PSL UNIVERSITÉ PARIS

THÈSE DE DOCTORAT DE L'UNIVERSITÉ PSL

Préparée à Université Paris Dauphine

Homogénéisation quantitative de milieux aléatoires: environnements dégénérés et modèle d'interface

Soutenue par

Paul Dario

Le 18 Juin 2019

École doctorale nº543

École Doctorale de Dauphine

Spécialité

Mathématiques

Composition du jury :

M. Pierre CARDALIAGUET Université Paris-Dauphine	*Président du jury*
M. Noam BERGER Technical University of Munich	*Rapporteur*
M. Thomas SPENCER Institute for Avanced Study	*Rapporteur*
M. Ron PELED Tel Aviv University	*Membre du jury*
Mme Arianna GIUNTI University of Bonn	*Membre du jury*
M. Julian FISCHER Institute of Science and Technology Austria	*Membre du jury*
M. Jean-Christophe MOURRAT ENS Paris	*Directeur de thèse*
M. Scott ARMSTRONG Courant Institute of Mathematical Sciences	*Directeur de thèse*

Dauphine UNIVERSITÉ PARIS | PSL

Remerciements:

Je tiens tout d'abord à adresser mes plus sincères remerciements à mes directeurs de thèse, Jean-Christophe Mourrat et Scott Armstrong. Ils ont su me guider durant ces années avec leurs encouragements, leur bienveillance, tout en faisant preuve à la fois d'une grande disponibilité et d'une grande générosité dans le partage de leurs idées. Leur créativité, leur connaissance et leur compréhension des mathématiques resteront pour moi une importante source d'admiration et d'inspiration. Je suis très honoré d'avoir eu la chance d'être leur étudiant et travailler avec eux fut pour moi un grand plaisir.

Je remercie vivement Noam Berger et Thomas Spencer d'avoir accepté de rapporter cette thèse. Leurs remarques et leurs commentaires m'ont été précieux et je leur témoigne ici mon respect et mon admiration mathématique. Je suis par ailleurs très reconnaissant à Pierre Cardaliaguet, Julian Fisher, Arianna Giunti et Ron Peled de me faire l'honneur de faire partie du jury.

Ces années de thèse ont été très riches en discussions mathématiques et je souhaite remercier toutes les personnes avec qui j'ai eu le plaisir d'interagir. Je tiens tout particulièrement à remercier Chenlin Gu et Wei Wu, avec qui j'ai eu et ai encore la chance de travailler. Je remercie Laure Saint-Raymond pour son invitation lors des journées d'équipes de l'UMPA et Felix Otto pour son invitation au Max Planck Institute. Je remercie également Ofer Zeitouni et Ron Peled de m'avoir invité au Weizmann Institute et à l'université de Tel Aviv où j'ai pu avoir des discussions très enrichissantes et découvrir une partie de la communauté mathématique israélienne. Je tiens aussi à remercier Vincent Millot qui a été mon tuteur pendant mes années à l'ENS et Sylvia Serfaty pour ses conseils en début de thèse.

J'ai eu la chance de réaliser cette thèse dans deux laboratoires, le Courant Institute et le DMA, dans lesquels j'ai bénéficié d'excellentes conditions de travail. Je remercie les équipes de ces deux laboratoires qui ont su m'accueillir chaleureusement. En particulier, j'adresse mes remerciements à Bénédicte Auffray, Amélie Castelain, Zaïna Elmir et Albane Trémeau pour leur sympathie et leur efficacité.

L'ambiance au sein des doctorants et doctorantes fut particulièrement sympathique au cours de ces trois années de thèse, et je tiens à saluer mes anciens co-bureaux Baptiste, Barbara, Chenlin, Halyun, Jérémy, Jessica, Maxence, Mickaël, Onur, Rémy, Shariar, Théophile et Yusuke pour de nombreuses discussions passionnantes. Merci aussi à tous les autres jeunes et/ou doctorants ou doctorantes avec qui j'ai eu la chance de discuter autour d'un repas ou d'un café, Aymeric, Ephrème, Guillaume, Louise, Maxime, Michel, Nicolas, Thomas, Tobias, Tunan, Yichao, Wei.

Ces dernières années ont été riches en rencontres et je remercie très chaleureusement Alejandro, Alexandre, Anna, Anne, Antoine, Arnaud, Axel, Benjamin, Charles, Christophe, Florent, Guillaume, Hugo, Jean, Jean, Luc, Orianne, Pierre, Pierre, Pierre-Yves, Quentin et sa famille, Raphaël, Raphaël, Romain, Salim, Simon, Sohrab, Thomas pour leur compagnie.

Un grand merci va tout particulièrement à ma famille, à mes parents et à mes sœurs pour leur soutien inestimable et sans qui rien de tout cela n'aurait été possible.

Contents

Résumé de la thèse

Contents

Cette thèse est consacrée à l'homogénéisation stochastique, qui cherche à étudier le comportement d'équations aux dérivées partielles présentant des coefficients aléatoires oscillants rapidement. L'étude de ces systèmes se révèle en général difficile et une idée classique en homogénéisation consiste à réaliser une analyse asymptotique pour obtenir, lorsque le rapport entre les échelles tend vers l'infini, une équation limite appelée équation homogène. Cette équation est typiquement déterministe et à coefficients constants ce qui en facilite l'étude.

Un des objets central de ce manuscrit est l'équation elliptique linéaire sous forme de divergence,

$$-\nabla \cdot (\mathbf{a}(x)\nabla u) = f \quad \text{in } U \subseteq \mathbb{R}^d$$

où le coefficient $\mathbf{a}$ est une variable aléatoire stationnaire à valeurs dans l'ensemble des matrices symétriques définies positives. L'étude de ces équations est reliée aux marches aléatoires en milieux aléatoires et peut également être utilisée pour mieux comprendre les propriétés effectives de matériaux composites, via par exemple la conduction thermique ou l'électromagnétisme.

Jusqu'à présent, la majorité des travaux réalisés en homogénéisation stochastique repose sur une hypothèse d'uniforme ellipticité sur l'environnement. Cependant de nombreux modèles physiques ne satisfont pas cette hypothèse et la question de l'extension de la théorie se pose alors. Un bon exemple de système dégénéré peut être obtenu en considérant le réseau discret $\mathbb{Z}^d$ et en autorisant les conductances aléatoires (i.e. l'équivalent du coefficient $\mathbf{a}$ dans le contexte discret) à prendre la valeur 0 - avec une probabilité strictement inférieure à $1-p_c$, où p_c est le seuil de percolation critique. Ce cadre d'étude est l'objet du Chapitre 2, où nous établissons un théorème d'homogénéisation quantitative ainsi qu'une théorie de la régularité à grande échelle dans ce contexte dégénéré. La principale nouveauté repose sur un argument de renormalisation pour la composante connexe infinie : en utilisant des résultats classiques de percolation en régime surcritique, nous construisons une partition de $\mathbb{Z}^d$ en bons cubes (en un sens à préciser) de tailles variables. Cette partition fournit une échelle aléatoire au-dessus de laquelle la géométrie de l'amas infini de percolation est similaire à celle du réseau euclidien $\mathbb{Z}^d$.

Dans le Chapitre 3, ces résultats sont approfondis et appliqués à l'étude d'un objet clé en homogénéisation stochastique : le correcteur. En combinant des outils de trois types différents : la structure de renormalisation pour l'amas infini, l'inégalité de Poincaré multi-échelle et une inégalité de concentration de type Efron-Stein, nous obtenons des estimées spatiales optimales pour le correcteur. De telles estimations peuvent par la suite être utilisées pour obtenir des information sur la marche aléatoire sur l'amas infini en percolation supercritique.

Dans le Chapitre 4, nous étudions un autre type d'environnement dégénéré : le cas des formes différentielles. En nous appuyant sur des résultats de la théorie de la régularité pour les formes différentielles, nous sommes en mesure d'étendre les outils d'analyse fonctionnelle utilisés en homogénéisation à ce cadre plus général. Ceci permet d'étendre la théorie de l'homogénéisation développée sous une hypothèse d'uniforme ellipticité et d'établir un théorème d'homogénéisation quantitative pour un système d'équations elliptiques dégénérées impliquant des formes différentielles.

Dans le Chapitre 5, nous appliquons les idées de l'homogénéisation à un modèle issu de la physique statistique : le modèle de Ginzburg-Landau discret. Dans ce chapitre, nous revisitons le début de la théorie de l'homogénéisation stochastique et l'adaptons à ce nouveau modèle. Nous obtenons un taux de convergence quantitatif pour la tension de surface en volume fini du modèle. Une fois ceci établi, nous en déduisons une estimation quantitative de sous-linéarité pour la norme L^2 de l'interface aléatoire avec une condition de bord de Dirichlet affine. L'argument repose sur des idées venant de la théorie du transport optimal, nouvelles dans l'étude du modèle de Ginzburg-Landau discret, ainsi que des techniques d'homogénéisation.

Le reste de cette introduction est organisé comme suit. Dans la Section 0.1, nous présentons une introduction générale à la théorie de l'homogénéisation stochastique. Dans la Section 0.2, nous présentons le modèle de la percolation de Bernoulli par arêtes et exposons quelques résultats importants du domaine. Nous expliquons ensuite comment construire une structure de renormalisation pour le l'amas infini et terminons cette section en présentant les résultats d'homogénéisation obtenus dans cette direction. La Section 0.3 est consacrée à la présentation du Chapitre 4 : nous motivons l'étude des formes différentielles, présentons le modèle étudié au Chapitre 4 et expliquons les difficultés rencontrées pour étendre la théorie à ce nouveau contexte. Dans la Section 0.4, nous présentons le modèle de Ginzburg-Landau discret et les principaux objets et outils utilisés dans le Chapitre 5, ainsi que le résultat principal que nous obtenons.

0.1. Introduction : Homogénéisation stochastique

0.1.1. Introduction générale. L'objectif général de la théorie de l'homogénéisation est de répondre à la question suivante : étant donné un milieu hétérogène présentant des variations microscopiques dans sa composition, peut-on dire que, à grande échelle, les propriétés de l'environnement sont similaires à celles d'un milieu homogène ? Ce phénomène, quand il se produit, s'appelle *homogénéisation.* Il repose sur l'intuition que si les variations dans la composition sont distribuées de suffisamment maniéres aléatoire, elles se compenseront au niveau macroscopique, suivant une sorte de loi des grands nombres. Il est intéressant de comprendre précisément quand et comment ce phénoméne se produit, en effet de nombreux modèles physiques, impliquant par exemple la conduction thermique ou l'électromagnétisme, traitent d'environnements hétérogènes. Les équations obtenues à partir de ces modèles physiques présentent des oscillations rapides et sont coûteuses à résoudre numériquement. Néanmoins, il est tout de même possible d'obtenir des informations pertinentes avec un coût numérique réduit grâce à la théorie de l'homogénéisation : une stratégie intéressante consiste à prouver qu'un système hétérogène donné homogénéise, puis à résoudre l'équation homogénéisée et enfin à déduire de ces calculs des informations sur le milieu hétérogène.

Un exemple de situation physiquement pertinente où l'on s'attend à observer un phénomène d'homogénéisation est la conduction thermique dans un environnement hétérogène. Prenons l'exemple d'un milieu composite constitué de différents matériaux ayant des propriétés thermiques différentes. Nous souhaitons étudier la conduction thermique dans ce matériau. Mathématiquement, ceci peut être modélisé par l'équation parabolique

$$\partial_t u - \nabla \cdot (\mathbf{a}(x) \nabla u) = 0,$$

avec des conditions de bord adaptées. En considérant la limite où le temps tend vers l'infini, la distribution d'énergie thermique stationnaire est solution de l'équation elliptique

$$\nabla \cdot (\mathbf{a}(x) \nabla u) = 0. \tag{0.1.1}$$

Le fait que le milieu soit hétérogène signifie que la matrice $\mathbf{a}(x)$ varie dans l'espace. L'équation (0.1.1) est le principal objet d'étude de cette thèse, et nous présentons maintenant un modèle précis d'*homogénéisation stochastique* pour étudier cette équation aux dérivées partielles. Considérons une dimension $d \geq 1$, et notons $\mathcal{S}(\mathbb{R}^d)$ l'ensemble des matrices symétriques de $\mathbb{R}^d$. Nous considérons ensuite une fonction aléatoire

$$\begin{cases} \mathbb{R}^d \to \mathcal{S}(\mathbb{R}^d) \\ x \mapsto \mathbf{a}(x), \end{cases}$$

qui est mesurable par rapport aux tribus boréliennes sur $\mathbb{R}^d$ et $\mathcal{S}(\mathbb{R}^d)$. Nous supposons par ailleurs qu'il existe deux constantes d'ellipticité $0 < \lambda \leq \Lambda < \infty$ telles que

$$\lambda I_d \leq \mathbf{a}(x) \leq \Lambda I_d. \tag{0.1.2}$$

La matrice $\mathbf{a}$ est appelée l'*environnement.* Comme il a déjà été mentionné, nous supposons que l'environnement $\mathbf{a}$ est aléatoire, désignons par $\mathbb{P}$ la loi et par $\mathbb{E}$ l'espérance associée à cette mesure de probabilité. On suppose par ailleurs que la loi $\mathbb{P}$ satisfait les propriétés suivantes :

- *Stationnarité:* la loi $\mathbb{P}$ est invariante par translation par un vecteur de $\mathbb{Z}^d$, c'est-à-dire pour tout $y \in \mathbb{Z}^d$, les lois de $\mathbf{a}$ et de $\mathbf{a}(y + \cdot)$ sont identiques.
- *Ergodicité:* la loi $\mathbb{P}$ de l'environnement est ergodique par rapport aux translations sur $\mathbb{Z}^d$.

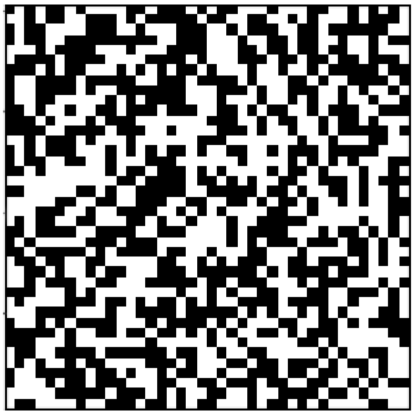

FIGURE 0.1.1. Un environnement typique satisfaisant les hypothèses de stationnarité et de dépendance à portée finie est le damier : on choisit deux matrices (déterministes) symétriques définies positives $\mathbf{a}_1$ et $\mathbf{a}_2$. L'environnement $\mathbf{a}$ est défini de telle sorte que $\mathbf{a}$ est égal à $\mathbf{a}_1$ sur les cellules noires et égal à $\mathbf{a}_2$ sur les cellules blanches. La couleur des cellules est choisie grâce à une percolation de Bernoulli par site de probabilité $p \in [0,1]$.

Grâce à ces deux hypothèses, il est possible développer une théorie de l'homogénéisation stochastique *qualitative*, i.e. établir des théorèmes de convergence sans aucun taux explicite. L'hypothèse d'ergodicité est de nature purement qualitative et on ne peut s'attendre à obtenir que des résultats qualitatifs sous cette hypothèse.

Au cours des dernières années, de nombreux progrès ont été réalisés dans le développement d'une théorie de l'homogénéisation stochastique *quantitative*, i.e. visant à établir des théorèmes de convergence avec des taux explicites. C'est l'objet d'étude de cette thèse et il est nécessaire, pour établir de tels résultats, de renforcer l'hypothèse d'ergodicité. Une possibilité, parmi tant d'autres, est de supposer une hypothèse de dépendance à portée finie sur l'environnement $\mathbf{a}$.

- *Dépendance à portée finie.* Si, pour tout ouvert $U \subseteq \mathbb{R}^d$, nous définissons $\mathcal{F}(U)$ comme étant la tribu borélienne engendrée par la famille d'applications,
$$\mathbf{a} \to \int_U \phi(x)\mathbf{a}(x)\,dx, \ \ \phi \in C_c^\infty(U),$$
alors les tribus $\mathcal{F}(U)$ et $\mathcal{F}(V)$ sont indépendantes dès que $\operatorname{dist}(U,V) \geq 1$.

L'objectif est alors d'étudier le comportement à grande échelle des solutions de l'équation elliptique (0.1.1). Pour ce faire, il est d'usage d'introduire un paramètre $0 < \varepsilon \ll 1$, qui représente le rapport entre l'échelle microscopique et l'échelle macroscopique, et de réaliser le changement d'échelles suivant :
$$\nabla \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon \right) = 0.$$
Un théorème d'homogénéisation typique que l'on souhaiterait montrer est le suivant : étant donné un domaine borné $U \subseteq \mathbb{R}^d$ et une fonction $f \in H^1(U)$, la famille des solutions u^ε des problèmes elliptiques
$$\left\{ \begin{array}{ll} \nabla \cdot \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon \right) = 0 & \text{in } U \\ u = f & \text{on } \partial U, \end{array} \right. \tag{0.1.3}$$
converge dans $L^2(U)$ lorsque ε tend vers 0 vers la solution $\overline{u}$ de l'équation elliptique
$$\left\{ \begin{array}{ll} \nabla \cdot (\bar{\mathbf{a}} \nabla \overline{u}) = 0 & \text{in } U \\ \overline{u} = f & \text{on } \partial U, \end{array} \right. \tag{0.1.4}$$

où la matrice $\bar{\mathbf{a}}$, appelée *environnement homogénéisé*, est déterministe et constante. Résoudre le problème elliptique (0.1.4) est équivalent, moyennant un changement de variables, à résoudre le problème de Poisson sur le domaine U et est moins coûteux à résoudre numériquement que l'équation hétérogène (0.1.1). Il est à noter que l'environnement homogénéisé $\bar{\mathbf{a}}$ dépend uniquement de la loi de l'environnement $\mathbf{a}$ (en particulier il ne dépend ni du domaine U ni de la condition de bord f). Cependant il dépend de manière complexe de la loi de l'environnement et il n'y a en général pas de formule explicite pour décrire cette quantité : en particulier, et sauf cas exceptionnels, $\bar{\mathbf{a}}$ n'est pas égal à l'espérance de $\mathbf{a}$.

0.1.2. Histoire de l'homogénéisation stochastique. Une théorie qualitative de l'homogénéisation stochastique a été initiée au début des années 80, avec les travaux de Kozlov [**107**], Papanicolaou et Varadhan [**135**] et Yurinskiĭ [**150**]. Ces résultats ont ensuite été étendus par Dal Maso et Modica in [**49, 50**], qui ont utilisé des arguments variationnels pour étudier des équations elliptiques non linéaires. Leurs preuves reposent sur une application du théorème ergodique et sont donc purement qualitatives.

Pour aller au-delà de la théorie qualitative et obtenir des taux de convergence quantitatifs en homogénéisation, il est nécessaire que la loi de l'environnement $\mathbf{a}$ satisfasse certaines conditions d'ergodicités quantitatives, telles que, par exemple, l'hypothèse de dépendance de portée finie mentionnée dans la section précédente. La principale difficulté de ce problème est que les solutions de l'équation elliptique $\nabla \cdot \mathbf{a} \nabla$ dépendent d'une manière très compliquée de l'environnement $\mathbf{a}$, et il n'est donc pas clair de la façon dont il faut procéder pour transférer les hypothèses d'ergodicités quantitatives de l'environnement aux solutions. Il s'agit d'un domaine de recherche actif qui a connu de nombreux progrès au cours des dernières années et la théorie est maintenant bien comprise, du moins dans le contexte uniformément elliptique.

Les premiers résultats quantitatifs ont été obtenus par Yurinskiĭ dans [**149**], où il démontre un taux de convergence algébrique sous-optimal pour l'erreur d'homogénéisation sous une hypothèse de mélange uniforme sur l'environnement, en dimension $d \geq 3$. Dix ans plus tard, dans [**133**], Naddaf et Spencer, à l'aide d'outils issus de la mécanique statistique, ont pu obtenir des taux de convergence optimaux dans le cadre d'un contraste d'ellipticité faible. Des résultats supplémentaires ont été obtenus dans cette direction par Conlon et Naddaf [**45**] et Conlon et Spencer dans [**46**], où les fonctions de Green ont été étudiées.

Dans le cas général, les premiers résultats quantitatifs satisfaisants ont été obtenus par Gloria et Otto in [**82, 83**]. Ils ont étudié le cas où l'environnement peut être décomposé en un nombre dénombrable de variables aléatoires indépendantes et identiquement distribuées. Leur approche, qui s'appuie sur les idées de Naddaf et Spencer [**133**], repose sur des inégalités de concentration, telles que les inégalités de trou spectral ou les inégalités de Sobolev logarithmiques. En particulier, ils ont obtenu des estimations sur le correcteur, les fluctuations de la densité d'énergie du correcteur et l'approximation de la matrice homogénéisée qui sont optimales en termes d'échelle spatiale et sous-optimales en termes d'intégrabilité stochastique. Puis, en collaboration avec Neukamm in [**81**], ils ont étendu les idées de [**82, 83**] et ont pu obtenir une estimée optimale sur la décroissance dans le temps de l'équation parabolique associée au correcteur, ce qui a permis d'en déduire des bornes sur les moments du correcteur.

Une autre approche a été initiée par Armstrong et Smart dans [**21**], qui ont étendu les techniques d'Avellaneda et Lin [**22, 23**] et celles de Dal Maso et Modica [**49, 50**]. Ils ont été en mesure d'obtenir une théorie de régularité $C^{0,1}$ à grande échelle sous une hypothèse de dépendance à portée finie. Armstrong, Kuusi et Mourrat l'ont ensuite généralisée à des conditions de mélange plus générales ainsi qu'à d'autres types d'équations [**20, 16**] et l'ont améliorée pour obtenir des taux de convergence optimaux [**17**]. La théorie de la régularité à grande échelle a également été étudiée dans les travaux de Gloria, Neukamm et Otto [**80**]. Ces résultats ont été étendus par Fischer et Otto in [**65**], qui ont développé une théorie de régularité d'ordre supérieur $C^{k,1}$. Dans [**84**], Gloria et Otto ont obtenu des estimées optimales sur la moyenne spatiale du gradient et du flux du correcteur et en ont déduit des estimées sur la croissance du correcteur ainsi que des estimations d'erreurs pour l'expansion à deux échelles.

La structure de corrélation et des fluctuations du correcteur a été étudiée par Mourrat et Otto [**131**], Mourrat et Nolen [**130**], Gu et Mourrat [**88**]. Les preuves reposent sur la formule de représentation d'Helffer-Sjöstrand, initialement introduite dans [**93, 145**] et utilisée par la suite par Naddaf et Spencer dans [**132**] afin d'obtenir un théorème central limit pour le modèle de Ginzburg-Landau discret (voir Section 0.4 pour une définition du modèle). Ces travaux s'appuient sur des idées présentes dans les travaux précités de Gloria, Neukamm, Otto ainsi que dans ceux de Gloria, Neukamm, Otto [**79**] et Marahrens et Otto [**110**]. Une théorie générale pour comprendre la structure des fluctuations en l'homogénéisation stochastique est établie par Duerinckx, Gloria et Otto dans [**60, 61**].

Dans la monographe [**18**], Armstrong Mourrat et Kuusi ont achevé le programme initié quelques années auparavant dans [**21**]. Ils ont pu obtenir des estimées optimales sur le correcteur ainsi que des estimations optimales pour l'erreur dans l'expansion à deux échelles. Ils ont également adapté la théorie au cas des équations paraboliques (voir aussi [**12, 27**]) et obtenu des estimées optimales d'homogénéisation pour les fonctions de Green elliptiques et paraboliques.

La théorie décrite dans les paragraphes précédents est principalement la théorie de l'homogénéisation stochastique des équations linéaires uniformément elliptiques, mais elle peut être généralisée à de nombreux autres contextes dans différentes directions, notamment :

- Traiter le cas des fonctionnelles uniformément convexes non linéaires, i.e. étudier les minimiseurs des problèmes

$$\min_u \int_U L(x, \nabla u(x))\, dx, \tag{0.1.5}$$

  où $U \subseteq \mathbb{R}^d$ est ouvert et $(x,p) \mapsto L(x,p)$ est aléatoire, convexe en la seconde variable. Cette situation a été étudiée par Dal Maso et Modica [**49**] et par Armstrong et Smart dans [**21**]. Le Chapitre 15 de [**98**] est consacré au cas des intégrandes ergodiques stationnaires et convexes. Armstrong et Mourrat dans [**20**] ont établi une théorie de régularité aux ordres supérieures et ont démontré des bornes Lipschitz pour les minimiseurs avec une intégrabilité stochastique optimale. Plus récemment, Armstrong, Ferguson et Kuusi ont prouvé dans [**14**] une estimée de régularité Lipschitz pour la différence de deux solutions, la principale difficulté étant que dans le cas non linéaire, l'ensemble des minimiseurs n'est pas un espace vectoriel et la différence de deux solutions n'est pas, en général, une solution. Dans [**59**], Duerinckx et Gloria ont établi un résultat d'homogénéisation qualitative pour une famille de fonctionnelles non convexes.
- Affaiblir l'hypothèse d'uniforme ellipticité : ceci nécessite d'affaiblir l'hypothèse

$$\forall x \in \mathbb{R}^d,\ \lambda I_d \le \mathbf{a}(x) \le \Lambda I_d.$$

La recherche dans cette direction a attiré beaucoup d'attention dernièrement, notamment en raison de la relation entre l'homogénéisation et les marches aléatoires en milieux aléatoires. Dans [**108**], Lamacz, Neukamm et Otto ont adapté la théorie de l'homogénéisation à un modèle de percolation de Bernoulli, où le modèle standard est modifié de sorte que toutes les arêtes dans une direction fixé sont déclarées ouvertes. Une autre façon commune d'étudier les environnements dégénérés est de supposer que les constantes d'ellipticité λ et Λ sont aléatoires et d'imposer une hypothèse d'intégrabilité de la forme suivante : il existe $p, q \in [1, \infty)$ tels que

$$\mathbb{E}\left[\lambda^{-p}\right] + \mathbb{E}\left[\Lambda^{q}\right] < \infty. \tag{0.1.6}$$

Ce cadre a été étudié par Andres, Deuschel, Slowik dans [**9**] (voir aussi [**10**]), puis par Chiarini et Deuschel dans [**44**]. Ils obtiennent un principe d'invariance *quenched* pour le processus de diffusion sous l'hypothèse $1/p + 1/q < 2/d$. Dans [**28**], Bella, Fehrman et Otto, travaillant toujours sous l'hypothèse $1/p + 1/q < 2/d$, ont obtenu un théorème de Liouville et une estimation de régularité à grande échelle $C^{1,\alpha}$ pour les fonctions $\mathbf{a}$-harmoniques. Andres, Chiarini, Deuschel et Slowik ont étendu ces résultats au cas des coefficients dépendants du temps dans [**8**]. La condition (1.1.19) exige que la valeur des conductances soit non nulle presque sûrement, une extension de ce modèle dans un cas où les conductances peuvent à la fois être nulles et petites (sous certaines conditions) a été étudiée par Deuschel, Nguyen et Slowik dans [**57**]. Dans [**75**], Giunti, Höfer et Velázquez ont étudié l'homogénéisation de l'équation de Poisson dans un domaine perforé aléatoirement.

0.2. Chapitres 2 et 3 : Percolation sur-critique

Les Chapitres 2 et 3 sont consacrés à l'adaptation de la théorie de l'homogénéisation stochastique dans le contexte de la percolation sur-critique. La principale difficulté vient du fait que l'hypothèse d'uniforme ellipticité (0.1.2) est mise en défaut, il faut donc trouver une substitution appropriée. Ceci est accompli en construisant une structure de renormalisation pour l'amas infini, et repose sur certains résultats du domaine (voir [**11, 136**]). Nous commençons cette section en définissant le modèle de la percolation de Bernoulli par arêtes et en passant en revue quelques résultats importants du domaine. Nous expliquons ensuite la structure de renormalisation en bons cubes, qui est un élément essentiel dans les Chapitres 2 et 3 et présentons finalement les principaux résultats obtenus dans ces deux chapitres.

0.2.1. Le modèle de la percolation de Bernoulli par arêtes.

0.2.1.1. *Définition du modèle et premières propriétés.* Le modèle de la percolation de Bernoulli a été introduit pour la première fois par Broadbent et Hammersley en 1957 dans l'article [**39**]. C'est l'un des modèles mathématiques les plus simples qui présente une transition de phase. Malgré son apparente simplicité, il a donné lieu à une théorie mathématique complexe et même si de nombreuses propriétés importantes relatives à ce modèle ont pu être comprises au cours des 70 dernières années, de nombreuses questions restent encore ouvertes. Avant de commencer à décrire le modèle, nous mentionnons les livres [**87, 102, 37, 147**], où l'on pourra trouver un aperçu plus complet du sujet.

On considère $\mathbb{Z}^d$ le réseau euclidien standard en dimension $d \geq 2$. Un point $x \in \mathbb{Z}^d$ est appelé un *sommet*. Nous équipons cet ensemble avec la norme 1 définie, pour chaque sommet $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$, par

$$\|x\|_1 := \sum_{i=1}^{d} |x_i|.$$

Nous disons que deux sommets $x, y \in \mathbb{Z}^d$ sont plus proches voisins si $\|x - y\|_1 = 1$. Une paire non orientée $\{x, y\}$ de plus proches voisins de $\mathbb{Z}^d$ est appelée une *arête*. On note E_d être l'ensemble des arêtes de $\mathbb{Z}^d$,

$$E_d := \left\{ \{x, y\} \; : \; x, y \in \mathbb{Z}^d \text{ and } \|x - y\|_1 = 1 \right\}.$$

Le modèle probabiliste de percolation de Bernoulli par arêtes est défini comme suit : nous considérons l'espace mesurable $(\Omega, \mathcal{F})$, où Ω est l'ensemble des fonctions de l'ensemble des arêtes E_d à valeurs dans $\{0, 1\}$, i.e. $\Omega := \{0, 1\}^{E_d}$, et $\mathcal{F}$ est la tribu engendrée par les événements dépendants d'un nombre fini d'arêtes. Une configuration de percolation est un élément $\omega \in \Omega$, et étant donné une arête $e \in E_d$, nous désignons par $\omega(e) \in \{0, 1\}$ la valeur de la configuration ω à l'arête e. Étant donné une configuration ω, on dit que l'arête e est *fermée* si $\omega(e) = 0$ et *ouverte* si $\omega(e) = 1$. Une composante connexe d'arêtes ouvertes est appelée un *amas*. Étant donné une probabilité $p \in [0, 1]$, nous notons $\mathbb{P}_p$ l'unique mesure de probabilité sur $(\Omega, \mathcal{F})$ telle que la famille de variables aléatoires $(\omega(e))_{e \in E_d}$ est indépendante et identiquement distribuée suivant une loi de Bernoulli de paramètre p.

