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Introductory notes on the model theory of valued fields

Zoé Chatzidakis[a] (CNRS - Université Paris 7)

These notes will give some very basic definitions and results from model theory. They contain many examples, and in particular discuss extensively the various languages used to study valued fields. They are intended as giving the necessary background to read the papers by Cluckers, Delon, Halupczok, Kowalski, Loeser and Macintyre in this volume. We also mention a few recent results or directions of research in the model theory of valued fields, but omit completely those themes which will be discussed elsewhere in this volume. So for instance, we do not even mention motivic integration.

People interested in learning more model theory should consult standard model theory books. For instance: D. Marker, *Model Theory: an Introduction*, Graduate Texts in Mathematics 217, Springer-Verlag New York, 2002; C.C. Chang, H.J. Keisler, *Model Theory*, North-Holland Publishing Company, Amsterdam 1973; W. Hodges, *A shorter model theory*, Cambridge University Press, 1997.

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1 Languages, structures, satisfaction

Languages and structures

1.1 Languages. A language is a collection $\mathcal{L}$, finite or infinite, of symbols. These symbols are of three kinds:

– *function* symbols,
– *relation* symbols,
– *constant* symbols.

To each function symbol f is associated a number $n(f) \in \mathbb{N}^{>0}$, and to each relation symbol R a number $n(R) \in \mathbb{N}^{>0}$. The numbers $n(f)$ and $n(R)$ are called the *arities* of the function f, resp., the relation R. In addition, the language will also contain variable symbols (usually denoted by $x, y, \ldots$), the equality relation $=$, as well as parentheses and logical symbols $\wedge$, $\vee$, $\neg$, $\rightarrow$ (and, or, negation, implies), quantifiers $\exists$, $\forall$ (there exists, for all).

1.2 $\mathcal{L}$-structures. We fix a language $\mathcal{L} = \{f_i, R_j, c_k \mid i \in I, j \in J, k \in K\}$, where the f_i's are function symbols, the R_j's are relation symbols, and the c_k's are constant symbols.
An *$\mathcal{L}$-structure* $\mathcal{M}$ is then given by

– A set M, called the *universe of* $\mathcal{M}$,

– For each function symbol $f \in \mathcal{L}$, a function $f^{\mathcal{M}} : M^{n(f)} \rightarrow M$, called the *interpretation of f in* $\mathcal{M}$,

– For each relation symbol $R \in \mathcal{M}$, a subset $R^{\mathcal{M}}$ of $M^{n(R)}$, called the *interpretation of R in* $\mathcal{M}$,

– For each constant symbol $c \in \mathcal{L}$, an element $c^{\mathcal{M}} \in M$, called the *interpretation of* c *in* $\mathcal{M}$.

The structure $\mathcal{M}$ is then denoted by

$$\mathcal{M} = (M, f_i^{\mathcal{M}}, R_j^{\mathcal{M}}, c_k^{\mathcal{M}} \mid i \in I, j \in J, k \in K).$$

In fact, the superscript $\mathcal{M}$ often disappears, and the structure and its universe are denoted by the same letter. This is when **no confusion is possible**, for instance when there is only one type of structure on $\mathcal{M}$.

1.3 Substructures. Let M be an $\mathcal{L}$-structure. An $\mathcal{L}$*-substructure* of M, or simply a substructure of M if no confusion is likely, is an $\mathcal{L}$-structure N, with universe contained in the universe of M, and such that the interpretations of the symbols of $\mathcal{L}$ in N are restrictions of the interpretation of these symbols in M, i.e.:

– If f is a function symbol of $\mathcal{L}$, then the interpretation of f in N is the restriction of f^M to $N^{n(f)}$,

– If R is a relation symbol of $\mathcal{L}$, then $R^N = R^M \cap N^{n(R)}$,

– If c is a constant symbol of $\mathcal{L}$, then $c^M = c^N$.

Hence a subset of M is the universe of a substructure of M if and only if it contains all the (elements interpreting the) constants of $\mathcal{L}$, and is closed under the (interpretation in M of the) functions of $\mathcal{L}$. Note that **if the language has no constant symbol**, then the empty set is the universe of a substructure of M.

1.4 Morphisms, embeddings, isomorphisms, automorphisms. Let M and N be two $\mathcal{L}$-structures. A map $s : M \to N$ is an ($\mathcal{L}$)-*morphism* if for all relation symbols $R \in \mathcal{L}$, function symbols $f \in \mathcal{L}$, constant symbols $c \in \mathcal{L}$, and tuples $\bar{a}, \bar{b}$ in M, we have:

$$\text{if } \bar{a} \in R, \text{ then } s(\bar{a}) \in R; \qquad s(f(\bar{b})) = f(s(\bar{b})); \qquad s(c) = c.$$

An *embedding* is an injective morphism $s : M \to N$, which satisfies in addition for all relation $R \in \mathcal{L}$ and tuple $\bar{a}$ in M, that

$$\bar{a} \in R \iff s(\bar{a}) \in R.$$

An *isomorphism* between M and N is a bijective morphism, whose inverse is also a morphism. Finally, an *automorphism* of M is an isomorphism $M \to M$.

1.5 Reducts and expansions Let $\mathcal{L} \subseteq \mathcal{L}'$ be languages, and M an $\mathcal{L}'$-structure. The *reduct of* M *to* $\mathcal{L}$ is the $\mathcal{L}$-structure (denoted by $M|_{\mathcal{L}}$

with universe the same as M, in which one forgets the interpretation of symbols of $\mathcal{L}' \setminus \mathcal{L}$. For instance, the ring of real numbers $(\mathbb{R}, +, \cdot, 0, 1)$ is a reduct of the ordered ring of real numbers $(\mathbb{R}, +, -, \cdot, 0, 1, <)$, which is a reduct of the exponential ordered ring of the real numbers $(\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$. Conversely, M is an *expansion of the* $\mathcal{L}$*-structure* $M|_{\mathcal{L}}$ to the language $\mathcal{L}'$.

Thus taking a reduct of a structure is forgetting some of the relations, constants or function symbols, while taking an expansion of a structure means adding new relations, constants or function symbols.

1.6 Examples of languages, structures, and substructures. The concrete structures considered in model theory all come from standard algebraic examples, and so the examples given below will be very familiar to you.

Example 1 - The language of groups (additive notation). The language of groups, $\mathcal{L}_G$, is the language $\{+, -, 0\}$, where $+$ is a 2-ary function symbol, $-$ is a unary function symbol, and 0 is a constant symbol.

Any group G has a natural $\mathcal{L}_G$-structure, obtained by interpreting $+$ as the group multiplication, $-$ as the group inverse, and 0 as the unit element of the group.

A substructure of the group G is then a subset containing 0, closed under multiplication and inverse: it is simply a subgroup of G. The notions of homomorphisms, embeddings, etc. between groups, have the usual meaning.

This is a good place to remark that the notion of substructure is sensitive to the language. While the inverse function and the identity element of the group G are retrievable (definable) from the group multiplication of G, the notion of "substructure" heavily depends on them. For instance, a $\{+, 0\}$-substructure of G is simply a submonoid of G containing 0, while a $\{+\}$-substructure of G can be empty.

If the group is not abelian, then one usually uses the multiplicative notation, i.e. one replaces $+$ by $\cdot$, $-$ by $^{-1}$ and 0 by 1. Here are some examples of $\mathcal{L}_G$-structures:

(1) $(\mathbb{Z}, +, -, 0)$, the natural structure on the additive group of the integers,

(2) $(\mathbb{R}, +, -, 0)$, the natural structure on the additive group of the reals,
(3) (multiplicative notation)
$(\mathbb{R}^{>0}, \cdot, ^{-1}, 1)$ the multiplicative group of the positive reals.
(4) (multiplicative notation), K a field, $n > 0$:
$(\mathrm{GL}_n(K), \cdot, ^{-1}, 1)$, the multiplicative group of invertible $n \times n$ matrices with coefficients in K.
(5) An $\mathcal{L}_G$-structure is not necessarily a group. E.g., define $+$ on $\mathbb{Z}$ by $a +^{\mathbb{Z}} b = 1$, $-^{\mathbb{Z}} a = 0$ for all $a, b \in \mathbb{Z}$, and $0^{\mathbb{Z}} = 2$. The resulting structure $(\mathbb{Z}, +^{\mathbb{Z}}, -^{\mathbb{Z}}, 0^{\mathbb{Z}})$ is not a classical structure.

Example 2 - The language of rings. The language of rings, $\mathcal{L}_R$, is the language $\{+, -, \cdot, 0, 1\}$, where $+$ and $\cdot$ are binary functions, $-$ is a unary function, 0 and 1 are constants.

A (unitary) ring S has a natural $\mathcal{L}_R$-structure, obtained by interpreting $+, -, \cdot$ as the usual ring operations of addition, subtraction and multiplication, 0 as the identity element of $+$, and 1 as the unit element of S.

A substructure of the $\mathcal{L}_R$-structure S is then simply a subring of S. Note that it will in particular contain the subring of S generated by 1, i.e., a copy of $\mathbb{Z}$ or of $\mathbb{Z}/n\mathbb{Z}$ for some integer n. Again homomorphisms and embeddings between rings have the usual meaning.

When one deals with fields, it is sometimes convenient to add a function symbol for the multiplicative inverse (denoted $^{-1}$). By convention $0^{-1} = 0$. Most of the time however, one studies fields in the language of rings.

Example 3 - The language of ordered groups, of ordered rings.

One simply adds to $\mathcal{L}_G$, resp. $\mathcal{L}_R$, a binary relation symbol, $\leq$ (or sometimes $<$). I will denote these languages by $\mathcal{L}_{og}$ and $\mathcal{L}_{or}$ respectively. I will also use the abbreviation $x < y$ for $x \leq y \wedge x \neq y$.

Example 4 - Valued fields. Here there are several possibilities.

Recall first that a *valued field* is a field K, with a map $v : K^{\times} \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group, and satisfying the following axioms:

- $\forall x\ v(x) = \infty \iff x = 0$,
- $\forall x, y\ v(xy) = v(x) + v(y)$,
- $\forall x, y\ v(x + y) \geq \min\{v(x), v(y)\}$.

By convention, ∞ is greater than all elements of Γ. Note that we do not assume that Γ is archimedean, e.g. $\mathbb{Z} \oplus \mathbb{Q}$ with the anti-lexicographical ordering is possible. (Recall that the *anti-lexicographical ordering* on a product $A \times B$ of ordered groups is defined by: $(a_1, b_1) \leq (a_2, b_2) \iff (b_1 < b_2) \vee [(b_1 = b_2) \wedge (a_1 \leq a_2)]$.

1. Maybe the most natural language (used in the definition) is the *two-sorted* language with a sort for the valued field and one for the value group; each sort has its own language (the language of rings for the field sort, and the language of ordered abelian groups with an additional constant symbol ∞ for the group sort); there is a function v from the field sort to the group sort. Thus our structure is

$$\big((K, +, -, \cdot, 0, 1), (\Gamma \cup \{\infty\}, +, -, 0, \infty, \leq), v\big).$$

Formulas are built as in classical first-order logic, except that variables come with their sort. Thus for instance, in the three defining axioms, all variables are of the field sort. To avoid ambiguity, one sometimes write $\forall x \in K$, or $\forall x \in \Gamma$. Or one uses a different set of letters. For instance, the axiom stating that the map v is surjective will involve both sorts, and can be written

$$\forall \gamma\, (\in \Gamma)\, \exists x\, (\in K)\, v(x) = \gamma.$$

2. Another natural language is the language $\mathcal{L}_{\rm div}$ obtained by adding to the language of rings a binary relation symbol $\,|\,$, interpreted by

$$x \,|\, y \iff v(x) \leq v(y).$$

Note that the valuation ring $\mathcal{O}_K$ is quantifier-free definable, by the formula $1 \,|\, x$, and that the group Γ is isomorphic to $K^\times/\mathcal{O}_K^\times$, the ordering been given by the image of $\,|\,$. Hence the ordered abelian group Γ is *interpretable* in $(K, +, -\cdot, 0, 1, \,|\,)$. See 1.17 for a formal definition.

3. A third possibility is to look at the field K in the language of rings augmented by a (unary) predicate for its valuation ring $\mathcal{O}_K$. The divisibility relation is then definable ($x \,|\, y \iff x \neq 0 \wedge (\forall z\ zx = 1 \to yz \in \mathcal{O}_K)$).

The following language has been used to study valued rings or fields with additional analytic structure.

4. The ring $\mathcal{O}_K$, in the language of rings augmented by a binary function Div, interpreted by:

$$\mathrm{Div}(x, y) = \begin{cases} xy^{-1} & \text{if } y \mid x, \\ 0 & \text{otherwise.} \end{cases}$$

In all four languages, the residue field k_K, as well as the residue map

$\mathcal{O}_K \to k_K$, are interpretable: k_K is the quotient of $\mathcal{O}_K$ by the maximal ideal $\mathcal{M}_K$ of $\mathcal{O}_K$. And $\mathcal{M}_K$ is of course definable by the formula expressing that the element x is not invertible in $\mathcal{O}_K$.

