# Measures on perfect PAC fields

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A conjecture, now disproved by Chernikov, Hrushovski, Kruckman, Krupinski, Pillay and Ramsey, asked whether any group with a simple theory is definably amenable. Recall that a group $G$ is definably amenable if its set of definable subsets (with parameters) can be endowed with a finitely additive measure, which is stable under translation. The question remains open for (bounded) perfect PAC fields, and our work tries to address it.

## Pseudofinite fields

The theory of finite fields and of its infinite models (the *pseudofinite fields*) was investigated by Ax in a 1968 paper, where he showed decidability of the theory, and gave an axiomatisation of the theory of finite fields and of pseudofinite fields. Pseudofinite fields are characterized by the following properties:
- they are perfect;
- their absolute Galois is isomorphic to $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$;
- they are pseudo-algebraically closed (PAC), i.e, every absolutely irreducible variety has a rational point.

Counting measure on definable sets of finite fields, with good definability properties.

One associates to any formula $\varphi(x, y)$ ($x, y$ finite tuples of variables) a finite subset $D$ of $\mathbb{N} \times (\mathbb{Q}^{>0} \cup \{0, 0\})$ and a constant $C_\varphi$, such that for any finite field $\mathbb{F}_q$ and any $b$ in $\mathbb{F}_q$, there is some pair $(d, e) \in D$ such that

$$(*) \qquad |\text{card}(\varphi(\mathbb{F}_q, b)) - eq^d| < C_\varphi q^{d-1/2}.$$

Furthermore, for each $(d, e) \in D$, there is a formula $\theta_{\varphi,d,e}(y)$ which defines in each $\mathbb{F}_q$ the set of $b$ satisfying $(*)$.

$\rightsquigarrow$ if $F$ is a pseudofinite field, to each definable set are associated a dimension (the algebraic dimension of its Zariski closure) and a measure relative to that definable set.

Hrushovski (2003) showed that this is the only function $\mu$ on definable subsets of $F^n$ with the following properties:

• $\mu$ is finitely additive on sets of the same dimension;

• if $V$ is an absolutely irreducible variety defined over $F$, then $\mu(V(F)) = 1$;

• the measure of a finite set is its cardinality;

• (Fubini) If $f : S \to U$ is a definable surjective map, with fibers of the same dimension and associated measure $m \in \mathbb{Q}^{>0}$, then $m\mu(U) = \mu(S)$.

The natural question arose whether other PAC fields possess nice measures. Unfortunately, the Fubini condition is very strong, and it implies that the absolute Galois group must be isomorphic to $\hat{\mathbb{Z}}$, and of course that the field must be perfect if of characteristic $p > 0$, since the map $x \mapsto x^p$ is injective. So - if PAC, it must be pseudo-finite.

Halupczok (2007) worked on the question, and relaxed the Fubini condition to the fact that the measure should be preserved under definable bijection. He was then able to show that in this case, the Galois group of the field should be procyclic, and that a perfect PAC field with procyclic group does possess a measure satisfying these conditions, and also showed that under certain hypotheses, this measure is unique. This gives a slightly weaker characterization of the "counting measure" on pseudofinite fields.

## Other measures

If $K$ is a field, the absolute Galois group of $K$, $Gal(K^{sep}/K)$, is denoted $G(K)$. It is a profinite group, compact and Hausdorff, and is endowed with a Haar measure $\mu$, defined by $\mu(G(K)) = 1$, and if $N \leq G$ is an open normal subgroup, then $\mu(N) = [G : N]^{-1}$; one then extends the measure so that it is stable under translation, etc.

Ax showed in his paper that with probability 1, if $\sigma \in G(\mathbb{Q})$, then Fix($\sigma$) is pseudo-finite. He also described all completions of Psf: they correspond to subfields of the algebraic closure of the prime field which have pro-cyclic absolute Galois group. In particular, any subfield of $\bar{\mathbb{F}}_p$ can occur as the field of algebraic numbers of a pseudofinite field.