Le but de la théorie de la percolation est d'étudier la géométrie des amas lorsque le paramètre p varie entre 0 et 1. Une première question intéressante est la suivante : existe-t-il un amas infini ? Une première étape pour étudier cette problématique est de remarquer que l'événement

$$E_\infty := \{\text{il existe un amas infini d'arêtes ouvertes}\}$$

est invariant par translation. Par une application de la loi du $0 - 1$, la probabilité de cet événement doit être 0 ou 1. Par ailleurs, nous définissons pour $p \in [0, 1]$,

$$\theta(p) := \mathbb{P}_p \left(0 \text{ appartient à une composante connexe infinie d'arêtes ouvertes} \right).$$

Par des arguments standards, nous savons que la fonction $p \mapsto \theta(p)$ est croissante et les trois énoncés suivants sont équivalents :

(i) $\theta(p) > 0$,
(ii) $\mathbb{P}_p (E_\infty) = 1$,
(iii) il existe un amas infini $\mathbb{P}_p$-presque sûrement.

Nous définissons donc la probabilité critique $p_c := p_c(d)$ comme étant

$$p_c := \inf \{ p \in [0, 1] \; : \; \theta(p) > 0 \}.$$

Si $0 < p_c < 1$, on dit que le modèle présente une transition de phase. Dans ce cas, on peut distinguer trois régimes distincts :

(i) la phase sous-critique, quand $0 \le p < p_c$, tous les amas sont finis,
(ii) la phase critique, quand $p = p_c$,
(iii) la phase sur-critique, quand $p_c < p \le 1$, il existe au moins un amas infini.

Cette distinction n'a lieu d'être que si la probabilité critique p_c est strictement comprise entre 0 et 1. Le premier résultat que nous aimerions énoncer est dû à Broadbent et Hammersley [**39**] et Hammersley [**89, 90**] prouve l'existence d'une transition de phase pour ce modèle.

PROPOSITION 0.2.1 (Existence d'une transition de phase [**39, 89, 90**]). *Pour toute dimension* $d \geq 2$*, nous avons*

$$0 < p_c(d) < 1.$$

La question de la valeur précise de p_c est épineuse. Il a été prouvé par Kesten que sa valeur en dimension 2 est 1/2.

THEOREM 0.2.1 (Kesten [**100**]). *Nous avons l'égalité* $p_c(2) = \frac{1}{2}$.

Il est peu probable que l'on puisse obtenir une formule explicite utilisable pour $p_c(d)$ en dimension supérieure à 3. La raison justifiant la valeur exacte en dimension 2 est que le réseau carré $\mathbb{Z}^2$ satisfait une propriété d'auto-dualité (voir [**87**, Chapitre 1]), qui est très spécifique à cette dimension. Cette propriété est un ingrédient clé dans la preuve du Théorème 0.2.1 et dans l'étude de la percolation en dimension 2 en général.

Dans cette thèse, nous nous intéressons principalement à l'application de la théorie de l'homogénéisation stochastique sur l'amas infini de percolation en phase sur-critique, et nous nous concentrons uniquement sur cette phase dans la suite de cette introduction.

0.2.1.2. *La phase sur-critique.* Dans cette phase, il existe, par définition, au moins un amas infini presque sûrement. Une première question intéressante est de déterminer le nombre d'amas infinis. Si on dénote par N ce nombre, qui est une variable aléatoire prenant ces valeurs dans $\mathbb{N}^* \cup \{\infty\}$, on remarque que celle-ci est invariante par translation. Par une application de la loi du $0-1$, elle doit être constante presque sûrement, et nous avons donc montré

$$\text{Il existe } k \in \mathbb{N}^* \cup \{\infty\} \text{ tel que } \mathbb{P}_p\left(N = k\right) = 1.$$

Ce résultat a été amélioré en 1987 par Aizenman, Kesten et Newman dans [**4, 5**] qui ont prouvé que le nombre k est égal à 1 pour chaque $p \in (p_c, 1]$. Ceci est résumé dans le théorème suivant.

THEOREM 0.2.2 (Unicité de l'amas infini [**4, 5**]). *Pour chaque* $p \in (p_c, 1]$*, nous avons*

$$\mathbb{P}_p\left(\textit{Il existe un unique amas infini}\right) = 1.$$

Dans la suite, nous désignons par $\mathscr{C}_\infty$ cet unique amas infini. Maintenant que l'existence et l'unicité de $\mathscr{C}_\infty$ sont établies, l'étape suivante consiste à comprendre sa géométrie. L'idée générale à garder à l'esprit est l'Ansatz suivante : dans la phase sur-critique, l'amas infini s'étend essentiellement dans tout l'espace. Sa géométrie est, au moins à grande échelle, similaire à celle de $\mathbb{Z}^d$ et il coexiste avec de petits amas isolés et finis. Ceci est illustré par la Figure 0.2.1.

En suivant cette philosophie, nous souhaitons construire une structure de renormalisation pour l'amas infini et procédons comme suit. La première étape, qui est l'une des idées clés des articles [**11, 136**] d'Antal, Penrose et Pisztora, est d'introduire une version l'Ansatz décrite précedemment en volume fini. Étant donné un domaine borné et connexe $D \subseteq \mathbb{Z}^d$, on souhaite dire qu'avec une grande probabilité :

(i) Il y a un grand amas dans D qui joue le rôle de l'amas infini,
(ii) tous les autres amas sont petits.

Pour des raisons de simplicité, nous nous restreignons à une famille spécifique de domaines de $\mathbb{Z}^d$: les cubes. Un énoncé mathématique précis est donné dans la définition et la proposition suivantes.

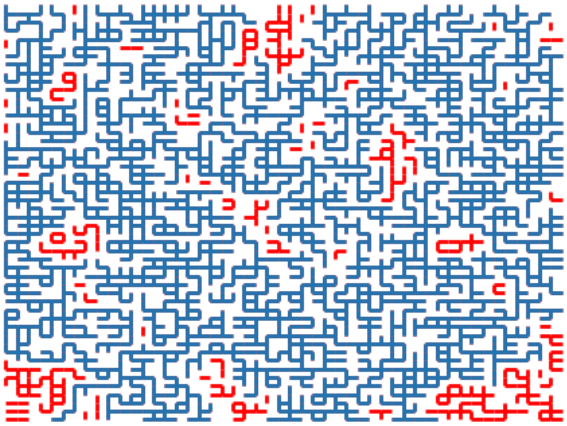

FIGURE 0.2.1. Un cube pré-bon. L'amas $\mathscr{C}_*(\square)$ est dessiné en bleu et touche les 4 côtés du carré. Il coexiste avec de petits amas isolés dessinés en rouge.

DEFINITION 0.2.2 (Cube pré-bon). On définit un cube de $\mathbb{Z}^d$ comme étant un sous-ensemble de la forme

$$[\![x, x+N]\!]^d, \ x \in \mathbb{Z}^d, \ N \in \mathbb{N}.$$

un cube générique est dénoté par $\square$ et l'entier N est appelé la taille du cube. Étant donné une configuration de percolation ω, on dit qu'un cube $\square$ de taille N est pré-bon s'il satisfait les propriétés suivantes:

- Il existe un amas qui intersecte les $2d$ faces du cube, cet amas est désigné par la notation $\mathscr{C}_*(\square)$,
- le diamètre de tous les autres amas est plus petit que $N/1000$.

Le principal résultat concernant cette notion est que, pour chaque $p > p_c$, la probabilité qu'un grand cube soit pré-bon est exponentiellement proche de 1 lorsque la taille du cube est grande. Ceci fut démontré par Penrose et Pisztora dans [**136**]. L'énoncé donné ci-dessous est une application de leur Théorème 5 avec $\phi_n = n/1000$.

THEOREM 0.2.3 (Theorem 5 of [**136**]). *Pour chaque dimension $d \geq 2$ and $p > p_c$, il existe une constante $c > 0$ telle que pour chaque cube $\square \subseteq \mathbb{Z}^d$ de taille n,*

$$\mathbb{P}_p(\square \textit{ est pré-bon}) \geq 1 - \exp(-cn).$$

L'objectif est maintenant de construire une partition de $\mathbb{Z}^d$ en cubes pré-bons. Une première exigence nécessaire à l'obtention d'une partition utilisable est la propriété de connectivité suivante : étant donné deux cubes pré-bons voisins $\square_1, \square_2$ et de tailles similaires, les amas $\mathscr{C}_*(\square_1)$ et $\mathscr{C}_*(\square_2)$ sont connectés dans $\square_1 \cup \square_2$. Malheureusement cette propriété ne peut se déduire directement de Definition 0.2.2, mais il est possible de trouver une solution : en utilisant que le Théorème 1.2.8 fournit un taux de convergence très rapide sur la probabilité qu'un cube soit pré-bon, on peut définir la notion suivante de bon cube.

DEFINITION 0.2.3 (Bon cube). Étant donné une configuration de percolation ω, on dit qu'un cube $\square$ de taille N est bon s'il satisfait les propriétés suivantes :

- Le cube $\square$ est pré-bon,
- tous les cubes $\square'$ de taille comprise entre $N/10$ et $10N$ et qui intersecte le cube $\square$ sont aussi pré-bons.

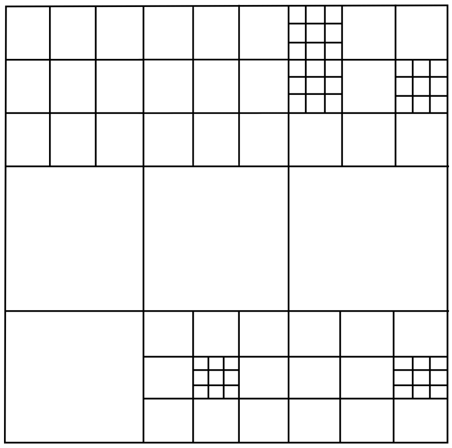

FIGURE 0.2.2. Une réalisation de la partition $\mathcal{P}$.

REMARK 0.2.4. Le nombre de cubes $\square'$ satisfaisant la deuxième hypothèse de la définition précédente est fini et peut être borné par CN^{d+1}. En utilisant le Théorème 0.2.3, nous pouvons montrer que la probabilité qu'un cube soit bon est exponentiellement proche de 1 : en réduisant la taille de l'exposant c, nous avons

$$\mathbb{P}_p(\square \text{ est bon}) \geq 1 - \exp(-cn). \tag{0.2.1}$$

Avec cette définition, il est possible de prouver la propriété de connectivité mentionnée ci-dessus. Nous ne détaillons pas la preuve, qui est assez directe et renvoyons au Chapitre 2, Lemme 2.2.8.

Nous soulevons maintenant la question suivante : étant donné une configuration de percolation ω, est-il possible de partitionner $\mathbb{Z}^d$ en bons cubes seulement ?

Notons d'abord que la probabilité qu'un cube soit bon n'est jamais exactement 1, il existe toujours un événement de faible probabilité où le cube ne satisfait pas les hypothèses de la Définition 1.2.3. Par le lemme de Borel-Cantelli, il n'est pas possible d'espérer construire une partition de $\mathbb{Z}^d$ en cubes qui sont à la fois bons et de même taille. La première propriété est la plus importante pour nous et donc nous renonçons à la seconde : l'objectif est dorénavant de construire une partition de $\mathbb{Z}^d$ en bons cubes de *tailles variables*.

Nous limitons notre réflexion à un sous-ensemble des cubes de $\mathbb{Z}^d$ qui est bien adapté à la construction de partitions, à savoir les *cubes triadiques*.

DEFINITION 0.2.5. Pour chaque entier $n \in \mathbb{N}$, nous définissons le cube $\square_n$ par

$$\square_n := \left[-\frac{3^n}{2}, \frac{3^n}{2} \right] \cap \mathbb{Z}^d.$$

Pour $n \in \mathbb{N}$, nous notons $\mathcal{T}_n$ l'ensemble des *cubes triadiques* de taille 3^n, défini par la formule

$$\mathcal{T}_n := \left\{ z + \square_n \ : \ z \in 3^n \mathbb{Z}^d \right\}.$$

L'ensemble $\mathcal{T}$ de tous les cubes triadiques est défini par

$$\mathcal{T} := \bigcup_{n=0}^{\infty} \mathcal{T}_n.$$

Ces cubes satisfont la propriété pratique suivante : deux cubes triadiques $\square$ et $\square'$ sont soit inclus l'un dans l'autre soit disjoints. Cette propriété rend ce sous-ensemble de cubes de $\mathbb{Z}^d$ bien adapté à la conception de partitions. En utilisant l'estimée (1.2.1) sur la probabilité qu'un cube soit bon, il est possible construire la partition suivante.

Proposition 0.2.6 (Chapitre 2, Proposition 2.2.1). *Pour chaque dimension $d \geq 2$ et chaque probabilité $p \in (p_c, 1]$, il existe, $\mathbb{P}_p$-presque sûrement, une partition $\mathcal{P}$ de $\mathbb{Z}^d$ en cubes triadiques de tailles variables telle que*

(i) *chaque cube $\square \in \mathcal{P}$ est un bon cube,*

(ii) *deux cubes adjacents $\square, \square' \in \mathcal{P}$ ont des tailles comparables,*

$$\frac{1}{3} \leq \frac{\mathrm{size}(\square)}{\mathrm{size}(\square')} \leq 3.$$

(iii) *Pour chaque sommet $x \in \mathbb{Z}^d$, si l'on dénote par $\square_{\mathcal{P}}(x)$ l'unique cube de la partition $\mathcal{P}$ contenant x, alors la taille du cube $\square_{\mathcal{P}}(x)$ est une variable aléatoire satisfaisant l'estimée suivante*

$$\mathrm{size}\left(\square_{\mathcal{P}}(x)\right) \leq \mathcal{O}_1\left(C\right).$$

La preuve de cette proposition peut être trouvée dans le Chapitre 2, et la Figure 1.2.5 illustre ce à quoi ressemble cette partition. Cette partition est un ingrédient crucial dans les preuves des Chapitres 2 et 3, puisqu'elle permet de développer un calcul fonctionnel sur l'amas infini. Plus spécifiquement, en utilisant cette partition il est possible de prouver des inégalités de Poincaré, des inégalités de Sobolev et une estimation de Meyers pour des fonctions définies sur l'amas infini et nous renvoyons à la Section 2.3 du Chapitre 2 pour plus de détails.

0.2.2. Homogénéisation quantitative sur l'amas de percolation. Dans cette section, nous présentons les résultats obtenus en adaptant la théorie de l'homogénéisation stochastique sur l'amas de percolation. C'est l'objet du Chapitre 2 ainsi que des articles [**13, 51**]. Décrivons tout d'abord le modèle. Étant donné un paramètre d'ellipticité fixé $\lambda \in (0, 1]$, nous définissons l'environnement $\mathbf{a}$ dans ce contexte discret comme étant une variable aléatoire

$$\mathbf{a} : E_d \to \{0\} \cup [\lambda, 1],$$

de sorte que la famille $(\mathbf{a}(e))_{e \in E_d}$ est indépendante et identiquement distribuée. Ces variables aléatoires sont appelées les conductances. Nous désignons par $\mathbb{P}$ la loi de l'environnement et supposons que, pour chaque arête $e \in E_d$,

$$\mathfrak{p} := \mathbb{P}\left(\mathbf{a}(e) \in [\lambda, 1]\right) > p_c\left(d\right),$$

de manière à assurer l'existence presque sûre d'une composante connexe infinie d'arêtes de conductance non nulle, qui sera dénotée par la suite $\mathscr{C}_\infty$. Nous définissons ensuite l'opérateur elliptique $\nabla \cdot \mathbf{a} \nabla$ comme suit, pour chaque fonction $u : \mathscr{C}_\infty \to \mathbb{R}$ et chaque $x \in \mathscr{C}_\infty$,

$$\nabla \cdot \mathbf{a} \nabla u(x) = \sum_{y \sim x} \mathbf{a}(\{x, y\})(u(y) - u(x)). \tag{0.2.2}$$

On dit qu'une fonction u est $\mathbf{a}$-harmonique si elle satisfait

$$\nabla \cdot (\mathbf{a} \nabla u) = 0 \text{ dans } \mathscr{C}_\infty.$$

Le premier résultat obtenu dans cette thèse fournit un théorème d'homogénéisation quantitative pour les problèmes elliptiques sur l'amas de percolation.

Theorem 0.2.4 (Chapitre 2, Théorème 2.1.1). *Fixons un exposant $p > 2$. Il existe deux exposants $s > 0$, $\alpha > 0$, une constante $C < \infty$ et une variable aléatoire positive $\mathcal{X}$ satisfaisant l'estimée sous-exponentielle*

$$\mathbb{P}\left(\mathcal{X} > t\right) \leq C \exp\left(-C^{-1} t^s\right), \tag{0.2.3}$$

*telle que les énoncés suivants sont vérifiés : pour chaque entier $m \in \mathbb{N}$ tel que $3^m \geq \mathcal{X}$, le cube $\square_m$ est un bon cube, et pour chaque fonction $u : \mathscr{C}_*(\square_m) \to \mathbb{R}$ qui est **a**-harmonique, c'est-à-dire qui satisfait*

$$-\nabla \cdot \mathbf{a} \nabla u = 0 \quad \textit{dans } \mathscr{C}_*(\square_m),$$

il existe une fonction harmonique u_{hom} qui est définie sur le cube triadique continu $\left[-3^m/2, 3^m/2\right]^d$ telle que

$$u = u_{\mathrm{hom}} \quad \textit{sur le bord } \mathscr{C}_*(\square_m) \cap \partial \square_m, \tag{0.2.4}$$

et

$$\left\| u - u_{\mathrm{hom}} \right\|_{\underline{L}^2(\mathscr{C}_*(\square_m))} \leq C 3^{-m(1-\alpha)} \left\| u \right\|_{\underline{L}^p(\mathscr{C}_*(\square_m))}. \tag{0.2.5}$$

Nous déduisons du théorème précédent une théorie de la régularité à grande échelle et une description précise de l'ensemble $\mathcal{A}_k(\mathscr{C}_\infty)$ des fonctions **a**-harmoniques sur l'amas infini qui croissent plus lentement qu'un polynôme de degré k, défini précisément par

$$\mathcal{A}_k(\mathscr{C}_\infty) := \left\{ u : \mathscr{C}_\infty \to \mathbb{R}^d \, : \, \nabla \cdot (\mathbf{a} \nabla u) = 0 \text{ in } \mathscr{C}_\infty \text{ and } \limsup_{R \to \infty} R^{-(k+1)} \left\| u \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)} = 0 \right\}.$$

Theorem 0.2.5 (Théorie de la régularité, Chapitre 2, Théorème 2.1.2). *Il existe deux exposants $s, \delta > 0$ et une variable aléatoire positive $\mathcal{X}$ satisfaisant l'estimée sous-exponentielle*

$$\mathbb{P}\left(\mathcal{X} > t\right) \leq C \exp\left(-C^{-1} t^s\right),$$

tels que:

(i) *Pour chaque entier $k \in \mathbb{N}$, il existe une constante $C < \infty$ telle que, pour chaque $u \in \mathcal{A}_k(\mathscr{C}_\infty)$, il existe un polynôme harmonique p de degré au plus k tel que, pour chaque $r \geq \mathcal{X}$,*

$$\left\| u - p \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \leq C r^{-\delta} \left\| p \right\|_{\underline{L}^2(B_r)}. \tag{0.2.6}$$

(ii) *Pour chaque entier $k \in \mathbb{N}$ et chaque polynôme harmonique p de degré au plus k, il existe $u \in \mathcal{A}_k$ tel que, pour chaque $r \geq \mathcal{X}$, l'inégalité* (0.2.6) *est vérifiée.*

(iii) *Pour chaque entier $k \in \mathbb{N}$, il existe une constante $C < \infty$ telle que, pour chaque $R \geq 2\mathcal{X}$ et chaque fonction **a**-harmonique $u : \mathscr{C}_\infty \cap B_R \to \mathbb{R}$, il existe $\phi \in \mathcal{A}_k(\mathscr{C}_\infty)$ telle que, pour chaque $r \in \left[\mathcal{X}, \frac{1}{2}R\right]$, nous avons*

$$\left\| u - \phi \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \leq C \left(\frac{r}{R} \right)^{k+1} \left\| u \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)}.$$

Une conséquence de ce résultat est que l'espace vectoriel (aléatoire) $\mathcal{A}_k(\mathscr{C}_\infty)$ est presque sûrement de dimension fini et sa dimension est donnée par

$$\dim \mathcal{A}_k\left(\mathscr{C}_\infty\right) = \binom{d+k-1}{k} + \binom{d+k-2}{k-1}.$$

Un cas particulier intéressant est celui de l'espace $\mathcal{A}_1\left(\mathscr{C}_\infty\right)$ qui grâce au théorème précédent est presque sûrement de dimension $(d+1)$. Nous pouvons par ailleurs déduire de (i) que chaque fonction **a**-harmonique $u \in \mathcal{A}_1\left(\mathscr{C}_\infty\right)$ peut s'écrire sous la forme

$$u(x) = p \cdot x + \phi_p(x) + c, \text{ pour certains } p \in \mathbb{R}^d, \ c \in \mathbb{R}.$$

La fonction ϕ_p s'appelle le correcteur et est une quantité essentielle en homogénisation. De (i), on en déduit que le correcteur a une croissance sous-linéaire : il existe un exposant $\delta > 0$ tel que

$$\lim_{R\to\infty} \frac{1}{R^{1-\delta}} \left\| \phi_p - (\phi_p)_{\mathscr{C}_\infty \cap B_R} \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)} = 0, \ \mathbb{P} - \text{presque sûrement.} \tag{0.2.7}$$

L'objet de la section suivante est d'améliorer l'estimée précédente afin d'obtenir une borne optimale pour la croissance du correcteur.

0.2.3. Estimée spatiale optimale pour le correcteur. Cette section est consacrée à la présentation du Chapitre 3. L'objectif principal est d'améliorer la borne sous-linéaire du correcteur (0.2.7) et d'obtenir les bornes spatiales optimales pour cette quantité. Le théorème que nous démontrons est le suivant.

THEOREM 0.2.6 (Estimée spatiale optimale pour le correcteur, Chapitre 3, Théorème 3.1.1). *Pour chaque dimension $d \geq 3$, il existe un exposant $s > 0$, une constante $C < \infty$ tels que, pour chaque $x, y \in \mathbb{Z}^d$, et chaque $p \in B_1$,*

$$|\phi_p(x) - \phi_p(y)| \mathbb{1}_{\{x,y \in \mathscr{C}_\infty\}} \leq \mathcal{O}_s(C).$$

En dimension 2, la croissance du correcteur se comporte comme la racine carrée d'un logarithme, i.e.,

$$|\phi_p(x) - \phi_p(y)| \mathbb{1}_{\{x,y \in \mathscr{C}_\infty\}} \leq \mathcal{O}_s\left(C \log^{\frac{1}{2}} |x - y|\right).$$

Dans les énoncés précédents, nous utilisons la notation $\mathcal{O}_s$ pour quantifier l'intégrabilité stochastique, elle est définie de la manière suivante : étant donnés une variable aléatoire positive X et un nombre réel positif θ, on écrit

$$X \leq \mathcal{O}_s(\theta) \ \textit{si et seulement si} \ \mathbb{E}\left[\exp\left(\left(\frac{X}{\theta}\right)^s\right)\right] \leq 2.$$

La preuve repose sur les inégalités de concentration, introduites en l'homogénéisation par Naddaf et Spencer [**133**]. Un exemple classique d'une telle inégalité est celle d'Efron-Stein qui peut s'énoncer comme suit : si $X = (X_1, \ldots, X_n)$ est une famille de variables aléatoires indépendantes et si $(X_1', \ldots, X_n')$ est une copie indépendante de X, alors pour toute fonction mesurable F,

$$\operatorname{var}[F] \leq \frac{1}{2} \sum_{i=1}^{n} \operatorname{var}\left[\left(F(X_1, \ldots, X_{i-1}, X_i', X_{i+1} \ldots, X_n) - F(X)\right)^2\right]. \tag{0.2.8}$$

Dans le Chapitre 3, nous souhaitons appliquer cette inégalité lorsque F est le correcteur et X l'environnement $\mathbf{a}$. Ceci soulève deux difficultés :

(1) l'environnement $\mathbf{a}$ est indexé sur les arêtes de $\mathbb{Z}^d$ et contient un nombre infini de variables aléatoires. Nous avons donc besoin d'une généralisation de (1.2.8) pour une famille infinie de variables aléatoires.
(2) l'inégalité d'Efron-Stein donne une estimation sur la variance de la variable aléatoire, alors que nous souhaitons obtenir une intégrabilité stochastique sous-exponentielle, il nous faut donc une version sous-exponentielle de cette inégalité.

Une telle généralisation existe, est énoncée dans la Proposition 3.2.16 du Chapitre 3 et provient de [**19**]. Un deuxième outil important est l'inégalité de Poincaré multi-échelle qui permet de transférer les informations des moyennes spatiales du gradient du correcteur au correcteur lui-même.

0.3. Chapitre 4 : Formes différentielles

Dans la section précédente, nous avons présenté une manière d'étendre la théorie de l'homogénéisation stochastique dans le cadre dégénéré de l'amas infini en percolation sur-critique. Il était possible d'adapter la théorie connue dans le cadre uniformément elliptique grâce à une structure de renormalisation de l'amas infini. Dans cette section, nous développons une autre manière d'étendre la théorie dans un autre contexte dégénéré, celui des formes différentielles.

Pour introduire le problème, nous fixons une dimension $d \geq 2$ et un entier $k \in [\![0, d]\!]$. Nous notons $\Lambda^k(\mathbb{R}^d)$ l'ensemble des applications k-multilinéaires alternées. C'est un espace vectoriel de dimension $\binom{d}{k}$. Une base canonique de cet espace est donnée par la famille

$$\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}, \; 1 \leq i_1 < \cdots < i_k \leq d. \tag{0.3.1}$$

Étant donné un domaine $U \subseteq \mathbb{R}^d$, une forme différentielle u sur U est alors définie comme une application de U dans $\Lambda^k(\mathbb{R}^d)$; en utilisant la base canonique $\Lambda^k(\mathbb{R}^d)$, elle peut être décomposée comme

$$u : \begin{cases} \mathbb{R}^d & \to \Lambda^k(\mathbb{R}^d) \\ x & \mapsto \sum_{1 \leq i_1 < \cdots < i_k \leq d} u_{i_1, \cdots, i_k}(x) \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}. \end{cases}$$

Nous supposons toujours que les fonctions $u_{i_1, \cdots, i_k}$ sont mesurables et nous supposons fréquemment des propriétés supplémentaires sur ces fonctions en imposant leur appartenance à certains espaces fonctionnels tels que $L^2(U), H^1(U), C^k(U)$ etc. Notons également que dans le cas particulier $k = 0$, l'ensemble des formes différentielles peut être identifié avec l'ensemble des fonctions de U dans $\mathbb{R}$ et dans le cas de $k = 1$, l'ensemble des formes différentielles 1 peut être identifié avec l'ensemble des champs de vecteurs de U dans $\mathbb{R}^d$.

