{when } T \to +\infty; \tag{9.6.1}$$

and:

SE$_2$: *for every* $\eta \in \mathbb{R}_+^*$,

$$\sum_{i=1}^{+\infty} e^{-\eta/\lambda_i} < +\infty. \tag{9.6.2}$$

Indeed, if (9.6.1) holds, the (9.6.2) holds for any η in $(\varepsilon, +\infty)$; conversely, if (9.6.2) holds, then (9.6.1) holds with $\varepsilon = \eta$.

When **SE**$_1$ and **SE**$_2$ hold, we shall say that $\|.\|'/\|.\|$ *satisfies* **SE**, or that $\|.\|'$ *satisfies* **SE** *with respect to* $\|.\|$.

Observe that condition **SE** is a very weak quantitative strengthening of the compactness of $\|.\|'$ with respect to $\|.\|$. It is notably satisfied when $N(T)$ grows at most polynomially as a function of T, or equivalently, when there exists $p \in \mathbb{R}_+^*$ such that

$$\operatorname{Tr}(\|.\|'^p/\|.\|^p) := \sum_{i=1}^{\infty} \lambda^p < +\infty. \tag{9.6.3}$$

9.6.2. The implication: $\lambda^{[0]}(\overline{M}) < +\infty$ and $\|.\|'/\|.\|$ satisfies SE $\Longrightarrow \overline{M}'$ θ^1-finite. Having introduced the condition **SE** on pairs of Euclidean seminorms on some $\mathbb{R}$-vector space, we may establish the following criterion for θ^1-finiteness:

PROPOSITION 9.6.1. *Let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf. If $\lambda^{[0]}(\overline{M}) < +\infty$, then $\overline{M}' := (M, \|.\|')$ is θ^1-finite for every Euclidean seminorm $\|.\|'$ on $M_{\mathbb{R}}$ such that $\|.\|'/\|.\|$ satisfies condition* **SE**.

PROOF. Assume that $\lambda^{[0]}(\overline{M}) < +\infty$. Let N be the subspace of M generated by those $m \in M$ with $\|m\| \leq 1 + \lambda^{[0]}(\overline{M})$. Then the image of N in $M_{\mathbb{R}}$ is dense in $(M_{\mathbb{R}}, \|.\|)$. In particular, N is dense in $(M_{\mathbb{R}}, \|.\|')$. As a consequence, Remark 8.5.6 shows that, if $(N, \|.\|')$ is θ^1-finite, then $(M, \|.\|')$ is θ^1-finite. Additionnally, the construction of N shows that $\gamma(N, \|.\|_{|N_{\mathbb{R}}})$ is finite. Therefore, after replacing M with N, we may assume that $\gamma(\overline{M})$ is finite, or equivalently that $\overline{M}$ is generated by bounded sections.

By definition, the invariant h^1_θ of an object $(\mathbb{Z}m, \|.\|)$ of $\overline{\mathbf{Coh}}_{\mathbb{Z}}$ with a cyclic underlying $\mathbb{Z}$-module satisfies:

$$\begin{aligned} h^1_\theta(\mathbb{Z}m, \|.\|) &= \log\theta(\|m\|^{-2}) && \text{if } \|m\| \neq 0, \\ &= 0 && \text{if } \|m\| = 0. \end{aligned}$$

where:

$$\begin{aligned} \log\theta(x) &:= \log \sum_{k\in\mathbb{Z}} e^{-\pi k^2 x} \\ &= 2e^{-\pi x} + O(e^{-4\pi x}) \quad \text{when } x \in \mathbb{R}_+^* \text{ goes to } +\infty. \end{aligned}$$

In particular, for every A in $\mathbb{R}_+$, there exist $c_2(A) > c_1(A) > 0$ such that:

$$c_1(A)\, e^{-\pi\|m\|^{-2}} \leq h^1_\theta(\mathbb{Z}m, \|.\|) \leq c_2(A)\, e^{-\pi\|m\|^{-2}} \quad \text{if } \|m\| \leq A, \tag{9.6.4}$$

where we define $e^{-\pi x^{-2}}$ to be 0 when $x = 0$.

When $\gamma(\overline{M})$ is finite, we may choose a family $(m_k)_{k\in\mathbb{N}}$ of generators of the $\mathbb{Z}$-module M such that

$$B := \sup_{k\in\mathbb{N}} \|m_k\| < +\infty,$$

and we may consider the exhausting filtration $C_\bullet := (C_i)_{i\in\mathbb{N}}$ of M by submodules in $\mathrm{coh}(M)$ defined by

$$C_i := \sum_{0\leq k\leq i} \mathbb{Z}m_i.$$

Let $\|.\|'$ be a Euclidean seminorm on $M_{\mathbb{R}}$. According to Proposition 8.5.1 and Definition 8.5.2, to show that $\overline{M}' := (M, \|.\|')$ is θ^1-finite, it is enough to prove that the condition

$$\mathbf{Sum}(C_\bullet \otimes \overline{\mathcal{O}}(-\delta)): \quad \sum_{i=0}^{\infty} h^1_\theta(\overline{C_i/C_{i-1}}' \otimes \overline{\mathcal{O}}(-\delta)) < +\infty$$

holds for every $\delta \in \mathbb{R}$.

For every $i \in \mathbb{N}$, the quotient module $\overline{C_i/C_{i-1}}'$ is cyclic, generated by the class $\overline{m}_i$ of m_i, and if $\|.\|'$ denotes the Euclidean seminorm on the subquotient $C_{i,\mathbb{R}}/C_{i-1,\mathbb{R}}$ deduced from $\|.\|'$, then according to (9.6.4), the θ-invariant

$$h^1_\theta(\overline{C_i/C_{i-1}}' \otimes \overline{\mathcal{O}}(-\delta)) = h^1_\theta(\mathbb{Z}\overline{m}_i, e^\delta \|.\|'_i)$$

satisfies the estimates:

$$c_1(e^\delta B)\exp(-\pi e^{2\delta}\|\overline{m_i}\|_i'^{-2}) \leq h^1_\theta(\overline{C_i/C_{i-1}}' \otimes \overline{\mathcal{O}}(-\delta)) \leq c_2(e^\delta B)\exp(-\pi e^{2\delta}\|\overline{m_i}\|_i'^{-2}).$$

Therefore, for every $\delta \in \mathbb{R}$, the condition $\mathbf{Sum}(C_\bullet \otimes \overline{\mathcal{O}}(-\delta))$ holds if and only if

$$\sum_{i\in\mathbb{N}} \exp(-\pi e^{2\delta}\|\overline{m_i}\|_i'^{-2}) < +\infty. \tag{9.6.5}$$

Let us define an increasing function $\varphi_\delta : \mathbb{R}_+ \to \mathbb{R}_+$ by:

$$\begin{aligned} \varphi_\delta(x) &:= \exp(-\pi e^{2\delta}/x) && \text{if } x > 0, \\ &:= 0 && \text{if } x = 0. \end{aligned}$$

It is convex on some neighborhood of 0 in $\mathbb{R}_+$; indeed, the second derivative of $(x \mapsto e^{-1/x})$ is $(x \mapsto (-2x^{-3} + x^{-4})e^{-1/x})$, and is positive on $(0, 1/2)$. Moreover, when $\|.\|'/\|.\|$ satisfies **SE**, the validity of $\mathbf{SE}_2$ with $\eta = \pi e^{2\delta}$ shows that, with the notation of Appendix A:

$$\mathrm{Tr}\, \varphi_\delta(\|.\|'^2/\|.\|^2) < +\infty.$$

Therefore, according to Corollary A.3.3 applied with V the $\mathbb{R}$-vector space $M_{\mathbb{R}}$, with v_i the image of m_i in $M_{\mathbb{R}}$, and with $\tilde{\varphi}$ the function φ_δ, we have:

$$\sum_{i\in\mathbb{N}} \varphi_\delta(\|\overline{m_i}\|_i'^{2}) < +\infty,$$

and consequently the summability condition (9.6.5) holds. □

9.7. *Coronidis loco:* Finiteness and Eventual Vanishing Properties of Euclidean Quasi-coherent Sheaves

In this final section, we sum up the implications established in this monograph between the finiteness and eventual vanishing properties of the elementary invariants $\rho(\overline{M})$, $\gamma(\overline{M})$ and $\lambda^{[0]}(\overline{M})$ attached to a Euclidean quasi-coherent sheaf $\overline{M} := (M, \|.\|)$, and of its theta invariants, notably $\overline{h}^1_\theta(\overline{M})$.

We also summarize the implications between these properties and the ones of the Euclidean quasi-coherent sheaf $\overline{M}' := (M, \|.\|')$ deduced from $\overline{M}$ by replacing the Euclidean semi-norm $\|.\|$ by a Euclidean semi-norm $\|.\|'$ on $M_{\mathbb{R}}$ that satisfies a suitable compactness or trace condition with respect to $\|.\|$.

We finally present a consequence of these implications concerning families $(M, \|.\|_\alpha)_{\alpha\in A}$ of Euclidean quasi-coherent sheaves defined by a fixed $\mathbb{Z}$-module M equipped with a family $(\|.\|_\alpha)_{\alpha\in A}$ of Euclidean seminorms satisfying a suitable nuclearity property.

To provide some hint of the Diophantine applications of our results, we also discuss how such families naturally arise in Diophantine geometry, when considering sections of vector bundles over schemes of finite type over $\mathbb{Z}$.

9.7.1. Let $\overline{M}$ be a Euclidean quasi-coherent sheaf.

Recall that, for each of the invariants $\varphi = \lambda^{[0]}$, γ, ρ, $\overline{\rho}$, or $\overline{h}^1_\theta$, the invariant $\mathrm{ev}\varphi$ is defined as follows:

$$\mathrm{ev}\varphi(\overline{M}) := \lim_{C\in\mathrm{coh}(M)} \varphi(\overline{M/C}) = \inf_{C\in\mathrm{coh}(M)} \varphi(\overline{M/C}),$$

where C runs over the directed set $(\mathrm{coh}(M), \subseteq)$ of finitely generated $\mathbb{Z}$-submodules of M, and satisfies:

$$\mathrm{ev}\varphi(\overline{M}) \leq \varphi(\overline{M}) \quad \text{and} \quad \mathrm{ev}\varphi(\overline{M}) < +\infty \Longleftrightarrow \varphi(\overline{M}) < +\infty.$$

Moreover, we have:

$$\lambda^{[0]}(\overline{M}) \leq \gamma(\overline{M}) \leq 2\,\rho(\overline{M}) \leq 2\,\overline{\rho}(\overline{M})$$

— only the second inequality is not trivial, and has been established in Proposition 6.4.10 — and consequently:

$$\mathrm{ev}\lambda^{[0]}(\overline{M}) \leq \mathrm{ev}\gamma(\overline{M}) \leq 2\,\mathrm{ev}\rho(\overline{M}) \leq 2\,\mathrm{ev}\overline{\rho}(\overline{M}).$$

In particular, we have:

$$\overline{\rho}(\overline{M}) < +\infty \Longrightarrow \rho(\overline{M}) < +\infty \Longrightarrow \gamma(\overline{M}) < +\infty \Longrightarrow \lambda^{[0]}(\overline{M}) < +\infty,$$

and:

$$\mathrm{ev}\overline{\rho}(\overline{M}) = 0 \Longrightarrow \mathrm{ev}\rho(\overline{M}) = 0 \Longrightarrow \mathrm{ev}\gamma(\overline{M}) = 0 \Longrightarrow \mathrm{ev}\lambda^{[0]}(\overline{M}) = 0.$$

Moreover, as observed in Proposition 6.3.11, the general formalism of Section 4.5 establishes the implication:

$$\overline{M}\ \rho^2\text{-summable} \Longrightarrow \mathrm{ev}\overline{\rho}(\overline{M}) = 0.$$

Similarly, as stated in Theorem 8.2.1, the following implication holds:

$$\overline{M}\ \theta^1\text{-summable} \Longrightarrow \mathrm{ev}h^1_\theta(\overline{M}) = 0$$

In Theorem 9.1.1, we have established the following estimate:

$$\overline{h}^1_\theta(\overline{M}) \leq \pi\,\overline{\rho}(\overline{M})^2,$$

which in turn implies:

$$\mathrm{ev}\overline{h}^1_\theta(\overline{M}) \leq \pi\,\mathrm{ev}\overline{\rho}(\overline{M})^2.$$

In particular, the following implications hold:

$$\overline{\rho}(\overline{M}) < +\infty \Longrightarrow \overline{h}^1_\theta(\overline{M}) < +\infty \quad \text{and} \quad \mathrm{ev}\overline{\rho}(\overline{M}) = 0 \Longrightarrow \mathrm{ev}\overline{h}^1_\theta(\overline{M}) = 0.$$

Moreover, in Proposition 9.4.6, we have established the implication:

$$\overline{M}\ \rho^2\text{-summable} \Longrightarrow \overline{M}\ \theta^1\text{-finite}.$$

9.7.2. As before, let $\overline{M} := (M, \|.\|)$ be a Euclidean quasi-coherent sheaf. Moreover let $\|.\|_c$ (resp. $\|.\|_{SE}$, resp. $\|.\|_{HS}$) be a Euclidean seminorm on $M_{\mathbb{R}}$ that is compact (resp. that satisfies the condition **SE**, resp. that is Hilbert-Schmidt) with respect to $\|.\|$.