5. In the same spirit as the first language, here are three more examples of natural many-sorted languages in which one can study valued fields. The first one has three sorts: the valued field, the value group and the residue field, in their natural language, together with the valuation map v, and the residue map res, which coincides with the usual residue map on the valuation ring, and 0 outside. Our valued field K will then be the structure

$$\big((K, +, -, \cdot, 0, 1), (\Gamma \cup \{\infty\}, +, -, 0, \infty, <), (k_K, +, -, \cdot, 0, 1), v, \mathrm{res}\big).$$

We will see later a variant of this language given by Denef-Pas. Another natural language is the following: given a valued field K, let $\mathrm{RV}(K) = K^\times/1 + \mathcal{M}_K$. We then have an exact sequence

$$0 \to k_K^\times \to \mathrm{RV}(K) \overset{val_{rv}}{\to} \Gamma \to 0,$$

where val_{rv} is the natural map induced by the valuation map. To K one associates the following two-sorted structure:

$$\big((K, +, -, \cdot, 0, 1), (\mathrm{RV}(K) \cup \{0\}, \cdot, /, 1, \leq, k^\times, +, 0), v, rv, \big),$$

where the map $rv : K \to \mathrm{RV}(K)$ is the natural quotient map on $K^\times$ and sends 0 to 0; $\cdot$, $/$ and 1 give the group structure of $\mathrm{RV}(K)$ (multiplication or division by 0 can be defined to be 0); $k^\times$ is a unary predicate (for a subgroup of $\mathrm{RV}(K)$), and $+$ is a binary operation on $k = k^\times \cup \{0\}$; finally $\leq$ is interpreted by $x \leq y \iff val_{rv}(x) \leq val_{rv}(y)$. You then see that while the residue field k is definable in $\mathrm{RV}(K)$, the value group Γ is only interpretable in it.
Finally, in mixed characteristic, it is sometimes convenient or necessary to work with

$$\mathrm{RV}_n(K) = K^\times/1 + n\mathcal{M}_K,$$

together with the natural maps $rv_n : K \to \mathrm{RV}_n(K) \cup \{0\}$. The language has now sorts indexed by the integers, and is similar to the one above.

1.7 Multi-sorted structures. Multi-sorted structures appeared naturally in example 4. The difference with the classical ("1-sorted") structures is that a structure will have several *sorts* or *universes*, say indexed by a set I which may be infinite. As already mentioned, each sort comes with its own language, and there may be relations or functions between different sorts or cartesian products of sorts. Some authors require that the universes of distinct sorts be disjoint, but it is not necessary.

Formulas

This subsection and the next are fairly boring, and I would recommend that the reader at first only reads paragraphs 1.10 and 1.13 which give examples. Formulas are built using some basic logical symbols (given below) and in a fashion which ensures unique readibility. Satisfaction is defined in the only possible manner. We give here the formal definitions, and the idea is that the reader can come back to them when he needs a precise definition.

1.8 Terms. We can start using the symbols of $\mathcal{L}$ to express properties of a given $\mathcal{L}$-structure. In addition to the symbols of $\mathcal{L}$, we will consider a set of symbols (which we suppose disjoint from $\mathcal{L}$), called the *set of logical symbols*. It consists of

– logical connectives $\wedge$, $\vee$, $\neg$, and also (for convenience) $\rightarrow$ and $\leftrightarrow$,
– parentheses (and),
– a (binary relation) symbol $=$ for equality,
– infinitely many variable symbols, usually denoted x, y, x_i, etc ...
– the quantifiers $\forall$ (for all) and $\exists$ (there exists).

Fix a language $\mathcal{L}$. An $\mathcal{L}$-formula will then be a string of symbols from $\mathcal{L}$ and logical symbols, obeying certain rules. We start by defining $\mathcal{L}$-*terms* (or simply, terms). Roughly speaking, terms are expressions obtained from constants and variables by applying functions. In any $\mathcal{L}$-structure M, a term t will then define uniquely a function from a certain cartesian power of M to M. Terms are defined by induction, as follows:

– a variable x, or a constant c, are terms.
– if $t_1, \dots, t_n$ are terms, and f is an n-ary function, then $f(t_1, \dots, t_n)$ is a term.

Given a term $t(x_1, \dots, x_m)$, the notation indicating that the variables occurring in t are among $x_1, \dots, x_m$, and an $\mathcal{L}$-structure M, we get a function $F_t : M^m \rightarrow M$. Again this function is defined by induction on the complexity of the term:

– if c is a constant symbol, then $F_c : M^0 \rightarrow M$ is the function $\emptyset \mapsto c^M$,
– if x is a variable, then $F_x : M \rightarrow M$ is the identity,
– if $t_1, \dots, t_n$ are terms in the variables $x_1, \dots, x_m$ and f is an n-ary

function symbol, then $F_{f(t_1,\dots,t_n)} : (x_1,\dots,x_m) \mapsto f(F_{t_1}(\bar{x}),\dots,F_{t_n}(\bar{x}))$ ($\bar{x} = (x_1,\dots,x_m)$).

1.9 Formulas. We are now ready to define formulas. Again they are defined by induction.

An *atomic formula* is a formula of the form $t_1(\bar{x}) = t_2(\bar{x})$ or $R(t_1(\bar{x}),\dots,t_n(\bar{x}))$, where $\bar{x} = (x_1,\dots,x_m)$ is a tuple of variables, $t_1,\dots,t_n$ are terms (of the language $\mathcal{L}$, in the variables $\bar{x}$), and R is an n-ary relation symbol of $\mathcal{L}$.

The set of *quantifier-free formulas* is the set of *Boolean combinations* of atomic formulas, i.e., is the closure of the set of atomic formulas under the operations of $\wedge$ (and), $\vee$ (or) and $\neg$ (negation, or not). So, if $\varphi_1(\bar{x})$, $\varphi_2(\bar{x})$ are quantifier-free formulas, so are $(\varphi_1(\bar{x})\wedge\varphi_2(\bar{x}))$, $(\varphi_1(\bar{x})\vee\varphi_2(\bar{x}))$, and $(\neg\varphi_1(\bar{x}))$.

One often uses $(\varphi_1 \to \varphi_2)$ as an abbreviation for $(\neg\varphi_1)\vee\varphi_2$, and $(\varphi_1 \leftrightarrow \varphi_2)$ as an abbreviation for $(\varphi_1\wedge\varphi_2)\vee[(\neg\varphi_1)\wedge(\neg\varphi_2)]$.

A *formula* ψ is then a string of symbols of the form

$$Q_1x_1Q_2x_2\dots Q_mx_m\,\varphi(x_1,\dots,x_n) \tag{1}$$

where $\varphi(\bar{x})$ is a quantifier-free formula, with variables among $\bar{x} = (x_1,\dots,x_n)$, and $Q_1,\dots,Q_m$ are quantifiers, i.e., belong to $\{\forall,\exists\}$. We may assume $m \leq n$.

Important: the variables $x_1,\dots,x_n$ are supposed distinct - $\forall x_1\exists x_1\dots$ is not allowed. If $m \leq n$, the variables $x_{m+1},\dots,x_n$ are called the *free variables of the formula* ψ. One usually writes $\psi(x_{m+1},\dots,x_n)$ to indicate that the free variables of ψ are among $(x_{m+1},\dots,x_n)$. The variables $x_1,\dots,x_m$ are called the *bound variables of* ψ. If $n = m$, then ψ has no free variables and is called a *sentence.*

If all quantifiers $Q_1,\dots,Q_m$ are $\exists$, then ψ is called an *existential formula*; if they are all $\forall$, then ψ is called a *universal formula.* One can define a hierarchy of complexity of formulas, by counting the number of alternations of quantifiers in the string $Q_1,\dots,Q_n$. Let us simply say that a Π_2*-formula*, also called a $\forall\exists$*-formula*, is one in which $Q_1\dots Q_n$ is a block of $\forall$ followed by a block of $\exists$, that a Σ_2*-formula*, also called a $\exists\forall$*-formula*, is one in which $Q_1\dots Q_n$ is a block of $\exists$ followed by a block of $\forall$. In these definitions, either block is allowed to be empty, so that an existential formula is both a Π_2 and a Σ_2-formula. Let us also mention that a *positive formula* is one of the form $Q_1x_1\dots Q_mx_m\varphi(x_1,\dots,x_n)$, where $\varphi(\bar{x})$ is a finite disjunction of finite conjunctions of *atomic* formulas.

1.10 Warning. This is not the usual definition of a formula. A formula as in (1) is said to be in *prenex form.* The set of formulas in prenex form is **not closed** under Boolean operations. One has however a notion of "*logical equivalence*", under which for instance the formulas $Q_1x_1 \dots Q_mx_m\, \varphi(x_1, \dots, x_m, x_{m+1}, \dots, x_n)$ and $Q_1y_1 \dots Q_my_m$ $\varphi(y_1, \dots, y_m, x_{m+1}, \dots, x_n)$ are logically equivalent. It is then quite easy to see that a Boolean combination of formulas in prenex form is logically equivalent to a formula in prenex form. E.g,

$$(Q_1x_1 \dots Q_mx_m\, \varphi_1(x_1, \dots, x_n)) \wedge (Q'_1x_1 \dots Q'_mx_m\, \varphi_2(x_1, \dots, x_n))$$

is logically equivalent to

$$Q_1x_1Q'_1y_1 \dots Q'_my_m(\varphi_1(x_1, \dots, x_n) \wedge \varphi_2(y_1, \dots, y_m, x_{m+1}, \dots, x_n)).$$

If one wants to be economical about the number of quantifiers, one notes that in general $\forall x\, \varphi_1(x, \dots) \wedge \forall x\, \varphi_2(x, \dots)$ is logically equivalent to $\forall x\, (\varphi_1(x, \dots) \wedge \varphi_2(x, \dots))$, and $\exists x\, \varphi_1(x, \dots) \vee \exists x\, \varphi_2(x, \dots)$ is logically equivalent to $\exists x\, (\varphi_1(x, \dots) \vee \varphi_2(x, \dots))$. For negations, one uses the logical equivalence of $\neg(Q_1x_1 \dots Q_mx_m\, \varphi(x_1, \dots, x_n))$ with $Q'_1x_1 \dots Q'_mx_m$ $\neg(\varphi(x_1, \dots, x_n))$, where $Q'_i = \exists$ if $Q_i = \forall$, $Q'_i = \forall$ if $Q_i = \exists$. Thus the negation of a Π_2-formula is a Σ_2-formula, etc.

Logical equivalence can also be used to rewrite Boolean combinations, and one can show that any quantifier-free formula $\varphi(\bar{x})$ is logically equivalent to one of the form $\bigvee_i \bigwedge_j \varphi_{i,j}(\bar{x})$, where the $\varphi_{i,j}$ are atomic formulas or negations of atomic formulas.

1.11 Adding constant symbols, diagrams. Let $\mathcal{L}$ be a language, M an $\mathcal{L}$-structure, and A a subset of M. The language $\mathcal{L}(A)$ is obtained by adding to $\mathcal{L}$ a new constant symbol symbol $\underline{a}$ for each element a in A. M has then a natural (expansion to an) $\mathcal{L}(A)$-structure: interpret each $\underline{a}$ by the corresponding a. The *basic diagram*, or *atomic diagram* of A in M, $\mathrm{Diag}(A)$ (or $\mathrm{Diag}_M(A)$), is the set of quantifier-free $\mathcal{L}(A)$-sentences satisfied by M.

Example. Let $\mathcal{L} = \mathcal{L}_G$, and $M = \mathbb{Z}$ with the usual group structure, and $A = \{n \in \mathbb{Z} \mid n \geq -1\}$. Then $\mathrm{Diag}(A)$ will contain $\mathcal{L}(A)$-sentences of the following form:

$$\underline{1} + \underline{1} = \underline{2}, \quad \underline{-1} = -\underline{1}, \quad \underline{1} + \underline{4} \neq \underline{3},$$

and so on. Thus, an $\mathcal{L}_G(A)$-structure which is a group and a model of $\mathrm{Diag}(A)$ will be a group in which we have named the elements of a copy of $\{-1\} \cup \mathbb{N}$.

One can also look at more complicated formulas: the *elementary diagram* of A in M, $\mathrm{Diag}_{\mathrm{el}}^M(A)$[1], is the set of all $\mathcal{L}(A)$-sentences which are true in M. Thus for instance with A and M as above, $\mathrm{Diag}_{\mathrm{el}}(A)$ will express the fact that $\underline{1}$ is not divisible by 2 ($\forall x\ x+x \neq \underline{1}$). So, the natural expansion of the group $\mathbb{Q}$ to an $\mathcal{L}_G(A)$-structure is a model of $\mathrm{Diag}(A)$, but not of $\mathrm{Diag}_{\mathrm{el}}(A)$.

In most (all?) situations, we omit the underline on the constant symbol, i.e., denote the same way the constant and its interpretation.

1.12 Examples of formulas. The definitions given above are completely formal. When considering concrete examples, they get very much simplified, to agree with current usage. The first thing to note is that the formula $\neg(x = y)$ is abbreviated by $x \neq y$.

