Pro-p groups acting on trees with finitely many maximal vertex stabilizers up to conjugation

Zoé Chatzidakis and Pavel Zalesskii *

November 18, 2020

Abstract

We prove that a finitely generated pro-p group G acting on a pro-p tree T splits as a free amalgamated pro-p product or a pro-p HNN-extension over an edge stabilizer. If G acts with finitely many vertex stabilizers up to conjugation we show that it is the fundamental pro-p group of a finite graph of pro-p groups $(\mathcal{G}, \Gamma)$ with edge and vertex groups being stabilizers of certain vertices and edges of T respectively. If edge stabilizers are procyclic, we give a bound on Γ in terms of the minimal number of generators of G. We also give a criterion for a pro-p group G to be accessible in terms of the first cohomology $H^1(G, \mathbb{F}_p[[G]])$.

1 Introduction

The dramatic advance of classical combinatorial group theory happened in the 1970's, when Bass-Serre theory of groups acting on trees changed completely the face of the theory.

The profinite version of Bass-Serre theory was developed by Luis Ribes, Oleg Melnikov and the second author because of the absence of the classical methods of combinatorial group theory for profinite groups. However it does not work in full strength even in the pro-p case. The reason is that if a pro-p group G acts on a pro-p tree T then a maximal subtree of the quotient graph $G\backslash T$ does not always exist and even if it exists it does not always lift to T. As a consequence the pro-p version of Bass-Serre theory does not give subgroup structure theorems the way it does in the classical Bass-Serre theory. In fact, for infinitely generated pro-p subgroups there are counter examples.

The objective of this paper is to study the situation when G has only finitely many vertex stabilizers up to conjugation and in this case we can prove the main Bass-Serre theory structure theorem.

Theorem 5.1. *Let G be a finitely generated pro-p group acting on a pro-p tree T with finitely many maximal vertex stabilisers up to conjugation. Then G is the fundamental group of a*

*The second author partially supported by CNPq and FAPDF.

reduced finite graph of finitely generated pro-p groups $(\mathcal{G}, \Gamma)$*, where each vertex group* $\mathcal{G}(v)$ *and each edge group* $\mathcal{G}(e)$ *is a maximal vertex stabilizer* $G_{\tilde{v}}$ *and an edge stabilizer* $G_{\tilde{e}}$ *respectively (for some* $\tilde{v}, \tilde{e} \in T$*).*

In the abstract situation, a finitely generated (abstract) group G acting on a tree has a G-invariant subtree D such that $G\backslash D$ is finite and so has automatically finitely many maximal vertex stabilizers up to conjugation. In the pro-p situation such an invariant subtree does not exist in general and the existence of it in the case of only finitely many stabilizers up to conjugation is not clear even if vertex stabilizers are finite. Nevertheless, for a finitely generated pro-p group acting on a pro-p tree we can prove a splitting theorem into an amalgamated product or an HNN-extension.

Theorem 4.2. *Let* G *be a finitely generated pro-p group acting on a pro-p tree* T *without global fixed points. Then* G *splits non-trivially as a free amalgamated pro-p product or pro-p HNN-extension over some stabiliser of an edge of* T*.*

This in turn allows us to prove that a non virtually cyclic pro-p group acting on a pro-p tree with finite edge stabilizers has more than one end.

Theorem 4.5. *Let* G *be a finitely generated pro-p group acting on a pro-p tree with finite edge stabilizers and without global fixed points. Then either* G *is virtually cyclic and* $H^1(G, \mathbb{F}_p[[G]]) \cong \mathbb{F}_p$ *(i.e.* G *has two ends) or* $H^1(G, \mathbb{F}_p[[G]])$ *is infinite (i.e.* G *has infinitely many ends).*

Theorem 4.2 raises naturally the question of accessibility; namely whether we can continue to split G into an amalgamated free product or HNN-extension forever, or do we reach the situation after finitely many steps where we can not split it anymore. The importance of this is underlined also by the following observation: if a pro-p group G acting on a pro-p tree T is accessible with respect to splitting over edge stabilizers, then by Theorem 4.2 this implies finiteness of the maximal vertex stabilizers up to conjugation and so Theorem 5.1 provides the structure theorem for G.

In the abstract situation, accessibility was studied by Dunwoody ([3], [4]) for splitting over finite groups, and by Bestvina and Feighn ([1]) over an arbitrary family of groups. In the pro-p case accessibility was studied by Wilkes ([24]) where a finitely generated not accessible pro-p group was constructed. For a finitely generated pro-p group acting faithfully and irreducibly on a pro-p tree (see Section 2 for definitions) no such example is known.

The next theorem gives a sufficient condition of accessibility for a pro-p group; we do not know whether the converse also holds (it holds in the abstract case).

Theorem 6.10. *Let* G *be a finitely generated pro-p group. If* $H^1(G, \mathbb{F}_p[[G]])$ *is a finitely generated* $\mathbb{F}_p[[G]]$*-module, then* G *is accessible.*[1]

We show here that finitely generated pro-p groups are accessible with respect to cyclic subgroups and in fact give precise bounds.

[1]proved by G. Wilkes independently in [25].

Theorem 6.6. *Let $G = \pi_1(\Gamma, \mathcal{G})$ be the fundamental group of a finite graph of pro-p-groups, with procyclic edge groups, and assume that $d(G) = d \geq 2$. Then (assuming that the graph is reduced), the vertex groups are finitely generated, the number of vertices of Γ is $\leq 2d - 1$, and the number of edges of Γ is $\leq 3d - 2$.*

Observe that Theorem 6.6 contrasts with the abstract groups situation where for finitely generated groups the result does not hold (see [5]).

As a corollary we deduce the bound for pro-p limit groups (pro-p analogs of limit groups introduced in [10], see Section 6 for a precise definition).

Corollary 6.7. *Let G be a pro-p limit group. Then G is the fundamental group of a finite graph of finitely generated pro-p groups $(\mathcal{G}, \Gamma)$, where each edge group $\mathcal{G}(e)$ is infinite procyclic. Moreover, $|V(\Gamma)| \leq 2d - 1$, and $|E(\Gamma)| \leq 3d - 2$, where d is the minimal number of generators of G.*

It is worth mentioning that for abstract limit groups the best known estimate for $|V(\Gamma)|$ is $1 + 4(d(G) - 1)$, proved by Richard Weidmann in [22, Theorem 1].

In Section 7 we investigate Howson's property for free products with procyclic amalgamation and HNN-extensions with procyclic associated subgroups. In Section 8.1 we apply the results of Section 7 to normalizers of procyclic subgroups.

Theorem 8.3. *Let C be a procyclic pro-p group and $G = G_1 \amalg_C G_2$ be a free amalgamated pro-p product or a pro-p HNN-extension $G = \mathrm{HNN}(G_1, C, t)$ of Howson groups. Let U be a procyclic subgroup of G and $N = N_G(U)$. Assume that $N_{G_i}(U^g)$ is finitely generated whenever $U^g \leq G_i$. If $K \leq G$ is finitely generated, then so is $K \cap N$.*

Section 2 contains basic notions and facts of the theory of pro-p groups acting on trees used in the paper. The following sections are devoted to the proofs of the results mentioned above.

Aknowledgement. This research was started while the second author was visiting the ENS as an invited professor, and the authors thank the ENS for this hospitality.

2 Notation, definitions and basic results

2.1. Notation. If a pro-p group G continuously acts on a profinite space X we call X a G-space. $H_1(G)$ denotes the first homology $H_1(G, \mathbb{F}_p)$ and is canonically isomorphic to $G/\Phi(G)$. If $x \in T$ and $g \in G$, then $G_{gx} = gG_xg^{-1}$. We shall use the notation $h^g = g^{-1}hg$ for conjugation. If H a subgroup of G, H^G will stand for the (topological) normal closure of H in G. If G is an abstract group $\widehat{G}$ will mean the pro-p completion of G.

2.2. Conventions. Throughout the paper, unless otherwise stated, groups are pro-p, subgroups will be closed and morphisms will be continuous. Finite graphs of groups will be proper and reduced (see Definitions 2.12 and 2.15). Actions of a pro-p group G on a profinite graph

Γ will a priori be supposed to be faithful (i.e., the action has no kernel), unless we consider actions on subgraphs of Γ.

Next we collect basic definitions, following [17].

2.1 Profinite graphs

Definition 2.3. A *profinite graph* is a triple (Γ, d_0, d_1), where Γ is a profinite (i.e. boolean) space and $d_0, d_1 : \Gamma \to \Gamma$ are continuous maps such that $d_i d_j = d_j$ for $i, j \in \{0, 1\}$. The elements of $V(\Gamma) := d_0(G) \cup d_1(G)$ are called the *vertices* of Γ and the elements of $E(\Gamma) := \Gamma \setminus V(\Gamma)$ are called the *edges* of Γ. If $e \in E(\Gamma)$, then $d_0(e)$ and $d_1(e)$ are called the initial and terminal vertices of e. A vertex with only one incident edge is called pending. If there is no confusion, one can just write Γ instead of (Γ, d_0, d_1).

Definition 2.4. A *morphism* $f : \Gamma \to \Delta$ of graphs is a map f which commutes with the d_i's. Thus it will send vertices to vertices, but might send an edge to a vertex.[2]

Definition 2.5. Every profinite graph Γ can be represented as an inverse limit $\Gamma = \varprojlim \Gamma_i$ of its finite quotient graphs ([17, Proposition 1.5]).

A profinite graph Γ is said to be *connected* if all its finite quotient graphs are connected. Every profinite graph is an abstract graph, but in general a connected profinite graph is not necessarily connected as an abstract graph.

2.6. Collapsing edges. If Γ is a graph and e an edge which is not a loop we can *collapse* the edge e by removing $\{e\}$ from the edge set of Γ, and identify $d_0(e)$ and $d_1(e)$ with a new vertex y. I.e., Γ' is the graph given by $V(\Gamma') = V(\Gamma) \setminus \{d_0(e), d_1(e)\} \cup \{y\}$ (where y is the new vertex), and $E(\Gamma') = E(\Gamma) \setminus \{e\}$. We define $\pi : \Gamma \to \Gamma'$ by setting $\pi(m) = m$ if $m \notin \{e, d_0(e), d_1(e)\}$, $\pi(e) = \pi(d_0(e)) = \pi(d_1(e)) = y$. The maps $d_i' : \Gamma' \to \Gamma'$ are defined so that π is a morphism of graphs. Another way of describing Γ' is that $\Gamma' = \Gamma/\Delta$, where Δ is the subgraph $\{e, d_0(e), d_1(e)\}$ collapsed into the vertex y.

2.2 Pro-p trees

2.7. An exact sequence. Let Γ be a profinite graph, with set of vertices $V(\Gamma)$ and $E(\Gamma) = \Gamma \setminus V(\Gamma)$. Let $(E^*(\Gamma), *) = (\Gamma/V(\Gamma), *)$ be the pointed profinite quotient space with $V(\Gamma)$ as distinguished point, and let $\mathbb{F}_p[[E^*(\Gamma), *]]$ and $\mathbb{F}_p[[V(\Gamma)]]$ be respectively the free profinite $\mathbb{F}_p$-modules over the pointed profinite space $(E^*(\Gamma), *)$ and over the profinite space $V(\Gamma)$ (cf. [15, section 5.2]). Note that when $E(\Gamma)$ is closed, then $\mathbb{F}_p[[E^*(\Gamma), *]] = \mathbb{F}_p[[E(\Gamma)]]$. Let the maps $\delta : \mathbb{F}_p[[E^*(\Gamma), *]] \to \mathbb{F}_p[[V(\Gamma)]]$ and $\varepsilon : \mathbb{F}_p[[V(\Gamma)]] \to \mathbb{F}_p$ be defined respectively by $\delta(e) = d_1(e) - d_0(e)$ for all $e \in E^*(\Gamma)$ and $\varepsilon(v) = 1$ for all $v \in V(\Gamma)$. Then we have the following complex of free profinite $\mathbb{F}_p$-modules

$$0 \longrightarrow \mathbb{F}_p[[E^*(\Gamma), *]] \xrightarrow{\delta} \mathbb{F}_p[[V(\Gamma)]] \xrightarrow{\varepsilon} \mathbb{F}_p \longrightarrow 0.$$

[2] It is called a *quasimorphism* in [13].

Definition 2.8. The profinite graph Γ is a *pro-p tree* if the above sequence is exact. If T is a pro-p tree, then we say that a pro-p group G *acts on* T if it acts continuously on T and the action commutes with d_0 and d_1. We say that G acts *irreducibly* on T if T does not have proper G-invariant subtrees and that it acts *faithfully* if the kernel of the action is trivial. If $t \in V(T) \cup E(T)$ we denote by G_t the stabilizer of t in G.
For a pro-p group G acting on a pro-p tree T we let $\tilde{G}$ denote the subgroup generated by all vertex stabilizers. Moreover, for any two vertices v and w of T we let $[v, w]$ denote the geodesic connecting v to w in T, i.e., the (unique) smallest pro-p subtree of T that contains v and w.

2.3 Finite graphs of pro-p groups

When we say that $\mathcal{G}$ is a finite graph of pro-p groups we mean that it contains the data of the underlying finite graph, the edge pro-p groups, the vertex pro-p groups and the attaching continuous maps. More precisely,

Definition 2.9. let Γ be a connected finite graph. A graph of pro-p groups $(\mathcal{G}, \Gamma)$ over Γ consists of specifying a pro-p group $\mathcal{G}(m)$ for each $m \in \Gamma$ (i.e. $\mathcal{G} = \dot{\bigcup}_{m\in\Gamma} \mathcal{G}(m)$), and continuous monomorphisms $\partial_i : \mathcal{G}(e) \longrightarrow \mathcal{G}(d_i(e))$ for each edge $e \in E(\Gamma)$, $i = 1, 2$.

Definition 2.10. (1) A *morphism* of graphs of pro-p groups: $(\mathcal{G}, \Gamma) \to (\mathcal{H}, \Delta)$ is a pair $(\alpha, \bar{\alpha})$ of maps, with $\alpha : \mathcal{G} \longrightarrow \mathcal{H}$ a continuous map, and $\bar{\alpha} : \Gamma \longrightarrow \Delta$ a morphism of graphs, and such that $\alpha_{\mathcal{G}(m)} : \mathcal{G}(m) \longrightarrow \mathcal{H}(\bar{\alpha}(m))$ is a homomorphism for each $m \in \Gamma$ and which commutes with the appropriate ∂_i. Thus the diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\alpha} & \mathcal{H} \\ \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial_i} \\ \mathcal{G} & \xrightarrow{\alpha} & \mathcal{H} \end{array}$$

is commutative.

