HENSEL FIELDS IN EQUAL CHARACTERISTIC $p > 0$

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This talk is an attempt to be a survey of the problem of Hensel fields in equal characteristic. No paper has appeared since the works of Ershov [E] and Ax - Kochen ([AK],[KO]). The only positive result in char. $p > 0$ is due to Ershov, based on an important algebraic work of Kaplansky [Ka] . Our talk doesn't countain new results but gives many examples and counter-examples which show the limits and difficulties of a possible generalization.

1.- We consider in this talk <u>valued fields</u>, that is fields with a surjection "the valuation" $K^* \to G$, where G is an ordered group, extended in 0 by putting $\mathrm{val}(0) = \infty$, an infinite element adjoined to G, and with the usual properties

$$\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$$

and the stronger form of triangular inequality

$$\mathrm{val}(x + y) \geq \mathrm{Min}\,[\mathrm{val}(x), \mathrm{val}(y)]\ .$$

In this field we define the valuation ring

$$A = \{x \in K\ ;\ \mathrm{val}(x) \geq 0\}$$

which is a local ring, with maximal ideal

$$I = \{x \in K\ ;\ \mathrm{val}(x) > 0\}$$

and the residue field $\overline{K} = A/I$. In our cas K and $\overline{K}$ have the same characteristic.

The Hensel property for the valued field K is the following : If $f(x) \in A[x]$ has a simple residual root, then it has a root in A, whose residue agrees with the root of f. It is a first-order property.

The important result of Ax-Kochen and Ershow is the following :

Proposition.- If K is a henselian field of equal characteristic 0, then $\mathcal{H} \cup Th(\overline{K}) \cup Th(\text{val } K) \vdash Th(K)$; where $\mathcal{H}$ are the axioms saying that K is henselian.

It is well known that this result doesn't generalize to the case of characteristic p. The following counter-example is often given :

Definition.- If k is a field and G an ordered group, we define the field of generalized power series with coefficients in k and exponents in G :

$$k((T^g \ ; \ g \in G)) = \{\Sigma \ a_i \ T^i \ ; \ a_i \in k, (i) \text{ is a well-ordered subset of } G\}$$

The operations are the usual over the series, multiplication being possible by the condition of well-ordered support :

$$(\sum_{k_0} a_k)(\sum_{\ell_0} b_\ell) = (\sum_{k+\ell=i} a_k . b_\ell)$$

then when k increases from k_0 to $i - \ell_0$, ℓ decreases from ℓ_0 to $i - k_0$ ans so takes only a finite number of values.

Example.- Let k be an algebraically closed field with characteristic $p > 0$. It is known that $k((T^g \ ; \ g \in \mathbb{Q}))$ is an algebraically closed field. Let us now look at the subfield :

$$K = \bigcup_{n \in \mathbb{N}} k((T^{\frac{1}{n!}}))$$

(K is generalization of the Puisieux series over $\mathbb{C}$) ; K is not algebraically closed as we see it by looking at the Artin Schreier equation $\quad x^p - x + \frac{1}{T} = 0$ whose solutions are

$$x_i = \frac{1}{T} + \frac{1}{T^p} + \ldots + \frac{1}{T^{p^n}} + \ldots + i, \quad i = 0,1,\ldots,p-1.$$

The fields $k((T^g\ ;\ g \in \mathbb{Q}))$ and K are then two Hensel fields with same residue field k and valuation group $\mathbb{Q}$, but they are not elementary equivalent.

We draw out of Ax and Kochen's proof two facts which are true for characteristic 0 but false for characteristic p :

1°.- We give first some definitions :

If $K \subset L$ are two valued fields, we can look at the residue extension and the group extension. We say that the extension is <u>immediate</u> when $\overline{K} = \overline{L}$ and val K = val L. It is the case for $k(T)$ and $k((T))$, and more generally for a field and its completion. A field is said to be <u>maximal</u> when it has no immediate proper extension ; an example is $k((T))$ or all generalized power series fields.

The valuation is an homomorphism $(K^*,.) \to (\text{val } K,+)$; a <u>cross-section</u> is a section of this mapping. It allows us to see the valuation group as included in K.

For the characteristic 0, we have an isomorphism theorem : two maximal fields with cross-section and having same residue field and same valuation group are isomorphic.