For each of the invariants $\varphi = \lambda^{[0]}$, γ, or ρ, the following implication holds:

$$\varphi(\overline{M}) < +\infty \Longrightarrow \mathrm{ev}\varphi(M, \|.\|_c) = 0,$$

and moreover:

$$\overline{M}\ \theta^1\text{-summable} \Longrightarrow (M, \|.\|_c)\ \theta^1\text{-finite.}$$

Finally, according to Propositions 9.6.1, 6.5.3 and 9.4.6, the following implications hold:

$$\lambda^{[0]}(\overline{M}) < +\infty \Longrightarrow\ (M, \|.\|_{SE})\ \theta^1\text{-finite,}$$

$$\lambda^{[0]}(\overline{M}) < +\infty \Longrightarrow \mathrm{ev}\rho(M, \|.\|_{HS}) = 0,$$

and:

$$\overline{M}\ \theta^1\text{-summable} \Longrightarrow (M, \|.\|_{HS})\ \rho^2\text{-summable.}$$

9.7.3. From the implications recalled in the previous subsections, we immediately derive the following proposition, concerning families of Euclidean quasi-coherent sheaves $(M, \|.\|_\alpha)_{\alpha\in A}$ defined by fixed countably generated $\mathbb{Z}$-module M, equipped with Euclidean semi-norms $(\|.\|_\alpha)_{\alpha\in A}$ that satisfy a suitable nuclearity condition.

PROPOSITION 9.7.1. *Let M be a countably generated $\mathbb{Z}$-module and let $(\|.\|_\alpha)_{\alpha\in A}$ a family of Euclidean seminorms over $M_{\mathbb{R}}$.*

If for every $\alpha \in A$, there exists $\beta \in A$ such that $\|.\|_\beta$ is Hilbert-Schmidt with respect to $\|.\|_\alpha$, then the validity of the condition:

$$\text{for some } \alpha \text{ in } A, \text{ the Euclidean coherent sheaf } \overline{M}_\alpha := (M, \|.\|_\alpha) \text{ satisfies } \mathbf{P} \tag{9.7.1}$$

does not depend on the choice of $\mathbf{P}$ *among the following properties, where φ denotes any of the invariants $\lambda^{[0]}$, γ, ρ, $\overline{\rho}$, or $\overline{h}^1_\theta$:*

(1) $\varphi(\overline{M}_\alpha) < +\infty$;
(2) $\mathrm{ev}\varphi(\overline{M}_\alpha) = 0$;
(3) $\overline{M}_\alpha$ is ρ^2-finite;
(4) $\overline{M}_\alpha$ is θ^1-finite;
(5) $\overline{M}_\alpha$ is θ^1-summable.

9.7.4. Families $(\overline{M}_\alpha)_{\alpha\in A}$ of Euclidean quasi-coherent sheaves as in Proposition 9.7.1 will play a key role in Diophantine applications of the formalism developed in this monograph. Actually such families naturally arise in Diophantine geometry as demonstrated by Example 9.7.2 below.

This example will rely on some basic facts concerning L^2-norms of analytic sections of complex analytic vector bundles on complex analytic manifolds which we briefly recall.

If $\overline{F} := (F, \|.\|)$ is a Hermitian vector bundle[23] on some complex analytic manifold V equipped with some positive volume form μ, we shall denote by $\|.\|_{L^2(V,\mu,\overline{F})}$ the L^2-quasinorm on the space $\mathcal{O}^{\mathrm{an}}(V, F)$ of complex analytic sections of F over V defined by the equality:

$$\|s\|^2_{L^2(V,\mu,\overline{F})} := \int_V \|s(x)\|^2\, d\mu(x).$$

In particular, for every relatively compact open subset U of V, we may consider the L^2-Hermitian seminorm on $\mathcal{O}^{\mathrm{an}}(V, F)$:

$$\|.\|_U := \|.\|_{L^2(U,\mu_{|U},\overline{F}_{|U})}.$$

[23]In other words, F is a complex analytic vector bundle over V, and $\|.\|$ a continuous Hermitian metric on F.

It is defined by the following equality, for every $s \in \mathcal{O}^{\mathrm{an}}(V, F)$:

$$\|s\|_U^2 := \int_U \|s(x)\|^2 \, d\mu(x).$$

If moreover U' is a relatively compact open subset of U, the seminorm $\|.\|_{U'}$ is well-known to be Hilbert-Schmidt with respect to $\|.\|_U$. Actually, for every $p \in \mathbb{R}_+^*$, these seminorms satisfy:

$$\operatorname{Tr}(\|.\|_{U'}^p / \|.\|_U^p) < +\infty. \tag{9.7.2}$$

EXAMPLE 9.7.2. Let X be a separated scheme of finite type over $\operatorname{Spec} \mathbb{Z}$ such that $X_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$. Then $X(\mathbb{C})$ is a complex manifold, and we may choose a positive volume form μ over $X(\mathbb{C})$, invariant under complex conjugation.

Moreover let $\overline{E} := (E, \|.\|)$ be a Hermitian vector bundle over X, defined by a vector bundle E over X and a continuous metric $\|.\|$, invariant under complex conjugation, on the complex analytic vector bundle $E_{\mathbb{C}}^{\mathrm{an}}$ on $X(\mathbb{C})$.

According to Proposition 1.1.6 (2), the $\mathbb{Z}$-module of sections of E over X:

$$M := \Gamma(X, E),$$

is countably generated. Moreover the complex (resp. real) vector space:

$$M_{\mathbb{C}} \simeq \Gamma(X_{\mathbb{C}}, E_{\mathbb{C}}) \quad (\text{resp. } M_{\mathbb{R}} \simeq \Gamma(X_{\mathbb{R}}, E_{\mathbb{R}}))$$

may be identified to a subspace of $\mathcal{O}^{\mathrm{an}}(X(\mathbb{C}), E_{\mathbb{C}}^{\mathrm{an}})$ (resp. of the subspace of $\mathcal{O}^{\mathrm{an}}(X(\mathbb{C}), E_{\mathbb{C}}^{\mathrm{an}})$ defined by analytic sections of $E_{\mathbb{C}}^{\mathrm{an}}$ over $X(\mathbb{C})$ that are invariant under complex conjugation).

Consequently, if U is a relatively compact subset of $X(\mathbb{C})$, invariant under complex conjugation, the Hermitian seminorm:

$$\|.\|_U := \|.\|_{L^2(X(\mathbb{C}), \mu_{|U}, \overline{E}_{|U}^{\mathrm{an}})}$$

defines a Hermitian seminorm on $M_{\mathbb{C}}$ which is invariant under complex conjugation, or equivalently a Euclidean seminorm on $M_{\mathbb{R}}$.

Therefore the pair $\overline{M}_U := (M, \|.\|_U)$ defines a Euclidean quasi-coherent sheaf.

Finally, let K be a compact subset of the complex manifold $X(\mathbb{C})$ that is invariant under complex conjugation, and let $\mathcal{N}(K)$ be the set of open neighborhoods of K in $X(\mathbb{C})$ which are invariant under complex conjugation and relatively compact in $X(\mathbb{C})$.

The nuclearity properties of the seminorms $\|.\|_U$ recalled in (9.7.2) above show that the family of Euclidean quasi-coherent sheaves $(\overline{M}_U)_{U \in \mathcal{N}(K)}$ satisfy the assumption of Proposition 9.7.1. Consequently the existence of some $U \in \mathcal{N}(K)$ such that one of the properties (1)–(5) in Proposition 9.7.1 is satisfied (with $\alpha = U$) does not depend of this property.

The existence of such a U is easily seen not to depend of the choice of the volume form μ and of the Hermitian metric $\|.\|$ on $E_{\mathbb{C}}^{\mathrm{an}}$ defining $\overline{E}$, and accordingly to be a property of the triple (X, E, K).

It is actually possible to extend the above construction of the Euclidean quasi-coherent sheaves $\overline{M}_U$ to the situation where $X_{\mathbb{Q}}$ is not assumed to be smooth anymore, and where E is an arbitrary coherent sheaf over X.

Part 4

Appendices

APPENDIX A

The Singular Values Attached to a Pair of Euclidean Seminorms

A.1. Pairs of Euclidean Seminorms: Continuity and Compactness

Let V be a real vector space, and let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on V, defined by some Euclidean scalar products $\langle .,. \rangle$ and $\langle .,. \rangle'$.

We may consider the associated "separated completions" $V^{\mathrm{cpt}}_{\|.\|}$ and $V^{\mathrm{comp}}_{\|.\|'}$ and the canonical morphisms

$$j_{V,\|.\|} : V \longrightarrow V^{\mathrm{comp}}_{\|.\|} \quad \text{and} \quad j_{V,\|.\|'} : V \longrightarrow V^{\mathrm{cpt}}_{\|.\|'}.$$

We shall still denote by $\|.\|$ and $\|.\|'$ the Hilbertian norms on $V^{\mathrm{comp}}_{\|.\|}$ and $V^{\mathrm{comp}}_{\|.\|'}$ deduced from the Euclidean seminorms $\|.\|$ and $\|.\|'$. By construction, for every $v \in V$, we have:

$$\|j_{V,\|.\|}(v)\| = \|v\| \quad \text{and} \quad \|j_{V,\|.\|'}(v)\| = \|v\|.'$$

A.1.1. Continuity. We shall say that $\|.\|'$ is *continuous*, or *bounded*, relatively to $\|.\|$ when the seminorm $\|.\|'$ defines a continuous function on the seminormed vector space $(V, \|.\|')$. This holds precisely when the identity map Id_V is continuous from $(V, \|.\|)$ to $(V, \|.\|')$, or equivalently, when the unit ball $B(V, \|.\|; 1)$ in $(V, \|.\|)$ is bounded in $(V, \|.\|')$, or when

$$\sup(\|.\|'/\|.\|) := \sup \{C \in \mathbb{R}_+ \mid \forall v \in V, \|v\|' \leq C\|v\|\}$$

is finite.

When these conditions are satisfied, the identity map Id_V of V defines a continuous linear map

$$I^{\|.\|}_{\|.\|'} : V^{\mathrm{cpt}}_{\|.\|} \longrightarrow V^{\mathrm{cpt}}_{\|.\|'},$$

characterized by the commutativity of the diagram:

$$\begin{tikzcd}
V \arrow[r, "\mathrm{Id}_V"] \arrow[d, "j_{V,\|.\|}"'] & V \arrow[d, "j_{V,\|.\|}"] \\
V^{\mathrm{cpt}}_{\|.\|} \arrow[r, "I^{\|.\|}_{\|.\|'}"] & V^{\mathrm{cpt}}_{\|.\|'},
\end{tikzcd}$$

between the pre-Hilbert spaces $(V^{\mathrm{cpt}}_{\|.\|}, \|.\|)$ and $(V^{\mathrm{cpt}}_{\|.\|'}, \|.\|')$, and the operator norm of $I^{\|.\|}_{\|.\|'}$ coincides with $\sup(\|.\|'/\|.\|)$.