(2) We say that $(\alpha, \bar{\alpha})$ is a *monomorphism* if both $\alpha, \bar{\alpha}$ are injective. In this case its image will be called a subgraph of groups of $(\mathcal{H}, \Delta)$. In other words, a *subgraph of groups* of a graph of pro-p-groups $(\mathcal{G}, \Gamma)$ is a graph of groups $(\mathcal{H}, \Delta)$, where Δ is a subgraph of Γ (i.e., $E(\Delta) \subseteq E(\Gamma)$ and $V(\Delta) \subseteq V(\Gamma)$, the maps d_i on Δ are the restrictions of the maps d_i on Γ), and for each $m \in \Delta$, $\mathcal{H}(m) \leq \mathcal{G}(m)$.

2.11. Definition of the fundamental group. The pro-p fundamental group

$$G = \Pi_1(\mathcal{G}, \Gamma)$$

of the graph of pro-p groups $(\mathcal{G}, \Gamma)$ is defined by means of a universal property: G is a pro-p group together with the following data and conditions:

(i) a maximal subtree D of Γ;

(ii) a collection of continuous homomorphisms

$$\nu_m : \mathcal{G}(m) \longrightarrow G \quad (m \in \Gamma),$$

and a continuous map $E(\Gamma) \longrightarrow G$, denoted $e \mapsto t_e$ $(e \in E(\Gamma))$, such that $t_e = 1$ if $e \in E(D)$, and

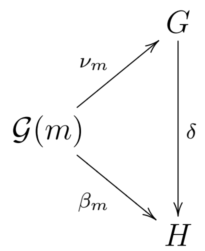
$$(\nu_{d_0(e)}\partial_0)(x) = t_e(\nu_{d_1(e)}\partial_1)(x)t_e^{-1}, \quad \forall x \in \mathcal{G}(e),\ e \in E(\Gamma);$$

(iii) the following universal property is satisfied:

whenever one has the following data

- H is a pro-p group,
- $\beta_m : \mathcal{G}(m) \longrightarrow H$, $(m \in \Gamma)$, a collection of continuous homomorphisms,
- a map $e \mapsto s_e$ $(e \in E(\Gamma))$ with $s_e = 1$ if $e \in E(D)$, and
- $(\beta_{d_0(e)}\partial_0)(x) = s_e(\beta_{d_1(e)}\partial_1)(x)s_e^{-1}, \forall x \in \mathcal{G}(e),\ e \in E(\Gamma)$,

then there exists a unique continuous homomorphism $\delta : G \longrightarrow H$ such that $\delta(t_e) = s_e$ $(e \in E(\Gamma))$, and for each $m \in \Gamma$ the diagram



commutes.

The main examples of $\Pi_1(\mathcal{G}, \Gamma)$ are an amalgamated free pro-p product $G_1 \amalg_H G_2$ and an HNN-extension $\mathrm{HNN}(G, H, t)$ that correspond to the case of Γ having one edge and two and one vertex respectively.

Definition 2.12. We call the graph of groups $(\mathcal{G}, \Gamma)$ *proper* (injective in the terminology of [13]) if the natural map $\mathcal{G}(v) \to \Pi_1(\mathcal{G}, \Gamma)$ is an embedding for all $v \in V(\Gamma)$.

Remark 2.13. In the pro-p case, a graph of groups $(\mathcal{G},\Gamma)$ is not always proper. However, the vertex and edge groups can always be replaced by their images in $\Pi_1(\mathcal{G},\Gamma)$ so that $(\mathcal{G},\Gamma)$ becomes proper and $\Pi_1(\mathcal{G},\Gamma)$ does not change. Thus throughout the paper we shall only consider proper graphs of pro-p groups. In particular, all our free amalgamated pro-p products are proper.

If $(\mathcal{G},\Gamma)$ is a finite graph of finitely generated pro-p groups, then by a theorem of J-P. Serre (stating that every finite index subgroup of a finitely generated pro-p group is open, cf. [15, §4.8]) the fundamental pro-p group $G=\Pi_1(\mathcal{G},\Gamma)$ of $(\mathcal{G},\Gamma)$ is the pro-p completion of the usual fundamental group $\pi_1(\mathcal{G},\Gamma)$ (cf. [19, §5.1]). Note that $(\mathcal{G},\Gamma)$ is proper if and only if $\pi_1(\mathcal{G},\Gamma)$ is residually p. In particular, edge and vertex groups will be subgroups of $\Pi_1(\mathcal{G},\Gamma)$.

2.14. Presentation of the fundamental group.
In [29, paragraph (3.3)], the fundamental group G is defined explicitly in terms of generators and relations associated to a chosen subtree D. Namely

$$G=\langle \mathcal{G}(v), t_e \mid v\in V(\Gamma), e\in E(\Gamma), t_e=1 \text{ for } e\in D, \partial_0(g)=t_e\partial_1(g)t_e^{-1}, \text{ for } g\in\mathcal{G}(e)\rangle \qquad (1)$$

I.e., if one takes the abstract fundamental group $G_0=\pi_1(\mathcal{G},\Gamma)$, then $\Pi_1(\mathcal{G},\Gamma)=\varprojlim_N G_0/N$, where N ranges over all normal subgroups of G_0 of index a power of p and with $N\cap\mathcal{G}(v)$ open in $\mathcal{G}(v)$ for all $v\in V(\Gamma)$. Note that this last condition is automatic if $\mathcal{G}(v)$ is finitely generated (as a pro-p-group). It is also proved in [29] that the definition given above is independent of the choice of the maximal subtree D.

Definition 2.15. A finite graph of pro-p groups $(\mathcal{G},\Gamma)$ is said to be *reduced*, if for every edge e which is not a loop, neither $\partial_1(e)\colon \mathcal{G}(e)\to\mathcal{G}(d_1(e))$ nor $\partial_0(e):\mathcal{G}(e)\to\mathcal{G}(d_0(e))$ is an isomorphism.

Remark 2.16. Any finite graph of pro-p groups can be transformed into a reduced finite graph of pro-p groups by the following procedure: If $\{e\}$ is an edge which is not a loop and for which one of ∂_0, ∂_1 is an isomorphism, we can collapse $\{e\}$ to a vertex y (as explained in 2.6). Let Γ' be the finite graph given by $V(\Gamma')=\{y\}\cup V(\Gamma)\setminus\{d_0(e),d_1(e)\}$ and $E(\Gamma')=E(\Gamma)\setminus\{e\}$, and let $(\mathcal{G}',\Gamma')$ denote the finite graph of groups based on Γ' given by $\mathcal{G}'(y)=\mathcal{G}(d_1(e))$ if $\partial_0(e)$ is an isomorphism, and $\mathcal{G}'(y)=\mathcal{G}(d_0(e))$ if $\partial_0(e)$ is not an isomorphism.

This procedure can be continued until $\partial_0(e),\partial_1(e)$ are not surjective for all edges not defining loops. Note that the reduction process does not change the fundamental pro-p group, i.e., one has a canonical isomorphism $\Pi_1(\mathcal{G},\Gamma)\simeq\Pi_1(\mathcal{G}_{red},\Gamma_{red})$. So, if the pro-$p$ group G is the fundamental group of a finite graph of pro-p groups, we may assume that the finite graph of pro-p groups is reduced.

2.17. Standard (universal) pro-p tree. Associated with the finite graph of pro-p groups $(\mathcal{G},\Gamma)$ there is a corresponding *standard pro-p tree* (or universal covering graph) $T=T(G)=$

$\dot\bigcup_{m\in\Gamma} G/\mathcal{G}(m)$ (cf. [29, Proposition 3.8]). The vertices of T are those cosets of the form $g\mathcal{G}(v)$, with $v \in V(\Gamma)$ and $g \in G$; its edges are the cosets of the form $g\mathcal{G}(e)$, with $e \in E(\Gamma)$; and the incidence maps of T are given by the formulas:

$$d_0(g\mathcal{G}(e)) = g\mathcal{G}(d_0(e)); \quad d_1(g\mathcal{G}(e)) = gt_e\mathcal{G}(d_1(e)) \;\; (e \in E(\Gamma), t_e = 1 \text{ if } e \in D).$$

There is a natural continuous action of G on T, and clearly $G\backslash T = \Gamma$. There is a standard connected transversal $s : \Gamma \to T$, given by $m \mapsto \mathcal{G}(m)$. Note that $s_{|D}$ is an isomorphism of graphs and the elements t_e satisfy the equality $d_1(s(e)) = t_e s(d_1(e))$. Using the map s, we shall identify $\mathcal{G}(m)$ with the stabilizer $G_{s(m)}$ for $m \in \Gamma$:

$$\mathcal{G}(e) = G_{s(e)} = G_{d_0(s(e))} \cap G_{d_1(s(e))} = \mathcal{G}(d_0(e)) \cap t_e\mathcal{G}(d_1(e))t_e^{-1} \tag{2}$$

with $t_e = 1$ if $e \in D$. Remark also that since Γ is finite, $E(T)$ is compact.

2.18. The fundamental group of a profinite graph. If all vertex and edge groups are trivial we get the definition of the pro-p *fundamental group* $\pi_1(\Gamma)$. It follows that $\pi_1(\Gamma)$ is a free pro-p group on the base $\Gamma \setminus D$ and so coincides with the pro-p completion $\widehat{\pi}_1^{abs}(\Gamma)$ of the abstract (usual) fundamental group $\pi_1^{abs}(\Gamma)$ that also can be defined traditionally by closed circuits. Therefore if Γ is connected profinite and $\Gamma = \varprojlim \Gamma_i$ is an inverse limit of finite graphs it induces the inverse system $\{\pi_1(\Gamma_i) = \widehat{\pi}_1^{abs}(\Gamma_i)\}$ and $\pi_1(\Gamma)$ is defined as $\pi_1(\Gamma) = \varprojlim_i \pi_1(\Gamma_i)$ in this case. The fundamental group $\pi_1(\Gamma)$ acts freely on a pro-p tree $\widetilde{\Gamma}$ (universal cover) such that $\pi_1(\Gamma)\backslash\tilde{\Gamma} = \Gamma$ (see [27] or [13, Chapter 3] for details).

We shall use frequently in the paper the following known results.

Proposition 2.19. *([17, Lemma 3.11]). Let G be a pro-p group acting on a pro-p tree T. Then there exists a non-empty minimal G-invariant subtree D of T. Moreover, if G does not stabilize a vertex, then D is unique.*

Theorem 2.20. *([17, Theorem 3.9]) Let G be a finite p-group acting on a pro-p tree T. Then G fixes a vertex of T.*

Theorem 2.21. *([9, Proposition 2.4] or [13, Theorem 9.6.1]) Let G be a pro-p group acting on a second countable (as a topological space) pro-p tree T with trivial edge stabilizers. Then there exist a continuous section $\sigma : G\backslash V(T) \longrightarrow V(T)$ and*

$$G = \coprod_{v\in G\backslash V(T)} G_{\sigma(v)} \amalg F,$$

where F is a free pro-p group naturally isomorphic to $G/\tilde{G}$.

Theorem 2.22. *([13, Theorem 7.1.2], [29, Theorem 3.10]) Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental pro-p group of a finite graph of pro-p groups $(\mathcal{G}, \Gamma)$. Then any finite subgroup K of G is conjugate into some vertex group $\mathcal{G}(v)$. In particular, if the groups $\mathcal{G}(v)$ are finite, they are exactly the maximal finite subgroups of G up to conjugation.*

We finish this section with a proposition that completes a missing case of [23, Corollary 3.3], and which is needed in our paper as well as in [23].

Proposition 2.23. *Let $(\mathcal{G}, \Gamma)$ be a reduced finite graph of finite p-groups, and suppose that $G = \Pi_1(\mathcal{G}, \Gamma, v_0)$ contains a free open subgroup H. Then there exist finitely many reduced finite graphs of finite p-groups $(\mathcal{G}', \Gamma')$ up to isomorphism such that $G \simeq \Pi_1(\mathcal{G}', \Gamma', w_0)$.*

Proof. The case of non-procyclic H was proved in [23, Corollary 3.3]. If H is trivial, then G is finite and so the reduced graph of groups $(\mathcal{G}, \Gamma)$ is a vertex v with $G = \mathcal{G}(v)$. So we assume that $H \neq 1$ is procyclic. Then G acts irreducibly on the standard pro-p tree $T(G)$ (see 2.17) and the kernel K of the action has order less than or equal to $[G : H]$. Hence G/K acts faithfully on $T(G)$ and so by [17, Theorem 3.15] either $G/K \cong \mathbb{Z}_p$ or $G/K = C_2 \amalg C_2$ is infinite dihedral. In the first case $G = K \rtimes \mathbb{Z}_p = HNN(K, K, t)$ and in the second case $G = G_1 \amalg_K G_2$, with $[G_1 : K] = 2 = [G_2 : K]$. Note that K is the unique finite normal subgroup of G, and the result follows easily, using Theorem 2.22. □

3 Preliminaries: Auxiliary results

In this section we shall prove several auxiliary results on profinite graphs and pro-p groups acting on trees and which are needed later in the paper.

Lemma 3.1. *(cf. [19, Corollary 2]) Let $\nu : \Delta \longrightarrow \Gamma$ be a morphism of finite connected graphs representing the collapse of an edge, not a loop. If T is a maximal subtree of Γ, then $\nu^{-1}(T)$ is a maximal subtree of Δ.*

Proof. Consider $\nu^{-1}(T)$. Since $V(\Gamma) \subset T$, $V(\Delta) \subset \nu^{-1}(T)$. Since $\nu^{-1}(T)$ contains a collapsed edge, $|E(\nu^{-1}(T))| = |E(T)| + 1$ and $\nu^{-1}(T)$ is connected. Thus $|E(\nu^{-1}(T))| = |E(T)| + 1 = |V(\Gamma)| = |V(\Delta)| - 1$. Since $\nu^{-1}(T)$ is connected, it must be a tree, as needed.

□

Lemma 3.2. *Let Γ be a profinite graph and Δ an abstract connected subgraph of finite diameter n (i.e. the shortest path between any two vertices has length at most n). Then the closure $\overline{\Delta}$ of Δ in Γ has diameter at most n.*

Proof. Write $\Gamma = \varprojlim \Gamma_i$ as an inverse limit of finite quotient graphs and let Δ_i be the image of Δ in Γ_i. Then Δ_i is finite and has diameter not more than n. Since $\overline{\Delta} = \varprojlim \Delta_i$, so does $\overline{\Delta}$. Indeed, pick two vertices v, w in $\overline{\Delta}$ and let v_i, w_i their images in Δ_i. The set Ω_i of paths of length at most n between v_i and w_i is finite and non-empty. Then $\Omega = \varprojlim \Omega_i$ consists of paths between v and w of length not greater than n and is non-empty.