Ax' results on pseudo-finite fields were extended by Jarden and Kiehne (1975) to $e$-free PAC fields. An $e$-free PAC field is a perfect PAC field with absolute Galois group isomorphic to $\widehat{F}_e$, the free profinite group on $e$ generators (i.e., the profinite completion of the free discrete group on $e$ generators). JK described the completions, and showed that for every $e \in \mathbb{N}^{>0}$, if $K$ is a countable hilbertian field, then for $\bar{\sigma} \in G(K)^e$ with probability 1, the (perfect closure of the) field $\mathrm{Fix}(\bar{\sigma})$ is $e$-free PAC.

Furthermore, to each sentence $\theta$ of $\mathcal{L}_{\mathrm{rings}}(K)$, they associate a rational number $r \in [0, 1]$ which is

$$\mu\{\bar{\sigma} \in G(K)^e \mid \mathrm{Fix}(\bar{\sigma}) \models \theta\}.$$

Examples of Hilbertian fields are finitely generated fields, and more generally, function fields.

What follows is work (very much) in progress with Nick Ramsey.

The idea is to use the results of Jarden and Kiehne to define a measure. First of all, let me recall the following, which we will very much use. Let $k$ be a perfect PAC field, with $e$-free absolute Galois group, and let $k^* \succ k$ be an elementary extension. Let $a, b$ be finite tuples in $k^*$. Then TFAE:

• $tp(a/k) = tp(b/k)$

• There is a $k$-isomorphism $\varphi : k(a)^{sep} \cap k^* \to k(b)^{sep} \cap k^*$ which sends $a$ to $b$.

Since $G(k)$ is small, the restriction map $G(k^*) \to G(k)$ is an isomorphism, and therefore so is the map $G(k(a)^{sep} \cap k^*) \to G(k)$.

Let $k$, $a$ be as before. The above description of types shows that $tp(a/k)$ is entirely determined by the isomorphism type (over $k(a)$) of the relative algebraic closure of $k(a)$ in $k^*$. Given an $e$-tuple $\bar{\sigma}$ generating $G(k)$, such a subfield is the fixed field by some $e$-tuple $\bar{\tau}$ of $G(k(a))$ extending $\bar{\sigma}$. So, for each finite Galois extension $L$ of $k(a)$, one needs to describe the isomorphism type (over $k(a)$) of Fix$(\bar{\tau}) \cap L$. Note that any field $K$ with $k(a) \subseteq K \subseteq L$, which is a regular extension of $k$, and is such that $Gal(L/K)$ is $e$-generated, can occur.

Let $V$ be an absolutely ireducible variety defined over $k$, let $a$ be a generic point of $V$. We will define a measure on $V$, as follows. Let $S \subset V$ be definable. If $\dim(S) < \dim(V)$, set $\mu_k(S) = 0$. By the description of types given above, there is some finite Galois extension $L$ of $k(a)$ such that the relative algebraic closure of $k(a)$ in $L$ will determine whether or not $a$ is in $S$. So it suffices to define measures for those sets.

Let $L$ be a finite Galois extension of $k(a)$, $k_L = k^{sep} \cap L$, and let $K_1, \dots, K_r$ enumerate (up to conjugation over $k(a)$) the subfields of $L$ containing $k(a)$ which are regular over $k$ and such that $Gal(L/K_i)$ is $e$-generated. For each $i$, there is a formula $\theta_i$ which expresses that the relative algebraic closure of $k(a)$ in $L$ is isomorphic to $K_i$ over $k(a)$. So if $S_i$ is the subset of $V$ defined by $\theta_i$, then $\dim(S_i) = \dim(V)$, and $\dim(V \setminus \bigcup S_i) < \dim(V)$, and we will have $\sum_i \mu_k(S_i) = 1$.

We take the measure defined by Jarden and Kiehne, and define $\mu_k(S_i)$ as follows:

$$\mu_k(S_i) = \frac{\mu(\theta_i)}{\mu(\{\bar{\sigma} \in G(k(a))^e \mid \mathsf{Fix}(\bar{\sigma}) \cap k_L = k\})},$$

or equivalently,

$$\mu_k(S_i) = \frac{|\{\bar{\sigma} \in Gal(L/k(a))^e \mid \mathsf{Fix}(\bar{\sigma}) \simeq_{k(a)} K_i\}|}{|\{\bar{\sigma} \in Gal(L/k(a))^e \mid \mathsf{Fix}(\bar{\sigma}) \cap k_L = k\}|}$$

One can show that this number does not depend on the choice of $L$: namely that if one works in a larger Galois extension $M$ containing $L$, one gets the same result. This does depend on the fact that $G(k)$ is free.

