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On the Zero Locus of Normal Functions and the Étale Abel–Jacobi Map

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In this paper, we investigate questions of an arithmetic nature related to the Abel–Jacobi map. We give a criterion for the zero locus of a normal function to be defined over a number field, and we give some comparison theorems with the Abel–Jacobi map coming from continuous étale cohomology.

1 Introduction

Let $X \to S$ be a family of complex smooth projective varieties over a quasi-projective base, and let $Z \hookrightarrow X$ be a flat family of cycles of pure codimension i which are homologically equivalent to zero in the fibers of the family. For any point s of S, the Abel–Jacobi map associates to the cycle Z_s a point in the intermediate Jacobian $J^i(X_s)$ of X_s, which is a complex torus (see part 2 for details). This construction works in family, yielding a bundle of complex tori, the Jacobian fibration $J^i(X/S)$, and a normal function ν_Z, which is the holomorphic section of $J^i(X/S)$ associated to $Z \hookrightarrow X$. We can attach to ν_Z an admissible variation H of mixed Hodge structures on S, see [22], fitting in an exact sequence

$$0 \to R^{2i-1} f_* \mathbb{Z}/(\text{torsion}) \to H \to \mathbb{Z}$$

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such that the zero locus of ν_Z is the locus where this exact sequence splits, that is, the projection on S of the locus of Hodge classes in H which map to 1 in $\mathbb{Z}$. In analogy with the celebrated Deligne–Cattani–Kaplan theorem of [8] which shows the algebraicity of Hodge loci for variations of pure Hodge structures, Green and Griffiths have stated the following conjecture, which deals with the mixed Hodge structure appearing above.

Conjecture 1. The zero locus of the normal function ν is algebraic. □

In the same way that Deligne–Cattani–Kaplan's result gives evidence for the Hodge conjecture, this would give evidence for Bloch–Beilinson-type conjectures on filtration on Chow groups, see Section 2.1. Actually, it has been recently obtained independently by Brosnan–Pearlstein and M. Saito in [6] and [23], see also [5] that if S has a smooth compactification with complement a smooth divisor, then the zero locus of ν is algebraic. Brosnan and Pearlstein have announced a proof of conjecture 1 in full generality, which should appear soon.

Assume everything is defined over a finitely generated field k. In line with general conjectures on algebraic cycles, one would expect that not only the zero locus of ν is algebraic, but it should be defined over k, hence the interest in trying to investigate number-theoretic properties of the zero locus of normal functions. In the context of pure Hodge structures, that is, that of Deligne–Cattani–Kaplan's theorem, Voisin shows in [28] how this question is related to the question whether Hodge classes are absolute and gives criteria for Hodge loci to be defined over number fields.

In this paper, we want to tackle such questions and also investigate the arithmetic counterpart of normal functions, namely the étale Abel–Jacobi map introduced by Jannsen using continuous étale cohomology. We give comparison results between the étale Abel–Jacobi map and Griffiths' Hodge-theoretic one. Recent work around the same circle of ideas can be found in [11]. The use of Terasoma's lemma in this context is very relevant to our work, and the results proved in that paper are closely related to our Theorem 2 (though up to torsion).

Let us state our main results precisely. Start with a subfield k of $\mathbb{C}$ which is generated by a finite number of elements over $\mathbb{Q}$, and let S be a quasi-projective variety over k. Let $\pi : X \to S$ be a smooth family of projective varieties of pure dimension n, and let $Z \hookrightarrow S$ be a family of codimension i algebraic cycles of X, flat over S. Assume that Z is homologically equivalent to 0 on the geometric fibers of π. In the paper [14], Jannsen defines continuous étale cohomology, which is a version of étale l-adic cohomology for varieties over fields which are not necessarily algebraically closed. There is a cycle map from Chow groups to continuous étale cohomology. For any point s of S with value in a finitely generated extension K of k, let $\overline{X_s}$ be the variety $X_s \times_K \overline{K}$, and $G_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K. The cycle class of Z_s in the continuous étale cohomology of X_s gives a class $aj_{\text{ét}}(Z_s) \in H^1(G_K, \tilde{H}^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}}(i)))$, where $\tilde{H}^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}})$ denotes the quotient of $H^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}})$ by its torsion subgroup, $\hat{\mathbb{Z}}$ being the profinite completion of $\mathbb{Z}$. This cohomology class is obtained by applying a Hochschild–Serre spectral sequence to continuous cohomology. Proposition 6 shows that the vanishing of this class is independent of the choice of K, that is, it vanishes in $H^1(G_K, \tilde{H}^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}}(i)))$ if and only if it vanishes in $H^1(G_L, \tilde{H}^{2i-1}(\overline{X_s}, \hat{\mathbb{Z}}(i)))$ for any finite type extension L of K. This observation appears in [24] and, according to one of the referees, is due to Nori.

It would follow from general conjectures on algebraic cycles, whether a combination of the Hodge and Tate conjectures for open varieties or versions of the Bloch–Beilinson conjectures on filtrations on Chow groups, that the zero locus of the normal function associated to the family of cycles induced by $Z_{\mathbb{C}}$ on $X_{\mathbb{C}} \to S_{\mathbb{C}}$ is precisely the vanishing set of the étale Abel–Jacobi map. For the latter, assuming Beilinson's conjecture on Chow groups of varieties over number fields and Lewis' Bloch–Beilinson conjecture of [19], one would know that the kernels of both Abel–Jacobi maps are equal to the second step of the unique Bloch–Beilinson filtration on Chow groups, hence that they coincide. Unfortunately, such comparison results seem to be known only for divisors and zero cycles, where the étale Abel–Jacobi map can be computed using the Kummer exact sequence for Picard or Albanese varieties. We are not aware of any result in other codimension. In this paper, we therefore try to tackle such comparison results. We don't prove the general case, but we prove results of two different flavors in the variational case. We obtain such results assuming the algebraicity of the components of the zero locus of normal functions, which does not seem to be known in full generality at the time being, although as explained before it might be obtained in the near future.

In the previous situation, consider the normal function ν_Z associated to the family $Z_{\mathbb{C}}$ in $X_{\mathbb{C}}$. Its zero locus is an analytic subvariety of $S(\mathbb{C})$. Our theorems are the following, where the expression "finitely generated field" denotes a field generated by a finite number of elements over its prime subfield—$\mathbb{Q}$ in our case.

Theorem 2.

(i) Let T be an irreducible component of the zero locus of ν_Z. Assume that T is algebraic and that $R^{2i-1}\pi_{\mathbb{C}*}\mathbb{C}$ has no non-zero global sections over T. Let k be a finitely generated field over which T is defined. Then for all point t of T

with value in a finitely generated field, the étale Abel–Jacobi invariant of Z_t is zero.