Moreover, there exists a unique selfadjoint continuous linear map

$$A : V^{\mathrm{comp}}_{\|.\|} \longrightarrow V^{\mathrm{comp}}_{\|.\|}$$

such that,

(A.1.1) $$\text{for every } v \in V^{\mathrm{comp}}_{\|.\|}, \quad \|v\|'^2 = \langle v, Av \rangle.$$

This operator coincide with the composition of $I^{\|.\|}_{\|.\|'}$ and its adjoint:

$$A = (I^{\|.\|}_{\|.\|'})^* . I^{\|.\|}_{\|.\|'}.$$

The operator A is positive, and its square root $B := A^{1/2}$ is the unique positive continuous linear map

$$B : V^{\mathrm{comp}}_{\|.\|} \longrightarrow V^{\mathrm{comp}}_{\|.\|}$$

such that,

$$\text{for every } v \in V^{\mathrm{comp}}_{\|.\|}, \quad \|v\|'^2 = \langle B(v), B(v)\rangle. \tag{A.1.2}$$

A.1.2. Compactness. We shall say that $\|.\|'$ *is compact with respect to* $\|.\|$ when the unit ball $B(V, \|.\|; 1)$ in $(V, \|.\|)$ is precompact in $(V, \|.\|')$. This implies the continuity of $\|.\|'$ with respect to $\|.\|$.

Observe that for any sequence $(v_n)_{n\in\mathbb{N}}$ in V, the following two conditions are equivalent:

(i) the sequence $(j_{V,\|.\|}(v_n))_{n\in\mathbb{N}}$ converges weakly to zero in the Hilbert space $(V^{\mathrm{cpt}}_{\|.\|}, \|.\|)$;
(ii) the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded and converges weakly to zero in $(V, \|.\|)$; in other words,

$$\sup_{n\in\mathbb{N}} \|v_n\| < +\infty \quad \text{and} \quad \lim_{n\to+\infty} \langle v_n, w\rangle = 0 \text{ for every } w \in V. \tag{A.1.3}$$

Proposition A.1.1. *If the Euclidean seminorm $\|.\|'$ is continuous relatively to $\|.\|$, then the following conditions are equivalent:*

(i) the Euclidean seminorm $\|.\|'$ on V is compact relatively to $\|.\|$;
(ii) the Euclidean seminorm $\|.\|' \circ I^{\|.\|}_{\|.\|'}$ is compact relatively to the Hilbert norm $\|.\|$ on $V^{\mathrm{cpt}}_{\|.\|}$;
(iii) the continuous linear map $I^{\|.\|}_{\|.\|'}$ from $V^{\mathrm{cpt}}_{\|.\|}$ to $V^{\mathrm{cpt}}_{\|.\|'}$ is compact;
(iv) the continuous endomorphism A of $V^{\mathrm{cpt}}_{\|.\|}$ is compact;
(v) the continuous endomorphism B of $V^{\mathrm{cpt}}_{\|.\|}$ is compact;
(vi) for every sequence $(x_n)_{n\in\mathbb{N}}$ converging weakly to zero in the Hilbert space $V^{\mathrm{cpt}}_{\|.\|}$, we have:

$$\lim_{n\to+\infty} \|I^{\|.\|}_{\|.\|'}(x_n)\|' = 0;$$

(vii) for every sequence $(v_n)_{n\in\mathbb{N}}$ that is bounded and converges weakly to zero in $(V, \|.\|)$, we have:

$$\lim_{n\to+\infty} \|v_n\|' = 0.$$

The equivalences between these conditions are either well known, or easy to establish. We leave the details to the reader.

If C is a $\mathbb{R}$-vector subspace of V, we may consider the seminorms $\|.\|_{V/C}$ and $\|.\|'_{V/C}$ on the quotient V/C defined as the quotient seminorms of $\|.\|$ and $\|.\|'$. Clearly, if $\|.\|'$ is bounded with respect to $\|.\|$, then $\|.\|'_{V/C}$ is bounded with respect to $\|.\|_{V/C}$, and we have:

$$\sup(\|.\|'_{V/C}/\|.\|_{V/C}) \leq \sup(\|.\|'/\|.\|).$$

More generally, $\sup(\|.\|'_{V/C}/\|.\|_{V/C})$ is a decreasing function of C.

In particular, $\sup(\|.\|'_{V/C}/\|.\|_{V/C})$ admits a well-defined limit when C varies over the directed set $(\mathrm{Fd}(V), \subseteq)$ of finite dimensional vector subspaces of V:

$$\lim_{C\in\mathrm{Fd}(V)} \sup(\|.\|'_{V/C}/\|.\|_{V/C}) = \inf_{C\in\mathrm{Fd}(V)} \sup(\|.\|'_{V/C}/\|.\|_{V/C}) \in [0, \sup(\|.\|'/\|.\|)]. \tag{A.1.4}$$

Proposition A.1.2. *The seminorm $\|.\|'$ is compact with respect to $\|.\|$ if and only if the limit* (A.1.4) *vanishes.*

If $(C_i)_{i\in\mathbb{N}}$ is an increasing sequence of finite dimensional vector subspaces of V such that $\bigcup_{i\in\mathbb{N}} C_i$ is dense in $(V, \|.\|')$, then $\|.\|'$ is compact with respect to $\|.\|$ if and only if:

$$\lim_{i\to+\infty} \sup(\|.\|'_{V/C_i}/\|.\|_{V/C_i}) = 0.$$

A.2. The Singular Values Attached to a Pair of Euclidean Seminorms

To any pair $(\|.\|, \|.\|')$ of Euclidean seminorms such that $\|.\|'$ is compact with respect to $\|.\|$, we may associate its sequence of singular values:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0. \tag{A.2.1}$$

It satisfies:

$$\lim_{i \to +\infty} \lambda_i = 0,$$

and is characterized by the following properties:

(i) It is the sequence of singular values of the compact operator $I_{\|.\|'}^{\|.\|}$, defined as the eigenvalues of B, in decreasing order, each one repeated according to its multiplicity.[1]

(ii) Assume that $V_{\|.\|}^{\mathrm{cpt}}$ is a separable infinite dimensional Hilbert space (resp. has finite dimension n). There exists a Hilbert basis $(e_i)_{i \in \mathbb{N}_{\geq 1}}$ (resp. $(e_i)_{1 \leq i \leq n}$) of $V_{\|.\|}^{\mathrm{cpt}}$ such that, for any $v \in V_{\|.\|}^{\mathrm{cpt}}$,

$$\|v\|'^2 = \sum_{i=1}^{\infty} \lambda_i^2 \langle v, e_i \rangle^2 \quad (\text{resp.} \quad \|v\|'^2 = \sum_{i=1}^{n} \lambda_i^2 \langle v, e_i \rangle^2.) \tag{A.2.2}$$

(iii) For any $k \in \mathbb{N}$, we may endowed the k-the exterior power $\Lambda^k V$ of the Euclidean seminorm $\|.\|_{\wedge^k}$ deduced from $\|.\|$: by definition, the Euclidean scalar product $\langle ., . \rangle$ associated to $\|.\|_{\wedge^k}$ satisfies

$$\langle v_1 \wedge \cdots \wedge v_k, w_i \wedge \cdots \wedge w_k \rangle_{\wedge^k} = \det(\langle v_i, w_j \rangle)_{1 \leq i,j \leq k}. \tag{A.2.3}$$

Then we have:

$$\prod_{i=1}^{k} \lambda_i = \sup(\|.\|'_{\wedge^k} / \|.\|_{\wedge^k}).$$

It is also the operator norm of the continuous linear map defined by taking the k-th exterior power $\wedge^k (I_{\|.\|'}^{\|.\|})$ of $I_{\|.\|'}^{\|.\|}$, which defines a continuous linear map from $\wedge^k V_{\|.\|}^{\mathrm{cpt}}$ to $\wedge^k V_{\|.\|'}^{\mathrm{cpt}}$.

(iv) We may consider the projective space

$$\mathbb{P}(V_{\|.\|}^{\mathrm{cpt}}) := (V_{\|.\|}^{\mathrm{cpt}} \setminus \{0\}) / \mathbb{C}^*,$$

which parametrizes the one-dimensional complex vector subspaces of $V_{\|.\|}^{\mathrm{cpt}}$, and the function

$$q : \mathbb{P}(V_{\|.\|}^{\mathrm{cpt}}) \longrightarrow \mathbb{R}_+, \quad [v] \longmapsto \|v\|'^2 / \|v\|^2.$$

It is a C^∞ function on the Hilbert manifold $\mathbb{P}(V_{\|.\|}^{\mathrm{cpt}})$, and the positive singular values of the pair $(\|.\|, \|.\|')$ are precisely the positive singular values of the function q, namely the elements of $\mathbb{R}_+^* \cap q(Dq^{-1}(0))$. Moreover a positive singular value λ of q occurs in the sequence (A.2.1) with a multiplicity

$$n = 1 + \dim_{\mathbb{C}} q^{-1}(\lambda).$$

Actually $q^{-1}(\lambda)$ is a complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$, $\mathbb{C}$-analytically embedded in the Hilbert manifold $\mathbb{P}(V_{\|.\|}^{\mathrm{cpt}})$.

For any function

$$\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+,$$

we shall define:

$$\operatorname{Tr} \varphi(\|.\|' / \|.\|) := \sum_{i=1}^{+\infty} \varphi(\lambda_i) \quad (\in [0, +\infty]).$$

[1] When $n := \dim_{\mathbb{R}} V$ is finite, this defines $\lambda_1, \ldots, \lambda_n$. Then $\lambda_i := 0$ for every $i > n$.

When φ is a Borel function, we may define the (possibly unbounded) self-adjoint operator $\varphi(B)$ by means of the Borel functional calculus and the following equality holds:

$$\operatorname{Tr}\varphi(\|.\|'/\|.\|) = \operatorname{Tr}\varphi(B).$$

We will use only the special instance of this definition where φ is a continuous function such that $\varphi(0) = 0$, in which case $\varphi(B)$ is a compact self-adjoint (symmetric) operator.

We shall also use the related notation:

$$\operatorname{Tr}\varphi(\|.\|'^2/\|.\|^2) := \operatorname{Tr}\varphi(B^2) = \operatorname{Tr}\varphi(A) = \sum_{i=1}^{+\infty}\varphi(\lambda_i^2).$$

In particular,

$$\operatorname{Tr}\|.\|'^2/\|.\|^2 := \operatorname{Tr} B^2 = \operatorname{Tr} A = \sum_{i=1}^{+\infty}\lambda_i^2.$$

A.3. The Peierls-Bogoliubov Inequality.

A.3.1. Traces of convex functions of positive compact operators. The following proposition shows how a convex function of the diagonal matrix elements of a positive compact operator may be bounded in terms of the trace of the convex function applied to the operator:

PROPOSITION A.3.1. *Let $(H, \|.\|)$ be a real Hilbert space, and let $A : H \to H$ be a positive compact operator.*

For every convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and for every orthonormal family $(e_i)_{i\in I}$ in $(H, \|.\|)$, the following inequality holds in $[0, +\infty]$:

$$\sum_{i\in I}\varphi(\langle e_i, Ae_i\rangle) \leq \operatorname{Tr}\varphi(A). \tag{A.3.1}$$

The inequality (A.3.1) appears in the literature of mathematical physics under the name of *Peierls-Bogoliubov inequality* (see for instance [**Sim79**, 8, (c)]). Indeed this inequality, or some variant, plays a key role in the work of Peierls and Bogoliubov on quantum statistical mechanics, who used it to derive estimates and variational principles for quantum partition functions (see [**Pei38**], and [**Hua87**, section 10.4]).

