Cloture définissable

Déf Soit $(R, \dots)$ une structure, et $A \subset R$.
On définit $dcl(A)$, l'ensemble des éléments de R qui sont définissables sur A, par la propriété suivante :
$a \in dcl(A)$ s'il existe une formule $\varphi(x, \bar{y})$ sans paramètres et un uplet $\bar{b}$ dans A ($|x| = 1$, $|\bar{y}|$ finie) tels que a est l'unique élément de R satisfaisant $\varphi(x, \bar{b})$.

Donc $dcl(A)$ n'est pas définissable, mais c'est une union d'ensembles définissables, et il contient A. On montre facilement que $dcl(A) = dcl(dcl(A))$.

Proposition 64 Soit $(R, +, -, 0, 1, <, \dots)$ o-minimale et $A \subset R$.
Alors $dcl(A) \prec R$

Dém Rappel du critère de Tarski

Soient $M \subseteq N$ des structures d'un langage $\mathcal{L}$.
$M \prec N$ veut dire : si $\varphi(\bar{x})$ est une formule de $\mathcal{L}$, et $\bar{a}$ un uplet de M alors $\varphi(\bar{a})$ est vraie dans M ($M \models \varphi(\bar{a})$) si et seulement si elle est vraie dans N.
Le critère dit : pour avoir $M \prec N$ il suffit de montrer, pour toute formule $\varphi(x, \bar{y})$ (sans paramètres) et uplet $\bar{a}$ dans M, s'il existe $b \in N$ tel que $N \models \varphi(b, \bar{a})$ alors il existe $b \in M$ tel que $N \models \varphi(b, \bar{a})$

Dém de la proposition : On utilise les fonctions de choix définissable et le critère de Tarski.

Setting:

$R = (R, +, -, \cdot, 0, 1, <, \dots)$ ordered field.

Study of differentiability.

Def: Let $I \subseteq R$ be open. A function $f : I \to R^n$ is differentiable at $x \in I$ with derivative $a \in R^n$ iff

$$\lim_{t \to 0} \frac{f(x+t) - f(x)}{t} = a.$$

Note: this implies f continuous at x, a is unique.

Write $a = f'(x)$.

Properties: The following are easy to show: let $f, g : I \to R^n$ be differentiable at x

Then $(f+g)'(x) = f'(x) + g'(x)$

$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

dot product in R^n: $(y_1, \dots, y_n) \cdot (z_1, \dots, z_n) = \sum y_i z_i$

If $n = 1$, $g(y) \neq 0$ $\forall y \in I$,

$$(f/g)'(x) = (f'(x) g(x) - g'(x) f(x)) / g(x)^2$$

constant maps have derivative 0.

the identity map has derivative 1.

$I, J \subset R$ open, $f : I \to R$ continuous differentiable at x, $g : J \to R$ continuous differentiable at $f(x) \in J$.

Then $g \circ f$, defined on $I \cap f^{-1}(J)$, is continuous differentiable at x, with

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

Directional derivatives

$f : U \to \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ open, $x \in U$, $v \in \mathbb{R}^m$

f is differentiable at x in the v-direction with derivative $a \in \mathbb{R}^n$, if $g(t) = f(x+tv)$ is differentiable at $0 \in \mathbb{R}$ with derivative a.

We write $d_x f(v) = a$.

usual $\frac{\partial f}{\partial x_i}(x) \rightsquigarrow v = (0, \dots, 1, 0 \dots 0)$ i-th place.

Differential of a map:

$f = (f_1, \dots, f_n) : U \to \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ open.

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear map, $x \in U$.

We say f is differentiable at x with differential T if for each $\varepsilon > 0$ we have some $\delta > 0$ s.t $|v| < \delta$ implies $|f(x+v) - f(x) - T(v)| < \varepsilon |v|$.

Then f is continuous at x, T is unique. Write $T = d_x f$.

$\rightsquigarrow d_x f(v) = T(v)$.

$m = 1$ $\quad d_x f(1) = f'(x)$.

