Mémoire

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Homogénéisation en physique statistique et systèmes de spins désordonnés

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1. Introduction

This thesis presents a survey of my research activity since the defense of my PhD in 2019. Broadly speaking, my research focuses on the quantitative study of the macroscopic properties of probabilistic models and statistical physics, and can be divided into three intertwined areas.

An important part of my research work is in stochastic homogenization, a field which aims at studying the behaviour of solutions of elliptic partial differential equations with rapidly oscillating random coefficients. The analysis of an equation with random coefficients exhibiting rapid oscillations is generally difficult; the classic idea of the theory of homogenization is to demonstrate that the solutions of these equations are approximated by solutions of an equation which depends on a small number of effective parameters. My research is in line with the recent development of a quantitative theory of stochastic homogenization by A. Gloria, F. Otto, S. Neukamm, S. Armstrong, J.-C. Mourrat, T. Kuusi, and has aimed to extend the theory in two directions: in collaboration with S. Armstrong and C. Gu, we developed a theory of quantitative homogenization on the infinite cluster in supercritical percolation, and, in a second direction and in collaboration with S. Armstrong and W. Wu, I extended some of the known results in the theory to a statistical physics model known as the $\nabla\varphi$ interface model. This second direction is the subject of the next paragraph.

A second direction of research on which my work has focused and which is closely related to the first has consisted in studying a model of random interfaces called the $\nabla\varphi$ interface model. A random interface is generally defined as a random function $\varphi : \mathbb{Z}^d \to \mathbb{R}$ whose law is defined on a microscopic scale by nearest-neighbour interactions. The aim is then to study the macroscopic properties of the model: convergence of the interface, fluctuations around the limiting shape, etc. My work (presented in this thesis) in this area has been structured around two axes. Using techniques developed within the framework of the theory of quantitative homogenization and in collaboration with S. Armstrong, we have obtained a quantitative version of the hydrodynamic limit for this model. In another direction, I obtained some localisation/delocalisation estimates for a class of $\nabla\varphi$ interface models with a degenerate potential.

A third line of research concerns the study of the macroscopic behaviour of different models of statistical physics. In this direction, I first studied, in collaboration with W. Wu, the large-scale behaviour of a spin system called the Villain model in dimensions 3 and higher and at low temperature (specifically, we studied the asymptotic behaviour of its two-point function). I then studied in various directions the properties of some spin systems in the presence of a random disorder. The first one pertains to the Imry-Ma phenomenon, which was conjectured by Y. Imry and S.-K. Ma and demonstrated by M. Aizenman and J. Wehr, and asserts that the addition of a random magnetic field can significantly alter the qualitative properties of these models by causing a disappearance of first-order phase transitions in low dimensions. The majority of results in the literature concerning the Imry-Ma phenomenon are qualitative, and it is only recently that the first quantitative results have been obtained in the case of the Ising model in dimension 2. In collaboration with M. Harel and R. Peled, we established a quantitative version of the Aizenman-Wehr theorem for general spin systems. We also studied the $\nabla\varphi$ interface model in the presence of a random magnetic field, and obtained quantitative upper and lower bounds characterising the typical height of the interface in any dimension. Finally I studied with C. Garban the effect that a different type of random disorder (on the underlying graph this time) can have on a spin system known as the XY model.

1.1. **List of the articles presented in this thesis.**

(1) P. Dario et C. Garban. *Phase transitions for the XY model in non-uniformly elliptic and Poisson-Voronoi environments.* Preprint, 48p.

(2) A. Bou-Rabee, W. Cooperman et P. Dario. *Rigidity of harmonic functions on the supercritical percolation cluster.* To appear in Transaction of the American Mathematical Society, 73p.

(3) P. Dario. *Upper bounds on the fluctuations of a class of degenerate convex grad-phi interface models.* ALEA – Latin American Journal of Probability and Mathematical Statistics, 21, 385–430 (2024).

(4) S. Armstrong et P. Dario. *Quantitative hydrodynamic limits of the Langevin dynamics for gradient interface models.* Electronic Journal of Probability, 29 (2024), paper no. 9, 93 pp.

(5) P. Dario. *Convergence of the thermodynamic limit for random-field random surfaces.* Annals of Applied Probability, 33(2): 1373-1395.

(6) P. Dario, M. Harel et R. Peled. *Random-field random surfaces.* Probability Theory and Related Fields, 186, 91–158 (2023).

(7) P. Dario, M. Harel et R. Peled. *Quantitative disorder effects in low-dimensional spin systems.* Communications in Mathematical Physics, 405, 212 (2024).

(8) P. Dario et W. Wu. *Massless Phases for the Villain model in* $d \geq 3$. *Astérisque* 447 (2024).

(9) P. Dario et C. Gu. *Quantitative homogenization of the parabolic and elliptic Green's functions on percolation clusters.* Annals of Probability, 49 (2021), 556–636.

Other articles (from my PhD)

(10) P. Dario. *Quantitative homogenization of the disordered* $\nabla\varphi$ *model.* Electronic Journal of Probability 24 (2019).

(11) P. Dario. *Quantitative homogenization of differential forms.* Annales de l'Institut Henri Poincaré, Vol. 57, No. 2, pp. 1157-1202.

(12) P. Dario. *Optimal corrector estimates on percolation clusters.* Annals of Applied Probability, 31 (2021), 377–431.

(13) S. Armstrong et P. Dario. *Elliptic regularity and quantitative homogenization on percolation clusters.* Communication in Pure and Applied Mathematics, 71 (2018), 1717-1849.

In this list, the items (10), (11), (12) and (13) correspond to my PhD thesis. The articles (12), (13) are (briefly) discussed in this thesis because the results there are important ingredients for the proofs in the articles (2) and (8).

1.2. **Outline of the thesis.** The rest of this thesis is organised as follows.

Chapter 2 is a brief introduction to the theory of stochastic homogenization, presenting the main motivations of the theory as well as some of the important tools and techniques (specifically, the first-order corrector and the two-scale expansion) which are used in my works.

Chapter 3 presents the main results obtained in the articles [20, 74, 78, 53] by adapting the theory of quantitative stochastic homogenization to the setting of supercritical Bernoulli bond percolation. We first provide an introduction to Bernoulli percolation in the supercritical regime and then present a renormalization technique for the infinite cluster developed in [20]. We then briefly present the main results of my first two articles [20, 74] and dedicate the rest of the section to the presentation of the results of my (post PhD) articles [78, 53].

Chapter 4 is devoted to the $\nabla\varphi$ interface model. We first give a brief introduction to the model, its main properties and some important results. We will focus our attention on three properties of this model: the localisation/delocalisation of the interface, the hydrodynamic limit and the scaling limit. We will then present the main tools which are used to establish these properties (especially, the Helffer-Sjöstrand representation formula) and will state two contributions: a quantitative version of the hydrodynamic limit (which is the result of [21] and makes use of techniques of stochastic homogenization) and a quantitative localisation/delocalisation estimate for the interface for a class of degenerate potentials established in [76]. We complete this chapter by presenting some ongoing work regarding the degenerate $\nabla\varphi$ interface model as well as some perspectives.

Chapter 5 is devoted to spin systems. We first present a result obtained in collaboration with W. Wu [81] regarding the asymptotic behaviour of a spin system called the Villain model in dimension $d \geq 3$. We then introduce a broad topic in mathematical physics which consists of studying the behaviour of spin systems in the presence of a random disorder. In this line, we first discuss the Imry-Ma phenomenon as well as my contribution with M. Harel and R. Peled [79] on the topic. We then present some results established in [80, 75] regarding the behaviour of the $\nabla\varphi$ interface model in the presence of a random external field. We complete this section by discussing the article [77] which is devoted to another spin system, called the XY model, and studies the impact that the addition of a random disorder (of a different nature than in the Imry-Ma phenomenon) can have on the properties of the model.

Table of notation:

The following list collects the most frequently used notation in the thesis.

Homogenization

- $e_1, \ldots, e_d$ is the canonical basis of $\mathbb{R}^d$.
- $x \cdot y$ is the Euclidean scalar product between $x, y \in \mathbb{R}^d$, $|x|$ and $|x|_1$ are the Euclidean norm and 1-norm respectively of $x \in \mathbb{R}^d$.
- $B_R(x)$ is the Euclidean ball of center $x \in \mathbb{R}^d$ and radius $R > 0$, we write B_R for $B_R(0)$.
- For $u : \mathbb{R}^d \to \mathbb{R}$ and $i \in \{1, \ldots, d\}$, we denote by $\partial_i u$ the partial derivative in the direction e_i and by ∇u its gradient.
- Given a bounded open set $U \subseteq \mathbb{R}^d$ and a measurable function $f : U \to \mathbb{R}$, we denote by $\|f\|_{L^2(U)}$ the L^2-norm of f, i.e., $\|f\|_{L^2(U)} := \left(\int_U |f(x)|^2 \, dx\right)^{1/2}$ and by $\|f\|_{L^\infty(U)}$ the (essential) supremum of f.
- Given a (sufficiently regular) open bounded set $U \subseteq \mathbb{R}^d$, we denote by $H^1(U)$ and $H^1_0(U)$ the standard Sobolev space and Sobolev space of functions with vanishing trace respectively. We denote by $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ the local Sobolev space. For $u \in H^1(U)$, we denote by $\|u\|_{H^1(U)} = \|u\|_{L^2(U)} + \|\nabla u\|_{L^2(U)}$.
- For $y \in \mathbb{R}^d$, we denote by τ_y the translation by y, i.e., $\tau_y \mathbf{a} : x \mapsto \mathbf{a}(x + y)$.
- Given an exponent $s > 0$, a constant $K > 0$ and a non-negative random variable X, we write

$$X \leq \mathcal{O}_s(K) \text{ if and only if } \forall t \in [1, \infty], \; \mathbb{P}[X \geq tK] \leq 2 \exp(-t^s).$$

Lattice and percolation

- Let $\mathbb{Z}^d$ be the standard Euclidean lattice. A point $x \in \mathbb{Z}^d$ is called a *vertex*. We say that two vertices $x, y \in \mathbb{Z}^d$ are nearest neighbours and denote it by $x \sim y$ if $|x - y| = 1$.
- An unoriented pair $\{x, y\}$ of nearest neighbours of $\mathbb{Z}^d$ is called an *edge*. We denote by $E(\mathbb{Z}^d)$ be the set of edges of $\mathbb{Z}^d$, and for $U \subseteq \mathbb{Z}^d$, by $E(U)$ the edges of U. We similarly denote by $\vec{E}(\mathbb{Z}^d)$ and $\vec{E}(U)$ the set of directed edges (i.e., oriented pairs of nearest neighbour) of $\mathbb{Z}^d$ and U.
- A box or a cube of $\mathbb{Z}^d$ is a set of the form

$$\square := [x, x + N]^d \cap \mathbb{Z}^d, \; x \in \mathbb{Z}^d, \; N \in \mathbb{N}.$$

  We call the integer N the size of the box and denote it by $\mathrm{size}(\square)$.
- For $n \in \mathbb{N}$, we let $\square_n$ be the discrete triadic cube

$$\square_n := \left[-\frac{3^n}{2}, \frac{3^n}{2}\right] \cap \mathbb{Z}^d.$$

- For $n \in \mathbb{N}$, we let $\mathcal{T}_n$ be the set of *triadic cubes* of size 3^n defined by $\mathcal{T}_n := \{z + \square_n \; : \; z \in 3^n \mathbb{Z}^d\}$. We let $\mathcal{T} := \bigcup_{n=0}^\infty \mathcal{T}_n$ be the set of triadic cubes.
- A Bernoulli bond (resp. site) percolation configuration on a subset $U \subseteq \mathbb{Z}^d$ is a function $\omega : E(U) \to \{0, 1\}$ (resp. $\omega : U \to \{0, 1\}$). Given a bond percolation configuration ω, we say that the edge e is *closed* if $\omega(e) = 0$ and *open* if $\omega(e) = 1$. We similarly define open and closed sites for a site percolation configuration.

$\nabla\varphi$ interface model

- For $L \in \mathbb{N}$, we let $\Lambda_L := \left[-L, L\right]^d \cap \mathbb{Z}^d = \{-L, \ldots, L\}^d$ be the box of sidelength $(2L+1)$ centered around 0. The external boundary of the box Λ_L is the set $\partial\Lambda_L := \Lambda_{L+1} \setminus \Lambda_L$.
- For $L \in \mathbb{N}$, we denote by $\frac{1}{L}\mathbb{Z}^d := \{x \in \mathbb{R}^d : Lx \in \mathbb{Z}^d\}$ and by $\frac{1}{L}\Lambda_L := [-1,1]^d \cap \frac{1}{L}\mathbb{Z}^d$.
- For $L \in \mathbb{N}$, we denote by $\mathbb{T}_L := (\mathbb{Z}/(2L+1)\mathbb{Z})^d$ the discrete torus of sidelength L (N.B. the torus is only used in this thesis to impose periodic boundary conditions on either solutions of elliptic equations or spin systems). We denote by $E(\mathbb{T}_L)$ the set of the edges of the torus.
- A time-dependent environment is a function $\mathbf{a} : (0,\infty) \times E(\mathbb{Z}^d) \to [0,\infty]$. We allow the environments to be defined on subsets of $(0,\infty) \times E(\mathbb{Z}^d)$ or to be defined on the edges of the torus $E(\mathbb{T}_L)$.
- Given a function $u : \mathbb{Z}^d \to \mathbb{R}$, its discrete gradient is the function $\nabla u : \vec{E}(\mathbb{Z}^d) \to \mathbb{R}$ defined by, for any directed edge $e = (x,y) \in \vec{E}(\mathbb{Z}^d)$, $\nabla u(e) = u(y) - u(x)$. We extend this definition to functions defined on a subset of $\mathbb{Z}^d$ and to functions depending on time, i.e., real-valued functions defined on a subset of $(0,\infty) \times \mathbb{Z}^d$. For $x \in \mathbb{Z}^d$, we will denote by $|\nabla u(x)| := \sum_{i=1}^d |u(x+e_i) - u(x)|$.
- Given an environment $\mathbf{a}$, we define the elliptic operator $\nabla \cdot \mathbf{a} \nabla$ according to the identity, for $u : (0,\infty) \times \mathbb{Z}^d \to \mathbb{R}$ and $(t,x) \in (0,\infty) \times \mathbb{Z}^d$,
$$\nabla \cdot \mathbf{a} \nabla u(t,x) = \sum_{y \sim x} \mathbf{a}(t, \{x,y\})(u(t,x) - u(t,y)).$$
- For $x \in \mathbb{Z}^d$, δ_x is the discrete Dirac mass defined in the discrete setting to be the function $\delta_x : \mathbb{Z}^d \to \mathbb{R}$ such that $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $y \neq x$.

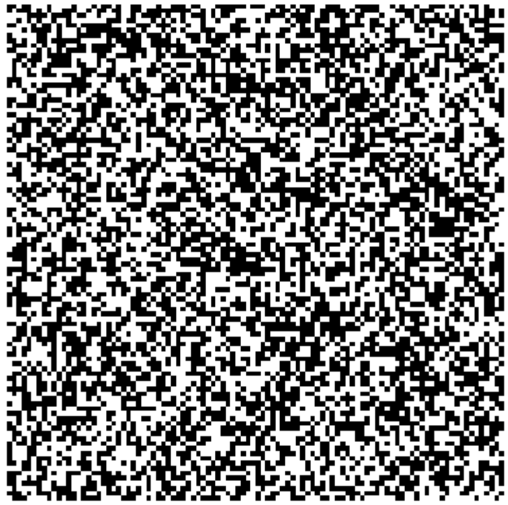

FIGURE 1. A typical environment satisfying the stationarity and ergodicity assumptions is the checkerboard: we choose two positive-definite symmetric (deterministic) matrices $\mathbf{a}_1$ and $\mathbf{a}_2$. The environment $\mathbf{a}$ is defined so that $\mathbf{a}$ is equal to $\mathbf{a}_1$ on the black cells and equal to $\mathbf{a}_2$ on the white cells. The colour of the cells is chosen by sampling i.i.d. Bernoulli random variables of probability $\mathfrak{p} \in [0,1]$.

2. An overview of stochastic homogenization

Stochastic homogenization aims at understanding partial differential equations with rapidly varying random coefficients. An typical problem which has been extensively investigated is the Dirichlet problem for the linear, uniformly elliptic equation in divergence form: given a smooth bounded domain $D \subseteq \mathbb{R}^d$ and a function $f \in C^\infty(D)$,

$$\begin{cases} -\nabla \cdot \mathbf{a}(x) \nabla u = f \text{ in } D, \\ \qquad\qquad u = 0 \text{ on } \partial D, \end{cases} \tag{2.1}$$

where the coefficient field (or environment) $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is assumed to satisfy the following properties:

- Ellipticity:

$$\exists \lambda, \Lambda \in (0,\infty),\ \forall x \in \mathbb{R}^d, \forall \xi \in \mathbb{R}^d,\ \xi \cdot \mathbf{a}(x)\xi \geq \lambda|\xi|^2 \text{ and } |\mathbf{a}(x)\xi|^2 \leq \Lambda \xi \cdot \mathbf{a}(x)\xi.$$

- $\mathbb{Z}^d$-stationarity and ergodicity: the law of $\mathbf{a}$ is stationary and ergodic with respect to the spatial translations τ_y for $y \in \mathbb{Z}^d$.

The equation (2.1) (even without introducing a random coefficient field) is used in many branches of physics, notably heat propagation, electrostatics, fluid mechanics etc.

As stated, the coefficient field a priori varies on a unit scale. To model a rapidly oscillating coefficient field, it is customary to rescale the problem by introducing a small parameter $0 < \varepsilon \ll 1$ which represents the ratio between the microscopic and macroscopic scales, and to rewrite the equation (2.1) as follows

$$\begin{cases} -\nabla\cdot\left(\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = f \text{ in } D, \\ \qquad\qquad\qquad u^\varepsilon = 0 \text{ on } \partial D. \end{cases} \tag{2.2}$$

The question is then to understand the behaviour of the function u^ε as $\varepsilon \to 0$.

It was proved in the 80s, under quite general assumptions over the coefficient field, that there exists a constant, deterministic and uniformly elliptic matrix denoted by $\bar{\mathbf{a}}$ and called the *homogenized coefficient* or *homogenized environment* (which depends on the law of the coefficient field $\mathbf{a}$) such that the function u^ε converges in L^2 to the solution of the elliptic equation

$$\begin{cases} -\nabla\cdot\bar{\mathbf{a}}\nabla\overline{u} = f \text{ in } D, \\ \qquad\quad u = 0 \text{ on } \partial D. \end{cases} \tag{2.3}$$

Specifically, the following theorem holds.

Theorem 1 (Homogenization theorem [138, 158, 176])**.** *Let $D \subseteq \mathbb{R}^d$ be smooth bounded domain and $f \in L^2(D)$, and $\varepsilon > 0$. Then, if we let $u^\varepsilon, \overline{u}$ be the solutions of (2.2) and (2.3), we have*

$$\|u^\varepsilon - \overline{u}\|_{L^2(D)} \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ almost surely.}$$

In the following section, we present an outline of the proof of Theorem 1 and introduce a function and a technique which play an important role in the rest of this thesis: the first-order corrector and the two-scale expansion.

2.1. **First-order corrector and two-scale expansion.** This section is split into two parts. In Section 2.1.1, we introduce the first-order corrector. Section 2.1.2 presents the Ansatz of the two-scale expansion. Let us note that the first-order corrector and the two-scale expansion originally appeared in the theory of periodic homogenization (see [13] or [44, Chapter 1]) and have become by-now standard tools and techniques in the theory of both periodic and stochastic homogenization. Different definitions have been used for the first-order corrector and some technicalities are required to make the general Ansatz of the two-scale expansion fully rigorous, the arguments are thus only presented below on a general level.