For the characteristic p, Kaplansky in [Ka] has studied the uniqueness of the immediate maximal extension of a valued field K ; he has given conditions on K, called Kaplansky conditions or conditions A guaranteeing the uniqueness :

- val K is p-divisible
- for all $a_{n-1},\ldots,a_o,\ b \in \overline{K}$ the equation $x^{p^n} + a_{n-1}x^{p^{n-1}} + \ldots + a_1x^p + a_ox + b = 0$ has a solution in $\overline{K}$.

With the same conditions the isomorphism theorem is true.

2°.- An henselian field of equal characteristic 0 doesn't admit any immediate algebraic extension. In the example of the Puisieux serie, at the opposite extreme, $K[x_o]$ where $x_o = \sum_{i\in\mathbb{N}} \frac{1}{T^{\frac{1}{p^i}}}$ is an immediate extension of K.

This difference is easy to be taken up : we remark that the property for a valued field of having no immediate algebraic proper extension - K is then called algebraically maximal - is first order ; K has only to satisfy for all integer n the sentence : "For all polynomial $P(X) \in K[X]$ of degree n

$$\{\forall v\ [\ \exists x(\mathrm{val}\ P(x) = v)] \to [\ \exists x'(\mathrm{val}\ P(x') > v)]\}$$

$$\to \{\exists y(P(y) = 0)\}\ \text{"}$$

Here we use again the work of Kaplansky ; if there is in K a pseudo-convergent sequence $(u_\alpha)_{\alpha<\alpha_0}$ and a polynomial $P(X)$ (of minimal degree) such that val $(P(u_\alpha))$ is not eventually constant, then K has an immediate extension $K[u]$ where $P(u) = 0$. With the terminology of Kaplansky, K is algebraically maximal iff all pseudo-convergent sequence of algebraic type have a pseudo-limit in K.

A direct proof allows us to avoid the reference to Kaplansky : algebraic maximality is equivalent to the sentences which say :

- "K is henselian" ; we then have uniqueness of the extension of the valuation in all algebraic extension of K ; if c is algebraic over K with minimal polynomial $x^m + c_{m-1}x^{m-1} + \ldots + c_1x + c_0$, we must have $\mathrm{val}(c) = \frac{1}{m}\mathrm{val}(c_0)$.

- For all n : "for each polynomial $P(X) \in K[X]$ of degree n, there is $a \in K(x) = K[X]/P(X)$ such that $\mathrm{val}(a) \notin \mathrm{val}\ K$ or $[\mathrm{val}(a) = 0$ and $\bar{a} \in \bar{K}]$. "

Now by elimination of x between P(x) and the decomposition of a in K(x), we know how to characterize a by its minimal polynomial over K. Hence the expression inverted commas is equivalent to :

"There exists $A(X) = X^r + a_{r-1}X^{r-1} + \ldots + a_1X + a_0$ minimal polynomial of an $a \in K(x)$ such that

$$[\bigwedge_{i=0}^{r-1} \mathrm{val}(a_i) \geq 0 \wedge \forall y\ \bar{A}(\bar{y}) \neq \bar{0}] \vee [r \nmid \mathrm{val}\ (a_0)]\text{"}$$

As far as we know, this notion of algebraic maximality is only studied in [Z].

With these two precisions, the same proof as for char. 0 works and gives the following result :

Proposition.- (Ershov) : When K_1 and K_2 are two algebraically maximal Kaplansky fields, we have $K_1 \equiv K_2$ iff $\overline{K}_1 \equiv \overline{K}_2$ and val $K_1 \equiv$ val K_2.

2.- On the contrary, if K is not Kaplansky, the system $Th(\overline{K}) \cup Th(\text{val } K) \cup$ ("K is algebraically maximal") is in general not complete ; we shall give a counterexample which shows other interesting facts .

Proposition.- Let k be a field, char. k = p > 0, and $a = a_0^p + T.a_1^p \in k((T))$ be such that $a_0, a_1 \in k((T))$ are algebraically independant over k(T) ; K is the relative algebraic closure for k(T,a) in k((T)).

Then K is algebraically maximal and $K \not\equiv k((T))$.

Proof.- The field K is valued in $\mathbb{Z}$ and hence its completion is its unique immediate maximal extension. It is easy to see that a valued field is algebraically maximal iff it is relatively algebraically closed in each immediate maximal extension ; therefore K is algebraically maximal by construction. The first-order property which distinguishes the theories of k((T)) and K is the algebraic completeness, notion introduced by Ershov : a valued field K is algebraically complete iff

1) it is henselian

2) each finite algebraic extension $L \supset K$ satisfies $[L : K] = [\overline{L} : \overline{K}]$ (val L : val K).