The inequality (A.3.1) also appears in the mathematical literature on a par with classical estimates on singular values due to Weyl, Horn, and Fan; see notably [**GK69**, Section II.5, Theorem 5.2], where a more general version of Proposition A.3.1 is established.

For the sake of completeness, we recall a simple proof of the estimate (A.3.1) (compare [**Sim79**, 8, (c)]).

PROOF. Observe that, for every vector $e \in H$ such that $\|e\| = 1$, we have:

$$\varphi(\langle e, Ae\rangle) \leq \langle e, \varphi(A)e\rangle. \tag{A.3.2}$$

Indeed this is Jensen's inequality applied to the convex function φ and to the probability measure μ on $\mathbb{R}_+$ defined as the spectral mesure[2] attached to the operator A and to the vector e.

By applying (A.3.2) to the vectors e_i and taking the sum over $i \in I$, we obtain:

$$\sum_{i\in I}\varphi(\langle e_i, Ae_i\rangle) \leq \sum_{i\in I}\langle e_i, \varphi(A)e_i\rangle \leq \operatorname{Tr}\varphi(A). \qquad \square$$

[2]If $(f_\alpha)_{\alpha\in\mathcal{A}}$ is an Hilbert basis of $(H, \|.\|)$ consisting of eigenvectors of A, and if $\lambda_\alpha \in \mathbb{R}_+$ is the eigenvalue of C attached to C, defined by $Cf_\alpha = \lambda_\alpha f_\alpha$, then $\mu := \sum_{\alpha\in\mathcal{A}}\langle e, f_\alpha\rangle^2\,\delta_{\lambda_\alpha}$.

COROLLARY A.3.2. *Let V be a real vector space, and let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on V, with $\|.\|'$ compact relatively to $\|.\|$.*

For every convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and every family $(e_i)_{i\in I}$ in V orthonormal for the scalar product $\langle .,. \rangle$ defining $\|.\|$, the following inequality holds in $[0, +\infty]$:

$$\sum_{i\in I} \varphi(\|e_i\|'^2) \leq \operatorname{Tr} \varphi(\|.\|'^2/\|.\|^2). \tag{A.3.3}$$

PROOF. With the notation of Proposition A.1.1, this is Proposition A.3.1 applied to the Hilbert space $H := V^{\mathrm{cpt}}_{\|.\|}$, the positive compact operator A defined by (A.1.1), and the orthonormal family $(j_{V,\|.\|}(e_i))_{i\in I}$ in H. □

A.3.2. Application: pair of compact Euclidean seminorms and seminorms on subquotients. In the next sections, we shall rely on the estimate (A.3.1) to establish finiteness properties of the θ-invariants of certain Euclidean quasi-coherent sheaves. To achieve this, we will use it in the following setting.

Let V be a real vector space and let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on V, with $\|.\|'$ compact relatively to $\|.\|$.

Let N be an element of $\mathbb{N} \cup \{+\infty\}$, and let $(v_i)_{0\leq i<N}$ be a sequence of elements in V. We let $V_{-1} := 0$, and, for every $i \in \mathbb{N} \cap [0, N)$,

$$V_i := \sum_{k=0}^{i} \mathbb{R}.v_i,$$

and we denote by $\bar{v}_i$ the class of v_i in the $\mathbb{R}$-vector space V_i/V_{i-1}, and by $\|.\|'_i$ the Euclidean seminorm on V_i/V_{i-1} defined as the quotient of the Euclidean seminorm $\|.\|'_{|V_i}$ on V_i. Clearly, the quotient V_i/V_{i-1} is either 0, or the line $\mathbb{R}.\bar{v}_i$.

COROLLARY A.3.3. *Let us keep the previous notation, and let us assume that the following condition is satisfied:*

$$\text{for every } i \in \mathbb{N} \cap [0, N), \quad \|v_i\| \leq 1. \tag{A.3.4}$$

Then, if $N = +\infty$, we have:

$$\lim_{i\to+\infty} \|\bar{v}_i\|'_i = 0. \tag{A.3.5}$$

Moreover, for any convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$, the following inequality holds:

$$\sum_{0\leq i<N} \varphi(\|\bar{v}_i\|'^2_i) \leq \operatorname{Tr} \varphi(\|.\|'^2/\|.\|^2). \tag{A.3.6}$$

When $\varphi = \mathrm{Id}_{\mathbb{R}_+}$, the inequality (A.3.6) reads:

$$\sum_{0\leq i<N} \|\bar{v}_i\|'^2_i \leq \operatorname{Tr} (\|.\|'^2/\|.\|^2). \tag{A.3.7}$$

In substance, the inequality (A.3.7) goes back to the work of Schur [**Sch09**].

PROOF. We divide the proof in successive steps, where the validity of Corollary A.3.3 is established under assumptions of increasing generality on the sequence $(v_i)_{0\leq i<N}$ and on the Euclidean seminorm $\|.\|$.

(i) Let us first assume that the v_i, $0 \leq i < N$, are linearly independent, and that the Euclidean seminorm $\|.\|$ is actually a norm.

Let us denote $\langle ., . \rangle$ the Euclidean scalar product on V that defines $\|.\|$, and let us consider the sequence $(e_i)_{0 \leq i < N}$ deduced from $(v_i)_{1 \leq i < N}$ by Gram-Schmidt orthonormalization in the pre-Hilbert space $(V, \|.\|)$. For every $i \in [0, N) \cap \mathbb{N}$, we may write:

$$e_i = t_i v_i + r_i,$$

with t_i in $\mathbb{R}_+^*$, and v_i in V_{i-1}. Moreover, $\langle e_i, r_I \rangle = 0$, and therefore:

$$t_i^2 \|v_i\|^2 = \|e_i\|^2 + \|r_i\|^2 \geq \|e_i\|^2 = 1.$$

Since $\|v_i\| \leq 1$, this shows that t_i belongs to $[1, +\infty)$.

The class $\bar{e}_i$ of e_i in V_i/V_{i-1} satisfies:

$$\bar{e}_i = t_i \bar{v}_i,$$

and therefore:

$$\|\bar{v}_i\|'_i = t_i^{-1} \|\bar{e}_i\|'_i \leq \|\bar{e}_i\|'_i \leq \|e_i\|'. \tag{A.3.8}$$

When $N = +\infty$, the sequence $(e_i)_{i \in \mathbb{N}}$ is bounded and converges weakly to zero in the pre-Hilbert space $(V, \|.\|)$. Since $\|.\|'$ is compact relatively to $\|.\|$, this implies:

$$\lim_{i \to +\infty} \|e_i\|' = 0.$$

Together with (A.3.8), this implies (A.3.5).

Finally, as φ is increasing, the estimates (A.3.8) also yield:

$$\sum_{0 \leq i < N} \varphi(\|\bar{v}_i\|_i'^2) \leq \sum_{0 \leq i < N} \varphi(\|e_i\|'^2).$$

According to (A.3.3), the last sum is bounded from above by $\operatorname{Tr} \varphi(\|.\|'/\|.\|)$.

(ii) Let us assume that the Euclidean seminorm $\|.\|$ is a norm.

Let us introduce

$$\tilde{N} := \dim_{\mathbb{R}} \sum_{0 \leq i < N} \mathbb{R}.v_i \quad (\in \mathbb{N} \cup \{+\infty\}),$$

and for every $k \in [0, \tilde{N}) \cap \mathbb{N}$, let:

$$i(k) := \min\{i \in [0, N) \cap \mathbb{N} \mid \dim_{\mathbb{R}} V_i = k + 1\}.$$

For any $i \in [0, N) \cap \mathbb{N}$, the quotient V_i/V_{i-1} is one-dimensional, or equivalently $\bar{v}_i \neq 0$, if and only if i occurs in the sequence $(i(k))_{0 \leq k < \tilde{N}}$. The vectors $(v_{i(k)})_{0 \leq k < \tilde{N}}$ are $\mathbb{R}$-linearly independent, and Step (i), applied with $(v_i)_{0 \leq i < N}$ replaced by $(v_{i(k)})_{0 \leq k < \tilde{N}}$, establishes the validity of Corollary A.3.5.

(iii) In general, we may consider the quotient map

$$q : V \longrightarrow V/\ker \|.\|.$$

The Euclidean seminorm $\|.\|$ (resp. $\|.\|'$) on V descends to a Euclidean norm $\|.\|$ on $V/\ker \|.\|$ (resp. to a Euclidean seminorm $\|.\|'$ on $V/\ker \|.\|$ that is compact relatively to $\|.\|$). Step (ii) applied to $V/\ker \|.\|$ equipped with these Euclidean (semi)-norms and to the sequence $(q(v_i))_{0 \leq i < N}$ completes the proof of Corollary A.3.5 □

Corollary A.3.4. *The following inequality holds:*

$$\sum_{0 \leq i < N} \|\bar{v}_i\|_i'^2 \leq \operatorname{Tr}(\|.\|'^2/\|.\|^2) \sup_{0 \leq i < N} \|v_i\|^2. \tag{A.3.9}$$

Proof. When $\sup_{0 \leq i < N} \|v_i\| = 1$, this is Corollary A.3.3 applied with $\varphi = \mathrm{Id}_{\mathbb{R}_+}$. The general case follows by homogeneity. □

Observe that the proof of (A.3.9) does not require the full strength of the Peierls-Bogoliubov inequality (A.3.6), and relies only on Schur's inequality (A.3.7).

Corollary A.3.5. *Let $\tilde{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}_{|[0,\varepsilon_0)}$ is convex for some $\varepsilon_0 \in \mathbb{R}_+^*$. If*

$$\operatorname{Tr} \tilde{\varphi}(\|.\|'^2/\|.\|^2) < +\infty,$$

then:

$$\sum_{0 \leq i < N} \tilde{\varphi}(\|\bar{v}_i\|_i'^2) < +\infty. \tag{A.3.10}$$

Proof. We may find a convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ that coincides with $\tilde{\varphi}$ on some neighborhood of 0 in $\mathbb{R}_+$. Then the finiteness of $\operatorname{Tr} \tilde{\varphi}(\|.\|'^2/\|.\|^2)$ and $\operatorname{Tr} \varphi(\|.\|'^2/\|.\|^2)$ are equivalent, and (A.3.10) directly follows from (A.3.6). $\square$

A.4. Intermediate Euclidean Seminorms

The following properties of converging sequences and series in $\mathbb{R}_+$ are elementary and well-known:

Proposition A.4.1. *Let $(\lambda_i)_{i \geq 1}$ be a sequence in $\mathbb{R}_+$ such that*

$$\lim_{i \to +\infty} \lambda_i = 0.$$

(1) There exist sequences $(\mu_i)_{i \geq 1}$ and $(\nu_i)_{i \geq 1}$ in $\mathbb{R}_+$ such that

$$\lambda_i = \mu_i \nu_i$$

for every $i \geq 1$, and

$$\lim_{i \to +\infty} \mu_i = \lim_{i \to +\infty} \nu_i = 0.$$

(2) If moreover

$$\sum_{i \geq 1} \lambda_i < +\infty,$$

then we may find $(\mu_i)_{i \geq 1}$ and $(\nu_i)_{i \geq 1}$ as in (1) that also satisfy:

$$\sum_{i \geq 1} \mu_i < +\infty.$$

Indeed (1) is satisfied if we let $\mu_i = \nu_i := \lambda_i^{1/2}$ for every $i \geq 1$. Moreover, when $\sum_{i \geq 1} \lambda_i < +\infty$,, we may choose a strictly increasing sequence of positive integer $(i_k)_{k \in \mathbb{N}}$ such that

$$\sum_{i \geq i_k} \lambda_i \leq 2^{-k}.$$

Then it is straightforward that (2) holds if we let, for every integer i such that $i_k \leq i < i_{k+1}$:

$$\mu_i := 2^{k/2} \lambda_i \quad \text{and} \quad \nu_i := 2^{-k/2}.$$

The following proposition may be seen as an avatar, concerning relatively compact pairs of Euclidean seminorms, of the above properties of sequences in $\mathbb{R}_+$.