2.1.1. *The first-order corrector.* The first order corrector is a fundamental ingredient in the theory of stochastic homogenization. It is formally defined below.

Definition 2.1 (First-order corrector, see, e.g., Theorem 2 of [158])**.** Let $\mathbf{a} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be an elliptic and ergodic coefficient field. Then, for almost every realization of the coefficient field and for any slope $p \in \mathbb{R}^d$, there exists a unique random function $\varphi_p \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ which is a weak solution of the equation

$$-\nabla\cdot\mathbf{a}(x)\,(p + \nabla\varphi_p) = 0$$

and satisfies the following two properties:

- Normalization: the function φ_p satisfies $\int_{(0,1)^d}\varphi_p(x)\,dx = 0$ and $\mathbb{E}\left[\int_{(0,1)^d}\nabla\varphi_p(x)\,dx\right] = 0$;
- Stationarity: If, for $y \in \mathbb{Z}^d$, we denote by $\tau_y\varphi_p$ the corrector under the environment $\tau_y\mathbf{a}$, then we have the identity

$$\nabla\tau_y\varphi_p = \nabla\varphi_p(y + \cdot).$$

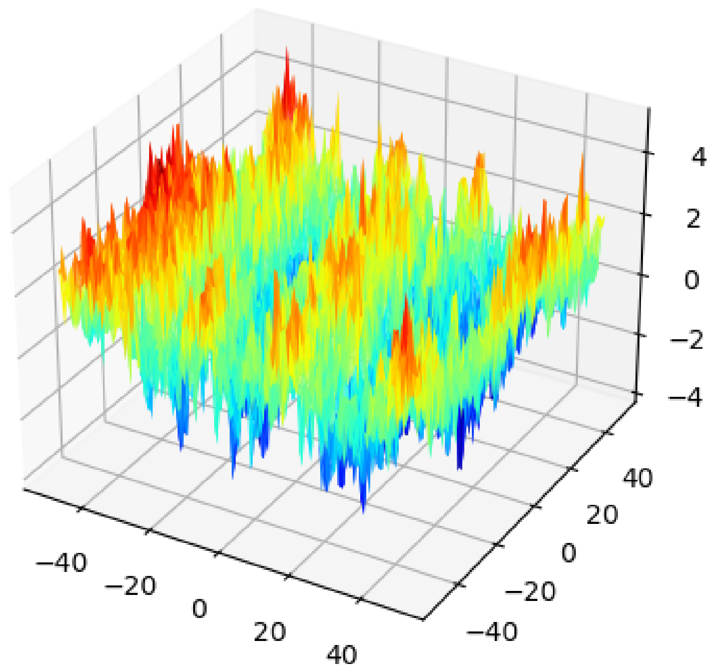

FIGURE 2. The first-order corrector.

Note that the map $p \mapsto \varphi_p$ is linear and that, by the De Giorgi-Nash-Moser regularity, the corrector is a continuous function (even α-Hölder continuous for a small exponent $\alpha > 0$). The homogenized environment can then be defined from the corrector as follows.

Definition 2.2 (Homogenized coefficient)**.** The homogenized coefficient is defined by the identity, for any $p \in \mathbb{R}^d$,

$$\bar{\mathbf{a}}p = \mathbb{E}\left[\int_{(0,1)^d} \mathbf{a}(x)(p + \nabla\varphi_p(x))\,dx\right].$$

One is then interested in establishing two properties on the first-order corrector:

- *Sublinearity:*

$$\varepsilon \left\| \varphi_p\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(D)} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{2.4}$$

- *Sublinearity for the flux:*

$$\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\left(p + \nabla\varphi_p\left(\frac{\cdot}{\varepsilon}\right)\right) \underset{\varepsilon\to 0}{\rightharpoonup} \bar{\mathbf{a}}p, \tag{2.5}$$

  where the arrow refers to the weak convergence in the space $H^1_{\mathrm{loc}}(\mathbb{R}^d)$.

The properties (2.4) and (2.5) can be established using Definition 2.1 and the stationarity of the environment.

All the results stated above are qualitative, and we complete this section about the first-order corrector by stating some (optimal) quantitative estimates which have been obtained in the line of the development of a quantitative theory of stochastic homogenization.

Before stating the result, we mention that, in order to obtain quantitative estimates, it is necessary to make a quantitative ergodicity assumption on the law of the environment. This can be done in a variety of ways, and the choice of the quantitative assumption may affect the scaling estimates on the corrector. We state below a result which holds under classical assumptions in the literature. For instance, it can be obtained by assuming that the law of the environment has (a suitable) spectral gap or logarithmic Sobolev inequality, or that it has a finite range of dependence (i.e., if U_1 and U_2 are two open sets of distance larger than 1, then the restrictions $\mathbf{a}_{|U_1}$ and $\mathbf{a}_{|U_2}$ are independent).

Theorem 2 (Optimal estimates on the first-order corrector, see [122, 121] or Chapter 4 of [28])**.** *Under some suitable ergodicity assumptions on the environment, one has the following estimates on the first-order corrector:*

- *Sublinearity of the corrector:*

$$\varepsilon \left\| \varphi_p \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(D)} \leq \mathcal{O}_2 \left(\varepsilon \big(1 + |\ln \varepsilon|^{\frac{1}{2}} \mathbf{1}_{\{d=2\}} \big) \right).$$

- *Sublinearity for the flux of the corrector:*

$$\left\| \mathbf{a} \left(\frac{\cdot}{\varepsilon} \right) \left(p + \nabla \varphi_p \left(\frac{\cdot}{\varepsilon} \right) \right) - \overline{\mathbf{a}} p \right\|_{H^{-1}(D)} \leq \mathcal{O}_2 \left(\varepsilon \big(1 + |\ln \varepsilon|^{\frac{1}{2}} \mathbf{1}_{\{d=2\}} \big) \right),$$

with

$$\|h\|_{H^{-1}(D)} := \sup \left\{ \int_D h(x) g(x)\, dx \, : \, g \in H^1_0(D) \text{ and } \|g\|_{H^1(D)} \leq 1 \right\}.$$

2.1.2. *The two-scale expansion.* Once equipped with the first-order corrector, the proof of Theorem 1 relies on the introduction of the two-scale expansion w^ε defined by the identity

$$w^\varepsilon(x) := \overline{u}(x) + \varepsilon \chi(x) \sum_{i=1}^d \partial_i \overline{u}(x) \varphi_{e_i} \left(\frac{x}{\varepsilon} \right),$$

where $\chi : D \to \mathbb{R}$ is a smooth cutoff function which is equal to 1 outside a small boundary layer around ∂D, is equal to 0 on ∂D (and thus ensures that $w^\varepsilon = 0$ on ∂D).

Two properties can then be proved on the two-scale expansion w^ε:

- From the definition of the first-order corrector and the property (2.4), we know that the map w^ε is close to the map $\overline{u}$. Specifically, we have

$$\|w^\varepsilon - \overline{u}\|_{L^2(D)} = \sum_{i=1}^d \|\partial_i \overline{u}\|_{L^\infty(D)} \, \varepsilon \left\| \varphi_{e_i} \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2(D)} \underset{\varepsilon \to 0}{\longrightarrow} 0, \tag{2.6}$$

where we used that the gradient of the function $\overline{u}$ is bounded over D, which is a consequence of a regularity estimate for the solutions of the constant coefficient equation (2.3).

- Using the definition of the two-scale expansion w^ε together with an explicit computation (which is not presented here to keep the details light), one can prove that the function w^ε is "almost" a solution of the equation $\nabla \cdot \mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \nabla w^\varepsilon = f$. Specifically, one can verify that the convergences (2.4) and (2.5) imply that the term

$$\mathcal{E}^\varepsilon := -\nabla \cdot \left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \nabla w^\varepsilon \right) - f$$

satisfies

$$\left\| \mathcal{E}^\varepsilon \right\|_{H^{-1}(D)} \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{2.7}$$

Moreover, the convergence (2.7) can be quantified using the results of Theorem 2. We can then use (2.2) to see that

$$\begin{cases} -\nabla \cdot \left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) (\nabla w^\varepsilon - \nabla u^\varepsilon) \right) = \mathcal{E}^\varepsilon & \text{in } D, \\ w^\varepsilon - u^\varepsilon = 0 & \text{on } \partial D. \end{cases} \tag{2.8}$$

Multiplying the equation (2.8) by the function $w^\varepsilon - u^\varepsilon$, integrating over the domain D and performing an integration by parts, we obtain

$$\begin{aligned} \int_D (\nabla w^\varepsilon - \nabla u^\varepsilon) \cdot \mathbf{a}\left(\frac{x}{\varepsilon}\right) (\nabla w^\varepsilon(x) - \nabla u^\varepsilon(x)) \, dx &= \int_D \mathcal{E}^\varepsilon \left(w^\varepsilon(x) - u^\varepsilon(x) \right) \\ &\leq \left\| \mathcal{E}^\varepsilon \right\|_{H^{-1}(D)} \left\| \nabla w^\varepsilon - \nabla u^\varepsilon \right\|_{L^2(D)}. \end{aligned}$$

The term on the left-hand side can be estimated from below using the ellipticity assumption

$$\lambda \left\| \nabla w^\varepsilon - \nabla u^\varepsilon \right\|_{L^2(D)}^2 \leq \int_D (\nabla w^\varepsilon - \nabla u^\varepsilon) \cdot \mathbf{a}\left(\frac{x}{\varepsilon}\right) (\nabla w^\varepsilon(x) - \nabla u^\varepsilon(x)) \, dx.$$

A combination of the two previous displays shows that

$$\left\| \nabla w^\varepsilon - \nabla u^\varepsilon \right\|_{L^2(D)} \leq C \left\| \mathcal{E}^\varepsilon \right\|_{H^{-1}(D)}.$$

By Poincaré's inequality, we deduce that

$$\left\| w^\varepsilon - u^\varepsilon \right\|_{L^2(D)} \leq C \left\| \nabla w^\varepsilon - \nabla u^\varepsilon \right\|_{L^2(D)} \leq C \left\| \mathcal{E}^\varepsilon \right\|_{H^{-1}(D)}.$$

Consequently

$$\begin{aligned} \left\| u^\varepsilon - \overline{u} \right\|_{L^2(D)} &\leq \left\| w^\varepsilon - u^\varepsilon \right\|_{L^2(D)} + \left\| w^\varepsilon - \overline{u} \right\|_{L^2(D)} \\ &\leq \left\| \nabla \overline{u} \right\|_{L^\infty(D)} \sum_{i=1}^d \varepsilon \left\| \varphi_{e_i}\left(\frac{\cdot}{\varepsilon}\right) \right\|_{L^2(D)} + C \left\| \mathcal{E}^\varepsilon \right\|_{H^{-1}(D)}. \end{aligned}$$

Applying (2.6) and (2.7) shows that the right-hand side tends to 0 as ε tends to 0.

To complete this section, we note that this technique is quantitative and explicit (and even optimal) rates of convergence can be obtained once good control over the fluctuations of the corrector and the weak norm of its flux have been obtained (as in, e.g., Theorem 2). The method can be adapted to many frameworks, e.g., various boundary conditions, different types of equations (such as parabolic and/or nonlinear equations), degenerate environments etc. It will be discussed again in this thesis in another setting in Section 4.2, where is it used to obtain a quantitative version of the hydrodynamic limit for the $\nabla\varphi$-interface model.

2.2. **Historical background.**

2.2.1. *Qualitative and quantitative stochastic homogenization.* A qualitative theory of stochastic homogenization was initiated in the early 1980s with the work of Kozlov [138], Papanicolaou and Varadhan [158] and Yurinskiĭ [176]. These results were then extended by Dal Maso and Modica in [72, 73] to the nonlinear setting using variational techniques. Their proofs are based on an application of the ergodic theorem and are therefore qualitative (except the notable exception of [176]).

In order to go beyond the qualitative theory and obtain quantitative convergence rates in homogenization, it is first necessary to make a quantitative ergodicity assumption on the law the environment $\mathbf{a}$. The main difficulty is then to transfer the quantitative ergodicity from the coefficient field $\mathbf{a}$ to the solutions of the elliptic equation $\nabla \cdot \mathbf{a} \nabla u = 0$. This turns out to be a difficult problem since the solutions depend in a complicated way on the environment $\mathbf{a}$. It is an active area of research which has seen much progress in recent years, and a brief account of the literature is given in the following paragraph.

The first satisfactory quantitative results were obtained by Gloria and Otto in [119, 120], which subsequently gave rise to an important series of works, notably [117, 118, 116], in collaboration with Neukamm, concerning the quantification of ergodicity, the regularity on large scales (a theory of regularity at higher orders was developed by Fischer and Otto [104]) and the optimisation of convergence rates in homogenization theorems, the work of Mourrat and Otto [155], Mourrat and Nolen [153], Gu and Mourrat [127] and of Duerinckx, Gloria, Otto [99, 98, 100] concerning the development of a theory of fluctuations in homogenization and the work of Gloria and Otto [122] concerning the corrector.

Another approach was initiated by Armstrong and Smart in [32], who extended and quantified the techniques of Avellaneda and Lin in [33, 34] and Dal Maso and Modica in [72, 73]. It was subsequently developed by Armstrong, Kuusi and Mourrat to deal with more general mixing conditions [29] and to improve the convergence rates [26, 27], and recently extended by Armstrong and Kuusi [25] to the high-contrast regime. This line of research has given rise to two monographs: [28] by Armstrong Mourrat and Kuusi and [24] by Armstrong and Kuusi.

Extensions of the quantitative theory to other types of equations include the nonlinear equations [32, 29, 23, 22, 103, 67], the parabolic equations [19], the $\nabla\varphi$ interface model [30, 174], interacting particles systems [114, 115, 126, 109], fluid mechanics [95, 94, 97, 96, 46], etc.

2.2.2. *The probabilistic aspect: the random conductance model.* The model has been presented above from an analytic perspective but can be reformulated in a probabilistic language. Indeed, given a realization of an environment $\mathbf{a}$, one can consider a Markov process $(X_t^{\mathbf{a}})_{t\geq 0}$ whose generator is the operator $\nabla \cdot \mathbf{a} \nabla$. This process is a diffusion process and it is interesting to study its properties over large times. In particular and on a high level, the homogenization theorem stated in Theorem 1 can be rephrased in the present context as follows: does it hold that, for almost every realization of the environment $\mathbf{a}$, a suitably rescaled version of the diffusion process $(X_t^{\mathbf{a}})_{t\geq 0}$ converges toward a Brownian motion (multiplied by a diffusion coefficient)? This result, when it holds, is known as *the invariance principle*. It is customary in the probabilistic literature to discretize the underlying space and consider a random walk defined on the lattice $\mathbb{Z}^d$ and evolving in a random environment rather than a diffusion process (defined on $\mathbb{R}^d$). The model defined this way is known as the random conductance model and has been an active field of research in the past decades. We review below some of the contributions, and refer to the survey article of Biskup [47] for a more detailed account of the literature.

In the uniformly elliptic setup, the quenched invariance principle was established by Osada in [157] (in the continuous setting) and by Sidoravicius and Sznitman in [167] (in the discrete setting).

In the setting when the conductances are only bounded from above, a quenched invariance principle was proved by Mathieu in [146] and by Biskup and Prescott in [49]. In the case when the conductances are bounded from below, a quenched invariance principle and heat kernel bounds are proved in [36] by Barlow and Deuschel. In [14], Andres, Barlow, Deuschel and Hambly established a quenched invariance principle in the general case when the conductances are allowed to take values in $[0, \infty)$.

The i.i.d. assumption on the environment can be relaxed: in [16], Andres, Deuschel and Slowik proved a quenched invariance principle for the random walk for general ergodic environments which are unbounded from above and below. We also refer to the works of Chiarini, Deuschel [66], Deuschel, Nguyen, Slowik [85] and Bella and Schäffner [41, 42] for additional quenched invariance principles in degenerate ergodic environments. The case of ergodic, time-dependent, degenerate environments is investigated by Andres, Chiarini, Deuschel, and Slowik in [15] where they establish a quenched invariance principle under some moment conditions on the environment.

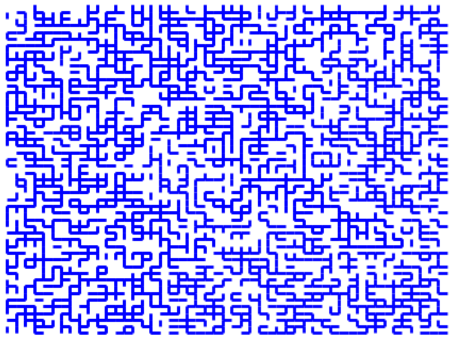

FIGURE 3. A realization of a bond percolation with $\mathfrak{p} = 0.5$.

FIGURE 4. A realization of a site percolation with $\mathfrak{p} = 0.5$.

3. Quantitative homogenization on the percolation cluster

An important part of my research has been devoted to the adaptation of the quantitative theory of stochastic homogenization to the degenerate framework of the supercritical Bernoulli bond percolation. We begin this section with a brief overview of Bernoulli bond percolation and then describe the main results that me and my collaborators obtained in this line of research.

3.1. The super-critical phase of Bernoulli percolation.

The Bernoulli bond percolation model was first introduced by Broadbent and Hammersley in 1957 [59]. It is one of the simplest mathematical exhibiting a phase transition. Despite its apparent simplicity, it gave rise to a deep mathematical theory for which we refer to the books [124, 136, 52, 173].

3.1.1. *The model.* Let us fix a dimension $d \geq 2$ and a probability $\mathfrak{p} \in (0,1)$. We define the Bernoulli bond (resp. site) percolation to be the probability space $(\Omega_{\mathrm{bond}}, \mathcal{F}, \mathbb{P}_{\mathfrak{p}})$ (resp. $(\Omega_{\mathrm{site}}, \mathcal{F}, \mathbb{P}_{\mathfrak{p}})$) where:

- $\Omega_{\mathrm{bond}} := \{0,1\}^{E(\mathbb{Z}^d)}$ is the space of bond percolation configurations (resp. $\Omega_{\mathrm{site}} := \{0,1\}^{\mathbb{Z}^d}$ is the space of site percolation configurations), we refer to Figures 3 and 4 for a visual description;
- $\mathcal{F}$ is the product σ-algebra;
- $\mathbb{P}_{\mathfrak{p}}$ is the i.i.d. measure of probability $\mathfrak{p} \in (0,1)$ on Ω_{bond} (resp. Ω_{site}).

In the rest of this section, we will focus on the bond percolation model (the Bernoulli site percolation plays an important role at the end of this thesis in Section 5.5) and we mention that all the statements and results stated here hold for both models.

First, it is known that this model exhibits a phase transition characterised by the existence of an infinite connected component of open edges. Specifically, we introduce the following notation: for $\mathfrak{p} \in [0,1]$

$$\theta(\mathfrak{p}) := \mathbb{P}_{\mathfrak{p}}(0 \leftrightarrow \infty),$$

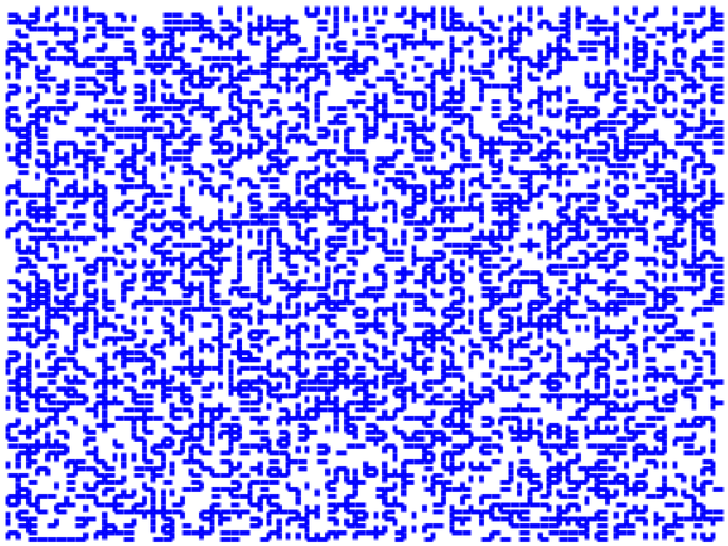

FIGURE 5. A percolation configuration in the subcritical regime with $\mathfrak{p} = 0.3$

the probability that the vertex 0 belongs to an infinite cluster of open edges. We then define the critical probability $p_c \in [0;1]$ according to the identity

$$p_c := \inf \{ \mathfrak{p} \in [0,1] : \theta(\mathfrak{p}) > 0 \} .$$

The first result we would like to state is due to Broadbent and Hammersley in 1957 [59] and Hammersley [128, 129] and shows the existence of a phase transition for Bernoulli percolation.