(ii) Assume that for every closed point s of S with value in a number field, the étale Abel–Jacobi invariant of Z_s is zero and that $R^{2i-1}\pi_{\mathbb{C}*}\mathbb{C}$ has no non-zero global sections over $S_{\mathbb{C}}$. Then, ν_Z is identically zero on $S_{\mathbb{C}}$. □

Theorem 3. Let T be an irreducible component of the zero locus of ν_Z. Assume that T is algebraic and that $R^{2i-1}\pi_{\mathbb{C}*}\mathbb{C}$ has no non-zero global sections over T. Then, T is defined over a finite extension of the base field k and for any automorphism σ of $\mathbb{C}$ over k, the image of T by σ is an irreducible component of ν_Z. □

Remark. In this result, we do not assume that all the components of the zero locus of ν_Z are algebraic. Furthermore, we are considering the image of T by σ as a subscheme of S, and we prove that it is, as a subscheme of S, a component of the zero locus of the holomorphic normal function ν_Z. This contrasts with the situation in [28], where similar results were obtained using only the reduced structure on the subschemes considered. The main difference in our setting is that the (mixed) Hodge structures we consider have at most one non-zero Hodge class, up to multiplication by a constant.

Theorem 4.

(i) Assume that for every closed point s of S with value in a number field, the étale Abel–Jacobi invariant of Z_s is zero and that there exists a complex point s of S such that $\nu_Z(s) = 0$. Then, ν_Z is identically zero on $S_{\mathbb{C}}$.

(ii) Let T be an irreducible component of the zero locus of ν_Z. Assume that T is algebraic and that there exists a point t of T such that $aj_{\text{ét}}(Z_t)$ is zero. Then for all points t of T with value in a finitely generated field, the étale Abel–Jacobi invariant of Z_t is zero. □

The lack of symmetry between both Abel–Jacobi maps in our results is frustrating. Indeed, while the local structure of zero locus of normal functions is well understood—it is an analytic variety, and its local description is well described, see [13, 27], Chapter 17—we have very few results on the “zero locus” of the étale Abel–Jacobi map. We feel that it would be very interesting to prove an étale counterpart of the results of [6] and [23].

In this paper, if X is any variety, the cohomology groups of X, whether singular or étale, will always be considered modulo torsion, so as to avoid cumbersome notations. The same convention goes for higher direct images.

2 Preliminary Results on Abel–Jacobi Maps

Let X be a smooth projective variety over a field k of characteristic zero, $\bar{k}$ an algebraic closure of k and $G_k = \mathrm{Gal}(\bar{k}/k)$ the absolute Galois group of k. In his paper [3], Beilinson constructs a conjectural filtration $F^\bullet$ on the Chow groups $CH^i(X) \otimes \mathbb{Q}$ of X with rational coefficients. It is obtained in the following way. Let $\mathrm{MM}(k)$ be the abelian category of mixed motives over k. There should exist a spectral sequence, Beilinson's spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathrm{MM}(k)}(\mathbf{1}, \mathfrak{h}^q(X)(i)) \Rightarrow \mathrm{Hom}_{D^b(\mathrm{MM}(k))}(\mathbf{1}, \mathfrak{h}(X)(i)[p+q])$$

where $\mathfrak{h}^q(X)$ denotes the weight q part of the image $\mathfrak{h}(X)$ of X in the category of pure motives. For $p+q = 2i$, the abutment of this spectral sequence should be canonically isomorphic with $CH^i(X)$, hence the filtration F. For weight reasons, a theorem of Deligne in [9] would imply that this spectral sequence degenerates at $E_2 \otimes \mathbb{Q}$. We get the formula

$$Gr^\nu_F CH^i(X)_\mathbb{Q} = \mathrm{Ext}^\nu_{\mathrm{MM}(k)}(\mathbf{1}, \mathfrak{h}^{2i-\nu}(X)(i)) \otimes \mathbb{Q}.$$

The existence of such a filtration is also a conjecture of Bloch and Murre.

2.1 Étale cohomology

The previous construction should have its reflection in the various usual cohomology theories. Let us first consider étale cohomology. In the paper [14], Jannsen constructs continuous étale cohomology groups with value in the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$ for varieties over a field. In his paper, Jannsen actually deals with $\mathbb{Z}_l$-coefficients. We define the cohomology groups with value in $\hat{\mathbb{Z}}$ as the product over all l of the cohomology groups with value in $\mathbb{Z}_l$. This is indeed a $\hat{\mathbb{Z}}$-module, which satisfies, since we work in characteristic zero, all the expected properties of an étale cohomology group. It would be easy to give a direct definition following [14].

Those continuous étale cohomology groups enjoy good functoriality properties and they are equal to the usual étale cohomology groups in case the base field is algebraically closed. In particular, there is a Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(G_k, H^q(X_{\overline{k}}, \hat{\mathbb{Z}}(i))) \Rightarrow H^{p+q}(X, \hat{\mathbb{Z}}(i)) \tag{1}$$

as well as a cycle map

$$\mathrm{cl}^X : CH^i(X) \to H^{2i}(X, \hat{\mathbb{Z}}(i)).$$

Those are compatible with the usual cycle map $\mathrm{cl}^{X_{\overline{k}}}$ to $H^{2i}(X_{\overline{k}}, \hat{\mathbb{Z}}(i))$. Let $CH^i(X)_{\mathrm{hom}}$ be the kernel of $\mathrm{cl}^{X_{\overline{k}}}$, that is, the part of the Chow group consisting of those cycles which are homologically equivalent to zero.

Definition 5. The map

$$aj_{\text{ét}} : CH^i(X)_{\mathrm{hom}} \to H^1(G_k, H^{2i-1}(X_{\overline{k}}, \hat{\mathbb{Z}}(i)))$$

induced by the spectral sequence (1) is called the étale Abel–Jacobi map. □

This map is expected to be the image by some realization functor of the analogous map coming from Beilinson's spectral sequence. As an evidence for this, we can cite Jannsen's result in [16], Lemma 2.7, stating that in the case k is finitely generated, and for any "reasonable" category of mixed motives, the filtrations on $CH^i(X) \otimes \mathbb{Q}$ induced by Beilinson's spectral sequence and by the Hochschild–Serre spectral sequence (1) coincide if cl^X is injective—which is also a conjecture of Bloch and Beilinson. More specifically, if any Bloch–Beilinson filtration (see [4, 16, 17]) exists on $CH^i(X) \otimes \mathbb{Q}$ and cl^X is injective, then it has to be the filtration induced by (1).