Proposition A.4.2. *Let V be a real vector space, and let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on V, with $\|.\|'$ compact relatively to $\|.\|$.*

(1) There exists a Euclidean seminorm $\|.\|^\sim$ on V such that $\|.\|^\sim$ is compact relatively to $\|.\|$ and $\|.\|$ is compact relatively to $\|.\|^\sim$.

(2) Let $(\varphi_\alpha)_{\alpha \in A}$ be a countable family of continuous increasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, vanishing at 0.

If, for very $\alpha \in A$,

$$\mathrm{Tr}\,\varphi_\alpha(\|.\|'/\|.\|) < +\infty, \tag{A.4.1}$$

then there exists $\|.\|_1^\sim$ and $\|.\|_2^\sim$ as in (1) such that, moreover, for every $\alpha \in A$,

$$\mathrm{Tr}\,\varphi_\alpha(\|.\|_1^\sim/\|.\|) < +\infty \tag{A.4.2}$$

and

$$\mathrm{Tr}\,\varphi_\alpha(\|.\|'/\|.\|_2^\sim) < +\infty. \tag{A.4.3}$$

Proof. By means of the canonical isometry from V to $V^{\mathrm{cpt}}_{\|.\|}$, one easily reduces to the situation where $(V, \|.\|)$ is a Hilbert space. Then we may introduce the compact positive operator $B : V \to V$ such that, for any $v \in V$,

$$\|v\|' = \|Bv\|.$$

With this notation, for any Borel function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, positive on $\mathbb{R}_+^*$, such that

$$\lim_{x \to 0_+} \psi(x) = \psi(0) = 0 \quad \text{and} \quad \lim_{x \to 0_+} \psi(x)/x = +\infty,$$

the Euclidean seminorm $\|.\|^\sim$ on V defined by:

$$\|v\|^\sim := \|\psi(B)v\| \tag{A.4.4}$$

satisfies the conditions in (1).

To prove (2), let us introduce the sequence $(\lambda_i)_{i \geq 1}$ of the singular values of $\|.\|'$ with respect to $\|.\|$, and let us first assume that A contains a single element $*$. Then the condition (A.4.1) is equivalent to:

$$\sum_{i \geq 1} \varphi_*(\lambda_i) < +\infty.$$

Moreover, if $\|.\|^\sim$ is defined by (A.4.4) with ψ as above, then the condition (A.4.2) (resp. (A.4.3)) holds if and only if

$$\sum_{i \geq 1} \varphi_*(\psi(\lambda_i)) < +\infty \tag{A.4.5}$$

(resp. if and only if

$$\sum_{i \geq 1, \lambda_i > 0} \varphi_*(\lambda_i/\psi(\lambda_i)) < +\infty). \tag{A.4.6}$$

The existence of $\|.\|_1^\sim$ (resp. of $\|.\|_2^\sim$) follows from the existence of ψ as above that satisfies (A.4.5) (resp. (A.4.6)) and

$$\psi(0) = 0 \quad \text{and} \quad \lim_{x \to 0_+} \psi(x)/x = +\infty$$

(resp. and

$$\psi(0) = 0 \quad \text{and} \quad \lim_{x \to 0_+} \psi(x) = 0.)$$

This is proved by a variant of the argument establishing part (2) of Proposition A.4.1, and we leave its details to the reader.

To establish (2) in full generality, we choose a family $(\varepsilon_\alpha)_{\alpha \in A}$ in $\mathbb{R}_+^{*A}$ such that the series of functions

$$\varphi_* := \sum_{\alpha \in A} \varphi_\alpha$$

converges normally on every compact subset of $\mathbb{R}_+$, and such that

$$\sum_{\alpha \in A} \mathrm{Tr}\,\varphi_\alpha(\|.\|'/\|.\|) < +\infty.$$

This last condition is equivalent to:

$$\operatorname{Tr}\varphi_*(\|.\|'/\|.\|) < +\infty.$$

Therefore the validity of (2) when A has a single element implies the existence of Euclidean seminorms $\|.\|_1^\sim$ and $\|.\|_2^\sim$ as in (1) such that

$$\operatorname{Tr}\varphi_*(\|.\|_1^\sim/\|.\|) < +\infty \tag{A.4.7}$$

and

$$\operatorname{Tr}\varphi_*(\|.\|'/\|.\|_2^\sim) < +\infty. \tag{A.4.8}$$

For every $\alpha \in A$, the estimate $\varphi_\alpha \leq \varepsilon_\alpha^{-1}\varphi_*$ shows that (A.4.2) (resp. (A.4.3)) follows from (A.4.7) (resp. (A.4.8)). $\square$

APPENDIX B

The Relative Trace of Two Euclidean Seminorms

In this work, we shall make use of the notion of relative trace of (the squares of) two Euclidean seminorms, at the level of generality considered in [**Bou69**, Annexe, 1]. In this section, we provide a short but self-contained discussion of this notion and of its various properties that will be used in the sequel.

B.1. Definitions

Let V be a real vector space, equipped with a Euclidean seminorm $\|.\|$; let us denote the Euclidean scalar product that defines $\|.\|$ by $\langle .,. \rangle$.

We may consider the subspace

$$\ker \|.\| := \{v \in V \mid \|v\| = 0\}$$

of V, and equip the quotient vector space $V/\ker\|.\|$ with the Euclidean norm induced by $\|.\|$, that we will still denote by $\|.\|$. The completion of the pre-Hilbert space $(V/\ker\|.\|, \|.\|)$ is a real Hilbert space, which we shall denote by $(V, \|.\|)^{\mathrm{comp}}$, or shortly V^{comp}.

The quotient map from V to $V/\ker\|.\|$ defines a $\mathbb{R}$-linear map

$$j_{V,\|.\|} : V \longrightarrow V^{\mathrm{comp}}.$$

It is an isometry, with dense image, from the seminormed vector space $(V, \|.\|)$ to the Hilbert space $(V, \|.\|)^{\mathrm{comp}}$.

For any $n \in \mathbb{N}$, we define $\mathrm{Orth}_n^{\leq 1}(V, \|.\|)$ as the set of the $\mathbb{R}$-linear maps

$$\varphi : \mathbb{R}^n \longrightarrow V$$

that satisfy the condition

$$\mathbf{Orth}^{\leq 1}: \qquad \|\varphi(x)\| \leq |x| \quad \text{for every } x \in \mathbb{R}^n,$$

where $|.|$ denotes the standard Euclidean norm on $\mathbb{R}^n$, defined by

$$|(x_1, \ldots, x_n)|^2 := x_1^2 + \cdots + x_n^2.$$

We shall also consider the subsets $\mathrm{Orth}^{<1}(V, \|.\|)$ and $\mathrm{Orth}(V, \|.\|)$ of $\mathrm{Orth}(V, \|.\|)$ defined respectively by replacing the condition $\mathbf{Orth}^{\leq 1}$ by the conditions

$$\mathbf{Orth}^{<1}: \qquad \|\varphi(x)\| < |x| \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\},$$

and

$$\mathbf{Orth}: \qquad \|\varphi(x)\| = |x| \quad \text{for every } x \in \mathbb{R}^n.$$

Finally, we shall consider the disjoint union[1]

$$\mathrm{Orth}^{\leq 1}(V, \|.\|) := \coprod_{n \in \mathbb{N}} \mathrm{Orth}_n^{\leq 1}(V, \|.\|),$$

[1] By definition, an element of $\mathrm{Orth}^{\leq 1}(V, \|.\|)$ is a pair (n, φ) with n in $\mathbb{N}$ and φ in $\mathrm{Orth}_n^{\leq 1}(V, \|.\|)$.

and its subsets

$$\mathrm{Orth}^{<1}(V, \|.\|) := \coprod_{n\in\mathbb{N}} \mathrm{Orth}_n^{<1}(V, \|.\|),$$

and

$$\mathrm{Orth}(V, \|.\|) := \coprod_{n\in\mathbb{N}} \mathrm{Orth}_n(V, \|.\|).$$

Observe that $\mathrm{Orth}(V, \|.\|)$ may be identified with the set of finite orthonormal sequences in the inner-product space $(V, \langle ., .\rangle)$, by the map which sends $(n, \varphi) \in \mathrm{Orth}(V, \|.\|)$ to $(\varphi(e_1), \dots, \varphi(e_n))$, where $(e_1, \dots, e_n)$ denotes the canonical basis of $\mathbb{R}^n$.

Definition B.1.1. Let V be a real vector space, and let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on V. The *relative trace of* $\|.\|'^2$ *with respect to* $\|.\|^2$ is defined as:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) := \sup_{(n,\varphi)\in\mathrm{Orth}^{\leq 1}(V,\|.\|)} \sum_{i=1}^{n} \|\varphi(e_i)\|'^2 \in [0, +\infty]. \tag{B.1.1}$$

B.2. Properties of the Relative Trace

The relative trace $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$ defined by (B.1.1) satisfies the following properties, which indeed provide alternative definitions of it.[2]

Proposition B.2.1. *Let V be a real vector space, and let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on V.*

(i) For any $v \in V$, we have:

$$\|v\|'^2 \leq \mathrm{Tr}(\|.\|'^2/\|.\|^2)\, \|v\|^2.$$

In particular, if the relative trace $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$ is finite, then the seminorm $\|.\|'$ is continuous on V equipped with the seminorm $\|.\|$.

(ii) If there exists $v \in V$ such that $\|v\| = 0$ and $\|v\|' \neq 0$, or equivalently if $\ker \|.\| \not\subseteq \ker \|.\|'$, then

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) = +\infty.$$

If $\ker \|.\| \subseteq \ker \|.\|'$, then

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) := \sup_{(n,\varphi)\in\mathrm{Orth}(V,\|.\|)} \sum_{i=1}^{n} \|\varphi(e_i)\|'^2.$$

(iii) The relative trace of $\|.\|'^2$ with respect to $\|.\|^2$ satisfies:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) := \sup_{(n,\varphi)\in\mathrm{Orth}^{<1}(V,\|.\|)} \sum_{i=1}^{n} \|\varphi(e_i)\|'^2. \tag{B.2.1}$$

(iv) If V_0 is a real vector subspace of V, dense in V for the topology defined by $\|.\|$, then the relative trace of $\|.\|'^2$ with respect to $\|.\|^2$ coincides with that of their restriction to V_0:

$$\mathrm{Tr}(\|.\|'^2_{|V_0}/\|.\|^2_{|V_0}) = \mathrm{Tr}(\|.\|'^2/\|.\|^2).$$

(v) If $(V, \|.\|)$ is a real Hilbert space, or equivalently if $\|.\|$ is a complete norm on V, then, for every Hilbert basis $(e_i)_{i\in I}$ of $(V, \|.\|)$, we have:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) = \sum_{i\in I} \|e_i\|'^2.$$

From the expressions (B.1.1) and (B.2.1) for the relative trace, one easily derives that it satisfies the monotonicity and continuity properties in the next two propositions.

[2] For instance, Property (ii) is used as a definition of $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$ in [**Bou69**, Annexe, 1].

PROPOSITION B.2.2. *Let V be a real vector space, and let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on V.*

(i) The relative trace $\mathrm{Tr}\left(\|.\|'^2/\|.\|^2\right)$ is an increasing (resp. decreasing) function of the Euclidean seminorm $\|.\|'$ (resp. of $\|.\|$).