Nous présentons ensuite quelques outils utiles en rapport avec cette notion. Premièrement, l'espace $\Lambda^k(\mathbb{R}^d)$ peut être équippé d'un produit scalaire $\langle \cdot, \cdot \rangle$ en déclarant que la base canonique (0.3.1) est une base orthonormale. Étant donné un domaine $U \subseteq \mathbb{R}^d$, ce produit scalaire peut être étendu à l'espace vectoriel des formes différentielles dont les coefficients $(u_{i_1, \cdots, i_k})_{1 \leq i_1 < \cdots < i_k \leq d}$ sont dans $L^2(U)$ grâce à la formule, pour deux formes u, v,

$$\langle u, v \rangle_{L^2(U)} = \sum_{1 \leq i_1 < \cdots < i_k \leq d} \int_U u_{i_1, \cdots, i_k}(x) v_{i_1, \cdots, i_k}(x) \, dx. \tag{0.3.2}$$

Un deuxième outil essentiel pour étudier les formes différentielles est la notion de dérivée extérieure : pour $k \in [\![0, d-1]\!]$, étant donné une forme différentielle u définie sur un domaine $U \subseteq \mathbb{R}^d$, nous dénotons par $\mathrm{d}u$ la $(k+1)$-forme définie formellement par

$$\mathrm{d}u = \sum_{i=1}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} \frac{\partial u_{i_1, \cdots, i_k}}{\partial x_i}(x) \mathrm{d}x_i \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

et étendons la définition de la dérivée extérieure aux d-formes en définissant $\mathrm{d}u = 0$ si u est une d-forme. Une autre opérateur intéressant est l'adjoint formel de la dérivée extérieure d par rapport au produit scalaire (0.3.2), appelé la codifferentielle et dénotée par δ. Cet opérateur différentiel envoie les k-formes sur les $(k-1)$-formes, suivant la formule

$$\delta u = \sum_{1 \leq i_1 < \cdots < i_k \leq d} \sum_{i \in \{i_1, \ldots, i_k\}} (-1)^i \frac{\partial u_{i_1, \cdots, i_k}}{\partial x_i}(x) \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

où la notation $\widehat{\mathrm{d}x_i}$ signifie que le terme $\mathrm{d}x_i$ est effacé du produit extérieur $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$. Nous notons également l'une des propriétés clés de ces opérateurs : ils satisfont les identités

$$\mathrm{d} \circ \mathrm{d} = 0 \text{ et } \delta \circ \delta = 0. \tag{0.3.3}$$

Un des principaux intérêt de ce formalisme est qu'il englobe la plupart des opérateurs différentiels fréquemment utilisés. Par exemple, les opérateurs différentiels suivants peuvent être exprimés dans le langage des formes différentielles : pour une k-forme f, nous avons

- si $k = 0$, alors la 1-forme (ou le champ de vecteurs) $\mathrm{d}f$ peut être identifiée avec ∇f,
- si $k = 1$, alors la 0-forme (ou la fonction) δf peut être identifiée avec $\operatorname{div} f$,
- en dimension 3, l'espace vectoriel $\Lambda^2(\mathbb{R}^3)$ est de dimension 3. L'espace des 2-formes peut donc être identifié avec l'espace des champs de vecteurs de $\mathbb{R}^3$. Si $k = 1$, alors $\mathrm{d}f$ est une 2-forme (ou un champ de vecteurs) et peut être identifiée avec $\overrightarrow{\operatorname{rot}} f$.
- pour $k \in [\![0, d]\!]$, nous retrouvons le Laplacien usuel grâce à la formule

$$(\mathrm{d}\delta + \delta\mathrm{d}) f(x) = \sum_{1 \le i_1 < \cdots < i_k \le d} \Delta f_{i_1,\cdots,i_k}(x)\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

L'objectif principal du Chapitre 4 est d'étendre la théorie de l'homogénéisation stochastique dans le contexte des formes différentielles. Étant donné un nombre entier $k \in [\![0, d]\!]$ et un domaine $U \subseteq \mathbb{R}^d$, nous définissons $H^1_{\mathrm{d}}\Lambda^k(U)$ comme étant l'adhérence des k-formes différentielles lisses à support compact par rapport à la norme

$$\|f\|_{H^1_{\mathrm{d}}\Lambda^k(U)} := \|f\|_{L^2(U)} + \|\mathrm{d}f\|_{L^2(U)}.$$

Nous introduisons aussi l'espace des matrices symétriques sur l'espace vectoriel euclidien $\Lambda^k(\mathbb{R}^d)$, cet espace sera dénoté $\mathcal{S}(\Lambda^k(\mathbb{R}^d))$ par la suite. Comme dans la théorie classique de l'homogénéisation stochastique, nous définissons un environnement $\mathbf{a}$ comme étant une variable aléatoire

$$\begin{cases} \mathbb{R}^d \to \mathcal{S}(\Lambda^k(\mathbb{R}^d)) \\ x \mapsto \mathbf{a}(x), \end{cases}$$

qui satisfait la même hypothèse d'uniforme ellipticité que celle énoncée en (0.1.2). Nous supposons par ailleurs que cet environnement est aléatoire et satisfait les hypothèses de stationnarité et de dépendance à portée finie énoncées dans la Section 0.1.1. L'objectif est alors d'étudier les solutions de l'équation

$$u \mapsto \mathrm{d}(\mathbf{a}\mathrm{d}u) = 0 \text{ in } U,$$

qui sont les points critiques associés à la fonctionnelle

$$\langle \mathrm{d}u, \mathbf{a}\mathrm{d}u \rangle_{L^2(U)}. \tag{0.3.4}$$

Nous remarquons que ce contexte est strictement plus général que celui présenté dans la Section 0.1.1. En effet, en choisissant de se concentrer sur le cas particulier $k = 0$, la dérivée extérieure $\mathrm{d}u$ peut être identifiée avec le gradient de u et on retrouve le cadre habituel. D'autre part, ce formalisme s'inscrit dans le cadre plus général des systèmes d'équations elliptiques : puisque les dimensions des espaces $\Lambda^k(\mathbb{R}^d)$ sont toujours finies, l'équation $\mathrm{d}\mathbf{a}\mathrm{d}u = 0$ peut être décrite par un système d'équations elliptiques.

Une motivation pour étudier ces systèmes vient du cas spécifique où $r = 1$ et l'espace sous-jacent est de dimension 4 : dans ce contexte le système d'équations (1.3.4) possède la même structure que *les équations de Maxwell* (voir par exemple [**109**, Section 1.2]), avec cependant une différence fondamentale : ici nous supposons que l'environnement $\mathbf{a}$

est riemannien, c'est-à-dire elliptique au sens de (4.1.21), alors que pour les équations de Maxwell la structure géométrique sous-jacente est lorentzienne. Le remplacement d'une structure lorentzienne par une structure riemannienne, une procédure parfois appelée "rotation de Wick", est très courante en théorie quantique des champs, voir par exemple [**78**, Section 6.1(ii)]. Bien que les objets que nous étudions ici sont des minimiseurs d'un Lagrangien aléatoire, nous espérons que les techniques développées dans le Chapitre 4 seront instructives pour étudier les mesures de Gibbs associées à ces Lagrangiens.

La principale difficulté de ce chapitre réside dans l'extension des inégalités fonctionnelles utiles en l'homogénéisation dans le contexte plus général des formes différentielles. Ceci est obtenu en utilisant les résultats de Mitrea, Mitrea, Monniaux [**120**], Mitrea, Mitrea, Mitrea, Shaw [**122**] et la monographe de Schwarz [**142**].

0.4. Chapitre 5 : Modèle de Ginzburg-Landau discret

0.4.1. Définition du modèle et de la tension de surface. De nombreux phénomènes physiques présentent une transition entre deux phases pures, particulièrement à basse température, comme c'est le cas par exemple pour l'eau liquide et la glace à température nulle. L'étude de la géométrie de l'interface qui assure la transition entre ces deux phases est un sujet d'étude pour les mathématiciens depuis le début du 20ème siècle. Le premier modèle mathématique permettant de décrire de telles interfaces fut introduit par Wulff en 1901 dans l'article [**148**] : les interfaces y sont caractérisées comme les minimiseurs d'une certaine fonctionnelle, dite fonctionnelle de Wulff, et définie, pour un sous-ensemble $E \subseteq \mathbb{R}^d$, suivant la formule

$$W(E) := \int_{\partial E} \sigma(\mathbf{n}(x))\, \mathrm{d}x,$$

où $\mathbf{n}(x)$ est la normale extérieure à ∂E au point x et σ est la tension de surface entre les deux phases. Le minimiseur de la fonctionnelle de Wulff est appelé la forme Wulff. D'un point de vue mathématique, les interfaces sont des objets macroscopiques, que l'on souhaiterait décrire à partir de modèles issus de la mécanique statistique qui sont définis au niveau microscopique. De nombreux résultats importants dans cette direction ont été obtenus dans les années 90 ; dans [**6**], Alexander, Chayes et Chayes ont obtenu une construction de Wulff dans le cas de la percolation de Bernoulli sur-critique en dimension 2. Dans la monographe [**58**], Dobrushin, Kotecký et Shlosman, établirent une construction de Wulff pour le modèle d'Ising ferromagnétique bidimensionel à basse température avec des conditions de bord périodiques. Ces résultats ont par la suite été étendus à toutes les températures sous le seuil critique, on pourra se référer aux travaux de Ioffe [**94, 95**], Schonmann, Shlosman [**141**] et Pfister, Velenik [**137**] et Ioffe et Schonmann in [**96**] pour plus d'informations.

En dimension 3, Cerf a démontré dans [**41**] une forme de construction de Wulff pour la percolation Bernoulli en régime sur-critique. Bodineau dans [**35**], a prouvé un résultat similaire pour le modèle Ising pour toutes les dimensions $d \geq 3$ et à basse température.

Dans cette section, nous considérons un modèle mathématique d'interface plus simple, à savoir le modèle de Ginzburg-Landau discret. Il modélise les interfaces comme des fonctions $\phi : \mathbb{R}^d \mapsto \mathbb{R}$, qui varient autour de l'interface plate $\phi = 0$. Pour être plus précis, nous discrétisons l'espace continu $\mathbb{R}^d$ et considérons les applications $\phi : \mathbb{Z}^d \mapsto \mathbb{R}$. L'interface discrète est alors représentée par l'ensemble $\{(x, \phi(x)) : x \in \mathbb{Z}^d\} \subseteq \mathbb{Z}^d \times \mathbb{R}$ et nous associons à une interface ϕ une énergie qui peut être calculée grâce au hamiltonien

$$H(\phi) := \sum_{|x-y|_1 = 1} V(\phi(x) - \phi(y)),$$

où V est un potentiel élastique satisfaisant les propriétés :

(1) V est pair : $V(x) = V(-x)$ pour chaque $x \in \mathbb{R}$,

(2) V est uniformément convexe : il existe $\lambda \in (0,1]$ tel que pour chaque $x, y \in \mathbb{R}$,

$$\lambda |x-y|^2 \leq V(x) + V(y) - 2V\left(\frac{x+y}{2}\right) \leq \frac{1}{\lambda}|x-y|^2. \tag{0.4.1}$$

La mesure d'équilibre formelle associée à ce modèle est donnée par la mesure de Gibbs

$$\frac{1}{Z}\exp\left(-H(\phi)\right)\prod_x \mathrm{d}\phi(x), \tag{0.4.2}$$

où Z est une constante de normalisation qui fait de la mesure ci-dessus une mesure de probabilité et que l'on appelle la fonction de partition. Un autre aspect important du modèle est son interprétation dynamique : on considère deux phases pures séparées par une interface $\phi_{t=0}$ au temps $t=0$ et on laisse cette interface évoluer dans le temps. Elle évoluera en cherchant à minimiser son énergie et, en l'absence de lois de conservation, cette évolution ne sera affectée que par un bruit. Ceci conduit à la dynamique de Langevin gouvernée par l'équation différentielle stochastique

$$\mathrm{d}\phi_t(x) = -\sum_{|y-x|_1=1} V'\left(\phi_t(x) - \phi_t(y)\right)dt + \sqrt{2}dB_t\left(x\right),$$

où $\left(B_t\left(x\right)\right)_{x\in\mathbb{Z}^d}$ est une famille de mouvements browniens standard indépendants. Formellement, la mesure (1.4.2) est invariante pour cette équation différentielle stochastique et elle rend la dynamique réversible.

Un résultat informel que l'on souhaiterait démontrer est qu'une version correctement rééchelonnée de l'interface, qui est a priori un objet aléatoire distribué selon la mesure de probabilité (0.4.2), contracte autour d'une forme déterministe, que nous souhaiterions caractériser dans l'esprit de la construction de Wulff. Pour cela, une quantité cruciale est la tension de surface qui est définie comme suit : étant donné un sous-ensemble connexe, borné et discret $U \subseteq \mathbb{Z}^d$, nous définissons la mesure de probabilité associée au modèle de Ginzburg-Landau dans le domaine U avec une condition de bord affine de pente $p \in \mathbb{R}^d$ suivant la formule

$$\mathbb{P}_{U,p}(\mathrm{d}\phi) := Z_{U,p}^{-1}\exp\left(-\sum_{x,y\in Q_n,|x-y|_1=1} V\left(\phi(x)-\phi(y)\right)\right)\prod_{x\in U}\mathrm{d}\phi(x)\prod_{y\in\partial U}\delta_{p\cdot y}\left(\phi_y\right).$$

La quantité qui nous intéresse est alors la fonction de partition $Z_{U,p}$ ou plus précisément, la version correctement redimensionnée de celle-ci,

$$\nu\left(U,p\right) := -\frac{1}{|U|}\log Z_{U,p}.$$

Cette valeur est appelée la tension de surface en volume fini pour le domaine U et est une quantité importante pour obtenir des informations sur le modèle. Un premier résultat qui peut être obtenu est le suivant : étant donné un entier $n \in \mathbb{N}$, nous dénotons par Q_n le cube discret $[-n,n]^d \cap \mathbb{Z}^d$, il est alors possible de démontrer, par un argument de sous-additivité, que la suite $\left(\nu\left(Q_n,p\right)\right)_{n\in\mathbb{N}}$ converge lorsque la taille du cube tend vers l'infini, et nous dénotons par $\overline{\nu}\left(p\right)$ sa limite, i.e.,

$$\overline{\nu}\left(p\right) := \lim_{n\to\infty}\nu\left(Q_n,p\right),$$

cette quantité est appelée la tension de surface du modèle. Nous référons à l'article de Funaki et Spohn [**70**] pour la démonstration originelle de ce résultat.

La tension de surface est fondamentale pour la compréhension des propriétés macroscopiques du modèle. Elle apparaît par exemple dans des principes de grandes déviations, comme cela a été étudié par Deuschel, Giacomin et Ioffe dans [**56**]. Leurs démonstrations reposent sur la formule de représentation d'Helffer-Sjöstrand qui fut introduite pour la première fois dans l'étude ce modèle par Naddaf et Spencer dans l'article [**132**], où ils sont en mesure de démontrer prouver un théorème central limite pour le champ de gradient en combinant cette formule avec des techniques de la théorie qualitative de l'homogénéisation. Dans le cadre dynamique, Funaki et Spohn dans [**70**] ont établi une loi de grand nombre pour l'évolution et le processus de limite est caractérisé par une équation aux dérivées partielles parabolique non linéaire définie à partir de la tension de surface. Les fluctuations de la dynamique ont été étudiées par Giacomin Olla et Spohn dans [**72**], où ils ont prouvé qu'elles sont gouvernées à grande échelle par un processus Ornstein-Uhlenbeck en dimension infinie. Tous ces résultats font intervenir la tension de surface et nous mentionnons la référence [**68**] pour plus d'information sur le sujet.

L'objectif du Chapitre 5 est revisiter le début de la théorie de l'homogénéisatiom stochastique telle que présentée dans le monographe [**18**]. Pour cela la première étape consiste à quantifier la vitesse de convergence de la tension de surface en volume fini et nous obtenons un taux algébrique énoncé dans le théorème suivant.

Theorem 0.4.1 (Chapitre 5, Théorème 5.1.1). *Il existe un exposant $\alpha > 0$ et une constante $C < \infty$ tels que pour chaque $p \in \mathbb{R}^d$,*

$$|\nu\left(Q_n, p\right) - \overline{\nu}(p)| \leq C n^{-\alpha}\left(1 + |p|^2\right).$$

La preuve de ce résultat repose sur des idées provenant de deux domaines différents: l'homogénisation stochastique d'une part et le transport optimal d'autre part.

CHAPTER 1

Introduction

Contents

This thesis is devoted to the study of stochastic homogenization, which aims at understanding the behavior of partial differential equations with highly heterogeneous, but statistically homogeneous, random coefficients. Analyzing an equation with heterogeneous random coefficients is very difficult, and the classical idea of homogenization is to perform an asymptotic analysis by deriving, as the ratio of the length scales tends to infinity, an effective or homogenized equation. This limiting equation is typically deterministic, has constant coefficients, and is therefore much simpler to analyze.

A central object of study in this thesis is the linear elliptic equation in divergence form,

$$-\nabla \cdot (\mathbf{a}(x)\nabla u) = f \quad \text{in } U \subseteq \mathbb{R}^d$$

where the coefficient $\mathbf{a}$ is a stationary random variable valued in the set of positive definite matrices. The study of these equations is related to random walks in random environments and also has applications to the study of effective properties of composite materials through, for instance, heat conduction or electromagnetism.

So far, most of the quantitative theory for stochastic homogenization has been developed under the assumption of uniform ellipticity. However, many physical models do not satisfy this assumption and it is important to develop a theory in randomly perforated media. Perhaps the best example is on the discrete lattice and concerns random conductances which may be zero with probability strictly less than $1 - p_c$, where p_c is the critical percolation threshold. This setting is studied in Chapter 2, where quantitative homogenization, large scale regularity and Liouville results are established and provide the first quantitative results in porous medium. The main novelty is a renormalization argument for the infinite cluster: building on standard percolation results, one is able to design a partition of $\mathbb{Z}^d$ into only good cubes (in some sense to be made precise) of varying sizes, which provides a large scale above which the geometry of the infinite percolation cluster is similar to the one of the euclidean lattice $\mathbb{Z}^d$.

In Chapter 3, these results are further extended and applied to study a key object in stochastic homogenization: the first-order corrector. By combining tools of three different types: the renormalization structure for the infinite cluster, the multiscale Poincaré inequality and an Efron-Stein type concentration inequality, we derive optimal spatial scaling estimates for the corrector. Such estimates can be used to derive information about the random walk on the supercritical percolation cluster.

In Chapter 4, we study another type of degenerate environment: the setting of differential forms. Building on results from the regularity theory of differential forms, we are able to extend the tools of functional analysis used in homogenization to this more general setting. We are then able to extend the theory of homogenization developed in the uniformly elliptic setting and establish a quantitative homogenization theorem for a degenerate elliptic system of equations involving differential forms.

In Chapter 5, we apply ideas from homogenization to a model of statistical physics: the discrete Ginzburg-Landau or $\nabla\phi$ model. In this chapter, we revisit the beginning of the theory of stochastic homogenization and adapt it to this new model. The result we obtain is a quantitative rate of convergence for the finite-volume surface tension. Once this is established, we deduce a quantitative sublinearity estimate for the L^2 norm of the random interface with Dirichlet boundary condition. The argument relies on ideas from optimal transport and the techniques from homogenization, which are new in the study of the discrete Ginzburg-Landau model.

The rest of this introduction is organized as follows. In Section 1.1, we present an introduction to the theory of stochastic homogenization and review some of the key notions and results of the theory. In Section 1.2, we introduce the model of Bernoulli bond percolation and present a few important results in this field. When then explain how to construct a renormalization structure for the infinite cluster and complete this section by introducing the results obtained in this direction. Section 1.3 is devoted to the presentation of Chapter 4: we motivate the study of differential forms, present the model studied in Chapter 4 and explain the difficulties encountered to extend to results of stochastic homogenization to this new setting. In Section 1.4, we present the $\nabla\phi$ model and the main objects and tools which are used in Chapter 5, as well as the main result we obtained.

Index of notation

The following list records the most frequently used notations in the introduction of this thesis. These notations are essentially consistent with the rest of the thesis.

General notation pertaining to homogenization.

$\mathcal{S}(\mathbb{R}^d)$: set of symmetric positive definite matrices of $\mathbb{R}^d$,
∇ : gradient operator,
$\nabla\cdot$ or div : divergence operator,
$\mathbf{a}$: heterogeneous environment,
$\bar{\mathbf{a}}$: homogeneous environment,
λ and Λ : lower and upper bound for the ellipticity of the environment,
$\mathbf{a}$-harmonic functions : functions $u : \mathbb{R}^d \to \mathbb{R}$ satisfying $\nabla \cdot (\mathbf{a}\nabla u) = 0$,
$\bar{\mathbf{a}}$-harmonic functions : functions $u : \mathbb{R}^d \to \mathbb{R}$ satisfying $\nabla \cdot (\bar{\mathbf{a}}\nabla u) = 0$,
ε : size of the microscopic scale,
U : a bounded domain of $\mathbb{R}^d$,
$\overline{U}$: closure of the set U,
∂U : the topological boundary of U,
Q_r : the cube of size r and centered in 0 defined by $Q_r := \left(-\frac{r}{2}, \frac{r}{2}\right)^d$,
B_r and $B_r(x)$: balls of radius r centered in 0 and x, with $x \in \mathbb{R}^d$, respectively,
osc_U : oscillation of a function over a set U defined by $\operatorname{osc}_U f := \sup_U f - \inf_U f$,
l_p : the affine function of slope $p \in \mathbb{R}^d$ defined by $l_p(x) = p \cdot x$,
$|U|$: Lebesgue measure of the Borel set U,
$\fint_U$: averaged integral over the Borel set U defined by $\fint_U = \frac{1}{|U|}\int_U$,
$(v)_U$: average value of a measurable function $v : U \to \mathbb{R}$, defined by $(v)_U := \fint_U v(x)\,dx$,
$\mathbb{1}_U$: indicator function of the set U,
$C^\infty(U), C_c^\infty(U)$: set of smooth and smooth compactly supported functions from U to $\mathbb{R}^d$,
$C^\infty\left(\overline{U}\right)$: set of the functions from $\overline{U}$ to $\mathbb{R}$, which are equal to the restriction to $\overline{U}$ of a function $f \in C^\infty(V)$, for some open set V such that $\overline{U} \subseteq V$,
$C^k(U)$: set of functions which are k-times continuously differentiable with $k \in \mathbb{N}$,
$C^\beta(U)$: set of β-Hölder functions from U to $\mathbb{R}^d$ with $\beta \in (0,1]$,
$L^p(U)$, for $p \in [1,\infty)$: set of measurable functions from U to $\mathbb{R}$ such that the p-th power of their absolute value is Lebesgue integrable,
$\|\cdot\|_{L^p(U)}$ and $\|\cdot\|_{\underline{L}^p(U)}$: the L^p and rescaled L^p norms on the space $L^p(U)$, defined by

$$\|f\|_{L^p(U)} = \left(\int_U |f(x)|^p \, dx\right)^{\frac{1}{p}} \text{ and } \|f\|_{\underline{L}^p(U)} = \left(\fint_U |f(x)|^p \, dx\right)^{\frac{1}{p}},$$

$L^\infty(U)$: set of essentially bounded measurable functions from U to $\mathbb{R}$,
$W^{k,p}(U)$: Sobolev space with regularity $k \in \mathbb{N}$ and integrability $p \in [1,\infty]$,
$W^{\alpha,p}(U)$: fractional Sobolev space with regularity $\alpha \in (0,\infty)$ and integrability $p \in [1,\infty]$,
$\|\cdot\|_{W^{k,p}}$, $\|\cdot\|_{W^{\alpha,p}(U)}$: Sobolev norm on the spaces $W^{k,p}(U)$ and $W^{k,p}(U)$,
$W_0^{k,p}(U)$, $W_0^{\alpha,p}(U)$: the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$, $W^{\alpha,p}(U)$.
$H^1(U)$ and $H_0^1(U)$: Sobolev spaces with regularity parameter $k = 1$ and integrability $p = 2$, i.e. $H^1(U) = W^{1,2}(U)$ and $H_0^1(U) = W_0^{1,2}(U)$,
$\|\cdot\|_{H^1(U)}$: Sobolev norm on the space $H^1(U)$.

Notation pertaining to percolation.

E_d : set of edges of $\mathbb{Z}^d$,
$\mathscr{C}_\infty$: unique maximal infinite cluster of open edges,
$\mathcal{C}(x)$: the maximal cluster containing a point $x \in \mathbb{Z}^d$,
$\operatorname{dist}(x,y)$: the distance between two points $x, y \in \mathbb{Z}^d$ given by $\operatorname{dist}(x,y) = \sum_{i=1}^d |x_i - y_i|$,
$\operatorname{dist}_{\mathscr{C}}(x,y)$: the graph distance between two points $x, y \in \mathscr{C}$, within the subgraph $\mathscr{C}$ of $\mathbb{Z}^d$,

U : generic discrete bounded and connected subset $U \subseteq \mathbb{Z}^d$,
∂U : discrete boundary of the set $U \subseteq \mathbb{Z}^d$,
$|U|$: cardinality of the set U,
$\square_n$: discrete triadic cube of size 3^n, defined by $\square_n := \left[-\frac{3^n}{2}, \frac{3^n}{2}\right] \cap \mathbb{Z}^d$,
∇ : the discrete gradient defined by, for each function $u : \mathbb{Z}^d \to \mathbb{R}$, and each edge $e = (x,y) \in E_d$,

$$\nabla u(e) = u(x) - u(y).$$

Notation pertaining to differential forms.
$\Lambda^k(\mathbb{R}^d)$: set of k-alterning mutilinear maps of $\mathbb{R}^d$,
$\wedge$: exterior product,
d : exterior derivative,
$\mathrm{d}x_i$: differential of the i-th coordinate,
δ : codifferential operator.

Notation pertaining to the $\nabla\phi$ model.
δ_z : the Dirac measure centered on $z \in \mathbb{R}$,
$\mathbb{R}^U$: for a discrete bounded subset $U \subseteq \mathbb{Z}^d$, $\mathbb{R}^U$ denotes the set of functions from U to $\mathbb{R}$,
$\mathcal{P}\left(\mathbb{R}^U\right)$: the set of probability measure on $\mathbb{R}^U$,
Leb : Lebesgue measure on $\mathbb{R}^U$,
$\mathbb{P} \ll \mathrm{Leb}$: means that the probability measure $\mathbb{P} \in \mathcal{P}\left(\mathbb{R}^U\right)$ is absolutely continuous with respect to the Lebesgue measure,
$\frac{\mathrm{d}\mathbb{P}}{\mathrm{dLeb}}$: the Radon-Nikodym derivative of $\mathbb{P}$ with respect to the Lebesgue measure, assuming that $\mathbb{P} \ll \mathrm{Leb}$,
$f_*\mathbb{P}$: pushforward of a measure $\mathbb{P} \in \mathcal{P}\left(\mathbb{R}^U\right)$ by a measurable function $f : \mathbb{R}^U \to \mathbb{R}^U$.

1.1. Stochastic homogenization

1.1.1. General introduction. The general goal of the theory of homogenization is to answer the following question: given a heterogeneous medium exhibiting microscopic variations in its composition, do the properties of the environment on a large scale behave like the ones of a homogeneous medium? This phenomenon, when it happens, is called *homogenization*. It relies on the intuition that if the variations in the composition are distributed randomly, they will average out on a large scale, following some sort of law of large numbers. Understanding precisely when and how homogenization happens is useful since many physical models, involving heat conduction or electromagnetism, are dealing with heterogeneous environments. The equations obtained from the physical models exhibit quick oscillations and are costly to solve numerically. Nevertheless, it is possible to derive valuable information with a much lower numerical cost thanks to the theory of homogenization: an interesting strategy is to prove that a given heterogeneous system homogenizes, to solve the homogenized equation, and deduce from these computations information on the heterogeneous medium.

An example of a physically relevant situation where homogenization is expected to happen is heat conduction in a heterogeneous environment. Consider a composite medium made of different materials with different heat conductivity properties. We wish to study heat conduction in this composite material. Mathematically, it is modeled by the parabolic equation

$$\partial_t u - \nabla \cdot (\mathbf{a}(x) \nabla u) = 0,$$

with suitable boundary conditions. Sending the time to infinity, the steady state energy profile satisfies the elliptic equation

$$\nabla \cdot (\mathbf{a}(x) \nabla u) = 0. \tag{1.1.1}$$

The fact that the medium is heterogeneous means that the diffusivity matrix $\mathbf{a}(x)$ varies in space. The equation (1.1.1) is the main object of study of this thesis, and we now introduce a precise model of *stochastic homogenization* to study this partial differential equation. Fix a dimension $d \geq 1$ and denote by $\mathcal{S}(\mathbb{R}^d)$ the set of symmetric matrices of $\mathbb{R}^d$. We then consider a random mapping

$$\begin{cases} \mathbb{R}^d \to \mathcal{S}(\mathbb{R}^d) \\ x \mapsto \mathbf{a}(x), \end{cases}$$

which is measurable with respect to the Borel sigma-algebras on $\mathbb{R}^d$ and $\mathcal{S}(\mathbb{R}^d)$. We assume additionally that there exist two ellipticity constants $0 < \lambda \leq \Lambda < \infty$

$$\lambda I_d \leq \mathbf{a}(x) \leq \Lambda I_d. \tag{1.1.2}$$

The matrix $\mathbf{a}$ is called the *environment*. As was mentioned above, we suppose that the environment $\mathbf{a}$ is random, denote by $\mathbb{P}$ its law and by $\mathbb{E}$ the expectation associated to this probability measure. The law $\mathbb{P}$ is assumed to satisfy the following properties:

- *Stationarity:* the law $\mathbb{P}$ is invariant under translations by any vector of $\mathbb{Z}^d$, i.e. for any $y \in \mathbb{Z}^d$, the laws of $\mathbf{a}$ and of $\mathbf{a}(y + \cdot)$ are the same.
- *Ergodicity:* the law $\mathbb{P}$ of the environment is ergodic with respect to the translations of $\mathbb{Z}^d$.

With these two assumptions, one can develop a *qualitative* theory of stochastic homogenization, i.e. establish convergence theorem without any explicit rate. The assumption of ergodicity is qualitative in nature and one can only expect to obtain qualitative results under this assumption.

Over the past few years, much progress was achieved to develop a *quantitative* theory of stochastic homogenization, i.e. establish convergence theorems, with explicit rates. It is the object of study of this thesis and one must strengthen the ergodicity assumption to obtain such results. A possibility, among many others, is to assume a finite range dependence on the coefficient field $\mathbf{a}$.

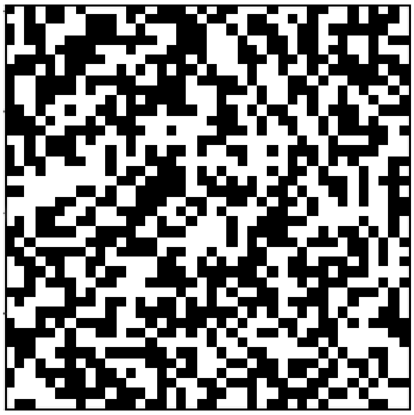

FIGURE 1.1.1. A typical environment satisfying the stationarity and finite range dependence assumption is the checkerboard: select two deterministic symmetric positive definite matrices $\mathbf{a}_1$ and $\mathbf{a}_2$. The environment $\mathbf{a}$ is defined such that such that $\mathbf{a}$ is equal to $\mathbf{a}_1$ on the black cells and is equal to $\mathbf{a}_2$ on the white cells. The color of the cells is chosen according to a Bernoulli site percolation of probability $p \in [0,1]$.

- *Finite range dependence.* If, for any open subset $U \subseteq \mathbb{R}^d$, we let $\mathcal{F}(U)$ be the sigma-algebra generated by the family of mappings,

$$\mathbf{a} \to \int_U \phi(x)\mathbf{a}(x)\,dx,\ \phi \in C_c^\infty(U),$$

then the sigma-algebras $\mathcal{F}(U)$ and $\mathcal{F}(V)$ are independent as soon as $\operatorname{dist}(U,V) \geq 1$.