Proposition 3.1 (Existence of a phase transition, [59, 128, 129])**.** *For each dimension $d \geq 2$, one has*

$$0 < p_c(d) < 1.$$

In the rest of this section, we will focus on the *supercritical phase*, i.e., we assume that $\mathfrak{p} > p_c(d)$.

3.1.2. *The supercritical phase.* In this regime, there exists at least one infinite cluster almost surely. An ergodicity argument shows that the number of infinite clusters is constant almost surely, and in fact it is easily shown that this number is either 1 or infinity. The fact that the infinite cluster is almost surely unique is more difficult and has been established by Aizenman, Kesten and Newman in [6, 7], and a flexible argument has been discovered by Burton and Keane [61].

Theorem 3 (Uniqueness of the infinite cluster [6, 7, 61])**.** *For each $\mathfrak{p} \in (p_c, 1]$, one has*

$$\mathbb{P}_{\mathfrak{p}} \left(\textit{There exists a unique infinite cluster} \right) = 1.$$

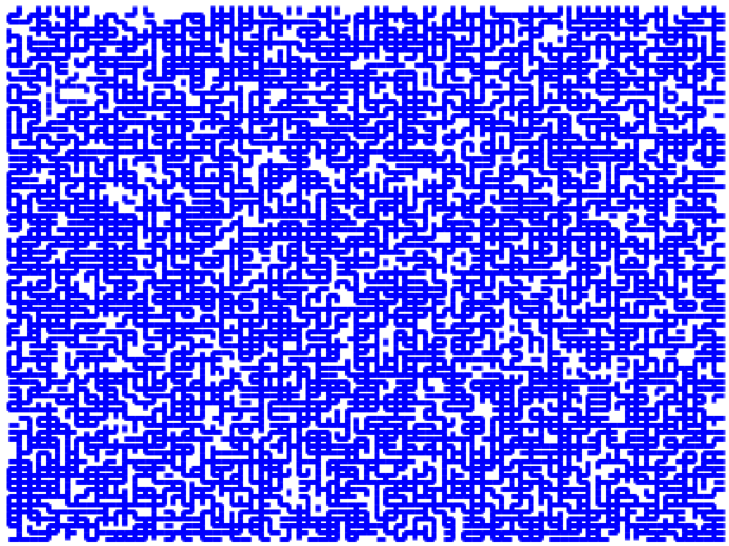

FIGURE 6. A percolation configuration in the supercritical regime with $\mathfrak{p} = 0.7$

From now on, we denote by $\mathcal{C}_\infty$ the unique infinite cluster. Once the existence and uniqueness of $\mathcal{C}_\infty$ are established, one would like to understand its geometry. The broad picture to keep in mind is the following Ansatz: in the supercritical phase the infinite cluster spreads in most of the space. Its geometry is, at least on large scales, similar to the one of $\mathbb{Z}^d$, and this infinite cluster coexists with small finite clusters. This can be seen in Figure 7 where the small clusters are coloured in red.

The formalization and proof of this Ansatz have lead to many important contributions including the ones of Grimmett, Marstrand [125], Aizenman, Delyon and Souillard [3], Chayes, Chayes, Grimmett, Kesten, and Schonmann [65] and Kesten, Zhang [137]. More details are presented in the following sections.

3.1.3. *The good boxes.* We first introduce a finite-volume version of the Ansatz described in Section 3.1.2 through a notion of good box.

Definition 3.2 (Pre-good and good box)**.** Given a percolation configuration ω, we say that a cube $\square$ of size N is pre-good if it satisfies the following properties (see Figure 7):

- There exists a cluster of open edges which intersects the $2d$ faces of the cube $\square$, this cluster is denoted by $\mathcal{C}_*(\square)$,
- The diameter of all the other clusters is smaller than $N/1000$.

It is said to be good if it satisfies the properties:

- The cube $\square$ is pre-good,
- Every cube $\square'$ whose size is between $N/10$ and N and which has non-empty intersection with $\square$ is also a pre-good cube.

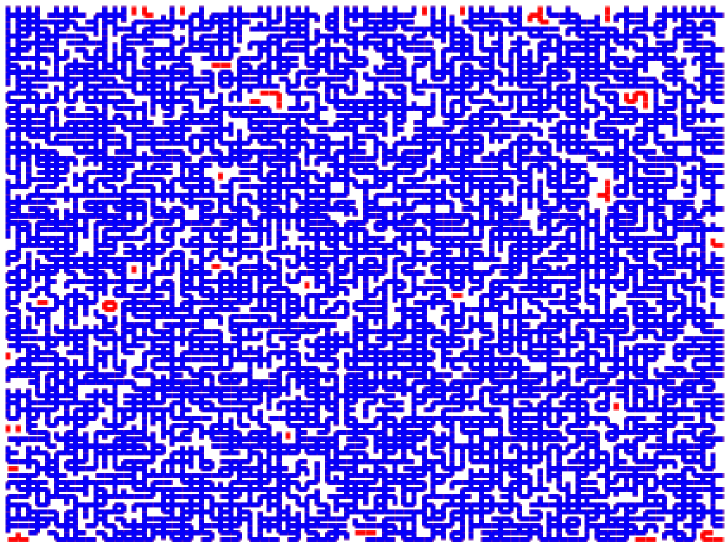

FIGURE 7. A good box. The cluster $\mathcal{C}_*(\square)$ is drawn in blue and touches the four faces of the cubes. It coexists with small isolated clusters drawn in red.

Remark 3.3. The reason to define the notion of good boxes in two steps is to ensure that they satisfy the following connectivity property: given two neighbouring pre-good cubes $\square_1, \square_2$ of "similar" sizes (e.g., the ratio of side lengths is between $1/3$ and 3), the clusters $\mathcal{C}_*(\square_1)$ and $\mathcal{C}_*(\square_2)$ are connected within $\square_1 \cup \square_2$.

The main result pertaining to this notion is that the probability of a large cube to be good is exponentially close to one as the size of the cube goes to infinity. This was proved by Penrose and Pisztora in [160] (see also [18]). The statement given here is an application of their Theorem 5 with $\varphi_n = n/1000$.

Theorem 4 (Theorem 5 of [160])**.** *For each dimension $d \geq 2$ and $\mathfrak{p} > p_c(d)$, there exists a constant $c > 0$ such that for each cube $\square \subseteq \mathbb{Z}^d$ of size n,*

$$\mathbb{P}_{\mathfrak{p}}\left(\square \text{ is good}\right) \geq 1 - \exp\left(-cn\right).$$

3.1.4. *A partition in good boxes.* Once equipped with the notion of good box, we are able to implement a renormalization argument for the infinite cluster by partitioning $\mathbb{Z}^d$ into good boxes (of necessarily different sizes, see Figure 8). This construction was performed in [20] and is explained in more details in the following statement.

Proposition 3.4 (Partition of good cubes [20])**.** *There exists, $\mathbb{P}_{\mathfrak{p}}$ almost surely, a partition $\mathcal{P}$ of $\mathbb{Z}^d$ into triadic cubes of varying sizes such that*

(i) *Every cube $\square \in \mathcal{P}$ is a good cube,*

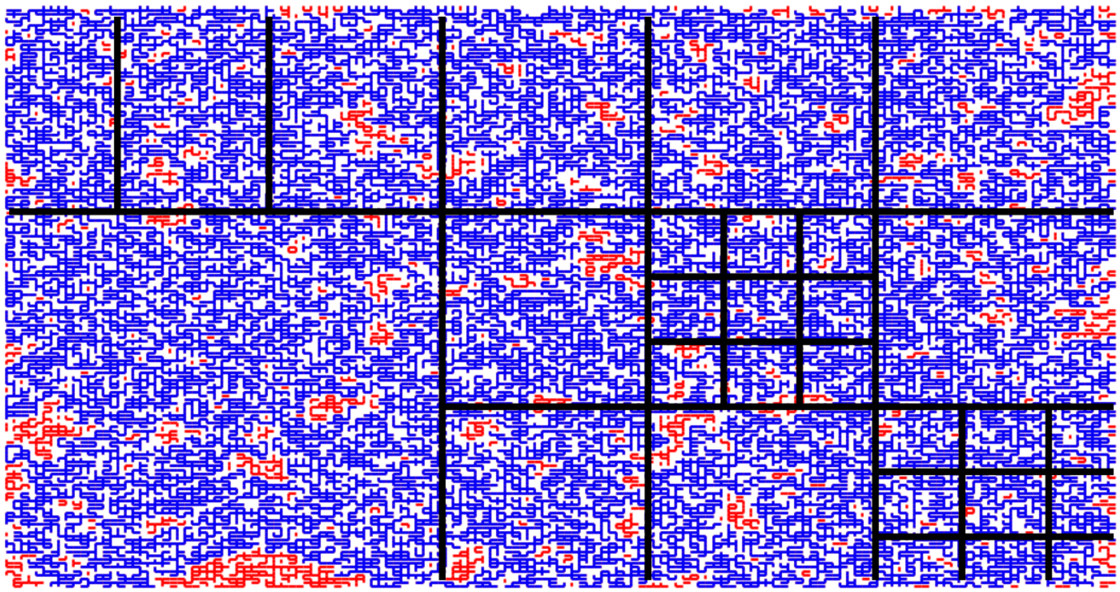

Figure 8. A realization of the partition $\mathcal{P}$.

(ii) Two neighbouring cubes $\square, \square' \in \mathcal{P}$ *have comparable sizes,*

$$\frac{1}{3} \leq \frac{\text{size}(\square)}{\text{size}(\square')} \leq 3.$$

(iii) For $x \in \mathbb{Z}^d$*, if we denote by* $\square_{\mathcal{P}}(x)$ *the unique cube of the partition* $\mathcal{P}$ *containing* x*, then the size of* $\square_{\mathcal{P}}(x)$ *is a random variable satisfying the following exponential tail estimate*

$$\text{size}\left(\square_{\mathcal{P}}(x)\right) \leq \mathcal{O}_1\left(C\right).$$

This partition is a crucial ingredient of my contributions on this topic [20, 74, 78, 53] as it provides a flexible tools to rigorously justify the Ansatz described in Section 3.1.2. In particular, it allows to develop a functional calculus on the infinite cluster: one can, for instance, use this partition to show the existence of Poincaré inequalities and Sobolev inequalities for functions defined on the infinite cluster.

3.1.5. *Harmonic functions on the percolation cluster.* In order to adapt the theory of stochastic homogenization to the infinite cluster, we will be interested in the properties (over large scales) of the harmonic functions on the infinite cluster. To be more precise, let us consider a fixed realization of the infinite cluster which we denote by $\mathcal{C}_\infty$. Given a vertex $x \in \mathcal{C}_\infty$, we define the discrete Laplace operator in the infinite cluster according to the identity, for any $x \in \mathcal{C}_\infty$ and any function $u : \mathcal{C}_\infty \to \mathbb{R}$, we define

$$\Delta_{\mathcal{C}_\infty} u(x) := \sum_{y \sim x} \left(u(y) - u(x)\right). \tag{3.1}$$

A function $u : \mathcal{C}_\infty \to \mathbb{R}$ is *harmonic* if it satisfies

$$\Delta_{\mathcal{C}_\infty} u = 0 \text{ in } \mathcal{C}_\infty.$$

We will also be interested in parabolic equations, and say that a function $u : (0, \infty) \times \mathcal{C}_\infty \to \mathbb{R}$ is *caloric* if it satisfies

$$\partial_t u - \Delta_{\mathcal{C}_\infty} u = 0 \text{ in } (0, \infty) \times \mathcal{C}_\infty.$$

Harmonic and caloric functions on the infinite cluster have been extensively studied in the probabilistic literature, notably due to the connection with the random walk evolving on a percolation cluster and defined as follows. On the positive probability event $\{0 \in \mathcal{C}_\infty\}$, we define a continuous time random walk $(X_t)_{t\geq 0}$ as follows: we start from 0 at time $t = 0$, i.e. $X_0 = 0$, we then equip each edge incident to 0 with an exponential clock of parameter 1 (the clocks are independent of each other). The random walk waits for the first clock to ring and jumps through the edge. After the jump, we consider a new collection of exponential clocks and iterate the procedure. One can then ask many questions on the behaviour of the random walker $(X_t)_{t\geq 0}$ over large times t, e.g., what is the typical size of $|X_t|$ for $t \geq 1$? Does it have the same behaviour as for the random walk on $\mathbb{Z}^d$? Is it true that the (suitably rescaled) random walk converges in law to a Brownian motion (as it is the case for the random walk on $\mathbb{Z}^d$)?

From an analytic perspective, the random walk $(X_t)_{t\geq 0}$ defined above is a continuous time Markov process whose generator is the discrete Laplacian (3.1). In particular, the law of the random walk is given by the heat kernel on the percolation cluster, i.e. the function $P(t,y) = \mathbb{P}[X_t = y]$ is the unique solution of the discrete parabolic equation

$$\begin{cases} \partial_t P - \Delta_{\mathcal{C}_\infty} P = 0 & \text{in } (0,\infty) \times \mathcal{C}_\infty, \\ P(0,\cdot) = \delta_0 & \text{on } \mathcal{C}_\infty, \end{cases} \tag{3.2}$$

where δ_0 is the discrete Dirac function.

The behaviour of the random walk on the percolation cluster and the large-scale properties of the harmonic and caloric functions (including the heat kernel) have been a fruitful line of research in the probabilistic literature. We present below some of the results which have been obtained by the community.

A fundamental property concerning the random walk is the existence of an invariance principle. In this direction, an annealed invariance principle was proved in [82] by De Masi, Ferrari, Goldstein and Wick. In [167], Sidoravicius and Sznitman proved a quenched invariance principle for the simple random walk in dimension $d \geq 4$. This result was extended to every dimension $d \geq 2$ by Berger and Biskup in [45] and by Mathieu and Piatnitski in [147]. Their result asserts that there exists a deterministic diffusivity constant $\overline{\sigma} > 0$ such that, for almost every realization of the infinite cluster, the following convergence holds in the Skorokhod topology

$$\varepsilon X_{\frac{\cdot}{\varepsilon^2}} \xrightarrow[\varepsilon \to 0]{\text{(law)}} \overline{\sigma} B_\cdot,$$

where $B_\cdot$ is a standard Brownian motion. More general models of random walks on percolation clusters with long range correlation, including random interlacements and level sets of the Gaussian free field, are studied by Procaccia, Rosenthal and Sapozhnikov in [162], where a quenched invariance principle is established.

The properties of the harmonic/caloric functions and the heat kernel on the infinite cluster have been investigated in the literature. In [148], Mathieu and Remy proved that, almost surely, the heat kernel decays as fast as $t^{-d/2}$. These bounds were extended in [35] by Barlow who established Gaussian lower and upper bounds for this function.

In the article [37], Barlow and Hambly proved a local central limit theorem for the random walk. Their main result can be stated as follows: if we define, for each $t \geq 0$ and

$x \in \mathbb{R}^d$,

$$\overline{P}(t,x) := \frac{1}{\left(2\pi\overline{\sigma}^2 t\right)^{d/2}} \exp\left(-\frac{|x|^2}{2\,\overline{\sigma}^2 t}\right),$$

the heat kernel with diffusivity $\overline{\sigma}$, then, for each time $T > 0$, the following convergence holds, $\mathbb{P}_\mathfrak{p}$-almost surely on the event $\{0 \in \mathcal{C}_\infty\}$,

$$\lim_{n\to\infty} \sup_{t\geq T} \sup_{x\in\mathbb{R}^d} \left| n^{d/2} P(nt, g_n(x), 0) - \theta(\mathfrak{p})^{-1} \overline{P}(t,x) \right| = 0,$$

where the notation $g_n(x)$ means the closest point to $\sqrt{n}x$ in the infinite cluster.

Regarding the behaviour of the harmonic functions on the percolation cluster, Barlow [35] proved a Harnack inequality (this result implies the Liouville property for harmonic functions: any bounded harmonic function on the percolation cluster is constant), and Barlow and Hambly [37] established the parabolic Harnack inequality. In the article [43], Benjamini, Duminil-Copin, Kozma and Yadin proved (among other results) that the space of linearly growing harmonic functions has almost surely dimension $(d+1)$.

The article [20] was then devoted to the adaptation of the theory of quantitative stochastic homogenization as developed in [32, 28] to the degenerate setting of the supercritical percolation cluster so as to establish a quantitative homogenization theorem as well as a large-scale regularity for harmonic functions on the infinite cluster (in particular, we identified the dimensions of harmonic functions growing like a polynomial of prescribed degree). The article [74] obtained quantitative estimate on the fluctuations of the corrector (in the spirit of Theorem 2).

3.2. **The heat kernel and Green's function on the infinite cluster.** In collaboration with C. Gu [78], we studied the heat kernel as defined in (3.2) and adapted the techniques of [28, Chapter 8] to the percolation setting. We obtained the following quantitative version of the local limit theorem for the random walk.

Theorem 5 (Homogenization for the heat kernel [78])**.** *For each exponent $\varepsilon > 0$, there exists a positive random variable $\mathcal{X}_\varepsilon$ satisfying the stochastic integrability estimate*

$$\mathcal{X}_\varepsilon \leq \mathcal{O}_s(C),$$

for some $s > 0$, so that, for each time $t \geq \mathcal{X}_\varepsilon$ and each point $x \in \mathcal{C}_\infty$ satisfying $|x| \leq t$, we have the upper bound

$$\left| P(t,x) - \theta(\mathfrak{p})\overline{P}(t,x) \right| \leq \frac{1}{t^{\frac{1}{2}-\varepsilon}} \frac{C}{t^{\frac{d}{2}}} \exp\left(-\frac{|x|}{C\sqrt{t}}\right).$$

The strategy of the proof, which follows the argument of [28, Chapter 8], is to use a two-scale expansion similar to the one presented in Section 2.1.2 but adapted to the parabolic and percolation setting.

3.3. **Some differences between the harmonic functions on the infinite cluster and on $\mathbb{Z}^d$.** All the theorems mentioned so far show that the behaviour of harmonic functions on the infinite cluster $\mathcal{C}_\infty$ is similar to that of harmonic functions on $\mathbb{R}^d$ or $\mathbb{Z}^d$. In the article [53], in collaboration with A. Bou-Rabee and W. Cooperman, we investigated the differences between these functions and identified some properties which are true for harmonic functions on the lattice and false for harmonic functions on the cluster.

Specifically, we were interested in the following three properties which hold true for harmonic functions:

- On $\mathbb{Z}^d$, there exists a function which is both Lipschitz, harmonic and non-constant (e.g, the function $x := (x_1, \ldots, x_d) \mapsto x_1$);
- On $\mathbb{Z}^d$, there exists a function which is harmonic, integer-valued and grows linearly fast (e.g., the same function as above);
- On $\mathbb{Z}^d$ and for each integer $k \in \mathbb{N}$, there exists a function $u : \mathbb{Z}^d \to \mathbb{R}$ such that
  - The function $-\Delta u : \mathbb{Z}^d \to \mathbb{R}$ is integer-valued and finitely supported;
  - The function u decays faster than the inverse polynomial $x \mapsto |x|^{-k}$;
  - The function u is not finitely supported.