Our definition of the étale Abel–Jacobi map may seem to be highly dependent on the base field k, which is not convenient since we expect that for an algebraic cycle Z homologically equivalent to zero, $aj_{\text{ét}}(Z)$ should reflect geometric properties of Z related to the image of $Z_{\mathbb{C}}$ by the Abel–Jacobi map. The following proposition shows that the vanishing of $aj_{\text{ét}}(Z)$ does not actually depend on the base field. This would be false had we considered in our definition the torsion part of the cohomology of X. The fact that we want the following result to hold is the reason why we have to ignore this torsion part, which is related to arithmetic properties of algebraic cycles, as opposed to their geometric properties. It has been attributed to Nori and appears in a very similar form in [24], Lemma 1.4.

Proposition 6. Let X be a smooth projective variety over a finitely generated field k, and let $Z \in CH^i_{\text{hom}}(X)$. Let K be a field which is finitely generated over k. Then, $aj_{\text{ét}}(Z_K) = 0$ if and only if $aj_{\text{ét}}(Z) = 0$. □

Proof. We can assume that K is Galois over k. We have an exact sequence of groups

$$1 \to G_K \to G_k \to \text{Gal}(K/k) \to 1,$$

hence the following exact sequence coming from the Hochschild–Serre spectral sequence

$$0 \to H^1(\text{Gal}(K/k), V^{G_K}) \to H^1(G_k, V) \to H^1(G_K, V)^{\text{Gal}(K/k)}, \tag{2}$$

with $V = H^{2i-1}(\overline{X}, \hat{\mathbb{Z}}(i))$. The definition of the étale Abel–Jacobi map from a Leray spectral sequence shows that $aj_{\text{ét}}(Z_K)$ is obtained from $aj_{\text{ét}}(Z)$ by the last map in (2). On the other hand, it is a consequence of the Weil conjectures that V^{G_K} is zero (recall V is torsion-free), which implies that the last map in (2) is an injection. ■

The next result is due to Jannsen in [15], being inspired by results from Carlson and Beilinson we will recall later, and gives a geometric meaning to the étale Abel–Jacobi map. We recall it shortly, as it is crucial to the results of our paper.

Start with X as before, and let Z be an algebraic cycle of pure codimension i in X. Let $|Z|$ be the support of Z, and U be the complement of $|Z|$ in X. By purity, we have an exact sequence of G_k-modules

$$0 \to H^{2i-1}(X_{\overline{k}}, \hat{\mathbb{Z}}(i)) \to H^{2i-1}(U_{\overline{k}}, \hat{\mathbb{Z}}(i)) \to H^0(|Z|_{\overline{k}}, \hat{\mathbb{Z}}) \to 0$$

and the class of Z gives a map $\hat{\mathbb{Z}} \to H^0(|Z|_{\overline{k}}, \hat{\mathbb{Z}})$. The pullback of the previous exact sequence by this map is an exact sequence of G_k-modules

$$0 \to H^{2i-1}(X_{\overline{k}}, \hat{\mathbb{Z}}(i)) \to H_{\text{ét}} \to \hat{\mathbb{Z}} \to 0. \tag{3}$$

This extension gives a class $\alpha(Z) \in H^1(G_k, H^{2i-1}(X_{\overline{k}}, \hat{\mathbb{Z}}(i)))$.

Proposition 7. We have $\alpha(Z) = aj_{\text{ét}}(Z)$. □

Let us note that this proposition immediately carries out to the variational setting for flat families of algebraic cycles. In this case, for all prime numbers l, we get an extension of locally constant $\mathbb{Z}_l$-sheaves over the base scheme which on every fiber is canonically isomorphic to the l-adic part of the extension (3). In this situation, it would be more convenient to refer to the collection of $\mathbb{Z}_l$-sheaves that we get as a $\hat{\mathbb{Z}}$-sheaf. One could indeed use [10] to get a suitable category of $\hat{\mathbb{Z}}$-sheaves. We won't use this terminology.

2.2 Hodge theory

The Hodge-theoretic picture is different. Indeed, the category of mixed Hodge structures has no higher extension groups as shown by Beilinson, so we cannot expect to construct directly a similar filtration on Chow groups through this means. The use of Leray spectral sequences in this setting has been studied by Nori, Saito, Green–Griffiths and others, and can be considered well understood. Even though we cannot expect to construct a Bloch–Beilinson filtration on Chow groups using Hodge theory, at least in a naive way, we can construct a two-term filtration using Deligne cohomology. We use this approach to make the similarity with the previous discussion more obvious, but in this paper we simply use Griffiths' Abel–Jacobi map, which was defined in [12]. Griffiths' work on normal functions is of course fundamental to our results.

Let us assume for this paragraph that the base field is $\mathbb{C}$. Recall that we have Deligne cohomology groups $H^i_{\mathcal{D}}(X, \mathbb{Z}(j))$. Those are the "absolute" version of Hodge cohomology groups in the same way that continuous étale cohomology is the absolute version of étale cohomology over an algebraic closure of the base field. There is an exact sequence

$$0 \to J^i(X) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)) \to H^{2i}(X, \mathbb{Z})(i) \to 0$$

as well as a cycle map

$$\mathrm{cl}^X : CH^i(X) \to H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)).$$

Those are compatible with the usual cycle map to $H^{2i}(X, \mathbb{Z})(i)$.

The cohomology group $H^{2i}(X, \mathbb{Z})(i)$ is, up to a Tate twist the usual singular cohomology of the complex manifold X with its canonical Hodge structure, and $J^i(X)$ is Griffiths' ith intermediate Jacobian, which is defined the following way.

Integration of differential forms gives a map from the singular homology group $H_{2n-2i+1}(X, \mathbb{Z})$ to $F^{n-i+1}H^{2n-2i+1}(X, \mathbb{C})^*$, n being the dimension of X and F the Hodge filtration. The quotient group

$$F^{n-i+1}H^{2n-2i+1}(X,\mathbb{C})^*/H_{2n-2i+1}(X,\mathbb{Z})$$

is a complex torus, canonically isomorphic to

$$J^i(X) := \frac{H^{2i-1}(X,\mathbb{C})}{F^iH^{2i-1}(X,\mathbb{C}) \oplus H^{2i-1}(X,\mathbb{Z})}.$$

There is a canonical isomorphism of abelian groups between $J^i(X)$ and the extension group $\mathrm{Ext}^1_{MHS}(\mathbb{Z}, H^{2i-1}(X,\mathbb{Z})(i))$ in the category of mixed Hodge structures, as noted by Carlson in [7]. One can also refer to [27], p. 463.