One then wishes to study the large scale behavior of the solutions of the equation (1.1.1). To this end, it is customary to introduce a parameter $0 < \varepsilon \ll 1$, which represents the ratio between the microscopic and the macroscopic scale, and to rescale the equation as

$$\nabla \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon \right) = 0.$$

A typical homogenization theorem one wishes to show is the following: given a bounded domain $U \subseteq \mathbb{R}^d$ and a function $f \in H^1(U)$, the family of solutions u^ε of the elliptic problems

$$\begin{cases} \nabla \cdot \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon \right) = 0 & \text{in } U \\ u = f & \text{on } \partial U, \end{cases} \tag{1.1.3}$$

converges in $L^2(U)$ as ε tends to 0 to the solution $\overline{u}$ of the elliptic equation

$$\begin{cases} \nabla \cdot (\bar{\mathbf{a}} \nabla \overline{u}) = 0 & \text{in } U \\ \overline{u} = f & \text{on } \partial U, \end{cases} \tag{1.1.4}$$

where the matrix $\bar{\mathbf{a}}$, called the *effective* or *homogenized environment*, is deterministic and constant in space. Solving the elliptic problem (1.1.4) is equivalent, up to a change of variables, to solving the standard Poisson problem on the domain U and is less costly to solve numerically than the heterogeneous equation (1.1.3). The homogenized environment $\bar{\mathbf{a}}$ depends only on the law of the coefficient field $\mathbf{a}$ (in particular it does not depend on the domain U nor on the boundary condition f) but it depends in a complicated manner on the law of the environment. There is in general no explicit formula to describe this quantity and $\bar{\mathbf{a}}$ is not equal to the expectation of $\mathbf{a}$ in general.

1.1.2. Two-scale expansion and the corrector. In this section we introduce one of the most important tools to study homogenization: the two-scale expansion. It can be described by the following Ansatz: we let u^ε be the solution to (1.1.3) and postulate that u^ε can be expanded, as ε tends to zero, according to

$$u^\varepsilon(x) \simeq u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots \tag{1.1.5}$$

where the function u_0, u_1, u_2, etc. are reasonable and do not depend on ε. This Ansatz is called the two-scale expansion and yields precious information about homogenization, as we shall now explain. A first computation one can perform is to differentiate the previous identity formally. We denote by ∇_x (resp. ∇_y) the gradient with respect to the first variable x (resp. the second variable $\frac{x}{\varepsilon}$). We differentiate (1.1.5) and sort the different terms obtained according to the powers of the factors ε. This yields

$$\nabla u^\varepsilon(x) = \frac{1}{\varepsilon}\nabla_y u_0\left(x, \frac{x}{\varepsilon}\right) + \left(\nabla_x u_0\left(x, \frac{x}{\varepsilon}\right) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right) + \varepsilon\left(\nabla_x u_1\left(x, \frac{x}{\varepsilon}\right) + \nabla_y u_2\left(x, \frac{x}{\varepsilon}\right)\right) + \cdots \tag{1.1.6}$$

Since u^ε is a solution of the elliptic equation (1.1.3), its gradient cannot be too large (it is at least bounded in $L^2(U)$) and thus we expect, at least heuristically,

$$\nabla_y u_0\left(x, \frac{x}{\varepsilon}\right) = 0. \tag{1.1.7}$$

With this new piece of information, the two-scale expansion can be rewritten

$$u^\varepsilon(x) \simeq u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

Sending ε to zero should imply

- $u^\varepsilon - u_0$ converges to 0 in $L^2(U)$ as ε goes to zero,
- $\nabla u^\varepsilon - \nabla u_0(x) - \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)$ converges to zero in $L^2(U)$.

These remarks already give some valuable information, indeed while we expect u^ε to converge to the function u_0, we do not expect that it will be the case for the gradient of u^ε: there should be a corrective term $\nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)$ preventing ∇u^ε from converging to ∇u_0 in L^2.

More information can be obtained from this Ansatz such as

(1) identifying the homogenized matrix $\bar{\mathbf{a}}$ and obtaining a heuristic argument to characterize u_0 as the solution of the equation (1.1.4),
(2) identifying the corrective term $u_1\left(x, \frac{x}{\varepsilon}\right)$,

as we shall now explain. The first step is to define a quantity of interest in homogenization, the *first-order corrector*. To this end, we simplify the problem (1.1.3) and consider a specific case: the situation where the boundary condition is affine. For $p \in \mathbb{R}^d$, we let l_p be the affine function of slope p, i.e. $l_p(x) = p \cdot x$. In this setting, u_0 is expected to be the affine function l_p, since the affine functions are $\bar{\mathbf{a}}$-harmonic (independently of the value of $\bar{\mathbf{a}}$). We define, for each $p \in \mathbb{R}^d$ and $\varepsilon > 0$,

$$\phi_p^{1/\varepsilon} : \begin{cases} \frac{1}{\varepsilon}U \to \mathbb{R} \\ x \mapsto \frac{1}{\varepsilon}u^\varepsilon(\varepsilon x) - p \cdot x, \end{cases}$$

This function is defined to verify the expansion

$$u^\varepsilon(x) = p \cdot x + \phi_p^{1/\varepsilon}\left(\frac{x}{\varepsilon}\right).$$

Note also that $\phi_p^{1/\varepsilon}$ belongs to the Sobolev space $H_0^1\left(\frac{1}{\varepsilon}U\right)$. By a change of variables, one can bound the rescaled L^2 norm of $\nabla \phi_p^{1/\varepsilon}$,

$$\left\|\nabla \phi_p^{1/\varepsilon}\right\|_{\underline{L}^2\left(\frac{1}{\varepsilon}U\right)} \le C|p|,$$

Applying the Poincaré inequality for functions in $H_0^1\left(\frac{1}{\varepsilon}U\right)$, we obtain an estimate on the averaged L^2 norm of $\phi_p^{1/\varepsilon}$,

$$\left\|\phi_p^{1/\varepsilon}\right\|_{\underline{L}^2\left(\frac{1}{\varepsilon}U\right)} \le C|p|\varepsilon^{-1}.$$

On a heuristic level, one can interpret these two inequalities the following way: the growth of the function $\phi_p^{1/\varepsilon}$ is at most linear and the gradient of $\phi_p^{1/\varepsilon}$ is of order 1. Following the heuristic of the two-scale expansion, one can postulate a stronger result on the growth of $\phi_p^{1/\varepsilon}$: since u^ε is expected to converge to l_p in $L^2(U)$, one expects

$$\left\|\phi_p^{1/\varepsilon}\right\|_{\underline{L}^2\left(\frac{1}{\varepsilon}U\right)} \ll \varepsilon^{-1},$$

which means that this function has sublinear growth.

Even though we are not justifying this fact in this section, it is possible to take the limit $\varepsilon \to 0$ for the mapping $\nabla\phi_p^{1/\varepsilon}$. This gives rise to a gradient field $\nabla\phi_p : \mathbb{R}^d \to \mathbb{R}^d$. From this gradient field, one recovers a function $\phi_p : \mathbb{R}^d \to \mathbb{R}$, which is only defined up to a constant since we only have access to information about its gradient. This map is called *the corrector* and plays a central role in homogenization. Since we need to use the corrector, and not only its gradient, in the rest of the introduction, and do not want to work with an ill-defined quantity, we select one corrector among all the possibilities according to the arbitrary criterion

$$\int_{[0,1]^d} \phi_p(x)\,dx = 0.$$

Note that the corrector depends on the environment $\mathbf{a}$ and is thus a random function. For simplification, this dependence is not displayed in the notation. We expose below, without proof and heuristically, a few properties of the corrector.

(i) The corrector is sublinear, for each $R > 0$, one has

$$\|\phi_p\|_{L^\infty(B_R)} \ll R, \tag{1.1.8}$$

(ii) It is solution of the elliptic equation

$$\nabla\cdot\left(\mathbf{a}\left(p+\nabla\phi_p\right)\right) = 0 \text{ in } \mathbb{R}^d. \tag{1.1.9}$$

(iii) It is unique up to a constant: if ϕ_p, ϕ_p' are two functions satisfying (i) and (ii) then there exists a constant $c \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R}^d,\ \phi_p(x) = \phi_p'(x) + c.$$

In particular, $\nabla\phi_p$ is uniquely defined.

(iv) The mapping $p \to \nabla\phi_p$ is linear.

(v) The gradient of the corrector is $\mathbb{Z}^d$-stationary, for each $y \in \mathbb{Z}^d$,

$$\nabla\phi_p \text{ and } \nabla\phi_p(y+\cdot) \text{ have the same law.}$$

Remark 1.1.1. The sublinearity of the corrector is a very important property even in the very definition of the corrector: if we allow the corrector to have linear growth then one can simply take $\phi_p = -l_p$ which is a solution to (1.1.9). Of course looking at this kind of solutions does not have any mathematical interest.

Remark 1.1.2. The linearity property (iv) is a consequence of the properties (i), (ii) and (iii). Indeed if one selects two vectors $p, q \in \mathbb{R}^d$, then the mapping $\phi_p + \phi_q$ satisfies $\nabla\cdot\left(\mathbf{a}\left(p+q+\nabla\phi_p+\nabla\phi_q\right)\right) = 0$ and has sublinear oscillations.

Remark 1.1.3. The stationarity property is also a consequence of the properties (i), (ii) and (iii). Given an environment $\mathbf{a}$, we let $\phi_p^{\mathbf{a}}$ the corrector for the environment $\mathbf{a}$. For $y \in \mathbb{Z}^d$, it is clear that

$$\nabla\cdot\left(\mathbf{a}\left(p+\nabla\phi_p^{\mathbf{a}}\right)\right) = 0 \text{ in } \mathbb{R}^d \implies \nabla\cdot\left(\mathbf{a}(y+\cdot)\left(p+\nabla\phi_p^{\mathbf{a}}(y+\cdot)\right)\right) = 0 \text{ in } \mathbb{R}^d.$$

The mapping $x \to \phi_p^{\mathbf{a}}(y+x)$ also has sublinear oscillations. Using the uniqueness property (iii), we obtain the following identity

$$\nabla \phi_p^{\mathbf{a}}(y+\cdot) = \nabla \phi_p^{\mathbf{a}(y+\cdot)}.$$

The stationarity property on the environment implies that $\nabla \phi_p$ and $\nabla \phi_p(y+\cdot)$ have the same law.

The corrector is a useful object and can be used to define the homogenized matrix $\bar{\mathbf{a}}$ and the corrective term u_1 in the two-scale expansion. This is the subject of the next paragraph and we first explain how to define the homogenized environment $\bar{\mathbf{a}}$ in terms of the corrector.

Definition 1.1.4. Using the linearity of the mapping $p \to \nabla \phi_p$, we define the matrix $\bar{\mathbf{a}}$ according to, for each $p \in \mathbb{R}^d$,

$$\bar{\mathbf{a}} p = \mathbb{E}\left[\int_{[0,1]^d} \mathbf{a}(x)\,(p+\nabla\phi_p)\,dx\right]. \tag{1.1.10}$$

From this formula, we see that the matrix $\bar{\mathbf{a}}$ depends on the environment $\mathbf{a}$ in a complicated fashion involving the corrector, in particular,

$$\text{we expect that } \bar{\mathbf{a}} \neq \mathbb{E}\left[\int_{[0,1]^d} \mathbf{a}(x)\,dx\right] \text{ in general.}$$

We next record, without proof, a few properties of the corrector and the homogenized matrix $\bar{\mathbf{a}}$.

Proposition 1.1.5. *The homogenized environment* $\bar{\mathbf{a}}$ *satisfies the following properties.*

- *It is symmetric positive definite and satisfies*

$$\lambda I_d \le \bar{\mathbf{a}} \le \Lambda I_d.$$

- *Using the stationarity and ergodicity assumptions on the environment, the equality in expectation* (1.1.10) *can be refined: one has the almost sure weak convergence,*

$$\mathbf{a}\left(\frac{x}{\varepsilon}\right)\left(p+\nabla\phi_p\left(\frac{x}{\varepsilon}\right)\right) \rightharpoonup \bar{\mathbf{a}} p \text{ in } L^2\left([0,1]^d\right) \text{ as } \varepsilon \to 0.$$

Remark 1.1.6. Computing the value of the homogenized matrix $\bar{\mathbf{a}}$ given the law of an environment is a very difficult problem in general and can only be solved exactly in some very specific cases:

- In dimension 1, in that case the differential equation (1.1.3) can be solved explicitly and the problem reduces to a law of large numbers. An explicit computation gives

$$\bar{\mathbf{a}} = \left(\mathbb{E}\left[\int_0^1 \mathbf{a}(x)^{-1}\,dx\right]\right)^{-1}.$$

- In dimension 2 under a specific assumption on the law of the environment (see [**98**, Section 7.3]). We do not make this condition explicit here but mention that it is satisfied by the checkerboard described in Figure 1.1.1 with $\mathbf{a}_1 = \alpha I_2$, $\mathbf{a}_2 = \beta I_2$, $\alpha, \beta > 0$ and probability $p = 1/2$. The homogenized matrix is given by

$$\bar{\mathbf{a}} = \sqrt{\alpha\beta} I_2.$$

We now come back to the two-scale expansion introduced at the beginning of this section, explain how the corrector can be used in this expansion and why the term u_0 satisfies the elliptic equation (1.1.4) for the definition of $\bar{\mathbf{a}}$ stated in (1.1.10).

Using the computation for the gradient of u^ε presented in (1.1.6) together with (1.1.7), gives

$$\nabla u^\varepsilon(x) = \nabla u_0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\left(\nabla_x u_1\left(x,\frac{x}{\varepsilon}\right) + \nabla_y u_2\left(x,\frac{x}{\varepsilon}\right)\right) + \cdots \tag{1.1.11}$$

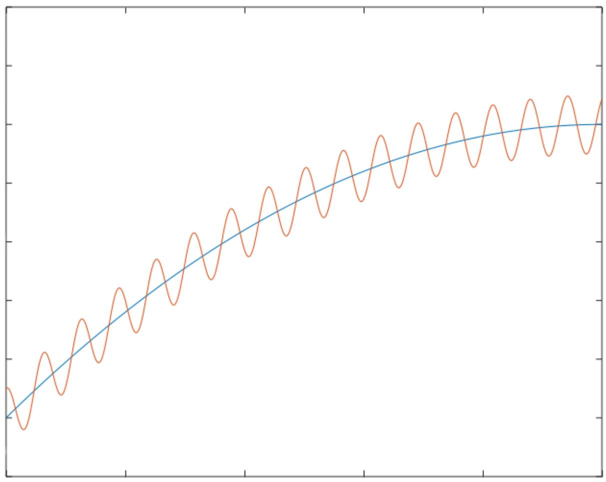

FIGURE 1.1.2. The function u^ε, drawn in red, is expected to fluctuate around the homogenized and smoother function u_0, drawn in blue.

The main contribution in the gradient of u^ε is given by the term $\nabla_x u_0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)$. Since u^ε solves the elliptic equation (1.1.3), we expect

$$\nabla \cdot \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right)\left(\nabla u_0(x) + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right)\right) \simeq 0. \tag{1.1.12}$$

The key idea is then to distinguish two scales, the macroscopic scale, which corresponds to the variable x, and the microscopic one, which corresponds to the variable x/ε. The function $\nabla_x u_0$ is a function of x and oscillates on the macroscopic scale of size typically 1. These oscillations are very slow compared to the ones of the maps $\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)$ and $\nabla_y u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)$, which happen on a scale of size ε (see Figure 1.1.2). This leads us to make the rough but fruitful assumption that the term $\nabla_x u_0(x)$ can be treated, in first order approximation, as a constant.

To emphasize this idea, we set the notation $p := \nabla_x u_0(x)$ in the next display. The identity (1.1.12) becomes

$$\nabla \cdot \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right)\left(p + \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)\right)\right) \simeq 0.$$

At this point of the argument, the corrector ϕ_p is a natural candidate for u_1 and we set

$$u_1\left(x, \frac{x}{\varepsilon}\right) = \phi_p\left(\frac{x}{\varepsilon}\right) = \phi_{\nabla u_0(x)}\left(\frac{x}{\varepsilon}\right).$$

To simplify the previous display, we decompose the gradient of u_0 along the canonical basis of $\mathbb{R}^d$ and write $\nabla u_0(x) = \sum_{i=1}^d \frac{\partial u_0}{\partial x_i}(x)\mathfrak{e}_i$. Together with the linearity of the mapping $p \to \phi_p$, this implies

$$u_1\left(x, \frac{x}{\varepsilon}\right) = \phi_{\nabla u_0(x)}\left(\frac{x}{\varepsilon}\right) = \sum_{i=1}^d \frac{\partial u_0}{\partial x_i}(x)\phi_{\mathfrak{e}_i}\left(\frac{x}{\varepsilon}\right).$$

Using the corrector, we were able to identify, at least heuristically, the term u_1 in the two-scale expansion of u^ε. This argument can be made rigorous as we will see later. It also highlights the importance of the corrector in homogenization: if one can understand precisely how this function behaves: one should be able to get precise information on the solutions of the elliptic equation $\nabla \cdot \mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla u = 0$, with very general boundary conditions.

An important point remains to be treated: we need to explain, at least heuristically, why u_0 solves the homogenized equation (1.1.4). Now that the first error term u_1 in the two-scale expansion has been identified, the argument is rather straightforward, indeed (1.1.11) becomes

$$\begin{aligned}\nabla u^{\varepsilon}(x) &= \nabla u_0(x) + \sum_{i=1}^{d} \frac{\partial u_0}{\partial x_i}(x) \nabla \phi_{\mathfrak{e}_i}\left(\frac{x}{\varepsilon}\right) + \text{lower order terms} \\ &= \sum_{i=1}^{d} \frac{\partial u_0}{\partial x_i}(x)\left(\mathfrak{e}_i + \nabla \phi_{\mathfrak{e}_i}\right) + \text{lower order terms,}\end{aligned}$$

and thus

$$0 = \nabla \cdot \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right) \nabla u^{\varepsilon}(x)\right) = \sum_{i=1}^{d} \nabla \cdot \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right)\left(\mathfrak{e}_i + \nabla \phi_{\mathfrak{e}_i}\right) \frac{\partial u_0}{\partial x_i}(x)\right) + \text{lower order terms.}$$

Taking the limit $\varepsilon \to 0$, we expect

$$\mathbf{a}\left(\frac{x}{\varepsilon}\right)\left(\mathfrak{e}_i + \nabla \phi_{\mathfrak{e}_i}\right) \rightharpoonup \bar{\mathbf{a}} \mathfrak{e}_i \text{ in } L^2(U),$$

and thus

$$\sum_{i=1}^{d} \nabla \cdot \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right)\left(\mathfrak{e}_i + \nabla \phi_{\mathfrak{e}_i}\right) \frac{\partial u_0}{\partial x_i}(x)\right) \to \sum_{i=1}^{d} \nabla \cdot \left(\bar{\mathbf{a}} \mathfrak{e}_i \frac{\partial u_0}{\partial x_i}(x)\right) = \nabla \cdot \left(\bar{\mathbf{a}} \nabla u_0\right),$$

where the convergence holds at least in the sense of distributions. The lower order terms are also expected to converge to 0, at least in the sense of distributions. This implies

$$\nabla \cdot \left(\bar{\mathbf{a}} \nabla u_0\right) = 0.$$

This series of ideas about the two-scale expansion can be made rigorous and is summarized in the following theorem.

Theorem 1.1.1 (Homogenization Theorem, qualitative version [**106, 135, 150**]). *Let U be a bounded smooth domain of $\mathbb{R}^d$ and let $f \in C^{\infty}\left(\overline{U}\right)$. Under the assumptions of stationarity and ergodicity of the environment, if we let u^{ε} be the solution of*

$$\left\{\begin{array}{ll}\nabla \cdot \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right) \nabla u^{\varepsilon}\right) = 0 & \text{in } U \\ u = f & \text{on } \partial U,\end{array}\right. \tag{1.1.13}$$

and u_0 be the deterministic solution of

$$\left\{\begin{array}{ll}\nabla \cdot \left(\bar{\mathbf{a}} \nabla u_0\right) = 0 & \text{in } U \\ u_0 = f & \text{on } \partial U,\end{array}\right. \tag{1.1.14}$$

then

$$\left\|u^{\varepsilon} - u_0\right\|_{L^2(U)} \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ almost surely.}$$

The gradient of u^{ε} does not converge to the gradient of u_0: there is a corrective term which can be identified thanks to the two-scale expansion and one has

$$\left\|\nabla u^{\varepsilon} - \nabla u_0 - \sum_{i=1}^{d} \frac{\partial u_0}{\partial x_i} \nabla \phi_{\mathfrak{e}_i}\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(U)} \underset{\varepsilon \to 0}{\longrightarrow} 0. \tag{1.1.15}$$

Remark 1.1.7. As was mentioned above, ∇u^{ε} does not converge to ∇u_0 strongly in L^2 but the convergence is true for the weak topology in L^2. This can be justified by the following arguments

- By the stationarity and ergodicity assumptions on the environment, together with the ergodic theorem, we have

$$\limsup_{\varepsilon \to \infty} \left\|\nabla \phi_p\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(U)} = \mathbb{E}\left[\int_{[0,1]^d} \left|\nabla \phi_p\right|^2\right] < \infty \text{ almost-surely.}$$

  In particular, almost surely, one can extract a subsequence of $\left(\nabla \phi_p\left(\frac{\cdot}{\varepsilon}\right)\right)_{\varepsilon > 0}$ which converges weakly in $L^2(U)$.

- Using the sublinearity of the corrector, one has, for each $g \in C_c^\infty(U)$,

$$\int_U g(x)\nabla\phi_p\left(\frac{x}{\varepsilon}\right)dx = \int_U \operatorname{div} g(x)\varepsilon\phi_p\left(\frac{x}{\varepsilon}\right)dx$$
$$\leq \|\operatorname{div} g\|_{L^2(U)}\,\varepsilon\left\|\phi_p\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(U)}.$$

Using the sublinearity of ϕ_p, we know that $\varepsilon\left\|\phi_p\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(U)}$ converges to zero almost surely. Consequently

$$\int_U g(x)\nabla\phi_p\left(\frac{x}{\varepsilon}\right)dx \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

- Using that u_0 is a solution of the elliptic equation (1.1.14), one knows, by regularity theory, that it is very smooth: precise pointwise bounds on u_0 as well as on all its derivatives are available. The previous argument for the weak convergence of the corrector can be adapted to show

$$\frac{\partial u_0}{\partial x_i}(x)\nabla\phi_{\mathfrak{e}_i}\left(\frac{x}{\varepsilon}\right) \rightharpoonup 0 \text{ in } L^2(U), \text{ almost surely.}$$

Combining this result with the strong L^2 convergence given in (1.1.15), shows

$$\nabla u^\varepsilon \rightharpoonup \nabla u_0 \text{ in } L^2(U), \text{ almost surely.}$$

Remark 1.1.8. The L^2 convergence can be upgraded into an L^∞ convergence. Indeed the De Giorgi-Nash-Moser theory (see [**54, 125, 126**] or [**74**, Chapter 8]) shows that the solutions of (1.1.13) and (1.1.14) are α-Hölder continuous, for some small exponent $\alpha > 0$, as well as the bounds

$$\|u^\varepsilon\|_{C^\alpha(U)} \leq C \text{ and } \|u_0\|_{C^\alpha(U)} \leq C.$$

One can then conclude by interpolating the L^∞ norm between the L^2 norm (which is small) and the C^α norm (which is bounded).

1.1.3. The probabilistic approach. The problem of stochastic homogenization was presented in the previous sections in an analytical framework. It can also have interesting applications in probability and more specifically in random walks in random environments.

Given a uniformly elliptic environment environment $\mathbf{a}$, we denote by $(X_t)_{t\geq 0}$ the diffusion process associated with the generator $-\nabla\cdot\mathbf{a}\nabla$, and by $\mathbb{P}_y^{\mathbf{a}}$ and by $\mathbb{E}_y^{\mathbf{a}}$ its law and expectation started from $y \in \mathbb{R}^d$.

1.1.3.1. *The elliptic problem.* Obtaining information on the generator $-\nabla\cdot\mathbf{a}\nabla$ yields information of the random walker itself. A first example of this can be the relation between the random walker and the solution of the Dirichlet problem: let $f \in C^\infty\left(\overline{B}_1\right)$, then the solution of the elliptic equation

$$\begin{cases} \nabla\cdot(\mathbf{a}\nabla u) = 0 \text{ in } B_1, \\ u = f \text{ on } \partial B_1 \end{cases} \tag{1.1.16}$$

satisfies the identity $u(x) = \mathbb{E}_x^{\mathbf{a}}\left[f\left(X_{\tau_{\partial B_1}}\right)\right]$, where $\tau_{\partial B_1}$ is the stopping time for the diffusion process X_t

$$\tau_{\partial B_1} := \inf\{t \in \mathbb{R}_+ \,:\, X_t \in \partial B_1\}.$$

Introducing a microscopic scale $\varepsilon > 0$ and performing a change of variables shows that the solution u^ε of the Dirichlet problem

$$\begin{cases} \nabla\cdot\left(\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = 0 \text{ in } B_1, \\ u^\varepsilon = f \text{ on } \partial B_1 \end{cases}$$

satisfies

$$u^\varepsilon(x) = \mathbb{E}_{\varepsilon^{-1}x}^{\mathbf{a}}\left[f\left(\varepsilon X_{\tau_{\partial B_{\varepsilon^{-2}}}}\right)\right].$$

The homogenization theorem stated in the previous section (Theorem 1.1.1) or more precisely the third point of the Remark 1.1.7 upgrading the L^2 convergence into an L^∞ convergence, shows that

$$u^\varepsilon(x) \underset{\varepsilon\to 0}{\longrightarrow} u_0(x), \ \forall x \in B_1, \text{ almost surely,}$$

where u_0 is the solution of the homogenized equation (1.1.14). If we let $B_t^{\bar{\mathbf{a}}}$ be a brownian motion of diffusivity $2\bar{\mathbf{a}}$ starting form x, we have the representation formula

$$u_0(x) = \mathbb{E}\left[f\left(B^{\bar{\mathbf{a}}}_{\tau_{\partial B_1}}\right)\right].$$

This arguments tells us that the rescaled process $\varepsilon X_{\varepsilon^{-2}t}$ behaves like a brownian motion of diffusivity $2\bar{\mathbf{a}}$, almost surely in the environment: we expect a quenched invariance principle for the diffusion process to hold. For a precise statement of the result, we refer to the work of Papanicolaou and Varadhan [**135**] or to the one of Osada [**134**].

1.1.3.2. *The parabolic problem.* The approach of the previous paragraph can be made much more precise by studying the following parabolic problem: for $f \in C_c^\infty(\mathbb{R}^d)$, we let $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be the solution of the parabolic equation

$$\left\{ \begin{array}{l} \partial_t u + \nabla \cdot (\mathbf{a}\nabla u) = 0 \text{ in } (0,\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = f \text{ in } \{0\}\times\mathbb{R}^d. \end{array} \right. \tag{1.1.17}$$

The Markov process X_t is related to the function u through the identity

$$u(t,x) := \mathbb{E}_x^{\mathbf{a}}\left[f(X_t)\right].$$

In particular, information on the solutions of the parabolic problem can be transferred into information on the random walker. The theory of stochastic homogenization introduced in the previous section was presented in the setting of elliptic equations mostly for the sake of simplicity: it can be extended to the setting of parabolic equations (see [**135**]). One can prove a parabolic version of Theorem 1.1.1 and expect that the rescaled process $\varepsilon X_{\varepsilon^{-2}t}$ satisfies

$$\mathbb{E}^{\mathbf{a}}_{\varepsilon^{-2}x}\left[f\left(\varepsilon X_{\varepsilon^{-2}t}\right)\right] \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}\left[f\left(B_t^{\bar{\mathbf{a}}}\right)\right].$$

A theorem associated to this heuristic result was established in [**135**] and requires to average on the starting point of the diffusion.

1.1.3.3. *Probabilistic use of the corrector.* Another, more probabilistic, approach to show a quenched invariance principle for the diffusion process, makes use of the corrector. If one denotes by $\chi := (\chi_{\mathfrak{e}_1},\ldots,\chi_{\mathfrak{e}_d})$ the vector-valued corrector, then the process

$$X_t + \chi(X_t) \text{ is a martingale.}$$

The strategy is to apply a standard martingale convergence theorem and then to derive a quenched invariance principle for the rescaled process $\varepsilon X_{t/\varepsilon^2} + \varepsilon\chi(X_{t/\varepsilon^2})$. Using the sublinearity of the corrector χ allows to show that the term $\chi(X_t)$ is negligible compared to the main term X_t, and eventually to deduce that the diffusion process X_t itself satisfies a quenched central limit theorem. This strategy was carried out by Kozlov in [**105**], Kipnis and Varadhan in [**104**] and by Boivin in [**36**].

1.1.4. History of stochastic homogenization. A qualitative theory of stochastic homogenization was initiated in the early 80s, with the works of Kozlov [**107**], Papanicolaou and Varadhan [**135**] and Yurinskiĭ [**150**]. These results were then extended by Dal Maso and Modica in [**49, 50**], who used variational arguments to study nonlinear elliptic equations. Their proofs rely on an application of the ergodic theorem and are thus purely qualitative.

To go beyond the qualitative theory and obtain quantitative rates of convergence for homogenization, it is necessary that the law of the environment $\mathbf{a}$ satisfies some quantitative ergodic conditions, such as, but not necessarily restricted to, the finite range dependence assumption mentioned in Section 1.1.1. The main difficulty in this problem is that the solutions of the elliptic equations $\nabla\cdot\mathbf{a}\nabla$ depend in a very complicated manner on the coefficient field $\mathbf{a}$, and it is thus not clear how to proceed to transfer the quantitative ergodic conditions from the environment to the solutions. This has been an active field of research over the past few years and the theory is now well-understood, at least in the uniformly elliptic setting.

The first quantitative results were achieved by Yurinskiĭ in [**149**], where he obtained an algebraic, suboptimal rate of convergence for the homogenization error under a uniform mixing condition on the environment, in dimension $d \geq 3$. Ten years later, in [**133**], Naddaf and Spencer, using tools from statistical mechanics were able to derive optimal rates of convergence in the setting of small ellipticity contrast. Additional results were obtained in this direction by Conlon and Naddaf [**45**] and Conlon and Spencer in [**46**], where Green's functions were studied.