In [53], we show that these three properties are false for the harmonic functions on the percolation cluster. Specifically, we proved the following result.

Theorem 6 ([53]). *The following properties hold for almost every realization of the infinite cluster $\mathcal{C}_\infty$:*

- *If $u : \mathcal{C}_\infty \to \mathbb{R}$ is Lipschitz and harmonic, then u is constant (N.B. we say that a function $u : \mathcal{C}_\infty \to \mathbb{R}$ is Lipschitz if there exists a constant $K > 0$ such that $|u(x) - u(y)| \leq K \operatorname{dist}_{\mathcal{C}_\infty}(x, y)$ for any $x, y \in \mathcal{C}_\infty$ and where $\operatorname{dist}_{\mathcal{C}_\infty}$ denotes the distance on the infinite cluster)*
- *If $u : \mathcal{C}_\infty \to \mathbb{R}$ is harmonic, integer-valued and grows linearly fast, then u is constant.*
- *In dimension $d = 2$, if $u : \mathcal{C}_\infty \to \mathbb{R}$ is a function satisfying:*
  - *The function $-\Delta_{\mathscr{C}_\infty} u$ is integer-valued and finitely supported,*
  - *The limit superior as R tends to infinity of the map $R \mapsto R\left(\sup_{B_R \setminus B_{R/2}} |u|\right)$ is equal to 0,*

  then the support of u is finite.

Roughly, the strategy of the proof is to proceed by contradiction, assume that such functions exist and find a specific, deterministic, finite subgraph of $\mathbb{Z}^d$ (which must appear infinitely many times in the infinite cluster due to the ergodicity of Bernoulli percolation) on which this property cannot hold. The proofs make important use of the results of quantitative homogenization on the percolation cluster established in [20, 74, 78].

4. The $\nabla\varphi$ interface model

4.1. **The model.** In this section, we consider a mathematical model of random interfaces called the $\nabla\varphi$ interface model. We define it formally below and then discuss its motivation from the point of view of statistical physics.

Throughout this section, we fix a dimension $d \geq 1$. Given an integer $L \in \mathbb{N}$, we consider the box $\Lambda_L := \{-L, \ldots, L\}^d \subseteq \mathbb{Z}^d$ and the space of functions

$$\Omega_L := \{\varphi : \Lambda_{L+1} \to \mathbb{R} : \varphi = 0 \text{ on } \partial\Lambda_L\} \simeq \mathbb{R}^{|\Lambda_L|}.$$

To each function $\varphi \in \Omega_L$, we associate an energy which is given by the formula

$$H(\varphi) := \sum_{x \sim y} V(\varphi(x) - \varphi(y)),$$

where $V : \mathbb{R} \to \mathbb{R}$ is an elastic potential satisfying the following properties:

(1) V is an even function, i.e., $V(x) = V(-x)$ for any $x \in \mathbb{R}$,
(2) V is $C^2(\mathbb{R})$ and uniformly convex, i.e., there exists $\lambda \in (0,1]$ such that,

$$\forall x \in \mathbb{R},\ \lambda \leq V''(x) \leq \frac{1}{\lambda}.$$

We then equip the space Ω_L with a Gibbs measure given by the identity

$$\mu_{\Lambda_L}(d\varphi) := \frac{1}{Z_L} \exp(-H(\varphi)) \prod_{x \in \Lambda_L} \mathrm{d}\varphi(x), \tag{4.1}$$

where Z_L is a normalisation constant which makes the above measure a probability measure and which is called the partition function. We denote by $\mathbb{E}_{\mu_{\Lambda_L}}$ and $\mathrm{Var}_{\mu_{\Lambda_L}}$ the expectation and variance with respect to μ_{Λ_L}. An important example of potential satisfying the assumptions above is the function $V(x) = x^2/2$. In that case, the measure (4.1) is the law of a Gaussian vector whose covariance matrix is explicitly computable and is known as the (discrete) Gaussian free field. This model plays an important role in statistical physics due to its universal properties: it is expected (and proved in a number of cases) that many models of statistical physics behave over large scales like a Gaussian free field. This is in particular the case for the model (4.1) defined above.

This model, known in the mathematical physics literature as the $\nabla\varphi$ interface model, has been the subject of extensive research in the past 50 years (see [108, 171] and Section 4.1.2 for an account of the literature on the topic). It belongs to the broader class of interface models in statistical physics, whose aim is to describe the separation of two phases. In this model, the interface is modeled as the graph of the function $\varphi \in \Omega_L$, and the value $\varphi(x)$ represents the height of the interface at $x \in \Lambda_L$ (see Figure 9). The interface is then a random function, whose law is defined on a microscopic scale (via the Hamiltonian, which is defined as a sum over nearest neighbour edges), and one would like to understand the macroscopic properties of the model. In this direction, many properties can be investigated and three of them are presented in Section 4.1.1 below: the localisation/delocalisation of the interface, the hydrodynamic limit and the scaling limit.

Before presenting in more details these three properties, we mention that the Gibbs measure (4.1) is naturally associated with a Langevin dynami which takes the form of the following system of stochastic differential equations:

$$\begin{cases} \mathrm{d}\varphi_t(x) = \sum_{y \sim x} V'(\varphi_t(y) - \varphi_t(x))\, dt + \sqrt{2} dB_t(x) & \text{for } x \in \Lambda_L, \\ \varphi_t(x) = 0 & \text{for } x \in \partial\Lambda_L, \end{cases} \tag{4.2}$$

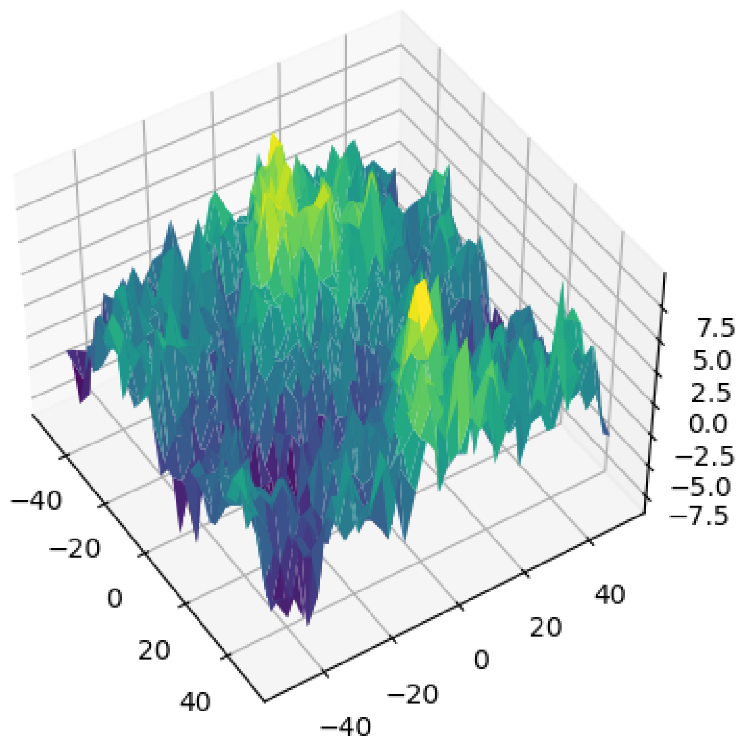

FIGURE 9. A representation of a random interface.

where $\{B_\cdot(x) : x \in \Lambda_L\}$ is a collection of independent Brownian motions. More specifically, the Langevin dynamic is stationary, ergodic and reversible under the Gibbs measure (4.1). This stochastic differential equation contains two terms which have a competing effect on the evolution of the interface. The drift term acts as a smoothing term and makes the dynamic evolve in a direction which minimises the energy $H(\varphi_t)$ (this minimum is uniquely attained for the flat interface $\varphi = 0$ with the assumptions imposed on the function V). This evolution is then affected by a noise incorporated to the model through the Brownian motions. This dynamic plays an important role in the study of the model for two reasons:

- The dynamic appears in the Helffer-Sjöstrand representation formula (see Proposition 4.1) which is one of the central tool to study the model.
- From an analytic perspective, the difference $\varphi_t(y) - \varphi_t(x)$ is a discrete gradient and the sum $\sum_{y \sim x}$ is a discrete divergence. The evolution (4.2) can thus be rewritten, using more analytic notation, as follows

$$d\varphi_t(x) = \nabla \cdot V'(\nabla \varphi_t(x))\, dt + \sqrt{2} dB_t(x)\,.$$

Written this way, the drift term is (the discrete version of) a non-linear elliptic operator in divergence form which can be studied using tools of elliptic regularity. This approach is discussed in more details in Section 4.2 below.

4.1.1. *Three results on the $\nabla\varphi$ interface model.* In this section, we present three results on the $\nabla\varphi$ interface model: the localisation/delocalisation of the interface, the hydrodynamic limit and the scaling limit.

1) The localisation/delocalisation of the interface.

A fundamental question in the topic of random interfaces in mathematical physics is the one of the localisation or delocalisation of the interface. In the setting considered here, a possible way to formulate the problem is as follows: how does the sequence $L \mapsto \mathrm{Var}_{\mu_{\Lambda_L}}[\varphi(0)]$ behave as L tends to infinity? When the sequence is bounded, the random interface is said to be localised, and when it is unbounded, the interface is said to be delocalised. The answer to this question may depend on the potential V and the dimension d of the underlying space.

We mention that the previous paragraph presents one way to formalize the problem, but that many aspects of the localisation or delocalisation of the random interfaces can be investigated (e.g., one can try to derive lower bounds or upper bounds, qualitative or quantitative estimates, investigate the same question for related interface models etc.). We refer to the lecture notes of Velenik [171] for (much) more information on this question.

In the case of the $\nabla\varphi$ interface model with even and uniformly elliptic potential considered in this section, the question can be answered precisely and we have the following result established in [56] (N.B. The proof of [56] applies to a broader class of potentials).

Theorem 7 (Brascamp, Lieb and Lebowitz [56]). *For any even and uniformly convex potential V, there exist two constants $c := c(d,V) > 0$ and $C := C(d,V) < \infty$ such that*

$$\begin{aligned} cL \le \mathrm{Var}_{\mu_{\Lambda_L}}[\varphi(0)] &\le CL && \text{if } d = 1, \\ c\ln L \le \mathrm{Var}_{\mu_{\Lambda_L}}[\varphi(0)] &\le C\ln L && \text{if } d = 2, \\ c \le \mathrm{Var}_{\mu_{\Lambda_L}}[\varphi(0)] &\le C && \text{if } d \ge 3. \end{aligned}$$

There exist various techniques which can be used to establish this result. We will present below one of them: the Helffer-Sjöstrand representation formula due to Helffer and Sjöstrand [130] and applied to the $\nabla\varphi$ model in [156, 119] (N.B. this is not the technique used in [56]).

Proposition 4.1 (Helffer-Sjöstrand representation formula [130, 156, 112]). *For any $L \in \mathbb{N}$, one has the identity*

$$\mathrm{Var}_{\mu_{\Lambda_L}}[\varphi(0)] = \mathbb{E}\left[\int_0^\infty P_{\mathbf{a}}(t,0)\,dt\right],$$

where $P_{\mathbf{a}}$ is the heat kernel associated with the equation

$$\begin{cases} \partial_t P_{\mathbf{a}} - \nabla\cdot\mathbf{a}\nabla P_{\mathbf{a}} = 0 & \text{in } [0,\infty]\times\mathbb{Z}^d, \\ P_{\mathbf{a}}(0,\cdot) = \delta_0 & \text{in } \Lambda_L, \\ P_{\mathbf{a}}(t,\cdot) = 0 & \text{in } [0,\infty]\times\partial\Lambda_L, \end{cases} \tag{4.3}$$

with the time dependent environment $\mathbf{a}(t,e) = V''(\nabla\varphi_t(e))$, *where* φ_t *is distributed according to the Langevin dynamic*

$$(4.4)\qquad \begin{cases} d\varphi_t(x) = \nabla\cdot V'(\nabla\varphi_t)(x)dt + \sqrt{2}dB_t(x) & x\in\Lambda_L,\\ \varphi_t(x) = 0 & x\in\partial\Lambda_L,\\ \varphi_0(x) = \varphi(x) & x\in\Lambda_L. \end{cases}$$

where the initial condition φ *is distributed according to* μ_{Λ_L} *and is independent of the Brownian motions.*

The Helffer-Sjöstrand representation formula can be used in conjunction with the following upper and lower bounds on the heat kernel.

Proposition 4.2 (On diagonal Nash-Aronson inequality)**.** *Under the assumption that* V *is uniformly convex, there exist two constants* $c := c(d,V) > 0$ *and* $C := C(d,V) < \infty$ *such that the heat kernel* $P_{\mathbf{a}}$ *defined in* (4.3) *satisfies*

$$\frac{c}{t^{d/2}}\exp\left(-\frac{t}{cL^2}\right) \le P_{\mathbf{a}}(t,0) \le \frac{C}{t^{d/2}}\exp\left(-\frac{t}{CL^2}\right).$$

Theorem 7 is then obtained as a combination of the two previous results, e.g., for the upper bound, we may write

$$(4.5)\qquad \mathrm{Var}_{\mu_{\Lambda_L}}[\varphi(0)] = \mathbb{E}\left[\int_0^\infty P_{\mathbf{a}}(t,0)\,dt\right] \le \int_0^\infty \frac{C}{t^{d/2}}\exp\left(-\frac{t}{CL^2}\right)dt \le \begin{cases} CL \text{ if } d=1,\\ C\ln L \text{ if } d=2,\\ C \text{ if } d\ge 3.\end{cases}$$

The Helffer-Sjöstrand representation formula plays a fundamental role in the theory and we remark that this formula is not restricted to the variance of the heigth $\varphi(0)$ but can be generalized to a large class of (both linear and nonlinear) functionals of the height. A more general version of this identity is stated in (4.10) below, and is one of the major ingredient in the identification of the scaling limit of the model by [156, 112].

An important part of my research on this topic relies on the study of the Langevin dynamics (4.4), and can be summarized by the following heuristic "One can obtain information on the behaviour Langevin dynamics by differentiating it with respect to the right parameters".

We illustrate this point by providing below a sketch of the proof of Proposition 4.1 (which does not follow the original argument [130, 156, 112]) and where the formula is obtained by computing the derivative of the same function in two different ways.

Sketch of proof of Proposition 4.1. Fix $L\in\mathbb{N}$ and for $h\in\mathbb{R}$, consider the *tilted* measure on Ω_L

$$\mu^h_{\Lambda_L}(d\varphi) := \frac{1}{Z^h_L}\exp\left(-\sum_{x\sim y}V(\varphi(x)-\varphi(y)) + h\varphi(0)\right)\prod_{x\in\Lambda_L}d\varphi(x),$$

and denote by $G(h) := \mathbb{E}_{\mu^h_{\Lambda_L}}[\varphi(0)] = \int\varphi(0)\mu^h_{\Lambda_L}(d\varphi)$. The derivative of the function G at the value $h=0$ can be explicitly computed and we have

$$G'(0) = \mathrm{Var}_{\mu_{\Lambda_L}}[\varphi(0)].$$

We next consider the tilted Langevin dynamic

$$(4.6)\qquad \begin{cases} d\varphi_t^h(x) := \nabla\cdot V'(\nabla\varphi_t(e))dt + h\mathbf{1}_{\{x=0\}} + \sqrt{2}dB_t(x) & x\in\Lambda_L,\\ \varphi_t^h(x) = 0 & x\in\partial\Lambda_L,\\ \varphi_0^h(x) = \varphi(y) & x\in\Lambda_L.\end{cases}$$

This dynamic is ergodic and has $\mu^h_{\Lambda_L}$ as an invariant measure. In particular, the law of φ_t^h converges, as $t\to\infty$, toward $\mu^h_{\Lambda_L}$ and thus

$$(4.7)\qquad \mathbb{E}\left[\varphi_t^h(0)\right] \underset{t\to\infty}{\longrightarrow} G(h).$$

Differentiating the dynamic (4.6) with respect to h at the value $h=0$, we obtain that the map $w(t,x) := \frac{d\varphi_t^h(x)}{dh}$ solves the linear parabolic equation

$$\begin{cases} \partial_t w(t,x) = \nabla\cdot\mathbf{a}\nabla w(t,x) + \mathbf{1}_{\{x=y\}} & x\in\Lambda_L,\\ w(t,x) = 0 & x\in\partial\Lambda_L,\\ w(0,x) = 0 & x\in\Lambda_L,\end{cases}$$

with $\mathbf{a}(t,e) := V''(\nabla\varphi_t(e))$. Applying Duhamel principle, we have that

$$w(t,0) := \int_0^t P_{\mathbf{a}}(t,0,s,0)\,ds.$$

Using the stationarity the dynamic, we obtain

$$\begin{aligned}\frac{d}{dh}\mathbb{E}\left[\varphi_t^h(0)\right] = \mathbb{E}\left[\frac{d}{dh}\varphi_t^h(0)\right] = \mathbb{E}\left[w(t,0)\right] &= \mathbb{E}\left[\int_0^t P_{\mathbf{a}}(t,0,s,0)\,ds\right]\\ &= \mathbb{E}\left[\int_0^t P_{\mathbf{a}}(t,0)\,ds\right].\end{aligned}$$

Assuming that derivative with respect to h and the limit $t\to\infty$ can be exchanged, we deduce the Helffer-Sjöstrand representation formula from (4.7) and the previous identity. □

2) The hydrodynamic limit.