□

Proposition 3.3. *Let Γ be a connected profinite graph of finite diameter. If $\pi_1(\Gamma)$ is finitely generated, then there exists a finite connected subgraph Δ of Γ such that $\pi_1(\Gamma) = \pi_1(\Delta)$.*

Proof. By [8, Corollary 4] Γ is connected as an abstract graph. Then by [21, Proposition 2.7] $\pi_1(\Gamma)$ is the pro-p completion of the usual fundamental group $\pi_1^{abs}(\Gamma)$ and so $\pi_1^{abs}(\Gamma)$ is a free group of the same rank n as $\pi_1(\Gamma)$. Let D be an abstract maximal subtree of the abstract graph Γ. Then $|\Gamma \setminus D| < \infty$. Let $e_1, \ldots, e_n$ be all edges from $\Gamma \setminus D$. Then $\pi_1(\Gamma))$ is free pro-p of rank n. Let Ω be a minimal subtree of D containing all vertices of $e_1, \ldots, e_n$. Since Γ has finite diameter, Ω is finite. Therefore $\Delta = \Omega \cup \{e_1 \cup \cdots \cup e_n\}$ is a finite connected subgraph of Γ and $\pi_1^{abs}(\Delta)$ is a free group of rank n. But the fundamental group of a subgraph is a free factor of the fundamental group of a graph, so $\pi_1^{abs}(\Delta) = \pi_1^{abs}(\Gamma)$ and so their pro-p completions $\pi_1(\Delta) = \pi_1(\Gamma)$.

□

Proposition 3.4. *Let G be a pro-p group acting on a pro-p tree T. Then $G/\tilde{G} = \pi_1(G\backslash T)$ is a free pro-p group acting freely on $\tilde{G}\backslash T$. Moreover, if $G\backslash T$ is finite, then the rank of $\pi_1(G\backslash T)$ is $|E(G\backslash T)| - |V(G\backslash T)| + 1$.*

Proof. Recall that $\tilde{G}$ is the closed subgroup of G generated by the vertex stabilisers G_v, $v \in T$. By [13, Corollary 3.9.3] $G/\tilde{G} = \pi_1(G\backslash T)$ is free pro-p and by Proposition [17, Proposition 3.5] $\tilde{G}\backslash T$ is a pro-p tree. If $\Gamma := G\backslash T$ is finite it has a maximal subtree D and by [13, Theorem 3.7.4] a basis of $\pi_1(G\backslash T)$ is $\Gamma \setminus D$. Since $V(D) = V(G\backslash T)$ the result follows.

□

Proposition 3.5. *Let G be a finitely generated pro-p group acting on a pro-p tree T such that $\Gamma = G\backslash T$ has finite diameter. Then T possesses a G-invariant subtree D such that $G\backslash D$ is finite.*

Proof. By Proposition 3.4, $G = \tilde{G} \rtimes \pi_1(\Gamma)$. We first show that there are finitely many vertices $w_1, \ldots, w_n$ such that $G = \langle G_{w_i}, \pi_1(\Gamma) \mid i = 1, \ldots, n\rangle$.

Indeed, let $f : G \longrightarrow G/\Phi(G)$ be the natural epimorphism to the quotient modulo the Frattini subgroup. Then $G/\Phi(G) = f(\tilde{G}) \oplus f(\pi_1(\Gamma))$ and since $f(\tilde{G})$ is finite (as $G/\Phi(G)$ is) there are vertices $w_1, \ldots, w_n$ of T such that $f(\tilde{G}) = \langle f(G_{w_1}), \ldots, f(G_{w_n})\rangle$. Hence $G = \langle G_{w_i}, \pi_1(\Gamma) \mid i = 1, \ldots, n\rangle$.

Now since G is finitely generated, so is $\pi_1(\Gamma)$ and therefore by Proposition 3.3, Γ contains a finite subgraph Δ such that $\pi_1(\Delta) = \pi_1(\Gamma)$. Let $v_1, \ldots, v_n$ be the images of $w_1, \ldots, w_n$ in Γ and Ω a minimal connected graph containing Δ and $v_1, \ldots, v_n$. Clearly (because Γ has finite diameter) Ω is finite and so there exists a connected transversal Σ of Ω in T. Let $w'_1, \ldots, w'_n$ be the vertices of Σ whose images in Ω are $v_1, \ldots v_n$ respectively. Since for each i we have $w'_i = g_i w_i$ for some $g_i \in G$ and so $G_{w'_i}$ is a conjugate of G_{w_i} in G, it follows that $G = \langle G_{w'_i}, \pi_1(\Gamma) \mid i = 1, \ldots n\rangle$. Let D be the connected component of the inverse image of Ω in T containing Σ. We show that D is G-invariant. Let $H = Stab(D)$ be the setwise stabilizer of D in G. Clearly, $G_{w'_i} \leq H$ for each i. By [2, Lemma 2.14], we have $H\backslash D = \Omega$. Note that $\Delta \subseteq \Omega \subseteq \Gamma$ and so $\pi_1(\Delta) = \pi_1(\Omega) = \pi_1(\Gamma)$. By Proposition 3.4 $H = \tilde{H} \rtimes \pi_1(\Omega)$, i.e. we may assume that $\pi_1(\Gamma) = \pi_1(\Omega)$ is contained in H. But then $G = H$ and $G\backslash D = \Omega$ is finite as desired.

□

Lemma 3.6. *Let $G = \Pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a finite graph of pro-p groups $(\mathcal{G}, \Gamma)$. Let D be a maximal subtree of Γ, n be the number of pending vertices of D. Then $n \leq 3d(G)$, where $d(G)$ is the minimal number of generators of G, and $|\Gamma \setminus D| \leq d(G)$.*

Proof. For every pending vertex v of Γ and the (unique) edge $e \in D$ connected to it, $\overline{\mathcal{G}}(v) = \mathcal{G}(v)/\mathcal{G}(e)^{\mathcal{G}(v)}$ is non-trivial, because the graph of groups $(\mathcal{G}, \Gamma)$ is reduced, and the groups are pro-p. Define the quotient graph of groups $(\overline{\mathcal{G}}, \Gamma)$ by putting $\overline{\mathcal{G}}(m) = 1$ if $m \in \Gamma$ is not a pending vertex and $\overline{\mathcal{G}}(v) = \mathcal{G}(v)/\mathcal{G}(e)^{\mathcal{G}(v)} \neq 1$ if v is a pending vertex. Then from the presentation (1) for $\Pi_1(\overline{\mathcal{G}}, \Gamma)$ it follows then that $\Pi_1(\overline{\mathcal{G}}, \Gamma) = \coprod_{v \in V(\Gamma)} \overline{\mathcal{G}}(v) \amalg \pi_1(\Gamma)$. The natural morphism $(\mathcal{G}, \Gamma) \longrightarrow (\overline{\mathcal{G}}, \Gamma)$ induces then the epimorphism $G = \Pi_1(\mathcal{G}, \Gamma) \longrightarrow \overline{G} = \Pi_1(\overline{\mathcal{G}}, \Gamma)$.

The number of pending vertices of Γ is at most $d(\overline{G}) \leq d(G)$. On the other hand the rank of $\pi_1(\Gamma)$ equals $|\Gamma \setminus D|$ and is not greater than $d(G)$ by Proposition 3.4. Every edge of $\Gamma \setminus D$ connects at most two pending vertices of D and so the number of pending vertices of D is at most $3d(\overline{G}) \leq 3d(G)$.

□

Lemma 3.7. *Let G be a pro-p group acting on a pro-p tree T with $|G\backslash T| < \infty$. Let H be a subgroup of G with an H-invariant subtree D of T such that the natural map $H\backslash D \longrightarrow G\backslash T$ is injective. Then $G = \Pi_1(\mathcal{G}, G\backslash T)$, $H = \Pi_1(\mathcal{H}, H\backslash D)$ and $(\mathcal{H}, H\backslash D)$ is a subgraph of groups of $(\mathcal{G}, G\backslash T)$.*

Proof. A maximal subtree of $H\backslash D$ can be extended to a maximal subtree of $G\backslash T$ and so we can choose a connected transversal S of $H\backslash D$ in D that extends to a connected transversal Σ of $G\backslash T$ in T. We may further suppose that if an edge e is in S or Σ, then so is $d_0(e)$. Let $\rho : T \longrightarrow G\backslash T$ be the natural epimorphism.

Then we can define the graph of groups $(\mathcal{H}, H\backslash D)$ and $(\mathcal{G}, G\backslash T)$ in the standard manner, as follows. If $s \in S$, define $\mathcal{H}(\rho(s)) = H_s$; if $e \in S$ is an edge, define $\partial_0 : \mathcal{H}_{\rho(e)} \to \mathcal{H}_{\rho(d_0(e))}$ to be the natural inclusion $H_e \to H_{d_0(e)}$, and if $k_e d_1(e) \in S$, define $\partial_1 : \mathcal{H}_{\rho(e)} \to \mathcal{H}_{\rho(d_1(e))}$ to be the natural inclusion $H_e \to H_{d_1(e)}$ followed by conjugation by k_e^{-1}: $H_{\rho(d_1(e))} \to H_{\rho(k_e d_1(e))}$. The definition is similar for $(\mathcal{G}, G\backslash T)$.

By [13, Proposition 3.10.4 and Theorem 6.6.1], we then have $G = \Pi_1(\mathcal{G}, G\backslash T)$, $H = \Pi_1(\mathcal{H}, H\backslash D)$.

□

4 Splitting of pro-p groups acting on trees

Lemma 4.1. *Let G be a finitely generated pro-p group acting on a pro-p tree T. Then $G = \varprojlim_{U \triangleleft_o G} G/\tilde{U}$ and $G/\tilde{U} = \Pi_1(\mathcal{G}_U, \Gamma_U)$ is the fundamental group of a finite reduced graph of finite p-groups. Moreover, the inverse system $\{G/\tilde{U}, \pi_{VU}\}$ can be chosen in such a way that it is linearly ordered and for each $\{G/\tilde{V}\}$ of the system with $V \leq U$ there exists a natural morphism $(\eta_{VU}, \nu_{VU}) : (\mathcal{G}_V, \Gamma_V) \longrightarrow (\mathcal{G}_U, \Gamma_U)$ where ν_{VU} is just a collapse of edges of Γ_V and $\eta_{VU}(\mathcal{G}_V(m)) = \pi_{VU}(\mathcal{G}_V(m))$; the induced homomorphism of the pro-p fundamental groups coincides with the canonical projection $\pi_{VU} : G/\tilde{V} \longrightarrow G/\tilde{U}$.*

Proof. Recall that $\tilde{U}$ is the closed subgroup of G generated by the vertex stabilizers U_v. Clearly $G/\tilde{U}$ and $U/\tilde{U}$ act on $\tilde{U}\backslash T$; by Proposition 3.4 $U/\tilde{U}$ is free pro-p. Thus $G_U := G/\tilde{U}$ is virtually free pro-p.

By [7, Theorem 1.1] it follows that G_U is the fundamental pro-p group $\Pi_1(\mathcal{G}_U, \Gamma_U)$ of a finite graph of finite p-groups. As mentioned in Section 2 we may assume that $(\mathcal{G}_U, \Gamma_U)$ is reduced.

Although the finite graph of finite p-groups $(\mathcal{G}_U, \Gamma_U)$ is not uniquely determined by U, the index U in the notation shall express that these objects are depending on U. Since the maximal finite subgroups of G_U are exactly the vertex groups of $(\mathcal{G}_U, \Gamma_U)$ up to conjugation (see Theorem 2.22), the number of vertices of Γ_U does not depend on the choice of $(\mathcal{G}_U, \Gamma_U)$, and since $\pi_1(\Gamma_U) = U/\tilde{U}$ is free pro-p of rank $|E(\Gamma_U)| - |V(\Gamma_U)| + 1$, the size of Γ_U is bounded in terms of possible decompositions as a reduced finite graph of finite p-groups of $U/\tilde{U}$.

Clearly we have $G = \varprojlim_U G_U$.

By [18, Prop. 1.10], viewing G_U as a quotient of G_V when $V \leq U$ (via the natural map $\pi_{VU} \colon G/\tilde{V} \longrightarrow G/\tilde{U}$), one has a natural decomposition of $G/\tilde{U}$ as the pro-p fundamental group $G/\tilde{U} = \Pi_1(\mathcal{G}_{VU}, \Gamma_V)$ of a finite graph of finite p-groups $(\mathcal{G}_{VU}, \Gamma_V)$, where the vertex and edge groups satisfy $\mathcal{G}_{VU}(x) = \pi_{VU}(\mathcal{G}_V(x))$, $x \in V(\Gamma_V) \cup E(\Gamma_V)$. Thus we have a morphism $\eta_{VU} \colon (\mathcal{G}_V, \Gamma_V) \longrightarrow (\mathcal{G}_{VU}, \Gamma_V)$ of graphs of groups such that the induced homomorphism on the pro-p fundamental groups coincides with the canonical projection π_{VU}.

If $(\mathcal{G}_{VU}, \Gamma_V)$ is not reduced, then collapsing some fictitious edges $e_i, i = 1, \ldots, k$, we arrive at a reduced graph of groups $((\mathcal{G}_{VU})_{red}, \Delta_V)$. By Proposition 2.23, the number of isomorphism classes of finite reduced graphs of finite p-groups $(\mathcal{G}', \Delta)$ which are based on a finite graph Δ and satisfy $G/\tilde{U} \simeq \Pi_1(\mathcal{G}', \Delta)$ is finite.

Using this remark, for each open normal subgroup V we let Ω_V be the (finite) set of reduced finite graphs of finite p-groups $(\mathcal{G}_V, \Gamma_V)$ with $G/\tilde{V} \simeq \Pi_1(\mathcal{G}_V, \Gamma_V)$. Let V_i, $i \in \mathbb{N}$, be a decreasing chain of open normal subgroups of G with $V_0 = U$ and $\bigcap_i V_i = (1)$. For $X \subseteq \Omega_{V_i}$ define $T(X)$ to be the set of all reduced graphs of groups in $\Omega_{V_{i-1}}$ that can be obtained from graphs of groups in X by the procedure explained in the preceding paragraph (note that T does not define a map on X). Define $\Omega_1 = T(\Omega_{V_1})$, $\Omega_2 = T(T(\Omega_{V_2}))$, $\ldots$, $\Omega_i = T^{(i)}(\Omega_{V_i})$ and note that Ω_i is a non-empty subset of Ω_U for every $i \in \mathbb{N}$. Clearly $\Omega_{i+1} \subseteq \Omega_i$ and since Ω_U is finite there is an $i_1 \in \mathbb{N}$ such that $\Omega_j = \Omega_{i_1}$ for all $j > i_1$ and we denote this Ω_{i_1} by Σ_U. Then $T(\Sigma_{V_i}) = \Sigma_{V_{i-1}}$ for all i, and so we can construct an infinite sequence of graphs of groups $(\mathcal{G}_{V_j}, \Gamma_j) \in \Omega_{V_j}$ such that $(\mathcal{G}_{V_{j-1}}, \Gamma_{j-1}) \in T(\mathcal{G}_{V_j}, \Gamma_j)$ for all j. This means that $(\mathcal{G}_{V_j V_{j-1}}, \Gamma_{V_j})$ can be reduced to $(\mathcal{G}_{V_{j-1}}, \Gamma_{j-1})$, i.e., that this sequence $\{(\mathcal{G}_{V_j}, \Gamma_j)\}$ is an inverse system of reduced graphs of groups satisfying the required conditions.