In the general case, the first satisfactory quantitative results were obtained by Gloria and Otto in [**82, 83**]. They studied the case when the environment can be decomposed into a countable number of independent identically distributed random variables. Their approach, which builds upon the ideas of Naddaf and Spencer [**133**], relies on concentration inequalities, such as Spectral Gap or Logarithmic Sobolev inequalities, to transfer quantitative information from the coefficient field to the solutions. In particular, they obtained estimates on the corrector, the fluctuations of the energy density of the corrector and the approximation of the homogenized matrix which are optimal in terms of spatial scaling and suboptimal in terms of stochastic integrability. Then in collaboration with Neukamm in [**81**], they extended the ideas of [**82, 83**] and were able to obtain an optimal estimate for the decay in time of the parabolic equation associated to the corrector and to deduce moments bounds for the corrector.

Another approach was initiated by Armstrong and Smart in [**21**], who extended the techniques of Avellaneda and Lin [**22, 23**] and the ones of Dal Maso and Modica [**49, 50**]. They were able to obtain a quenched large scale $C^{0,1}$ regularity theory under an assumption of finite range dependence on the environment. This was then generalized by Armstrong, Kuusi and Mourrat to general mixing conditions and to other types of equations [**20, 16**] and improved to obtain optimal rates of convergence [**17**]. The large scale regularity theory was also studied in the works of Gloria, Neukamm and Otto [**80**]. These results were extended by Fischer and Otto in [**65**], who developed a higher-order $C^{k,1}$ regularity theory, in the spirit of Theorem 1.1.4 below. In [**84**] Gloria and Otto obtained optimal bounds on the spatial average of the gradient and flux of the corrector and deduced from it bounds on the growth of the corrector as well as error estimates for the two-scale expansion.

The structure of correlations and fluctuations of the corrector were studied by Mourrat and Otto [**131**], Mourrat and Nolen [**130**], Gu and Mourrat [**88**]. The proofs rely on the Helffer-Sjöstrand representation formula, initially introduced in [**93, 145**] and then used by Naddaf and Spencer in [**132**] to derive a central limit theorem for the $\nabla\phi$ model (see Section 1.4 for a definition of this model). These works build on ideas present in the aforementioned works of Gloria, Neukamm, Otto as well as the ones of Gloria, Neukamm, Otto [**79**] and Marahrens and Otto [**110**]. A general theory to understand the structure of fluctuations in stochastic homogenization is established by Duerinckx, Gloria and Otto in [**60, 61**].

In the monograph [**18**], Armstrong Mourrat and Kuusi completed the program initiated a few years ago in [**21**] and were able to obtain optimal bounds on the first-order corrector, and optimal error estimates for the two-scale expansion. They also adapted the theory to the setting of parabolic equations (see also [**12, 27**]) and obtained optimal homogenization estimates for the elliptic and parabolic Green's functions.

The theory described in the previous paragraphs is mostly the theory of stochastic homogenization of uniformly elliptic linear equations, but it can be generalized to many other settings in different directions including:

- Treating nonlinear uniformly convex functionals, i.e. studying the minimizers of the problems

$$\min_u \int_U L(x, \nabla u(x))\, dx, \tag{1.1.18}$$

where $U \subseteq \mathbb{R}^d$ is open and $(x,p) \mapsto L(x,p)$ is random, convex in the second variable. This setting was considered by Dal Maso and Modica [**49**] and by Armstrong and Smart in [**21**]. Chapter 15 of [**98**] is devoted to the case of convex stationary ergodic integrands. Armstrong and Mourrat in [**20**] established a higher regularity theory and proved Lipschitz bounds for the minimizers with optimal stochastic integrability. More recently, Armstrong, Ferguson and Kuusi proved in [**14**] a Lipschitz regularity estimate for the differences of solutions, the main difficulty being that in the non-linear setting, the set of minimizers, or equivalently the set of solutions of the equation $\operatorname{div}(\nabla L(x, \nabla u(x))) = 0$, is not a vector space and the difference of two solutions is not, in general, a solution. In [**59**], Duerinckx and Gloria managed to prove qualitative homogenization for a family of nonconvex functionals with convex growth.

- Relaxing the uniform ellipticity assumption: this requires to weaken the uniform ellipticity assumption

$$\forall x \in \mathbb{R}^d, \ \lambda I_d \leq \mathbf{a}(x) \leq \Lambda I_d.$$

Research in this direction has attracted a lot of attention lately, in particular due to the relation between homogenization and random walks in random environments described in Section 1.1.3. In [**108**], Lamacz, Neukamm and Otto adapted the theory of homogenization to a model of Bernoulli bond percolation, where the standard model is modified such that all the bonds in a fixed direction are open. Another common way to study degenerate environments is to make the assumption that the ellipticity constants λ and Λ are random and to assume a moment condition: there exist $p, q \in [1, \infty]$ such that

$$\mathbb{E}\left[\lambda^{-p}\right] + \mathbb{E}\left[\Lambda^{q}\right] < \infty. \tag{1.1.19}$$

This setting was first considered by Andres, Deuschel, Slowik in [**9**] (see also [**10**]), and then by Chiarini and Deuschel in [**44**]. They are able to derive a quenched invariance principle for the diffusion process under the assumption $1/p + 1/q < 2/d$. In [**28**], Bella, Fehrman and Otto, still working under the assumption $1/p + 1/q < 2/d$, obtained a first-order Liouville theorem and a large scale $C^{1,\alpha}$ estimate for $\mathbf{a}$-harmonic functions. An extension of these results to the case of time-dependent coefficients has been carried out by Andres, Chiarini, Deuschel, and Slowik in [**8**]. The condition (1.1.19) requires the value of the conductances to be non-zero almost surely, an extension of this model in a case when the conductances are allowed to be zero and to be small (under some moment conditions) was investigated by Deuschel, Nguyen and Slowik in [**57**]. In [**76**], Giunti and Mourrat proved relaxation decay for the solutions of the parabolic equation, in the discrete setting, under the assumptions that the environment is bounded from above, i.e. $\mathbf{a}(x) \leq I_d$, and satisfies the degenerate lower bound condition $\mathbb{E}\left[\mathbf{a}(x)^{-p}\right] < \infty$. In [**75**], Giunti, Höfer and Velázquez studied homogenization for the Poisson equation in a randomly perforated domain.

1.1.5. The additive structure of stochastic homogenization. This section focuses on the theory of stochastic homogenization developed by Armstrong, Kuusi and Mourrat in [**18**] which contains most of the tools needed in this thesis. The approach relies on studying the energy quantities associated to the elliptic problem. In [**49**], Dal Maso and Modica were the first to study homogenization through the energy quantities. Given a bounded domain $U \subseteq \mathbb{R}^d$, we define, for $p \in \mathbb{R}^d$,

$$\nu(U,p) = \inf_{v \in l_p + H_0^1(U)} \fint_U \frac{1}{2} \nabla v \cdot \mathbf{a} \nabla v, \tag{1.1.20}$$

where l_p is the affine function given by $l_p(x) := p \cdot x$. By the assumption of uniform ellipticity made on the environment $\mathbf{a}$, one knows that this quantity is well-defined and that there exists a unique minimizer which is the solution to the elliptic equation

$$\begin{cases} \nabla \cdot (\mathbf{a}\nabla u) = 0 & \text{in } U, \\ u = l_p & \text{on } \partial U. \end{cases} \tag{1.1.21}$$

This quantity satisfies a number of convenient properties, one of the simplest and most remarkable is its subadditivity: consider $U_1, \ldots, U_n$ a partition of the open bounded set $U \subseteq \mathbb{R}^d$ into open sets, i.e. $U_i \subseteq U$ and $|U \setminus \bigcup_{i=1}^n U_i| = 0$, then

$$\nu(U,p) \le \sum_{i=1}^n \frac{|U_i|}{|U|} \nu(U_i,p). \tag{1.1.22}$$

The proof of this inequality is rather short: for each $i \in \{1, \ldots, n\}$, we let u_i be the solution of the equation (1.1.21) in the domain U_i, and u be the solution in the full domain U. The function

$$x \mapsto \sum_{i=1}^n u_i(x) \mathbb{1}_{U_i}(x) \tag{1.1.23}$$

belongs to the space $l_p + H_0^1(U)$ and can be used as a test function in the definition of $\nu(U,p)$, which implies the subadditivity result. For later purpose, we record that this statement can be quantified: by using the uniform ellipticity assumption on the environment $\mathbf{a}$, one derives

$$\sum_{i=1}^n \frac{|U_i|}{|U|} \|u_i - u\|_{\underline{L}^2(U_i)}^2 \le \frac{2}{\lambda} \sum_{i=1}^n \frac{|U_i|}{|U|} (\nu(U_i,p) - \nu(U,p)), \tag{1.1.24}$$

following the conventions of the calculus of variations, we refer to this identity as the second variation.

Remark 1.1.9. This shows in particular that the term on the left-hand side of the previous inequality is positive, which is (1.1.22).

If we restrict our consideration to cubes and introduce the notation

$$Q_r := \left(-\frac{r}{2}, \frac{r}{2}\right)^d,$$

by the subadditive ergodic theorem the sequence $\nu(Q_r,p)$ converges as r tends to infinity almost surely and in $L^1(\mathbb{P})$ to a deterministic value, which we denote by $\overline{\nu}(p)$. Since the mapping $p \mapsto \nu(Q_r,p)$ is quadratic (because the solution u of the equation (1.1.21) seen as a function of p is linear), we obtain that the limit $\overline{\nu}(p)$ is also quadratic. The following proposition relates $p \mapsto \overline{\nu}(p)$ to the homogenized matrix $\bar{\mathbf{a}}$.

Proposition 1.1.10. *For each $p \in \mathbb{R}^d$, one has the equality*

$$\overline{\nu}(p) = \frac{1}{2} p \cdot \bar{\mathbf{a}} p.$$

Remark 1.1.11. In [**18**], the matrix $\bar{\mathbf{a}}$ is defined by the previous identity, the corrector is then defined later in the proofs, the identity presented in Definition 1.1.4 becomes a proposition.

As was already mentioned in Section 1.1.4, the main difficulty to develop a quantitative theory of stochastic homogenization is to transfer information from the coefficient field to the solutions of the elliptic equation, since these solutions depend on the environment in a complicated, non-local and non-linear way. The core idea is to use the energy as an intermediate quantity and to split the argument into two steps:

- First we transfer information from the coefficient field to the energy, essentially by establishing a quantitative rate of convergence of $\nu(Q_r,p)$ toward $\frac{1}{2} p \cdot \bar{\mathbf{a}} p$.
- Second, we transfer information from the energy quantity $\nu(Q_r,p)$ to the solutions, this is explained in Section 1.1.8.

To investigate the first point, we note that the quantity $\nu(Q_r, p)$ is local, as can be immediately seen from its definition. Moreover as the size of the domain diverges, ν is expected to converge and the inequality (1.1.22) will become an equality. Under the finite range dependence assumption, this means that $\nu(U, p)$ can be written, up to a small error, as a sum of independent random variables. But sum of independent random variables are very well-understood and thanks to this, one can deduce precise quantitative estimates on the energy quantity. Once its behavior is well-understood, one can transfer information from it to the solutions of the elliptic PDE: one has used the energy as an intermediate quantity between the coefficient field and the solutions of the PDE.

The problem of the convergence of the energy can be separated into two distinct problems:

(1) Establishing a quantitative rate of convergence of $\mathbb{E}[\nu(Q_r, p)]$ to $p \cdot \bar{\mathbf{a}} p$ as r tends to infinity,
(2) Studying the fluctuations of $\nu(Q_r, p)$ around its expectation.

In the next two sections, we give a heuristic description of the proofs for the points (1) and (2). The precise arguments can be found in [**18**, Chapter 2].

1.1.6. Convergence of the expectation of the energy. To summarize what has been achieve so far, we know by a subadditivity argument that

$$\mathbb{E}[\nu(Q_r, p)] \text{ is decreasing and converges toward } \frac{1}{2} p \cdot \bar{\mathbf{a}} p.$$

Quantifying this convergence requires to solve the following difficulty: the subadditive ergodic theorem is intrinsically qualitative and cannot be quantified in general. In particular, it only relies on the ergodicity assumption on the environment which is not sufficient to derive a quantitative theory. To develop such a theory, one needs to find a replacement for this theorem which requires to make use of stronger mixing conditions on the law of the environment.

The main idea to quantify the rate of convergence is to note that ν is quadratic, uniformly convex in the p variable and to use its Legendre-Fenchel transform defined by

$$\nu^*(Q_r, q) := \sup_{p \in \mathbb{R}^d} (p \cdot q - \nu(Q_r, p)). \tag{1.1.25}$$

We know that ν^* is quadratic (since ν is quadratic), and it satisfies

$$\forall p, q \in \mathbb{R}^d,\ \nu^*(Q_r, q) + \nu(Q_r, q) \geq p \cdot q, \text{ and } \forall p \in \mathbb{R}^d, \exists q \in \mathbb{R}^d,\ \nu^*(Q_r, q) + \nu(Q_r, q) = p \cdot q. \tag{1.1.26}$$

Moreover, since $\nu(Q_r, p)$ converges, as r tends to infinity to $\frac{1}{2} p \cdot \bar{\mathbf{a}} p$, one expects $\nu^*(Q_r, q)$ to converge to $\frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q$.

To obtain properties on ν^* which do not only come from the definition (1.1.25), one would like to find another characterization of this quantity, and more specifically, one would like to obtain a variational formulation for ν^* in the spirit of (1.1.20). The definition of ν is a variational formulation of an elliptic PDE with Dirichlet boundary condition, to find a dual quantity, it is natural to try to solve an elliptic PDE with Neumann boundary condition. The correct definition is given by

$$\mu(U, q) := \sup_{v \in H^1(U)} \fint_U -\frac{1}{2} \nabla v \cdot \mathbf{a} \nabla v + q \cdot \nabla v. \tag{1.1.27}$$

This quantity was first introduced by Armstrong and Smart in [**21**] and is a good candidate in view of the properties it satisfies.

(1) It corresponds to the energy of a Neumann problem: the maximizer of (1.1.27) exists, is unique up to a constant and solves the Neumann boundary problem

$$\begin{cases} \nabla \cdot (\mathbf{a} \nabla v) = 0 & \text{in } U \\ \mathbf{n} \cdot \mathbf{a} \nabla v = \mathbf{n} \cdot q & \text{on } \partial U. \end{cases} \tag{1.1.28}$$

(2) Testing the minimizer of the definition of ν into the variational formulation for μ yields

$$\forall p, q \in \mathbb{R}^d, \mu(Q_r, q) + \nu(Q_r, q) \geq p \cdot q, \tag{1.1.29}$$

which is similar to the first property of (1.1.26). The second property of (1.1.26) is not exactly true and we do not have $\mu = \nu^*$, but as we will see below, the main objective will be to quantify the defect between these two quantities and to show that this defect vanishes as the size r of the cube Q_r diverges.

The quantity μ satisfies a number of properties which are similar to the ones of ν: it is subadditive, if one considers a partition of an open set U into open sets $U_1, \ldots, U_n$,

$$\mu(U, q) \leq \sum_{i=1}^{n} \frac{|U_i|}{|U|} \mu(U_i, q), \tag{1.1.30}$$

it is quadratic and by an application of the ergodic subadditive theorem, there exists a symmetric positive definite matrix $\mathbf{b} \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\mu(Q_r, q) \underset{r \to \infty}{\longrightarrow} \frac{1}{2} q \cdot \mathbf{b} q \text{ almost surely and in } L^1(\mathbb{P}).$$

In particular, one deduces from the subadditivity property

$$\mathbb{E}[\mu(Q_r, q)] \quad \text{decreases and converges toward } \frac{1}{2} q \cdot \mathbf{b} q.$$

Based on the heuristic that μ is a good substitute for ν^*, one can postulate

$$\mathbf{b} = \bar{\mathbf{a}}^{-1},$$

which is correct as will be explained below. At this point of the argument, we cannot obtain the previous equality, but from (1.1.29), one still has the lower bound

$$\forall p, q \in \mathbb{R}^d, \ \frac{1}{2} q \cdot \mathbf{b} q + \frac{1}{2} p \cdot \bar{\mathbf{a}} p \geq p \cdot q \implies \mathbf{b} \geq \bar{\mathbf{a}}^{-1}, \tag{1.1.31}$$

where the inequality is understood in the sense of symmetric matrices.

The core idea of [**18**, Chapter 2] is to show that μ is indeed a good approximation of ν^*. Informally speaking, they show that for each $p \in \mathbb{R}^d$, there exists $q \in \mathbb{R}^d$ such that, for each $r \geq 0$,

$$\mathbb{E}[\mu(Q_r, q)] + \mathbb{E}[\nu(Q_r, p)] - p \cdot q \leq \tau_r, \tag{1.1.32}$$

where τ_r is a sequence converging to 0 as r tends to infinity, which will be made explicit below. This inequality implies two important properties: first, letting r tend to infinity, we get

$$\forall p \in \mathbb{R}^d, \exists q \in \mathbb{R}^d, \ \frac{1}{2} q \cdot \mathbf{b} q + \frac{1}{2} p \cdot \bar{\mathbf{a}} p - p \cdot q = 0. \tag{1.1.33}$$

Together with (1.1.31), this shows $\mathbf{b} = \bar{\mathbf{a}}^{-1}$. The second important property can be deduced from the first one: by subtracting (1.1.33) from (1.1.32), one obtains

$$\underbrace{\left(\mathbb{E}[\mu(Q_r, q)] - \frac{1}{2} q \cdot \mathbf{b} q\right)}_{\geq 0} + \underbrace{\left(\mathbb{E}[\nu(Q_r, p)] - \frac{1}{2} p \cdot \bar{\mathbf{a}} p\right)}_{\geq 0} \leq \tau_r, \tag{1.1.34}$$

which implies in particular

$$0 \leq \mathbb{E}[\nu(Q_r, p)] - \frac{1}{2} p \cdot \bar{\mathbf{a}} p \leq \tau_r. \tag{1.1.35}$$

In a word, a precise understanding of the quantity $(p, q) \mapsto \mathbb{E}[\mu(Q_r, q)] + \mathbb{E}[\nu(Q_r, p)] - p \cdot q$ gives precise information on the quantity $p \mapsto \mathbb{E}[\nu(Q_r, p)] - \frac{1}{2} p \cdot \bar{\mathbf{a}} p$. This is important for the following reasons:

(1) A major problem to quantify the subadditive ergodic theorem was that very little information is available on the homogenized environment $\bar{\mathbf{a}}$. One had to look at the environment in the entire space $\mathbb{R}^d$ to know its value, which made the task of understanding $\mathbb{E}\left[\nu\left(Q_r,p\right)\right] - \frac{1}{2}p\cdot\bar{\mathbf{a}}p$ complicated. By using the dual quantity μ, it is enough to understand the environment in Q_r to deduce how close $\mathbb{E}\left[\nu\left(Q_r,p\right)\right]$ is from its limit.
(2) The right-hand sides of (1.1.32) and (1.1.35) are the same, this means that the proof is quantitative: if one obtains a rate of convergence for τ_r in (1.1.32), it can be transferred into a rate of convergence in (1.1.35).

In [**18**, Chapter 2], one does not obtain directly a quantified rate of convergence for the error term τ_r but one has the formula

$$\tau_r := C\left(\sup_{p\in B_1}\left(\mathbb{E}\left[\nu\left(Q_r,p\right)\right] - \mathbb{E}\left[\nu\left(Q_{2r},p\right)\right]\right) + \sup_{q\in B_1}\left(\mathbb{E}\left[\mu\left(Q_r,q\right)\right] - \mathbb{E}\left[\mu\left(Q_{2r},q\right)\right]\right)\right). \tag{1.1.36}$$

This is still enough to obtain a quantified rate of convergence: if one denotes by

$$A_r := \sup_{p,q\in B_1}\left(\mathbb{E}\left[\mu\left(Q_r,q\right)\right] - \frac{1}{2}q\cdot\bar{\mathbf{a}}^{-1}q + \mathbb{E}\left[\nu\left(Q_r,p\right)\right] - \frac{1}{2}p\cdot\bar{\mathbf{a}}p\right),$$

then (1.1.34) can be reformulated into

$$0 \leq A_r \leq C\left(A_r - A_{2r}\right).$$

This last inequality requires an argument to prove $\tau_r \leq C\left(A_r - A_{2r}\right)$ and is achieved by using the convexity of the mappings μ and ν and the fact that $\mathbb{R}^d$ is finite dimensional, we refer to [**18**, Proposition 2.11] for the details. The inequality can then be rewritten into

$$A_{2r} \leq \frac{C}{C+1}A_r.$$

Iterating this argument gives

$$A_{2^n r} \leq \left(\frac{C}{C+1}\right)^n A_r.$$

Going down the scales and using the bound $A_r \leq C$, which can be obtained by using the variational definitions of ν and ν^*, one deduces

$$\forall r \geq 0,\ A_r \leq Cr^{-\alpha},$$

with $\alpha := -\log\left(\frac{C}{C+1}\right) > 0$. This is an algebraic, quantitative rate of convergence.

1.1.7. Fluctuations around the expectation. Now that we have established an algebraic suboptimal rate of convergence for the energy $\nu\left(Q_r,p\right)$ of the form

$$\left|\mathbb{E}\left[\nu\left(Q_r,p\right)\right] - \frac{1}{2}p\cdot\bar{\mathbf{a}}p\right| \leq Cr^{-\alpha}, \tag{1.1.37}$$

one would like to understand the fluctuations of $\nu\left(Q_r,p\right)$ around its average. This is achieved by combining the four following ingredients:

(1) For any domain $U\subseteq\mathbb{R}^d$ and any $p\in\mathbb{R}^d$, the random variables $\nu\left(U,p\right)$ are bounded:

$$\nu\left(U,p\right) \leq C|p|^2,\ \mathbb{P}-\text{almost-surely}.$$

(2) The subadditivity property (1.1.22): if $U_1,\ldots,U_n$ is a partition of open sets of U then

$$\nu\left(U,p\right) \leq \sum_{i=1}^n \frac{|U_i|}{|U|}\nu\left(U_i,p\right).$$

(3) The observation that for each open bounded domain $U\subseteq\mathbb{R}^d$,

$$\nu\left(U,p\right)\ \text{is}\ \mathcal{F}\left(U\right)-\text{measurable}.$$

(4) The finite range dependence assumption: if U, V are two domains of $\mathbb{R}^d$ such that $\operatorname{dist}(U, V) \geq 1$ then

$$\mathcal{F}(U) \text{ and } \mathcal{F}(V) \text{ are independent.}$$

The idea is then to combine the four previous ingredients to obtain that $\nu(U, p)$ is bounded from above by a sum of independent bounded random variables, for which one can apply concentration inequalities. The precise result provides an exponential moment bound:

$$\forall t \geq 0,\ \mathbb{P}\left(\nu(Q_r, p) - \frac{1}{2} p \cdot \bar{\mathbf{a}} p > C r^{-\alpha}(1+t)\right) \leq \exp(-t). \tag{1.1.38}$$

Using similar arguments for the dual subadditive quantity μ and the one-sided convex duality (1.1.29), one derives a control for the fluctuations of ν from below,

$$\forall t \geq 0,\ \mathbb{P}\left(\nu(Q_r, p) - \frac{1}{2} p \cdot \bar{\mathbf{a}} p < -C r^{-\alpha}(1+t)\right) \leq \exp(-t). \tag{1.1.39}$$

A combination of (1.1.38) and (1.1.39) implies

$$\forall t \geq 0,\ \mathbb{P}\left(\left|\nu(Q_r, p) - \frac{1}{2} p \cdot \bar{\mathbf{a}} p\right| < C r^{-\alpha}(1+t)\right) \leq \exp(-t). \tag{1.1.40}$$

This highlights an important fact in quantitative stochastic homogenization: there are two types of quantity to quantify, the spatial variables, such as the convergence of the expectation of the energy, the sublinearity of the corrector (see (1.1.8) for a qualitative statement and (1.1.42) or Theorem 1.1.6 below for quantitative statements), and stochastic integrability such as (1.1.40). Exponential stochastic integrability is a standard type of stochastic integrability and appears in many results, we thus introduce a specific notation to describe this phenomenon.

For a non-negative random variable X and a constant $\theta \in (0, \infty)$, we denote by

$$X \leq \mathcal{O}_1(\theta) \text{ if and only if } \mathbb{E}\left[\exp\left(\frac{X}{\theta}\right)\right] \leq 2,$$

which means that the random variable X is in average of size θ and has an exponential tail. With this notation, the estimate (1.1.40) can be rewritten

$$\left|\nu(Q_r, p) - \frac{1}{2} p \cdot \bar{\mathbf{a}} p\right| \leq \mathcal{O}_1(C r^{-\alpha}). \tag{1.1.41}$$

Exponential integrability is not the only one appearing and we need to generalize the $\mathcal{O}_1$ notation: for $s \in (0, \infty)$, we define

$$X \leq \mathcal{O}_s(\theta) \text{ if and only if } \mathbb{E}\left[\exp\left(\left(\frac{X}{\theta}\right)^s\right)\right] \leq 2.$$

Remark 1.1.12. The larger s is, the better the stochastic integrability is, and one can show the following statement: given $s, s' \in (0, \infty)$ with $s' \leq s$, there exists a constant $C := C(s') < \infty$ such that

$$X \leq \mathcal{O}_{s'}(\theta) \implies X \leq \mathcal{O}_s(C\theta).$$

Remark 1.1.13. The case $s = 2$ corresponds to the Gaussian stochastic integrability.

1.1.8. Convergence of the minimizers and the multiscale Poincaré inequality. In the previous sections, we established a quantitative rate of convergence for the energy ν, but the final goal of the theory of stochastic homogenization is to study the behavior of the solutions of the elliptic PDE (1.1.1). In this section, we show how to transfer information from the energy quantities to the solutions.

For the sake of simplicity, we focus on the energy ν and denote by u_r the minimizer associated to the variational formulation (1.1.20). It is the solution of the equation

$$\begin{cases} \nabla \cdot (\mathbf{a} \nabla u_r) = 0 & \text{in } Q_r, \\ u_r = l_p & \text{on } \partial Q_r. \end{cases}$$

Reusing the notation of Section 1.1.2, we introduce

$$\phi_p^r = u_r - l_p.$$

Studying this quantity is interesting because, as was already mentioned in Section 1.1.2, the gradient $\nabla \phi_p^r$ is expected to converge to the gradient of corrector ϕ_p as the size r of the cube tends to infinity. In particular, one would like to obtain a quantified version of the sublinearity of the corrector.

PROPOSITION 1.1.14. *The mapping ϕ_p^r satisfies the quantitative L^2 sublinearity property*

$$\left\| \phi_p^r \right\|_{\underline{L}^2(Q_r)} \leq \mathcal{O}_1 \left(C r^{1-\alpha} \right). \tag{1.1.42}$$

The proof of this fact relies on a refinement of the Poincaré inequality, called the multiscale Poincaré inequality. The reason behind the introduction of this new inequality is the following: for a function $v \in H_0^1(Q_r)$, the standard Poincaré inequality reads

$$\|v\|_{L^2(Q_r)} \leq Cr \|\nabla v\|_{L^2(Q_r)}.$$

In our setting, we wish to apply this inequality to the approximation ϕ_p^r of the corrector. It is a function which

(1) oscillates quickly on a microscopic scale, so its gradient is expected to be constantly of size 1,
(2) has small macroscopic oscillations, this is what we aim at proving.

Consequently, the L^2 norm of the gradient of the approximation of the corrector ϕ_p^r should be, at least heuristically, bounded from above and below

$$c \leq \left\| \nabla \phi_p^r \right\|_{L^2(Q_r)} \leq C.$$

Applying the Poincaré inequality can only provide a linear bound on the L^2 norm of the corrector of the form

$$\left\| \phi_p^r \right\|_{L^2(Q_r)} \leq Cr,$$

which shows that the corrector has at most linear growth. This is obviously a much weaker statement than (1.1.42) and is not satisfactory.

The idea is then to define a new tool to treat functions which are oscillating quickly on a microscopic scale but have sublinear global oscillations. A first observation is that, by the Stokes formula, the spatial averages of the gradient of these functions must be small: for $x \in Q_r$ and $1 \ll s \leq r$ such that the ball $B(x,s)$ is included in Q_r, the spatial average of the gradient of ϕ_p^r can be estimated by the macroscopic oscillations of the same function on the ball $\partial B(x,s)$,

$$\begin{aligned} \fint_{B(x,s)} \nabla \phi_p^r &= \frac{1}{|B(x,s)|} \int_{\partial B(x,s)} \phi_p^r(y) \mathbf{n}(y) \, dy \\ &\leq \frac{C}{s} \operatorname*{osc}_{\partial B(x,s)} \phi_p^r, \end{aligned}$$

where the oscillation of a function f is defined according to the formula

$$\operatorname*{osc}_{\partial B(x,s)} f := \sup_{\partial B(x,s)} f - \inf_{\partial B(x,s)} f.$$

When s is large, the oscillation term on the right-hand side is small, since the mapping ϕ_p^r has sublinear oscillations.

To use this idea, one would like to obtain a new version of the Poincaré inequality, where the right-hand side contains spatial averages of the gradient. This is achieved by the following proposition.

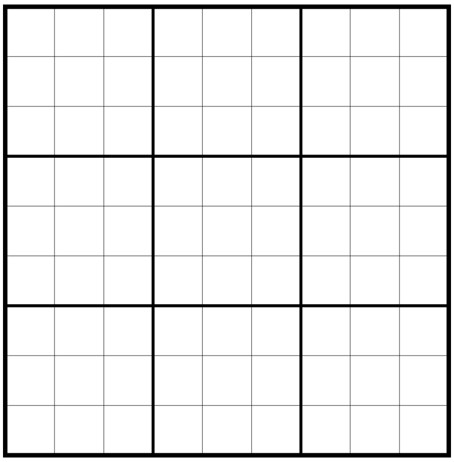

FIGURE 1.1.3. A cube Q_r together with the collections of subcubes $\left(z+Q_{r/3}\right)_{z\in\mathcal{Z}_1}$ and $\left(z+Q_{r/9}\right)_{z\in\mathcal{Z}_2}$.