The hydrodynamic limit is one of the fundamental result on the $\nabla\varphi$ interface model which plays the role of the law of large number. To be more specific, a possible way to formalize the result is as follows. Consider a sufficiently regular function $f:[-1,1]^d\to\mathbb{R}$ (e.g., Lipschitz), and, given a large integer $L\in\mathbb{N}$, denote by Ω_L the space of functions $\varphi:\Lambda_L\to\mathbb{R}$ which are equal to $Lf\left(\frac{\cdot}{L}\right)$ on the boundary of Λ_L, i.e.,

$$\Omega_L^f := \left\{\varphi:\Lambda_{L+1}\to\mathbb{R} \,:\, \varphi = Lf\left(\frac{\cdot}{L}\right) \text{ on } \partial\Lambda_L\right\}.$$

We then consider the Gibbs measure $\mu^f_{\Lambda_L}$ to be the probability distribution on the space Ω_L^f given by the formula (i.e., this is the same measure as in (4.1) but we impose the boundary condition to be equal to $Lf\left(\frac{\cdot}{L}\right)$ instead of 0)

$$\mu^f_{\Lambda_L}(d\varphi) := \frac{1}{Z_L^f}\exp\left(-H\left(\varphi\right)\right)\prod_{x\in\Lambda_L} \mathrm{d}\varphi(x),$$

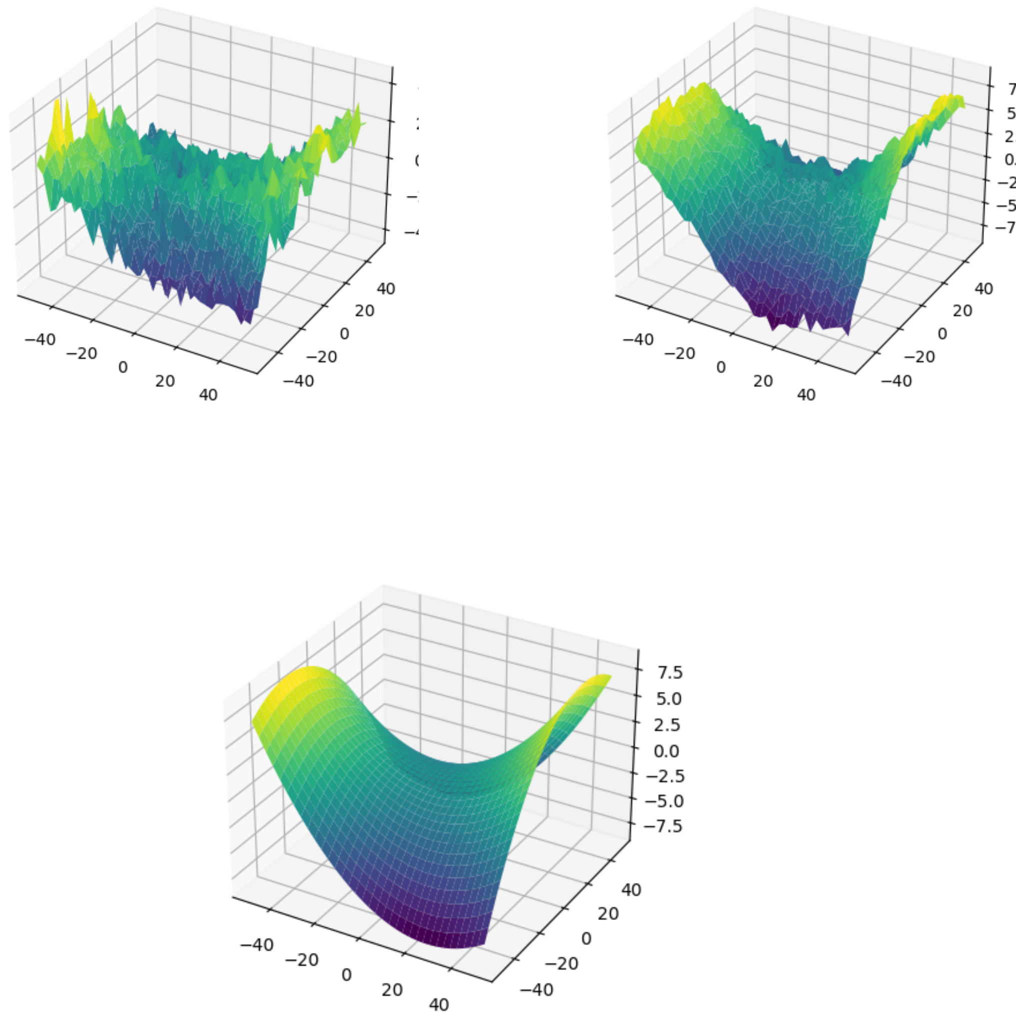

FIGURE 10. An illustration of Theorem 8: the two pictures on the first line represent a random interface sampled according to the Gibbs measure $\mu^f_{\Lambda_L}$ with $L = 50$ and $L = 100$ (and suitably rescaled). As $L \to \infty$, the interfaces concentrate around a smooth deterministic profile drawn on the second line.

and denote by $\mathbb{E}_{\mu^f_{\Lambda_L}}$ the expectation with respect to $\mu^f_{\Lambda_L}$. The hydrodynamic limit for the $\nabla\varphi$ interface model can then be stated as follows.

Theorem 8 (Hydrodynamic limit for the $\nabla\varphi$ interface model)**.** *There exists a uniformly convex function $\overline{\sigma} : \mathbb{R}^d \to \mathbb{R}$ such that if we let $\overline{u} \in f + H^1_0([-1,1]^d)$ be the minimiser of the variational problem*

$$\inf_{u \in f + H^1_0([-1,1]^d)} \int_{[-1,1]^d} \overline{\sigma}(\nabla u(x))\, dx,$$

then we have the convergence

$$\mathbb{E}_{\mu^f_{\Lambda_L}}\left[\frac{1}{|\Lambda_L|}\sum_{x\in\frac{1}{L}\Lambda_L}\left|\frac{1}{L}\varphi(Lx)-\overline{u}(x)\right|^2\right]\underset{L\to\infty}{\longrightarrow}0.$$

Remark 4.3. The map $\overline{\sigma}$ is called the surface tension and can be characterized by the following identity: if, for $p\in\mathbb{R}^d$, we denote by $\ell_p:\mathbb{R}^d\to\mathbb{R}$ the affine function satisfying $\ell_p(x)=p\cdot x$, then we have

$$\overline{\sigma}(p)=\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\ln Z_L^{\ell_p}.$$

Theorem 8 states that a suitably rescaled random interface sampled according to the Gibbs measure $\mu^f_{\Lambda_L}$ concentrates around a deterministic interface which can be characterised as the minimiser of a certain strictly convex functional.

Statements in the form of Theorem 8 play a fundamental role in the theory of random interfaces. In fact, the first mathematical model to describe these interfaces was introduced by Wulff in 1901 in the article [175]: the interfaces are characterised as the minimisers of a certain functional, known as the Wulff functional, and defined, for a subset $E\subseteq\mathbb{R}^d$, according to the formula

$$W(E):=\int_{\partial E}\sigma(\mathbf{n}(x))\,\mathrm{d}x, \tag{4.8}$$

where $\mathbf{n}(x)$ is the exterior normal to ∂E at point x and σ is the surface tension between the two phases.

Besides the $\nabla\varphi$ interface model, many important results establishing the concentration of a random interface around a deterministic shape minimising a functional of the form (4.8) have been established in the mathematical physics literature (and the result frequently goes by the name "Wulff construction"). For instance in [12], Alexander, Chayes and Chayes obtained a Wulff construction for supercritical Bernoulli percolation in 2 dimensions. In the monograph [93], Dobrushin, Kotecký and Shlosman established a Wulff construction for the two-dimensional ferromagnetic Ising model at low temperature with periodic boundary conditions. These results were subsequently extended to all temperatures below the critical threshold, and we refer to the work of Ioffe [132, 133], Schonmann, Shlosman [164], Pfister, Velenik [161] and Ioffe and Schonmann in [134] for further information. In dimension 3, Cerf has demonstrated in [63] a form of Wulff construction for Bernoulli percolation in the supercritical regime. Bodineau in [51] proved a similar result for the Ising model for all dimensions $d\geq 3$ and at low temperature.

We complete this section by mentioning that a version of the hydrodynamic limit can be established for the Langevin dynamic associated with the $\nabla\varphi$ interface model, and that this result implies the one stated in Theorem 8 (which concerns the Gibbs measure). The hydrodynamic limit for the Langevin dynamic is the subject of the article of Funaki and Spohn [110] and is discussed in more details in Section 4.2 below.

3) The scaling limit.

The third result pertaining to the $\nabla\varphi$ interface model is the scaling limit of the model, which corresponds to the central limit theorem. In the case of the $\nabla\varphi$ interface model the scaling limit is a Gaussian free field. The precise theorem is stated below and we introduce the following notation: for a smooth function $f \in C_c^\infty((-1,1)^d)$, we let $u \in H_0^1([-1,1]^d)$ be the solution of the Laplace equation

$$\begin{cases} -\Delta u_f = f \text{ in } [-1,1]^d, \\ \quad u_f = 0 \text{ on } [-1,1]^d. \end{cases}$$

Theorem 9 (Scaling limit for the $\nabla\phi$ interface model [156, 112])**.** *For $L \in \mathbb{N}$, we let $\varphi_L \in \Omega_L$ be a random interface sampled according to the measure μ_{Λ_L}. Then, there exists a constant $\overline{\mathbf{a}} := \overline{\mathbf{a}}(d,V) > 0$ such that, for any $f \in C_c^\infty((-1,1)^d)$,*

$$\frac{1}{L^{\frac{d}{2}+1}} \sum_{x \in \mathbb{Z}^d} f\left(\frac{x}{L}\right) \varphi_L(x) \xrightarrow[L\to\infty]{\text{(law)}} \mathcal{N}\left(0, \overline{\mathbf{a}}^{-1} \int_{[-1,1]^d} f(x) u_f(x)\, dx\right).$$

This result was originally proved (in infinite volume) by Naddaf-Spencer [156] and Giacomin-Olla-Spohn [112]. We give below a brief sketch of the proof which follows the argument of [112].

Sketch of proof. For $L \in \mathbb{N}$ and $f \in C_c^\infty(\mathbb{R}^d)$, we introduce the notation

$$S_L := \frac{1}{L^{\frac{d}{2}+1}} \sum_{x \in \mathbb{Z}^d} f\left(\frac{x}{L}\right) \phi(x).$$

The proof is decomposed in two steps:

(i) Proving convergence of the variance, i.e., showing that

$$\operatorname{var}_\mu[S_L] \xrightarrow[L\to\infty]{} \overline{\mathbf{a}}^{-1} \int_{[-1,1]^d} f(x) u_f(x)\, dx. \tag{4.9}$$

(ii) Proving that S_L converges in distribution to a Gaussian random variable.

We only sketch the proof of (4.9). By applying the (suitable version of) the Helffer-Sjöstrand representation we have the identity

$$\operatorname{var}_\mu[S_L] = \int_0^\infty \frac{1}{L^d} \sum_{x \in \frac{1}{L}\mathbb{Z}^d} f(x) \mathbb{E}\left[H_L(L^2 t, Lx)\right] dt. \tag{4.10}$$

where the function $H_L : (0,\infty) \times \mathbb{Z}^d \to \mathbb{R}$ is defined to be the solution of the discrete parabolic equation

$$\begin{cases} \partial_t H_L - \nabla \cdot \mathbf{a} \nabla H_L = 0 & \text{in } (0,\infty] \times \Lambda_L, \\ H_L(0,\cdot) = f\left(\frac{\cdot}{L}\right) & \text{in } \Lambda_L, \\ H_L = 0 & \text{on } (0,\infty) \times \partial\Lambda_L, \end{cases}$$

where $\mathbf{a}$ is as in Proposition 4.1. It can then be proved that the random environment $\mathbf{a}$ is ergodic and that the parabolic operator $\partial_t - \nabla \cdot \mathbf{a} \nabla$ can be homogenized. Specifically, one can prove that there exists a deterministic coefficient $\overline{\mathbf{a}} := \overline{\mathbf{a}}(d,V) > 0$ such that, if we let $\overline{H}$ be the solution of the (continuous) parabolic equation

$$\begin{cases} \partial_t \overline{H} - \overline{\mathbf{a}} \Delta \overline{H} = 0 & \text{in } (0,\infty] \times [-1,1]^d, \\ \overline{H}(0,\cdot) = f & \text{in } [-1,1]^d, \\ \overline{H} = 0 & \text{on } (0,\infty) \times \partial[-1,1]^d, \end{cases}$$

then

$$\int_0^\infty \frac{1}{L^d} \sum_{x \in \frac{1}{L}\mathbb{Z}^d} \left| \mathbb{E}\left[H_L(L^2 t, Lx) \right] - \overline{H}(t,x) \right|^2 \, dt \underset{L\to\infty}{\longrightarrow} 0.$$

Combining the previous result with (4.10) implies

$$\mathrm{var}_\mu \left[S_L \right] \underset{L\to\infty}{\longrightarrow} \int_0^\infty \int_{[-1,1]^d} f(x) \overline{H}(t,x)\, dx = \bar{\mathbf{a}}^{-1} \int_{[-1,1]^d} f(x) u_f(x)\, dx.$$

□

4.1.2. *Historical background.* We mention in this section some of the important results about the $\nabla\varphi$ interface model, but the list is certainly not exhaustive and we refer the interested reader to the review articles [108, 171] on the topic. In the uniformly convex setting, the scaling limit of the model was identified by Brydges and Yau [60] in a perturbative setting based on a renormalization group approach. After the groundbreaking works of Funaki, Spohn [110], Naddaf, Spencer [156] and Giacomin, Olla, Spohn [112], large deviation estimates and concentration inequalities were established by Deuschel, Giacomin and Ioffe [84], and sharp decorrelation estimates for the discrete gradient of the field were obtained by Delmotte and Deuschel [83]. The scaling limit of the field in finite-volume was established by Miller [152]. More recently, Armstrong and Wu [30] proved the C^2 regularity of the surface tension and the fluctuation-dissipation relation (see also the recent subsequent work of Wu [174]), and Deuschel and Rodriguez [87] identified the scaling limit of the square of the gradient field.

The case of non-uniformly convex potentials was studied in the high temperature regime by Cotar, Deuschel and Müller [69], who established the strict convexity of the surface tension, and by Cotar and Deuschel [68] who proved the uniqueness of ergodic Gibbs measures, obtained sharp estimates on the decay of covariance and identified the scaling limit of the model (see also [86] for the hydrodynamic limit). The strict convexity of the surface tension in the low temperature regime was established by Adams, Kotecký and Müller [2] through a renormalization group argument. This renormalization group approach was further developed in [1] to obtain a (form of) verification of the Cauchy-Born rule for these models. In [48], Biskup and Kotecký showed the possible non-uniqueness of infinite-volume, shift-ergodic gradient Gibbs measures for some nonconvex interaction potentials, and Biskup and Spohn [50] proved that, for an important class of nonconvex potentials, the scaling limit of the model is a Gaussian free field (see Armstrong and Wu [31] for the scaling limit of the SOS-model using a similar strategy). We finally mention the recent works of Magazinov and Peled [145], who established sharp localisation and delocalisation estimates for a class of convex degenerate potentials V, the one of Andres and Taylor [17] who identified the scaling limit of the model for a class of convex potentials satisfying the assumption $\inf V'' \geq \lambda > 0$, and the recent work of Sellke [165] who obtained sharp upper bounds for the localisation/delocalisation of the interface for a broad class of potentials. We finally refer to the thesis of Sheffield [166] for many additional results and techniques on this class of models (including large deviations principles for the random interface, proof of the strict convexity of the surface tension, the introduction of the cluster swapping etc.).

4.2. **The hydrodynamic limit as a nonlinear homogenization problem.** In the article [21], S. Armstrong and I established a quantitative version of the hydrodynamic limit proved in [110]. Our approach is different from that of [110] and is based on quantitative stochastic homogenization methods.

Before stating the result, we mention that we will make use of the notation $D_p\overline{\sigma}$ for the gradient of the surface tension $\overline{\sigma}$ and introduce the notation for the parabolic cylinder of $(0,\infty)\times\Lambda_L$

$$\partial_{\mathrm{par}}((0,\infty)\times\Lambda_L) := (\{0\}\times\Lambda_L)\times((0,\infty)\times\partial\Lambda_L).$$

We will use similarly

$$\partial_{\mathrm{par}}((0,\infty)\times(-1,1)^d) := \left(\{0\}\times(-1,1)^d\right)\times\left((0,\infty)\times\partial(-1,1)^d\right).$$

Theorem 10 ([21]). *Let $f\in C^\infty([0,\infty)\times[-1,1]^d)$, $L\in\mathbb{N}$, let $\varphi:(0,\infty)\times\Lambda_L\to\mathbb{R}$ be the solution of the system of stochastic differential equations*

$$\begin{cases} d\varphi_t(x) = \sum_{y\sim x} V'\left(\varphi_t(y)-\varphi_t(x)\right)dt + \sqrt{2}dB_t(x) & \text{for } (t,x)\in(0,\infty)\times\Lambda_L, \\ \varphi_t(x) = Lf\left(\frac{t}{L^2},\frac{x}{L}\right) & \text{for } (t,x)\in\partial_{\mathrm{par}}((0,\infty)\times\Lambda_L), \end{cases}$$

and let $\overline{u}:(0,\infty)\times(-1,1)^d\to\mathbb{R}$ be the solution of the parabolic equation

$$\begin{cases} \partial_t\overline{u} - \nabla\cdot D_p\overline{\sigma}(\nabla\overline{u}) = 0 & \text{in } (0,\infty)\times(-1,1)^d, \\ \overline{u} = f & \text{on } \partial_{\mathrm{par}}((0,\infty)\times(-1,1)^d). \end{cases}$$

Then we have

$$\int_0^1 \frac{1}{|\Lambda_L|}\sum_{x\in\frac{1}{L}\Lambda_L}\left|\frac{1}{L}\varphi_{L^2t}(Lx) - \overline{u}(t,x)\right|^2 \leq \mathcal{O}_2\left(C_f L^{-\frac{1}{2}}\left(1+|\log L|^{\frac{1}{2}}\mathbf{1}_{\{d=2\}}\right)\right).$$

We complete this section with a discussion of how homogenization theory (as described in Section 2) can play a role in understanding the macroscopic behaviour of this interface model.

Some elements of proof :

The standard problem of stochastic homogenization of nonlinear elliptic equations, following [32, 23, 22, 103], is defined as follows. We consider a Lagrangian $L:(x,p)\mapsto L(x,p)$ with $x,p\in\mathbb{R}^d$ and assume that, for all $x\in\mathbb{R}^d$, the function $p\mapsto L(x,p)$ is uniformly convex. We also assume that the Lagrangian L is random and that its distribution is stationary and ergodic with respect to spatial translations. We are then interested in studying the large-scale behaviour of the solutions of the non-linear elliptic equation

$$\nabla\cdot D_pL(x,\nabla u) = 0 \text{ in } \mathbb{R}^d. \tag{4.11}$$

The starting point of the analysis is the observation that the Langevin dynamics (4.2) can be considered as a non-linear (discrete) parabolic equation with noise. As mentioned above, one can make the replacement

$$\sum_{e\ni x} V'(\nabla\varphi_t(e)) \quad \leftrightsquigarrow \quad \nabla\cdot D_pL(\nabla u),$$

and consider (4.2) as being similar to

$$\partial_t u - \nabla\cdot D_pL(\nabla u) = \text{“noise”}, \tag{4.12}$$

where the noise corresponds to the term involving Brownian motion (4.2). The equation (4.12) can then be interpreted as a discrete and parabolic version of the equation (4.11) where the randomness is not encoded in the Lagrangian but externally by random noise. A similar observation was made by Cardaliaguet, Dirr and Souganidis [62], who proved a qualitative homogenization result for a continuous version of the Langevin dynamics.

The standard homogenization theorem for non-linear elliptic equations [72, 73] states that there exists a uniformly convex deterministic function $p \mapsto \overline{L}(p)$, called *homogeneous or effective Lagrangian*, such that any solution of (4.11) is well approximated on large scales by a solution $\overline{u}$ of the equation

$$\nabla \cdot D_p \overline{L}(\nabla \overline{u}) = 0 \text{ in } \mathbb{R}^d.$$

Compared with the previous statement, the hydrodynamic limit for the $\nabla\varphi$ interface model [110] indicates that the solutions of the Langevin dynamics (4.2) are well approximated on large scales by the solution of the deterministic equation

$$\partial_t \overline{u} - \nabla \cdot D_p \overline{\sigma}(\nabla \overline{u}) = 0.$$

The *hydrodynamic limit* can therefore be thought of as constituting a *homogenization theorem*, where the surface tension $\overline{\sigma}$ plays the same role as the effective Lagrangian. This comparison can be extended and the hydrodynamic limit can be proved with the same technique used to prove homogenization theorems, namely the two-scale expansion presented in Section 2.1.2. We do not present here the details of the mathematical construction and refer the interested reader to [21].

4.3. **A localisation/delocalisation estimate in for a class of degenerate potentials.** All the results discussed until now have been stated under the assumption that the potential V is uniformly convex, i.e., there exists a constant $\lambda \in (0,1)$ such that, for any $x \in \mathbb{R}$,

$$\lambda \leq V''(x) \leq \frac{1}{\lambda}, \tag{4.13}$$

but one can make sense of the Gibbs measure (4.1) under much less restrictive assumptions on the potential V, for instance, it is sufficient to assume that the function $x \mapsto e^{-V(x)}$ is bounded and integrable over $\mathbb{R}$. It is thus an interesting question to investigate to what extent the results mentioned above can be extended to potentials V which do not satisfy the uniform ellipticity assumption (4.13). A long-standing prediction asserts that the behaviour of the $\nabla\varphi$ interface model should be similar to the one of the Gaussian free field for a large class of potentials, and a number of results have been obtained in this line of research (some of them have been discussed in Section 4.1.2). We discuss here one of them which is the subject of [76].