$f = (f_1 \dots f_n)$ is differentiable at x if each f_i is, and then the matrix $\left(\frac{\partial f_i}{\partial x_j}(x)\right)$ is the matrix of T relative to the standard basis $\rightsquigarrow$ $n \times m$ matrix

Properties ; usual ones :

$f, g : U \to R^n$ differentiable at $x \in U$, U open in R^m

$d_x(f+g) = d_x f + d_x g$.

$d_x cf = c \cdot d_x f \qquad c \in R$.

$h : V \to R^n$ diff^{ble} at $f(x) \in V$, f continuous

then $h \circ f$, defined on $U \cap f^{-1}(V)$, is differentiable at x, $d_x(h \circ f) = d_{f(x)}(h) \cdot d_x f$.

Now assume

$(R, +, \cdot, -, 0, 1, <, \cdots)$ O-minimal.

So real closed.

<u>Lemma</u> (Rolle) Let $a < b$, and spse the function $f : [a,b]$ is definable, continuous, $f(a) = f(b)$ and f is differentiable at each pt of (a,b). Then there is $c \in (a,b)$ such that $f'(c) = 0$.

<u>Pf</u> Let $c \in (a,b)$ such that $f(c)$ is minimum/maximum

Show $f'(c) = 0$.

(Exercise)

<u>Mean Value theorem</u> $a < b$, $f : [a,b] \to R$ definable, continuous, differentiable on (a,b). Then for some $c \in (a,b)$,

$$f(b) - f(a) = (b-a) f'(c).$$

<u>Pf</u> look at $g(t) : [0,1] \to R$ defined by

$$g(t) = f(a + t(b-a)) - \qquad t(f(b) - f(a))$$

$$g(0) = f(a) \qquad = 0.$$

$$g(1) = f(b) - \qquad - (f(b) - f(a)) = 0.$$

$$\leadsto g'(t) = f'(a + t(b-a))(b-a) + f(a) - f(b)$$

Lemma

$f : [a,b] \to \mathbb{R}$ continuous, diff^{ble} on (a,b).
If $f'(x) = 0$ for all $x \in (a,b)$ then $f(x)$ is constant.

Goal: Let $f : I \to \mathbb{R}$ be definable, $I \subseteq \mathbb{R}$ an interval. Then f is differentiable at all but finitely many points of I.

Need several lemmas

$x \in I$. Define $f(x^+) = \lim_{t \to 0^+} f(x+t) - f(x)/t \quad \in \mathbb{R} \cup \{\pm\infty\}$

$$f(x^-) = \lim_{t \to 0^-} f(x+t) - f(x)/t.$$

f differentiable at x: $f(x^+) = f(x^-) \in \mathbb{R}$.

Lemma: Assume f is continuous, $f'(x^+) > 0$ for all $x \in I$. Then f is strictly increasing, $f^{-1} : f(I) \to \mathbb{R}$, satisfies $(f^{-1})'(y^+) = 1/f'(x^+)$ for $x \in I$, $f(x) = y$. ($1/{+\infty} = 0$).

Pf: If f were not strictly increasing, then there would be a subinterval J on which f is constant ($f' = 0$), or strictly decreasing, which contradicts $f'(x^+) > 0$.
Give $\varepsilon > 0$
for t sufficiently small >0, we have

$$\lim_{t \to 0^+} \frac{f(x+t) - f(x)}{t} = \lim_{t \to 0^+} \left(\frac{t}{f(x+t) - f(x)} \right)^{-1}$$

$$= \lim_{u \to 0^+} \left(\frac{f^{-1}(y+u) - f^{-1}(u)}{u} \right)^{-1}.$$

Lemma : $f : I \to R$ def., continuous, $x \mapsto f'(x^+)$, and $x' \mapsto f'(x^-)$ are R-valued, continuous on I. Then f is differentiable on I, and f' is continuous

Pf Enough to show : $f'(a^+) = f'(a^-)$ for all $a \in I$.
Otherwise, say $f'(a^+) > f'(a^-)$. Let $c \in R$ be between $f'(a^+), f'(a^-)$, let $J \subset I$ be such that $f'(x^+) > c > f'(x^-)$ on J. Then $g : J \to R$, $g(x) = f(x) - cx$ satisfies : $g'(x^+) > 0$, $g'(x^-) < 0$ for all $x \in J$. So g is both strictly increasing, strictly decreasing on J #
(Lemme précédent appliqué à $-g$)