PROPOSITION 1.1.15 (Multiscale Poincaré inequality, Proposition 1.12 of [**18**]). *Fix $r\geq 1$, for each integer $n\in[0,\log_3 r]$, we define $\mathcal{Z}_n := 3^{-n}r\mathbb{Z}^d\cap Q_r$ so that the family $(z+Q_{3^{-n}r})_{z\in\mathcal{Z}_n}$ is a partition of Q_r (see Figure (1.1.3)). Then for each function v satisfying either $v\in H^1_0(Q_r)$ or $v\in H^1(Q_r)$ and $(v)_{Q_r}=0$,*

$$\|v\|^2_{\underline{L}^2(Q_r)} \leq C\|\nabla v\|^2_{\underline{L}^2(Q_r)} + C\sum_{n=0}^{\lfloor \log_3 r\rfloor} r3^{-n}|\mathcal{Z}_n|^{-1}\sum_{z\in\mathcal{Z}_n}\left|\fint_{z+Q_{3^{-n}r}}\nabla v(x)\,dx\right|^2. \tag{1.1.43}$$

REMARK 1.1.16. By an application of the Jensen inequality, the right-hand side of the previous display is bounded by $Cr\fint_{Q_r}|\nabla v|^2$, and one recovers the standard Poincaré inequality.

We now explain how to apply this inequality to derive (1.1.42). This requires to make use of the second variation: we fix $n\in\mathbb{N}$ and denote by $u_1,\ldots,u_{3^{dn}}$ the minimizers of $\nu(Q^n_1,p),\ldots,\nu\left(Q^n_{3^{dn}},p\right)$ respectively, by the second variation (1.1.24), one has

$$\begin{aligned}3^{-dn}\sum_{i=1}^{3^{dn}}\|\nabla u_r-\nabla u_i\|^2_{\underline{L}^2(Q^n_i)} &\leq C3^{-dn}\sum_{i=1}^{3^{dn}}\left(\nu(Q^n_i,p)-\nu(Q_r,p)\right)\\ &\leq C3^{-dn}\left(\sum_{i=1}^{3^{dn}}\nu(Q^n_i,p)-p\cdot\bar{\mathbf{a}}p\right)-\left(\nu(Q_r,p)-p\cdot\bar{\mathbf{a}}p\right).\end{aligned}$$

We then use that, since $u_i\in l_p+H^1_0(Q^n_i)$,

$$\fint_{Q^n_i}\nabla u_i(x)\,dx = p.$$

Consequently, by the Jensen inequality

$$3^{-dn}\sum_{i=1}^{3^{dn}}\left|\fint_{Q^n_i}\nabla u_r - p\right|^2 \leq C3^{-dn}\left(\sum_{i=1}^{3^{dn}}\nu(Q^n_i,p)-p\cdot\bar{\mathbf{a}}p\right)-\left(\nu(Q_r,p)-p\cdot\bar{\mathbf{a}}p\right).$$

This inequality shows that the spatial averages of the gradient of the approximation of the corrector ϕ^r_p can be estimated by the difference between the energies ν on different scales. Since in the previous section we obtained a quantitative rate of convergence for the energy, it can be transferred to the spatial averages of the gradients: using (1.1.41), one obtains

$$3^{-dn}\sum_{i=1}^{3^{dn}}\left|\fint_{Q^n_i}\nabla\phi^r_p\right|^2 \leq \mathcal{O}_1\left(C\left(3^{-n}r\right)^{-\alpha}\right).$$

Applying the multiscale Poincaré inequality to the function ϕ_p^r and using the almost sure bound on the L^2 norm of its gradient (which is an immediate consequence of its definition),

$$\begin{aligned}\left\|\phi_p^r\right\|^2_{\underline{L}^2(Q_r)} &\leq C\left\|\nabla\phi_p^r\right\|^2_{\underline{L}^2(Q_r)} + Cr\sum_{n=0}^{\lfloor\log_3 r\rfloor} 3^{-n}\mathcal{O}_1\left(C\left(3^{-n}r\right)^{-\alpha}\right)\\ &\leq C + Cr\sum_{n=0}^{\lfloor\log_3 r\rfloor} 3^{-n}\mathcal{O}_1\left(C\left(3^{-n}r\right)^{-\alpha}\right).\end{aligned}$$

Using the standard properties of the $\mathcal{O}_s$ notation, see [**18**, Appendix A], one has

$$\begin{aligned}\left\|\phi_p^r\right\|^2_{\underline{L}^2(Q_r)} &\leq C + Cr^{1-\alpha}\mathcal{O}_1\left(\sum_{n=0}^{\lfloor\log_3 r\rfloor} 3^{-(1-\alpha)n}\right)\\ &\leq \mathcal{O}_1\left(Cr^{1-\alpha}\right),\end{aligned}$$

which proves (1.1.42) and establishes a quantitative sublinearity estimate for the approximate corrector ϕ_p^r.

1.1.9. Homogenization of the Dirichlet problem. In the previous sections, we saw how to obtain an algebraic rate of convergence for the energy quantity and how to derive from it information on the finite volume approximation of the corrector ϕ_p^r. The next step of the theory is to generalize Proposition 1.1.14: now that one has obtained quantitative estimates for solutions on cubes with affine boundary conditions, is it possible to derive information for general domains with general boundary conditions?

The key idea here is that thanks to the heuristic of the two-scale expansion, it is enough to have information on the corrector (or its approximation ϕ_p^r). By combining the technique of the two-scale expansion together with the bounds obtained on the approximate corrector, one can prove a quantitative version of Theorem 1.1.1.

Theorem 1.1.2 (Homogenization theorem, Theorem 2.18 of [**18**]). *Fix a bounded Lipschitz domain $U \subseteq \mathbb{R}^d$ and let $f \in W^{1,2+\delta}(U)$. For $\varepsilon \in (0,1]$, consider the solutions u^ε of the elliptic boundary value problems*

$$\begin{cases}\nabla\cdot\left(\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = 0 & \text{in } U\\ u = f & \text{on } \partial U,\end{cases}$$

and $\overline{u}$ the solution of

$$\begin{cases}\nabla\cdot\left(\bar{\mathbf{a}}\nabla\overline{u}\right) = 0 & \text{in } U\\ \overline{u} = f & \text{on } \partial U,\end{cases}$$

then there exists an exponent $\beta > 0$ such that the quantitative convergence for the L^2 norm holds

$$\left\|u^\varepsilon - u\right\|_{\underline{L}^2(U)} \leq \mathcal{O}_1\left(C\left\|f\right\|_{W^{1,2+\delta}(U)}\varepsilon^\alpha\right),$$

and one has the following estimate in H^1,

$$\left\|u^\varepsilon - \overline{u} - \sum_{i=1}^d \phi_i^{1/\varepsilon}\partial_i\overline{u}\right\|_{H^1(U)} \leq \mathcal{O}_1\left(C\left\|f\right\|_{W^{1,2+\delta}(U)}\varepsilon^\alpha\right).$$

This statement is quite general and we allow for somewhat rough boundary conditions: the domain U can be Lipschitz and the boundary condition f has to be only slightly better than $H^1(U)$, namely $W^{1,2+\delta}(U)$. Going beyond these regularity assumptions, and in particular assuming $f \in H^1(U)$, is however not possible in general, the problem is that for the two-scale expansion to work, one needs to have some regularity estimates on the homogenized function. Since this function is solution to a constant-coefficient elliptic equation, the theory of elliptic regularity (see [**74**, Chapter 2]) provides such estimates in the interior of the domain U. If the boundary condition f is too irregular, then the solution u may be also irregular in a boundary layer close to ∂U, and it can be enough to invalidate the quantitative convergence estimate.

By requiring some regularity on the boundary condition f, one derives some regularity on the function $\overline{u}$ up to the boundary: in the case presented here we assume an $L^{2+\delta}$ estimate on f and its gradient, to obtain, by applying the Meyers estimate (see [**118**]), some additional integrability on the gradient of $\overline{u}$ which eventually allows to neglect the boundary layer on which homogenization does not occur.

1.1.10. Large scale regularity theory. The next step to develop a robust theory of homogenization is to improve the rates of convergence. At this point of the analysis, we are able to provide a quantitative rate of convergence with an exponent $\alpha > 0$ whose value is tiny and suboptimal. To improve this rate, it is necessary to obtain a deeper understanding on the behavior of $\mathbf{a}$-harmonic functions. For these solutions, it is known, by the De Giorgi-Nash-Moser theory, that they are β-Hölder continuous for some small $\beta > 0$. One would like to improve this regularity by using the specific properties of stochastic homogenization. The idea is the following: by performing the change of variables $y = \frac{x}{\varepsilon}$ in Theorem 1.1.2, one knows that on large scales, an $\mathbf{a}$-harmonic function is well approximated by an $\bar{\mathbf{a}}$-harmonic function, for which an extensive theory of regularity is available. It is then possible to use this proximity to borrow the regularity of the $\bar{\mathbf{a}}$-harmonic functions and transfer it to the rougher $\mathbf{a}$-harmonic function.

It has to be noted that this strategy only provides regularity for $\mathbf{a}$-harmonic functions on large scales and highlights an important point: the theory of stochastic homogenization can only provide information on scales which are greater than the correlation length, and will not deliver any relevant information about what is happening on small scales.

A first estimate which is satisfied for $\bar{\mathbf{a}}$-harmonic functions but not for $\mathbf{a}$-harmonic function is the pointwise bound on the gradient of the solutions: for every $\bar{\mathbf{a}}$-harmonic function $\overline{u} : \mathbb{R}^d \to \mathbb{R}$, and every $R > 0$,

$$|\nabla \overline{u}(0)| \leq \frac{C}{R} \left\| \overline{u} - (\overline{u})_{B_R} \right\|_{\underline{L}^2(B_R)} . \tag{1.1.44}$$

The strategy presented above can be implemented to prove a suitable version of this bound on the gradient of $\mathbf{a}$-harmonic functions on large scales, as stated in the following theorem.

THEOREM 1.1.3 (Quenched $C^{0,1}$-type estimate, Theorem 3.3 of [**18**]). *There exists a non-negative random variable*

$$\mathcal{X} \leq \mathcal{O}_1(C),$$

such that for every $R \geq \mathcal{X}$ and every weak solution $u \in H^1(B_R)$ of

$$-\nabla \cdot (\mathbf{a} \nabla u) = 0 \text{ in } B_R,$$

one has, for every $r \in [\mathcal{X}, R]$, the estimate

$$\frac{1}{r} \left\| u - (u)_{B_r} \right\|_{\underline{L}^2(B_r)} \leq \frac{C}{R} \left\| u - (u)_{B_R} \right\|_{\underline{L}^2(B_R)}$$

which by the Caccioppoli and Poincaré inequalities is equivalent to

$$\|\nabla u\|_{\underline{L}^2(B_r)} \leq \frac{C}{R} \left\| u - (u)_{B_R} \right\|_{\underline{L}^2(B_R)} .$$

As was the case in Section 1.1.9, there is a set of small probability where the environment $\mathbf{a}$ behaves very differently from the homogenized environment $\bar{\mathbf{a}}$. On this set of small probability, the previous result will not apply. To remedy this, essentially two options are available: either write inequalities with a random right-hand side, as was done in Theorem 1.1.2, or introduce a random variable $\mathcal{X}$ which represents a minimal scale above which the result applies. When the environment $\mathbf{a}$ behaves badly, the random variable $\mathcal{X}$ is large. One cannot expect this minimal scale to be bounded, but fortunately good stochastic integrability is available: it has exponential moments and can only be large on events of exponentially small probability.

By taking $r = \mathcal{X} \vee 1$ in the previous theorem, we obtain

$$\|\nabla u\|_{\underline{L}^2(B_1)} \le \frac{C(\mathcal{X} \vee 1)^{\frac{d}{2}}}{R} \left\| u - (u)_{B_R} \right\|_{\underline{L}^2(B_R)},$$

and since the random variable $\mathcal{X}$ has exponential moments, one can morally think of it as being bounded: in this case we recover an inequality similar to the $C^{0,1}$ regularity for $\bar{\mathbf{a}}$-harmonic functions stated in (1.1.44). As was already mentioned, homogenization cannot provide pointwise information and only studies behavior of $\mathbf{a}$-harmonic functions on scales larger than the correlation length. Here the correlation length is of order 1, and thus one can only obtain meaningful results on length scales larger than 1. This justifies why there is a left-hand side of the form $\|\nabla u\|_{\underline{L}^2(B_1)}$ instead of the pointwise bound $|\nabla u(0)|$.

The regularity theory can be further extended thanks to the following remark. For $k \in \mathbb{N}$, we let $\overline{\mathcal{A}}_k$ be the set of $\bar{\mathbf{a}}$-harmonic polynomials of degree less than k. This space can be equivalently characterized as the $\bar{\mathbf{a}}$-harmonic functions with controlled L^2 growth: one has the identity

$$\overline{\mathcal{A}}_k = \left\{ p \in H^1_{\mathrm{loc}}\left(\mathbb{R}^d\right) \, : \, -\nabla \cdot (\bar{\mathbf{a}} \nabla p) = 0, \text{ and } \lim_{r \to \infty} r^{-k-1} \|p\|_{\underline{L}^2(B_r)} = 0 \right\}.$$

This characterization of the $\bar{\mathbf{a}}$-harmonic polynomials of degree less than k is the correct definition to establish a large scale regularity theory: to develop such a theory, one needs to find a suitable version of the set $\overline{\mathcal{A}}_k$ to work with $\mathbf{a}$-harmonic functions. Since the environment $\mathbf{a}$ is not constant in space, an $\mathbf{a}$-harmonic function which does not grow faster than a polynomial of degree k is not in general a polynomial. The characterization by the growth of the L^2 norm can however be extended (see (1.1.46) below) and is a crucial ingredient in the statement of Theorem 1.1.4.

From the theory of regularity of $\bar{\mathbf{a}}$-harmonic functions, one can estimate the $(k+1)$-th derivative of an $\bar{\mathbf{a}}$-harmonic function $\overline{u}$ according to the formula, for each $r > 0$,

$$\left|\nabla^{k+1} \overline{u}(0)\right| \le \frac{C}{r^{k+1}} \inf_{p \in \overline{\mathcal{A}}_k} \|\overline{u} - p\|_{\underline{L}^2(B_r)}. \tag{1.1.45}$$

Note that the specific case $k = 0$ is precisely (1.1.44). For a general $\mathbf{a}$-harmonic function, one cannot make sense of $\nabla^k u$ as a function, but one can make sense of the right-hand side of the previous display: instead of subtracting an $\bar{\mathbf{a}}$-harmonic polynomial, one can subtract an $\mathbf{a}$-harmonic function which does not grow faster than a polynomial of degree k. We let $\mathcal{A}_k$ be the set of such $\mathbf{a}$-harmonic functions,

$$\mathcal{A}_k := \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \, : \, -\nabla \cdot (\mathbf{a} \nabla u) = 0 \text{ and } \lim_{r \to \infty} r^{-k-1} \|u\|_{\underline{L}^2(B_r)} = 0 \right\}. \tag{1.1.46}$$

Then one would like to

(1) characterize the random vector space $\mathcal{A}_k$, for instance one would to know whether it is finite dimensional almost surely and compute its dimension.
(2) Prove a large scale $C^{k,1}$-regularity estimate of the form: there exists a random variable $\mathcal{X}$ with exponential stochastic integrability such that for each $\mathbf{a}$-harmonic function $u : \mathbb{R}^d \to \mathbb{R}$ and each $R, r \in [1, \infty)$ satisfying $\mathcal{X} \le r \le R$,

$$\inf_{p \in \mathcal{A}_k} \|u - p\|_{\underline{L}^2(B_r)} \le C \left(\frac{r}{R}\right)^{k+1} \inf_{p \in \mathcal{A}_k} \|u - p\|_{\underline{L}^2(B_r)}.$$

The previous questions can be answered by the following theorem.

Theorem 1.1.4 (Higher-order large-scale regularity, Theorem 3.8 of [**18**]). *There exist an exponent $\delta > 0$ and a non-negative random variable $\mathcal{X}$ satisfying the moment bound condition*

$$\mathcal{X} \le \mathcal{O}_1(C),$$

such that the following statements hold: for each $k \in \mathbb{N}$,

(i) *Every element of $\mathcal{A}_k$ is well approximated by a polynomial of $\overline{\mathcal{A}}_k$: for each $u \in \mathcal{A}_k$, there exists $p \in \overline{\mathcal{A}}_k$ such that for every $r \geq \mathcal{X}$,*

$$\|u - p\|_{\underline{L}^2(B_r)} \leq Cr^{-\delta} \|u\|_{\underline{L}^2(B_r)}$$

(ii) *Every polynomial of $\overline{\mathcal{A}}_k$ is well approximated by an element of $\mathcal{A}_k$: for each $p \in \overline{\mathcal{A}}_k$, there exists $u \in \mathcal{A}_k$ such that for every $r \geq \mathcal{X}$,*

$$\|p - u\|_{\underline{L}^2(B_r)} \leq Cr^{-\delta} \|p\|_{\underline{L}^2(B_r)}$$

(iii) *For every radius $R \geq \mathcal{X}$, and every weak solution $u \in H^1(B_R)$ of*

$$-\nabla \cdot (\mathbf{a}\nabla u) = 0 \text{ in } B_R,$$

there exists an element $\phi \in \mathcal{A}_k$ such that for every $r \in [\mathcal{X}, R]$, one has the estimate

$$\|u - \phi\|_{\underline{L}^2(B_r)} \leq C\left(\frac{r}{R}\right)^{k+1} \|u\|_{\underline{L}^2(B_R)}.$$

REMARK 1.1.17. A combination of (i) and (ii) shows that the space $\mathcal{A}_k$ is almost surely finite-dimensional, and its dimension is the same as the one of $\overline{\mathcal{A}}_k$:

$$\dim \mathcal{A}_k = \dim \overline{\mathcal{A}}_k = \binom{d+k-1}{k} + \binom{d+k-2}{k-1}, \quad \mathbb{P} - \text{almost surely}.$$

The case $k = 0$ is a Liouville theorem: the dimension of the space $\mathcal{A}_0$ is equal to 1 almost surely and is reduced to the constant functions. The case $k = 1$ is also interesting, since it establishes the existence and sublinearity of the correctors defined on the full space: the dimension of $\mathcal{A}_1$ is equal to $(d+1)$ almost surely, and by properties (i) and (ii), every function $u \in \mathcal{A}_1$ can be written

$$u = l_p + \phi_p + c, \text{ for some } p \in \mathbb{R}^d, c \in \mathbb{R},$$

where the function ϕ_p is the corrector. It is defined up to a constant and satisfies the sublinear growth estimate

$$\left\|\phi_p - (\phi_p)_{B_r}\right\|_{\underline{L}^2(B_r)} \leq C|p|r^{1-\delta}. \tag{1.1.47}$$

As in Section 1.1.2, we select one corrector arbitrarily thanks to the criterion

$$\int_{B_1} \phi_p = 0,$$

so that the corrector is a well-defined quantity. Note that with this definition, the corrector is not stationary.

REMARK 1.1.18. As was done for the $C^{0,1}$ estimate, taking the value $r = \mathcal{X} \vee 1$ in (iii) leads to, for every $r \geq 1$,

$$\inf_{p \in \mathcal{A}_k} \|u - p\|_{\underline{L}^2(B_1)} \leq \frac{C(\mathcal{X} \vee 1)^{d/2+k+1}}{r^{k+1}} \|u\|_{\underline{L}^2(B_r)},$$

which is a large-scale version of the C^{k+1}-regularity estimate for $\bar{\mathbf{a}}$-harmonic functions stated in (1.1.45).

1.1.11. Optimal rates of convergence. Now that one has developed a regularity theory for $\mathbf{a}$-harmonic functions, it is possible to improve the suboptimal rates of convergence given in (1.1.41) and Theorem 1.1.2. Obtaining optimal rates of convergence is accomplished through a bootstrap argument: the idea is to start from the suboptimal exponents obtained in the previous sections and to improve them until they reach optimality. In this section, we record the optimal results one can obtain in stochastic homogenization without detailing the proofs, we refer the reader to [**18**, Chapters 4 and 6] for the details.

1.1.11.1. *Optimal rates of convergence for the energy quantities ν and ν^*.* In Section 1.1.6, we obtained an algebraic rate of convergence for $\mathbb{E}\left[\nu\left(Q_r,p\right)\right]$ and $\mathbb{E}\left[\nu^*\left(Q_r,q\right)\right]$. Improving the exponent α which appears in (1.1.37) is interesting because it can then be transferred to obtain information on the **a**-harmonic functions, as was explained in the previous sections. Unfortunately, when one tries to improve this exponent a problem appear: on a boundary layer of size of order 1 around ∂Q_r, one cannot control the behavior of the energy. Because of this, one cannot expect to obtain better than

$$\left|\mathbb{E}\left[\nu\left(Q_r,p\right)\right]-\frac{1}{2}p\cdot\bar{\mathbf{a}}p\right|\leq Cr^{-1},$$

which provides the exponent $\alpha=1$. This limitation does not come from the intrinsic structure of stochastic homogenization but rather from a boundary layer effect, and one can hope to improve this rate by solving this issue. To this end, the first step is to find a new variational formulation for $\nu\left(Q_r,p\right)$. As in Section 1.1.8, we let u_r be the minimizer in the variational formulation (1.1.20) of $\nu\left(Q_r,p\right)$, it is characterized as the unique solution of

$$\begin{cases}\nabla\cdot\left(\mathbf{a}\nabla u_r\right)=0 & \text{in } Q_r\\ u_r=l_p & \text{on } \partial Q_r.\end{cases}$$

In particular u_r-l_p belongs to the space $H_0^1(U)$ thus, for every **a**-harmonic function $v\in\mathcal{A}\left(Q_r\right)$,

$$\fint_{Q_r}\left(\nabla u_r-p\right)\mathbf{a}\nabla v=0.$$

Moreover since $u_r\in\mathcal{A}\left(Q_r\right)$, we obtain that it is the unique maximizer of the variational problem

$$\sup_{v\in\mathcal{A}(U)}\fint_{Q_r}\left(-\frac{1}{2}\nabla v\cdot\mathbf{a}\nabla v+p\cdot\mathbf{a}\nabla v\right).$$

The previous variational problem is written with a supremum so as to have the following property, by the first and second variation,

$$\sup_{v\in\mathcal{A}(U)}\fint_{Q_r}-\frac{1}{2}\nabla v\cdot\mathbf{a}\nabla v+p\cdot\mathbf{a}\nabla v=\fint_{Q_r}\frac{1}{2}\nabla u\cdot\mathbf{a}\nabla u=\nu\left(Q_r,p\right).$$

This formulation is more convenient because the set $\mathcal{A}\left(Q_r\right)$ does not involve looking at the values of the function on the boundary, as was the case with $l_p+H_0^1\left(Q_r\right)$.

To remove the boundary layer, we denote by

$$\Phi_r(x):=\frac{1}{\pi^{\frac{d}{2}}r^d}\exp\left(-\frac{|x|^2}{r^2}\right)\ \text{for } x\in\mathbb{R}^d,\ r>0,$$

so that it satisfies the three following properties

(1) Φ_r is approximately equal to r^{-d} on the cube Q_r,
(2) Φ_r is decaying fast outside the cube Q_r and is approximately 0 far away from Q_r,
(3) Φ_r is smooth.

With these new notation, we define a new version of the energy, for $p\in\mathbb{R}^d$,

$$\nu\left(\Phi_r,p\right):=\inf_{\mathcal{A}_1}\int_{\mathbb{R}^d}-\Phi_r\left(\frac{1}{2}\nabla v\cdot\mathbf{a}\nabla v+p\cdot\mathbf{a}\nabla v\right),$$

where we are minimizing over the set $\mathcal{A}_1$ of **a**-harmonic function with at most linear growth to ensure that the term on the right-hand side is well-defined. Since the minimizer of the energy $\nu(Q_r,p)$ is expected to converge to a function of the form $l_p+\phi_p+c$ the choice of space $\mathcal{A}_1$ is sensible.

Note that the definition of the quantity $\nu\left(Q_r,p\right)$ can be rewritten as

$$\nu\left(Q_r,p\right)=\inf_{\mathcal{A}(B_r)}\int_{\mathbb{R}^d}-\frac{\mathbb{1}_{B_r}}{r^d}\left(\frac{1}{2}\nabla v\cdot\mathbf{a}\nabla v+p\cdot\mathbf{a}\nabla v\right).$$

To define $\nu(\Phi_r, p)$ we have replaced the function $\mathbb{1}_{B_r}$ by the function Φ_r. The point is that the latter function is much more regular than the indicator $\mathbb{1}_{B_r}$, one can show that with this new definition, the boundary layer effect disappear and obtain a much faster rate of convergence.

PROPOSITION 1.1.19 (Optimal rate for the convergence of the energy, Theorem 4.6 of [**18**]). *One has the following optimal rates of convergence:*

- *Control of the expectation: there exists a constant $C < \infty$ such that, for every $p \in B_1$,*
$$|\mathbb{E}[\nu(\Phi_r, p)] - p \cdot \bar{\mathbf{a}} p| \leq C r^{-d},$$
- *CLT scaling of the fluctuations: for every $s < 2$, one has, for every $p \in B_1$,*
$$|\nu(\Phi_r, p) - \mathbb{E}[\nu(\Phi_r, p)]| \leq \mathcal{O}_s\left(C r^{-d/2}\right).$$

REMARK 1.1.20. The CLT scaling for the energy is a predictable result: if one follows the heuristic according to which the inequality in the subadditivity (1.1.22) of ν becomes an equality when r is large, then the random variable $\nu(\Phi_r, p)$ can essentially be written as the average of r^d independent and bounded random variables, for which one has a Gaussian bound on the fluctuations $\mathcal{O}_2\left(r^{-d/2}\right)$. Here we cannot obtain exactly the Gaussian stochastic inegrability $\mathcal{O}_2$, but we have the slightly weaker result with $\mathcal{O}_s$ stochastic integrability, for each $s < 2$.

1.1.11.2. *Optimal rates for the Homogenization Theorem.* In this section, we record the optimal version of Theorem 1.1.2. As it was the case in the previous section, the error caused by the boundary layer is the limiting factor and one obtains an optimal rate of order $\varepsilon^{1/2}$, with an additional logarithmic factor in dimension 2. Since the boundary layer is the limiting factor, one has to be careful with the regularity of the boundary condition. The result is summarized in the following theorem.

THEOREM 1.1.5 (Quantitative two-scale expansion, Theorem 6.11 of [**18**]). *Consider a Lipschitz domain $U \subseteq \mathbb{R}^d$, and a function $\overline{u} \in W^{3/2,p}(U)$, for some $\alpha \in (0,\infty)$ and $p \in (2,\infty]$. Then for each $\varepsilon \in \left(0, \frac{1}{2}\right]$ and each $s \in (0,2)$, there exists a non-negative random variable $\mathcal{X}_\varepsilon$ satisfying the stochastic integrability estimate*

$$\mathcal{X}_\varepsilon \leq \begin{cases} \mathcal{O}_s\left(C\varepsilon^{\frac{1}{2}}|\log\varepsilon|^{\frac{1}{2}}\right) & \text{if } d = 2, \\ \mathcal{O}_2\left(C\varepsilon^{\frac{1}{2}}\right) & \text{if } d > 2, \end{cases}$$

such that if we let w^ε be the two-scale expansion of $\overline{u}$ defined according to

$$w^\varepsilon := \overline{u}(x) + \varepsilon \sum_{k=1}^{d} \left(\partial_{x_k}\overline{u} * \zeta_\varepsilon\right)(x)\phi_{e_k}\left(\frac{x}{\varepsilon}\right),$$

where $\zeta_\varepsilon := \varepsilon^{-d}\zeta\left(\frac{\cdot}{\varepsilon}\right)$ and ζ is a standard smooth compactly supported mollifier and if we let u^ε be the solution of the Dirichlet problem

$$\begin{cases} -\nabla \cdot \left(\mathbf{a}\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = -\nabla \cdot \left(\bar{\mathbf{a}}\nabla\overline{u}\right) & \text{in } U, \\ u^\varepsilon = u & \text{on } \partial U, \end{cases}$$

then we have the H^1 estimate

$$\|u^\varepsilon - w^\varepsilon\|_{H^1(U)} \leq \mathcal{X}_\varepsilon \|u\|_{W^{1+\alpha,p}(U)}. \tag{1.1.48}$$

REMARK 1.1.21. The two-scale expansion presented in the previous theorem requires to use the mollifier ζ_ε on the function u. This is necessary to have a well-defined two-scale expansion: since we only assumed $u \in W^{3/2,p}(U)$, the mapping $\partial_{x_k}\overline{u}(x)\phi_{e_k}\left(\frac{x}{\varepsilon}\right)$ may not be in the Sobolev space $H^1(U)$ so that (1.1.48) is meaningful.

1.1.11.3. *Optimal bounds on the corrector.* By optimizing the bound (1.1.47) on the corrector, one can show a much more precise bound: it is essentially bounded in dimension 3 and higher, and grows like the square root of logarithm in dimension 2.

THEOREM 1.1.6 (Optimal bounds on the corrector, Theorem 4.1 of [**18**]). *For every dimension $d \geq 3$, there exists a constant $C < \infty$ such that, for each $p \in B_1$, and each $x \in \mathbb{R}^d$*

$$\|\phi_p\|_{L^2(B_1(x))} \leq \mathcal{O}_2(C).$$

In dimension 2, the situation is different both in terms of stochastic integrability and spatial behavior: for each integrability parameter $s < 2$, there exists a constant $C < \infty$ such that for each $x \in \mathbb{R}^d$ and each $p \in \mathbb{R}^d$,

$$\|\phi_p\|_{L^2(B_1(x))} \leq \mathcal{O}_s\left(C \log^{\frac{1}{2}} |x|\right).$$

REMARK 1.1.22. These estimates are optimal both in terms of scaling and stochastic integrability. In particular the square root of the logarithm which appears in dimension 2 is optimal, a reason to justify this scaling is that the properly rescaled version of the corrector is known to converge to the Gaussian free field (see [**18**, Chapter 5]). Since in dimension 2, the variance of this field behaves like a logarithm, we can expect the field to grow like the square root of the logarithm.

REMARK 1.1.23. We recall that to select one corrector among all the possibilities, we made the additional assumption $\fint_{B_1} \phi_p = 0$.