Specifically, the article [76] is concerned with the potentials $V : \mathbb{R} \to \mathbb{R}$ satisfying the following assumptions (see Figure 11):

(i) *Regularity and convexity:* we assume that V is twice-continuously differentiable and convex;
(ii) *Growth of the second derivative:* we assume that the second derivative of V satisfies a power-law growth condition: there exist an exponent $r > 2$ and two constants $c_+, c_- \in (0, \infty)$ such that

$$0 < c_- \leq \liminf_{|x| \to \infty} \frac{V''(x)}{|x|^{r-2}} \leq \limsup_{|x| \to \infty} \frac{V''(x)}{|x|^{r-2}} \leq c_+ < \infty.$$

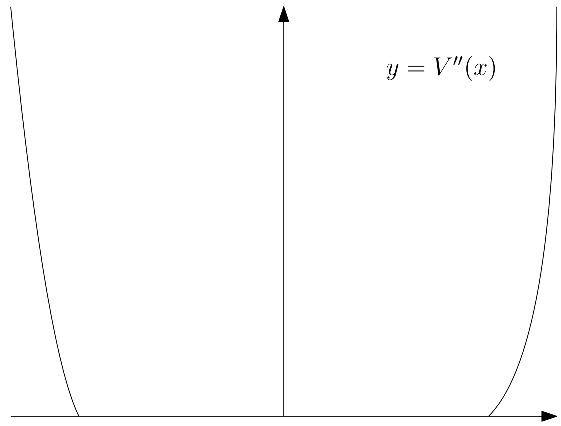

FIGURE 11. An example for the second derivative of the potential V under Assumptions (i) and (ii).

This assumption relaxes both the upper and lower bounds in (4.13) by allowing the second derivative of V to vanish on a finite but arbitrarily large interval, and imposing that it eventually grows like a polynomial (typical examples of potentials satisfying the assumptions above are the functions $x \mapsto |x|^p$ for $p > 2$). In [76], I investigated the question of the localisation/delocalisation of the interface.

In order to state the result, we have to introduce a periodic version of the finite-volume random interfaces introduced in Section 4.1. Specifically, we introduce:

- The discrete torus $\mathbb{T}_L := (\mathbb{Z}/(2L+1)\mathbb{Z})^d$, and write $x \sim y$ to mean that $x, y \in \mathbb{T}_L$ are neighbours in $\mathbb{T}_L$,
- The space of of periodic functions $\Omega_L^\circ := \left\{ \varphi : \mathbb{T}_L \to \mathbb{R} : \sum_{x \in \mathbb{T}_L} \varphi(x) = 0 \right\}$ and denote by $\mathrm{d}\varphi$ the Lebesgue measure on Ω_L°,
- The Gibbs measure $\mu_{\mathbb{T}_L}$ which is the probability distribution on the space Ω_L° given by the formula (N.B. this amounts to imposing a periodic boundary condition instead of the Dirichlet boundary condition)

$$\mu_{\mathbb{T}_L}(d\varphi) := \frac{1}{Z_{\mathrm{per},L}} \exp\left(- \sum_{\substack{x,y \in \mathbb{T}_L \\ x \sim y}} V\left(\varphi(x) - \varphi(y)\right) \right) \mathrm{d}\varphi.$$

Theorem 11 (Localisation and delocalisation [76])**.** *Under the Assumptions (i) and (ii), there exists a constant $C := C(d, V) < \infty$ such that, for any $L \geq 2$,*

$$\mathrm{Var}_{\mu_{\mathbb{T}_L}}\left[\varphi(0)\right] \leq \begin{cases} CL & \text{if } d = 1 \\ C \ln L & \text{if } d = 2, \\ C & \text{if } d \geq 3. \end{cases}$$

Remark 4.4. Matching lower bounds are known: the result is easily obtained in dimension 1 (where the question amounts to studying a sum of independent random variables) and in dimensions 3 and higher. In dimension 2, it can be deduced from an adaptation of the Mermin-Wagner argument (see e.g., [150]).

Some elements of proof :

The strategy follows the one presented in Section 4.1.1 and makes use of the Helffer-Sjöstrand representation formula (N.B. The formula was stated for uniformly convex potentials and with Dirichlet boundary condition, but a similar result holds in the setting of Theorem 11). The main difficulty is that, since the second derivative of V can vanish, the environment $\mathbf{a}(t,e) = V''(\nabla\varphi_t(e))$ can also vanish and the Nash-Aronson estimate stated in Proposition 4.2 cannot be applied. The strategy is then to rely on the article of Mourrat and Otto [154] which provides a sufficient condition on a random and degenerate environment $\mathbf{a}$ for the Nash-Aronson estimate to hold. Specifically, their result is the following (N.B. their statement has been adapted to the periodic setting).

Theorem 12 (Anchored Nash estimate [154])**.** *Fix an integer $L \in \mathbb{N}$ and let $\mathbf{a} : E(\mathbb{T}_L) \times [0,\infty) \to [0,1]$ be a random environment whose law is invariant under space and time translations and the symmetries of the torus. Then there exists an exponent $p \in [1,\infty]$ and a constant $C < \infty$ such that, if*

$$\mathbb{E}\left[\left(\int_t^\infty \frac{\mathbf{a}(s,e)}{(1+(s-t))^4}\,ds\right)^{-p}\right] < \infty, \tag{4.14}$$

then

$$\mathbb{E}\left[P_{\mathbf{a}}(t,0)\right] \leq \frac{C}{t^{d/2}} \exp\left(-\frac{t}{CL^2}\right).$$

Remark 4.5. The assumptions on the law of the environment imply that the left-hand side of (4.14) does not depend on the value of the time $t \in (0,\infty)$ and the edge $e \in E(\mathbb{T}_L)$.

Remark 4.6. A similar results holds when the conductance $\mathbf{a}$ can take values larger than 1 (with suitable assumptions on the probability of the conductance to be large)

The assumption (4.14) is typically satisfied for environments $\mathbf{a}$ which can take the value 0 but do not remain equal to 0 for a long time; this is for instance the case if the law of the environment satisfies a finite-range dependence assumption in the time variable.

The strategy is then to combine Theorem 12 with the Helffer-Sjöstrand representation formula so that the same computation as in (4.5) can be used to deduce Theorem 11. The main difficulty is to verify that, if we set $\mathbf{a} := V''(\nabla\varphi_t(e))$ where φ is the Langevin dynamic (4.4) (or, to be precise, the Langevin dynamic with periodic boundary conditions), then the assumption (4.14) is satisfied. This is the main result of [76], where it is shown that the gradient of the Langevin dynamic cannot remain in the compact set $\{x \in \mathbb{R} : V''(x) = 0\}$ for a very long time. Specifically, the following result is proved: for any $e \in E(\mathbb{T}_L)$ and any $T \geq 1$,

$$\mathbb{P}\left[\forall t \in [0,T] : V''(\nabla\varphi_t(e)) = 0\right] \leq C \exp\left(-c\left|\ln T\right|^{\frac{r}{r-2}}\right). \tag{4.15}$$

The inequality (4.15) asserts that the probability that the environment $\mathbf{a}(t,e) := V''(\nabla\varphi_t(e))$ remains equal to 0 for a time longer than T decays super-polynomially fast in T. Once this estimate is established, the inequality (4.14) can be fairly easily verified.

The core of the argument is thus to show (4.15). As in Section 4.1.1, the proof of this inequality is obtained by differentiating the Langevin dynamic with respect to the suitable parameter; specifically, with respect to the noise driving the dynamic. To give a few more details, we introduce the following decomposition: for the times $t \in [0,1]$, we decompose the Brownian motion $B(0)$ into a sum of a Gaussian random variable

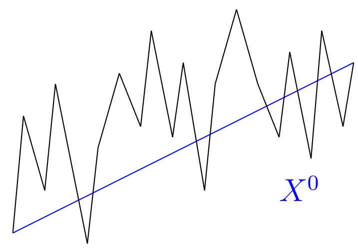

FIGURE 12. A realization of a Brownian motion (in black) and an increment (in blue).

(referred to as the increment) and a Brownian bridge, i.e.,

$$B_t(0) := \underbrace{tB_1(0)}_{X_1} + \underbrace{B_t(0) - tB_1(0)}_{W_t},$$

where the random variable X_1 is independent of the Brownian bridge W. We then compute the derivative of the random variable $\varphi_1(0)$ with respect to the increment X_1. An explicit computation yields the identity

$$\frac{\partial \varphi_1(0)}{\partial X_1} = \int_0^1 P_{\mathbf{a}}(1, 0; s)\, ds,$$

where we denote by $P_{\mathbf{a}}(\cdot, \cdot; s)$ the same heat kernel as in (4.3), except that we start the equation from the discrete Dirac δ_0 at time $s > 0$ instead of time 0. It is then possible to prove that this derivative is larger than a strictly positive real number, which implies that the function $\varphi_1(0)$ is sensitive to the increment X_1. We may then iterate the argument by fixing an integer $k \in \mathbb{N}$ and differentiating the random variables $\varphi_1(0), \ldots, \varphi_k(0)$ with respect to the increments $X = B_1(0)$, $X_2 = B_2(0) - B_1(0)$, ..., $X_k = B_k(0) - B_{k-1}(0)$ respectively. Using that the increments are independent random variables, we may deduce that the probability that the dynamic $\varphi_t(0)$ remains in the compact set $\{x \in \mathbb{R} : V''(x) = 0\}$ for a long time is small. An adaptation (to treat the discrete gradient of the dynamic) and a quantification of this argument yields the estimate (4.15).

4.4. **Ongoing projects and perspective.** Various directions can be investigated on the $\nabla\varphi$ interface model, especially regarding the extensions of the theory developed in the uniformly convex setting to more general potentials. In this direction, in the article [76] upper bounds for the localisation/delocalisation of the interface were established for potentials satisfying the Assumptions (i) and (ii) above. In an ongoing project, I am investigating the hydrodynamic limit for the potentials satisfying these assumptions. Follow-up questions could be to establish a large deviation principle for the Gibbs measure, identify the scaling limit, the decay of the covariance of the field and of the discrete gradient of the field. More generally, the problems discussed in [108] can be investigated for the potentials satisfying Assumptions (i) and (ii) (e.g., entropic repulsion, pinning and wetting transitions etc.).

5. Spin systems

5.1. **General definitions.** The third line of research is the study of spin systems. A general spin system in finite volume is defined through the three quantities:

- *A base space:* we consider a box $\Lambda_L \subseteq \{-L, \ldots, L\}^d \subseteq \mathbb{Z}^d$ and denote its outer boundary by $\partial\Lambda_L := \Lambda_{L+1} \setminus \Lambda_L$.
- *A spin space:* A spin space is a set equipped with a σ-algebra and a measure. We will consider either a finite set equipped with the counting measure, or the circle $\mathbb{S}^1$ equipped with the uniform measure, or the real line $\mathbb{R}$ equipped with the Lebesgue measure λ; in all of the cases, we will denote the spin space by $\mathcal{S}$ and the measure by κ. A configuration is a function $\sigma : \Lambda_{L+1} \to \mathcal{S}$. A boundary condition $\tau : \partial\Lambda_L \to \mathcal{S}$ can be prescribed by restricting our attention to the configurations σ which are equal to τ on the external boundary $\partial\Lambda_L$.
- *A Hamiltonian and an inverse temperature:* We consider a function H_{Λ_L} defined on the space of configurations $\sigma : \Lambda_{L+1} \to \mathcal{S}$ and valued in $\mathbb{R}$, assume that this map is bounded, and consider an inverse temperature $\beta > 0$.

We equip the space of configurations $\sigma : \Lambda_{L+1} \to \mathcal{S}$ with a probability measure (also called Gibbs measure) given by the formula

$$\mu_{\Lambda_L,\beta}(d\sigma) := \frac{1}{Z_{\Lambda_L,\beta}} \exp\left(-\beta H_{\Lambda_L}(\sigma)\right) \prod_{v \in \Lambda_{L+1}} \kappa(d\sigma_v) \tag{5.1}$$

where $Z_{\Lambda_L,\beta}$ is the normalization constant chosen so that $\mu_{\Lambda_L,\beta}$ is a probability distribution (N.B. in the case when the measure κ is not a finite measure, we only consider Hamiltonians H_{Λ_L} so that (5.1) is a finite measure). The measure (5.1) is referred to as the Gibbs measure with free boundary condition.

In order to prescribe a boundary condition $\tau : \partial\Lambda_L \to \mathcal{S}$ to the system, we will consider the Gibbs measure on the space of configurations with boundary condition $\tau : \partial\Lambda_L \to \mathcal{S}$,

$$\mu^\tau_{\Lambda_L,\beta}(d\sigma) := \frac{1}{Z^\tau_{\Lambda_L,\beta}} \exp\left(-\beta H_{\Lambda_L}(\sigma)\right) \prod_{v \in \Lambda_L} \kappa(d\sigma_v),$$

where $Z^\tau_{\Lambda_L,\beta}$ is the normalization constant.

This fairly general formalism has the advantage of containing a number of statistical physics models. We present below three important examples of spin systems: the Ising model, the XY model and the Villain model.

5.2. **Three examples of spin systems: the Ising, XY and Villain models.**

5.2.1. *Definitions.* We specify the previous (general) definition to three important models in statistical mechanics (see Figures 13 and 14):

- *The Ising model:* We consider the spin space $\mathcal{S} = \{1, -1\}$ and let $\kappa := \delta_1 + \delta_{-1}$ be the counting measure. The Gibbs measure with free boundary condition is given by the formula

$$\mu^{\mathrm{Is}}_{\Lambda_L,\beta}(d\sigma) := \frac{1}{Z^{\mathrm{Is}}_{\Lambda_L,\beta}} \exp\left(\beta \sum_{\substack{x,y \in \Lambda_{L+1} \\ x \sim y}} \sigma_x \cdot \sigma_y\right) \prod_{v \in \Lambda_L} \kappa(d\sigma_v).$$

We denote by $\langle \cdot \rangle^{\mathrm{Is}}_{\Lambda_L,\beta}$ the expectation with respect to the measure $\mu^{\mathrm{Is}}_{\Lambda_L,\beta}$.

- *The XY model:* We consider the spin space $\mathcal{S} = \mathbb{S}^1$ and let κ be the uniform measure on the circle (simply denoted by $d\sigma_x$ below). The Gibbs measure with free boundary condition is given by the formula

$$\mu^{\mathrm{XY}}_{\Lambda_L,\beta}(d\sigma) := \frac{1}{Z^{\mathrm{XY}}_{\Lambda_L,\beta}} \exp\left(\beta \sum_{\substack{x,y\in\Lambda_{L+1}\\ x\sim y}} \sigma_x \cdot \sigma_y\right) \prod_{x\in\Lambda} d\sigma_x. \tag{5.2}$$

We denote by $\langle \cdot \rangle^{\mathrm{XY}}_{\Lambda_L,\beta}$ the expectation with respect to the measure $\mu^{\mathrm{XY}}_{\Lambda_L,\beta}$.

- *The Villain model:* We consider the spin space $\mathcal{S} = \mathbb{S}^1$ and identify it with the interval $[0,2\pi)$ through the bijection $\theta \in [0,2\pi) \mapsto e^{i\theta} \in \mathbb{S}^1$. We let κ be the Lebesgue measure on $[0,2\pi)$ (and simply denote it by $d\theta_x$ below). Given an inverse temperature $\beta > 0$, we denote by $v_\beta : [0,2\pi) \mapsto (0,\infty)$ the heat kernel on the circle defined by

$$v_\beta(\theta) := \sum_{n\in\mathbb{Z}} \exp\left(-\frac{\beta}{2}\left(\theta - 2\pi n\right)^2\right).$$

The Gibbs measure with free boundary condition is given by the formula

$$\mu^{\mathrm{Vil}}_{\Lambda_L,\beta}(d\theta) := \frac{1}{Z^{\mathrm{Vil}}_{\Lambda_L,\beta}} \prod_{\substack{x,y\in\Lambda_{L+1}\\ x\sim y}} v_\beta(\theta_x - \theta_y) \prod_{x\in\Lambda_{L+1}} d\theta_x. \tag{5.3}$$

We denote by $\langle \cdot \rangle^{\mathrm{Vil}}_{\Lambda_L,\beta}$ the expectation with respect to the measure $\mu^{\mathrm{Vil}}_{\Lambda_L,\beta}$.

5.2.2. *Phase transitions.* A fundamental feature of these models is that they exhibit a phase transition in dimension $d \geq 2$ as discussed below. We first fix an integer $L \in \mathbb{N}$, consider the (finite-volume) *two-point function* defined for the Ising model as follows

$$\left|\begin{array}{l} \Lambda_L \to [-1,1], \\ \quad x \mapsto \langle \sigma_0 \sigma_x \rangle^{\mathrm{Is}}_{\Lambda_L,\beta}. \end{array}\right.$$

In the case of the XY and Villain model, this function is defined as follows

$$\text{XY:}\left|\begin{array}{l} \Lambda_L \to [-1,1], \\ \quad x \mapsto \langle \sigma_0 \cdot \sigma_x \rangle^{\mathrm{XY}}_{\Lambda_L,\beta}. \end{array}\right. \qquad \text{Villain:}\left|\begin{array}{l} \Lambda_L \to [-1,1], \\ \quad x \mapsto \langle \cos(\theta_0 - \theta_x) \rangle^{\mathrm{Vil}}_{\Lambda_L,\beta}. \end{array}\right.$$

Below we set out two properties of the correlation function. These properties are stated for the Ising model (for notational convenience), but also hold for the XY and Villain models. They are direct consequences of the correlation inequalities of Griffith [123] and Ginibre [113]:

- For each $L \in \mathbb{N}$, each $x \in \Lambda_L$ and each $\beta \geq 0$, $\langle \sigma_0 \sigma_x \rangle^{\mathrm{Is}}_{\Lambda_L,\beta} \geq 0$,
- For each $x \in \mathbb{Z}^d$ and each $\beta \geq 0$, the sequence $(\langle \sigma_0 \sigma_x \rangle^{\mathrm{Is}}_{\Lambda_L,\beta})_{L\in\mathbb{N}}$ is increasing and thus converges as L tends to infinity to a real number which we denote by $\langle \sigma_0 \sigma_x \rangle^{\mathrm{Is}}_{\beta}$. We refer to this quantity as *the two-point function*.

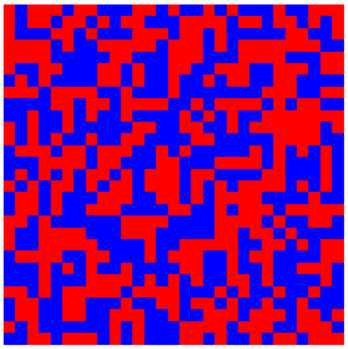

FIGURE 13. A configuration for the Ising model (the values "+1" are depicted in red and the values "-1" are in blue).

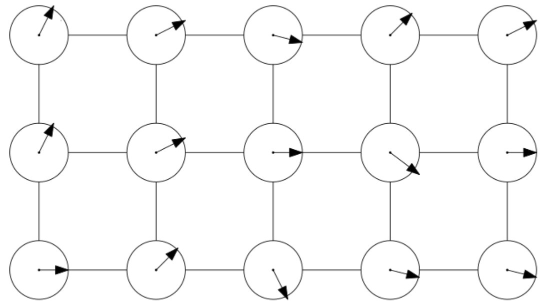

FIGURE 14. A configuration for the XY and/or Villain model.

A major research effort was then devoted to studying the asymptotic behaviour of the two-point function $x \mapsto \langle \sigma_0 \sigma_x \rangle_\beta^{\mathrm{Is}}$ (the latter may depend on the dimension and the inverse temperature β). Its behaviour is summarised in the following table where c represents a strictly positive constant which is allowed to depend on the dimension d and the inverse temperature β.