□

Note that in the classical Bass-Serre theory, a finitely generated group G acting irreducibly on a tree T has finitely many orbits, i.e. $G\backslash T$ is finite. This is not the case in the pro-p case; this fact highlights the complementary difficulties that appear in the pro-p case. The next result partially overcomes this.

Theorem 4.2. *Let G be a finitely generated pro-p group acting on a pro-p tree T without global fixed points. Then G splits non-trivially as a free amalgamated pro-p product or pro-p*

HNN-extension over some stabilizer of an edge of T.

Proof. By Lemma 4.1 $G = \varprojlim_{U \lhd_o G} G/\tilde{U}$, where $G/\tilde{U} = \Pi_1(\mathcal{G}_U, \Gamma_U)$ is the fundamental group of a finite reduced graph of finite p-groups and for each $V \lhd_o G$ contained in U, one has a natural morphism $(\eta_{VU}, \nu_{VU}) : (\mathcal{G}_V, \Gamma_V) \longrightarrow (\mathcal{G}_U, \Gamma_U)$ such that ν_{VU} is just a collapse of edges of Γ_V. Moreover, the induced homomorphism of the pro-p fundamental groups coincides with the canonical projection $\pi_{VU} \colon G/\tilde{V} \longrightarrow G/\tilde{U}$.

Note that $U/\tilde{U}$ is non-trivial for some U, since otherwise $G/\tilde{U}$ is finite for every U and by Theorem 2.20 stabilizes a vertex v_U; hence by an inverse limit argument G would stabilize a vertex v in T contradicting the hypothesis. Hence Γ_U contains at least one edge.

Case 1. There exists U and an edge e_U in Γ_U such that $\Gamma_U \setminus \{e_U\}$ is disconnected.

Let e_V be an edge of Γ_V such that $\nu_{VU}(e_V) = e_U$. Since Γ_U is obtained from Γ_V by collapsing edges, $\Gamma_V \setminus \{e_V\}$ is disconnected as well. Thus we may write $G_V = A_V \amalg_{\mathcal{G}_V(e_V)} B_V$, where A_V, B_V are the fundamental groups of the graphs of groups $(\mathcal{G}_V, \Gamma_V)$ restricted to the connected components of $\Gamma_V \setminus \{e_V\}$, and we have an inverse limit of free amalgamated products that gives a decomposition $G = A \amalg_{\mathcal{G}(e)} B$ for some $e \in E(T)$, with $A = \varprojlim_V A_V$, $B = \varprojlim_V B_V$.

Case 2. For all U and each edge e_U of Γ_U the graph $\Gamma_U \setminus \{e_U\}$ is connected.

By Lemma 3.1 (applied inductively) the preimage in Γ_V of a maximal subtree D_U of Γ_U is a maximal subtree D_V of Γ_V. Therefore for each V we have

$$G/\tilde{V} = \mathrm{HNN}(L_V, \mathcal{G}_V(e), t_e, e \in \Gamma_V \setminus D_V),$$

where $L_V = \Pi_1(\mathcal{G}_V, D_V)$. Note that the image of $\tilde{G}$ in $G/\tilde{V}$ is $\widetilde{G/\tilde{V}}$ and since G is finitely generated, so is $G/\tilde{G}$. Therefore, by Proposition 3.4, $\pi_1(\Gamma_V) = F(\Gamma_V \setminus D_V)$ is a free pro-p group of rank $|\Gamma_V \setminus D_V| \leq rank(G/\tilde{G})$, i.e. we can assume that $\Gamma_V \setminus D_V$ is a constant set E. Then we can view E as a finite subset of $E(T)$ and putting $L = \varprojlim_V L_V$ we have $G = \mathrm{HNN}(L, G_e, t_e, e \in E)$ for some $e \in E(T)$ as required.

□

Remark 4.3. By Lemma 4.1, a finitely generated pro-p group G acting on a pro-p-tree T can be represented as an inverse limit of fundamental groups of finite reduced graphs of pro-p-groups. There is however no bound on the size of the finite graphs. The next result will show that given any finite graph Γ_U occuring in this inverse limit, G admits a representation as $\Pi_1(\mathcal{G}, \Gamma_U)$.

Corollary 4.4. *Let G be a finitely generated pro-p-group G which acts on a pro-p tree T, and let $U, G_U, \Gamma_U, \mathcal{G}_U$ satisfy the conclusion of Lemma 4.1. Then G splits as the pro-p fundamental group of a reduced finite graph of pro-p groups $G = \Pi_1(\mathcal{G}, \Gamma_U)$ with edge groups being stabilizers of some edges of T.*

Proof. We use induction on the size of Γ_U. If $|\Gamma_U| = 1$, there is nothing to prove. If $|\Gamma_U| > 1$, then $G/\tilde{U}$ is infinite. Let $\pi_U : G \longrightarrow G/\tilde{U}$ be the natural projection. Pick $e_U \in E(\Gamma_U)$. If $\Gamma_U \setminus \{e_U\} = \Delta \cup \Omega$ is disconnected with two connected components Δ and Ω, then from the proof of Theorem 4.2 it follows that G splits as an amalgamated free product $A \amalg_{G_e} B$

with $\pi_U(G_e) = \mathcal{G}_U(e_U)$, where $\pi_U(A)$ and $\pi_U(B)$ are the fundamental groups of the reduced graphs of groups $(\mathcal{G}_U, \Delta)$ and $(\mathcal{G}_U, \Omega)$ that are the restrictions of $(\mathcal{G}_U, \Gamma_U)$ to these connected components. Moreover, assuming w.l.o.g $d_0(e) \in \Delta$, $d_1(e) \in \Omega$ we have embeddings $\partial_0 : \mathcal{G}_U(e) \longrightarrow \mathcal{G}_U(d_0(e)) \leq \pi_U(A)$, $\partial_1 : \mathcal{G}_U(e) \longrightarrow \mathcal{G}_U(d_1(e)) \leq \pi_U(B)$ for each U that induce the natural embeddings $\mathcal{G}(e) \longrightarrow \mathcal{G}(d_0(e)) \leq A$, $\mathcal{G}(e) \longrightarrow \mathcal{G}(d_1(e)) \leq B$. Hence from the induction hypothesis $A = \Pi_1(\mathcal{G}, \Delta)$, $B = \Pi_1(\mathcal{G}, \Omega)$ and the result follows.

If $\Gamma_U \setminus \{e_U\}$ is connected then again from the proof of Theorem 4.2 it follows that G_U splits as an HNN-extension $G_U = \mathrm{HNN}(L, \mathcal{G}_U(e), t_e, e \in \Gamma_U \setminus D_U)$, where D_U is a maximal subtree of $\Gamma_U \setminus \{e_U\}$ and $\pi_U(G_e) = \mathcal{G}_U(e_U)$, $\pi_U(L) = \pi_1(\mathcal{G}_U, D_U)$. Moreover, we have embeddings $\partial_i : \mathcal{G}_U(e) \longrightarrow \mathcal{G}_U(d_i(e)) \leq \pi_U(L)$, for each U that induce the natural embeddings $\mathcal{G}(e) \longrightarrow \mathcal{G}(d_i(e)) \leq L$. Then by induction hypothesis $L = \Pi_1(\mathcal{G}, D_U)$ and $G = \Pi_1(\mathcal{G}, \Gamma_U)$ as needed.

Finally we observe that $(\mathcal{G}, \Gamma_U)$ is reduced since $(\mathcal{G}_U, \Gamma_U)$ is.

□

A.A. Korenev [11] defined the number of pro-p ends $e(G)$ for an infinite pro-p group G as $e(G) = 1 + dim H^1(G, \mathbb{F}_p[[G]])$. The next theorem shows that similar to the abstract case a pro-p group acting irreducibly on an infinite pro-p tree with finite edge stabilizers has more than one end.

Theorem 4.5. *Let G be a finitely generated pro-p group acting on a pro-p tree T with finite edge stabilizers and without global fixed points. Then either G is virtually cyclic and $H^1(G, \mathbb{F}_p[[G]]) \cong \mathbb{F}_p$ (i.e. G has two ends) or $H^1(G, \mathbb{F}_p[[G]])$ is infinite (i.e. G has infinitely many ends).*

Proof. By Theorem 4.2, G splits either as an amalgamated free pro-p product or an HNN-extension over an edge stabilizer G_e and so acts on the standard pro-p tree $T(G)$ associated with this splitting. Let H be an open normal subgroup of G intersecting G_e trivially. Then H acts on $T(G)$ with trivial edge stabilizers and so by Theorem 2.21 H is a non-trivial free pro-p product $H = H_1 \amalg H_2$.

Then we have the following exact sequence (associated to the standard pro-p tree) for this free product decomposition:

$$0 \to \mathbb{F}_p[[H]] \xrightarrow{\delta} \mathbb{F}_p[[H/H_1]] \oplus \mathbb{F}_p[[H/H_2]] \xrightarrow{\varepsilon} \mathbb{F}_p \to 0 \qquad (*)$$

Claim. The augmentation ideal $I(H)$ is decomposable as an $\mathbb{F}_p[[H]]$-module.

Proof. Let M_1 and M_2 be the kernels of the restrictions of ε to $\mathbb{F}_p[[H/H_1]]$ and $\mathbb{F}_p[[H/H_2]]$ respectively. We will show that $\delta(I(H)) = M_1 \oplus M_2$. Since $\delta(\mathbb{F}_p[[H]]) = \ker(\varepsilon)$, $M_1 \oplus M_2$ is a submodule of $\delta(\mathbb{F}_p[[H]])$ and since the middle term of $(*)$ modulo $M_1 \oplus M_2$ is $\mathbb{F}_p \oplus \mathbb{F}_p$, it is of index p in $\ker(\varepsilon)$. But $\mathbb{F}_p[[H]]$ is a local ring and so has a unique maximal left ideal, hence $\delta(I(H)) = M_1 \oplus M_2$ as needed. The claim is proved.

Now applying $Hom_{\mathbb{F}_p[[H]]}(-, \mathbb{F}_p[[H]])$ to

$$0 \to I(H) \to \mathbb{F}_p[[H]] \to \mathbb{F}_p \to 0$$

and observing that by [11, Lemma 3]

$$Hom_{\mathbb{F}_p[[H]]}(\mathbb{F}_p, \mathbb{F}_p[[H]]) = (\mathbb{F}_p[[H]])^H = 0, \quad Hom_{\mathbb{F}_p[[H]]}(\mathbb{F}_p[[H]], \mathbb{F}_p[[H]]) = \mathbb{F}_p[[H]]$$

and $Ext^1_{\mathbb{F}_p[[H]]}(\mathbb{F}_p[[H]], M) = 0$ since $\mathbb{F}_p[[H]]$ is a free pro-p module, we obtain the exact sequence

$$0 \to \mathbb{F}_p[[H]] \xrightarrow{\varphi} Hom_{\mathbb{F}_p[[H]]}(I(H), \mathbb{F}_p[[H]]) \to H^1(H, \mathbb{F}_p[[H]]) \to 0.$$

(Here we also use that $Ext^1_{\mathbb{F}_p[[H]]}(\mathbb{F}_p, M) = H^1(H, M)$ for an $\mathbb{F}_p[[H]]$-module M). Since $\mathbb{F}_p[[H]]$ is indecomposable and

$$Hom_{\mathbb{F}_p[[H]]}(I(H), \mathbb{F}_p[[H]]) \cong Hom_{\mathbb{F}_p[[H]]}(M_1, \mathbb{F}_p[[H]]) \oplus Hom_{\mathbb{F}_p[[H]]}(M_2, \mathbb{F}_p[[H]])$$

(from the claim), φ is not onto and so $H^1(H, \mathbb{F}_p[[H]]) \neq 0$.

Then by [11, Theorems 1 and 2], the dimension of $H^1(H, \mathbb{F}_p[[H]])$ is either infinite or 1 and in the latter case H is virtually cyclic. By [11, Lemma 2], $H^1(H, \mathbb{F}_p[[H]]) \cong H^1(G, \mathbb{F}_p[[G]])$, hence the result.

□

5 Subgroups of fundamental groups of graphs of pro-p groups

In the classical Bass-Serre theory of groups acting on trees a finitely generated group G acting on a tree T is the fundamental group of a finite graph of groups whose edge and vertex groups are G-stabilizers of edges and vertices of T respectively. This is due to the fact that for finitely generated G there exists a G-invariant subtree D such that $G\backslash D$ is finite. In the pro-p situation this is not always the case. Note that $G\backslash D$ finite implies that there are only finitely many maximal stabilizers of vertices of T in G up to conjugation. In this section we prove the above mentioned result in the pro-p case under the assumption of finitely many maximal vertex stabilizers up to conjugation.

Theorem 5.1. *Let G be a finitely generated pro-p group acting on a pro-p tree T with finitely many maximal vertex stabilizers up to conjugation. Then G is the fundamental group of a reduced finite graph of pro-p groups $(\mathcal{G}, \Gamma)$, where each vertex group $\mathcal{G}(v)$ and each edge group $\mathcal{G}(e)$ is a maximal vertex stabilizer $G_{\tilde v}$ and an edge stabilizer $G_{\tilde e}$ respectively (for some $\tilde v, \tilde e \in T$).*

Proof. By Lemma 4.1, $G = \varprojlim_{U \triangleleft_o G} G/\tilde U$, where $G/\tilde U = \Pi_1(\mathcal{G}_U, \Gamma_U)$ is the fundamental group of a finite reduced graph of finite p-groups and for each $V \triangleleft_0 G$ contained in U one has a natural morphism $(\eta_{VU}, \nu_{VU}) : (\mathcal{G}_V, \Gamma_V) \longrightarrow (\mathcal{G}_U, \Gamma_U)$ such that ν is just a collapse of edges of Γ_V. Moreover, the induced homomorphism of the pro-p fundamental groups coincides with the canonical projection $\pi_{VU} \colon G/\tilde V \longrightarrow G/\tilde U$.

We claim now that the number of vertices and edges of Γ_U is bounded independently of U. Let $G_{v_1}, \dots, G_{v_n}$ be the maximal vertex stabilizers of G up to conjugation. Then G_U and

$U/\tilde{U}$ act on $\tilde{U}\backslash T$; by Proposition 3.4, the quotient group $U/\tilde{U}$ acts freely on the pro-p tree $\tilde{U}\backslash T$. Thus all vertex stabilizers of $G/\tilde{U}$ are finite and are the images of the corresponding vertex stabilizers of G. Note that any finite subgroup of $G/\tilde{U}$ stabilizes a vertex (Theorem 2.20) and since the maximal finite subgroups of G_U are exactly the vertex groups of $(\mathcal{G}_U, \Gamma_U)$ up to conjugation (see Theorem 2.22), we see that the number of vertices of Γ_U is bounded by n. Since $\pi_1(\Gamma_U) \cong G/\tilde{G}$ is free pro-p of rank $|E(\Gamma_U)| - |V(\Gamma_U)| + 1$ by Proposition 3.4, the number of edges of Γ_U is bounded by $n + d(G) - 1$ and so the size of Γ_U is bounded independently of U. Hence, for some U and all $V \leq U$, the maps $\nu_{VU} : \Gamma_V \to \Gamma_U$ are isomorphisms, and we will denote this graph by Γ.