Example 1. $\mathcal{L}_{og} = \{+, -, 0, <\}$. A term is built up from 0, +, −, and some variables. E.g., $+(0, -(+(x_1, -(x_1))))$ is a term, in the variable x_1. If we work in an arbitrary $\mathcal{L}_{og}$-structure, i.e., not necessarily a group, this expression cannot be simplified. If we work in a group, then we will first of all switch to the usual notation of $x + y$ instead of $+(x, y)$, $-x$ instead of $-(x)$ and $x - y$ instead of $x + (-y)$; then we allow ourselves to use the associativity of the group law to get rid of extraneous parentheses. The term above then becomes $0 - (x_1 - x_1)$, which can be further simplified to 0 (we are now using the fact that in all groups, the sentence $\forall x\ x - x = 0$ holds. I.e., this reduction is only valid because we are working *modulo the theory of groups*).

From now on, we will assume that our $\mathcal{L}_{og}$-structures are **commutative groups**. We add to the language some new symbols of constants, $c_1, \ldots, c_n$.

Here are some terms: $x + x$, $x + x + x, \ldots, nx$, $-nx$ ($n \in \mathbb{N}$), $c_1 + c_2$, $2c_3$. General form of a term $t(x_1, \ldots, x_m)$:

$$\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{n} \ell_j c_j,$$

where the n_i, ℓ_j belong to $\mathbb{Z}$. This notation can be a little dangerous,

[1] When the theory T is complete, one often writes $T(A)$ instead of $\mathrm{Diag}_{\mathrm{el}}^M(A)$

as it suggests a uniformity in the coefficients. *One should insists on the fact that if n and m are distinct integers, then the terms nx and mx are different.* [So, in general, the set of torsion elements of a group is not definable in the group G, since an element g is torsion if and only if **for some** n **in** $\mathbb{N}$, $ng = 0$. There are of course exceptions, e.g., if the order of torsion elements is bounded.]

Quantifier-free-formulas: apply relations and Boolean connectives to terms: $\bar{x} = (x_1, \ldots, x_m)$, $t_1(\bar{x}), \ldots, t_4(\bar{x})$ terms:

$$\big(t_1(\bar{x}) = t_2(\bar{x}) \wedge t_3(\bar{x}) < t_4(\bar{x})\big) \vee \quad \big(t_1(\bar{x}) < t_2(\bar{x})\big).$$

Example 2. $\mathcal{L}_R = \{+, -, \cdot, 0, 1\}$. Again, terms as defined formally, are extremely ugly. But, in case all $\mathcal{L}_R$-structures considered are commutative rings, they can be rewritten in a more natural fashion. From now on, **all $\mathcal{L}_R$-structures are commutative rings**.

If $n \in \mathbb{N}^{>1}$ the term $1 + 1 + \cdots + 1$ (n times) will simply be denoted by n. Similarly $x + x + \cdots + x$ (n times) is denoted by nx, and $x \cdot \ldots \cdot x$ (n times) by x^n. An arbitrary term will then be of the form $f(x_1, \ldots, x_n)$, where $f(X_1, \ldots, X_n) \in \mathbb{Z}[X_1, \ldots, X_n]$.

Quantifier-free formulas are finite disjunctions of finite conjunctions of equations and inequations. Thus, in the ring $\mathbb{C}$, they will define the usual *constructible sets* which are defined over $\mathbb{Z}$. If we want to get all constructible sets, we should work in the language $\mathcal{L}_R(\mathbb{C})$, obtained by adding constant symbols for the elements of $\mathbb{C}$.

If one adds $\leq$ to the language, and assumes that our structures are **ordered rings**, then quantifier-free formulas can be rewritten as finite conjunctions of finite disjunctions of formulas of the form

$$f(\bar{x}) = 0, \quad g(\bar{x}) > 0, \tag{2}$$

where f, g are polynomials over $\mathbb{Z}$. Here, $x < y$ stands for $x \leq y \wedge x \neq y$, and one uses the equivalences $x \neq 0 \iff x < 0 \vee x > 0$, $x > 0 \iff (-x) < 0$. If M is an ordered ring, then $\mathcal{L}_{or}(M)$-quantifier-free formulas will be as above, except that f and g are polynomials over M. In case M is the ordered field $\mathbb{R}$, one then gets the usual *semi-algebraic sets*.

Satisfaction

1.13 Satisfaction. Let M be an $\mathcal{L}$-structure, $\varphi(\bar{x})$ an $\mathcal{L}$-formula, where $\bar{x} = (x_1, \ldots, x_n)$ is a tuple of variables occurring freely in φ, and $\bar{a} = (a_1, \ldots, a_n)$ an n-tuple of elements of M. We wish to define the notion M *satisfies* $\varphi(\bar{a})$, (or $\bar{a}$ *satisfies* φ *in* M, or $\varphi(\bar{a})$ *holds in* M, *is true in* M), denoted by

$$M \models \varphi(\bar{a}).$$

(The negation of $M \models \varphi(\bar{a})$ is denoted by $M \not\models \varphi(\bar{a})$, or ... by $M \models \neg\varphi(\bar{a})$.) Satisfaction is what it should be if you read the formula aloud. Here is a formal definition, by induction on the complexity of the formulas. It is fairly boring, and if you wish you can skip it. Let $\bar{a}$, $\bar{b}$ be tuples in M,

– If $\varphi(\bar{x})$ is the formula $t_1(\bar{x}) = t_2(\bar{x})$, where t_1, t_2 are $\mathcal{L}$-terms in the variable $\bar{x}$, then

$$M \models t_1(\bar{a}) = t_2(\bar{a}) \text{ if and only if } F_{t_1}(\bar{a}) = F_{t_2}(\bar{a}).$$

– If $\varphi(\bar{x})$ is the formula $R(t_1(\bar{x}), \ldots, t_m(\bar{x}))$, where $t_1, \ldots, t_m$ are terms and R is an m-ary relation, then

$$M \models R(t_1(\bar{a}), \ldots, t_m(\bar{a})) \text{ if and only if } (F_{t_1}(\bar{a}), \ldots, F_{t_m}(\bar{a})) \in R^M.$$

– If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \vee \varphi_2(\bar{x})$, then

$$M \models \varphi(\bar{a}) \text{ if and only if } M \models \varphi_1(\bar{a}) \textbf{ or } M \models \varphi_2(\bar{a}).$$

– If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$, then

$$M \models \varphi(\bar{a}) \text{ if and only if } M \models \varphi_1(\bar{a}) \textbf{ and } M \models \varphi_2(\bar{a}).$$

– If $\varphi(\bar{x}) = \neg\varphi_1(\bar{x})$, then

$$M \models \varphi(\bar{a}) \text{ if and only if } M \not\models \varphi_1(\bar{a}).$$

– If $\varphi(\bar{x}) = \exists y\, \psi(\bar{x}, y)$, where the free variables of ψ are among $\bar{x}, y$, then

$$M \models \varphi(\bar{a}) \text{ if and only if } \textbf{there is } c \in M \text{ such that } M \models \psi(\bar{a}, c).$$

– If $\varphi(\bar{x}) = \forall y\, \psi(\bar{x}, y)$, then

$$\begin{aligned} M \models \varphi(\bar{a}) &\text{ if and only if } M \models \neg(\exists y\, \neg\psi(\bar{a}, y)) \\ &\text{ if and only if } \textbf{for all } c \text{ in } M, \quad M \models \varphi(\bar{a}, c). \end{aligned}$$

Note that of course, for all $\bar{a}$ in M, one has

$$M \models \forall y\, \psi(\bar{a}, y) \text{ if and only if } M \models \neg(\exists y\, \neg\psi(\bar{a}, y)).$$

1.14 Parameters, definable sets. Let M be an $\mathcal{L}$-structure, $\varphi(\bar{x}, \bar{y})$ a formula ($\bar{x}$ an n-tuple of variables, $\bar{y}$ an m-tuple of variables), and $\bar{a} \in M^n$. Then the set $\{\bar{b} \in M^m \mid M \models \varphi(\bar{a}, \bar{b})\}$ is called a *definable set*. We also say that it is *defined over* $\bar{a}$ by the formula $\varphi(\bar{a}, \bar{y})$, or that it is $\bar{a}$*-definable*. The tuple $\bar{a}$ is a *parameter of the formula* $\varphi(\bar{a}, \bar{y})$. When $\bar{a}$ varies over M^n, the sets $\{\bar{b} \in M^m \mid M \models \varphi(\bar{a}, \bar{b})\}$, which are sometimes denoted by $\varphi(\bar{a}, M^m)$ or by $\varphi(\bar{a}, M)$, form a *family of uniformly definable sets*.

Let M be an $\mathcal{L}$-structure. The set of $\mathcal{L}(M)$-definable subsets of M^n is clearly closed under unions, intersections and complements (corresponding to the use of the logical connectives $\vee$, $\wedge$ and $\neg$). If $S \subseteq M^{n+1}$ is defined by the formula $\varphi(\bar{x}, \bar{a})$, $\bar{x} = (x_1, \dots, x_{n+1})$, and $\pi : M^{n+1} \to M$ is the projection on the first n coordinates, then $\pi(S)$ is defined by the formula $\exists x_{n+1}\, \varphi(\bar{x}, \bar{a})$, and the complement of $\pi(S)$ by the formula $\forall x_{n+1}\, \neg\varphi(\bar{x})$.

Thus an alternate definition of $\mathcal{L}$-subsets of M is as follows: it is the smallest collection $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$, where each $\mathcal{S}_n$ is a set of subsets of M^n, which satisfies the following conditions:

- $\mathcal{S}_1$ contains all singletons of constants; if $f \in \mathcal{L}$ is an n-ary function symbol, then the graph of f is in $\mathcal{S}_{n+1}$; if R is an n-ary function symbol, then the interpretation of R is in $\mathcal{S}_n$; $\mathcal{S}_2$ contains the diagonal.
- Each $\mathcal{S}_n$ is closed under Boolean operations $\cup$, $\cap$, and complement.
- $\mathcal{S}$ is closed under (finite) cartesian products.
- If $\pi : M^{n+1} \to M^n$ is a projection on an n-subset of the coordinates, and $S \in \mathcal{S}_{n+1}$, then $\pi(S) \in \mathcal{S}_n$.

1.15 An example. Consider the $\mathcal{L}_{og}$-formula

$$\varphi(x, y) := x < y \wedge (\forall z\, x < z \to z = y \vee y < z),$$

where as usual $x < y$ is an abbreviation for $x \leq y \wedge x \neq y$. In an ordered group G, this formula expresses that y is an immediate successor of x. Thus, in $(\mathbb{Z}, +, -, 0, \leq)$, the formula will define the graph of the successor function. But in $(\mathbb{Q}, +, -, 0, \leq)$ it will define the empty set, as $\mathbb{Q}$ is a dense ordering.

Definability, interpretability

1.16 Definability of a structure in another one. Let M be an

$\mathcal{L}$-structure, and N an $\mathcal{L}'$-structure. We say that N is *definable in M* if there are

– an $\emptyset$-definable set $S \subseteq M^n$ for some n,

– for each m-ary relation R of the language $\mathcal{L}'$, an $\emptyset$-definable subset R^* of S^m,

– for each m-ary function f of $\mathcal{L}'$, an $\emptyset$-definable subset Γ_f of S^{m+1} such that Γ_f is the graph of a function $f^* : S^m \to S$,

– for each constant c, an $\emptyset$-definable tuple $c^* \in S$,

– and a bijection $F : N \to S$, which defines an $\mathcal{L}'$-isomorphism between the structure $N = (N, R \ldots, f, \ldots, c, \ldots)$ and the structure

$$N^* = (S, R^*, \ldots, f^*, \ldots, c^* \ldots).$$

1.17 Interpretability of a structure in another one. Let M be an $\mathcal{L}$-structure, N an $\mathcal{L}'$-structure. We say that N is *interpretable in N* if there are

– an $\emptyset$-definable set S of some M^ℓ,

– an $\emptyset$-definable equivalence relation E on S,

– for each m-ary relation R of the language $\mathcal{L}'$, an $\emptyset$-definable subset R' of S^m, which projects to a subset R^* of $(S/E)^m$,

– for each m-ary function f of $\mathcal{L}'$, an $\emptyset$-definable subset Γ_f of S^{m+1} such that Γ_f induces the graph of a function $f^* : (S/E)^m \to S$,

– for each constant c, an $\emptyset$-definable E-equivalence class $c^* \in S/E$,

– and a bijection $F : N \to S/E$, which defines an $\mathcal{L}'$-isomorphism between the structure $N = (N, R \ldots, f, \ldots, c, \ldots)$ and the structure

$$N^* = (S/E, R^*, \ldots, f^*, \ldots, c^* \ldots).$$

1.18 Adding parameters. In both definitions, if instead of working in the $\mathcal{L}$-structure M, one works in the $\mathcal{L}(A)$-structure M for some $A \subset M$, one will say that N is A-definable, resp. A-interpretable, in M.

1.19 Bi-interpretability of two structures. Let M be an $\mathcal{L}$-structure, N an $\mathcal{L}'$-structure. We say that M and N are *bi-interpretable* if

(i) N is interpretable in M, and M is interpretable in N,
(ii) the bijections F and F' which give the interpretations of N in M and of M in N respectively, can be chosen so that the maps $F \circ F'$ and $F' \circ F$ are $\emptyset$-definable in M and N respectively.