$\langle \sigma_0 \sigma_x \rangle_\beta^{\mathrm{Is}}$	$d = 1$	$d \geq 2$
$\beta \gg 1$	$e^{-c\lvert x\rvert}$	c
$\beta \ll 1$	$e^{-c\lvert x\rvert}$	$e^{-c\lvert x\rvert}$

In particular, the model exhibits a phase transition in dimensions 2 and higher between a low temperature phase where the two-point function is always strictly larger than a positive constant (this is known as *long-range order*) and a high temperature phase where the two-point function decays exponentially fast (see Figures 15 and 16).

In the case of the XY and Villain models, the situation is different, and the two-point function decays according to the following table (N.B. the behaviour of two-point functions of the two models follows the same pattern)

$\langle \sigma_0 \cdot \sigma_x \rangle_\beta^{\mathrm{XY}}$	$d = 1$	$d = 2$	$d \geq 3$
$\beta \gg 1$	$e^{-c\lvert x\rvert}$	$\lvert x\rvert^{-c}$	c
$\beta \ll 1$	$e^{-c\lvert x\rvert}$	$e^{-c\lvert x\rvert}$	$e^{-c\lvert x\rvert}$

$\langle \sigma_0 \cdot \sigma_x \rangle_\beta^{\mathrm{Vil}}$	$d = 1$	$d = 2$	$d \geq 3$
$\beta \gg 1$	$e^{-c\lvert x\rvert}$	$\lvert x\rvert^{-c}$	c
$\beta \ll 1$	$e^{-c\lvert x\rvert}$	$e^{-c\lvert x\rvert}$	$e^{-c\lvert x\rvert}$

In particular, the two-point function converges to 0 (as $\lvert x\rvert \to \infty$) at every temperature in two dimensions, but the rate of convergence depends on the temperature: a polynomial decay is observed at low temperature and an exponential decay at high temperature. This phenomenon is known as the Berezinskii–Kosterlitz–Thouless (BKT) transition (see Figures 17 and 18). We add below some bibliographical details concerning the previous results:

- The exponential decay of the two-point function at high temperature is a standard phenomenon in statistical physics (generalisable to many spin systems). We refer, for example, to Dobrushin's argument [92].

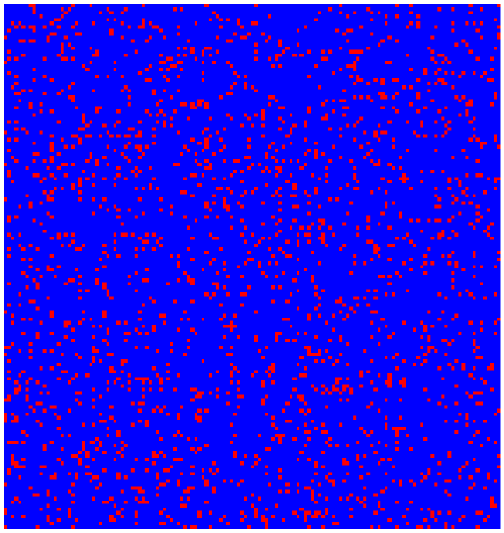

FIGURE 15. Low temperature Ising model.

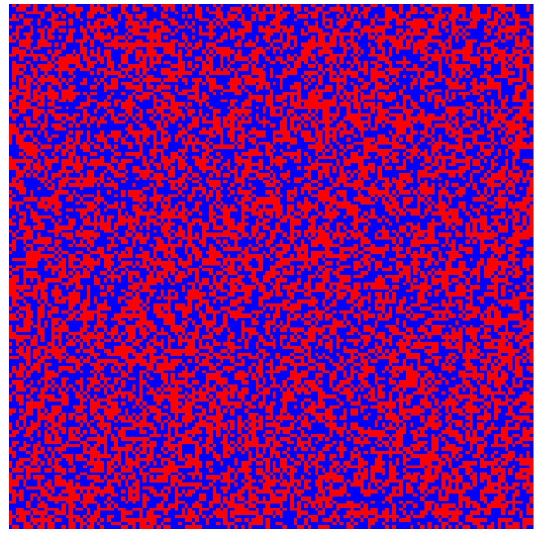

FIGURE 16. High temperature Ising model.

- In the case of the Ising model, the long-range order at low temperature is (essentially) a consequence of Peierls argument [159].
- As mentioned above, the phase transition in dimension $d = 2$ for the XY and Villain models is of a specific nature and is known as the Berezinskii-Kosterlitz-Thouless (BKT) transition. We give in the following paragraph a more detailed (although not exhaustive) account of the literature on this question. The theorem of Mermin-Wagner [150] and the result of McBryan and Spencer [149] show that the correlation function decreases at least like the inverse of a polynomial at any temperature. Fröhlich and Spencer [106] obtained the existence of a polynomial lower bound at low temperature for the two-point function, thus showing the existence of the BKT phase transition for this model. New proofs and generalisations of this result have recently been obtained in a series of works by Lammers [143, 142, 140], Lammers and Ott [141], van Engelenburg and Lis [168, 169] as well as Aizenman, Harel, Peled and Shapiro [5]. To go further in the understanding of the XY model, an interesting question is the one of the identification of the effective exponent which encodes the decay of the two-point function at low temperature; in this direction we mention the recent work of Bauerschmidt, Park and Rodriguez [39, 40] who identified the scaling limit of a related model, the high temperature discrete Gaussian free field.
- In dimension $d \geq 3$, the existence of an ordered phase at low temperature was demonstrated by Fröhlich, Simon and Spencer in [105]. Other proofs were subsequently obtained, notably by Kennedy and King [135] and Garban and Spencer [111].

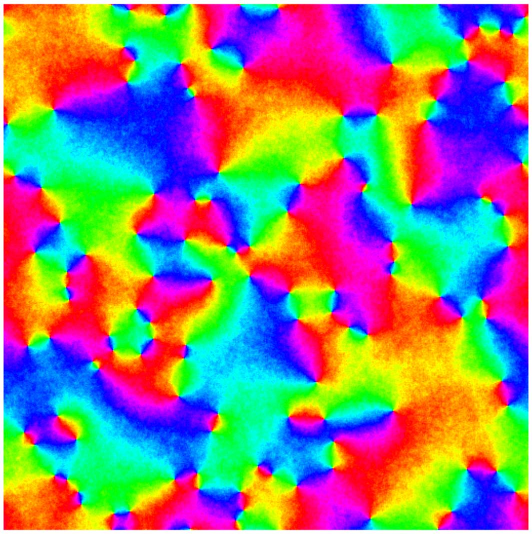

Figure 17. Low temperature XY/Villain model.

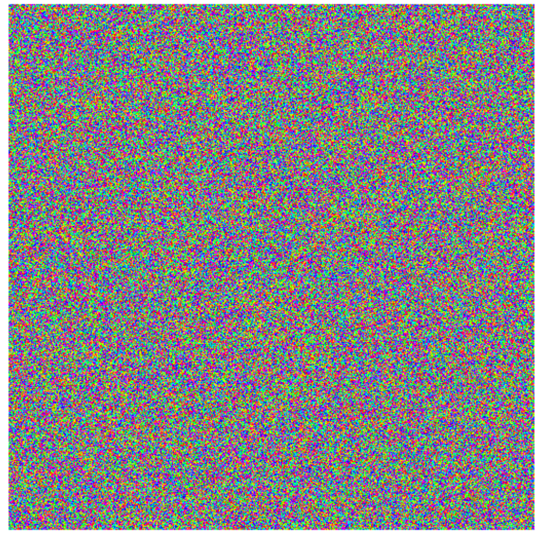

Figure 18. High temperature XY/Villain model.

Figure 19. Simulations from https://www.ibiblio.org/e-notes/Perc/xy.htm

5.3. **Massless phases for the Villain model.** In this section, we will be interested in the Villain model (see (5.3)) in dimension $d \geq 3$ and in the low temperature regime ($\beta \gg 1$). Under these assumptions, the results of [105, 135, 111] imply that the two-point function is always larger than a strictly positive number, i.e., if we set

$$c_\beta := \inf_{x \in \mathbb{Z}^d} \langle \sigma_0 \cdot \sigma_x \rangle_\beta^{\mathrm{Vil}}$$

then $c_\beta > 0$ (N.B. it can be shown that this infimum converges to 1 as $\beta \to \infty$). In fact, one can prove that the two-point function $x \mapsto \langle \sigma_0 \cdot \sigma_x \rangle_\beta^{\mathrm{Vil}}$ converges to c_β as $|x| \to \infty$ (this is a consequence of the Messager-Miracle-Sole correlation inequality [151]). In the article [81] in collaboration with W. Wu, we investigated the rate of convergence of the two-point function to the value c_β; the *spin-wave conjecture* [102, 150] postulates that the difference $\langle \sigma_0 \sigma_x \rangle_\beta^{\mathrm{Vil}} - c_\beta$ decays like (a suitable multiple of) the Green's function, i.e., that there exists a constant $\overline{c}_\beta > 0$ depending on d and β such that

$$\langle \sigma_0 \cdot \sigma_x \rangle_\beta^{\mathrm{Vil}} = c_\beta + \frac{\overline{c}_\beta}{|x|^{d-2}} + o\left(\frac{1}{|x|^{d-2}} \right).$$

We proved this result when the temperature is sufficiently low. Specifically, we obtained the following theorem.

Theorem 13 ([81])**.** *Fix a dimension $d \geq 3$. Then there exists an inverse temperature $\beta_0 := \beta_0(d) \gg 1$ such that for any $\beta \geq \beta_0$, there exists a constant $\overline{c}_\beta > 0$ and an exponent $\alpha := \alpha(d) \in (0,1)$ such that*

$$\langle \sigma_0 \cdot \sigma_x \rangle_\beta^{\mathrm{Vil}} = c_\beta + \frac{\overline{c}_\beta}{|x|^{d-2}} + O\left(\frac{1}{|x|^{d-2+\alpha}} \right). \tag{5.4}$$

Remark 5.1. To be precise, the result is proved in [81] for the thermodynamic limit of the Villain model with *wired* boundary conditions (i.e., we impose $\theta_x = 0$ on the boundary of the box Λ_L in (5.3) and then take the limit $L \to \infty$ to compute the two-point function).

Some elements of proof :

The proof starts from the work of Fröhlich and Spencer [107] (see also [38]) who observed that the Villain model can be mapped, by duality, to a model of lattice Coulomb gas. In the low temperature regime, it is possible to implement a cluster expansion to obtain the following identity for the two-point function

$$\langle \sigma_0 \cdot \sigma_x \rangle_\beta^{\mathrm{Vil}} = \left\langle e^{i(\varphi(0)-\varphi(x))} \right\rangle_{\mathrm{GFF}} \langle X_x \rangle_{\mu_\beta} , \tag{5.5}$$

where the terms on the right-hand side have the following definitions:

- In the first term on the right-hand side, the random variable φ is a Gaussian free field, i.e., the limit $L \to \infty$ of the measure (4.1) with $V(x) = \beta x^2/2$ (this limit is well-defined in dimension $d \geq 3$). This term can be computed explicitly and we have
$$\left\langle e^{i(\varphi(0)-\varphi(x))} \right\rangle_{\mathrm{GFF}} = e^{-\frac{1}{\beta}(G(0)-G(x))}$$
where $G(x)$ is the discrete Green's function on the lattice $\mathbb{Z}^d$.
- The second term is more technically involved and contains two terms which are briefly described below:
  - The measure μ_β is a measure of the form (4.1) (in particular, it belongs to the class of random interfaces and is not a spin system valued in the circle $\mathbb{S}^1$). This allows to use the techniques introduced in Section 4, and in particular the Helffer-Sjöstrand representation formula, to study it.
  - The random variable X_x is a technically involved function of the random interface whose explicit formula is not made explicit here (see [81, Proposition 3.10]) but which can still be analysed using the representation formula mentioned above.

It is thus possible to analyse both terms on the right-hand side of (5.5), using either explicit computations or the Helffer-Sjöstrand representation formula (combined with quantitative homogenization techniques) to derive the expansion (5.4).

5.4. **Disordered spin systems I: the Imry-Ma phenomenon.** In this section, we discuss an example of *disordered spin systems*: the general philosophy of this topic is to incorporate a random disorder to the spin system and to investigate if (and how) the properties of the spin system are modified by the addition of the disorder. The incorporation of a disorder can be done in various ways and two possibilities (among many others) are presented in this thesis: we either add a random external field to the spin systems (in this section) or study their behaviour when the base space is a random graph (in Section 5.5). In all this section, we will use the general notation introduced in Section 5.1.

To model the addition of a random external field, we consider a function $\eta : \Lambda_L \to \mathbb{R}$, an observable $f : \mathcal{S} \to \mathbb{R}$, a parameter $\varepsilon > 0$. We define the disordered measure on the space of configurations with boundary condition $\tau : \partial\Lambda_L \to \mathcal{S}$ by the formula

$$\mu_{L,\beta}^{\eta,\tau} := \frac{1}{Z_{\Lambda_L,\beta}^{\eta,\tau}} \exp\left(-\beta H_{\Lambda_L}(\sigma) + \varepsilon \sum_{v \in \Lambda_L} \eta_v f(\sigma_v) \right) \prod_{v \in \Lambda_L} \kappa(d\sigma_v), \tag{5.6}$$

and denote by $\langle \cdot \rangle_{\Lambda_L}^\tau$ the expectation with respect to the measure (5.6) (N.B. this quantity depends on the disorder η). We typically assume that the disorder $(\eta_v)_{v \in \mathbb{Z}^d}$ is a family of independent and identically distributed random variables of variance 1, and encode the strength of the disorder by the parameter ε. We use the notation $\mathbb{E}$ to refer to the expectation with respect to the disorder η.

The first works on this subject were those of Imry and Ma [131] in 1975, who predicted that the addition of a random magnetic field leads to the disappearance of phase transitions in $d \leq 2$ dimensions for general spin systems for any value of the disorder strength ε. This result was rigorously established by Aizenman and Wehr in the articles [10] and [11]. Their demonstration is based on ergodicity arguments and is therefore qualitative. The question of the quantification of the Imry-Ma phenomenon has given rise to significant progress in the case of the Ising model in the presence of a random magnetic field in dimension 2 [64, 89, 90, 8, 4]; the best results currently known establish an exponential decay in the length L of the effect of the boundary condition on the magnetization $m_L^\tau := \langle \sigma_0 \rangle_{\Lambda_L}^\tau$ at the centre of the box Λ_L. They were initially obtained by Ding and Xia [89] at zero temperature and then extended by Ding and Xia [90] as well as by Aizenman, Harel, Peled [4] to any positive temperature. The proofs are based on the existence of correlation inequalities for the Ising model with a random field, and thus leave open the question of the quantification of the Imry-Ma phenomenon for general spin systems. This is the subject of the two articles [80] and [79] presented below.

5.4.1. *General spin systems.* In the article [79], we assume that the law of the disorder η is Gaussian. We obtain a quantitative result, and show that the effect of the boundary condition on the spatially averaged magnetization

$$M_L^\tau := \frac{1}{|\Lambda_L|} \sum_{v \in \Lambda_L} \langle f(\sigma_v) \rangle_{\Lambda_L}^\tau$$

in the box Λ_L in dimension 2 decays faster than the map $L \mapsto (\ln \ln L)^{-\frac{1}{4}}$. The main theorem of the article [79] is stated below.

Theorem 14 ([79])**.** *Let $\beta > 0$ be an inverse temperature and $\lambda > 0$ be a disorder strength. Then there exists a constant $C > 0$ such that, for any $L \geq 3$,*

$$\mathbb{E}\left[\sup_{\tau_1, \tau_2 \in \mathcal{S}^{\partial \Lambda_L}} \left| \frac{1}{|\Lambda_L|} \sum_{v \in \Lambda_L} \left(\langle f_v(\sigma) \rangle_{\Lambda_L}^{\tau_1} - \langle f_v(\sigma) \rangle_{\Lambda_L}^{\tau_2} \right) \right| \right] \leq \frac{C}{\sqrt[4]{\ln \ln L}}.$$

Remark 5.2. It is known that a result of this nature does not hold in dimension $d \geq 3$ as the random-field Ising model is known to exhibit a phase transition in the presence of a weak disorder [57, 91, 88].

It is also an interesting problem to study the behaviour of the XY and Villain models in the presence of a random external field. In that case we make the following modifications to the formalism introduced above:

- We assume that $f(\sigma) = \sigma \in \mathbb{S}^1 \subseteq \mathbb{R}^2$;
- We assume that η is a Gaussian disorder valued in $\mathbb{R}^2$, i.e., for any $x \in \mathbb{Z}^d$, $\eta_x = (\eta_x^1, \eta_x^2) \in \mathbb{R}^2$ where η_x^1 and η_x^2 are independent Gaussian random variables;
- We replace the Hamiltonian H_{Λ_L} by the Hamiltonian of the XY (or Villain) model in (5.6), and we replace the term $\eta_v f(\sigma_v)$ by the scalar product $\eta_v \cdot \sigma_v$.

Under these assumptions, the qualitative behaviour of the system is different from the one described above (as it was the case without disorder) and the critical dimension under which the Imry-Ma phenomenon manifests itself is increased from $d = 2$ to $d = 4$.

The following theorem provides a quantitative estimate on the effect of the boundary condition on the spatially averaged magnetization.

Theorem 15 ([79])**.** *Consider the XY (or Villain) model with a random field. Let $\beta > 0$ be an inverse temperature and $\lambda > 0$ be a disorder strength. Then there exists a constant $C > 0$ such that, for any $L \geq 3$:*

- *If $d \in \{1,2,3\}$, then*
$$\left| \mathbb{E}\left[\sup_{\tau \in \mathcal{S}^{\partial \Lambda_L}} \frac{1}{|\Lambda_L|} \sum_{v \in \Lambda_L} \langle \sigma_v \rangle_{\Lambda_{2L}}^{\tau} \right] \right| \leq C L^{-\frac{4-d}{2(8-d)}}.$$
- *If $d = 4$, then*
$$\mathbb{E}\left[\sup_{\tau \in \mathcal{S}^{\partial \Lambda_L}} \left| \frac{1}{|\Lambda_L|} \sum_{v \in \Lambda_L} \langle \sigma_v \rangle_{\Lambda_{2L}}^{\tau} \right| \right] \leq \frac{C}{\sqrt{\ln \ln L}}.$$

Some elements of proof :

The proof is based on a thermodynamic approach and requires the introduction of the finite volume free energy of the system defined as follows: for each integer $L \geq 2$, each boundary condition $\tau \in \mathcal{S}^{\partial \Lambda_L}$, inverse temperature $\beta > 0$, and magnetic field $\eta : \Lambda_L \to \mathbb{R}$, we define the finite volume free energy as the renormalised logarithm of the partition function
$$\mathrm{FE}_{\Lambda_L}^{\tau}(\eta) := -\frac{1}{\beta |\Lambda_L|} \ln Z_{\Lambda_L,\beta}^{\eta,\tau}.$$
The proof is then based on the following standard observations:

- We may decompose the random field $\eta = (\hat{\eta}_L, \eta_L^{\perp})$ with $\hat{\eta}_L := |\Lambda_L|^{-1} \sum_{v \in \Lambda_L} \eta_v$ and $\eta_L^{\perp} := \eta - \hat{\eta}_L$ and observe that the observable $|\Lambda_L|^{-1} \sum_{v \in \Lambda_L} \langle f_v(\sigma) \rangle_{\Lambda_L}^{\tau}$ can be characterised as the derivative of the free energy with respect to the quantity $\hat{\eta}_L$, i.e.,
$$\frac{\partial}{\partial \hat{\eta}_L} \mathrm{FE}_{\Lambda_L}^{\tau}(\eta) = -\frac{1}{|\Lambda_L|} \sum_{v \in \Lambda_L} \langle f_v(\sigma) \rangle_{\Lambda_L}^{\tau}.$$
- For each realization of $\eta_L^{\perp}$, the map $\hat{\eta}_L \mapsto -\mathrm{FE}_{\Lambda_L}^{\tau}(\hat{\eta}_L, \eta_L^{\perp})$ is convex, differentiable and 1-Lipschitz. Moreover, for any pair of boundary conditions $\tau_1, \tau_2 \in \mathcal{S}^{\partial \Lambda_L}$, one has the upper bound
$$\left| \mathrm{FE}_{\Lambda_L}^{\tau_1}(\eta) - \mathrm{FE}_{\Lambda_L}^{\tau_2}(\eta) \right| \leq \frac{C}{L}. \tag{5.7}$$
- The random variable $\hat{\eta}_L$ is a Gaussian random variable whose variance is equal to $|\Lambda_L|^{-1}$. In two dimensions, its fluctuations are of order L^{-1}, which is comparable to the right-hand side of (5.7).