Then for $(\mathcal{G}, \Gamma) = \varprojlim (\mathcal{G}_U, \Gamma)$ we have $\mathcal{G}(x) = \varprojlim \mathcal{G}_U(x)$ if x is either a vertex or an edge of Γ, and $(\mathcal{G}, \Gamma)$ is a reduced finite graph of pro-p groups satisfying $G \simeq \Pi_1(\mathcal{G}, \Gamma)$. This finishes the proof of the theorem.

□

Corollary 5.2. *The number up to conjugation of maximal vertex stabilizers in G equals $|V(\Gamma)|$.*

Proof. Since $(\mathcal{G}, \Gamma) = \varprojlim_U (\mathcal{G}_U, \Gamma)$, the result follows from Theorem 5.1.

□

One of the main consequences of the main theorem of Bass-Serre theory is an extension of the Kurosh subgroup theorem to a group G acting on a tree T. Namely if H is a subgroup of G then $H = \pi_1(\mathcal{H}, \Delta)$ is the fundamental group of a graph of groups constructed as follows. Let $\Delta = H\backslash T$ and if Σ is a connected transversal of Δ in T then $\mathcal{H}$ consists of the stabilizers of the edges and vertices of Σ.

In the pro-p situation such a theorem does not hold in general ([6, Theorem 1.2]). Our next objective is to prove it for H acting acylindrically and having finitely many maximal vertex stabilizers up to conjugation.

Definition 5.3. The action of a pro-p group G on a pro-p tree T is said to be *k-acylindrical*, for k a constant, if for every $g \neq 1$ in G, the subtree T^g of fixed points has diameter at most k.

Theorem 5.4. *Let G be a finitely generated pro-p group acting n-acylindrically on a pro-p tree T with finitely many maximal vertex stabilizers up to conjugation. Then*

(i) The closure $\overline{D}$ of $D = \{t \in T \mid G_t \neq 1\}$ is a profinite G-invariant subgraph of T having finitely many connected components Σ_i, $i = 1, \ldots, m$, up to translation.

(ii) for the setwise stabilizer $G_i = Stab_G(\Sigma_i)$ the quotient graph $G_i\backslash\Sigma_i$ has finite diameter and Σ_i contains a G_i-invariant subtree D_i such that $G_i\backslash D_i$ is finite.

(iii) $G = \coprod_{i=1}^m G_i \amalg F$ is a free pro-p product, where F is a free pro-p group acting freely on T.

Proof. We follow the idea of the proof of [21, Theorem 3.5].

Since the action is n-acylindrical, T^{G_t} has diameter at most n for every non-trivial edge or vertex stabilizer G_t. Note that $D = \bigcup_{G_t\neq 1} T^{G_t}$. We first show that $G\backslash D$ has finite diameter (as an abstract graph). Indeed, since there are only finitely many maximal vertex stabilizers up to conjugation, say $G_{v_1}, \ldots G_{v_k}$, it suffices to show that for a maximal vertex stabilizer G_{v_i}, the tree $\bigcup_{1\neq G_t\leq G_{v_i}} T^{G_t}$ has finite diameter (if non-empty). But for $1 \neq G_t \leq G_{v_i}$ the geodesic $[t, v_i]$ is stabilized by G_t (cf. [17, Corollary 3.8]) and so has length at most n. Thus $G\backslash D$ as an abstract graph has finite diameter (at most $2nk$) and finitely many connected components (at most k).

It follows that the closure Δ of $G\backslash D$ in $G\backslash T$ has also finitely many (profinite) connected components (at most k) and finite diameter (at most $2nk$), see Lemma 3.2. Note that the preimage of Δ in T is exactly $\overline{D}$. Since Δ has finite diameter and so all its connected components are connected as abstract graphs by [8, Corollary 4], it is immediate that connected components of $\overline{D}$ are mapped surjectively onto corresponding connected components of Δ (Indeed, if Σ is a connected component of $\overline{D}$ and Ω is its image in Δ, let e be an edge with say $d_0(e) \in \Omega$. Then for an edge $e' \in \overline{D}$ which is mapped onto e there exists $g \in G$ with $d_0(ge') \in \Sigma$ so that $ge' \in \Sigma$ and so $e \in \Omega$). Thus the number of connected components of $\overline{D}$ up to translation equals the number of connected components of Δ ($\leq k$). This proves (i).

Collapsing all connected components of $\overline{D}$, by Proposition on page 486 of [28] or [13, Proposition 3.9.1, as well as Cor. 3.10.2 and Prop. 3.10.4(b)], we get a pro-p tree $\bar{T}$ on which G acts with trivial edge stabilizers, and vertex stabilizers being the setwise stabilizers of the connected components of $\bar{D}$. In particular, we have only finitely many vertices $v'_1, \ldots, v'_m$ up to translation whose stabilizers $G_{v'_i}$ are non-trivial (and $m \leq k$). So by Theorem 2.21, G is a free pro-p product

$$G = \coprod_{i=1}^{m} G_{v'_i} \amalg F,$$

where F is naturally isomorphic to $G/\tilde{G}$ with $\tilde{G}$ taken with respect to the action on $\bar{T}$, i.e. $\widetilde{G} = \langle G_{v'_1}, \ldots, G_{v'_m}\rangle^G$. Then F acts freely on T and (iii) is proved.

By [2, Lemma 2.14], for any connected component Σ_i of $\overline{D}$ and its setwise stabilizer $G_i = Stab_G(\Sigma_i)$ we have $G_i\backslash\Sigma_i \subseteq \Delta$ and so $G_i\backslash\Sigma_i$ is a connected component Δ_i of Δ. So Δ_i has finite diameter and by Proposition 3.5 Σ_i possesses a G_i-invariant subtree D_i such that $G_i\backslash D_i$ is finite. This proves (ii). □

Corollary 5.5. *Let G be a finitely generated pro-p group which is the fundamental group of a finite graph $(\mathcal{G}, \Gamma)$ of pro-p groups, and let $T = T(G)$ its standard pro-p tree. Let H be a finitely generated subgroup of G that acts n-acylindrically on T, with finitely many maximal vertex stabilizers up to conjugation. Then $H = \coprod_{i=1}^{m} H_i \amalg F$ (possibly with one factor) with F free pro-p and there exists an open subgroup U of G containing H such that*

(i) The natural map $F \longrightarrow U/\tilde{U}$ is injective.

(ii) $U = \Pi_1(\mathcal{U}, U\backslash T)$*,* $H_i = \Pi_1(\mathcal{H}_i, H_i\backslash D_i)$ *(where* D_i *is a minimal* H_i*-invariant subtree of* T*) and* $(\mathcal{H}_i, H_i\backslash D_i)$ *are disjoint subgraphs of groups of* $(\mathcal{U}, U\backslash T)$*.*

Moreover, the latter statements (i) and (ii) hold for any open subgroup V *of* U *containing the group* H*.*

Proof. By Theorem 5.4 applied to the action of H on T, there are subgroups H_i $(i = 1, \dots, m)$ and F of H, with F free pro-p, such that $H = \coprod_{i=1}^{m} H_i \amalg F$. Furthermore there are H_i-invariant subtrees D_i of T with $H_i\backslash D_i$ finite, and $H_i = Stab_H(D_i)$, and $F = H/\langle H_1, \dots, H_m\rangle^H$. Note that the $H_i\backslash D_i$ are disjoint subgraphs of $H\backslash T$, and are contained in $\bar{D}$ (notation as in Theorem 5.4). Choose an open subgroup U of G containing H such that the map $\bigcup_i (H_i\backslash D_i) \to U\backslash T$ is injective and the map $F \to U/\tilde{U}$ is injective (this is possible since $\bigcup_i H_i\backslash D_i$ is finite, F is finitely generated and $U/\tilde{U}$ is free pro-p). Then the $H_i\backslash D_i$ are disjoint in $U\backslash T$, whence if we choose maximal subtrees T_i of $H_i\backslash D_i$, their union extends to a maximal subtree of $U\backslash T$. As $G\backslash T$ is finite, so is $U\backslash T$, and we can then apply (the proof of) Lemma 3.7 to get the result. □

6 Generalized accessible pro-p groups

We apply here the results of the previous section to finitely generated generalized accessible pro-p groups in the sense of the following definition.

Definition 6.1. Let $\mathcal{F}$ be a family of pro-p groups. A pro-p group G will be called $\mathcal{F}$*-accessible* if there is a number $n = n(G)$ such that any finite, proper, reduced graph of pro-p groups with edge groups in $\mathcal{F}$ having fundamental group isomorphic to G has at most n edges.

The definition generalizes the definition of accessibility given in [24], where the edge groups are finite. In fact if $\mathcal{F}$ is the class of all finite p-groups, an $\mathcal{F}$-accessible pro-p group will simply be called *accessible*.

Proposition 6.2. *Let* $\mathcal{F}$ *be a family of pro-p groups and* G *a finitely generated* $\mathcal{F}$*-accessible pro-p group acting on a pro-p tree* T *with edge stabilizers in* $\mathcal{F}$*. Then* G *has only finitely many maximal vertex stabilizers up to conjugation, and in fact the number of such stabilizers does not exceed* $n(G) + 1$*.*

Proof. Let $G_{v_1}, \dots G_{v_m}$ be maximal vertex stabilizers which are non-conjugate. We will show that m is bounded. If $U \lhd_0 G$, then $G/\tilde{U}$ acts on $\tilde{U}\backslash T$ and by Lemma 4.1, $G = \varprojlim_{U \lhd_o G} G/\tilde{U}$, where $G/\tilde{U} = \Pi_1(\mathcal{G}_U, \Gamma_U)$ is the fundamental group of a finite reduced graph of finite p-groups. Thus starting from a certain U the stabilizers $G_{v_1}\tilde{U}/\tilde{U}, \dots G_{v_m}\tilde{U}/\tilde{U}$ of the images of $v_1, \dots, v_m$ in $\tilde{U}\backslash T$ are still maximal and distinct. So they are maximal finite subgroups of $G/\tilde{U} = \Pi_1(\mathcal{G}_U, \Gamma_U)$ and so are conjugate to vertex groups of $(\mathcal{G}_U, \Gamma_U)$ (see Theorem 2.22). Therefore Γ_U has at least m vertices. Then by Corollary 4.4, G admits a decomposition as the fundamental group of a reduced finite graph of pro-p groups $\Pi_1(\mathcal{G}, \Gamma_U)$ with edge groups in $\mathcal{F}$ and so $m \le n(G) + 1$. □

Theorem 6.3. *Let G be a finitely generated $\mathcal{F}$-accessible pro-p group acting on a pro-p tree T with edge stabilizers in $\mathcal{F}$. Then G is the fundamental pro-p group of a finite graph of pro-p groups $(\mathcal{G}, \Gamma)$, where each vertex group $\mathcal{G}(v)$ and each edge group $\mathcal{G}(e)$ is a vertex stabilizer $G_{\tilde{v}}$ and an edge stabilizer $G_{\tilde{e}}$ respectively (for some $\tilde{v}, \tilde{e} \in T$). Moreover, the size of $V(\Gamma)$ is bounded by $n(G) + 1$ for every such $(\mathcal{G}, \Gamma)$.*

Proof. By Proposition 6.2 the number of maximal vertex stabilizers of G is bounded by $n(G)+1$. Therefore the result follows from Theorem 5.1 and Corollary 5.2.

□

Remark 6.4. In the classical Bass-Serre theory of groups acting on trees, structure theorems like Theorem 6.3 are used to obtain structure results on subgroups of fundamental groups of graphs of groups (see for example [19, §5]). In our situation, to use Theorem 6.3 for this purpose one needs to assume that $\mathcal{F}$ is closed under subgroups. A relevant to the context such general example is the class of small pro-p groups. Namely, one can follow the approach of [1] in the abstract case and call a pro-p group G *small* if whenever G acts on a pro-p-tree T, and $K \leq G$ acts freely on T, then K is procyclic. If the action on T is associated with splitting G into a free amalgamated product $G = G_1 \amalg_H G_2$ or an HNN-extension $G = \mathrm{HNN}(G_1, H, t)$, it means that H is normal in G with G/H either procyclic or infinite dihedral (see [17, Theorem 3.15]). The class of small pro-p groups $\mathcal{S}$ is closed under subgroups. Then one can use Theorem 6.3 to prove the following statement.

Let G be a pro-p group acting on a pro-p tree T with small edge stabilizers. Let H be a finitely generated $\mathcal{S}$-accessible subgroup of G. Then $H = \Pi_1(\mathcal{H}, \Gamma)$ is the fundamental group of a finite graph of pro-p groups $(\mathcal{H}, \Gamma)$, where each vertex group $\mathcal{H}(v)$ and each edge group $\mathcal{H}(e)$ is a vertex stabilizer $H_{\tilde{v}}$ and an edge stabilizer $H_{\tilde{e}}$ respectively (for some $\tilde{v}, \tilde{e} \in T$). Moreover, the cardinality of $E(\Gamma)$ is bounded by the accessibility number $n(H)$.

Note also that a finitely generated free pro-p group F is $\mathcal{S}$-accessible, since a free small pro-p group has to be pro-cyclic.

Example 6.5. If $\mathcal{C}$ is a class of pro-cyclic pro-p groups then any finitely generated pro-p group is $\mathcal{C}$-accessible ([21, Lemma 3.2]).

In fact we can bound the $\mathcal{C}$-accessibility number $n(G)$ in terms of the minimal number of generators $d(G)$ of G.

Theorem 6.6. *Let $G = \pi_1(\Gamma, \mathcal{G})$ be the fundamental group of a finite graph of pro-p-groups, with procyclic edge groups, and assume that $d(G) = d \geq 2$. Then (assuming that the graph is reduced), the vertex groups are finitely generated, the number of vertices of Γ is $\leq 2d - 1$, and the number of edges of Γ is $\leq 3d - 2$.*

Proof. Let T be a maximal subtree of Γ and $H = \Pi_1(\mathcal{G}, T)$ be the fundamental group of the tree of groups $(\mathcal{G}, T)$ obtained by restricting $(\mathcal{G}, \Gamma)$ to T. Then $G = \mathrm{HNN}(H, C_1, \ldots, C_\ell, t_1, \ldots, t_\ell)$, where $\ell = |E(\Gamma)| - |E(T)|$. Note that the quotient of G by the normal subgroup generated by H is free on $t_1, \dots, t_\ell$, so $d(G) \geq \ell$.

Since $|V(\Gamma)| = |V(T)|$, and $|E(T)| = |V(T)| - 1$ we have $|E(\Gamma)| = |V(T)| - 1 + \ell \leq |V(T)| + d - 1$. It therefore suffices to show that $|V(T)| = |V(\Gamma)| \leq 2d - 1$.