1.20 Example. Let R be an integral domain, in the usual ring language, and let Q be its field of fractions. Then the ring Q is interpretable in R: indeed, we know that Q is the set of quotients a/b with $a \in R$,

$0 \neq b \in R$, so we can identify it with the set of elements of $R \times R^*$ quotiented by the equivalence relation $(a,b) \sim (c,d) \iff ad = bc$. In general, Q is not definable in R, because there is no way of selecting a particular pair in each equivalence class. However, $\mathbb{Q}$ is definable in $\mathbb{Z}$: $\mathbb{Z}$ is a principal ideal domain, in which an ordering is definable, and with only units -1 and 1.

2 Theories, and some important theorems

In this section we will introduce many definitions and important concepts. We will also mention the very important *Compactness theorem*, one of the crucial tools of model theory.

Theories and models

2.1 Theories, models of theories, etc.. Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-theory* (or simply, a theory), is a set of sentences of the language $\mathcal{L}$. A *model of a theory* T is an $\mathcal{L}$-structure M which satisfies all sentences of T, denoted by $M \models T$. The class of all models of T is denoted $\mathrm{Mod}(T)$. If $\mathcal{K}$ is a class of $\mathcal{L}$-structures, then $\mathrm{Th}(\mathcal{K})$ denotes the set of all sentences true in all elements of $\mathcal{K}$, and $\mathrm{Th}(\{M\})$ is denoted by $\mathrm{Th}(M)$.

A theory T is *consistent* iff it has a model. If φ is a sentence which holds in all models of T, this is denoted by $T \models \varphi$. Two $\mathcal{L}$-structures M and N are *elementarily equivalent*, denoted $M \equiv N$, iff they satisfy the same sentences, iff $\mathrm{Th}(M) = \mathrm{Th}(N)$. A theory is *complete* iff given a sentence φ, either $T \models \varphi$ or $T \models \neg\varphi$. Equivalently, if any two models of T are elementarily equivalent. (Observe that if M is an $\mathcal{L}$-structure, then necessarily $\mathrm{Th}(M)$ is complete).

Elementary equivalence is an equivalence relation between $\mathcal{L}$-structures. Two isomorphic $\mathcal{L}$-structures are clearly elementarily equivalent, however the converse only holds **for finite $\mathcal{L}$-structures**. A famous theorem (of Keisler-Shelah) states that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers, see definition in Section 2.14.

2.2 Elementary substructures, extensions, embeddings, etc. Let $M \subseteq N$ be $\mathcal{L}$-structures. We say that M is an *elementary substructure of* N, or that N *is an elementary extension of* M, denoted by

$M \prec N$, iff for any formula $\varphi(\bar{x})$ and tuple $\bar{a}$ from M,

$$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a}).$$

A map $f : M \to N$ is an *elementary embedding* iff it is an embedding, and if $f(M) \prec N$. In other words, if for any formula $\varphi(\bar{x})$ and tuple $\bar{a}$ from M, $M \models \varphi(\bar{a})$ if and only $N \models \varphi(f(\bar{a}))$.

Using the language of diagrams introduced in 1.11,

$$M \prec N \iff N \models \mathrm{Diag}^M_{\mathrm{el}}(M).$$

Similarly, an *elementary partial map from M to N* is a map f defined on some substructure A of M, with range included in N, and which preserves the formulas in $\mathrm{Diag}^M_{\mathrm{el}}(A)$, i.e., for any formula $\varphi(\bar{x})$ and tuple $\bar{a}$ from A, $M \models \varphi(\bar{a})$ if and only $N \models \varphi(f(\bar{a}))$. A map f which only preserves $\mathrm{Diag}^M(A)$ is called a *partial isomorphism*.

2.3 Comments. (1) Note that one can have $M \subseteq N$ and $M \equiv N$ without having $M \prec N$. Consider the group $\mathbb{Z}$ and its subgroup $2\mathbb{Z}$: they are isomorphic, but the inclusion $2\mathbb{Z} \subset \mathbb{Z}$ is not an elementary map, since 2 is divisible by 2 in $\mathbb{Z}$ but not in $2\mathbb{Z}$.
(2) Similarly, not every partial isomorphism is elementary. Again, the inclusion of $2\mathbb{Z}$ into $\mathbb{Z}$ provides an example.

Some classical results

2.4 Tarski's test. *Let M be a substructure of N. Then $M \prec N$ if and only if, for every formula $\varphi(\bar{x}, y)$ and tuple $\bar{a}$ in M, if $N \models \exists y\, \varphi(\bar{a}, y)$, then there exists $b \in M$ such that $N \models \varphi(\bar{a}, b)$.*

Note that while the element b is in M, the satisfaction is taken in N. This theorem is proved using induction on the complexity of formulas.

2.5 Soundness and completeness theorem. Given a set of sentences, there is a notion of proof, i.e., which statements are deducible from the given statements using some formal rules of deduction, such as modus ponens (from A and $A \to B$ deduce B), and some substitution rules (from a sentence of the form $\varphi(c)$ where c is a constant, deduce $\exists x\, \varphi(x)$). A proof can be thought of therefore as a finite sequence of sentences, each being obtained from the previous ones by applying some deduction rules. We use the notation

$$T \vdash \varphi$$

to indicate that there is a proof of φ from T. This is not to be confused with the notation

$$T \models \varphi$$

which means that φ is true in all models of T. The first result, the *soundness theorem*, tells us that our notion of satisfaction is well-defined: *If a theory T has a model, then one cannot derive a contradiction from T, i.e., one cannot prove from T the sentence $\forall x(x \neq x)$.*

In other words

$$T \vdash \varphi \Rightarrow T \models \varphi.$$

Gödel's completeness theorem then states the converse:
If from a given theory T, one cannot derive the sentence $\forall x(x \neq x)$, then the theory T has a model.

Another way of stating this result is by saying that
the set of sentences deducible from a given theory T is **exactly** *the set of sentences true in* **all models of** *T, i.e., in the notation introduced above, it coincides with* $\mathrm{Th}(\mathrm{Mod}(T))$.

2.6 Decidability. A theory T is *decidable*, if there is an algorithm allowing to decide whether a sentence φ holds in all models of T or not. If one can enumerate a theory T and one knows (somehow) that T is complete, then T is decidable: given a sentence φ, start enumerating the proofs from T; eventually you reach a proof of either φ or $\neg\varphi$.

2.7 Compactness theorem. *Let T be a set of sentences in a language $\mathcal{L}$. If every finite subset of T has a model, then T has a model.*

We will present later a proof of this theorem using ultraproducts. Note that it is a consequence of the completeness theorem, since any proof involves only finitely many elements of T. It also has for consequence the first half of the next theorem.

2.8 Löwenheim-Skolem Theorems. *Let $\mathcal{L}$ be a language, T a theory, and let M be an infinite model of T.*

(1) Let κ be an infinite cardinal, $\kappa \geq |M| + |\mathcal{L}|$. Then M has an elementary extension N with $|N| = \kappa$.
(2) Let X be a subset of M. Then M has an elementary substructure N containing X, with $|N| \leq |X| + |\mathcal{L}| + \aleph_0$.

2.9 Comments. These results allow us to use large models with good properties. For instance, assume that we have a set $\Sigma(x_1, \ldots, x_n)$ of formulas in the variables $(x_1, \ldots, x_n)$, and that we know that every finite fragment of $\Sigma(x_1, \ldots, x_n)$ is satisfiable in some model M of T, i.e., there is a tuple $\bar{a}$ of M which satisfies all formulas of that finite fragment. Then there is a model N of T containing a tuple $\bar{b}$ which satisfies simultaneously all formulas of $\Sigma(\bar{x})$. This is connected to saturation, see below for a definition.

Using other techniques, one can show that if $\bar{a}$ and $\bar{b}$ are tuples of an $\mathcal{L}$-structure M, which satisfy the same formulas in M, then M has an elementary extension M^*, in which there is an automorphism which sends $\bar{a}$ to $\bar{b}$.

2.10 Craig's interpolation theorem. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages. Let φ be a sentence of $\mathcal{L}_1$ and ψ a sentence of $\mathcal{L}_2$. If $\varphi \models \psi$, then there is a sentence θ of $\mathcal{L}_1 \cap \mathcal{L}_2$ such that $\varphi \models \theta$ and $\theta \models \psi$.*

A somewhat different interpolation theorem is given by Robinson:
Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages, and $\mathcal{L}_0 = \mathcal{L}_1 \cap \mathcal{L}_2$. Assume that T_1 and T_2 are theories in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, such that $T_0 = T_1 \cap T_2$ is complete. Then $T_1 \cup T_2$ is consistent.

Types, saturated models

Fix a complete theory T in a language $\mathcal{L}$, a subset A of a model M of T. A *(partial) n-type over A* (in the variables $\bar{x} = (x_1, \ldots, x_n)$) is a collection $p(\bar{x})$ of $\mathcal{L}(A)$-formulas which is finitely consistent in M. A *complete type over A* is an n-type $p(\bar{x})$ which is maximally consistent, i.e., given an $\mathcal{L}(A)$-formula $\varphi(\bar{x})$, one of $\varphi(\bar{x})$, $\neg\varphi(\bar{x})$ belongs to $p(\bar{x})$. The set of complete n-types over A is denoted $S_n(A)$. Here is an example: let $\bar{a}$ be an n-tuple in M. Then

$$tp(\bar{a}/M) := \{\varphi(\bar{x}) \in \mathcal{L}(A) \mid M \models \varphi(\bar{a})\},$$

the *type of $\bar{a}$ over A*, is a complete type.

Warning: depending on the context a *type* can mean either a partial type, or a complete type. There is no set usage.

Given an n-type $p(\bar{x})$ over A, a *realisation of p in M* is an n-tuple a in M which satisfies all formulas of $p(\bar{x})$. In any case, there will be an elementary extension N of M in which $p(\bar{x})$ will be realised.

2.11 Topology on the space of types. Given $A \subset M$ and $n > 0$ as above, one defines a topology on $S_n(A)$, whose basic open sets are

$$\langle \varphi(\bar{x}) \rangle = \{p(\bar{x}) \in S_n(A) \mid \varphi(\bar{x}) \in p(\bar{x})\}.$$

Then $S_n(A)$ is compact, totally disconnected. A type $p(\bar{x}) \in S_n(A)$ is *isolated* if and only if there is an $\mathcal{L}(A)$-formula which implies all formulas in $p(\bar{x})$.

2.12 Saturated models. Let κ be an infinite cardinal, M an $\mathcal{L}$-structure. We say that M is *κ-saturated* if for every subset A of M of cardinality $< \kappa$, every n-type over A is realised in M. We say that M is *saturated* if it is $|M|$-saturated. Observe that an infinite $\mathcal{L}$-structure M can never be $|M|^+$-saturated: consider the set of $\mathcal{L}(M)$-formulas $\{x \neq m \mid m \in M\}$.

Expressed in terms of definable sets: let $\mathcal{S}_n$ be the set of $\mathcal{L}(A)$-definable subsets of M^n. Then the $|A|^+$-saturation[2] of M means that if $(D_i)_{i \in I} \subset \mathcal{S}_n$ is such that the intersection of any finite collection of D_i's is non-empty[3], then there is a tuple $\bar{a}$ in the intersection of all D_i's.

Non-example. Consider the ordered group $(\mathbb{R}, +, -, 0 <)$. It is not $\aleph_0$-saturated: take $A = \{1\}$, and consider

$$\Sigma(x) = \{x > n \mid n \in \mathbb{N}\}.$$

This set of formulas is finitely consistent: for any n, the finite fragment $\{x > m \mid 0 \leq m \leq n\}$ is satisfied in $\mathbb{R}$ by $n+1$. However, no element of $\mathbb{R}$ is greater than all elements of $\mathbb{N}$. In fact, a (non-trivial) ordered abelian group which is $\aleph_0$-saturated cannot be archimedean. Note that this argument only works because the elements of $\mathbb{N}$ can be obtained as terms in $\mathcal{L}_{og}(A)$; one can show that the ordered **set** $(\mathbb{R}, <)$ is $\aleph_0$-saturated (but not $\aleph_1$-saturated, since $\mathbb{N}$ is countable and cofinal in $\mathbb{R}$).

2.13 Important results concerning saturated models:

Let κ be an infinite cardinal, M an infinite $\mathcal{L}$-structure. Then M has an elementary extension M^* which is κ-saturated.

In contrast, given an infinite cardinal κ and a theory T, there does not always exist a saturated model of T of cardinality κ. Under GCH[4], a theory T with infinite models has uncountable saturated models of any

[2] if A is finite, one considers instead $\aleph_0$-saturation.
[3] One then says that $\{D_i \mid i \in I\}$ has the *finite intersection property.*
[4] The General Continuum Hypothesis, which says that given an infinite cardinal κ,

cardinality.
A saturated model M of T has the following properties:
(i) (Universality) Any model of T of cardinality $< |M|$ embeds elementarily into M.
(ii) (Homogeneity). If $f : A \to B$ is an elementary partial map between subsets A and B of M of cardinality $< |M|$, then f extends to an automorphism of M.