(To be sure this property is first order the reader can use the same kind of proof that we gave directly for the algebraic maximality). The field k((T)) is algebraically complete as it is a maximal field (see for example [R]). On the other hand the extension $L = K[T^{\frac{1}{p}} ; a^{\frac{1}{p}}]$ of K satisfies $[L : K] = p^2$, $\overline{L} = \overline{K}$, (val L : val K) = p.

Remarks : 1°) We draw as a lesson from this example the fact that an algebraic extension, even a finite one, of an algebraically maximal field is not necessarly algebraically maximal. So $a^{1/p}$ is immediate over $K[T^{\frac{1}{p}}]$ but is not in this field.

2°) We see the limits of the algebraic maximality : we have definitely the implication

$$\begin{cases} K \text{ alg. maximal} \subset L \\ \text{val } K \text{ pure subgroup of val } L \\ \overline{K} \text{ relat. alg. closed in } \overline{L} \end{cases}$$

$$\Rightarrow K \text{ relativ alg. closed in } L,$$

but nothing works for transcendental extensions ; for example $L = K[a_o, a_1]$ is an immediate but inseparable extension of K (it is known that an elementary extension is separable).

3°) The previous remark gives another first-order property distinguishing K and k((x)). In this particular case (where the valuation group is $\mathbb{Z}$) , the following sentences express the fact that there is no immediate inseparable extension :

$$\text{(for all n)} \quad "\forall\, k_1,\ldots,k_n\ [\ \forall\, v\ \exists\, x_1,\ldots,x_n\ (\mathrm{val}(\Sigma k_i x_i^p) > v)]$$

$$\rightarrow [\ \exists\ y_1,\ldots,y_n\ (\Sigma k_i y_i^p = 0)]".$$

3.- We have defined different properties of maximal fields. If we refine the algebraic maximality into separable or inseparable alg. max., we have the implications :

alg. complete $\Rightarrow$ alg. max. $\Rightarrow$ sep. alg. max. $\Rightarrow$ henselian

with coïncidence of all there notions in char. 0, of the two first for Kaplansky fields and of the two last when the valuation group is $\mathbb{Z}$:

<u>Proposition</u>.- <u>Let</u> K <u>be a field valued in</u> $\mathbb{Z}$, <u>then</u> K <u>is henselian iff it is separably algebraically maximal</u>.

<u>Proof.</u>- One of the implications is obvious. Conservely let K be a field valued in $\mathbb{Z}$ with an immediate algebraic extension K(a), where the minimal polynomial A of a over K is separable ; a is then a limit of a Cauchy sequence $(a_\alpha)_{\alpha<\alpha_o}$ in K, such that $\mathrm{val}(P(a_\alpha))$ increases with

α with no limit. Now the only initial segment ($\neq \emptyset$) of $\mathbb{Z}$ without supremum is $\mathbb{Z}$; hence A admits a root approached to each order. In particular, since $A'(a) \neq 0$ we have eventually

$$\mathrm{val}(A(a_\alpha)) > 2 \ \mathrm{val}\ A'(a) = 2 \ \mathrm{val}\ A'(a_\alpha)$$

and then, by the strong form of Hensel lemma, A has a root in K.

Q.E.D.

This equivalence doesn't generalize when the valuation group is finitely generated or has the same theory as $\mathbb{Z}$.

<u>Example of a Hensel field valued in a $\mathbb{Z}$ -group, with an immediate Artin Schreier extension</u>

$$K = \bigcup_{i \in \mathbb{N}} k((T^g \ ; \ g \in \mathbb{Z}[\tfrac{\alpha}{i!}])) \subset k((T^g \ ; \ g \in \mathbb{Z}^*))$$

where k is a field with characteristic p, $\mathbb{Z}^*$ is a non-standard model of $\mathbb{Z}$ and $\alpha \in \mathbb{Z}^* - \mathbb{Z}$ is positive and divisible by all standard integers ; K is an henselian field as it is an increasing union of henselian fields ; val K is the divisible envelope of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}^*$ and is hence a $\mathbb{Z}$ -group. Now the root $\sum_{i \in \mathbb{N}} T^{-\frac{\alpha}{p^i}}$ of the equation $x^p - x + T^{-1} = 0$ is not in K.