Definition 8. The map

$$aj : CH^i(X)_{\mathrm{hom}} \to J^i(X)$$

induced from the exact sequence above defining Deligne cohomology is called the (transcendental) Abel–Jacobi map, or Griffiths' Abel–Jacobi map. □

In the light of the isomorphism above, Beilinson has shown in [2] (see also [7] for the case of divisors on curves) the following way of computing the Abel–Jacobi map, which is very similar to its étale counterpart—and has been proved earlier. Let Z be an algebraic cycle of pure codimension i in X. Let $|Z|$ be the support of Z, and U be the complement of $|Z|$ in X. We have an exact sequence of mixed Hodge structures

$$0 \to H^{2i-1}(X,\mathbb{Z}(i)) \to H^{2i-1}(U,\mathbb{Z}(i)) \to H^0(|Z|,\mathbb{Z}) \to 0$$

and the class of Z gives a map $\mathbb{Z} \to H^0(|Z|,\mathbb{Z})$. The pullback of the previous exact sequence by this map is an exact sequence of mixed Hodge structures

$$0 \to H^{2i-1}(X,\mathbb{Z}(i)) \to H \to \mathbb{Z} \to 0. \tag{4}$$

This extension gives a class $\alpha(Z) \in \mathrm{Ext}^1_{MHS}(\mathbb{Z}, H^{2i-1}(X,\mathbb{Z})(i)) = J^i(X)$. One can find in [2] the following.

Proposition 9. We have $\alpha(Z) = aj(Z)$. □

The vanishing of the Abel–Jacobi map has a simple interpretation in these terms. Indeed, recall that if H is a mixed Hodge structure (of weight 0) with weight filtration $W_\bullet$ and Hodge filtration $F^\bullet$, a Hodge class of weight k in H is an element of $W_{2k}H \cap F^k H_{\mathbb{C}}$. In this terminology, it is straightforward to see that the extension (4) splits if and only if there exists a Hodge class (which has to be of weight 0) in H mapping to 1 in $\mathbb{Z}$.

Again, in case of a flat family of algebraic cycles which are homologous to zero in the fibers, we get an extension of variations of mixed Hodge structures corresponding point by point to (4). It satisfies Griffiths' transversality, see [22], Lemma 1.3. We also get the Jacobian fibration $J^i(X/S)$, and a section ν_Z of it is obtained by applying the relative Abel–Jacobi map. The preceding remark shows that the zero locus of ν_Z is a Hodge locus for the variation of mixed Hodge structures above.

Definition 10. The section of $J^i(X/S)$ attached to the cycle Z is the normal function ν_Z attached to Z. □

Normal functions have been extensively studied, see [12, 22], etc. See also [27], Chapter 19. It is a fundamental fact that normal functions are holomorphic. In particular, their zero locus is analytic. In this paper, our results will take into account this analytic structure. Indeed, while this zero locus is non-reduced in general, the results we get are valid without passing first to the reduced analytic subspace, as opposed for instance to [28], Theorem 0.6 (2).

It will be very important to us, though straightforward, that if we start with a family over a finitely generated base field, the extension of local systems coming from the étale Abel–Jacobi map and from the transcendental one, after base change to $\mathbb{C}$, are compatible in the obvious way, as Artin's comparison theorem between étale and singular cohomology readily shows.

3 Proof of the Theorems

3.1 Zero loci for large monodromy groups

This section is devoted to showing how assuming the family $X \to S$ has a large monodromy that can help study the vanishing locus of the Abel–Jacobi map and deduce Theorems 2 and 3. This kind of argument is very much inspired by [28], where it appears in the pure case as a criterion for Hodge classes to be absolute.

The main idea is the following : start with an extension $0 \to H' \to H \to \mathbb{Z} \to 0$ of variations of mixed Hodge structures on a quasi-projective variety S. If the monodromy representation on H' has no non-trivial invariant part, then any global section of H is in $F^0 H$, the filtration F being the Hodge filtration. This remark allows us to reduce the question of the splitting of the previous exact sequence to a geometric question, and allows for comparison theorems.

In the setting of normal functions, this is equivalent to the following, which has been observed a long time ago. Under this assumption, the normal function with value in the ith intermediate Jacobian is determined by its Hodge class, see [13]. This has been used for instance by Manin in the proof of Mordell's conjecture over function fields in [20]. Our argument does not proceed along these lines for convenience, but part of it could easily be translated using Griffiths' results and the Leray spectral sequence.

Recall the notations of the introduction. We have a smooth projective family over a quasi-projective base $\pi : X \to S$, together with a flat family of algebraic cycles $Z \to X$ of pure codimension i. Everything is defined over a finitely generated field k of characteristic zero. As far as our results are concerned, and taking into account Proposition 6, standard spreading techniques allow us to assume without loss of generality that k is a number field. Suppose that for any geometric point s of S, Z_s is homologically equivalent to zero in X_s. Fix an embedding of k in $\mathbb{C}$. We get the normal function ν_Z on $S(\mathbb{C})$, which is a holomorphic section of the bundle of intermediate Jacobians over $S(\mathbb{C})$.

For every prime number l, we have the following exact sequence of local systems of $\mathbb{Z}_l$-sheaves on S, canonically attached to the family Z of algebraic cycles

$$0 \to R^{2i-1}\pi_*\mathbb{Z}_l(i) \to H_l \to \mathbb{Z}_l \to 0. \tag{5}$$

Since $\mathbb{Z}_l$ is flat over $\mathbb{Z}$, the pullback to $S_{\mathbb{C}}$ of this sequence of sheaves is the tensor product by $\mathbb{Z}_l$ of the exact sequence

$$0 \to R^{2i-1}(\pi_{\mathbb{C}})_*\mathbb{Z}(i) \to H \to \mathbb{Z} \to 0 \tag{6}$$

of local systems used to compute Griffiths' Abel–Jacobi map. Those local systems are underlying to variations of mixed Hodge structures. Saying that ν_Z vanishes on $S_{\mathbb{C}}$ is equivalent to saying that S is equal to the locus of Hodge classes of H which map to 1 in $\mathbb{Z}$.

We will deduce our theorems from the following result.