(ii) If $(\|.\|_i)_{i\in\mathbb{N}}$ is a decreasing sequence of Euclidean seminorms on V (resp. $(\|.\|'_i)_{i\in\mathbb{N}}$ an increasing sequence) of Euclidean seminorms on V such that

$$\lim_{i\to+\infty}\|v\|_i=\|v\| \quad \textit{and} \quad \lim_{i\to+\infty}\|v\|'_i=\|v\|'$$

for every $v\in V$, then the increasing sequence $\left(\mathrm{Tr}(\|.\|'^2_i/\|.\|^2)\right)_{i\in\mathbb{N}}$ satisfies:

$$\lim_{i\to+\infty}\mathrm{Tr}(\|.\|'^2_i/\|.\|^2_i)=\mathrm{Tr}(\|.\|'^2/\|.\|^2).$$

PROPOSITION B.2.3. *Let V be a real vector space, and let $\|.\|$ and $\|.\|'$ be two Euclidean seminorms on V.*

(i) For any $\mathbb{R}$-vector subspace W of V, we have:

$$\mathrm{Tr}(\|.\|'^2_{|W}/|.\|^2_{|W})\leq\mathrm{Tr}\left(\|.\|'^2/\|.\|^2\right)$$

and

$$\mathrm{Tr}(\|.\|'^2_{V/W}/\|.\|^2_{V/W})\leq\mathrm{Tr}\left(\|.\|'^2/\|.\|^2\right),$$

where $\|.\|_{V/W}$ and $\|.\|'^2_{V/W}$ denote the quotient Euclidean seminorms on V/W associated to $\|.\|$ and $\|.\|'$ respectively.

(ii) If $(W_i)_{i\in\mathbb{N}}$ is an increasing sequence of vector subspaces of V such that $\bigcup_{i\in\mathbb{N}}W_i=V$, then the increasing sequence $\left(\mathrm{Tr}(\|.\|'^2_{|W_i}/\|.\|^2_{|W_i})\right)_{i\in\mathbb{N}}$ satisfies:

$$\lim_{i\to+\infty}\mathrm{Tr}(\|.\|'^2_{|W_i}/\|.\|^2_{|W_i})=\mathrm{Tr}(\|.\|'^2/\|.\|^2).$$

If moreover $\mathrm{Tr}\left(\|.\|'^2/\|.\|^2\right)$ is finite, the decreasing sequence $\left(\mathrm{Tr}(\|.\|'^2_{V/W_i}/\|.\|^2_{V/W_i})\right)_{i\in\mathbb{N}}$ satisfies:

$$\lim_{i\to+\infty}\mathrm{Tr}(\|.\|'^2_{V/W_i}/\|.\|^2_{V/W_i})=0.$$

DEFINITION B.2.4. Let V be a real vector space, and let $\|.\|$ and $\|.\|'$ be two euclidean seminorms on V. We say that the quadratic form $\|.\|'^2$ is *nuclear with respect to the quadratic form* $\|.\|^2$ or that $\|.\|'$ is *Hilbert-Schmidt with respect to* $\|.\|$ if the relative trace $\mathrm{Tr}(\|.\|'^2/\|.\|^2)$ is finite.

One easily checks that $\|.\|'$ is Hilbert-Schmidt with respect to $\|.\|$ if and only if the seminorm $\|.\|'$ is continuous on V equipped with the topology defined by $\|.\|$ and if the continuous linear map of Hilbert spaces

$$\iota:(V,\|.\|)^{\mathrm{comp}}\longrightarrow(V,\|.\|')^{\mathrm{comp}},$$

induced by the identity map of V, is a Hilbert-Schmidt operator.

APPENDIX C

Measure Theory on some Infinite Dimensional Topological Vector Spaces

Let V be a $\mathbb{R}$-vector space of countable dimension. The choice of a $\mathbb{R}$-basis $(e_i)_{i\in I}$ of V, indexed by some countable set I, is equivalent to the datum of an isomorphism of $\mathbb{R}$-vector spaces:

$$\iota : V \xrightarrow{\sim} \mathbb{R}^{(I)}. \tag{C.0.1}$$

The dual $\mathbb{R}$-vector space

$$V^\vee := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}),$$

when equipped with the topology of pointwise convergence, or equivalently with the topology $\sigma(V^\vee, V)$, is a Fréchet space. Indeed the isomorphism (C.0.1) determines an isomorphism of topological vector spaces:

$$\iota^{\vee -1} : V^\vee \xrightarrow{\sim} \mathbb{R}^{(I)\vee} = \mathbb{R}^I,$$

where $\mathbb{R}^I$ is equipped with the product topology defined by endowing each factor $\mathbb{R}$ with its usual topology.

In this Appendix, we present various results concerning positive Borel measures on $V^\vee$ and their Fourier transform on V that play a key role in Chapter 8, in the study of the function $B_{\overline{F}}$ and the measure $\beta_{\overline{F}^\vee}$ associated to an object $\overline{F}$ in $\mathbf{q}\overline{\mathbf{Coh}}_{\mathbb{Z}}$.

These results are variations on some classical results of Bochner [**Boc55**], Prokhorov [**Pro56**], Sazonov [**Saz58**], and Minlos [**Min59**]. Our aim in this Appendix is twofold: to present these results in the specific form required for their applications in Chapter 8; and to make them accessible with only some familiarity with the basic tools of measure theory.

For more results and references on this circle of questions, we refer the reader to the masterly introduction by Gelfand and Vilenkin [**GV64**] and to the more thorough presentation in [**Bou69**] which also contains an illuminating "Note historique."[1] Useful expositions of this material also appear in [**Sch73**], [**SF76**], and [**VTC87**].

C.1. The Theorems of Bochner and P. Lévy

We shall denote by $\mathcal{M}_+^{\mathrm{fin}}(V^\vee)$ the convex cone of positive Borel measure of finite mass on the topological space $V^\vee$. To any measure μ in $\mathcal{M}_+^{\mathrm{fin}}(V^\vee)$ we may attach its Fourier transform, namely the function

$$\mathcal{F}\mu : V^\vee \longrightarrow \mathbb{C}$$

defined by the equality:

$$\mathcal{F}\mu(x) := \int_{V^\vee} e^{-2\pi i \langle \xi, x\rangle}\, d\mu(\xi). \tag{C.1.1}$$

[1] The reader should be aware that the use of Euclidean norms, and not of more general seminorms, in [**GV64**] make their exposition not directly suitable in the context of the present monograph, and of the idiosyncratic approach to measure theory in [**Bou69**].

The function $\mathcal{F}\mu$ is continuous on $V^\vee$ equipped with the inductive topology. In other words, its restriction to every finite dimensional $\mathbb{R}$-vector space W of V is continuous[2]. This is indeed a straightforward consequence of dominated convergence theorem.

Moreover the function $\mathcal{F}\mu$ is easily checked to be of positive type.[3] In particular, for every $(v,w) \in V^2$, the following relations hold:

$$\Phi(-v) = \overline{\Phi(v)}, \tag{C.1.2}$$

$$|\Phi(v)| \leq \Phi(0), \tag{C.1.3}$$

and:

$$|\Phi(v) - \Phi(w)|^2 \leq 2\Phi(0)\left(\Phi(0) - \operatorname{Re}\Phi(v-w)\right). \tag{C.1.4}$$

Observe also that the following equality holds:

$$\Phi(0) := \mu(V^\vee),$$

and that, for every $x \in V$, we have:

$$\operatorname{Re}\Phi(x) = \int_{V^\vee} \cos(2\pi\langle \xi, x\rangle)\, d\mu(\xi) \in [-\Phi(0), \Phi(0)] = [-\mu(V^\vee), \mu(V^\vee)]. \tag{C.1.5}$$

In particular, when μ is a probability measure, $1 - \operatorname{Re}\Phi(x)$ lies in the interval $[0,2]$.

THEOREM C.1.1. *The map* $(\mu \longmapsto \mathcal{F}\mu)$ *defined by* (C.1.1) *establishes a bijection from* $\mathcal{M}_+^{\mathrm{fin}}(V^\vee)$ *onto the convex cone* $\mathcal{P}_+(V)$ *of continuous functions of positive type on* V *equipped with the inductive topology.*

When V is finite dimensional, it is a classical theorem of Bochner; see [**Boc33**]. When V has infinite countable dimension, it is a special case of Minlos theorem, and actually directly follows from Bochner theorem concerning measures on finite dimensional $\mathbb{R}$-vector spaces combined with the identification of $\mathcal{M}_+^{\mathrm{fin}}(V^\vee)$ with the space of finite positive promeasures on $V^\vee$, which is a classical result of Kolmogorov. This extension of the original Bochner theorem in [**Boc33**] to the infinite dimensional setting is actually due to Bochner himself, and appears in his monograph [**Boc55**, Chapter 5].

Observe also that, as a consequence of the estimates (C.1.3) and (C.1.4), a function Φ of positive type on V is continuous on V equipped with the inductive topology if and only if it is continuous at the point 0 of V.

From the injectivity of $\mathcal{F}$, we immediately deduce:

COROLLARY C.1.2. *A measure* $\mu \in \mathcal{M}_+^{\mathrm{fin}}(V^\vee)$ *is a symmetric probability measure — in other words* μ *satisfies the conditions:*

$$\mu(V^\vee) = 1$$

and

$$\mu(-E) = \mu(E) \quad \textit{for every Borel subset } E \textit{ of } V^\vee$$

— if and only if the function $\Phi := \mathcal{F}\mu$ *satisfies:*

$$\Phi(0) = 1 \quad \textit{and} \quad \Phi(V) \subseteq \mathbb{R}.$$

The injectivity of $\mathcal{F}$ also admits the following consequence, which we leave as an easy exercise:

[2]when W equipped with its standard Hausdorff locally convex topology.

[3]Recall that a function $\Phi : A \to \mathbb{C}$ defined on some abelian group $(A,+)$ is called *of positive type* when, for every finite family $(a_\alpha)_{\alpha\in F}$ of elements of A, the matrix $(\Phi(a_\alpha - a_\beta))_{(\alpha,\beta)\in F^2}$ is Hermitian and semi-positive.

COROLLARY C.1.3. *Let W be a vector subspace of V and let*

$$W^{\perp} := \{\xi \in V^{\vee} \mid \langle \xi, x \rangle = 0\}$$

be the associated closed vector subspace of $V^{\vee}$. For every measure $\mu \in \mathcal{M}_{+}^{\mathrm{fin}}(V^{\vee})$ of Fourier transform $\Phi := \mathcal{F}\mu$, the following conditions are equivalent:

(i) μ *is supported by* $W^{\perp}$;
(ii) $\Phi_{|W} = \Phi(0)$;
(iii) *for every* $(v, w) \in V \times W$, $\Phi(v + w) = \Phi(v)$.

The space $\mathcal{M}_{+}^{\mathrm{fin}}(V^{\vee})$ is equipped with the topology of *narrow convergence*[4]. By definition a family $(\mu_\alpha)_{\alpha \in A}$ of elements of $\mathcal{M}_{+}^{\mathrm{fin}}(V^{\vee})$ indexed by a directed set $(A, \preceq)$ converges to $\mu \in \mathcal{M}_{+}^{\mathrm{fin}}(V^{\vee})$ in this topology when, for every bounded continuous function φ on $V^{\vee}$:

$$\lim_{\alpha \in A} \int_{V^{\vee}} \varphi \, d\mu_\alpha = \int_{V^{\vee}} \varphi \, d\mu. \tag{C.1.6}$$

The topology of narrow convergence on $\mathcal{M}_{+}^{\mathrm{fin}}(V^{\vee})$ is actually metrizable and makes $\mathcal{M}_{+}^{\mathrm{fin}}(V^{\vee})$ a Polish space; see [**Pro56**, Â§1.4], [**Bou69**, Â§5.4], or [**Str11**, Â§9.1].