Consider the Mayer-Vietoris sequence

$$\to \oplus_{e\in E(T)} H_1(\mathcal{G}(e)) \to \oplus_{v\in V(T)} H_1(\mathcal{G}(v)) \to H_1(G) \to \mathbb{F}_p[[E(T)]] \to \mathbb{F}_p[[V(T)]] \to \mathbb{F}_p \to 0$$

(see [13, Thm 9.4.1]). Since T is a tree,

$$0 \to \mathbb{F}_p[[E(T)]] \to \mathbb{F}_p[[V(T)]] \to \mathbb{F}_p \to 0$$

is exact (see Subsection 2.2) and so $H_1(G) \to \mathbb{F}_p[[E(T)]]$ is the zero map, and $\oplus_{v\in V(T)} H_1(\mathcal{G}(v)) \to H_1(G)$ is onto.

Let n be the number of vertices of T, let m be the number of vertices whose vertex groups are cyclic and k be the number of vertices whose vertex groups are not cyclic, so that $n = m+k$ and the number of edges of T is $n-1$. Then

$$d(G) = dim(H_1(G)) \geq m + 2k - (n-1) = m + 2k - n + 1 = k + 1$$

(here we are using that all edge groups are cyclic). On the other hand if the vertex group $\mathcal{G}(v)$ is cyclic and e is incident to v then the natural map $H_1(\mathcal{G}(e)) \to H_1(\mathcal{G}(v))$ is the zero map (because $\mathcal{G}(e) \leq \Phi(\mathcal{G}(v))$). Denoting by V_c the set of vertices with cyclic vertex group, it follows that $\oplus_{v\in V_c} H_1(\mathcal{G}(v))$ intersects trivially the image of $\oplus_{e\in E(T)} H_1(\mathcal{G}(e))$ and therefore maps injectively into $H_1(G)$. Therefore $m \leq d$. Thus $n = k+m \leq d(G) - 1 + d(G) = 2d(G) - 1$ as required.

Finally, since $\oplus_{e\in E(T)} H_1(\mathcal{G}(e))$ and $H_1(G)$ are finite, so is $\oplus_{v\in V(T)} H_1(\mathcal{G}(v))$, i.e. $\mathcal{G}(v)$ is finitely generated for every v.

□

We now apply Theorem 6.6 to the pro-p analogue of a limit groups defined in [10]. It is worth recalling their definition.

Denote by $\mathcal{G}_0$ the class of all free pro-p groups of finite rank. We define inductively the class $\mathcal{G}_n$ of pro-p groups G_n in the following way: G_n is a free amalgamated pro-p product $G_{n-1} \amalg_C A$, where G_{n-1} is any group from the class $\mathcal{G}_{n-1}$, C is any self-centralizing procyclic pro-p subgroup of G_{n-1} and A is any finite rank free abelian pro-p group such that C is a direct summand of A. The class of pro-p groups $\mathcal{L}$ (*pro-p limit groups*) consists of all finitely generated pro-p subgroups H of some $G_n \in \mathcal{G}_n$, where $n \geq 0$. Then H is a subgroup of a free amalgamated pro-p product $G_n = G_{n-1} \amalg_C A$, where $G_{n-1} \in \mathcal{G}_{n-1}$, $C \cong \mathbb{Z}_p$ and $A = C \times B \cong \mathbb{Z}_p^m$. By Theorem 3.2 in [14], this amalgamated pro-p product is proper. Thus H acts naturally on the pro-p tree T associated to G_n and its edge stabilizers are procyclic.

An immediate application of Theorem 6.6 then gives a bound on the $\mathcal{C}$-accessibility number of a limit pro-p group.

Corollary 6.7. *Let G be a pro-p limit group. Then G is the fundamental group of a finite graph of finitely generated pro-p groups $(\mathcal{G}, \Gamma)$, where each edge group $\mathcal{G}(e)$ is infinite procyclic. Moreover, $|V(\Gamma)| \leq 2d-1$, and $|E(\Gamma)| \leq 3d-2$, where d is the minimal number of generators of G.*

Accessible pro-p groups

This subsection is dedicated to accessible pro-p groups. Note that there exists a finitely generated non-accessible pro-p group (see [24]) and that it is an open question of whether a finitely presented pro-p group is accessible.

The next proposition gives a characterization of accessible pro-p groups.

Proposition 6.8. *Let G be a finitely generated pro-p group. Then G is accessible if and only if it is a virtually free pro-p product of finitely many virtually freely indecomposable pro-p groups.*

Proof. Let H be an open subgroup of G that splits as a free pro-p product of virtually freely indecomposable pro-p groups. Replacing H by the core of H in G and applying the Kurosh subgroup theorem for open subgroups (cf. [15, Thm. 9.1.9]), we may assume that H is normal in G. Refining the free decomposition if necessary and collecting free factors isomorphic to $\mathbb{Z}_p$ we obtain a free decomposition

$$H = F \amalg H_1 \amalg \cdots \amalg H_s, \tag{3}$$

where F is a free subgroup of rank t, and the H_i are virtually freely indecomposable finitely generated subgroups which are not isomorphic to $\mathbb{Z}_p$. By [23, Theorem 3.6] $G = \Pi_1(\mathcal{G}, \Gamma)$ is the fundamental pro-p group of a finite graph of pro-p groups with finite edge groups. Moreover, it follows from its proof (step 2) that H intersects all edge groups trivially. Then by [24, Theorem 3.1] Γ has at most $\frac{p[G:H]}{p-1}(d(G)-1)+1$ edges. So G is accessible.

Conversely, suppose G is accessible. Write $G = \Pi_1(\mathcal{G}, \Gamma)$, where $(\mathcal{G}, \Gamma)$ is a finite graph of pro-p groups with finite edge groups, and such that Γ is of maximal size. Choosing an open normal subgroup H intersecting all edge groups of G trivially we have

$$H = \coprod_{v \in V(\Gamma)} \coprod_{g_v \in H \backslash G / \mathcal{G}(v)} H \cap \mathcal{G}(v)^{g_v} \amalg F,$$

where g_v runs through double cosets representatives of $H \backslash G / \mathcal{G}(v)$ and F is free pro-p of finite rank (see Theorem 2.21 with the action of H on the standard pro-p tree $T(G)$). Since Γ is of maximal size, $\mathcal{G}(v)$ does not split as an amalgamated free pro-p product or HNN extension over a finite p-group, so by [23, Theorem A] $\mathcal{G}(v)$ is not a virtual free pro-p product, in particular $H \cap \mathcal{G}(v)^{g_v}$ is freely indecomposable. Since F is a free pro-p product of $\mathbb{Z}_p$'s the result follows. □

Question 6.9. Let G be a finitely generated pro-p group acting on a pro-p tree with finite vertex stabilizers. Is G accessible?

Note that $H^1(G, \mathbb{F}_p[[G]])$ is a right $\mathbb{F}_p[[G]]$-module. The next theorem gives a sufficient condition of accessibility for a pro-p group in terms of this module; we do not know whether the converse also holds (it holds in the abstract case).

Theorem 6.10. *Let G be a finitely generated pro-p group. If $H^1(G, \mathbb{F}_p[[G]])$ is a finitely generated $\mathbb{F}_p[[G]]$-module, then G is accessible.*

Proof. Suppose $G = \Pi_1(\mathcal{G}, \Gamma)$ is the fundamental group of a reduced finite graph $(\mathcal{G}, \Gamma)$ of pro-p groups with finite edge groups. We will first do the case where if v is any vertex of Γ, then $\mathcal{G}(v)$ is infinite and $H^1(\mathcal{G}(v), \mathbb{F}_p[[\mathcal{G}(v)]]) \neq 0$. The group G acts on the standard pro-p tree T associated to $(\mathcal{G}, \Gamma)$, and we get

$$0 \longrightarrow \oplus_{e\in E(\Gamma)}\mathbb{F}_p[[G/\mathcal{G}(e)]] \longrightarrow \oplus_{v\in V(\Gamma)}\mathbb{F}_p[[G/\mathcal{G}(v)]] \longrightarrow \mathbb{F}_p \longrightarrow 0$$

Applying $Hom_{\mathbb{F}_p[[G]]}(-, \mathbb{F}_p[[G]])$ to this exact sequence and taking into account that

$$Hom_{\mathbb{F}_p[[G]]}(\mathbb{F}_p, \mathbb{F}_p[[G]]) = (\mathbb{F}_p[[G]])^G = 0$$

([11, Lemma 3]), we get

$$0 \to \oplus_{v\in V(\Gamma)}Hom_{\mathbb{F}_p[[G]]}(\mathbb{F}_p[[G/\mathcal{G}(v)]], \mathbb{F}_p[[G]]) \to \oplus_{e\in E(\Gamma)}Hom_{\mathbb{F}_p[[G]]}(\mathbb{F}_p[[G/\mathcal{G}(e)]], \mathbb{F}_p[[G]]) \to$$

$$H^1(G, \mathbb{F}_p[[G]]) \to \oplus_{v\in V(\Gamma)}Ext^1_{\mathbb{F}_p[[G]]}(\mathbb{F}_p[[G/\mathcal{G}(v)]], \mathbb{F}_p[[G]]) \to \oplus_{e\in E(\Gamma)}Ext^1_{\mathbb{F}_p[[G]]}(\mathbb{F}_p[[G/\mathcal{G}(e)]], \mathbb{F}_p[[G]])$$

By Shapiro's lemma

$$Hom_{\mathbb{F}_p[[G]]}(\mathbb{F}_p[[G/\mathcal{G}(v)]], \mathbb{F}_p[[G]]) = H^0(\mathcal{G}(v), Res_{\mathbb{F}_p[[\mathcal{G}(v)]]}\mathbb{F}_p[[G]]) = (\mathbb{F}_p[[G]])^{\mathcal{G}(v)} = 0,$$

the latter equality since $\mathcal{G}(v)$ is infinite ([11, Lemma 3]); similarly, by Shapiro's lemma,

$$Ext^1_{\mathbb{F}_p[[G]]}(\mathbb{F}_p[[G/\mathcal{G}(v)]], \mathbb{F}_p[[G]]) = H^1(\mathcal{G}(v), Res_{\mathbb{F}_p[[\mathcal{G}(v)]]}\mathbb{F}_p[[G]]),$$

$$Hom_{\mathbb{F}_p[[G]]}(\mathbb{F}_p[[G/\mathcal{G}(e)]], \mathbb{F}_p[[G]]) = H^0(\mathcal{G}(e), Res_{\mathbb{F}_p[\mathcal{G}(e)]}\mathbb{F}_p[[G]]),$$

$$Ext^1_{\mathbb{F}_p[[G]]}(\mathbb{F}_p[[G/\mathcal{G}(e)]], \mathbb{F}_p[[G]]) = H^1(\mathcal{G}(e), Res_{\mathbb{F}_p[\mathcal{G}(e)]}\mathbb{F}_p[[G]]).$$

Now since $\mathcal{G}(e)$ is finite, G has a system of open normal subgroups U intersecting $\mathcal{G}(e)$ trivially and so

$$Res_{\mathbb{F}_p[\mathcal{G}(e)]}\mathbb{F}_p[[G]] = \varprojlim_U Res_{\mathbb{F}_p[\mathcal{G}(e)]}\mathbb{F}_p[[G/U]] = \varprojlim_U (\bigoplus_{\mathcal{G}(e)\backslash G/U} \mathbb{F}_p[\mathcal{G}(e)]).$$

Moreover, since Hom commutes with projective limits in the second variable we have

$$H^0(\mathcal{G}(e), Res_{\mathbb{F}_p[\mathcal{G}(e)]}\mathbb{F}_p[[G]]) = \varprojlim_U (\bigoplus_{\mathcal{G}(e)\backslash G/U} H^0(\mathcal{G}(e), \mathbb{F}_p[\mathcal{G}(e)])).$$

But $H^0(\mathcal{G}(e),\mathbb{F}_p[\mathcal{G}(e)])\cong\mathbb{F}_p$. Thus

$$\varprojlim_U(\bigoplus_{\mathcal{G}(e)\backslash G/U}H^0(\mathcal{G}(e),\mathbb{F}_p[\mathcal{G}(e)]))=\varprojlim_U(\bigoplus_{\mathcal{G}(e)\backslash G/U}\mathbb{F}_p)=\varprojlim_U\mathbb{F}_p[[\mathcal{G}(e)\backslash G/U]])=\mathbb{F}_p[[\mathcal{G}(e)\backslash G]].$$

Since $\mathbb{F}_p[[G]]$ is a free $\mathbb{F}_p[\mathcal{G}(e)]$ and $\mathbb{F}_p[[\mathcal{G}(v)]]$-module, (see [26, Proposition 7.6.3]) and so is projective by [26, Proposition 7.6.2], and since $\mathbb{F}_p[[G]]$ is a local ring, by [26, Proposition 7.5.1] and [26, Proposition 7.4.1 (b)]

$$Res_{\mathbb{F}_p[\mathcal{G}(e)]}\mathbb{F}_p[[G]])=\prod_I\mathbb{F}_p[\mathcal{G}(e)]$$

and

$$Res_{\mathbb{F}_p[[\mathcal{G}(v)]]}\mathbb{F}_p[[G]])=\prod_{j\in J}\mathbb{F}_p[[\mathcal{G}(v)]]$$

for some infinite sets of indices I,J.

Note also that Ext commutes with direct products on the second variable and $H^1(\mathcal{G}(e),\mathbb{F}_p[\mathcal{G}(e)])=0$, since a free $\mathbb{F}_p[\mathcal{G}(e)]$-module is injective. So

$$H^1(\mathcal{G}(e),Res_{\mathbb{F}_p[\mathcal{G}(e)]}\mathbb{F}_p[[G]])=\prod_I H^1(\mathcal{G}(e),\mathbb{F}_p[\mathcal{G}(e)]))=0$$

Thus the above long exact sequence can be rewritten as

$$0\longrightarrow\oplus_{e\in E(\Gamma)}\mathbb{F}_p[[G/\mathcal{G}(e)]]\longrightarrow H^1(G,\mathbb{F}_p[[G]])\longrightarrow\oplus_{v\in V(\Gamma)}H^1(\mathcal{G}(v),Res_{\mathbb{F}_p[[\mathcal{G}(v)]]}\mathbb{F}_p[[G]])\longrightarrow 0$$

We show now that $H^1(\mathcal{G}(v),Res_{\mathbb{F}_p[[\mathcal{G}(v)]]}\mathbb{F}_p[[G]])\neq 0$ for each v. Indeed, since Ext commutes with the direct product on the second variable we have $H^1(\mathcal{G}(v),Res_{\mathbb{F}_p[[\mathcal{G}(v)]]}\mathbb{F}_p[[G]])=\prod_{j\in J}H^1(\mathcal{G}(v),\mathbb{F}_p[[\mathcal{G}(v)]])$. But for every v the groups $H^1(\mathcal{G}(v),\mathbb{F}_p[[\mathcal{G}(v)]])\neq 0$ by the assumption at the beginning of the proof. So $H^1(\mathcal{G}(v),Res_{\mathbb{F}_p[[\mathcal{G}(v)]]}\mathbb{F}_p[[G]])\neq 0$ for every v.

