Examen du cours spécialisé *Corps réels clos et structures o-minimales* - Partie 2
(M2 MathFonda, printemps 2022).

The second part of the homework is to be done at home, and hand back on **May 6** *(2022). You are allowed to look at your course notes and at mine, but not anything else. Do not hesitate to ask me questions, even dumb ones. In all questions, you can suppose known the results of the preceding questions.*
All rings and fields are commutative.

Problem 8. Let $f, g : \mathbb{R} \to \mathbb{R}$ two functions. They *have the same germ at* ∞ iff there is $a \in \mathbb{R}$ such that for all $x \in (a, +\infty)$, $f(a) = g(a)$. We denote it by $f \sim g$.

(a) Show that $\sim$ is an equivalence relation.
(b) Let S be a ring of functions $\mathbb{R} \to \mathbb{R}$. Show that $S/\sim$ is a ring. Give an example to show that it is not necessarily a domain.

Let $\mathcal{R} = (\mathbb{R}, +, -, \cdot, 0, 1, <, \ldots)$ be an o-minimal structure on the field of real numbers (i.e., $+, -, \cdot$ are the usual operations, but there is maybe some additional structure).

(c) Let $\mathrm{Def}(\mathcal{R})$ be the set of functions $\mathbb{R} \to \mathbb{R}$ which are definable in the o-minimal structure $\mathcal{R}$. Show that $\mathrm{Def}(\mathcal{R})/\sim$ is an ordered field, and is closed under derivation.

Problem 9. Let $\mathcal{R} = (\mathbb{R}, +, -, \cdot, 0, 1, <, \ldots)$ be an o-minimal structure on the field of real numbers. We assume that if $f : \mathbb{R} \to \mathbb{R}$ is definable (i.e., $f \in \mathrm{Def}(\mathcal{R})$), then there is $n \in \mathbb{N}$ such that for $x \gg 0$, one has $|f(x)| \leq x^n$.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be definable in $\mathcal{R}$, and which is non-zero for $x \gg 0$. One can show that there exists $r \in \mathbb{R}$ such that $\lim_{x\to+\infty} f(x)/x^r \in \mathbb{R} \setminus \{0\}$ (This uses the hypothesis on the growth of definable functions). Show that this element r is unique.
(b) Consider the set Λ of real numbers r such that for some $f \in \mathrm{Def}(\mathcal{R})$ we have $\lim_{x\to+\infty} \frac{f(x)}{x^r} \in \mathbb{R}^\times$. Show that Λ is a field.
(c) Show that if $r \in \Lambda$, then the function $(0, +\infty) \to \mathbb{R}$, $x \mapsto x^r$, is definable in $\mathcal{R}$. (Hint: $y^r = (xy)^r/x^r$).