THEOREM C.1.4. *For every family $(\mu_\alpha)_{\alpha \in A}$ of elements of $\mathcal{M}_{+}^{\mathrm{fin}}(V^{\vee})$ indexed by a directed set $(A, \preceq)$ possessing a countable directed subset and every μ in $\mathcal{M}_{+}^{\mathrm{fin}}(V^{\vee})$, the following conditions are equivalent:*

(i) $(\mu_\alpha)_{\alpha \in A}$ *converges to* μ *in the topology of narrow convergence;*
(ii) *for every finite dimensional $\mathbb{R}$-vector subspace W of $V^{\vee}$ and every compact subset K of W, the family $(\mathcal{F}\mu_\alpha)_{\alpha \in A}$ converges to $\mathcal{F}\mu$ uniformly on K;*
(iii) *for every* $x \in V$,

$$\lim_{\alpha \in A} \mathcal{F}\mu_\alpha(x) = \mathcal{F}\mu(x).$$

When V is finite dimensional, Theorem C.1.4 is a classical result of P. Lévy.[5] The infinite dimensional case covered in Theorem C.1.4 follows easily by using the basic properties of narrow convergence.[6]

C.2. The Theorems of Minlos and of Prokhorov-Sazonov

In this section, we assume that the $\mathbb{R}$-vector space V is equipped with some Euclidean seminorm $\|.\|$.

C.2.1. Definitions. Statements of the theorems.

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[4] We follow the terminology of [**Bou69**, Â§5.3] and [**Sch73**]. This topology, or the induced topology on the set $\mathcal{M}_{+}^{1}(V^{\vee})$ of probability measure on $V^{\vee}$, is often called the topology of *weak convergence*, see for instance [**Pro56**], [**Bil99**] or [**Str11**]. When V is finite dimensional, this topology does *not* coincide with the topology induced by the weak topology on the space $\mathcal{D}'(V^{\vee})$ of distributions on $V^{\vee}$; the latter coincides with the topology called topology of vague convergence in [**Bou69**].

[5] See for instance [**Bou69**], Â§5, Exercice 13, or [**Str11**, Â§3.1].

[6] See for instance [**Bil99**], Example 2.4. In Chapter 8, we use only the finite dimensional version of Theorem C.1.4. The reader may refer to [**Fer67**, III.6] for more general versions of P. Lévy's theorem in the infinite dimensional setting.

C.2.1.1. To the seminorm $\|.\|$ is attached by duality a definite quasinorm on $V^\vee$,

$$\|.\|^\vee : V^\vee \longrightarrow [0, +\infty],$$

defined by the equality:

$$\|\xi\|^\vee := \sup_{x \in V, \|x\| \leq 1} |\langle \xi, x \rangle| \quad \text{for every } \xi \in V^\vee.$$

This quasinorm is lower semicontinuous on the Fréchet space $V^\vee$, and if we let:

$$V^{\vee \mathrm{Hilb}} := \{\xi \in V^\vee \mid \|\xi\|^\vee < +\infty\},$$

then $(V^{\vee\mathrm{Hilb}}, \|.\|^\vee)$ is a Hilbert space and the inclusion $V^{\vee\mathrm{Hilb}} \hookrightarrow V^\vee$ is a continuous linear map from this Hilbert space to the Fréchet space $V^\vee$. Moreover the space

$$V^{\vee\mathrm{Hilb}} = \bigcup_{n \in \mathbb{N}} \{\xi \in V^\vee \mid \|\xi\|^\vee \leq n\}$$

is a countable union of closed subsets, and therefore a Borel subset, of the Fréchet space $V^\vee$. Actually each "closed ball"

$$B_n := \{\xi \in V^\vee \mid \|\xi\|^\vee \leq n\} \tag{C.2.1}$$

is easily seen to be a compact subset of the Fréchet space $V^\vee$, and a subset E of $V^{\vee\mathrm{Hilb}}$ to be a Borel subset of the Hilbert space $(V^{\vee\mathrm{Hilb}}, \|.\|^\vee)$ if and only if it is a Borel subset of the Fréchet space $V^\vee$.

Moreover a simple application of the Hahn-Banach theorem shows that the closure $\overline{V^{\vee\mathrm{Hilb}}}$ of $V^{\vee\mathrm{Hilb}}$ in the Fréchet space $V^\vee$ satisfies:

$$\overline{V^{\vee\mathrm{Hilb}}} = K^\perp := \left\{\xi \in V^\vee \mid \xi_{|K} = 0\right\},$$

where K denotes the "kernel" of the seminorm $\|.\|$:

$$K := \{x \in V \mid \|x\| = 0\}.$$

In particular $V^{\vee\mathrm{Hilb}}$ is dense in $V^\vee$ if and only if $\|.\|$ is a norm.

Definition C.2.1. The *Sazonov topology* of $(V, \|.\|)$ is the locally convex topology on V defined by the Euclidean seminorms $\|.\|'$ on V such that:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty.$$

C.2.1.2. When $(V, \|.\|)$ is a finite dimensional Euclidean vector space, the following result appears in the literature under the name of Minlos lemma; see for instance [**Bou69**, Â§ 6.9]. It constitutes the key technical point underlying the proof of Minlos theorem which extend Bochner theorem to probability measures on the topological dual of nuclear spaces; see [**Min59**] and [**Bou69**, Â§ 6.10].

Theorem C.2.2. *Let μ be a Borel probability measure on $V^\vee$ and let $\Phi := \mathcal{F}\mu$ be its Fourier transform. Assume that, for some Euclidean seminorm $\|.\|'$ on V and some $\varepsilon \in \mathbb{R}_+^*$, the following implication holds for every $x \in V$:*

$$\|x\|' < 1 \Longrightarrow 1 - \mathrm{Re}\,\Phi(x) < \varepsilon. \tag{C.2.2}$$

Then the following estimate holds for every $C \in \mathbb{R}_+^*$:

$$\mu\left(\left\{\xi \in V^\vee \mid \|\xi\|^{\vee 2} > C\right\}\right) \leq (1 - e^{-1/2})^{-1}\left[\varepsilon + (2\pi^2 C)^{-1}\mathrm{Tr}(\|.\|'^2/\|.\|^2)\right]. \tag{C.2.3}$$

The following theorem is basically due to Prokhorov and Sazonov; see [**Pro56**, Â§1.6] and [**Saz58**]; see also [**Bou69**, Â§6.11].

Theorem C.2.3. *Let μ be a Borel probability measure on $V^\vee$ and let*

$$\Phi := \mathcal{F}\mu : V \longrightarrow \mathbb{C}$$

be its Fourier transform. The following conditions are equivalent:

(i) The measure μ is supported by $V^{\vee\mathrm{Hilb}}$.[7]

(ii) For every $\varepsilon \in \mathbb{R}_+^*$, there exists a Euclidean seminorm $\|.\|'$ on V which satisfies the following conditions:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty,$$

and, for every $x \in V$,

$$1 - \mathrm{Re}\,\Phi(x) \leq \varepsilon + \|x\|'^2.$$

(iii) The function Φ is continuous on V endowed with the Sazonov topology of $(V, \|.\|)$.

(iv) The function $\mathrm{Re}\,\Phi$ is continuous at the point 0 of V equipped with the Sazonov topology. Equivalently, for every $\varepsilon \in \mathbb{R}_+^*$, there exists a Euclidean seminorm $\|.\|'$ on V which satisfies the following conditions:

$$\mathrm{Tr}(\|.\|'^2/\|.\|^2) < +\infty,$$

and, for every $x \in V$,

$$\|x\|' < 1 \Longrightarrow 1 - \mathrm{Re}\,\Phi(x) \leq \varepsilon.$$

In the next subsections, we present a selfcontained derivation of Theorems C.2.2 and C.2.3. Our line of arguments originates in the lucid short note by Kolmogorov [**Kol59**], which gives a central role to "Fourier duality" formulae like (C.2.5) below, and thus provides a simpler proof of Minlos lemma and elucidates its relations to the earlier contributions of Prokhorov and Sazonov.

C.2.2. Proof of Theorem C.2.2.

C.2.2.1. The proof of "Minlos estimate" (C.2.3) will rely on the following integral formulae concerning finite dimensional Euclidean vector spaces.

Proposition C.2.4. *Let W be a finite dimensional $\mathbb{R}$-vector space, $\|.\|$ a Euclidean norm on W, and $\lambda_{\overline{W}}$ the Lebesgue measure on W attached to the Euclidean vector space $\overline{W} := (W, \|.\|)$.*

(1) For every Euclidean seminorm $\|.\|'$ on W, the following equality holds:

$$2\pi \int_W \|x\|'^2\, e^{-\pi\|x\|^2}\, d\lambda_{\overline{W}}(x) = \mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{C.2.4}$$

(2) For every Borel probability measure ν on the dual $W^\vee$ of W, of Fourier transform $\Psi := \mathcal{F}_{W^\vee}\nu$, the following equality holds:

$$\int_{W^\vee} \left(1 - e^{-\pi\|\xi\|^2_{\overline{W}^\vee}}\right) d\nu(\xi) = \int_W [1 - \mathrm{Re}\,\Psi(x)]\, e^{-\pi\|x\|^2}\, d\lambda_{\overline{W}}(x) \quad (\in [0,1]). \tag{C.2.5}$$

We have denoted by $\|.\|_{\overline{W}^\vee}$ the Euclidean norm on $W^\vee$ dual of the norm $\|.\|$. By definition of the Fourier transform $\mathcal{F}_{W^\vee}$, we have, for every $x \in W$:

$$\Psi(x) := \int_{W^\vee} e^{-2\pi i\langle \xi, x\rangle}\, d\nu(\xi).$$

Proof. Let us introduce the Gaussian probability measure $\gamma_{\overline{W}}$ on W:

$$d\gamma_{\overline{W}}(x) = e^{-\pi\|x\|^2}\, d\lambda_{\overline{W}}.$$

To prove (1), observe that the formation of $\gamma_{\overline{W}}$ is compatible with direct sums of Euclidean vector spaces. By considering a basis of W that is orthogonal both for the scalar product defining the quadratic form $\|.\|^2$ and the one defining $\|.\|'^2$, we are reduced to establish the identity (C.2.4) when W has dimension 1. In this case, it reduces to the identity:

$$2\pi \int_{\mathbb{R}} x^2 e^{-\pi x^2}\, dx = \int_{\mathbb{R}} t^2 e^{-t^2/2}\, \frac{dt}{\sqrt{2\pi}} = 1,$$

which expresses that the standard normal distribution has variance one.

[7] Namely μ satisfies: $\mu(V^\vee \setminus V^{\vee\mathrm{Hilb}}) = 0$, or equivalently: $\mu(V^{\vee\mathrm{Hilb}}) = 1$.