Hence the number of vertices in Γ cannot exceed the minimal number of generators of $\mathbb{F}_p[[G]]$-module $H^1(G,\mathbb{F}_p[[G]])$. The number of edges of Γ cannot exceed $d(G)+|V(\Gamma)|-1$ since the rank of $\pi_1(\Gamma)=G/\tilde{G}$ equals $|E(\Gamma)|-|V(\Gamma)|+1$, where the equality $\pi_1(\Gamma)=G/\tilde{G}$ follows from [13, Corollary 3.9.3] combined with [13, Proposition 3.10.4 (b)].

We will now do the general case. First observe that if $G=\Pi_1(\mathcal{G},\Gamma)$, where $(\mathcal{G},\Gamma)$ is a reduced finite graph of pro-p-groups with finite edge groups, then, letting T be a maximal subtree of Γ, there are at most $d:=d(G)$ edges in $\Gamma\setminus T$, and therefore there are at most $3d$ pending vertices in Γ, see Lemma 3.6. We will now bound the size of T.

Suppose now that some vertex group $\mathcal{G}(v)$ is either finite or has $H^1(\mathcal{G}(v),\mathbb{F}_p[[G(v)]])=0$. If $e\in E(T)$ is adjacent to v, with other extremity w, then collapsing $\{e,v,w\}$ into a new vertex y, and putting on top of y the group $\mathcal{G}(y)=\mathcal{G}(v)\amalg_{\mathcal{G}(e)}\mathcal{G}(w)$, by Theorem 4.5 we have $H^1(\mathcal{G}(y),\mathbb{F}_p[[\mathcal{G}(y)]])\neq 0$, and $\mathcal{G}(y)$ is infinite. Let M be the number of generators of $H^1(G,\mathbb{F}_p[[G]])$.

Claim. The diameter of T is at most $2M$.

Indeed, if not, it contains a path with $2M+2$ distinct vertices. But applying the above procedure to get rid of the bad vertices on the path, produces at least $M+1$ vertices y with $H^1(\mathcal{G}(y), \mathbb{F}_p[[\mathcal{G}(y)]]) \neq 0$ and $\mathcal{G}(y)$ infinite, which contradicts the first part.

The result now follows, as there is a bound on the size of trees with at most $3d$ pending vertices and diameter $\leq 2M$, and $|\Gamma \setminus T| \leq d$ (see Proposition 3.4).

□

7 Howson's property

Definition 7.1. We say that a pro-p group G *has Howson's property*, or *is Howson*, if whenever H and K are two finitely generated closed subgroups of G, then $H \cap K$ is finitely generated.

Free and Demushkin pro-p-groups are Howson [12, 20] and the Howson property is preserved under free (pro-p) products, see [20, Thm 1.9]. In this section we investigate the preservation of Howson's property under various (free) constructions.

Lemma 7.2. *Let G be a finitely generated pro-p group acting on a profinite space Y such that the number of maximal point stabilizers G_a, up to conjugation, is finite and represented by the elements $a \in A \subseteq Y$. Let H be a subgroup of G such that H_y is finitely generated for each $y \in Y$ and the kernel of the natural homomorphism $\beta : \mathbb{F}_p[[H \backslash Y]] \longrightarrow \mathbb{F}_p[[G \backslash Y]]$ is finite. Then the image of the natural homomorphism $\eta : H_1(H, \mathbb{F}_p[[Y]]) \longrightarrow H_1(H)$ is finite.*

Proof. We use the characterisation $H_1(G) = G/\Phi(G)$. By Shapiro's lemma $H_1(G, \mathbb{F}_p[[G/G_y]]) = H_1(G_y) = G_y/\Phi(G_y)$ and so the image of $\gamma : H_1(G, \mathbb{F}_p[[Y]]) \longrightarrow H_1(G)$ coincides with the smallest closed subgroup containing all images of the $H_1(G_y)$'s. Observe now that if $G_y \leq G_a$ and $g \in G$, then $G_y\Phi(G) \leq G_a\Phi(G) = G_a^g\Phi(G)$, whence the image of $H_1(G_y)$ in $H_1(G)$ is contained in the image of $H_1(G_a)$, which equals the image of $H_1(G_a^g)$. Thus the image of γ coincides with the subgroup of $H_1(G)$ generated by the images of $H_1(G_a)$, $a \in A$.

A given H-orbit $H/H_y \subset Y$ is sent by β to a subset of a G-orbit G/G_y in Y. If for $y \in Y$ the stabilizer G_y is not maximal, then there exists a maximal G_a, $a \in A$, and $g \in G$, such that $G_y \leq G_a^g$. Hence $H_y \leq H_a^g$. Since $\ker(\beta)$ is finite, the set B of H-orbits in Y that map into some G-orbit Ga with $a \in A$, is finite. Note that $H_1(H, \mathbb{F}_p[[H/H_a]]) = H_1(H_a)$ (by Shapiro's lemma). Thus the image of $\eta : H_1(H, \mathbb{F}_p[[Y]]) \longrightarrow H_1(H)$ coincides with the group generated by the images of $H_1(H_b)$, $b \in B$. But H_a is finitely generated, so each $H_1(H_a^g)$ is finite, and since B is finite, the image of η is also finite.

□

Theorem 7.3. *Let G be a finitely generated pro-p group acting on a pro-p tree T such that $G \backslash T$ is finite. Let H be a finitely generated subgroup of G such that $H \backslash T(H)$ is finite, where $T(H)$ is a minimal H-invariant subtree of T, and $[G_e : H_e] < \infty$ for all $e \in E(T(H))$. If K is a finitely generated subgroup of G, then $H \cap K$ is finitely generated in each of the following cases:*

(i) K intersects trivially all vertex stabilizers H_v, $v \in V(T(H))$;

(ii) K acts on T with finitely many maximal vertex stabilizers up to conjugation, the vertex stabilizers H_v, K_v are finitely generated and the G_v are Howson, $v \in V(T)$.

Proof. The proof follows the idea of the proof of Theorem 1.9 in [20]. Put $\Delta = H\backslash T(H)$. Observe that if $u \in T(H)$, then Hu is the intersection of all Uu with U an open subgroup of G containing H. As Δ is finite, there is some open subgroup U of G containing H and such that we have an injection $H\backslash T(H) \to U\backslash T$. By Lemma 3.7, passing to an open subgroup of G containing H, we may therefore assume that $G = \Pi_1(\mathcal{G}, \Gamma)$ with Γ finite, and that $(\mathcal{H}, \Delta)$ is a subgraph of groups of $(\mathcal{G}, \Gamma)$ such that $H = \Pi_1(\mathcal{H}, \Delta)$. Note that then $[\mathcal{G}(e) : \mathcal{H}(e)] < \infty$ for all $e \in E(\Delta)$, and, since Δ is finite, replacing G by an open subgroup containing H, we may assume that $\mathcal{G}(e) = \mathcal{H}(e)$ for every $e \in E(\Delta)$.

Let

$$0 \longrightarrow \mathbb{F}_p[[E(T)]] \longrightarrow \mathbb{F}_p[[V(T)]] \longrightarrow \mathbb{F}_p \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{F}_p[[E(T(H))]] \longrightarrow \mathbb{F}_p[[V(T(H))]] \longrightarrow \mathbb{F}_p \longrightarrow 0$$

be the short exact sequences associated with T and $T(H)$ (note that $E(T)$, $E(T(H))$ are compact since $G\backslash T$ is finite).

Applying to them $\mathbb{F}_p\hat{\otimes}_{\mathbb{F}_p[[K]]}$ and $\mathbb{F}_p\hat{\otimes}_{\mathbb{F}_p[[K\cap H]]}$ respectively we get the following long exact sequences (see [13, (9.6), page 265]):

$$H_1(K, \mathbb{F}_p[[V(T)]]) \to H_1(K, \mathbb{F}_p) \to \mathbb{F}_p\hat{\otimes}_{\mathbb{F}_p[[K]]}\mathbb{F}_p[[E(T)]] \to \\ \xrightarrow{\delta} \mathbb{F}_p\hat{\otimes}_{\mathbb{F}_p[[K]]}\mathbb{F}_p[[V(T)]] \to \mathbb{F}_p \to 0 \quad (4)$$

and

$$H_1(K \cap H, \mathbb{F}_p[[V(T(H))]]) \to H_1(H \cap K, \mathbb{F}_p) \to \\ \mathbb{F}_p\hat{\otimes}_{\mathbb{F}_p[[K]]}\mathbb{F}_p[[E(T(H))]] \xrightarrow{\sigma} \mathbb{F}_p\hat{\otimes}_{\mathbb{F}_p[[K]]}\mathbb{F}_p[[V(T)]] \to \mathbb{F}_p \to 0. \quad (5)$$

Using the definitions of T and $T(H)$ (see 2.17), we can rewrite the long exact sequences as follows:

$$\bigoplus_{v\in G\backslash V(T)} H_1(K, \mathbb{F}_p[[G/\mathcal{G}(v)]]) \to H_1(K, \mathbb{F}_p) \to \bigoplus_{e\in G\backslash E(T)} \mathbb{F}_p[[K\backslash G/\mathcal{G}(e)]] \to \\ \xrightarrow{\delta} \bigoplus_{v\in G\backslash V(T)} \mathbb{F}_p[[K\backslash G/\mathcal{G}(v)]] \to \mathbb{F}_p \to 0 \quad (6)$$

and

$$H_1(K\cap H, \bigoplus_{v\in H\backslash V(T(H))} \mathbb{F}_p[[H/\mathcal{H}(v)]]) \to H_1(H\cap K, \mathbb{F}_p)\to$$
$$\bigoplus_{e\in H\backslash E(T(H))} \mathbb{F}_p[[K\cap H\backslash H/\mathcal{H}(e)]] \overset{\sigma}{\to} \bigoplus_{v\in H\backslash V(T(H))} \mathbb{F}_p[[(K\cap H)\backslash H/\mathcal{H}(v)]] \to \mathbb{F}_p \to 0. \quad (7)$$

We then have the following commutative diagramme:

$$\begin{array}{ccc}
\bigoplus_{e\in G\backslash E(T)} \mathbb{F}_p[[K\backslash G/\mathcal{G}(e)]] & \xrightarrow{\delta} & \bigoplus_{v\in G\backslash V(T)} \mathbb{F}_p[[K\backslash G/\mathcal{G}(v)]] \\
\uparrow \alpha & & \uparrow \beta \\
\bigoplus_{e\in H\backslash E(T(H))} \mathbb{F}_p[[K\cap H\backslash H/\mathcal{H}(e)]] & \xrightarrow{\sigma} & \bigoplus_{v\in H\backslash V(T(H))} \mathbb{F}_p[[K\cap H\backslash H/\mathcal{H}(v)]]
\end{array}$$

We want to show that $\ker \beta\circ\sigma$ is finite, or equivalently that $\ker \delta\circ\alpha$ is finite. The dimension (as an $\mathbb{F}_p$-v.s) of $\ker\delta$ is $\leq \dim(H_1(K))$, i.e., less than or equal to the number of generators of K. So, we need to show that $\ker(\alpha)$ is finite, and if possible bound its size. We know that there is an inclusion of $(K\cap H)\backslash H$ in $K\backslash G$, and we need to see what happens when we quotient by the action of $\mathcal{G}(e)$ (on the right).

The inclusion map $\mathbb{F}_p[[(K\cap H)\backslash H]] \to \mathbb{F}_p[[K\backslash G]]$ is a map of right $\mathbb{F}_p[[H_e]]$-modules for any $e\in E(T(H))$, and note that it sends distinct H_e-orbits to distinct G_e-orbits (because $\mathcal{G}(e) = \mathcal{H}(e)$ for $e\in E(\Delta)$ and so $G_e = H_e$ for every $e\in E(T(H))$). Hence α is an injection!!

To summarise: $\delta\circ\alpha = \beta\circ\sigma$ have finite kernel, of dimension bounded by $d(K)$. Furthermore, as the image of σ in $\bigoplus_{v\in H\backslash V(T(H))} \mathbb{F}_p[[K\cap H\backslash H/\mathcal{H}(v)]]$ has codimension 1 (by the exact sequence (7)), it follows that $\ker(\beta)$ is also finite, and we have $\dim(\ker(\sigma)) + \dim(\ker(\beta)) \leq d(K)+1$.

(i) if K intersects trivially all conjugates of H_v then the left term of (7) is

$$H_1(K\cap H, \bigoplus_{v\in H\backslash V(T(H))} \mathbb{F}_p[[H/\mathcal{H}(v)]])$$

and equals 0 because $\bigoplus_{v\in H\backslash V(T(H))} \mathbb{F}_p[[H/\mathcal{H}(v)]]$ is a free $K\cap H$-module, so (i) follows from the injectivity of α.

(ii) Since the $\mathcal{G}(v), v\in V(\Delta)$, are Howson, and K_v, H_v are finitely generated, $K_v\cap H_v$ is finitely generated for any $v\in V(T)$. Since $\ker(\beta)$ is finite, we can apply Lemma 7.2 to

$K \cap H \leq K$ to deduce that the image of $H_1(K \cap H, \bigoplus_{v \in H \backslash V(T(H))} \mathbb{F}_p[[H/\mathcal{H}(v)]])$ in $H_1(H \cap K)$ is finite.

Combining this with the finiteness of $\ker(\sigma)$ we deduce that $H_1(H \cap K)$ is finite, i.e. $H \cap K$ is finitely generated. □

Corollary 7.4. *Let G be a finitely generated pro-p group acting on a pro-p tree T with procyclic edge stabilizers and Howson's vertex stabilizers. Let H be a finitely generated subgroup of G that does not split as a free pro-p product and whose action on T is n-acylindrical. If K is a finitely generated subgroup of G then $H \cap K$ is finitely generated.*

Proof. If H is procyclic, there is nothing to prove. Otherwise, by Corollary 5.5 there exists a minimal H-invariant subtree D of T with $\Delta = \mathcal{H}\backslash D$ finite and $H = \Pi_1(\mathcal{H}, \Delta)$. Moreover, since H is not a non-trivial free pro-p product, $\mathcal{H}(e) \neq 1$ for all $e \in E(\Delta)$ (indeed, if $\mathcal{H}(e) = 1$ then as explained in the proof of Theorem 4.2 either $G = G_1 \amalg G_2$, where G_1, G_2 are the fundamental groups of graphs of groups restricted to the connected components of $\Delta \setminus \{e\}$, or $G = G_1 \amalg \mathbb{Z}_p$, where G_1 is the fundamental groups of the graph of groups restricted to $\Delta \setminus \{e\}$). Hence for each $e \in E(\Delta)$ we have $[\mathcal{G}(e) : \mathcal{H}(e)] < \infty$. Vertex stabilizers K_v, H_v are finitely generated by Theorem 6.6, and by Example 6.5 and Corollary 6.2 the set of maximal vertex K-stabilizers is finite up to conjugation. Thus all hypotheses of Theorem 7.3 are satisfied from which we deduce the result.