Theorem 11. In the above setting, assume that the locally constant sheaf $R^{2i-1}(\pi_{\mathbb{C}})_*\mathbb{C}$ has no non-zero global section over $S_{\mathbb{C}}$. Then, the following are equivalent :

(i) The normal function ν associated to $Z_{\mathbb{C}}$ vanishes on $S_{\mathbb{C}}$.
(ii) For every closed point s of S with value in a finitely generated field K, the image of Z_s by the étale Abel–Jacobi map vanishes in the group $H^1(G_K, H^{2i-1}(\overline{X_s}, \widehat{\mathbb{Z}}(i)))$.
(iii) For any automorphism σ of $\mathbb{C}$, the normal function ν^{σ} associated to $Z^{\sigma} = Z_{\mathbb{C}} \times_{\sigma} \mathrm{Spec}(\mathbb{C})$ vanishes on S^{σ}. □

Proof of $(i) \Rightarrow (ii)$**.** Fix a prime number l. Fix a point s of S with value in a finitely generated field L, and let $\bar{s}$ be a complex point of S lying over s. Under our hypothesis, we have an injective map $(H_{l,\bar{s}})^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\bar{s})} \to \mathbb{Z}_l$. This is actually an isomorphism. Indeed, Baire's theorem applied to the locus of Hodge classes of H in $S_{\mathbb{C}}$ mapping to 1 in $\mathbb{Z}$ shows that in order for $S_{\mathbb{C}}$ to be equal to this locus, which is a countable union of analytic subvarieties, there has to be a non-zero global section of H which is a Hodge class in every fiber of H—and maps to 1 in $\mathbb{Z}$. The image in $H_{\bar{s}} \otimes \mathbb{Z}_l = H_{l,\bar{s}}$ of this section lies in $(H_{l,\bar{s}})^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\bar{s})}$ and maps to 1 in $\mathbb{Z}_l$.

Now let G_L be the absolute Galois group of L. We have an exact sequence $1 \to \pi_1^{\text{ét}}(S_{\mathbb{C}}, \bar{s}) \to \pi_1^{\text{ét}}(S \times \mathrm{Spec}(L), \bar{s}) \to G_L \to 1$, together with a splitting of this exact sequence. The full algebraic fundamental group acts on $H_{l,\bar{s}}$, and the map $H_{l,\bar{s}} \to \mathbb{Z}_l$ is equivariant with respect to the trivial action on $\mathbb{Z}_l$. It follows that the group G_L acts trivially on $(H_{l,\bar{s}})^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\bar{s})} \xrightarrow{\sim} \mathbb{Z}_l$. This being true for any l, it proves that the étale Abel–Jacobi invariant of Z_s is zero. ■

Proof of $(ii) \Rightarrow (iii)$**.** It is enough to prove the case where σ is the identity. Fix a prime number l. Let $\bar{s}$ be a geometric point of S. Using the same notation as in the previous proof, the algebraic fundamental group $\pi_1^{\text{ét}}(S, \bar{s})$ acts on $H_{l,\bar{s}}$. For any point s' of S with value in a field L, the absolute Galois group G_L of L maps into $\pi_1^{\text{ét}}(S, \bar{s})$. According to a lemma by Terasoma appearing in [26], Theorem 2, there exists such an L-valued point s', with L a number field, such that G_L and $\pi_1^{\text{ét}}(S, \bar{s})$ have the same image in the linear group $\mathrm{GL}(H_{l,\bar{s}})$. Since by assumption G_L fixes an element mapping to $1 \in \mathbb{Z}_l$, we get an element of $H_{l,\bar{s}}$, mapping to $1 \in \mathbb{Z}_l$, which is fixed by the whole monodromy group. In other words, the exact sequence (5) splits over S. This being true for any prime number l, the exact sequence (5) splits over S.

This means that the local system H_l on S has a non-zero global section for any prime number l. As a consequence, $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q}) \neq 0$, and as before we get an isomorphism $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q}) \simeq \mathbb{Q}$ as local systems, the map being induced by the morphism

$H \otimes \mathbb{Q} \to \mathbb{Q}$ of variations of mixed Hodge structures over $S_{\mathbb{C}}$. It is a result of Steenbrink and Zucker in [25], Theorem 4.1, generalizing Deligne's global invariant cycles theorem which is a fundamental tool of [28], that the $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ carries a canonical mixed Hodge structure which makes it a constant subvariation of mixed Hodge structures of H. The isomorphism of $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ with $\mathbb{Q}$ is a morphism of mixed Hodge structures, which proves $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ consists of Hodge classes.

Remark. The result of Steenbrink and Zucker is stated for variations of mixed Hodge structures of geometric origin—which is our case—over a one-dimensional base. The fact that $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ carries a canonical Hodge structure for S of any dimension is straightforward by restricting to a curve which is an intersection of hyperplane sections and using Lefschetz' hyperplane theorem.

This shows that the exact sequence (6) of variations of mixed Hodge structures splits rationally. We want to prove that it splits over $\mathbb{Z}$. We just proved that a splitting of the subjacent extension of local systems over $S_{\mathbb{C}}$ gives a splitting of (6), so we just have to prove that the exact sequence of local systems splits.

Let $\alpha \in H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ be the class mapping to $1 \in \mathbb{Q}$. For any prime number l, the image of $\alpha \in H^0(S_{\mathbb{C}}, H \otimes \mathbb{Q})$ is the only class mapping to $1 \in \mathbb{Q}_l$, which shows that this image belongs to $H^0(S_{\mathbb{C}}, H \otimes \mathbb{Z}_l)$, since the exact sequence (5) is split over $S_{\mathbb{C}}$. The only way for this to be true is that α belongs to $H^0(S_{\mathbb{C}}, H)$, which precisely means that the exact sequence we are considering splits. ■

Proof of $(iii) \Rightarrow (i)$**.** This is obvious. ■

Let us now use the notations of the introduction. The equivalence $(i) \Leftrightarrow (ii)$ we just proved readily implies Theorem 2 by restriction to the component T of the zero locus of ν_Z, which is assumed to be algebraic, under the assumption that the local system $R^{2i-1}(\pi_{\mathbb{C}})_*\mathbb{C}$ has no non-zero global section.

Proof of Theorem 3. Let $\bar{k}$ be an algebraic closure of k, and let T' be the Zariski closure of $T(\mathbb{C})$ in the k-scheme S. The previous theorem shows that the orbit of $T(\mathbb{C})$ in S under the action of the Galois group $\mathrm{Aut}(\mathbb{C}/k)$ is included in the zero locus of ν_Z. Since this orbit is dense in $T'(\mathbb{C})$ for the usual topology, it follows that ν_Z vanishes on $T'(\mathbb{C})$. By assumption, T is an irreducible component of the zero locus of ν_Z. It follows that T is an irreducible component of the algebraic variety T' defined over k, which proves that T is defined over a finite extension of k.