1.2. Supercritical percolation

Chapters 2 and 3 are devoted to adapting the theory of stochastic homogenization developed in Section 1.1.5 to the setting of supercritical percolation. The main difficulty is that the uniform ellipticity assumptions (1.1.2) does not hold in this new setting, and one needs to find a proper replacement for it. This is achieved by constructing a renormalization structure, or coarse graining procedure, for the infinite cluster, building upon existing results in the field (see [**11, 136**]). We begin this section by defining the model of Bernoulli bond percolation and reviewing a few results in the field. We then explain the renormalization structure, which is a critical input in Chapters 2 and 3 and eventually present the main results obtained in these two chapters.

1.2.1. The model of Bernoulli bond percolation. The Bernoulli bond percolation model was first introduced by Broadbent and Hammersley in 1957 [**39**]. It is one of the simplest mathematical models which exhibits a phase transition. Despite its apparent simplicity, it gave rise to a deep mathematical theory and even though mathematicians managed to understand a number of important properties pertaining to this model over the past 70 years, many open questions remain to be solved. Before we start describing the model, we refer to the books [**87, 102, 37, 147**] for a more complete overview of the topic.

Let $\mathbb{Z}^d$ be the standard euclidean lattice in dimension $d \geq 2$. A point $x \in \mathbb{Z}^d$ is called a *vertex*. We equip this set with the standard 1-norm defined, for each $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, by

$$\|x\|_1 := \sum_{i=1}^{d} |x_i|.$$

We say that two vertices $x, y \in \mathbb{Z}^d$ are nearest neighbors if $\|x - y\|_1 = 1$. An unoriented pair $\{x, y\}$ of nearest neighbors of $\mathbb{Z}^d$ is called an *edge*. We let E_d be the set of edges of $\mathbb{Z}^d$,

$$E_d := \left\{ \{x, y\} \ : \ x, y \in \mathbb{Z}^d \text{ and } \|x - y\|_1 = 1 \right\}.$$

The probabilistic model of Bernoulli bond percolation is defined as follows: we consider the measurable space $(\Omega, \mathcal{F})$, where Ω is the set of functions defined from the space E_d to $\{0,1\}$, i.e. $\Omega := \{0,1\}^{E_d}$, and $\mathcal{F}$ is the σ-algebra generated by the events depending on finitely many edges. A percolation configuration is an element $\omega \in \Omega$, and given an edge $e \in E_d$, we denote by

$\omega(e) \in \{0,1\}$ the value of the configuration at the edge e. Given a configuration, we say that the edge e is *closed* if $\omega(e) = 0$ and *open* if $\omega(e) = 1$. A connected component of open edges is called a *cluster*. Given a probability $p \in [0,1]$, we let $\mathbb{P}_p$ be the unique probability measure on $(\Omega, \mathcal{F})$ such that the family of random variables$(\omega(e))_{e \in E_d}$ is a collection of i.i.d Bernoulli random variables of parameter p.

The goal of the theory of percolation is to study the geometry of the clusters as the parameter p varies between 0 and 1. A first question of interest is: does there exist an infinite cluster? A first step to study this issue is to note that the event

$$E_\infty := \{\text{there exists an infinite connected cluster of open edges}\}$$

is translation invariant. By an application of the $0-1$ law, the probability of this event must be 0 or 1. Additionally, we define for $p \in [0,1]$,

$$\theta(p) := \mathbb{P}_p \left(0 \text{ belongs to an infinite cluster}\right).$$

By standard arguments, the mapping $p \mapsto \theta(p)$ is nondecreasing and the three following statements are equivalent:

(i) $\theta(p) > 0$,
(ii) $\mathbb{P}_p \left(E_\infty\right) = 1$,
(iii) there exists an infinite cluster of open edges $\mathbb{P}_p$-almost surely.

We then define the critical probability $p_c := p_c(d)$ as

$$p_c := \inf \left\{ p \in [0,1] \ : \ \theta(p) > 0 \right\}.$$

If $0 < p_c < 1$, then one says that the model exhibits a phase transition. In this case, one can distinguish three different regimes:

(i) the subcritical phase, when $0 < p < p_c$, all the clusters are finite,
(ii) the critical phase, when $p = p_c$,
(iii) the supercritical phase, when $p_c < p < 1$, there exists at least one infinite cluster.

This distinction occurs only if the critical probability p_c is strictly between 0 and 1. The first result we would like to record is due to Broadbent and Hammersley in 1957 [**39**] and Hammersley [**89, 90**] shows the existence of a phase transition.

Proposition 1.2.1 (Existence of a phase transition, [**39, 89, 90**]). *For each dimension $d \geq 2$, one has*

$$0 < p_c(d) < 1.$$

The question of the precise value of p_c is a thorny issue. It was proved by Kesten that its value in dimension 2 is 1/2.

Theorem 1.2.1 (Kesten [**100**]). *One has the identity $p_c(2) = \frac{1}{2}$.*

It is unlikely that one can obtain a useful explicit formula for $p_c(d)$ in dimension more than 3. The reason behind the exact value in dimension 2 is that the square lattice $\mathbb{Z}^2$ satisfies a self-duality property (see [**87**, Chapter 1]), which is very specific to this dimension and is a key ingredient in the proof of Theorem 1.2.1.

We now gather some information about what is known for each of the three phases. In this thesis, we are mainly interested in applying the theory of quantitative stochastic homogenization on the infinite percolation cluster in the supercritical phase, and we will review the results more precisely in this setting.

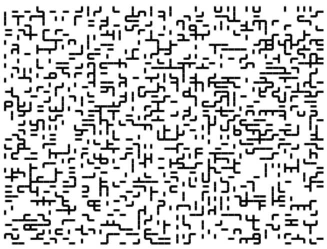

FIGURE 1.2.1. A percolation configuration in the subcritical regime with $p = 0.3$

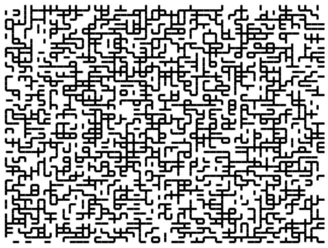

FIGURE 1.2.2. A percolation configuration in the critical regime with $p = 0.5$

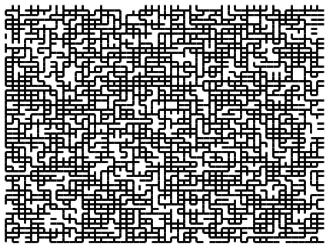

FIGURE 1.2.3. A percolation configuration in the supercritical regime with $p = 0.7$

1.2.1.1. *The critical phase.* In this phase very little information is known and most of the open questions of percolation are about trying to understand it better. The first question is the one of the existence of an infinite cluster at p_c. In dimension 2, Harris proved in [**92**] that $\theta(1/2) = 0$, which together with the result of Kesten $p_c = 1/2$ implies that there is no infinite cluster at criticality. In [**91**] Hara and Slade answered the question negatively in dimension larger than 19. The result was later improved by Fitzner and van der Hofstad [**66**] to prove that there is no infinite cluster at criticality in dimension larger than 11.

THEOREM 1.2.2. *In dimension* 2 *and dimension larger than* 11*, there is no infinite cluster at criticality*

$$\theta(p_c) = 0 \text{ for } d = 2 \text{ and } d \geq 11.$$

It is conjectured that there is no infinite cluster at criticality in every dimension, but the question remains open.

1.2.1.2. *The subcritical phase.* In this phase, all clusters are finite almost surely. The general philosophy of the subcritical phase is that there are only small isolated and finite clusters. In particular, the size of a cluster containing a given fixed vertex $x \in \mathbb{Z}^d$ is known to have an exponential tail. This is illustrated by the following Theorem. We refer to the works of Menshikov [**117**], Aizenman and Barsky [**2**] and Kesten [**101**] for the proofs.

THEOREM 1.2.3 ([**117, 2, 101**]). *For each* $p \in [0, p_c)$*, there exists a constant* $c := c(d,p) > 0$ *such that for each* $n \in \mathbb{N}$

$$\mathbb{P}_p\left(0 \text{ is connected to } \partial B(0,n)\right) \leq e^{-cn}$$

and

$$\mathbb{P}_p\left(|\mathcal{C}(0)| \geq n\right) \leq e^{-cn},$$

where, as in the previous section, $|\mathcal{C}(0)|$ *denotes the cardinality of the cluster containing* 0*.*

1.2.1.3. *The supercritical phase.* In this phase there exists at least one infinite cluster almost surely. A first interesting question is to determine the number of such infinite clusters. If one denotes by N the number of infinite clusters, which almost surely belongs to $\mathbb{N}^* \cup \{\infty\}$, one sees that this random variable is translation invariant. By an application of the $0-1$ law, it must be constant almost surely,

$$\text{there exists } k \in \mathbb{N}^* \cup \{\infty\} \text{ such that } \mathbb{P}_p(N = k) = 1.$$

This result was refined in 1987 by Aizenman, Kesten and Newman in [**4, 5**] who proved that the number k is equal to 1 for every $p \in (p_c, 1]$. This is summarized in the following theorem.

THEOREM 1.2.4 (Uniqueness of the infinite cluster [**4, 5**]). *For each $p \in (p_c, 1]$, one has*

$$\mathbb{P}_p(\textit{There exists a unique infinite cluster}) = 1.$$

From now on, we denote by $\mathscr{C}_\infty$ the unique infinite cluster. Now that the existence and uniqueness of $\mathscr{C}_\infty$ are established, one would like to understand its geometry. The broad picture to keep in mind is the following Ansatz: in the supercritical phase the infinite cluster spreads in most of the space. Its geometry is, at least on large scales, similar to the one of $\mathbb{Z}^d$. Additionally this infinite cluster coexists with small isolated and finite clusters. This can be seen in the Figures 1.2.3 and 1.2.4.

To illustrate this fact mathematically, we record a few classical results. The first result was proved by Chayes, Chayes, Grimmett, Kesten and Schonmann in 1989 [**43**]. It provides an exponential bound on the tail of the radius of a large finite cluster.

THEOREM 1.2.5 (Chayes, Chayes, Grimmett, Kesten and Schonmann [**43**]). *For each dimension $d \geq 2$ and each $p \in (p_c(d), 1]$, the limit*

$$\sigma(p) := \lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}_p(0 \textit{ is connected to } \partial B(0,n) \textit{ and } 0 \notin \mathscr{C}_\infty)$$

exists, and satisfies $0 < \sigma(p) < \infty$. Moreover, there exists a constant $C < \infty$ such that for each $n \in \mathbb{N}$

$$\mathbb{P}_p(0 \textit{ is connected to } \partial B(0,n) \textit{ and } 0 \notin \mathscr{C}_\infty) \leq C n^d \exp(-\sigma(p) n).$$

The size of the finite cluster satisfies a subexponential bound. A proof of this result can be found in [**3**] and [**103**].

THEOREM 1.2.6 ([**3, 103**]). *For each dimension $d \geq 2$ and each probability $p \in (p_c(d), 1]$, there exist two constants $c_1, c_2 > 0$ such that, for every $n \in \mathbb{N}$,*

$$\exp\left(-c_1 n^{\frac{d-1}{d}}\right) \leq \mathbb{P}_p(|\mathcal{C}(0)| = n) \leq \exp\left(-c_2 n^{\frac{d-1}{d}}\right).$$

By a summation, one obtains the following upper bound, for some constants $c_3 > 0$ and $C < \infty$,

$$\mathbb{P}_p(n \leq |\mathcal{C}(0)| < \infty) \leq C \exp\left(-c_3 n^{\frac{d-1}{d}}\right).$$

To illustrate that the infinite cluster spreads in the entire space, we record a result which is an almost immediate consequence of the result of Penrose and Pisztora [**136**, Theorem 1].

PROPOSITION 1.2.2. *For each $p \geq p_c$, there exist two constants $c, C \in (0, \infty)$ such that*

$$\mathbb{P}_p(\operatorname{dist}(0, \mathscr{C}_\infty) \geq n) \leq C \exp\left(-c n^{d-1}\right).$$

This proposition tells us that the probability for 0 to be far from the infinite cluster is exponentially small. It confirms the idea that the infinite cluster spreads over the entire lattice $\mathbb{Z}^d$: typically, in a ball of size R, no point will be at distance more than $\log R$ from the infinite cluster. To illustrate that the geometry of the infinite cluster resembles the one of $\mathbb{Z}^d$, a quantity of interest can be the chemical distance: given two points $x, y \in \mathscr{C}_\infty$, we define the chemical distance $\operatorname{dist}_{\mathscr{C}_\infty}(x, y)$ to be the graph distance between x and y inside the infinite cluster. In [**11**] and building upon the results of [**138, 136**], Antal and Pisztora proved that the chemical distance

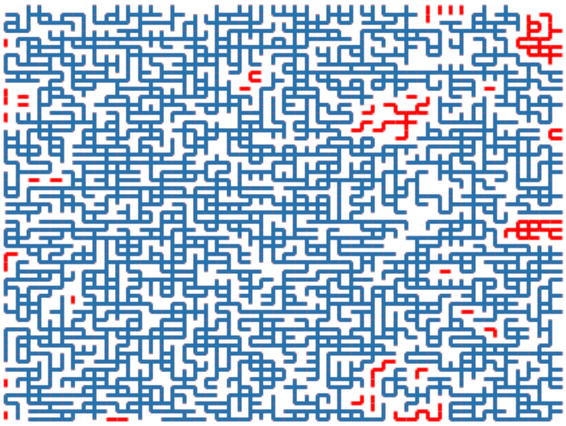

FIGURE 1.2.4. A pre-good box. The cluster $\mathscr{C}_*(\square)$ is drawn in blue and touches the four faces of the cubes. It coexists with small isolated clusters drawn in red.

of the infinite cluster behaves like the euclidean distance of $\mathbb{Z}^d$. More specifically, they proved a slightly stronger version of following theorem.

THEOREM 1.2.7 (Antal, Pisztora [**11**]). *For each dimension $d \geq 2$ and each probability $p \in (p_c, 1]$, there exist three constants $C, c, \rho \in (0, \infty)$ such that for each $y \in \mathbb{Z}^d$,*

$$\mathbb{P}_p\left(0, y \in \mathscr{C}_\infty \text{ and } \operatorname{dist}_{\mathscr{C}_\infty}(0, y) \geq \rho|y|\right) \leq C \exp\left(-c|y|\right).$$

1.2.2. Partitioning the infinite cluster into good cubes. In this section we define a renormalization structure of the infinite supercritical percolation cluster. This renormalization is instrumental in order to establish a theory of quantitative homogenization of the infinite cluster. The first step, which is one of the key ideas in the articles [**11, 136**] of Antal, Penrose and Pisztora, is to introduce a finite-volume version of the Ansatz described in Section 1.2.1.3. Given a bounded and connected subset $D \subseteq \mathbb{Z}^d$, one wishes to say that with large probability:

(i) there is one large cluster of open edges in D which morally plays the part of the infinite cluster,
(ii) except for this large cluster, there are only small isolated clusters.

For the sake of simplicity, we only consider a specific family of domains of $\mathbb{Z}^d$: the cubes. A precise mathematical statement is given in the following definition and proposition.

DEFINITION 1.2.3 (Pre-good cube). We define a cube of $\mathbb{Z}^d$ to be a set of the form

$$[\![x, x+N]\!]^d, \; x \in \mathbb{Z}^d, \; N \in \mathbb{N}.$$

A generic cube will be denoted by $\square$ and the integer N will be referred to as the size of the cube. Given a percolation configuration ω, we say that a cube $\square$ of size N is pre-good if it satisfies the following properties:

- There exists a cluster of open edges which intersects the $2d$ faces of the cube $\square$, this cluster is denoted by $\mathscr{C}_*(\square)$,
- The diameter of all the other clusters is smaller than $N/1000$.

REMARK 1.2.4. The value 1000 is arbitrary, we only need the constant to be large.

REMARK 1.2.5. The Figure 1.2.4 is a good illustration of what a pre-good cube looks like, the blue cluster is the large clusters and the red ones are the small isolated clusters.

The main result pertaining to this notion of pre-good cube is that, for each $p > p_c$, the probability of a large cube to be pre-good is exponentially close to one as the size of the cube goes to infinity. This was proved by Penrose and Pisztora in [**136**]. The statement given here is an application of their Theorem 5 with $\phi_n = n/1000$.

THEOREM 1.2.8 (Theorem 5 of [**136**]). *For each dimension $d \geq 2$ and $p > p_c$, there exists a constant $c > 0$ such that for each cube $\square \subseteq \mathbb{Z}^d$ of size n,*

$$\mathbb{P}_p\left(\square \text{ is a pre-good cube}\right) \geq 1 - \exp\left(-cn\right).$$

The objective is now to construct a partition of $\mathbb{Z}^d$ into pre-good cubes. A first important requirement to obtain a usable partition is the following connectivity property: given two neighboring pre-good cubes $\square_1, \square_2$ of similar sizes, the clusters $\mathscr{C}_*(\square_1)$ and $\mathscr{C}_*(\square_2)$ are connected within $\square_1 \cup \square_2$. Unfortunately this property does not follow directly from Definition 1.2.3, but there is still a solution: using that Theorem 1.2.8 provides a very strong rate of convergence on the probability of a cube to be pre-good, one can define the following notion of good cubes.

DEFINITION 1.2.6 (Good cube). Given a percolation configuration ω, we say that a cube $\square$ of size N is good if it satisfies the following properties:

- the cube $\square$ is pre-good,
- every cube $\square'$ whose size is between $N/10$ and $10N$ and which has non-empty intersection with $\square$ is also a pre-good cube.

REMARK 1.2.7. Note that the first point of the previous definition is implied by the second one since the cube $\square$ has a size between $N/10$ and $10N$ and has nonempty intersection with $\square$.

REMARK 1.2.8. The number 10 is arbitrary and could be replaced by another constant.

REMARK 1.2.9. The number of cubes $\square'$ satisfying the second assumption mentioned in the previous definition is finite and can be bounded by CN^{d+1}. Using Theorem 1.2.8 and a union bound, we can show that the probability of a cube to be good is exponentially close to 1: by reducing the size of the exponent c, one has

$$\mathbb{P}_p\left(\square \text{ is good}\right) \geq 1 - \exp\left(-cn\right). \tag{1.2.1}$$

With this definition, one can prove the connectivity property mentioned above. We do not detail the proof, which is quite straightforward and refer to Chapter 2, Lemma 2.2.8.

We now raise the following question: given a percolation configuration ω, is it possible to partition $\mathbb{Z}^d$ into only good cubes?

We first note that the probability of a cube to be good is never exactly 1, there is always an event of small probability where the cube does not satisfy the assumptions of Definition 1.2.3. By an application of the Borel-Cantelli Lemma, one cannot hope to have a partition of $\mathbb{Z}^d$ into cubes which are both all goods and all have the same size. The first property is the most important to us and so we renounce the second one: we will build a partition of $\mathbb{Z}^d$ into good cubes of *varying sizes*.

To this end, we restrict our consideration to a subset of the cubes of $\mathbb{Z}^d$ which are well-suited to construct partitions, namely the *triadic cubes*.

DEFINITION 1.2.10. For each $n \in \mathbb{N}$, we let $\square_n$ be the discrete cube

$$\square_n := \left[-\frac{3^n}{2}, \frac{3^n}{2}\right] \cap \mathbb{Z}^d.$$

For $n \in \mathbb{N}$, we let $\mathcal{T}_n$ be the set of *triadic cubes* of size 3^n defined by

$$\mathcal{T}_n := \left\{z + \square_n \ : \ z \in 3^n\mathbb{Z}^d\right\}.$$

The set $\mathcal{T}$ of triadic cubes is defined by

$$\mathcal{T} := \bigcup_{n=0}^{\infty} \mathcal{T}_n.$$

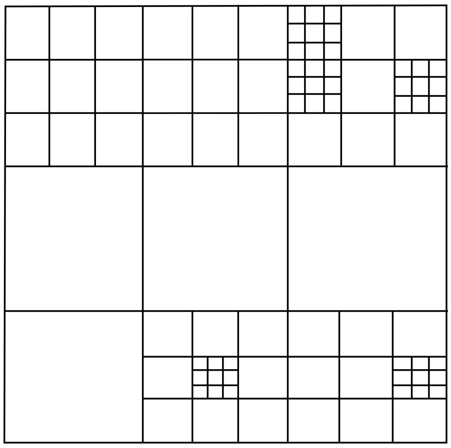

FIGURE 1.2.5. A realization of the partition $\mathcal{P}$.

These cubes satisfy the following convenient properties: two triadic cubes $\square$ and $\square'$ are either included in one another or disjoint. This property makes this subset of cubes of $\mathbb{Z}^d$ well-suited to design partitions. Using the bound (1.2.1) on the probability of a cube to be good, one can construct the following partition.

PROPOSITION 1.2.11 (Chapter 2, Proposition 2.2.1). *For each dimension $d \geq 2$ and each probability $p \in (p_c, 1]$, there exists, $\mathbb{P}_p$ almost surely, a partition $\mathcal{P}$ of $\mathbb{Z}^d$ into triadic cubes of varying sizes such that*

(i) *every cube $\square \in \mathcal{P}$ is a good cube,*
(ii) *two neighboring cubes $\square, \square' \in \mathcal{P}$ have comparable sizes,*

$$\frac{1}{3} \leq \frac{\operatorname{size}(\square)}{\operatorname{size}(\square')} \leq 3.$$

(iii) *For $x \in \mathbb{Z}^d$, if we denote by $\square_{\mathcal{P}}(x)$ the unique cube of the partition $\mathcal{P}$ containing x, then the size of $\square_{\mathcal{P}}(x)$ is a random variable satisfying the following exponential tail estimate*

$$\operatorname{size}\left(\square_{\mathcal{P}}(x)\right) \leq \mathcal{O}_1\left(C\right).$$

REMARK 1.2.12. The second condition ensures that with the definition of good cubes given in Definition 1.2.6, for any two neighboring cubes $\square, \square'$ of the partition $\mathcal{P}$, the clusters $\mathscr{C}_*(\square)$ and $\mathscr{C}_*(\square')$ are connected.

REMARK 1.2.13. Since the main clusters of the cubes of the partition are connected to one another, this implies that they belong to the infinite cluster,

$$\forall \square \in \mathcal{P},\ \mathscr{C}_*\left(\square\right) \subseteq \mathscr{C}_\infty.$$

REMARK 1.2.14. From the previous remark, we note that this partition is not local: given point $x \in \mathbb{Z}^d$, in order to know the size of $\square_{\mathcal{P}}(x)$, one needs to look at the percolation configuration in the entire $\mathbb{Z}^d$. We also note that the set of triadic cubes is not translation invariant: for each $z \in \mathbb{Z}^d$, one has

$$z + \mathcal{T} \neq \mathcal{T}.$$

A consequence of this remark is that the partition $\mathcal{P}$ is not stationary.

The proof of this proposition can be found in Chapter 2, and Figure 1.2.5 illustrates what this partition typically looks like. It can be used in the following way: given two points $x, y \in \mathbb{Z}^d$, we choose a deterministic path between x and y. We consider a configuration ω such that $x, y \in \mathscr{C}_\infty$ and look at the partition $\mathcal{P}$ for this configuration. Then, using that the main clusters of the cubes of the partition are connected, one can find a path going from x to y which is included in the infinite cluster $\mathscr{C}_\infty$ and which lies within the boxes crossed by the deterministic path. Since one has an exponential tail on the size of the cubes, one can, at least heuristically, think of these cubes as being bounded. This gives the following heuristic result: given a deterministic path between two points $x, y \in \mathscr{C}_\infty$, there exists a path in the infinite cluster which stays at (almost) bounded distance from the deterministic path. This is, in spirit, very similar to the result of Antal and Pisztora [**11**]. Of course, it is reasonable to obtain such a result since we used the same definition of good cubes they used to prove Theorem 1.2.7.

This partition is a crucial ingredient of the proofs in Chapters 2 and 3, since it allows to develop a functional calculus on the infinite cluster. More specifically, using this partition one can prove Poincaré inequalities, Sobolev inequalities and a Meyers estimate for functions defined on the infinite cluster, see Section 2.3 of Chapter 2.

1.2.3. Quantitative homogenization on the percolation cluster. In this section, we present the results obtained by adapting the theory of stochastic homogenization to the percolation cluster. This is the subject of Chapter 2 of this thesis and of the articles [**13, 51**]. Let us first describe the model. Given a fixed ellipticity parameter $\lambda \in (0,1]$, we define the environment $\mathbf{a}$ in the discrete setting as a random map

$$\mathbf{a} : E_d \to \{0\} \cup [\lambda, 1],$$

such that the collection $(\mathbf{a}(e))_{e \in E_d}$ forms a family of i.i.d. random variables and refer to these random variables as the conductances. We denote by $\mathbb{P}$ the law of the environment and additionally require that, for each edge $e \in E_d$,

$$\mathfrak{p} := \mathbb{P}\left(\mathbf{a}(e) \in [\lambda, 1]\right) > p_c\left(d\right),$$

so that there exists almost surely an infinite connected component of edges with nonzero conductances, denoted by $\mathscr{C}_\infty$. We then define the elliptic operator $\nabla \cdot \mathbf{a} \nabla$ as follows, for each function $u : \mathscr{C}_\infty \to \mathbb{R}$ and each $x \in \mathscr{C}_\infty$,

$$\nabla \cdot \mathbf{a} \nabla u(x) = \sum_{y \sim x} \mathbf{a}(\{x,y\})(u(y) - u(x)). \tag{1.2.2}$$

This is the discrete version of the theory presented in Section 1.1. Switching from the continuous setting to the discrete setting does not impact the theory of stochastic homogenization and all the results obtained in the former setting remain valid in the latter. We refer to [**82, 83**] for some works in stochastic homogenization of discrete elliptic equations. We also note that the assumptions of stationarity and finite range dependence are replaced by their discrete analogues: the assumption i.i.d. on the environment. A major difference here is that the environment is degenerate: the value 0 is allowed.

We say that a function $u : \mathscr{C}_\infty \to \mathbb{R}$ is $\mathbf{a}$-harmonic if it satisfies

$$\nabla \cdot (\mathbf{a} \nabla u) = 0 \text{ in } \mathscr{C}_\infty.$$

The first result presented in this thesis provides quantitative homogenization estimates for the elliptic problems on the percolation clusters. It is reminiscent of Theorem 1.1.2. This statement is slightly different from Theorem 1.1.2 due to a boundary layer effect: in the continuous and uniformly elliptic setting, it is possible to prove that the impact of the boundary layer is negligible, by assuming some additional regularity on the boundary condition. This is not the case on the percolation clusters and one needs to find a way around it. To treat the boundary layer effect, we assumed some additional integrability on the gradient of u, namely $\nabla u \in L^p$ for $p > 2$, and obtain a homogenization error (right-hand side of (1.2.5)) in terms of the L^p norm of u.

Theorem 1.2.9 (Chapter 2, Theorem 2.1.1). *Fix an exponent $p > 2$. There exist two exponents $s > 0$, $\alpha > 0$, a constant $C < \infty$ and a non-negative random variable $\mathcal{X}$ satisfying the subexponential estimate*

$$\mathcal{X} \le \mathcal{O}_s\left(C\right), \tag{1.2.3}$$

such that the following statement holds: for every $m \in \mathbb{N}$ such that $3^m \geq \mathcal{X}$, the cube $\square_m$ is a good cube, and for every function $u : \mathscr{C}_*(\square_m) \to \mathbb{R}$ which is $\mathbf{a}$-harmonic, i.e. which satisfies

$$-\nabla \cdot \mathbf{a} \nabla u = 0 \quad \text{in } \mathscr{C}_*(\square_m),$$

there exists a continuous harmonic function u_{hom} which is defined on the continuous triadic cube $[-3^m/2, 3^m/2]^d$ such that

$$u = u_{\mathrm{hom}} \quad \text{on the boundary } x \in \mathscr{C}_*(\square_m) \cap \partial \square_m, \tag{1.2.4}$$

and

$$\| u - u_{\mathrm{hom}} \|_{\underline{L}^2(\mathscr{C}_*(\square_m))} \leq C 3^{-m(1-\alpha)} \| u \|_{\underline{L}^p(\mathscr{C}_*(\square_m))} . \tag{1.2.5}$$

Remark 1.2.15. The stochastic integrability of the minimal scale $\mathcal{X}$ is subexponential: the exponent s in (1.2.3) is strictly positive but very small. The reason behind this subexponential decay is mainly technical: the renormalization structure must be taken into account in the computations and the fact that the size of the cubes of the partition is not bounded, but rather has an exponential moment bound, prevents us from obtaining optimal stochastic integrability.

Remark 1.2.16. The homogenized function u_{hom} is not discrete but continuous. The reason behind it is that this homogenization result is valid on large scales, and on these scales the lattice $\mathbb{Z}^d$ is a good approximation of the continuum $\mathbb{R}^d$. We also note that, contrary to Theorem 1.1.2, there is no homogenized matrix $\bar{\mathbf{a}}$ in this statement, or more precisely the homogenized matrix is equal to a multiple of the identity I_d. This is due to the symmetries of the problem: every linear transformation which preserves the lattice $\mathbb{Z}^d$ also preserves the law of the environment $\mathbf{a}$. These symmetries can be transferred to the homogenized coefficient $\bar{\mathbf{a}}$, and there are sufficiently many of them to prove that $\bar{\mathbf{a}}$ has to be a multiple of the identity.

Remark 1.2.17. The randomness here is in the minimal scale $\mathcal{X}$, and not in the right-hand side of the homogenization estimate as was the case in Theorem 1.1.2. As was already discussed in the Section 1.1.10, both choices are essentially equivalent. A reason to justify this particular choice here is that in this degenerate setting, the domain of the functions is the infinite cluster, which is random. To prove a homogenization theorem on a cube requires at least that the infinite cluster intersects the cube. The choice of the minimal scale ensures that the cube belongs to the partition $\mathcal{P}$ and settles the difficulty.