Lemma Let $f : I \to R$ be definable. There are only finitely many $x \in I$ at which $f'(x) \in \pm\infty$

Pf Suppose $A = \{x \in I, \; f'(x^+) = +\infty\}$ is infinite. Then A contains an interval, and wlog $A = I$, f is continuous on I. Then f is strictly increasing, and therefore $f'(x^-) \geq 0$ for all $x \in I$.
(We want to reach a contradiction, so we are allowed to shrink I to non-empty subintervals). So after shrinking I, we may assume we are in one of the following two cases

(i) $f'(x^-) = +\infty$ for all $x \in I$

(ii) $f'(x^-) \in R$ for all $x \in I$, and $x \mapsto f'(x^-)$ is continuous on I.

In subcase (i), we have $(f^{-1})'(y^-) = 0 = (f^{-1})'(y^+)$.
I.e, f^{-1} is constant. This contradicts $f'(x^+) > 0$.
In subcase (ii), let $a \in I$, and $c > f'(a^-)$. Then there is a subinterval $J \subset I$ on which $f'(x^-) < c$.

So looking at $g(x) = f(x) - cx$ we have:

$$g'(x^+) = f'(x^+) - c = f'(x^+) = +\infty$$

$$g'(x^-) = f'(x^-) - c < 0.$$

Contradiction. So $|A| < +\infty$.

Replacing $f(x)$ by $f(-x)$, we get that the set of $x \in I$, $f(x^-) = \pm\infty$ is finite.

Proof of the Proposition: if $f: I \to R$ is definable, then there are only finitely many points $\in I$ at which f is not differentiable.

Pf: We saw in the previous lemma that the set of points A such that one of $f'(x^+)$, $f'(x^-)$ is $\pm\infty$, is finite. Furthermore, throwing away finitely many points, we may assume that on each subinterval of $I \setminus A$, the maps $f'(x^+)$ and $f'(x^-)$ are continuous. Hence, at all points of $I \setminus A$, $f'(x^+) = f'(x^-)$ and f is differentiable

Aim Inverse function theorem, and implicit function theorem:

If the Jacobian is invertible at a point $a \in R^m$, then f is locally a homeo around a.

and IFT.

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and IFT.

For that we need more lemmas.

Setting definable $(f_1, \dots, f_n) = f : U \to R^n$, $U \subseteq R^m$ open

Def We call f a C^1-map if the partial derivatives $\partial f_i / \partial x_j$ are defined on U and continuous.

One shows easily that:

If f is C^1 then f is differentiable at each point of U, and the map $x \mapsto d_x f \in R^{n \times m} = \mathrm{Lin}(R^m, R^n)$ is continuous. And conversely.

(Usual proof for $\rightarrow$).

If $T : R^m \to R^n$ is R-linear (ie $\in R^{n \times m}$) define $|T| = \max \{ |Tx| \quad |x| \leq 1, x \in R^m \}$.

Then $|T(x)| \leq |T| |x|$.

Lemma Let $f : U \to R^n$ be C^1, $[a,b] = \{ (1-t)a + tb \mid 0 \leq t \leq 1 \}$ be a line segment contained in U.

Then $|f(b) - f(a)| \leq |b-a| \max\limits_{y \in [a,b]} |d_y f|$

Pf Let $g(t) : [0,1] \to U$, $g(t) = f((1-t)a + tb)$

Then $g'(t) =$ directional derivative of f at $(1-t)a + tb$, in direction $(b-a)$.

$= d_y f(b-a)$ when $y = (1-t)a + tb$

So $|g'(t)| \leq M$, $M = |b-a| \max\limits_{y \in [a,b]} d_y f$.

By MVT, we have $|f(b) - f(a)| = |g(1) - g(0)| \leq M$.