Moreover, if $f : V \to W$ is a birational map betwen two varieties defined over $k$, then the measure is preserved by $f$. However, if $e > 1$, definable bijections do not necessarily preserve the measure. When $e = 1$, the measure is preserved under definable bijection, and this implies that our measure coincides with the "counting" measure, by Halupczok's result.

Recall that a group $G$ is *definably amenable* if there is a finite additive measure $\mu$ on the definable subsets of $G$, which furthermore is invariant under translation.

Our result has the following consequence:

**Theorem**. *Let $k$ be an $e$-free PAC field, let $G$ be a group definable in $k$. Then $G$ is definably amenable.*

*Proof*. Results of Hrushovski and Pillay say that there are a definable subgroup $G_0$ of finite index in $G$, a connected algebraic group $H$ defined over $k$, and a definable homomorphism $f : G_0 \to H(k)$ with finite kernel and with $f(G_0)$ of finite index in $H(k)$.

Note that if $S \subset G$ is definable, then for each $n \leq |\mathrm{Ker}(f)|$, the set $\{g \in f(G_0) \mid |f^{-1}(g) \cap S| = n\}$ is definable. From this the result follows easily, using the measure we defined on $H(k)$, and which is clearly preserved by translation (translation by an element of $H(k)$ is a birational map).

What is next?

There are other ways of computing the measure, but they give the same result.

Extend to other projective profinite groups.

Ideally, we would like to extend these results to other bounded projective profinite groups. Existence of a measure is not difficult: just take the $\{0, 1\}$-measure associated to some complete type. But, if we want to obtain amenability of definable groups, we need to have a certain canonicity of the measure. Sofar, we succeeded in doing so when $G(k)$ is free pro-$p$ of finite rank.

$\omega$-free perfect PAC fields. There are two ways of defining a measure on definable subsets of an algebraic variety $V$. One way is simply to take a non-principal ultraproduct of the meaures $\mu_{V,e}$, as any non-principal ultraproduct of $e$-free perfect PAC fields is $\omega$-free. It has advantages and drawbacks: it only takes the values 0 and 1, and the "generic type" on $V$ is the one which says that $a$ is a generic of $V$ over $k$, and $k(a)$ is relatively algebraically closed.
There is another measure for $\omega$-free PAC fields, introduced by Jarden in another paper, which has the defect that $\mu_V(V) = +\infty$. There are however some definable sets of positive measure, so it is in a sense more precise than the other limit measure.

However, we still get amenability of groups definable in perfect $\omega$-free PAC fields, using the ultraproduct construction: if $\varphi(x)$ defines a group in the $\omega$-free PAC field $k$, then it defines a group $G_e$ in $e$-free PAC fields $k_e$ for $e \gg 0$. Hence each group $G_e$ has a measure $\mu_{G,e}$ which is stable under translation. Hence the ultraproduct $\mu_G$ is also stable under translation.
Similarly, from the definition of the measure in perfect PAC fields with absolute Galois group free pro-$p$ on $e$ generators, one gets that definable groups are definably amenable, and so are those definable in perfect PAC fields with absolute Galois group free pro-$p$ on infinitely many generators.

A simple computation.

Consider the definable subset $S$ of $\mathbb{G}_m(k)$ consisting of squares (in characteristic $\neq 2$). If $k$ is $e$-free, then $\mu_e(S) = 2^{-e}$: indeed, the probability that an $e$-tuple in $G(k(a))$ lies in $G(k(\sqrt{a}))$ is $2^{-e}$.

Hence, the limit measure of $S$ in an $\omega$-free PAC field $k$ will be 0. This is to be expected if one has $\mu(\mathbb{G}_m) = 1$, since $[k^\times : (k^\times)^2] = \infty$.

The other measure introduced by Jarden (on $\mathcal{G} = \coprod G(k)^e$) would give $\mu'(S) = \sum_{e=1}^{\infty} 2^{-e} = 1$, and $\mu'(k^\times) = +\infty$. So $\mu'$ will give information about small subgroups, but none about large ones.