To prove (2), observe that, using that ν and $\gamma_{\overline{W}}$ are probability measures and that Ψ satisfies the relation (C.1.2), the identity (C.2.5) may be written:

$$\int_{W^\vee} e^{-\pi\|\xi\|^2_{\overline{W}^\vee}}\, d\nu(\xi) = \int_W \Psi(x)\, d\gamma_{\overline{W}}(x). \tag{C.2.6}$$

Moreover the Fourier transform of $\gamma_{\overline{W}}$ is a Gaussian function. Namely, for every $\xi \in W^\vee$, we have:

$$\mathcal{F}_W \gamma_{\overline{W}}(\xi) := \int_W e^{-2\pi i\langle \xi, x\rangle}\, d\gamma_{\overline{W}}(x) = e^{-\pi\|\xi\|^2}.$$

Accordingly, (C.2.6) is nothing but the Fourier duality formula:

$$\int_{W^\vee} \mathcal{F}_W \gamma_{\overline{W}}\, d\nu = \int_W \mathcal{F}_{W^\vee}\nu\, d\gamma_{\overline{W}} \tag{C.2.7}$$

for the measures $\gamma_{\overline{W}}$ and ν on the dual vector spaces W and $W^\vee$. In turn (C.2.7) is a straightforward consequence of Fubini theorem, which shows that both sides of (C.2.7) equal the integral:

$$\int_{W\times W^\vee} e^{-2\pi\langle \xi, x\rangle}\, d\gamma_{\overline{W}}(x)\, d\nu(\xi).$$

□

C.2.2.2. To prove the estimate (C.2.3), after replacing the seminorm $\|.\|$ by $\sqrt{2\pi C}\|.\|$, it is sufficient to handle the case where $C = (2\pi)^{-1}$, namely to establish that if condition (C.2.2) is satisfied, then the following estimate holds:

$$\mu\left(\left\{\xi \in V^\vee \mid \|\xi\|^{\vee 2} > (2\pi)^{-1}\right\}\right) \leq (1 - e^{-1/2})^{-1}\left[\varepsilon + \pi^{-1}\mathrm{Tr}(\|.\|'^2/\|.\|^2)\right]. \tag{C.2.8}$$

We may choose a decreasing sequence $(\|.\|_i)_{i\in\mathbb{N}}$ of Euclidean norms on V that converges pointwise to $\|.\|$. Then the sequence of dual quasinorms $(\|.\|_i^\vee)_{i\in\mathbb{N}}$ on $V^\vee$ is increasing and converges pointwise to $\|.\|^\vee$. Consequently the sequence $\left(\left\{\xi \in V^\vee \mid \|\xi\|_i^{\vee 2} > (2\pi)^{-1}\right\}\right)_{i\in\mathbb{N}}$ of subsets of $V^\vee$ is increasing, and its union is $\left\{\xi \in V^\vee \mid \|\xi\|^{\vee 2} > (2\pi)^{-1}\right\}$. Therefore:

$$\lim_{i\to+\infty} \mu\left(\left\{\xi \in V^\vee \mid \|\xi\|^{\vee 2} > (2\pi)^{-1}\right\}\right) = \mu\left(\left\{\xi \in V^\vee \mid \|\xi\|^{\vee 2} > (2\pi)^{-1}\right\}\right). \tag{C.2.9}$$

Moreover, for every $i \in \mathbb{N}$, we have:

$$\mathrm{Tr}(\|.\|'^2/\|.\|_i^2) \leq \mathrm{Tr}(\|.\|'^2/\|.\|^2). \tag{C.2.10}$$

From (C.2.9) and (C.2.10), it follows that the validity of (C.2.8) follows from its validity when $\|.\|$ is replaced by each of the Euclidean norms $\|.\|_i$. Consequently, to establish (C.2.8), we may and will assume that the Euclidean seminorm $\|.\|$ is a norm.

C.2.2.3. Let us introduce the directed set $(\mathrm{coh}(V), \subseteq)$ of finite dimensional vector subspaces of V. The following proposition provides an extension to the infinite dimensional setting of the identity (C.2.5).

PROPOSITION C.2.5. *For every $W \in \mathrm{coh}(V)$, the integral*

$$\int_W [1 - \mathrm{Re}\,\Phi(x)]\, e^{-\pi\|x\|^2}\, d\lambda_{\overline{W}}(x)$$

belongs to $[0,1]$ and defines an increasing function of W in $\mathrm{coh}(V)$. Its limit over the directed set $\mathrm{coh}(V, \subseteq)$:

$$\lim_{W\in\mathrm{coh}(V)} \int_W [1 - \mathrm{Re}\,\Phi(x)]\, e^{-\pi\|x\|^2}\, d\lambda_{\overline{W}}(x) = \sup_{W\in\mathrm{coh}(V)} \int_W [1 - \mathrm{Re}\,\Phi(x)]\, e^{-\pi\|x\|^2}\, d\lambda_{\overline{W}}(x),$$

satisfies the equality:

$$\int_{V^\vee} \left(1 - e^{-\pi\|\xi\|^{\vee 2}}\right) d\mu(\xi) = \lim_{W\in\mathrm{coh}(V)} \int_W [1 - \mathrm{Re}\,\Phi(x)]\, e^{-\pi\|x\|^2}\, d\lambda_{\overline{W}}(x). \tag{C.2.11}$$

Proof. For every $W \in \operatorname{coh}(V)$, we may consider the inclusion map:

$$i_W : W \hookrightarrow V$$

and its transpose:

$$p_W : V^\vee \hookrightarrow W^\vee.$$

As a straightforward consequence of the definitions, the Fourier transform of the Borel probability measure $p_{W*}\mu$ satisfies:

$$\mathcal{F}_{W^\vee} p_{W*}\mu = i_W^* \mathcal{F}\mu = \Phi_{|W}.$$

Consequently Proposition C.2.4 (2), applied to $\overline{W} := (W, \|.\|_{|W})$, shows that the following equalities hold:

$$\begin{aligned}\int_{V^\vee} \left(1 - e^{-\pi\|p_W(\xi)\|^2_{\overline{W}^\vee}}\right) d\mu(\xi) &= \int_{W^\vee} \left(1 - e^{-\pi\|\xi\|^2_{\overline{W}^\vee}}\right) dp_{W*}\mu(\xi) \\ &= \int_W \left[1 - \operatorname{Re}\Phi(x)\right] e^{-\pi\|x\|^2} d\lambda_{\overline{W}}(x).\end{aligned} \tag{C.2.12}$$

Moreover for every ξ in $V^\vee$, $\|p_W(\xi)\|_{\overline{W}^\vee}$ is an increasing function of W in $\operatorname{coh}(V)$, and satisfies:

$$\lim_{W \in \operatorname{coh}(V)} \|p_W(\xi)\|_{\overline{W}^\vee} = \|\xi\|^\vee.$$

According to the monotonicity of the integral and to Lebesgue monotone convergence theorem, together with (C.2.12) this observation establishes the proposition. □

When condition (C.2.2) holds, for every $W \in \operatorname{coh}(V)$ and every $x \in W$, we have:

$$1 - \operatorname{Re}\Phi(x) \leq \varepsilon \qquad \text{if } \|x\|' < 1,$$

and:

$$1 - \operatorname{Re}\Phi(x) \leq 2 \leq 2\|x\|'^2 \quad \text{if } \|x\|' \geq 1.$$

Consequently, using Proposition (C.2.4) (1), we obtain:

$$\begin{aligned}\int_W \left[1 - \operatorname{Re}\Phi(x)\right] e^{-\pi\|x\|^2} d\lambda_{\overline{W}}(x) &\leq \int_W (\varepsilon + 2\|x\|'^2)\, e^{-\pi\|x\|^2} d\lambda_{\overline{W}}(x) \\ &= \varepsilon + \pi^{-1} \operatorname{Tr}(\|.\|'^2_{|W}/\|.\|^2_{|W}).\end{aligned} \tag{C.2.13}$$

Using (C.2.11) and Proposition C.2.3, we finally obtain the estimate:

$$\begin{aligned}\int_{V^\vee} \left(1 - e^{-\pi\|\xi\|^{\vee 2}}\right) d\mu(\xi) &\leq \sup_{W \in \operatorname{coh}(V)} \left(\varepsilon + \pi^{-1} \operatorname{Tr}(\|.\|'^2_{|W}/\|.\|^2_{|W})\right) \\ &= \varepsilon + \pi^{-1} \operatorname{Tr}(\|.\|'^2/\|.\|^2).\end{aligned} \tag{C.2.14}$$

Observe that, for every $\xi \in V^\vee$, the following equivalence holds:

$$\|\xi\|^{\vee 2} \geq (2\pi)^{-1} \Longleftrightarrow 1 - e^{-\pi\|\xi\|^{\vee 2}} \geq 1 - e^{-1/2}.$$

This immediately implies the following estimate:

$$(1 - e^{-1/2})\, \mu\left(\left\{\xi \in V^\vee \mid \|\xi\|^{\vee 2} \geq (2\pi)^{-1}\right\}\right) \leq \int_{V^\vee} \left(1 - e^{-\pi\|\xi\|^{\vee 2}}\right) d\mu(\xi). \tag{C.2.15}$$

The estimate (C.2.3) follows from (C.2.14) and (C.2.15).

C.2.3. Proof of the theorem of Prokhorov-Sazonov. In this subsection, we finally establish the Theorem of Prokhorov-Sazonov (Theorem C.2.3) by means of Theorem C.2.2.

C.2.3.1. The implications $(ii) \Rightarrow (iv)$ and $(iii) \Rightarrow (iv)$ in Theorem C.2.3 are straightforward.[8]

The estimates (C.1.3) and (C.1.4) show that the function Φ is continuous on V endowed with the Sazonov topology if $\operatorname{Re}\Phi$ is continuous at the point 0 of V endowed with the Sazonov topology, and therefore establish the implication $(iv) \Rightarrow (iii)$.

C.2.3.2. To prove the implication $(i) \Rightarrow (ii)$ in Theorem C.2.3, let us assume that μ is supported by $V^{\vee\mathrm{Hilb}}$ and let us choose $\varepsilon \in \mathbb{R}_+^*$.

If the integer n is large enough, then the ball B_n defined by (C.2.1) satisfies:

$$\mu(V^\vee \setminus B_n) \leq \varepsilon/2. \tag{C.2.16}$$

Indeed:

$$\lim_{n \to +\infty} \mu(V^\vee \setminus B_n) = \mu(V^\vee \setminus \bigcup_{n \in \mathbb{N}} B_n) = \mu(V^\vee \setminus V^{\vee\mathrm{Hilb}}) = 0.$$

Then, using the estimate:

$$1 - \cos t \leq \min(2, t^2/2) \quad \text{for every } t \in \mathbb{R},$$

for obtain the following upper bound, for every $x \in V$:

$$\begin{aligned} 1 - \operatorname{Re}\Phi(x) &= \int_{V^\vee} [1 - \cos(2\pi\langle \xi, x\rangle)]\, d\mu(\xi) \\ &\leq \int_{B_n} 2\pi^2 \langle \xi, x\rangle^2\, d\mu(\xi) + \int_{V^\vee \setminus B_n} 2\, d\mu(\xi) \\ &\leq \|x\|'^2 + \varepsilon, \end{aligned}$$

where:

$$\|x\|'^2 := 2\pi^2 \int_{B_n} \langle \xi, x\rangle^2\, d\mu(\xi). \tag{C.2.17}$$

To complete the proof of (iii), we are left to show that the Euclidean seminorm $\|.\|'$ on V defined by (C.2.17) is such that $\operatorname{Tr}(\|.\|'^2/\|.\|^2)$ is finite. To achieve this, observe that for every finite family $(e_i)_{i\in F}$ of vector of V orthonormal with respect to the scalar product defining $\|.\|$ and for every $\xi \in B_n$, we have:

$$\sum_{i\in I} \langle \xi, e_i\rangle^2 \leq \|\xi\|^{\vee 2} \leq n^2.$$

Consequently we have:

$$\sum_{i \in I} \|e_i\|'^2 = 2\pi^2 \int_{B_n} \sum_{i\in I} \langle \xi, e_i\rangle^2\, d\mu(\xi) \leq 2\pi^2 n^2.$$

This establishes the upper bound:

$$\operatorname{Tr}(\|.\|'^2/\|.\|^2) \leq 2\pi^2 n^2.$$

C.2.3.3. The implication $(iv) \Rightarrow (i)$ in Theorem C.2.3, follows from Theorem C.2.2. Indeed this theorem shows that, when (iv) holds, the measure

$$\mu\left(\{\xi \in V^\vee \mid \|\xi\|^{\vee 2} \geq C\}\right)$$

admits the limit 0 when $C \in \mathbb{R}_+^*$ goes to infinity. Since this limit coincides with

$$\mu\left(\{\xi \in V^\vee \mid \|\xi\|^{\vee 2} = +\infty\}\right) = \mu(V^\vee \setminus V^{\vee\mathrm{Hilb}}),$$

this establishes that μ is supported by $V^{\vee\mathrm{Hilb}}$.

[8] The implication $(iv) \Rightarrow (ii)$ also is straightforward, but will not be used directly in this proof. Condition (ii) has been included in Theorem C.2.3 for the convenience of the proof, and to facilitate the comparison with other formulations of the theorem of Prokhorov-Sazonov, for instance in [**Bou69**, Â§6.11].

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