Lemma Same assumptions, $x \in U$.

$$|f(b) - f(a) - d_x f(b-a)| \leq |b-a| \max_{y \in [a,b]} |d_y f - d_x f|.$$

Pf Consider $h(y) = f(y) - d_x f(y)$.

Then $d_y h = d_y f - d_x f$.

Lemma Same assumptions, $m = n$, $a \in U$, and assume that $d_a f$ is invertible. Then there are $\varepsilon > 0$, $C > 0$ in $\mathbb{R}$ s.t

$$|f(x) - f(y)| > C|x-y| \text{ for all } x, y \in U \text{ with } |x-a|, |y-a| < \varepsilon.$$

In particular, f is invertible on a nbhd of a.

Pf Let $\varepsilon > 0$ be small enough so that $B(a,\varepsilon) \subset U$.

By the previous lemma, we have

$$|f(x) - f(y) - d_a f(x-y)| \leq |x-y| \max_{z \in [x,y]} |d_z f - d_a f|$$

$$\geq$$

$$|d_a f(x-y)| - |f(x) - f(y)|$$

$$\leadsto |f(x) - f(y)| \geq d_a f(x-y) - |x-y| \max_{z \in [x,y]} |d_z f - d_a f|$$

As $d_a f$ is invertible, there is c', not depending on x, y, such that

$$|d_a f(x-y)| \geq c'|x-y|.$$

Indeed, we have $|z| = |d_a f^{-1} d_a f(z)| \leq |d_a f^{-1}| |d_a f(z)|$

So $|d_a f(z)| \geq |z| |d_a f^{-1}|^{-1}$!

Decreasing ε, we may assume that

$$|d_b f - d_a f| < c'/2 \text{ for all } b \in B(a,\varepsilon)$$

hence

$$|f(x) - f(y)| \geq c'|x-y| - c'/2|x-y| \geq c'/2|x-y|.$$

Inverse function theorem

Let $f: U \to R^m$ be a definable C^1 map on a definable open set $U \subseteq R^m$, $a \in U$ s.t $d_a f : R^m \to R^m$ is invertible.

Then there are: a definable open $U' \ni a$, $U' \subseteq U$, and a definable nbhd V' of $f(a)$ such that f maps U' homeomorphically onto V', and $f^{-1}: V' \to U'$ is also C^1.

Pf Since they define the same topology, we may replace $|\ |$ on R^m by $\|\ \|$, $\|(x_1,\ldots,x_m)\| = \sqrt{x_1^2+x_2^2+\cdots+x_m^2}$

We can find $c, \varepsilon > 0$ such that

$$\|x-a\| \leq \varepsilon \to x \in U \text{ and } d_x f \text{ is invertible.}$$

$$\|x-a\|, \|y-a\| \leq \varepsilon \to \|f(x)-f(y)\| \geq c\|x-y\|.$$

Claim $\{y \mid \|y-f(a)\| < \frac{1}{2}c\varepsilon\} \subseteq \{f(x) \mid \|x-a\| \leq \varepsilon\}$

Assume $\|y-f(a)\| < \frac{1}{2}c\varepsilon$

Consider $P(x) = \|f(x)-y\|^2$ on the ball $\|x-a\| \leq \varepsilon$.

As the ball is closed and bounded, $P(x)$ assumes its minimum value on it. However, if $\|x-a\| = \varepsilon$, then

$$P(x) = \|(f(x)-f(a)) - (y-f(a))\|^2$$

$$\geq \big(\underbrace{\|f(x)-f(a)\|}_{\geq c\varepsilon} - \underbrace{\|y-f(a)\|}_{< \frac{1}{2}c\varepsilon}\big)^2 > \left(\tfrac{1}{2}c\varepsilon\right)^2 > \|y-f(a)\|^2 = P(a)$$

So the minimum value is attained at b, $\|b-a\| < \varepsilon$.

So $0 = \dfrac{\partial P}{\partial x_j}(b) = \sum_{i=1}^{m} 2(f_i(b)-y_i)\left(\partial f_i/\partial x_j\,(b)\right)$ $\forall j$

i.e., $d_b f\,(f(b)-y) = 0$

$d_a f$ invertible implies $f(b) = y$, which proves the claim.
So the image by f of the open set $\{\|x-a\| < \varepsilon\}$ contains the open set $\{\|y - f(a)\| < \frac{1}{2} c\varepsilon\}$.