2.14 Definable and algebraic closures. Let T be a complete $\mathcal{L}$-theory, A a subset of a model M of T. We say that an element $a \in M$ is *algebraic over* A, noted $a \in \mathrm{acl}(A)$, if there is an $\mathcal{L}(A)$-formula $\varphi(x)$ which defines a finite subset of M containing a. We say that a is *definable over* A, noted $a \in \mathrm{dcl}(A)$, if there is such a formula $\varphi(x)$ which defines $\{a\}$. An *algebraic, resp. definable, tuple* is one whose elements are algebraic, resp. definable. If $\bar{a} \in \mathrm{acl}(A)$, then $tp(\bar{a}/A)$ is isolated. Clearly one has

$$\mathrm{dcl}(A) \subseteq \mathrm{acl}(A),\ \mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A),\ \mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A).$$

An alternate way of defining definable and algebraic closures is via automorphism groups: let M be a saturated model of cardinality $> |A|$, and $G = \mathrm{Aut}(M/A)$. Then $a \in \mathrm{dcl}(A)$ if and only if the G-orbit of a has only one element, and $a \in \mathrm{acl}(A)$ if and only if the G-orbit of a is finite.

Ultraproducts, Łos Theorem

In this section we introduce an important tool: ultraproducts. They are at the centre of many applications, within and outside model theory.

2.15 Filters and ultrafilters. Let I be a set. A *filter on I* is a subset $\mathcal{F}$ of $\mathcal{P}(I)$ (the set of subsets of I), satisfying the following properties:

(1) $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$.
(2) If $U \in \mathcal{F}$ and $V \supseteq U$, then $V \in \mathcal{F}$.
(3) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

An *ultrafilter on I* is a filter on I which is maximal for inclusion. Equivalently, it is a filter $\mathcal{F}$ such that for any $U \in \mathcal{P}(I)$, either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$.

a set I of cardinality κ, the successor cardinal of κ is the cardinality (2^κ) of the set of subsets of I. That is: $\kappa^+ = 2^\kappa$ for all $\kappa \geq \aleph_0$.

2.16 Remarks. (1) Note that condition (1) above forbids that both U and $I \setminus U$ belong to the same filter on I.

(2) Using Zorn's lemma (and therefore the axiom of choice), every filter on I is contained in an ultrafilter.

(3) If $\mathcal{G} \subset \mathcal{P}(I)$ has the *finite intersection property* (i.e., the intersection of finitely many elements of $\mathcal{G}$ is never empty), then $\mathcal{G}$ is contained in a filter. The *filter generated by* $\mathcal{G}$ is then the set of elements of $\mathcal{P}(I)$ containing some finite intersection of elements of $\mathcal{G}$.

2.17 Principal and non-principal ultrafilters, Fréchet filter. Let I be a set. An ultrafilter $\mathcal{F}$ on I is *principal* if there is $i \in I$ such that $\{i\} \in \mathcal{F}$ (and then we will have: $U \in \mathcal{F} \iff i \in U$). An ultrafilter is *non-principal* if it is not principal. Note that if I is finite, then every ultrafilter on I is principal.

Let I be infinite. The *Fréchet filter on* I is the filter $\mathcal{F}_0$ consisting of all cofinite subsets of I. An ultrafilter $\mathcal{F}$ on I is then non-principal if and only if contains the Fréchet filter on I. Note that if $S \subseteq I$ is infinite, then $\mathcal{F}_0 \cup \{S\}$ has the finite intersection property, so that it is contained in an ultrafilter.

2.18 Cartesian products of $\mathcal{L}$-structures. Fix a language $\mathcal{L}$. Let I be an index set, and (M_i), $i \in I$, a family of $\mathcal{L}$-structures. We define the $\mathcal{L}$-structure $M = \prod_{i\in I} M_i$ as follows:

— The universe of M is simply the cartesian product of the M_i's, i.e., the set of sequences $(a_i)_{i\in I}$ such that $a_i \in M_i$ for each $i \in I$. One sometimes uses the functional notation $a(i)$ instead of a_i.

— If c is a constant symbol of $\mathcal{L}$, then $c^M = (c^{M_i})_{i\in I}$.

— If R is an n-ary relation symbol, then $R^M = \prod_{i\in I} R^{M_i}$.

— If f is an n-ary function symbol and $((a_{1,i})_i, \dots, (a_{n,i})_i) \in M^n$, then

$$f^M((a_{1,i})_i, \dots, (a_{n,i})_i) = (f^{M_i}(a_{1,i}, \dots, a_{n,i}))_{i\in I}.$$

2.19 Reduced products of $\mathcal{L}$-structures. Let I be a set, $\mathcal{F}$ a filter on I, and (M_i), $i \in I$, a family of $\mathcal{L}$-structures. The *reduced product of the M_i's over $\mathcal{F}$*, denoted by $\prod_{i\in I} M_i/\mathcal{F}$, is the $\mathcal{L}$-structure defined as follows:

— The universe of $\prod_{i\in I} M_i/\mathcal{F}$ is the quotient of $\prod_{i\in I} M_i$ by the equivalence relation $\equiv_{\mathcal{F}}$ defined by

$$(a_i)_i \equiv_{\mathcal{F}} (b_i)_i \iff \{i \in I \mid a_i = b_i\} \in \mathcal{F}.$$

We denote by $(a_i)_{\mathcal{F}}$ the equivalence class of the element $(a_i)_i$ for this equivalence relation.

The structure on $\prod_{i\in I} M_i/\mathcal{F}$ is then simply the "quotient structure", i.e.,

- The interpretation of c is $(c^{M_i})_{\mathcal{F}}$, for c a constant symbol of $\mathcal{L}$.
- If R is an n-ary relation symbol, and if $a_1,\dots,a_n \in \prod_{i\in I} M_i/\mathcal{F}$ are represented by $(a_{1,i})_i,\dots,(a_{n,i})_i \in \prod_{i\in I} M_i$, then we set
$$\prod_{i\in I} M_i/\mathcal{F} \models R(a_1,\dots,a_n) \iff \{i\in I \mid (a_{1,i},\dots,a_{n,i}) \in R^{M_i}\} \in \mathcal{F}.$$
- If f is an n-ary function symbol and if $a_1,\dots,a_n \in \prod_{i\in I} M_i/\mathcal{F}$ are represented by $(a_{1,i})_i,\dots,(a_{n,i})_i \in \prod_{i\in I} M_i$, then we set
$$f^M(a_1,\dots,a_n) = (f^{M_i}(a_{1,i},\dots,a_{n,i}))_{\mathcal{F}}.$$

The properties of filters guarantee that the quotient structure is well-defined. Note that the quotient map : $\prod_{i\in I} M_i \to \prod_{i\in I} M_i/\mathcal{F}$, $(a_i)_i \mapsto (a_i)_{\mathcal{F}}$, is a morphism of $\mathcal{L}$-structures.

Definitions. If all structures M_i are equal to the same structure M, then we write $M^I/\mathcal{F}$ instead of $\prod_i M_i/\mathcal{F}$, and the structure is called a *reduced power of* M. If the filter $\mathcal{F}$ is an ultrafilter, then $\prod_i M_i/\mathcal{F}$ is called the *ultraproduct of the* M_i*'s* (with respect to $\mathcal{F}$), and $M^I/\mathcal{F}$ the *ultrapower of* M (with respect to $\mathcal{F}$).

2.20 Łos Theorem. *Let I be a set, $\mathcal{F}$ an ultrafilter on I, and (M_i), $i\in I$, a family of $\mathcal{L}$-structures. Let $\varphi(x_1,\dots,x_n)$ be an $\mathcal{L}$-formula, and let $a_1,\dots,a_n \in \prod_{i\in I} M_i/\mathcal{F}$ be represented by $(a_{1,i})_i,\dots,(a_{n,i})_i \in \prod_{i\in I} M_i$. Then*
$$\prod_{i\in I} M_i/\mathcal{F} \models \varphi(a_1,\dots,a_n) \iff \{i\in I \mid M_i \models \varphi(a_{1,i},\dots,a_{n,i})\} \in \mathcal{F}.$$

2.21 Corollary. *Let I be a set, $\mathcal{F}$ an ultrafilter on I, and M an $\mathcal{L}$-structure. Then the natural map $M \to M^I/\mathcal{F}$, $a \mapsto (a)_{\mathcal{F}}$, is an elementary embedding. (Here $(a)_{\mathcal{F}}$ is the equivalence class of the sequence with all terms equal to a).*

2.22 Remarks and comments. Let I be an infinite index set, and $\mathcal{F}$ an ultrafilter on I.

(1) If $\mathcal{F}$ is principal, say $\{j\} \in \mathcal{F}$, then $\prod_{i\in I} M_i/\mathcal{F} \simeq M_j$ for any family of $\mathcal{L}$-structures M_i, $i\in I$.

(2) Suppose that the M_i's are fields, with maybe additional structure (e.g., an ordering, new functions, etc.). Consider the ideal $\mathcal{M}$ of $\prod_i M_i$ generated by all elements $(a_i)_i$ such that $\{i \in I \mid a_i = 0\} \in \mathcal{F}$. Then $\mathcal{M}$ is a maximal ideal of $\prod_i M_i$, and quotienting by the equivalence relation $\equiv_{\mathcal{F}}$ is equivalent to quotienting by the maximal ideal $\mathcal{M}$. The strength of Łos theorem is to tell you that the elementary properties of the M_i's, including the ones depending on the additional structure, are preserved. E.g., that $\mathbb{R}^I/\mathcal{F}$ is a real closed field.

2.23 Keisler and Shelah's isomorphism theorem. *Let M and N be two $\mathcal{L}$-structures. Then $M \equiv N$ if and only if there is an ultrafilter $\mathcal{F}$ on a set I such that $M^I/\mathcal{F} \simeq N^I/\mathcal{F}$.*

Note the following immediate consequence: if $M \equiv N$, then there is M^* in which both M and N embed elementarily.

2.24 Application 1: another proof of the compactness theorem. *Let T be a theory in a language $\mathcal{L}$, and assume that every finite subset s of T has a model M_s. Then T has a model.*

Proof. If T is finite, there is nothing to prove, so we will assume that T is infinite. Let I be the set of all finite subsets of T. For every $\varphi \in T$, let $S(\varphi) = \{s \in I \mid \varphi \in s\}$. Then the family $\mathcal{G} = \{S(\varphi) \mid \varphi \in T\}$ has the finite intersection property, and therefore is contained in an ultrafilter $\mathcal{F}$. We claim that $\prod_{s\in I} M_s/\mathcal{F}$ is a model of T: let $\varphi \in T$. Then, by assumption, $\{s \in I \mid M_s \models \varphi\}$ contains $S(\varphi)$, and therefore belongs to $\mathcal{F}$. By Łos's theorem, $\prod_{s\in I} M_s/\mathcal{F} \models \varphi$.

2.25 Application 2: $\aleph_1$-saturated models. *If I is an infinite countable set, $\mathcal{U}$ is a non-principal ultrafilter on I, and $(M_i)_{i\in I}$ is a family of $\mathcal{L}$-structures where $\mathcal{L}$ is a countable language, then the ultraproduct $\prod_{i\in I} M_i/\mathcal{U}$ is $\aleph_1$-saturated.*

Proof. If there is a finite bound on the cardinalities of the M_i's, then $M^* = \prod_{i\in I} M_i/\mathcal{U}$ is finite, and there is nothing to prove, so assume this is not the case. Let $A \subset \prod_{i\in I} M_i/\mathcal{U}$ be countable, and $\Sigma(x)$ be a set of $\mathcal{L}(A)$-formulas which is finitely consistent. Then $\Sigma(x)$ is countable, and we choose an enumeration $\varphi_n(x, \bar{a}_n)$, $n \in \mathbb{N}$, of $\Sigma(\bar{x})$ ($\varphi(x, \bar{y}) \in \mathcal{L}$, $\bar{a}_n$ a finite tuple in A, represented by $(\bar{a}_n(i))_i \in \prod_i M_i$). We may assume that $I = \mathbb{N}$. For each n, let

$$S(n) = \{j \in I \mid M_j \models \exists x \bigwedge_{i\leq n} \varphi_i(x, \bar{a}_i(j))\}.$$

By assumption, each $S(n)$ is in $\mathcal{U}$, and $S(n)$ contains $S(n+1)$. For

$n \in I = \mathbb{N}$, we choose $b_n \in M_n$ in the following fashion: if $n \in S(n)$, take some $b_n \in M_n$ such that $M_n \models \bigwedge_{i\leq n} \varphi_i(b_n, \bar{a}_i(n))$; if $n \notin S(n)$, take for b_n any element of M_n. Then, for each n,

$$\{j \in I \mid M_j \models \varphi_n(b_j, \bar{a}_n(j))\} \supseteq S(n) \cap [n, +\infty),$$

and is therefore in $\mathcal{U}$. Hence, $M^* \models \varphi_n((b_j)_{\mathcal{U}}, \bar{a}_n)$.

Elimination of quantifiers

2.26 Elimination of quantifiers. Formulas with more than two alternations of quantifiers are fairly awkward, and usually difficult to decide the truth of. One therefore tries to "eliminate quantifiers".