<u>Example of a Hensel field valued in $\mathbb{Z}[\beta]$, with an immediate Artin-Schreier extension</u>

Let K be the henselisation of $k(T,T^\beta,a^p)$, where k is a field of characteristic p, β a non-standard positive integer, p doesn't divide β and $a = a'.T^{-\beta}$, with $a' \in k((T))$ transcendental series over k(T). The solutions of the equation $x^p - x + a^p = 0$ are

$$x_i = a + a^{\frac{1}{p}} + \ldots + a^{\frac{1}{p^n}} + \ldots + i, \ i = 0,1,\ldots,p-1 \ ;$$

this notation is valid because in the decomposition of each term $a^{\frac{1}{p^n}} = T^{-\frac{\beta}{p^n}} . a'^{\frac{1}{p^n}}$ as a series in T with exponents in $\mathbb{Z}[\frac{\beta}{p^n} \ ; \ n \in \mathbb{N}]$

all the monomials have valuation at standard distance from $-\beta p^{-n}$ (the reader will note that the support of the series x_i has order-type ω^2). Because of the inclusion $K \subset k((T^g, g \in \mathbb{Z}[\beta]))$ x_i is not in K ; to have that x_i is immediate over K, it is enough to note that a is not in K, as it is not in $k(T)(T^\beta)(a^p)$ and as it is radical over this field.

4.- We can ask ourselves whether algebraic completeness is the good first-order characterization of completeness or not. The fact that we do not know of two algebraically complete fields K and L satisfying $\overline{K} \equiv \overline{L}$, val K $\equiv$ val L and $K \not\equiv L$ may tempt us to give a positive answer. On the other hand we have the following result (unpublished result of Kochen and Jacob ; see for example a proof in [BDL]) :

Proposition.- In the language of valued fields with cross-section, $\mathbb{F}_p((T))$ is undecidable.

So in this enriched language, not only is the system "$\overline{K} \equiv \mathbb{F}_p$" $\cup$ "val K $\equiv \mathbb{Z}$" $\cup$ "K is algebraically complete" incomplete but but so also are all systems obtained by replacing algebraic maximality by a recursively enumerable system of axioms.

Along the same lines, we may have two maximal fields with the same residual and valuational theories but not elementary equivalent in this language :

Proposition.- In the language of valued fields with cross-section, if char. $k = p > 0$, $k((T)) \not\equiv k((T^g ; g \in \mathbb{Z}^*))$

Proof.- Let π be the cross-section, let us consider the sentence

$$\exists\, a\, [\exists\, b\ a = \pi(b)] \wedge [\mathrm{val}(a) < 0] \wedge [\exists\, x (x^p - x + a = 0)]$$

which is false in $k((T))$ but true in $k((T^g ; g \in \mathbb{Z}^*))$. We have only to take $a = T^{-\alpha}$ where α is a positive non-standard integer, infinitely divisible by p.

The question is now to determine the importance of cross-section in the language. But it remains true that in characteristic 0, even if we adjoin it in the language, $k((X))$ is decidable iff k is.

BIBLIOGRAPHY

[AK] J. AX and S. KOCHEN : Diophantine problems over local fields I Am. Journal of Math., vol 187 (1965), pp. 605-630, pp. 631-648.

Diophantine problems over local fields III, decidable fields, Annals of Math. 83 (1966), pp.437-456.

[BDL] J. BECKER, J. DENEF and L. LIPSCHITZ : Further remarks on the elementary theory of formal power series rings, this volume.

[E] J.L. ERSHOV : On the elementary theory of maximal normed fields Doklady 1965, Tome 165 N°1, pp.1390-1393.

[Ka] I. KAPLANSKY : Maximal fields with valuation, Duke Math. Journal 9 (1942), pp. 303-321.

[Ko] S. KOCHEN : The model theory of local fields, Logic Conference, Kiel 1974, Lecture notes in Mathematics, 499 Berlin, Springer Verlag 1975.

[Z] M. ZIEGLER : Die elementare Theorie der henselschen Körper, Inaugural Dissertation Köln 1972.

A basic book about valued fields is :

[R] P. RIBENBOIM : Théorie des valuations, Les Presses de l'Université de Montréal, (1964).