This shows that for any automorphism σ of $\mathbb{C}$ fixing k, the set $\sigma(T(\mathbb{C}))$ is included in the zero locus of ν_Z. Now consider the subscheme T^σ of S, which has $\sigma(T(\mathbb{C}))$ as set of complex points. We just showed that its reduced subscheme is included in the zero locus of ν_Z as an analytic subvariety, and it is irreducible. Let V be the irreducible component of the zero locus of ν_Z containing $\sigma(T(\mathbb{C}))$. We want to show $V = T^\sigma$ as analytic varieties. Let n be a non-negative integer. We can consider the artinian local rings $\mathcal{O}_{V,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$ and $\mathcal{O}_{T^\sigma,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$, $\mathfrak{m}^n_{\sigma(t)}$ denoting indifferently the maximal ideals of both local rings.

The rings $\mathcal{O}_{T^\sigma,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$ and $\mathcal{O}_{T,t}/\mathfrak{m}^n_t$ are canonically isomorphic, because the schemes T and T^σ are. On the other hand, we can explicitly describe $\mathcal{O}_{T,t}/\mathfrak{m}^n_t$ and $\mathcal{O}_{V,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$, as loci of Hodge classes, using the Gauss–Manin connection on $\mathcal{H}$ and Griffiths' transversality. This is explained in [27] in the case of pure Hodge structures, and explicitly stated for $n = 1$, see Lemma 17.16. Our case follows *mutatis mutandis*. As a consequence, since the Gauss–Manin connection is algebraic, see [18], we have a canonical isomorphism between $\mathcal{O}_{T,t}/\mathfrak{m}^n_t$ and $\mathcal{O}_{V,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$.

This discussion shows that $\mathcal{O}_{T^\sigma,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$ and $\mathcal{O}_{V,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$ are isomorphic as subrings of $\mathcal{O}_{S,\sigma(t)}/\mathfrak{m}^n_{\sigma(t)}$. Since this holds for all n, and since the reduced subscheme of T^σ is included in V, we get an equality $V = T^\sigma$, which shows that T^σ is an irreducible component of the zero locus of ν_Z. ■

3.2 Application

As in [28], there are many situations where one can easily check the conditions of Theorems 2 and 3. Let us give one example.

Theorem 12. Let $\pi : X \to S$ be a smooth projective family of complex Calabi–Yau threefolds over a quasi-projective base such that the induced map from S to the corresponding moduli space is finite, and let $Z \hookrightarrow X$ be a flat family of curves in X which are homologous to 0 in the fibers of π. Assume everything is defined over a finitely generated field k. Let ν be the associated normal function.

(i) Let T be an irreducible component of the zero locus of ν which is algebraic. Assume that T is of positive dimension and that for a general complex point t of T, the intermediate jacobian $J^2(X_t)$ has no abelian factor. Then, T is defined over a finite extension of k, all its conjugates are irreducible components of the zero locus of ν, and for every closed point t of T with value in a finitely generated field, the étale Abel–Jacobi invariant of Z_t is zero.

(ii) Let T be a subvariety of S of positive dimension defined over a finitely generated field. Assume that for a general complex point t of T, the intermediate jacobian $J^2(X_t)$ has no abelian factor and that for any point t of T with value in a finitely generated field, the étale Abel–Jacobi invariant of Z_t is zero. Then, ν vanishes on T. □

Proof. In order to apply our preceding results, we only have to check that in both situations above, the local system $R^3\pi_*\mathbb{Z}$ has no global section over $T(\mathbb{C})$. First of all, since the restriction of π to T is a non-trivial family of Calabi–Yau threefolds, the Hodge structure on $H^0(T(\mathbb{C}), R^3\pi_*\mathbb{Z})$ is of type $\{(2,1),(1,2)\}$. Indeed, the infinitesimal Torelli theorem for Calabi–Yau varieties, see [27], Theorem 10.27, shows that the fixed part of $R^3\pi_*\mathbb{Z}$ cannot have a part of type $(3,0)$. Now this proves that the invariant part of $R^3\pi_*\mathbb{Z}$ corresponds to a constant abelian subvariety of the Jacobian fibration $J^2(X_T/T)$, which has to be zero by assumption. This shows that the local system $R^3\pi_*\mathbb{Z}$ has no global section. ■

3.3 Hodge classes of normal functions and their étale counterpart

Let $\mathcal{H}$ be a variation of pure Hodge structures of weight -1 over an irreducible complex variety S, and let ν be a normal function on S. The Hodge class of ν is defined the following way. Let $H_\mathbb{Z}$ be the integral structure of $\mathcal{H}$. We have an exact sequence of sheaves on S

$$0 \to H_\mathbb{Z} \to \mathcal{H}/F^0\mathcal{H} \to \mathcal{J}(\mathcal{H}) \to 0,$$

$\mathcal{J}(\mathcal{H})$ being the sheaf of holomorphic sections of the Jacobian fibration. This gives a map $H^0(S, J(\mathcal{H})) \to H^1(S, H_\mathbb{Z})$. The normal function ν is a holomorphic section of $\mathcal{J}(\mathcal{H})$. Its image in $H^1(S, H_\mathbb{Z})$ is called its Hodge class and is denoted by $[\nu]$.

The homological interpretation of intermediate Jacobians suggests another way of defining Hodge classes of normal functions. Indeed, a normal function ν defines an extension of variations of mixed Hodge structures over S

$$0 \to H \to H' \to \mathbb{Z} \to 0.$$

The long exact sequence of sheaf cohomology gives a map $\delta : H^0(S, \mathbb{Z}) \to H^1(S, \mathcal{H})$. The following is straightforward, but we have been unable to find a reference.

Proposition 13. We have $[\nu] = \delta(1)$. □

Proof. Let us start by briefly recalling the explicit form given in [7] of the isomorphism $\phi : \mathrm{Ext}^1_{MHS}(\mathbb{Z}, H) \simeq J(H)$ when S is a point. Choose an isomorphism of abelian groups

$$H'_{\mathbb{Z}} \simeq H_{\mathbb{Z}} \oplus \mathbb{Z}.$$

There exists an element $\alpha \in H_{\mathbb{C}}$ such that $\alpha \oplus 1 \in F^0 H'_{\mathbb{C}}$. The class of α in

$$\frac{H_{\mathbb{C}}}{H_{\mathbb{Z}} \oplus F^0 H_{\mathbb{C}}} = J(H)$$

is well defined and is the image of the extension

$$0 \to H \to H' \to \mathbb{Z} \to 0$$

by ϕ.