Definition. A theory T *eliminates quantifiers* if for any formula $\varphi(\bar{x})$ there is a quantifier-free formula $\psi(\bar{x})$ which is *equivalent to* $\varphi(\bar{x})$ *modulo* T, i.e., is such that

$$T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Note that the set of free variables in φ and ψ are the same. Thus if φ is a sentence, so is ψ. (If the language has no constant symbol, then one allows ψ to be either $\top$ (true) or $\perp$ (false); if the language contains a constant symbol c, then one can use instead the formulas $c = c$ or $c \neq c$).

Expressed in terms of definable sets, this means: whenever M is a model of T, $S \subset M^{n+1}$ is *quantifier-free definable* (i.e., definable by a formula without quantifiers), and if $\pi : M^{n+1} \to M^n$ is the projection on the first n coordinates, then $\pi(S)$ is also quantifier-free definable.

Expressed in terms of diagrams, this is equivalent to: whenever M is a model of T and $A \subset M$, then $T \cup \mathrm{Diag}^M(A)$ is complete (in the language $\mathcal{L}(A)$).

2.27 Criterion for quantifier elimination: back and forth arguments. *Let T be a theory in a language $\mathcal{L}$, and Δ a set of $\mathcal{L}$-formulas, closed under finite conjunctions and disjunctions. The following are equivalent:*

(1) Every $\mathcal{L}$-formula is equivalent modulo T to a formula of Δ.

(2) Whenever M and N are $\aleph_1$-saturated models of T, $A \subset M$ and $B \subset N$ are countable (non-empty) substructures and $f : A \to B$ is a morphism which preserves the formulas in Δ (i.e., if $\bar{a}$ is a tuple in M, and $\varphi(\bar{x}) \in \Delta$, then $M \models \varphi(\bar{a}) \Rightarrow N \models \varphi(f(\bar{a}))$), then

- *(forth) for any $a \in M$ there is an extension of f with a in its domain and which preserves the formulas in Δ,*
- *(back) for any $b \in M$, there is an extension of f with b in its range and which preserves the formulas in Δ.*

2.28 Preservation theorems.

Let T be a theory in a language $\mathcal{L}$, and Δ a set of formulas in the (free) variables $(x_1, \dots, x_n)$, closed under finite disjunctions. Let $\Sigma(x_1, \dots, x_n)$ be a set of formulas in the free variables $(x_1, \dots, x_n)$, such that every finite fragment of $\Sigma(x_1, \dots, x_n)$ is satisfiable in a model of T. The following conditions are equivalent:

(1) There is a subset $\Gamma(\bar{x})$ of Δ such that, if $\bar{c} = (c_1, \dots, c_n)$ are new constant symbols, then

$$T \cup \Gamma(\bar{c}) \models \Sigma(\bar{c}), \qquad T \cup \Sigma(\bar{c}) \models \Gamma(\bar{c}).$$

(2) For all models M and N of T, and n-tuples $\bar{a}$ in M and $\bar{b}$ in N, if $N \models \Sigma(\bar{b})$ and $\bar{a}$ satisfies (in M) all formulas of Δ that are satisfied by $\bar{b}$ (in N), then $M \models \Sigma(\bar{a})$.

Remark. If the set $\Sigma(\bar{x})$ is finite, then so is $\Gamma(\bar{x})$. Hence, taking $\varphi(\bar{x})$ to be the conjunction of the formulas of $\Sigma(\bar{x})$, one obtains that $\varphi(\bar{x})$ is equivalent, modulo T, to a finite conjunction of formulas of Δ.

2.29 These two results allow to prove classical preservation theorems. Here are a few:

(1) A sentence [formula] is preserved under extensions if and only if it is equivalent to an existential sentence [formula].
(2) A sentence [formula] is preserved under substructures if and only if it is equivalent to a universal sentence [formula].
(3) A sentence [formula] is preserved under union of chains if and only it is equivalent to a $\forall\exists$-sentence [formula].
(4) A sentence [formula] is preserved under homomorphisms if and only if it is equivalent to a positive sentence [formula].

Comments. First a word of explanation of what it means for a formula to be preserved. For instance, the formula $\varphi(\bar{x})$ is preserved under union of chains if whenever $\bar{a} \in M_0$, and $(M_i)_{i \in \mathbb{N}}$ is an increasing chain of $\mathcal{L}$-structures such that for each i, $M_i \models \varphi(\bar{a})$, then $\bigcup_{i \in \mathbb{N}} M_i \models \varphi(\bar{a})$.
If in the above definition, one restrict one's attention to models of T, one will obtain equivalences modulo the theory T.

Examples of complete theories, of quantifier elimination

Here are some complete and incomplete theories, together with an axiomatisation.

2.30 Divisible ordered abelian groups. One has the obvious axioms. It is complete and eliminates quantifiers (in $\mathcal{L}_{og}$). Here is a proof that it eliminates quantifiers, using the criterion 2.27.

Let M and N be two ordered abelian divisible groups, which we assume $\aleph_1$-saturated. In particular, their dimension as $\mathbb{Q}$-vector spaces is $\geq \aleph_1$. We assume that $A \subset M$ and $B \subset N$ are countable substructures, and that $f : A \to B$ is an $\mathcal{L}_{og}$-isomorphism. Let $a \in M$. We want to show that there is $b \in N$ such that by setting $f(a) = b$, we define an isomorphism between the ordered groups $\langle A, a\rangle$ and $\langle B, b\rangle$. This will give us the forth direction, and the back direction is symmetric.

Case 1. There is an integer $n > 0$ such that $na \in A$. Take the smallest such n; because N is divisible, there is some $b \in N$ such that $nb = nf(a)$. One verifies easily that setting $f(a) = b$ gives us the desired extension of f. Indeed the elements of $\langle A, a\rangle$ are of the form $c + ma$, where $c \in A$, $0 \leq m < n$, and if $c' + m'a$ is another such element with $m \leq m'$, and $\Box$ is one of $=$, $<$, or $>$, we have

$$c + ma \,\Box\, c' + m'a \iff nc + mna \,\Box\, nc' + m'na.$$

This remark implies easily that we have an $\mathcal{L}_{og}$ isomorphism.

Case 2. Not case 1. Then, as a group, we have $\langle A, a\rangle = A \oplus \langle a\rangle \simeq A \oplus \mathbb{Z}$. First, using case 1, we may assume that A is divisible. Let $C = \{c \in A \mid c < a\}$, and consider the following set of formulas:

$$\Sigma(x) = \{x > f(c) \mid c \in C\} \cup \{x < f(c) \mid c \in A \setminus C\}.$$

This set is finitely consistent, since the ordering on N is dense. As A is countable and N is $\aleph_1$-saturated, there is some $b \in N$ which satisfies all formulas of Σ. We define $f(a) = b$. Then, as $b \notin f(A)$, $f(A)$ is divisible, and N is torsion free, this f defines a group isomorphism $A \oplus \langle a\rangle \to B \oplus \langle b\rangle$. It remains to show that it preserves the ordering: use the same type of argument as in case 1.

2.31 Ordered Z-groups. An *ordered Z-group* is an $\mathcal{L}_{og}$-structure G which is an ordered abelian group, with a (unique) smallest positive element, which we denote by 1; moreover it satisfies that $[G : nG] = n$

for any integer $n > 1$: we use the axiom

$$\forall x \bigvee_{i=0}^{n-1} \exists y\ x = ny - i.$$

Clearly, $\mathbb{Z}$ is a model of these axioms. This theory does not eliminate quantifiers: note that $2\mathbb{Z}$ is also a model of this theory, and the smallest element of $2\mathbb{Z}$ is $2 \neq 1$.

To eliminate quantifiers, one needs to augment the language, first by adding a constant symbol for 1 (the smallest positive element), and binary relation symbols $\equiv_n$ for congruence modulo n. This language is called the *Presburger* language, $\mathcal{L}_{\text{Pres}}$. The $\mathcal{L}_{\text{Pres}}$-theory of $\mathbb{Z}$ is then obtained by adding to the above axioms the following:

$$\forall x\ x > 0 \to x \geq 1,$$
$$\forall x, y\ x \equiv_n y \leftrightarrow \exists z\ x - y = nz,$$

for all $n > 1$.

2.32 Algebraically closed fields. The theory ACF of algebraically closed fields (in the language $\mathcal{L}_R$ of rings) is axiomatised by saying that the structure is a (commutative) field, and for each $n > 1$, by adding the axiom

$$\forall x_0, x_1, \ldots, x_n\, \exists y\ (x_n = 0 \vee \sum_{i=0}^{n} x_n y^n = 0).$$

(Every polynomial of degree $n > 1$ has a root). Note that this theory is not complete. It becomes complete if one specifies the characteristic: ACF_p says $p = 0$; ACF_0 says that $p \neq 0$ for all primes p. The completeness of ACF_0 is also known as the Lefschetz principle. Note that by compactness, if a sentence φ holds in the field $\mathbb{C}$, it will hold in all algebraically closed fields of characteristic p for p sufficiently large.

The theory ACF eliminates quantifiers, this is a classical result of algebraic geometry: quantifier-free definable sets are called *constructible sets* by geometers; a famous theorem states that the projection of a constructible set is constructible.

It can also be easily proved using a back and forth argument.

2.33 Real closed fields. We will first look at real closed fields in the language of rings. The theory RCF of real closed fields (in $\mathcal{L}_R$) is axiomatised by saying that the structure is a (commutative) field; $\forall x\, \exists y\ y^4 = x^2$; for all $n \geq 1$ the axiom $\forall x_0, \ldots, x_{2n+1},\ (x_{2n+1} = 0 \vee$

$\exists y\ \sum_{i=0}^{2n+1} x_n y^n = 0$). (Every polynomial of odd degree has a root). This is a complete theory. Observe that the ordering is definable: an element is positive if and only if it is a non-zero square. However this definition needs a quantifier (existential; or universal: say that $-x$ is not a square), and the $\mathcal{L}_R$-theory of $\mathbb{R}$ does not eliminate quantifiers. For instance, there are two $\mathcal{L}_R$-embeddings of $\mathbb{Q}(\sqrt{2})$ into $\mathbb{R}$, but inside $\mathbb{R}$, the two square roots of 2 do not satisfy the same formulas (since one of them is a square, while the other is not).

However, if one looks at real closed fields in the language $\mathcal{L}_{or}$ of ordered rings, then their theory eliminates quantifiers. This is a consequence of Sturm's algorithm. The $\mathcal{L}_{or}$-theory of real closed fields is obtained by adding to the above axioms the definition of the ordering: $x < y \leftrightarrow \exists z\ (x - y) = z^2 \wedge x \neq y$.

2.34 Algebraically closed valued fields. Let $\mathcal{L}_{\mathrm{div}} = \{+, -, \cdot, 0, 1, \mid\}$, and view algebraically closed valued fields as $\mathcal{L}_{\mathrm{div}}$-structures. The axiomatisation is the obvious one: the theory ACVF says that the structure is an algebraically closed field, and that $\mid$ is the divisibility relation coming from a valuation.

Theorem. *The theory ACVF eliminates quantifiers in the language $\mathcal{L}_{\mathrm{div}}$. Its completions are obtained by specifiying the characteristics of the valued field and of the residue field.*

Going back to the usual 2-sorted language, this means that every formula (of $\mathcal{L}_{\mathrm{div}}$ or even of the 2-sorted language introduced in Example 4 of 1.6 as long as the free variables are all of the valued field sort) is equivalent to a Boolean combination of formulas of the form

$$v(f(\bar{x})) \leq v(g(\bar{x})), \quad h(\bar{x}) = 0,$$

where f, g and h are polynomials over $\mathbb{Z}$. Note that we can work in either language, as we have a direct translation of atomic formulas in one language by quantifier-free formulas of the other language:

$$v(x) \geq v(y) \iff y \mid x.$$

The proof of quantifier-elimination can be done using a back-and-forth argument, see 2.27. We are given two $\aleph_1$-saturated algebraically closed valued fields M and N, and a valued field isomorphism f between two countable non-empty subrings A and B of M and N respectively. Note that A and B both contain 1, and therefore: they have the same characteristic, and the same residual characteristic (since in a valued field of characteristic 0 with residual characteristic $p > 0$ we have $p \neq 0 \wedge \neg p \mid 1$).

We are also given $c \in M$, and wish to extend f to $A[c]$. First note that an $\mathcal{L}_{\rm div}$-isomorphism between two domains extends uniquely to an isomorphism of their field of fractions which respects the valuation. Furthermore, elementary properties of valuations on fields imply that f extends to an isomorphism of valued fields between the algebraic closures of A and B (in M and N respectively). We may therefore assume that A and B are algebraically closed, and if $c \in A$, there is nothing to do.

Let $C = A(c)$. Then the extension C/A is of one of the following type:

a. C/A pure residual,
b. C/A totally ramified,
c. C/A *immediate* (same value group, same residue field).