□

Corollary 7.5. *Let $G = G_1 \amalg_C G_2$ be a free amalgamated pro-p product of free or Demushkin not soluble pro-p groups with C maximal procyclic in G_1 or G_2. Let H be a finitely generated subgroup of G that does not split as a free pro-p product. Then $H \cap K$ is finitely generated for any finitely generated subgroup K of G.*

Proof. In this case the action of G on its standard pro-p tree T is 2-acylindrical. This follows from the fact that if C is maximal procyclic in say G_1, then C is malnormal in G_1, i.e. $C \cap C^{g_1} = 1$ for any $g_1 \in G_1 \setminus C$. Indeed, if $C \cap C^{g_1} \neq 1$ then $\langle C, C^{g_1} \rangle$ normalizes this intersection. But every 2-generated subgroup of G_1 is free and so can not have procylic normal subgroups, so $C = C^{g_1}$ and so g_1 normalizes C. But then the same applies to $\langle C, g_1 \rangle$ so it is procyclic contradicting the maximality of C in G_1.

Thus using that free and Demushkin pro-p groups are Howson [12, 20], by Corollary 7.4 we obtain the result.

□

Theorem 7.6. *Let $G = G_1 \amalg_C G_2$ be a free pro-p product with procyclic amalgamation. Let $H_i \leq G_i$, be finitely generated such that $C \cap H_1 \cap H_2 \neq 1$, $H = \langle H_1, H_2 \rangle$ and $K \leq G$ a finitely generated subgroup of G. Then $K \cap H$ is finitely generated in each of the following cases*

(i) K intersects all conjugates of the H_i trivially. Moreover, if $C \leq H_i$ $(i = 1, 2)$ then $d(H \cap K) \leq d(K)$.

(ii) The G_i's are Howson pro-p.

Proof. The group G acts on its standard pro-p tree $T = T(G)$ and so we can apply Theorem 7.3 to deduce the first part of (i). Vertex stabilizers K_v, H_v are finitely generated by Theorem 6.6, and by Example 6.5 and Corollary 6.2 the set of maximal vertex K-stabilizers is finite up to conjugation. Thus all hypotheses of Theorem 7.3 are satisfied to deduce (ii).

Thus, we only need to show the second part of (i).

The proof of Theorem 7.3 involved replacing G by a subgroup of finite index, and so to obtain the precise bound $d(K)$, we need to show that this is not necessary. Towards that, our hypothesis $C \leq H_1 \cap H_2$ implies that, if $T(H)$ is a minimal H-invariant subtree of T, then $H \backslash T(H) \simeq G \backslash T$ and $\mathcal{G}(e) = \mathcal{H}(e)$.

As was observed in the proof of Theorem 7.3, $\delta \circ \alpha = \beta \circ \sigma$ have finite dimensional kernel, of dimension bounded by $d(K)$.

If K does not intersect the conjugates of the H_i then the left term of equation (7) (in the proof of 7.3) is 0, so from the injectivity of α one deduces that the natural map $H_1(K \cap H) \longrightarrow H_1(K)$ is an injection. □

Theorem 7.7. *Let $G = \mathrm{HNN}(G_1, C, t)$ be a pro-p HNN extension, G_1 a finitely generated and $C \neq 1$ procyclic. Let H_1 be a finitely generated subgroup of G_1 such that $C \cap H_1 \neq 1$ and $H = \langle H_1, t \rangle$. Then for a finitely generated subgroup K of G the intersection $K \cap H$ is finitely generated in each of the following cases*

(i) K intersects trivially every conjugate of H_1. Moreover, if $C \leq H_1$ then $d(H \cap K) \leq d(K)$.

(ii) G_1 satisfies Howson's property.

Proof. The proof is identical to the proof of Theorem 7.6. The group G acts on its standard pro-p tree $T = T(G)$ and so we can apply Theorem 7.3 to deduce the first part of (i). Vertex stabilizers K_v, H_v are finitely generated by Theorem 6.6, and by Example 6.5 and Corollary 6.2 the set of maximal vertex K-stabilizers is finite up to conjugation. Thus all hypotheses of Theorem 7.3 are satisfied to deduce (ii).

Thus, we only need to show the second part of (i), and this is done exactly as in Theorem 7.6. □

8 Normalizers

Proposition 8.1. *Let C be a procyclic pro-p group and $U \leq C$ a subgroup of C.*

(a) Let $G = G_1 \amalg_C G_2$ and $N = N_G(U)$. Then $N = N_{G_1}(C) \amalg_C N_{G_2}(C)$.

(b) Let $G = \mathrm{HNN}(G_1, C, t)$ be a proper pro-p HNN-extension and $N = N_G(U)$.

(i) If there is some $g \in G_1$ such that $U^g = U^t$, then $N = \mathrm{HNN}(N_{G_1}(U), C, t')$.

(ii) If U and U^t are not conjugate in G_1 then $N = N_1 \amalg_C N_2$, where $N_1 = N_{G_1^{t^{-1}}}(U)$ and $N_2 = N_{G_1}(U)$.

Proof. The proof follows the strategy of proofs in the ordinary profinite case of Proposition 2.5 in [16] or Proposition 15.2.4 (b) [13]. Let T be the standard pro-p tree for G and recall that $E(T) = G/C$, $V(T) = G/G_1 \cup G/G_2$ in Case (a) and $V(T) = G/G_1$ in Case (b). By [17, Theorem 3.7] the subset $Y = T^U$ of T consisting of points fixed by U is a pro-p subtree. Observe that if $g \in G$, then U fixes gC if and only if $U^g \leq C$.

Then N acts on Y continuously. Indeed, if $g \in N$, $y \in Y$ and $u \in U$, then $ug = gu'$ for some $u' \in U$, and therefore $ugy = gu'y = gy$. This being true for all u in U, we get that N acts on Y.

Consider the natural epimorphism $\varphi : T \to G\backslash T$. Then the natural map $\psi : Y \to N\backslash Y$ is the restriction of φ to Y. To see this pick $h \in G$ such that $hC \in E(Y)$; so $U^h \leq C$ and therefore $U, U^h \leq C$. As C is procyclic, we get $U = U^h$, i.e., $h \in N$ (work in finite quotients of G where the equality is obvious). This shows that ψ coincides with the restriction of φ to Y.

Thus $N\backslash E(Y)$ consists of one edge only, and therefore $N\backslash Y$ has at most two vertices. According to Proposition 4.4 in [30] (or [13, Theorem 6.6.1]), we have $N = N_1 \amalg_C N_2$, where $N_1 = N_{G_1}(U)$ and $N_2 = N_{G_2}(U)$, or $N = \mathrm{HNN}(N_{G_1}(U), C, t')$, depending on whether Y has two vertices or just one vertex. (Note that N contains C). In Case (a) $\varphi(gG_1) \neq \varphi(gG_2)$, so $\psi(Y)$ has two vertices.

In Case (b) $N\backslash Y$ has one vertex only iff $d_1(C) = tG_1$ is in the N-orbit of $d_0(C) = G_1$, i.e., if $G_1^t = G_1^n$ for some $n \in N$ iff $G_1 = G_1^{nt^{-1}}$ iff $g = nt^{-1} \in G_1$, in which case $U^g = U^t$ as required. □

Proposition 8.2. *Let G be a pro-p group acting on a pro-p tree T and U be a procyclic subgroup of G that does not stabilize any edge. Then one of the following happens:*

(1) *For some vertex v, $U \leq G_v$: then $N_G(U) = N_{G_v}(U)$.*

(2) *For all vertices v, $U \cap G_v = \{1\}$. Then $N_G(U)/K$ is either isomorphic to $\mathbb{Z}_p$ or to a generalized dihedral group $\mathbb{Z}_2 \amalg_{2\mathbb{Z}_2} \mathbb{Z}_2 = \mathbb{Z}_2 \rtimes \mathbb{Z}_2$, where K is some normal subgroup of $N_G(U)$ contained in the stabilizer of an edge.*

Proof. Let $N = N_G(U)$ and let D be a minimal U-invariant subtree of T (that exists by Proposition 2.19).

Case 1. $|D| = 1$, i.e., U stabilizes a vertex v. If $v \neq nv$ for some $n \in N$, then by [17, Corollary 3.8], U stabilizes all edges in $[v, nv]$, contradicting our hypothesis. So $N_G(U)$ fixes v and we have (1).

Case 2. D is not a vertex. Then U acts irreducibly on D and so by Proposition 2.19 it is unique. Note that if $n \in N$, then nD is also D-invariant, and therefore must equal D. Hence N acts irreducibly on D and by Lemma 4.2.6(c) in [13], $C_N(U)K/K$ is free pro-p, where K is the kernel of the action (and is the intersection of all stabilizers). Hence $C_N(U)K/K$ is procyclic (because UK/K is procyclic, $\neq 1$) and so N/K is either $\mathbb{Z}_p$ or $C_2 \amalg C_2$, since $Aut(U) \cong \mathbb{Z}_p \times C_{p-1}$ for $p > 2$ or $\mathbb{Z}_2 \times C_2$ for $p = 2$. □

Combining Theorem 7.6 and Propositions 8.1 and 8.2 we deduce the following

Theorem 8.3. *Let C be a procyclic pro-p group and $G = G_1 \amalg_C G_2$ be a free amalgamated pro-p product or a pro-p HNN-extension $G = \mathrm{HNN}(G_1, C, t)$ of Howson groups. Let U be a procyclic subgroup of G and $N = N_G(U)$. Assume that $N_{G_i}(U^g)$ is finitely generated whenever $U^g \leq G_i$. If $K \leq G$ is finitely generated, then so is $K \cap N$.*

Proof. Let T be the standard pro-p tree for G. If U does not stabilize any edge then by Proposition 8.2 either $N_G(U) = N_{G_i^g}(U)$ for some i and g, and so $K \cap N = K \cap N_{G_i^g}(U)$ is finitely generated, or $N_G(U)$ is metacyclic and therefore so is $K \cap N$.

If U stabilizes an edge then $U \leq C^g$ for some $g \in G$ and so we may assume that $U \leq C$. Then by Proposition 8.1 N satisfies the hypothesis of either Theorem 7.6 or Theorem 7.7 and so by one of these theorems $N \cap K$ is finitely generated. □

References

[1] M. Bestvina and M. Feighn, Bounding the complexity of simplicial group actions on trees. Inventiones Mathematicae, 103 (1991), no. 3, 449-469.

[2] S. C. Chagas and P. A. Zalesskii, The Figure Eight Knot group is Conjugacy Separable, Journal of Algebra and its Applications **8** (2009) 1-19.

[3] M. J. Dunwoody, The accessibility of finitely presented groups. Inventiones mathematicae, **81**(3) (1985) 449-457.

[4] M. J. Dunwoody, An inaccessible group. Geometric group theory, London Math. Society Lecture Notes, 1:75-78, 1993.

[5] M. J. Dunwoody and J.M. Jones, A group with strange decomposition properties. J. Group Theory **1** (1998) 301-305.

[6] W. Herfort and P.A. Zalesskii, A virtually free pro-p group need not be the fundamental group of a profinite graph of finite groups, Arch. d. Math. **94** (2010) 35–41.

[7] W. Herfort and P. A. Zalesskii, Virtually free pro-p groups, Publ. Inst. Hautes Études Sci. **118** (2013) 193–211.

[8] W. N. Herfort and P. A. Zalesskii, Addendum: Virtually free pro-p groups whose torsion elements have finite centralizer, arXiv:1204.6386.

[9] W. Herfort, P. A. Zalesskii and Th. Zapata, Splitting theorems for pro-p groups acting on pro-p trees. Selecta Math. (N.S.) **22** (2016) 1245 – 1268.

[10] D. Kochloukova and P. Zalesskii, On pro-p analogues of limit groups via extensions of centralizers, Math. Z. **267** (2011) 109–128.

[11] A.A. Korenev, Pro-p groups with a finite number of ends, Mat. Zametki **76** (2004) 531–538; translation in Math. Notes **76** (2004) 490–496.

[12] A. Lubotzky, Combinatorial group theory for pro-p groups, Journal of Pure and Applied Algebra **25** (1982) 311-325.

[13] L. Ribes, *Profinite graphs and groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics, volume 66, Springer, 2017.

[14] L. Ribes, On amalgamated products of profinite groups, Math. Z. **123** (1971) 357–364.

[15] L. Ribes and P. Zalesskii, *Profinite groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. volume 40, Springer-Verlag, Berlin, second edition, 2010.

[16] L. Ribes and P. A. Zalesskii, Conjugacy separability of amalgamated free products of groups. J. Algebra **179** (1996) 751 – 774.

[17] L. Ribes and P. Zalesskii, Pro-p trees and applications, in: *New horizons in pro-p groups*, volume 184 of Progr. Math., pages 75–119. Birkhäuser Boston, Boston, MA, 2000.

[18] L. Ribes and P. A. Zalesskii, Normalizers in groups and in their profinite completions. Rev. Mat. Iberoam. **30** (2014) 165 – 190.

[19] J-P. Serre, *Trees*. Springer-Verlag, Berlin, 1980.

[20] M. Shusterman and P. Zalesskii, Virtual retraction and Howson's theorem in pro-p-groups. Transaction of AMS **373** (2020) 1501-1527.

[21] I. Snopce and P. Zalesskii, Subgroup properties of pro-p extensions of centralizers. Selecta Mathematica, New Series (Printed ed.), **20** (2014) 465-489.

[22] R. Weidmann, The Nielsen method for groups acting on trees. Proc. London Math. Soc. (3) **85** (2002) 93-118.

[23] Th. Weigel and P.A. Zalesskii, Virtually free pro-p products. Israel J. Math. **221** (2017) 425-434.

[24] G. Wilkes, On accessibility for pro-p groups. J. Algebra **525** (2019) 1-18.

[25] G. Wilkes, A sufficient condition for accessibility of pro-p groups, arXiv 2007.07681.

[26] J. Wilson, *Profinite Groups*. Clarendon Press. Oxford. 1998.

[27] P. A. Zalesskii, Geometric characterization of free constructions of profinite groups. *Siberian Math. J.*, **30** (2) (1989) :227–235.

[28] P. A. Zalesskii, *Open Subgroups of Free Profinite Products*, Proceedings of the International Conference on Algebra, Part 1 (Novosibirsk, 1989), 473–491, Contemp. Math., **131**, Part 1, Amer. Math. Soc., Providence, RI, 1992.

[29] P.A. Zalesskii and O.V. Melnikov, Subgroups of profinite groups acting on trees, Math. USSR Sbornik **63** (1989) 405-424.

[30] P. A. Zalesskii and O. V. Mel'nikov, *Fundamental Groups of Graphs of Profinite Groups*, Algebra i Analiz **1** (1989); translated in: Leningrad Math. J., **1** (1990) 921–940.

P. A. Zalesskii
Department of Mathematics
University of Brasilia
70.910 Brasilia DF
Brazil
pz@mat.unb.br