Now let us work over a general complex quasi-projective base S as before. Let us choose a covering of $S(\mathbb{C})$ by open subsets U_i (for the usual topology) such that the exact sequence

$$0 \to H_{\mathbb{Z}} \to H'_{\mathbb{Z}} \to \mathbb{Z} \to 0$$

splits over U_i. Splittings correspond to sections $\sigma_i \in H^0(U_i, H'_{\mathbb{Z}})$ mapping to 1 in $\mathbb{Z}$. The cohomology class $\delta(1)$ is represented by the cocycle $\sigma_i - \sigma_j$.

For each i and each $s \in U_i$, let $\tau_i(s)$ be the image in $H_{s,\mathbb{C}}/F^0 H_{s,\mathbb{C}}$ of an element $\alpha_s \in H_{s,\mathbb{C}}$ such that $\sigma_i(s) + \alpha_s \in F^0 H'_{s,\mathbb{C}}$. The Hodge class of the normal function ν is represented by the cocycle $\tau_i - \tau_j \in H_{\mathbb{Z}}/(H_{\mathbb{Z}} \cap F^0 H_{\mathbb{Z}}) = H_{\mathbb{Z}}$. Since $\tau_i - \tau_j = \sigma_i - \sigma_j$ through this identification, this concludes the proof. ■

Let us now investigate the étale side. Let S be a quasi-projective variety over a finitely generated subfield k of $\mathbb{C}$, with extensions ν_l

$$0 \to H_l \to H'_l \to \mathbb{Z}_l \to 0$$

of $\mathbb{Z}_l$-sheaves over S for all prime numbers l. We get an extension class in $H^1(S_{\mathbb{C}}, H_l)$ by pulling back to $S_{\mathbb{C}}$, which we will denote by $[\nu_l]$. Let us denote by

$$[\nu_{\text{ét}}] \in H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\hat{\mathbb{Z}}(i)) := \prod_l H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}_l(i))$$

the class with components $[\nu_l]$.

The importance of the class $[\nu_{\text{ét}}]$ relies on the following result, which shows that it controls the extensions ν_l in the same way that the Hodge class of a normal function controls the normal function itself. It is indeed an analog of Griffiths' result in [13] which proves that a normal function with zero Hodge class is constant in the fixed part of the intermediate Jacobian.

Theorem 14. In the previous setting, assume that there exists a smooth projective family $\pi : X \to S$ such that $H_l = R^{2i-1}\pi_*\mathbb{Z}_l(i)$, and that $[\nu_{\text{ét}}] = 0$. Let s be any closed point of S, and let $\bar{s}$ be a geometric point over s. Then for every prime number l, the extension

$$0 \to H_l \to H_l' \to \mathbb{Z}_l \to 0 \tag{7}$$

splits over S if and only if the extension

$$0 \to H_{l,\bar{s}} \to H_{l,\bar{s}}' \to \mathbb{Z}_l \to 0$$

of $G_{k(s)}$-modules splits, where $G_{k(s)}$ is the absolute Galois group of the residue field of s. □

Proof. Using the argument of Proposition 6, we can make a finite extension of the base field and assume that s is a k-point of S. Assume that the extension

$$0 \to H_{l,s} \to H_{l,s}' \to \mathbb{Z}_l \to 0$$

of $G_{k(s)}$-modules splits. We need to prove that the exact sequence (7) splits over S for any prime number l. Fix a prime number l. We have an exact sequence

$$1 \to \pi_1^{\text{ét}}(S_{\mathbb{C}}, \bar{s}) \to \pi_1^{\text{ét}}(S, \bar{s}) \to G_k \to 1.$$

The last arrow admits a section σ coming from the rational point s.

The vector spaces $H_{l,\overline{s}}$ and $H'_{l,\overline{s}}$ are $\pi_1^{\text{ét}}(S,\overline{s})$-modules, and the extension (7) corresponds to the extension

$$0 \to H_{l,\overline{s}} \to H'_{l,\overline{s}} \to \mathbb{Z}_l \to 0$$

of $\pi_1^{\text{ét}}(S,\overline{s})$-modules. Now by assumption, the following sequence is exact.

$$0 \to (H_{l,\overline{s}})^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\overline{s})} \to (H'_{l,\overline{s}})^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\overline{s})} \to \mathbb{Z}_l \to 0. \tag{8}$$

Indeed, the vanishing of $[\nu_l]$ implies that there exists a global section of $H_{l,\overline{s}}$ over $S_{\mathbb{C}}$ mapping to 1 in $\mathbb{Z}_l$, which implies the surjectivity of the last arrow.

The Galois group G_k acts on $(H'_{l,\overline{s}})^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\overline{s})}$, either through σ or through the previous exact sequence—those are the same actions. We want to prove that G_k fixes an element mapping to 1 in $\mathbb{Z}_l$. Since the exact sequence of G_k-modules

$$0 \to H_{l,\overline{s}} \to H'_{l,\overline{s}} \to \mathbb{Z}_l \to 0$$

is split by assumption, G_k acting through σ, there exists $h' \in H'_{l,\overline{s}}$, mapping to 1 in $\mathbb{Z}_l$, such that $g(h') = h'$ for any $g \in G_k$.

On the other hand, let $\mathfrak{p}$ be a finite place of k that does not divide l and such that X_s has good reduction at $\mathfrak{p}$. Fix a Frobenius element $F_{\mathfrak{p}}$ in a decomposition group of $\mathfrak{p}$. Since $F_{\mathfrak{p}}$ acts trivially on $\mathbb{Z}_l$ and has weight -1 on $(H_{l,\overline{s}})^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\overline{s})} = (H^{2i-1}(X_{\overline{s}}, \mathbb{Z}_l(i)))^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\overline{s})}$ by the Weil conjectures, there exists $h'' \in (H'_{l,\overline{s}})^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\overline{s})} \otimes \mathbb{Q}$, mapping to 1 in $\mathbb{Z}_l$, such that $F_{\mathfrak{p}}(h'') = h''$. Since $(H_{l,\overline{s}})^{F_{\mathfrak{p}}} = 0$ by the Weil conjectures again, we have $h'' = h'$, which shows that h' lies in $(H'_{l,\overline{t}})^{\pi_1^{\text{ét}}(S_{\mathbb{C}},\overline{s})}$. This proves that the exact sequence (8) splits, which concludes the proof. ■

In case we start with a smooth projective family $\pi : X \to S$, together with a flat family of algebraic cycles $Z \hookrightarrow X$ of algebraic cycles of codimension i which are homologically equivalent to zero on the fibers of π, we get an extension of variations of mixed Hodge structures over $S_{\mathbb{C}}$ corresponding to ν_Z, and an extension ν_l of $\mathbb{Z}_l$-sheaves over S induced by the étale Abel–Jacobi map for every prime number l, with $H_l = R^{2i-1}\pi_*\mathbb{Z}_l(i)$. The pullback of the latter to $S_{\mathbb{C}}$ is the extension of local systems induced by ν. As a consequence of Artin's comparison theorem between étale and singular cohomology in [1], Exposé XI, we get the following "easy" part of the comparison theorems between Abel–Jacobi maps.