Extending f in each case follows from general results on valuation theory (in the immediate case, use Kaplansky's results on pseudo-convergent sequences [56]). $\square$

The original proof of this result by A. Robinson [75] is slightly different, and uses a 2-sorted language. Even though the theory ACVF is not complete, ACVF is decidable. Indeed, let φ be a sentence, we wish to decide whether φ holds in all algebraically closed valued fields. Let $\mathrm{ACVF}_{(0,0)}$ be the completion of ACVF obtained by saying that the residual characteristic is 0. Either φ is false in all (some) algebraically closed fields of residue characteristic 0, and we find a proof of $\neg\varphi$ from $\mathrm{ACVF}_{(0,0)}$; else, we find a proof of φ from $\mathrm{ACVF}_{(0,0)}$; this proof uses only finitely many axioms expressing that the residual characteristic is 0, i.e., for some integer N, if the residual characteristic is $p > N$, then φ is true in all algebraically closed fields of residual characteristic p. It now remains to check if all of the finitely many theories $\mathrm{ACVF}_{(0,p)}$, $\mathrm{ACVF}_{(p,p)}$, $p < N$, prove φ, and if they do, then we can give a positive answer: φ is true in all algebraically close valued fields. (Here $\mathrm{ACVF}_{(0,p)}$, $\mathrm{ACVF}_{(p,p)}$, denote the theory of algebraically closed fields whose residue field is of characteristic p, and which are of characteristic 0, resp. p. And of course, if one of these theories does not prove φ, then it will prove $\neg\,\varphi$.) This reasoning is of course absolutely non-effective. S.S. Brown [12] has some effective results on bounds on transfer principles for algebraically closed and complete discretely valued fields.

Imaginary elements

2.35 Definition. Let M be an $\mathcal{L}$-structure, let n be an integer, and E an $\emptyset$-definable equivalence relation on M^n. The E-equivalence class of an n-tuple $\bar{a}$, denoted $\bar{a}/E$, will be called an *imaginary element* of M.

To M we associate a structure M^{eq}, in the multi-sorted language $\mathcal{L}^{eq}$ whose set of sorts is indexed by the $\emptyset$-definable equivalence relations on cartesian powers of M. On the *home sort* M, we have the original $\mathcal{L}$-structure, on the new sorts M^n/E no structure other than the one induced by the natural projections $\pi_E : M^n \to M^n/E$ which are also in the language. So our structure is

$$M^{eq} = \big((M, \mathcal{L}), M^n/E, \ldots, \pi_E, \ldots\big).$$

Clearly, each finite cartesian product of sorts is interpretable in the original structure M, and if $T = \mathrm{Th}(M)$, then we obtain a theory T^{eq} in the language $\mathcal{L}^{eq}$. One shows that $(M^{eq})^{eq}$ is definable in M^{eq}, and that if $M \prec N$ then $M^{eq} \prec N^{eq}$.

2.36 Examples

1. This first example is fundamental. Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula, $\bar{x}$ an m-tuple of variables, $\bar{y}$ an n-tuple of variables, M an $\mathcal{L}$-structure. Define the equivalence relation E_φ on M^n by

$$E_\varphi(\bar{y}_1, \bar{y}_2) := \forall \bar{x}(\varphi(\bar{x}, \bar{y}_1) \leftrightarrow \varphi(\bar{x}, \bar{y}_2)).$$

This is clearly an equivalence relation, and it associates to the subset of M^m defined by the formula $\varphi(\bar{x}, \bar{a})$ the class $\bar{a}/E$, i.e., a *canonical parameter*, or *code*, for the set $\varphi(M, \bar{a})$. It is sometimes denoted by $\ulcorner \varphi(\bar{x}, \bar{a}) \urcorner$

2. Let M be a structure. Then the n-tuples are imaginary elements: M^n quotiented by the trivial equivalence relation. But also, any n-element subset of M is an imaginary element: consider the subset S of M^n consisting of n-tuples of distinct elements, and quotient by the (action of the) symmetric group on n elements.

3. In general, anything interpretable in a structure will be imaginary. For instance, let G be a group, H a definable subgroup (in any language containing the language of groups). Then any left-coset of H in G will be an imaginary element. I.e., the quotient G/H with an action of G by left translation, lives in G^{eq}.

4. In the particular case of valued fields, we already saw two examples of imaginary elements: note that the two-sorted language we introduced in Example 4 of 1.6 can be obtained from one of the basic languages by adding sorts of M^{eq}; the same will be true of the language of Pas that we will introduce later. There are other imaginaries we didn't add, e.g., the elements $K/\mathcal{O}_\alpha$, where $\mathcal{O}_\alpha$ is the set $\{x \in K \mid v(x) \geq \alpha\}$ (the *closed ball of radius α centered at* 0; also noted $B(0; \geq \alpha)$). There are many other imaginaries, for a description of imaginaries of algebraically closed fields, see below 2.39.5.

2.37 Elimination of imaginaries. Let T be a complete theory in a language $\mathcal{L}$. We say that T *eliminates imaginaries* if whenever M is a model of T, E is a $\emptyset$-definable equivalence relation on M^n, then there is a $\emptyset$-definable function $f : M^n \to M^\ell$ for some $\ell > 0$, such that the fibers of f are exactly the E-equivalence classes.

An equivalent statement is as follows: a theory T eliminates imaginaries if whenever M is a saturated model of T (hence, having many automorphisms), and $D \subseteq M^r$ an M-definable set, there is a finite tuple $\bar{c}$ in M such that for any $\sigma \in \mathrm{Aut}(M)$, $\sigma(D) = D$ if and only if σ fixes the elements of the tuple $\bar{c}$. In other words: if D is defined over $\bar{a}$ and over $\bar{b}$, then it is defined over $\mathrm{dcl}(\bar{a}) \cap \mathrm{dcl}(\bar{b})$. Working in M^{eq} this becomes: $\mathrm{dcl}^{eq}(\ulcorner D \urcorner) = \mathrm{dcl}^{eq}(\bar{c})$.

The theory T *weakly eliminates imaginaries* if given any model M of T and M-definable set D, there is a smallest algebraically closed set $A \subset M$ over which D is defined. In other words: if D is defined over $\bar{a}$ and over $\bar{b}$, then it is defined over $\mathrm{acl}(\bar{a}) \cap \mathrm{acl}(\bar{b})$. Working in M^{eq}, this becomes: $\mathrm{acl}^{eq}(\ulcorner D \urcorner) = \mathrm{acl}^{eq}(\bar{c})$.

Elimination of imaginaries implies weak elimination of imaginaries. This is enough for many applications. The property of (weakly) eliminating imaginaries is preserved under adjunction of constants to the language: if the $\mathcal{L}$-theory T (weakly) eliminates imaginaries, and A is a subset of a model M of T, then so does the $\mathcal{L}(A)$-theory $\mathrm{Diag}^M_{\mathrm{el}}(A)$. If one knows that a theory T weakly eliminates imaginaries, then to show that it eliminates imaginaries, it suffices to show that, for all $n, m > 0$, one can code m-element subsets of M^n.

2.38 Galois theory. If a theory T eliminates imaginaries, then, given $A \subset M \models T$, if G is the profinite group consisting of all $\mathcal{L}(A)$-automorphisms of $\mathrm{acl}(A)$ which are elementary in M, then there is a Galois correspondence between closed subgroups of G and definably closed subsets of $\mathrm{acl}(A)$. See e.g. B. Poizat [72].

2.39 Examples.

1. Clearly the theory T^{eq} eliminates imaginaries in the language $\mathcal{L}^{eq}$.

2. Consider the theory T of an infinite set, in the empty language $\mathcal{L}$. This theory eliminates quantifiers: any definable set will be defined by a Boolean combination of formulas of the form $x = y$, or $x = a$. In this language, T does not eliminate imaginaries: let M be an infinite set, $a \neq b$ two elements of M, and consider the definable set $\{a, b\}$; consider any permutation σ of M which sends a to b, b to a, and has no fixed point. One can however show that T weakly eliminates imaginaries.

3. If K is a field (maybe with extra structure), then any finite subset of a cartesian power of K has a code. Indeed, let $\bar{a}_i = (a_{i,1}, \dots, a_{i,n})$, $1 \leq i \leq m$, n-tuples in K. Consider the polynomial $g(\bar{X}) = \prod_{i=1}^{m}(X_0 + \sum_{j=1}^{n} a_{i,j}X_j)$. Then the tuple of coefficients of $g(\bar{X})$ is a code for the finite set $\{\bar{a}_1, \dots, \bar{a}_m\}$.

4. Many theories of fields eliminate imaginaries:
– the theory of algebraically closed fields of a given characteristic,
– the theory of real closed fields,
– the theory of differentially closed fields of characteristic 0,
– any complete theory of pseudo-finite field, in the language of fields to which one adds enough constant symbols to be able to describe for each $n > 1$ the unique algebraic extension of degree n.
– the theory of separably closed fields of characteristic $p > 0$ and finite degree of imperfection e, in the language of fields to which one adds e new constant symbols, which will be interpreted by the elements of a p-basis.

5. Let T be a complete theory of algebraically closed valued fields, in one of the languages $\mathcal{L}$ introduced before (in 1.6). We already saw examples of imaginaries which did not have real representatives in that language. D. Haskell, E. Hrushovski and H.D. Macpherson describe in [50] a language $\mathcal{L}_{\mathcal{G}}$ in which the natural expansion of T eliminates imaginaries. Let K be a model of T, $\mathcal{O}$ its valuation ring, $\mathcal{M}$ its maximal ideal, and $k = \mathcal{O}/\mathcal{M}$. $\mathcal{L}_{\mathcal{G}}$ is obtained by adding to $\mathcal{L}$ two sets of sorts: for each $n > 0$,
(i) S_n is the set of $\mathcal{O}$-submodules of K^n which are free of rank n. Thus

an element of S_n corresponds to the $\mathrm{GL}_n(\mathcal{O})$-orbit of a basis of the K-vector space K^n.
(ii) If $N \in S_n$, define $\mathrm{red}(N) = N/\mathcal{M}N$. Thus $\mathrm{red}(N)$ is isomorphic to k^n. Then T_n is the disjoint union of all $\mathrm{red}(N)$, $N \in S_n$. We add to the language the natural projection $T_n \to S_n$, $(a + \mathcal{M}N) \mapsto N$.

T. Mellor shows in [68] that the theory of real closed valued fields eliminates imaginaries in the language $\mathcal{L}_\mathcal{G}$. E. Hrushovski and B. Martin ([55]) show that the field of p-adic numbers eliminates imaginaries in a sublanguage of $\mathcal{L}_\mathcal{G}$, and they use this result to show that certain p-adic integrals are rational functions. One should also be able to use the language $\mathcal{L}_\mathcal{G}$ to eliminate imaginaries in other valued fields.

3 The results of Ax, Kochen, and Ershov

In this section we will briefly state some early results by Ax and Kochen, and independently by Ershov. These results are the inspiration for the later study of the model theory of valued fields. Recall that a valued field is *Henselian* if it satisfies Hensel's lemma (or equivalently, the valuation has a unique extension to the algebraic closure of the field). The references are [2], [3], [4] and [40], [34]–[39].

3.1 Theorem. *Let $\mathcal{U}$ be any non-principal ultrafilter on the set P of prime numbers. The valued fields $\prod_{p\in P} \mathbb{Q}_p/\mathcal{U}$ and $\prod_{p\in P} \mathbb{F}_p((t))/\mathcal{U}$ are elementarily equivalent.*

In fact, the proof of Ax and Kochen gives more: assuming CH (the continuum hypothesis, which states that the smallest uncountable cardinal $\aleph_1$ is $2^{\aleph_0}$), they prove that these two valued fields are isomorphic. Note that these two fields already have isomorphic residue field ($\prod_{p\in P} \mathbb{F}_p/\mathcal{U}$) and value group ($\mathbb{Z}^P/\mathcal{U}$). Under CH, these fields are furthermore saturated, and the proof uses this fact.

3.2 Consequences of Ax and Kochen. One of the motivations for their study was Artin's conjecture, that the fields $\mathbb{Q}_p$ are C_2, i.e., for every d, a form of degree d in $> d^2$ variables has a non-trivial zero. While the conjecture was later proved to be false (see [78]), their result shows that for every d, there is a number N such that whenever $p > N$, the statement holds for all forms of degree d. Furthermore, they obtain that the theory of all $\mathbb{Q}_p$ is decidable, using results of Ax on the theory of all finite fields [1].

3.3 Other results: the AKE-principle. The AKE-principle is fairly easy to state:
Two Henselian valued fields K and L are elementarily equivalent if and only if their residue fields are elementarily equivalent and their value groups are elementarily equivalent. The henselianity condition (or some additional condition) is necessary: $\mathbb{Q}$ with the p-adic valuation is not elementarily equivalent to $\mathbb{Q}_p$, even though they have isomorphic residue field and valuation group. However, the AKE principle does not always work. Here is a more precise statement of the results of Ax and Kochen [2] – [4], which were also obtained independently by Eršov:

Theorem. *Let K and L be valued fields, with residue fields k_K and k_L respectively, and value groups Γ_K, Γ_L respectively. Assume they satisfy one of the following set of conditions:*

(a) The residue fields of K and L are of characteristic 0.
(b) K and L are of characteristic 0, the residue fields are of characteristic $p > 0$, the value groups have a smallest positive element, and in both fields the value of p is a finite multiple e of this smallest positive element.