By combining the three previous observations with a general property of 1-Lipschitz convex functions, we were able to obtain the following inequality:
$$\mathbb{P}\left(\sup_{\tau_1,\tau_2 \in \mathcal{S}^{\partial \Lambda_L}} \frac{1}{|\Lambda_L|} \sum_{v \in \Lambda_L} \left(\langle f_v(\sigma) \rangle_{\Lambda_L}^{\tau_1} - \langle f_v(\sigma) \rangle_{\Lambda_L}^{\tau_2} \right) < \delta \right) \geq c_\delta, \tag{5.8}$$
where the constant $c_\delta > 0$ depends only on δ.

The lower bound stated in (5.8) is weaker than the statement in Theorem 14, which is obtained by iterating (5.8) over the scales.

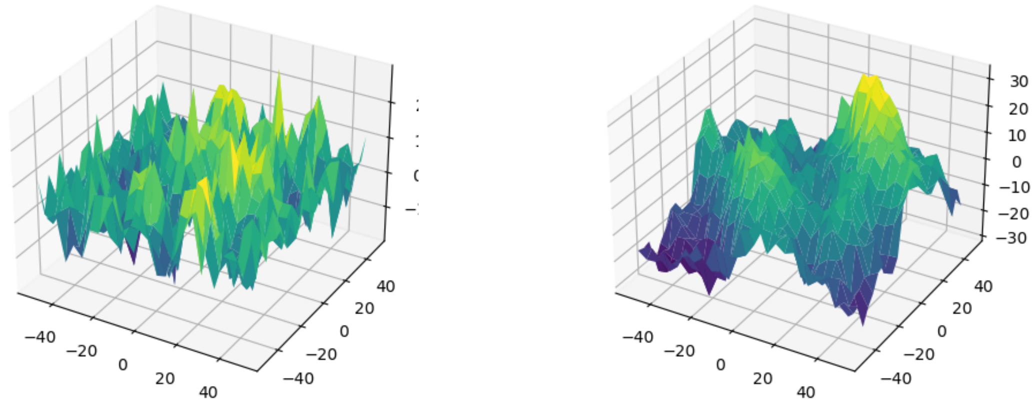

FIGURE 20. Left: a random interface without disorder. Right: a random interface with a random disorder.

5.4.2. *Random field random surfaces.* In another direction, in collaboration with M. Harel and R. Peled, we studied the behaviour of the $\nabla\varphi$ interface model in the presence of a random magnetic field. To define the model, we give ourselves an integer $L \in \mathbb{N}$, a function $\eta : \Lambda_L \to \mathbb{R}$, and define the Gibbs measure on the space of interfaces $\varphi : \Lambda_L \to \mathbb{R}$ satisfying the boundary condition $\varphi \equiv 0$ on $\partial\Lambda_L$,

$$\mu^{\eta}_{\Lambda_L}(d\varphi) := \frac{1}{Z^{\eta}_{\Lambda_L}} \exp\left(- \sum_{\substack{x,y\in\Lambda_{L+1} \\ x\sim y}} V(\varphi(x) - \varphi(y)) + \varepsilon \sum_{v\in\Lambda_L} \eta(v)\varphi(v) \right) \prod_{v\in\Lambda_L} d\varphi(v),$$

where $V : \mathbb{R} \to \mathbb{R}$ is a uniformly convex potential. This model has received some attention in the literature, notably by Külske and E. Orlandi [139] and van Enter and Külske [170] regarding the localisation/delocalisation of the interface and Cotar and Külske [70, 71] regarding the existence and uniqueness of translation-covariant gradient Gibbs measure, and more recently by Sagakawa [163] regarding the maximum of the interface.

In the article [80, 75], we investigated the question of the localisation/delocalisation of the random interface in the presence of a random disorder. By using concentration inequalities combined with elliptic regularity estimates, we were able to obtain upper and lower bounds, sharp up to a multiplicative constant, on the typical size of the interface in any dimension, showing in particular the delocalisation of the interface in dimension $d \le 4$ and its localisation in dimension $d \ge 5$. This is another occurrence of the Imry-Ma phenomenon as this behaviour is different from that of the model without random field described in Theorem 7. We also obtained similar estimates on the typical size of the discrete gradient of the interface in all dimensions.

Theorem 16 ([80])**.** *There exist two constants $C, c > 0$ such that*

$$\begin{aligned} 1 \le d \le 3: \quad & cL^{4-d} \le \mathbb{E}\left[\left\langle \varphi(0)^2 \right\rangle_{\mu^\eta_{\Lambda_L}} \right] \le CL^{4-d}, \\ d = 4: \quad & c \ln L \le \mathbb{E}\left[\left\langle \varphi(0)^2 \right\rangle_{\mu^\eta_{\Lambda_L}} \right] \le C \ln L, \\ d \ge 5: \quad & c \le \mathbb{E}\left[\left\langle \varphi(0)^2 \right\rangle_{\mu^\eta_{\Lambda_L}} \right] \le C. \end{aligned}$$

For the discrete gradient of the field (recalling the notation from the beginning of the manuscript), the situation also differs from the case without random field (where the variance of the discrete gradient is bounded in every dimension $d \ge 1$), and the following estimates have been proved in [80, 75].

Theorem 17 ([80, 75])**.** *There exist two constants $C, c > 0$ such that*

$$\begin{aligned} d = 1: \quad & cL \le \mathbb{E}\left[\frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \left\langle |\nabla \varphi(x)|^2 \right\rangle_{\mu^\eta_{\Lambda_L}} \right] \le CL, \\ d = 2: \quad & c \ln L \le \mathbb{E}\left[\frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \left\langle |\nabla \varphi(x)|^2 \right\rangle_{\mu^\eta_{\Lambda_L}} \right] \le C \ln L, \\ d = 3: \quad & c \le \mathbb{E}\left[\frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \left\langle |\nabla \varphi(x)|^2 \right\rangle_{\mu^\eta_{\Lambda_L}} \right] \le C, \\ d \ge 4: \quad & c \le \mathbb{E}\left[\left\langle |\nabla \varphi(0)|^2 \right\rangle_{\mu^\eta_{\Lambda_L}} \right] \le C. \end{aligned}$$

Remark 5.3. The spatially averaged estimates stated in dimensions $d = 1, 2, 3$ are strictly weaker than the pointwise estimate stated in dimension $d \ge 4$. It should be the case (but has not been proven) that a pointwise version of these upper and lower bounds holds.

Finally, in the article [75], I combined the results of [80] with the De Giorgi-Nash-Moser regularity in order to show that, in dimension $d \ge 5$, for almost any realisation of the disorder η, the sequence of measures $\mu^\eta_{\Lambda_L}$ converges in law, when L tends to infinity, towards a limiting measure (called the thermodynamic limit). Regarding the law of discrete gradient of the field, the same result holds in dimension $d \ge 4$.

To formally state the result, we introduce the notation:

- We denote by τ_y the translation by $y \in \mathbb{Z}^d$ and by $(\tau_y)_* \mu$ the image measure of μ by τ_y.
- For $L \in \mathbb{N}$ and $\eta : \Lambda_L \to \mathbb{R}$, we will denote by $\mu^\eta_{\nabla, \Lambda_L}$ the law of the discrete gradient of the interface $(\nabla \varphi(e))_{e \in E(\Lambda_L)}$ where φ is distributed according to $\mu^\eta_{\Lambda_L}$.

Theorem 18 ([75])**.** *Assume $d \ge 5$. Then, for almost every realisation of the disorder η, the sequence of measures $(\mu^\eta_{\Lambda_L})_{L \ge 0}$ converges weakly to an infinite volume translation-covariant measure μ^η, i.e., a measure satisfying, for any $y \in \mathbb{Z}^d$,*

$$\mu^{\tau_y \eta} = (\tau_y)_* \mu^\eta.$$

Assume $d \ge 4$. Then, for almost all realisations of the disorder η, the sequence of gradient measures $(\mu^\eta_{\nabla, \Lambda_L})_{L \ge 0}$ converges weakly to an infinite volume translation-covariant gradient measure μ^η_∇, i.e., a measure satisfying, for any $y \in \mathbb{Z}^d$,

$$\mu^{\tau_y \eta}_\nabla = (\tau_y)_* \mu^\eta_\nabla.$$

Remark 5.4. It is expected that the sequence of measures $(\mu^{\eta}_{\Lambda_L})_{L\geq 0}$ does not converge in dimensions $d = 1, 2, 3, 4$ (this result should essentially be a consequence of Theorem 16).

On the other hand, the gradient Gibbs measures $(\mu^{\eta}_{\nabla,\Lambda_L})_{L\geq 0}$ should converge in dimension $d = 3$ (and it is known that they do not converge in dimensions $d = 1, 2$ [170]). This question is closely related to the existence of a pointwise estimate on the discrete gradient of the field in dimension 3 discussed in Remark 5.3.

Some elements of proof :

To highlight the main ideas of proof of Theorem 16, we describe the strategy for the upper bounds in the case of the ground state of the disordered Hamiltonian (see below for the formal definition). In addition, we set the strength of the disorder ε to 1 and assume that the law of the disorder is Gaussian with expectation 0 and variance 1. The ground state is defined as the minimiser of the Hamiltonian

$$H^{\eta}_{\Lambda_L}(v) \coloneqq \sum_{x\sim y} V\left(v(x) - v(y)\right) - \sum_{x\in\Lambda_L} \eta(x)v(x)$$

among all the functions $v : \Lambda_L^+ \to \mathbb{R}$ which are equal to 0 on the boundary $\partial\Lambda_L$. We denote the minimiser of $H^{\eta}_{\Lambda_L}$ by $v_{L,\eta} : \Lambda_L^+ \to \mathbb{R}$; equivalently, it can be characterised as the unique solution of the discrete non-linear elliptic equation

$$\begin{cases} -\sum_{x\ni y} V'\left(v_{L,\eta}(x) - v_{L,\eta}(y)\right) = \eta(y) & \text{for } y \in \Lambda_L, \\ v_{L,\eta}(y) = 0 & \text{for } y \in \partial\Lambda_L. \end{cases} \tag{5.9}$$

We wish to estimate the variance (with respect to the random field η) of the random variable $v_{L,\eta}(0)$ and prove that it satisfies the bounds stated in Theorem 16.

The proof of the upper bounds is based on the Gaussian Poincaré inequality stated below

$$\mathrm{Var}\left[v_{L,\eta}(0)\right] \leq \sum_{x\in\Lambda_L} \mathbb{E}\left[\left(\frac{\partial v_{L,\eta}}{\partial \eta(x)}(0)\right)^2\right].$$

Consequently, to obtain the desired upper bounds, it is sufficient to prove the inequalities for any $x \in \Lambda_L$,

$$\mathbb{E}\left[\left(\frac{\partial v_{L,\eta}}{\partial \eta(x)}(0)\right)^2\right] \leq \begin{cases} CL^2 & d = 1, \\ C\left(\ln \dfrac{L}{(1+|x|)}\right)^2 & d = 2, \\ \dfrac{C}{(1+|x|)^{2d-4}} & d \geq 3. \end{cases} \tag{5.10}$$

To prove the upper bounds (5.10), we note that, by differentiating both sides of (5.9) with respect to $\eta(x)$, the derivative $w^x \coloneqq \frac{\partial v_{L,\eta}}{\partial \eta(x)}$ solves a discrete linear elliptic equation of the form

$$\begin{cases} -\nabla\cdot\mathbf{a}\nabla w^x = \delta_x & \text{in } \Lambda_L, \\ w^x = 0 & \text{on } \partial\Lambda_L, \end{cases}$$

with $\mathbf{a}(e) = V''\left(\nabla v_{L,\eta}(e)\right)$. The function w^x is thus equal to the Green's function associated with the environment $\mathbf{a}$ and satisfying a Dirichlet boundary condition on the boundary $\partial\Lambda_L$. Under the assumption that the environment $\mathbf{a}$ is uniformly elliptic, regularity estimates (e.g., the off-diagonal version of the Nash-Aronson inequality stated in Proposition 4.2) provides upper bounds on Green's function, the square of which is of the same order of magnitude as the right-hand side of (5.10).

5.5. **Disordered spin systems II: the XY model on a percolation cluster.** In the article [77], in collaboration with C. Garban, we studied the behaviour of the XY model when the underlying graph is not $\mathbb{Z}^d$ but the infinite cluster of a supercritical Bernoulli percolation (defined in Section 3 above). More precisely, given a dimension $d \geq 2$, we consider a supercritical probability $\mathfrak{p} > p_c(d)$ and a realisation of the infinite bond percolation cluster $\mathcal{C}_\infty$. We can then define the two-point function of the XY model on the infinite cluster as follows:

- For each integer $L \in \mathbb{N}$, we consider the intersection $\Lambda_L \cap \mathcal{C}_\infty$ of the infinite cluster with the box, and consider the two-point function $\langle \sigma_0 \cdot \sigma_x \rangle_{\Lambda_L \cap \mathcal{C}_\infty, \beta}$ (N.B. to define this two-point function, we use the same definition as in (5.2) except that, inside the exponential, we sum over the pairs of vertices of $\Lambda_L \cap \mathcal{C}_\infty$ which are neighbours in this graph).
- Using the same argument as the one mentioned above for the Ising model, the Ginibre correlation inequality [113] implies that the sequence $(\langle \sigma_0 \cdot \sigma_x \rangle_{\Lambda_L \cap \mathcal{C}_\infty, \beta})_{L \in \mathbb{N}}$ is increasing in L (this holds for any realization of the infinite cluster), and thus converges as L tends to infinity to a value which we denote by $\langle \sigma_0 \cdot \sigma_x \rangle_{\mathcal{C}_\infty, \beta}$.

A natural line of inquiry is to study the asymptotic behaviour of the two-point function for the XY model on the percolation cluster. In particular, it is interesting to know whether incorporating a random disorder into the model (in the form of a random underlying graph) changes its qualitative properties. The following theorem, which is the main result of [77], answers this question negatively.

Theorem 19 ([77])**.** *For the model XY in a random environment given by a Bernoulli bond percolation of parameter* $\mathfrak{p}$*, we have the following results:*

- *In dimension* $d = 2$*, for each* $\mathfrak{p} > p_c(2)$*, there exists an inverse temperature* $\beta_{BKT}(\mathfrak{p}) < \infty$*, such that, for each* $\beta \geq \beta_{BKT}(\mathfrak{p})$*, the function* $x \mapsto \mathbb{E}_{\mathfrak{p}}[\langle \sigma_0 \cdot \sigma_x \rangle_{\mathcal{C}_\infty, \beta}]$ *decays polynomially fast in* $|x|$*.*
- *In dimension* $d \geq 3$*, for each* $\mathfrak{p} > p_c(d)$*, there exists an inverse temperature* $\beta_c(d, \mathfrak{p}) < \infty$ *such that, for each* $\beta \geq \beta_c(d, \mathfrak{p})$*, the function* $x \mapsto \mathbb{E}_{\mathfrak{p}}[\langle \sigma_0 \cdot \sigma_x \rangle_{\mathcal{C}_\infty, \beta}]$ *remains bounded away from* 0*.*

Some elements of proof :

We give here a sketch of the proof of the result in the case of a highly supercritical site percolation cluster, i.e., we show the result for the model defined on a site percolation cluster when the probability $\mathfrak{p}$ is very close to 1 (a numerical application gives a probability $\mathfrak{p} \simeq 0.9998$ for the argument below in dimension $d = 2$).

The main idea is to transfer the a priori difficult analysis of the XY model on a lattice with *quenched* disorder to an *annealed* version of the XY model using an inequality called the Wells inequality [172]. This inequality was recently highlighted in the work of Madrid, Simon and Wells [144], but does not seem to have received much attention before that (except in [9, 58, 101]).

We do not state exactly Wells' inequality (for which we refer to [172, 58]) but we write below one of its almost immediate consequence: given an integer $L \in \mathbb{N}$ and an inverse temperature β, there exists an explicit probability distribution ν on the set of site percolation configurations $r \in \{0,1\}^{\Lambda_L}$ such that, for any pair of vertices $x \in \Lambda_L$,

$$\langle \sigma_0 \cdot \sigma_x \rangle_{\mu_{\Lambda_L,\beta/2}} \leq \mathbb{E}_\nu \left[\langle \sigma_0 \cdot \sigma_x \rangle_{\mu_{\Lambda_L,\beta,r}} \right], \tag{5.11}$$

where, given a site percolation configuration $r \in \{0,1\}^{\Lambda_L}$, $\langle \cdot \rangle_{\mu_{\Lambda_L,\beta,r}}$ is the expectation with respect to the measure

$$\mu_{\Lambda_L,\beta,r}(d\sigma) := \frac{1}{Z_{\Lambda_L,\beta,r}} \exp\left(\beta \sum_{\substack{x,y \in \Lambda_L \\ x \sim y}} r_x r_y \sigma_x \cdot \sigma_y \right) \prod_{x \in \Lambda_L} d\theta_x.$$

The exact formula for the probability measure ν is obtained by multiplying the i.i.d. Bernoulli percolation measure of parameter $\mathfrak{p} = 1/2$ by the partition function $Z_{\Lambda_L,\beta,r}$. Despite the apparent complexity of this definition, two observations can be deduced from this inequality:

- If the inverse temperature β is chosen sufficiently large, then the left-hand side of (5.11) decreases polynomially fast in $|x|$ (resp. remains far from 0 for $d \geq 3$).
- Using a standard criterion for stochastic domination by product measures, we can prove that the measure ν is stochastically dominated by an i.i.d. Bernoulli percolation measure with probability $\mathfrak{p}_0 := \mathfrak{p}_0(\beta) < 1$ (but close to 1). Combining this observation with Ginibre's correlation inequality (which implies that the two-point function is an increasing function of the underlying percolation configuration), we obtain
$$\mathbb{E}_\nu \left[\langle \sigma_0 \cdot \sigma_x \rangle_{\mu_{\Lambda,\beta,r}} \right] \leq \mathbb{E}_{\mathfrak{p}_0} \left[\langle \sigma_0 \cdot \sigma_x \rangle_{\mu_{\Lambda,\beta,r}} \right].$$

The combination of these two results implies the existence of a phase transition for the XY model on a highly supercritical percolation cluster.

5.6. **Ongoing projects and perspectives.** Regarding the disordered spin systems, an interesting open question is to improve the rate of convergence for general spin systems obtained in Theorem 14 and it is conjectured that the exponential decay should hold for a general class of spin systems (including, for instance, the q-states Potts model with an external field). It would also be an interesting question to study quantitatively other models of statistical physics in the presence of a random disorder such as the random interfaces studied by Bovier and Külske [54, 55].

Regarding the random field random surface model, a first interesting question would be to obtain pointwise estimates on the typical size of the discrete gradient of the interface in dimensions $d = 1, 2, 3$ and to prove the convergence of the finite volume gradient Gibbs measures in dimension $d = 3$. It would additionally be interesting to investigate the same questions as the ones presented in Section 4 for the $\nabla\varphi$ interface model and in particular to identify the hydrodynamic and scaling limits of this model.

Finally, various directions can be explored regarding the existence of phase transitions for spin systems on a supercritical percolation cluster, for instance the extension of Theorem 19 to the Villain model, or the existence of a roughening phase transition for the dual model: the integer-valued Gaussian free field in a random environment.

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