Proposition 15. The class $[\nu_{\text{ét}}]$ is the image in $H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\hat{\mathbb{Z}}(i))$ of the Hodge class $[\nu] \in H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}(i))$. As a consequence, $[\nu] = 0$ if and only if $[\nu_{\text{ét}}] = 0$. □

Proof. The first statement is a direct consequence of Proposition 13 and Artin's theorem which identifies the cohomology group $H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}_l(i))$ with $H^1(S_{\mathbb{C}}, R^{2i-1}\pi_* Z(i)) \otimes \mathbb{Z}_l$ in a functorial way. This proves that the vanishing of $[\nu]$ implies the vanishing of $[\nu_{\text{ét}}]$. Now if $[\nu_l] = 0$, then the class $[\nu] \in H^1(S_{\mathbb{C}}, R^{2i-1}\pi_* Z(i))$ vanishes in $H^1(S_{\mathbb{C}}, R^{2i-1}\pi_* Z(i)) \otimes \mathbb{Z}_l$, which proves that there exists an integer α prime to l such that $\alpha[\nu] = 0$. If $[\nu_{\text{ét}}] = 0$, this is true for all l, which implies that $[\nu] = 0$. ■

Remark 1. There are of course different ways of computing the value of $[\nu_{\text{ét}}]$. Indeed, Leray spectral sequences still exist in continuous étale cohomology, working in the category of l-adic sheaves as defined by Ekedahl in [10]. The cycle class of Z induces from the Leray spectral sequence attached to the morphism π an element in $H^1_{\text{ét}}(S, R^{2i-1}\pi_*\mathbb{Z}_l(i))$. This cohomology class maps to an element in $H^1_{\text{ét}}(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}_l(i)) = H^1(S_{\mathbb{C}}, R^{2i-1}\pi_*\mathbb{Z}_l(i))$. Now this class is equal to the l-adic component of $[\nu_{\text{ét}}]$. This can either be proved directly or using Proposition 15 and applying the corresponding well-known result for Griffiths' Abel–Jacobi map (see [27], Lemma 20.20). □

Remark 2. What we just showed implies the fact that for cycles *algebraically equivalent to zero*, the vanishing of Griffiths' Abel–Jacobi invariant is equivalent to the vanishing of the étale Abel–Jacobi invariant. This result is well known, and for zero cycles it is an easy consequence of the result of Raskind in the appendix of [21], but it does not appear to have been stated explicitly in the literature. We can easily reduce it to the case of divisors on curves by the following functoriality argument.

We work over a finitely generated subfield k of $\mathbb{C}$. Let $Z \hookrightarrow X$ be a flat family of cycles of codimension i over a smooth curve C such that the fiber of Z over a geometric point 0 of C is zero. Changing base to $\mathbb{C}$, the normal function $\nu_Z : C \to J^i(X)$ takes value in the algebraic part $J^i_{\text{alg}}(X)$ of the intermediate Jacobian of X. From the Kummer exact sequence on $J^i_{\text{alg}}(X)$, we get a map from the group of k-points of $J^i_{\text{alg}}(X)$ to $H^1(C, H^{2i-1}(X, \hat{\mathbb{Z}}(i))) = \prod_l H^1(C, H^{2i-1}(X, \mathbb{Z}_l(i)))$. Actually, we would need to change the base field to a field of definition of $J^i_{\text{alg}}(X)$ to do so, which we are allowed to do by Proposition 6, but $J^i_{\text{alg}}(X)$ is actually defined over k. This map corresponds to a collection of extensions ν'_l of sheaves on S for all l which we claim are the ν_l coming from the étale Abel–Jacobi map applied to the Z_s.

The comparison result we need comes from this claim and the Mordell–Weil theorem. Indeed, the Mordell–Weil theorem implies that the group of rational points of $J^i_{\text{alg}}(X)$ is finitely generated. On the other hand, the kernel of the map from the group of rational points of $J^i_{\text{alg}}(X)$ to $H^1(C, H^{2i-1}(X, \mathbb{Z}_l(i)))$ induced by the Kummer exact sequence is equal to the group of rational points of $J^i_{\text{alg}}(X)$ which are infinitely l-divisible. As such, this kernel is contained in the group of torsion points with torsion prime to l. It follows that the map from the group of k-points of $J^i_{\text{alg}}(X)$ to $H^1(C, H^{2i-1}(X, \hat{\mathbb{Z}}(i)))$ is injective, and that, assuming the claim, the vanishing of Griffiths' Abel–Jacobi invariant is equivalent to the vanishing of the étale Abel–Jacobi invariant in our situation.

Now since for all l, the extensions ν_l and ν'_l obviously split at the point 0, we just have to prove that $[\nu'_{\text{ét}}] = [\nu_{\text{ét}}]$. Indeed, it will then be enough to apply Theorem 14 to the extension $\nu_l - \nu'_l$. An easy functoriality argument reduces this to the case when X is the curve C itself, which concludes using Raskind results on zero cycles. We could also have used functoriality for $\nu_{\text{ét}}$ itself, but this is a little more cumbersome and is not necessary. □

3.4 **Proof of Theorem 4**

Let us now prove Theorem 4. It is actually an application of general results about normal functions and their Hodge classes and of their étale counterparts we just proved. We go back to the notation of Section 3.1.

Proof of (i)**.** In the situation of the theorem, we can use Terasoma's lemma as before to see that the exact sequence ν_l of $\hat{\mathbb{Z}}$-sheaves on S associated to $Z \hookrightarrow X$ is split for every prime number l, which implies that $[\nu_{\text{ét}}]$ is zero, and shows that the Hodge class $[\nu_Z]$ of ν_Z is zero by Proposition 15.

According to fundamental results of Griffiths, see [13], a normal function with zero Hodge class is constant in the fixed part of the intermediate Jacobian. In our case, since ν_Z vanishes at some complex point of S, this shows that $\nu_Z = 0$. ■

Proof of (ii)**.** After pulling back to T, we can assume that the normal function ν_Z is identically 0 on S. It follows from Proposition 15 that $[\nu_{\text{ét}}] = 0$. The assumptions of Theorem 14 are henceforth satisfied, which proves that the exact sequence (5) splits, and concludes the proof.

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