Then

(1)

$$K \equiv L \iff k_K \equiv k_L \text{ and } \Gamma_K \equiv \Gamma_L.$$

(2) *If K is a valued subfield of L, then*

$$K \prec L \iff k_K \prec k_L \text{ and } \Gamma_K \prec \Gamma_L.$$

Here K and L are equipped with any of the languages we discussed before, the residue fields are equipped with the ring structure, and the value group with the ordered group structure (in the languages $\mathcal{L}_R$ and $\mathcal{L}_{og}$ respectively).

3.4 Valued fields of positive characteristic. Note that all fields in the above result are of characteristic 0. Results in characteristic $p > 0$ are few, except for the algebraically closed case. An early result was obtained by Y. Ershov and states that the AKE-principle holds for valued fields of positive characteristic which satisfy *Kaplansky's condition A* (see [56] for a definition) and are *defectless* (i.e., if L is a finite extension of K, then L has no proper algebraic immediate extension). There are a few other positive results, see the work of F. Delon [20] and of F.V. Kuhlmann [57]. And undecidability results if one adds for instance a section of the valuation to the language.

When the residue characteristic is positive, but the field is of characteristic 0, the nicest results are for bounded ramification. For unbounded ramification, there are still some elementary equivalence results, see section 4 of [80], and structure results for definable sets in [17].

4 More results on valued fields

In this section, we will introduce the languages of Macintyre and of Denef – Pas. The Macintyre language is a language in which the theory of the field of p-adic numbers $\mathbb{Q}_p$ eliminates quantifiers[5]. This result is instrumental in subsequent proofs of rationality of Poincaré series (see [22]). The Denef – Pas language is a language which is 3-sorted, and in which one obtains relative quantifier-elimination, from which an AKE-principle can be reobtained.

Results on the p-adics, the language of Macintyre

4.1 The language of Macintyre. One of the languages in which the field of p-adic numbers eliminates quantifiers is the language of Macintyre, $\mathcal{L}_{\rm Mac}$, which is obtained by adding to $\mathcal{L}_{\rm div}$ predicates P_n, $n > 1$, which are interpreted by

$$P_n(x) \leftrightarrow \exists y\ y^n = x \wedge x \neq 0.$$

In fact, the relation $|$ is unnecessary, as it is quantifier-free definable in $\mathbb{Q}_p$: for instance, if $p \neq 2$, we have:

$$v(x) \leq v(y) \iff y = 0 \vee P_2(x^2 + py^2).$$

The definition however depends on p, and for uniformity questions it is better to include $|$ in the language.

4.2 Axioms for the p-adics. The $\mathcal{L}_{\rm div}$ theory of the valued field $\mathbb{Q}_p$ is axiomatised by expressing the following properties:

K is a Henselian valued field of characteristic 0, with residue field $\mathbb{F}_p$.

[5] Other people gave languages in which $\mathbb{Q}_p$ eliminates quantifiers, e.g. Ax and Kochen [4] and Cohen [19].

Its value group is an ordered Z-group, with $v(p)$ the smallest positive element.

4.3 Comments.

- Let K be a subfield of $\mathbb{Q}_p$, relatively algebraically closed in $\mathbb{Q}_p$. Then $K \prec \mathbb{Q}_p$. This follows from quantifier elimination in $\mathcal{L}_{\mathrm{Mac}}$. Thus, the relative algebraic closure of $\mathbb{Q}$ inside $\mathbb{Q}_p$ is an elementary substructure.
- By adding constant symbols, one may obtain a quantifier-elimination result for the theory of a finite algebraic extension of $\mathbb{Q}_p$.
- The elimination is uniform in p, see [64].
- A valued field satisfying the axioms given above is said to be *p-adically closed*.

A detailed study of formally p-adic fields and p-adically closed fields appears in [73].

The language of Denef – Pas

4.4 The splitting of the proof of the back-and-forth argument into three cases, residual, ramified and immediate, is also apparent in the proofs of the results of Ax and Kochen, and of Ershov. This suggests passing to three sorts: the valued field, the value group, and the residue field, with additional maps the valuation and the residue map. It turns out that for quantifier-elimination results this is not quite enough. One language, which is quite convenient, is the language $\mathcal{L}_{\mathrm{Pas}}$:

- It has three sorts: the valued field, the value group and the residue field.
- The language of the field sort is the language of rings.
- The language of the value group is any language containing the language of ordered abelian groups (and ∞).
- The language of the residue field is any language containing the language of rings.
- In addition, we have a map v from the field sort to the value group (the valuation), and a map $\overline{\mathrm{ac}}$ from the field sort to the residue field (*angular component map*).

4.5 Definition. The *angular component map* is a map $\overline{\mathrm{ac}} : K \to k_K$ (where k_K is the residue field of K), which is multiplicative, sends 0 to 0, and on the valuation ring $\mathcal{O}_K^\times$ coincides with the residue map. It therefore suffices to know this map on a set of representatives of the value group.

If the valuation map has a *cross-section*, i.e., a map $s : \Gamma_K \to K^\times$ satisfying $s(\gamma+\delta) = s(\gamma)s(\delta)$ and $vs = id_{\Gamma_K}$, then the natural way of defining an angular component map is by setting

$$\overline{\mathrm{ac}}(x) = xsv(x^{-1}).$$

In all natural examples, there is a natural coefficient map:

- On the valued field $k((t))$, ($v(t) = 1$, v trivial on k), define $\overline{\mathrm{ac}}(0) = 0$, and $\overline{\mathrm{ac}}(t) = 1$. Thus, if $a_j \neq 0$ then
$$\overline{\mathrm{ac}}(\sum_{i\geq j} a_i t^i) = a_j.$$
- On $\mathbb{Q}_p$, define $\overline{\mathrm{ac}}$ by $\overline{\mathrm{ac}}(p) = 1$.

In most cases we do strengthen the language by adding the angular component map. In the case of $\mathbb{Q}_p$ however, $\overline{\mathrm{ac}}$ is definable in the valued field $\mathbb{Q}_p$: indeed, $\overline{\mathrm{ac}}$ equals 1 on the $(p-1)$-th powers. It therefore suffices to specify the values of $\overline{\mathrm{ac}}$ on a system of generators of the finite group $\mathbb{Q}_p^\times/{\mathbb{Q}_p^\times}^{p-1}$. A similar result holds for finite algebraic extensions of $\mathbb{Q}_p$.

It is not true that every valued field has an angular component map. However, every valued field K has an *elementary extension* K^* which has an angular component map. This is because in every $\aleph_1$-saturated valued field L, the group morphism $v : L^\times \to \Gamma_L$ has a cross-section, and we know that every valued field has an elementary extension which is $\aleph_1$-saturated. See also [70] for a discussion on the non-definability of angular component maps.

4.6 Theorem. *Let (K, Γ_K, k_K) be an $\mathcal{L}_{\mathrm{Pas}}$-structure, where K is a Henselian valued field, and k_K has characteristic 0. Then every formula $\varphi(x, \xi, \bar{x})$ (x, ξ, $\bar{x}$, tuples of variables of the valued field, valued group, residue field sort) of the language is equivalent to a Boolean combination of formulas*

$$\varphi_1(x) \wedge \varphi_2(v(f(x)), \xi) \wedge \varphi_3(\overline{\mathrm{ac}}(f(x)), \bar{x}),$$

where $f(x)$ is a tuple of elements of $\mathbb{Z}[x]$, φ_1 is a **quantifier-free** *formula of the language of rings, φ_2 is a formula of the language of the group sort, and φ_3 is a formula of the language of the residue field sort.*

4.7 Other results of Pas include a cell decomposition of definable sets. See [69]. Even though the theorem speaks about valued fields of residual characteristic 0, by compactness, it also applies to valued fields $\mathbb{Q}_p$ for p sufficiently large. See also [71].

4.8 Adding angular component maps to $\mathbb{Q}_p$. If one wishes to study the p-adics in a three-sorted language with angular component maps, one is obliged to add angular component maps of higher order, namely, for each n, a multiplicative map $\overline{\mathrm{ac}}_n : K \to \mathbb{Z}/p^n\mathbb{Z}$, which on $\mathbb{Z}_p$ coincides with the usual $\mod p^n$ reduction. We also require that $\overline{\mathrm{ac}}_n = \overline{\mathrm{ac}}_{n+1} \mod p^n$. Note that this requires introducing many countably many new sorts, but that, as we saw above, these maps are interpretable in the valued field $\mathbb{Q}_p$. See [8], [9] for details.

4.9 Application to power series. Consider the natural $\overline{\mathrm{ac}}$ map on $k((t))$, where k is a field of characteristic 0, and look at the $\mathcal{L}_{\mathrm{Pas}}$-structure

$$(K, \mathbb{Z} \cup \{\infty\}, k)$$

where the language of the group sort is the Presburger language $\mathcal{L}_{\mathrm{Pres}}$, see 2.31. Then in the above the formula φ_2 will be a formula without quantifiers.

A definable (with parameters) function $K \to \Gamma_K$ will therefore be locally defined by expressions of the form

$$(\sum_i m_i v(f_i(x)) + \alpha)/N$$

where the f_i's are polynomials over K, $\alpha \in \Gamma_K$, and N is some integer. By locally, I mean that there is a partition of K into definable sets, and on each definable set of the partition, the function is given by an expression as above.

Assume now in addition that the theory of the residue field k eliminates quantifiers in the language of the residue field. Then, in the theorem, the formula φ_3 can also be taken to be quantifier-free. Thus we then obtain full elimination of quantifiers. Useful examples are: $\mathbb{C}$ in the language of rings; $\mathbb{R}$ in the language of ordered rings; and also … $\mathbb{Q}_p$ in the language $\mathcal{L}_{\mathrm{Mac}}$ of Macintyre. Thus we know a language in which the valued field $\mathbb{Q}_p((t))$ eliminates quantifiers.

Further reading

The (model) theory of valued fields is extremely rich, and has grown in several directions. We will here indicate some of the existing literature. Omissions are most of the time due to the writer's ignorance.

Quantifier-elimination. Many people worked on quantifier elimination for valued fields, sometimes with a cross-section for the valuation map, to cite a few: Ax and Kochen [4], Ziegler [81], Basarab [7], Delon [20], Weispfenning [80]. The paper of Weispfenning contains a very good bibliography.

Analytic structures. Complete valued fields can be endowed with an analytic structure, and several model theorists studied these enriched valued fields. On the field of p-adic numbers this was done first by Denef and Van den Dries [24]. On other fields, one of the earliest papers is by L. Lipshitz, and to-date, the most complete treatment is probably the one by R. Cluckers and L. Lipshitz [15], which in particular encompasses earlier results by Lipshitz, Robinson, Schoutens, ...; the paper contains an excellent bibliography. J. Denef gives in [23] an excellent survey of results obtained using quantifier-elimination. (One should be aware that later on, a mistake was discovered in the quantifier-elimination result of Gardener and Schoutens; see [63]).

Valued differential fields. Valued differential fields occur naturally in analysis. Work on differential valued fields was done by N. Guzy, F. Point and C. Rivière, see [42] – [47], and also by Bélair [10], Scanlon [76].

Valued difference fields. Classical examples of difference valued fields are the maximal unramified extension $\mathbb{Q}_p^{unr}$ of $\mathbb{Q}_p$ or its completion $W(\mathbb{F}_p^{alg})$, with a lifting of the Frobenius automorphism on the residue field. The theory of these difference fields was studied by Bélair, Macintyre and Scanlon [11], who prove a relative quantifier-elimination result for $W(\mathbb{F}_p^{alg})$, as well as an AKE-principle. Azgin and Van den Dries [5] improve slightly their result. In that connection we should also mention earlier work by Van den Dries [29] on $W(\mathbb{F}_p^{alg})$ with a predicate for the set of Teichmüller representatives. Also, Scanlon studies the model theory of D-valued fields (here D is an operator, which on the valued field originates from an automorphism via $\sigma(x) = eDx + x$ for some e in the valuation ring, and which on the residue field defines either a derivation or again yields an automorphism), and obtains AKE-type results, see [76]. Let me also mention an unpublished result of Hrushovski [52] on (algebraically closed) valued fields with an automorphism which is ω-increasing, such as a "non-standard Frobenius".

Various notions of minimality. Searching to generalize the properties of strong minimality (of algebraically closed fields) and of o-minimality (of real closed fields), several notions of minimality were studied. First by Macpherson and Steinhorn [67], then by other people, see e.g. [49], [17], [53]. See also the paper by Delon in this volume for more details.

Main omissions. I have not at all spoken about some of the main developments of the model theory of valued fields, which are taking place at this very moment and are in constant progress. Some of these developments started with the work of Denef and Loeser on motivic integration [25], or, should I say already with the work of Denef on the rationality of the Poincaré series [22]? This initial work was followed by many others, by Denef and Loeser, then joined by Cluckers, Hrushovski, Kazhdan, ... I should also mention on-going work around NIP, metastable theories, etc., which has already given important results (e.g., the space of types of Hrushovski and Loeser [54]). Other people are better qualified to talk about them.

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