Let $U' = \{x \mid \|x-a\| < \varepsilon\}$. Then, reasoning in the same way with all points $a' \in U'$ we get that f is open on U'. It is also injective, hence it is a homeomorphism between U' and $V' = f(U')$.
It remains to show that $f^{-1} : V' \to U'$ is C^1.

By definition

$$\left(f(b) - f(a) - d_a f(b-a)\right) / \|b-a\| \to 0 \text{ as } b \to a.$$

Also $\|b - a\| \leq c^{-1} \|f(b) - f(a)\|$. Apply $(d_a f)^{-1}$

$$(d_a f)^{-1}\left(f(b) - f(a)\right) - (b-a) \Big/ \|f(b) - f(a)\| \to 0$$

as $f(b) \to f(a)$.

ie: f^{-1} is differentiable at $f(a)$, with differential $d_{f(a)} f^{-1} = (d_a f)^{-1}$

This reasoning works at every point $a' \in U'$, and therefore f^{-1} is C^1 ($x \mapsto (d_x f)^{-1}$ is continuous)

__Corollary__ (Implicit function theorem)

Let $U \subseteq R^{m+n}$ be definable open, $f_1, \dots, f_n : U \to R$ be definable C^1. Let $(x_0, y_0) \in R^{m+n}$ be s.t.
m n

$f_1(x_0, y_0) = \dots = f_n(x_0, y_0) = 0$, and the matrix $\left(\partial f_j / \partial y_k (x_0, y_0) \right)$ is invertible. Then there is an open definable nbhd V of x_0 in R^m, and W of y_0 in R^n, and a definable C^1-map $\phi : V \to W$ such that $V \times W \subseteq U$, and for all $(x, y) \in V \times W$ we have

$$f_1(x,y) = \dots = f_n(x,y) = 0 \iff y = \phi(x).$$

Pf Apply the inverse function theorem to the map

$g: (x, y) \longmapsto (x, f_1(x,y), \dots, f_n(x,y))$.
$U \longrightarrow R^{m+n}$.

Consider the map $\pi g^{-1}(x, 0)$ near x_0.
$\pi : R^{m+n} \to R^m$

—

What else:

L'Hopital's rule.

$I \subset R$ interval, $f, g : I \to R$ definable
$g'(x) \neq 0$ for all $x \in I$ in a nbhd of a
$\lim_{x \to a} g(x) = 0 = \lim_{x \to a} f(x)$. Then $\lim f/g (x) = \lim f'/g' (x)$.

Taylor's formula: If $f : I \to R$ is $(n+1)$-times differentiable on I, and $a < b$ are in I. Then

$$f(b) = f(a) + f'(a)(b-a) + \frac{f^{(2)}(a)}{2}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some $c \in (a,b)$.

Def Let $A \subseteq R^m$, $f : A \to R^n$ a definable map. Then f is a C^1-map if there are a definable open $U \supseteq A$, and a definable C^1-map $F : U \to R^n$, such that $F|_A = f$.

A C^1-cell is a cell in which all defining functions are C^1.

Thm (C^1-cell decomposition)

(I_m) If $A_1, \dots, A_k \subseteq R^m$ are definable, there is a decomposition of R^m into C^1-cells partitioning $A_1, \dots, A_k$.

(II_m) If $A \subseteq R^m$ and $f : A \to R$ is definable, then there is a decomposition $\mathcal{D}$ of R^m into C^1-cells, partitioning A and such that if $C \in \mathcal{D}$, $C \subseteq A$ then $f|_C$ is C^1.

If f and A are as in II_m, $p \in \mathrm{Int}(A)$, write

$$\nabla f(p) = \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_m}(p) \right)$$ provided they exist.

$A' = \{ p \in A \mid p \in \mathrm{int}(A)$ and ∇f is defined at $p \}$

Then

(III_m) $A \setminus A'$ has empty interior.

[On utilise III_m pour prouver II_m]