As was the case in stochastic homogenization in the uniformly elliptic setting, we can deduce from the previous theorem a large scale regularity theory and a precise description of the set $\mathcal{A}_k(\mathscr{C}_\infty)$ of $\mathbf{a}$-harmonic functions on the infinite cluster that grow more slowly than a polynomial of degree $k+1$, precisely defined by

$$\mathcal{A}_k(\mathscr{C}_\infty) := \left\{ u : \mathscr{C}_\infty \to \mathbb{R}^d \, : \, \nabla \cdot (\mathbf{a} \nabla u) = 0 \text{ in } \mathscr{C}_\infty \text{ and } \limsup_{R \to \infty} R^{-(k+1)} \| u \|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)} = 0 \right\}.$$

The following result is the percolation version of Theorem 1.1.4.

Theorem 1.2.10 (Regularity theory, Chapter 2, Theorem 2.1.2). *There exist two exponents $s, \delta > 0$ and a nonnegative random variable $\mathcal{X}$ satisfying the subexponential estimate*

$$\mathcal{X} \leq \mathcal{O}_s(C),$$

such that the following hold:

(i) *For each $k \in \mathbb{N}$, there exists a constant $C < \infty$ such that, for every $u \in \mathcal{A}_k(\mathscr{C}_\infty)$, there exists a harmonic polynomial p of degree less than k such that, for every $r \geq \mathcal{X}$,*

$$\| u - p \|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \leq C r^{-\delta} \| p \|_{\underline{L}^2(B_r)} . \tag{1.2.6}$$

(ii) *For every $k \in \mathbb{N}$ and every harmonic polynomial p of degree less than k, there exists $u \in \mathcal{A}_k$ such that, for every $r \geq \mathcal{X}$, the inequality (1.2.6) holds.*

(iii) *For each $k \in \mathbb{N}$, there exists $C < \infty$ such that, for every $R \geq 2\mathcal{X}$ and every* **a**-*harmonic function $u : \mathscr{C}_\infty \cap B_R \to \mathbb{R}$, there exists $\phi \in \mathcal{A}_k(\mathscr{C}_\infty)$ such that, for every $r \in \left[\mathcal{X}, \frac{1}{2}R\right]$, we have*

$$\|u - \phi\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \leq C \left(\frac{r}{R}\right)^{k+1} \|u\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)}.$$

A consequence of this result is that the (random) vector space $\mathcal{A}_k(\mathscr{C}_\infty)$ is almost surely finite-dimensional, and its dimension is given by

$$\dim \mathcal{A}_k\left(\mathscr{C}_\infty\right) = \binom{d+k-1}{k} + \binom{d+k-2}{k-1}.$$

A specific space of interest is the space $\mathcal{A}_1\left(\mathscr{C}_\infty\right)$ which thanks to the previous theorem is almost surely of dimension $(d+1)$. We note that this result, in the case $k = 1$, had already been proved by Benjamini, Duminil Copin, Kozma and Yadin in [**29**]. Theorem 1.2.10 generalizes their result and answers one of their open questions.

We can also deduce from (i) that every **a**-harmonic function $u \in \mathcal{A}_1\left(\mathscr{C}_\infty\right)$ can be written as

$$u(x) = p \cdot x + \phi_p(x) + c, \text{ for some } p \in \mathbb{R}^d, \ c \in \mathbb{R},$$

where ϕ_p is the percolation version of the corrector already introduced in the uniformly elliptic setting. From (i), one also deduces that the corrector has sublinear growth: there exists an exponent $\delta > 0$ such that

$$\lim_{R \to \infty} \frac{1}{R^{1-\delta}} \left\| \phi_p - \left(\phi_p\right)_{\mathscr{C}_\infty \cap B_R} \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)} = 0, \ \mathbb{P}-\text{almost surely.} \tag{1.2.7}$$

1.2.4. Optimal scaling estimates for the corrector. This section is devoted to the presentation of Chapter 3. The main goal is to improve the sublinear bound on the corrector (1.2.7) and to derive the optimal scaling estimates, similar to the ones of Theorem 1.1.6.

Before stating the result, we remark that a specificity of the discrete setting is that the microscopic scale is of size 1. As was noted above, homogenization can only provide information on scales which are larger than 1 (or more precisely larger the correlation length of the coefficient field). This was the source of some technical difficulties in the continuous setting: one cannot present pointwise bounds but rather has to rely on average versions of these estimates (see Theorem 1.1.6). Here this issue disappears and one derives pointwise bounds on the corrector.

Theorem 1.2.11 (Optimal scaling estimates for the corrector, Chapter 3, Theorem 3.1.1). *For each dimension $d \geq 3$, there exist an exponent $s > 0$ and a constant $C < \infty$ such that, for each $x, y \in \mathbb{Z}^d$, and each $p \in B_1$,*

$$\left|\phi_p(x) - \phi_p(y)\right| \mathbb{1}_{\{x, y \in \mathscr{C}_\infty\}} \leq \mathcal{O}_s\left(C\right).$$

In dimension 2, the growth of the corrector behaves like the square root of the logarithm, i.e.,

$$\left|\phi_p(x) - \phi_p(y)\right| \mathbb{1}_{\{x, y \in \mathscr{C}_\infty\}} \leq \mathcal{O}_s\left(C \log^{\frac{1}{2}} |x - y|\right).$$

The proof bypasses the route presented in Section 1.1.11 and does not rely on an improvement of the rate of convergence of the subadditive quantities. Instead we rely on concentration inequalities, introduced in homogenization by Naddaf and Spencer [**133**]. A typical example of concentration inequality is the Efron-Stein inequality, which states that if $X = (X_1, \ldots, X_n)$ is a family of independent random variables and if $(X_1', \ldots, X_n')$ is an independent copy of X, then for any measurable function F,

$$\operatorname{var}\left[F\right] \leq \frac{1}{2} \sum_{i=1}^{n} \operatorname{var}\left[\left(F\left(X_1, \ldots, X_{i-1}, X_i', X_{i+1} \ldots, X_n\right) - F(X)\right)^2\right]. \tag{1.2.8}$$

In Chapter 3, we wish to apply this inequality when F is the corrector and X is the environment **a**. This raises two difficulties:

(1) the environment $\mathbf{a}$ is indexed on the edges of $\mathbb{Z}^d$ and contains an infinite number of random variables. We thus need a generalization of (1.2.8) for infinite family of random variables.
(2) the Efron-Stein inequality gives an estimate on the variance, while we are aiming at subexponential integrability, we thus need a subexponential version of this inequality.

Such a generalization exists, is stated in Proposition 3.2.16 of Chapter 3 and comes from [**19**]. A second important tool is the multiscale Poincaré inequality, already introduced in Proposition 1.1.15, which allows to transfer information from the spatial averages of the gradient of the corrector to the corrector itself.

1.3. Stochastic homogenization of differential forms

In the previous section, we presented a way to extend the theory of stochastic homogenization to the degenerate environment of the supercritical percolation cluster. It was possible to adapt the theory developed in the uniformly elliptic setting thanks to a renormalization structure for the infinite cluster. In this section, we develop another way to extend the theory to a degenerate setting: the case of differential forms.

To introduce the problem, we fix a dimension $d \geq 2$ and an integer $k \in [\![0, d]\!]$, and let $\Lambda^k\left(\mathbb{R}^d\right)$ be the set of k-alternating multilinear maps. It is a finite-dimensional vector space of dimension $\binom{d}{k}$. A canonical basis for this space is given by the family

$$\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}, \ 1 \leq i_1 < \cdots < i_k \leq d. \tag{1.3.1}$$

Given a domain $U \subseteq \mathbb{R}^d$, a k-differential form u on U is then defined as a mapping from U to $\Lambda^k\left(\mathbb{R}^d\right)$; using the canonical basis of $\Lambda^k\left(\mathbb{R}^d\right)$, it can be decomposed as

$$u : \begin{cases} \mathbb{R}^d & \rightarrow \quad \Lambda^k\left(\mathbb{R}^d\right) \\ x & \mapsto \quad \sum_{1\leq i_1<\cdots<i_k\leq d} u_{i_1,\cdots,i_k}(x)\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}. \end{cases}$$

We always assume that the mappings $u_{i_1,\cdots,i_k}$ are measurable and we frequently assume some additional properties on these functions by requiring them to belong to some functional spaces such as $L^2(U), H^1(U), C^k(U)$ etc. We also note that in the specific case $k = 0$, the set of 0-differential forms can be identified with the set of functions from U to $\mathbb{R}$ and in the case $k = 1$, the set of 1-differential forms can be identified with the set of vector fields from U to $\mathbb{R}^d$.

We next introduce a few useful tools pertaining to this notion. First, the space $\Lambda^k\left(\mathbb{R}^d\right)$ can be endowed with a scalar product $\langle \cdot, \cdot \rangle$ by declaring the canonical basis (1.3.1) to be an orthonormal basis. Given a domain $U \subseteq \mathbb{R}^d$, this scalar product can be extended to the vector space of differential forms whose coefficients $(u_{i_1,\cdots,i_k})_{1\leq i_1<\cdots<i_k\leq d}$ are in $L^2(U)$ according to the formula, for any two such forms u, v,

$$\langle u, v \rangle_{L^2(U)} = \sum_{1\leq i_1<\cdots<i_k\leq d} \int_U u_{i_1,\cdots,i_k}(x) v_{i_1,\cdots,i_k}(x)\, dx. \tag{1.3.2}$$

A second essential tool to study differential forms is the notion of exterior derivative: for $k \in [\![0, d-1]\!]$, given a differential form u defined on a domain $U \subseteq \mathbb{R}^d$, we denote by $\mathrm{d}u$ the $(k+1)$-form formally defined by

$$\mathrm{d}u = \sum_{i=1}^{d} \sum_{1\leq i_1<\cdots<i_k\leq d} \frac{\partial u_{i_1,\cdots,i_k}}{\partial x_i}(x) \mathrm{d}x_i \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

and extend the definition of the exterior derivative to d-forms by setting $\mathrm{d}u = 0$ if u is a d-form. Another quantity of interest is the formal adjoint of d with respect to the scalar product (1.3.2), called the codifferential and denoted by δ. This differential operator sends k-forms onto $(k-1)$-forms, according to the formula

$$\delta u = \sum_{1\leq i_1<\cdots<i_k\leq d} \sum_{i\in\{i_1,\ldots,i_k\}} (-1)^i \frac{\partial u_{i_1,\cdots,i_k}}{\partial x_i}(x) \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

where the notation $\widehat{\mathrm{d}x_i}$ means that the term $\mathrm{d}x_i$ is removed from the exterior product $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$. We also record one of the key properties of these operators: they satisfy the identities

$$\mathrm{d} \circ \mathrm{d} = 0 \text{ and } \delta \circ \delta = 0. \tag{1.3.3}$$

This formalism is interesting because it encompasses most of the differential operators which are commonly used. For instance, the following differential operators can be expressed in the language of differential forms: for a k-form f, one has

- if $k = 0$, then the 1-form (or vector field) $\mathrm{d}f$ can be identified with ∇f,
- if $k = 1$, then the 0-form (or function) δf can be identified with $\operatorname{div} f$,
- in dimension 3, the space $\Lambda^2(\mathbb{R}^3)$ is of dimension 3. From this we deduce that the vector space of 2-forms can be identified with the space of vector fields of $\mathbb{R}^3$. If $k = 1$, then $\mathrm{d}f$ is a 2-form (or a vector field) and can be identified with $\mathrm{curl} f$.
- for $k \in [\![0, d]\!]$, one recovers the laplacian thanks to the formula
$$(\mathrm{d}\delta + \delta\mathrm{d}) f(x) = \sum_{1 \le i_1 < \cdots < i_k \le d} \Delta f_{i_1,\cdots,i_k}(x)\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

The main objective of this chapter is to extend the theory of stochastic homogenization to the setting of differential forms. Given an integer $k \in [\![0, d]\!]$ and a domain $U \subseteq \mathbb{R}^d$, we let $H^1_{\mathrm{d}}\Lambda^k(U)$ be the closure of the set of k-differential forms with smooth compactly supported coefficients with respect to the norm

$$\|f\|_{H^1_{\mathrm{d}}\Lambda^k(U)} := \|f\|_{L^2(U)} + \|\mathrm{d}f\|_{L^2(U)}.$$

We also let $\mathcal{S}(\Lambda^k(\mathbb{R}^d))$ be the set of symmetric matrices on the euclidean vector space $\Lambda^k(\mathbb{R}^d)$. As was the case in the classical theory of stochastic homogenization, an environment $\mathbf{a}$ is then defined as a random mapping

$$\begin{cases} \mathbb{R}^d \to \mathcal{S}(\Lambda^k(\mathbb{R}^d)) \\ x \mapsto \mathbf{a}(x), \end{cases}$$

which satisfies the same uniform ellipticity assumption as (1.1.2). We assume that this environment is random and satisfies the assumptions of stationarity and finite range dependence stated in Section 1.1.1. The goal is then to study the solutions of the equation

$$u \mapsto \mathrm{d}(\mathbf{a}\mathrm{d}u) = 0 \text{ in } U,$$

which are the critical points associated to the functional

$$\langle \mathrm{d}u, \mathbf{a}\mathrm{d}u \rangle_{L^2(U)}. \tag{1.3.4}$$

Note that working in this setting strictly contains the framework of Section 1.1.5. Indeed by choosing to work with $k = 0$, the exterior derivative $\mathrm{d}u$ can be identified with the gradient of u and one recovers the usual framework. On the other hand, this formalism is part of the more general framework of the elliptic systems of partial differential equations: since the dimensions of the spaces $\Lambda^k(\mathbb{R}^d)$ are always finite, the equation $\mathrm{d}\mathbf{a}\mathrm{d}u = 0$ can be written as an elliptic system.

A motivation to study these systems comes from the specific case when $r = 1$ and the underlying space is 4-dimensional: in this setting the system of equations in (1.3.4) has the same structure as *Maxwell's equations* (see e.g. [**109**, Section 1.2]), with yet a fundamental difference: here we assume $\mathbf{a}(x)$ to be Riemannian, that is, elliptic in the sense of (1.1.2), while for Maxwell's equations the underlying geometric structure is Lorentzian. Replacing a Lorentzian geometry by a Riemannian one, a procedure sometimes referred to as "Wick's rotation", is very common in constructive quantum field theory, see e.g. [**78**, Section 6.1(ii)]. While the objects we study here are minimizers of a random Lagrangian, we believe that the techniques developed in Chapter 4 will be equally informative for the study of the Gibbs measures associated with such Lagrangians.

In the theory presented in Section 1.1.5, we did not make use of tools which are known to be true for elliptic equations and false for elliptic systems, such as the maximum principle or the De Giorgi-Nash-Moser regularity theory. All the results of Section 1.1.5 are valid for elliptic systems, and it is only for notational convenience that the theory was presented for elliptic equations.

One can even go beyond the results of Section 1.1.5: most of the results of the book [**18**] are valid for elliptic systems, up to a few sections which require to use De Giorgi-Nash-Moser theory. We refer to [**18**, Frequently Asked Questions] for details.

However there is still a crucial difference to be noted: the theory can be adapted only to uniformly elliptic systems. Such a requirement is *not* verified here: one has

$$\langle \mathrm{d}u, \mathbf{a}\mathrm{d}u \rangle_{L^2(U)} = 0 \text{ implies } \mathrm{d}u = 0, \text{ which does not imply that } u \text{ is constant.}$$

Indeed by the property (1.3.3), every k-form u which can be written $u = \mathrm{d}f$ satisfies $\mathrm{d}u = 0$ but is not necessarily constant. In Chapter 5, we show how to adapt the theory and are able to prove a version of the quantitative convergence of the energy (Theorem 4.1.1) and of the homogenization theorem (Theorem 4.1.2).

The main difficulty is to extend all the functional inequalities useful in homogenization, such as the Poincaré inequality or the multiscale Poincaré inequality (Proposition 1.1.15), to the setting of differential forms. This is achieved by using results of Mitrea, Mitrea, Monniaux [**120**], Mitrea, Mitrea, Shaw [**122**] and the monograph of Schwarz [**142**].

1.4. Stochastic homogenization applied to the $\nabla\phi$ model

1.4.1. Definition of the model and the surface tension. Many physical phenomena exhibit a transition between two pure phases, especially at low temperature, such is for instance the case for liquid water and ice at zero temperature. This transition motivates studying the interface between two phases. It has been a topic of interest for mathematicians since the dawn of the 20th century. The first mathematical model to describe interfaces was introduced by Wulff in 1901 in [**148**]: it characterizes interfaces as minimizers of the Wulff functional, defined, for a subset $E \subseteq \mathbb{R}^d$, by

$$W(E) := \int_{\partial E} \sigma(\mathbf{n}(x))\, \mathrm{d}x,$$

where $\mathbf{n}$ is the outward normal to ∂E at x and σ is a surface tension between the two phases. The minimizer of the Wulff functional is called the Wulff shape. From a mathematical point of view, the interfaces are macroscopic objects, and one would like to describe them using models from statistical mechanics which are defined on a microscopic level. Many important results in this direction were obtained in the 90s; in [**6**], Alexander, Chayes and Chayes derived a Wulff construction for the two dimensional supercritical Bernoulli bond percolation. In the monograph [**58**], Dobrushin, Kotecký and Shlosman, studied the two dimensional ferromagnetic Ising model at low temperature with periodic boundary conditions. These results were later extended to every temperature below the critical one, one can refer to the works of Ioffe [**94, 95**], Schonmann, Shlosman [**141**] and Pfister, Velenik [**137**] and by Ioffe and Schonmann in [**96**].

In dimension 3, Cerf proved in [**41**] a form of Wulff construction for the supercritical Bernoulli bound percolation. Bodineau in [**35**], proved a similar result for the Ising model in any dimension $d \geq 3$ at low temperature. Cerf and Pisztora in [**42**] proved a Wulff construction for Ising in dimension larger than 3 for temperatures below a limit of slab-thresholds.

In this section, we consider a simpler mathematical model of interfaces, namely the $\nabla\phi$ model. It encodes deviations from a perfectly flat interface by modeling it as a scalar field $\phi : \mathbb{R}^d \mapsto \mathbb{R}$, which fluctuates around the $\phi = 0$ interface. To be more precise, we discretize $\mathbb{R}^d$ and consider mappings $\phi : \mathbb{Z}^d \mapsto \mathbb{R}$. The discretized interface is represented by the set $\left\{(x, \phi(x)) : x \in \mathbb{Z}^d\right\} \subseteq \mathbb{Z}^d \times \mathbb{R}$. We call $\phi(x)$ the height variable at x, and associate to a configuration ϕ an energy computed through the hamiltonian,

$$H(\phi) := \sum_{|x-y|_1 = 1} V(\phi(x) - \phi(y)),$$

where $V : \mathbb{R} \to \mathbb{R}$ is an elastic potential satisfying the properties

(1) V is even: $V(x) = V(-x)$ for each $x \in \mathbb{R}$,

(2) V is uniformly convex: there exists $\lambda \in (0,1]$ such that for each $x, y \in \mathbb{R}$,

$$\lambda |x-y|^2 \leq V(x) + V(y) - 2V\left(\frac{x+y}{2}\right) \leq \frac{1}{\lambda}|x-y|^2. \tag{1.4.1}$$

The formal equilibrium measure associated to this model is given by the Gibbs measure

$$\frac{1}{Z}\exp\left(-H\left(\phi\right)\right)\prod_x \mathrm{d}\phi(x), \tag{1.4.2}$$

where Z is a normalization constant that makes the above measure a probability measure. Another important aspect of the model is its dynamical interpretation: one considers two pure phases separated by an interface $\phi_{t=0}$ at time $t = 0$ and let this interface evolve through time. The interface will then try to relax slowly over time to minimize its energy and, in the absence of conservation laws, this evolution will only be affected by a noise. This leads to the Langevin dynamics governed by the stochastic differential equation

$$\mathrm{d}\phi_t(x) = -\sum_{|y-x|_1=1} V'\left(\phi_t(x) - \phi_t(y)\right) dt + \sqrt{2} dB_t\left(x\right),$$

where $\left(B_t\left(x\right)\right)_{x\in\mathbb{Z}^d}$ is a family of independent standard Brownian motions. Formally, the measure (1.4.2) is invariant for this SDE and makes the dynamics reversible.

A typical result one wishes to prove is that a properly rescaled version of the interface, which is a priori a random object distributed according to the probability measure (1.4.2), approaches a deterministic shape over large scales and characterize this object in the spirit of a Wulff construction. This kind of result is similar in spirit to Theorem 1.1.2 in stochastic homogenization.

To characterize the limiting deterministic interface, a quantity of interest is the surface tension. It is defined as follows: given a discrete bounded connected subset $U \subseteq \mathbb{Z}^d$, we define the probability measure associated to the $\nabla\phi$ model in U with affine Dirichlet boundary condition of slope $p \in \mathbb{R}^d$ to be

$$\mathbb{P}_{U,p}(\mathrm{d}\phi) := Z_{U,p}^{-1} \exp\left(-\sum_{x,y\in Q_n, |x-y|_1=1} V\left(\phi(x) - \phi(y)\right)\right)\prod_{x\in U}\mathrm{d}\phi(x)\prod_{y\in\partial U}\delta_{p\cdot y}\left(\phi_y\right).$$

This corresponds to the equilibrium measure of the interface in a finite volume box, where one has enforced an affine boundary condition of slope p for the interface. The quantity of interest is then the partition function $Z_{U,p}$ or more precisely, the properly rescaled version of it,

$$\nu\left(U,p\right) := -\frac{1}{|U|}\log Z_{U,p}.$$

This quantity is called the finite-volume surface tension. We use the same notation for this quantity as the one for the energy associated to the Dirichlet problem with affine boundary condition in stochastic homogenization (see (1.1.20)) on purpose: these quantities share a number of common properties and play similar role in both models. Note however that there is an important difference: the finite volume surface tension $\nu\left(Q_n, p\right)$ is a deterministic number, while the energy in homogenization is a random variable, depending on the environment $\mathbf{a}$.

A first common property is that both quantities satisfy a subadditivity property. For the energy of stochastic homogenization, the inequality was stated in (1.1.22). For the finite volume surface tension, Funaki and Spohn in [**70**] essentially proved that, given a bounded connected set $U \subseteq \mathbb{Z}^d$ partitioned into connected subsets $U_1, \ldots, U_n$, and a slope $p \in \mathbb{R}^d$, one has

$$\nu\left(U,p\right) \leq \sum_{i=1}^n \frac{|U_i|}{|U|}\nu\left(U_i, p\right) + C\frac{\sum_{i=1}^n |\partial U_i|}{|U|},$$

which is similar to the homogenization setting, with an additional term on the right-hand side. Nevertheless, if one considers a partition of a set into subsets which are not too irregular, i.e. which satisfy $|\partial U_i| \ll |U_i|$, then this additional term can be neglected.

As a consequence of this subadditivity property, Funaki and Spohn deduced the convergence of the finite volume surface tension.

THEOREM 1.4.1 (Proposition 1.1 of [**70**]). *Let Q_n be the discrete cube of size n defined by $Q_n := [-n,n]^d \cap \mathbb{Z}^d$. For each $p \in \mathbb{R}^d$, the sequence $\nu(Q_n,p)$ converges as n tends to infinity. We write*

$$\overline{\nu}(p) := \lim_{n\to\infty} \nu(Q_n,p).$$

This limit is called the infinite volume surface tension.

The surface tensions, in finite or infinite volume, satisfy another property similar to the energy in stochastic homogenization: they are uniformly convex in the p variable. It is proved in [**70**, Proposition 1.1] that the surface tension ν belongs to $C^1(\mathbb{R}^d)$, its derivative is Lipschitz and it is an even function. In [**56, 72**] it is proved that the surface tension is uniformly convex from above and below: for each $p, q \in \mathbb{R}^d$,

$$\frac{1}{2}\lambda|p-q|^2 \leq \nu(p) - \nu(q) - (p-q)\cdot\nabla\nu(q) \leq \frac{1}{2\lambda}|p-q|^2,$$

where λ is the constant of (1.4.1). Contrary to the setting of stochastic homogenization, we do not have that ν is quadratic. This property was strongly related to the linear structure of the elliptic equation $\nabla\cdot(\mathbf{a}\nabla u) = 0$ and no such structure is available here. Nevertheless, the uniform ellipticity of the elastic potential shows that the surface tension has a quadratic growth: one has the inequalities

$$\nu(0) + \lambda|p|^2 \leq \nu(p) \leq \nu(0) + \frac{1}{\lambda}|p|^2.$$

The infinite volume surface tension plays a role similar to the homogenized matrix $\bar{\mathbf{a}}$ in stochastic homogenization. It is fundamental to the understanding of the macroscopic properties of the model. It appears for instance as the rate functional in large deviations principles, as was investigated by Deuschel, Giacomin and Ioffe in [**56**]. Their proofs relies on the Helffer-Sjöstrand representation formula which was first introduced to study the $\nabla\phi$ model by Naddaf and Spencer in [**132**]. In this article they were able to combine this formula with techniques from the qualitative theory of homogenization to prove a central limit theorem for the gradient field. In [**69**], Funaki and Sakagawa established a large deviation principle in the presence of a weak self potential. In the dynamic setting, Funaki and Spohn [**70**] established a law of large number for the evolution and the limiting process is characterized by a nonlinear parabolic PDE defined in terms of the surface tension. The fluctuations of the dynamics were studied by Giacomin Olla and Spohn in [**72**], who proved that they are governed on large scales by an infinite dimensionnal Ornstein-Uhlenbeck process. All these results involve the surface tension and we refer to [**68**] for a general review of the topic.

1.4.2. Quantitative convergence of the finite volume surface tension. The main goal of Chapter 5 is to revisit the beginning of the theory of quantitative stochastic homogenization following [**18**], applied to the $\nabla\phi$ model. More precisely, we have already established that the finite volume surface tension $\nu(Q_n,p)$ is a good analogue to the Dirichlet energy with affine boundary conditions in stochastic homogenization. The first result one would like to establish is an equivalent statement to the quantitative bound (1.1.37), which derives an algebraic rate of convergence for the surface tension. This is carried out in Chapter 5 where we obtain the following result.

THEOREM 1.4.2 (Chapter 5, Theorem 5.1.1). *There exist an exponent $\alpha > 0$ and a constant $C < \infty$ such that for each $p \in \mathbb{R}^d$,*

$$|\nu(Q_n,p) - \overline{\nu}(p)| \leq Cn^{-\alpha}\left(1+|p|^2\right).$$

The first step to implement this program is to find a dual quantity, similar to μ in stochastic homogenization. In view of its definition given in (1.1.27), and of the definition of the finite volume surface tension, it is natural to define, for a bounded connected subset $U \subseteq \mathbb{Z}^d$, and $q \in \mathbb{R}^d$,

$$\mu(U,q) := \frac{1}{|U|} \log \int_{\mathring{h}^1(U)} \exp\left(- \sum_{x,y \in U, |x-y|_1 = 1} V(\psi(x) - \psi(y)) - q \cdot (x-y)(\psi(x) - \psi(y)) \right) \mathrm{d}\psi.$$

where the space $\mathring{h}^1(U)$ is the space of functions from U to $\mathbb{R}$, with vanishing average

$$\mathring{h}^1(U) := \left\{ \psi : U \mapsto \mathbb{R} \ : \ \sum_{x \in U} \psi(x) = 0 \right\},$$

and the notation $\mathrm{d}\psi$ stands for the Lebesgue measure on $\mathring{h}^1(U)$. This space is natural since in stochastic homogenization, we know that the dual quantity μ is the energy associated to an elliptic problem with Neumann boundary condition (see (1.1.28)). The space naturally associated to the Neumann boundary condition is the space of functions in H^1 with zero average: this is the space we are trying to replicate here. This definition is actually a good candidate and satisfies all the properties we want it to satisfy, as is summarized in the following proposition.

Proposition 1.4.1. *The dual quantity μ satisfies the following properties.*

- *Quadratic growth and convexity: for each connected bounded subset $U \subseteq \mathbb{Z}^d$,*

$$q \mapsto \mu(U,q) \text{ is convex and for each } q \in \mathbb{R}^d, \ \mu(0) + \lambda |q|^2 \leq \mu(U,q) \leq \mu(0) + \frac{1}{\lambda}|q|^2.$$

- *Subadditivity: given a bounded connected set $U \subseteq \mathbb{Z}^d$ partitioned into connected subsets $U_1, \ldots, U_n$, for each $q \in \mathbb{R}^d$,*

$$\mu(U,p) \leq \sum_{i=1}^n \frac{|U_i|}{|U|} \mu(U_i,p) + C \frac{\sum_{i=1}^n |\partial U_i|}{|U|}.$$

- *Convex duality from below: For each $p, q \in \mathbb{R}^d$,*

$$\mu(Q_r, q) + \nu(Q_r, q) \geq p \cdot q - \frac{C}{r}\left(1 + |p|^2 + q^2\right).$$

  This statement is a good analogue to (1.1.29)*, there is an additional error term on the right-hand side which is small as r tends to infinity and which does not not impact the analysis.*

Following Section 1.1.6, the core of the analysis is to show the convex duality from above. If one denotes by, for $q \in \mathbb{R}^d$,

$$\tau_r^*(q) := \mu(Q_r, p) - \mu(Q_{2r}, p),$$

then one would like to prove a result similar to (1.1.32), namely that for each $q \in \mathbb{R}^d$, there exists $p \in \mathbb{R}^d$ such that

$$\mu(Q_r, q) + \nu(Q_r, p) - p \cdot q \leq \tau_r^*(q),$$

in order to derive an algebraic rate of convergence. The precise statement for this proposition can be found in Chapter 5, Proposition 5.4.5.

We complete this introduction by reviewing two important tools in the proofs. First we note that in stochastic homogenization, the energy quantity with affine boundary condition has a variational formulation (1.1.20). This formulation is very useful to develop the theory and one of the key results it provides is the second variation formula (1.1.24). This formula is crucial in the derivation of the proofs: if one wishes to apply these techniques to the $\nabla\phi$ model, it is necessary to find an alternative version of this statement.

The existence of a variational formulation for the finite volume surface tension is given by a large deviation principle: it is the infimum of an energy-entropy minimization problem. To be more specific, given a bounded connected subset $U \subseteq \mathbb{Z}^d$ and a probability measure $\mathbb{P}$ on $\mathbb{R}^U$, we define its entropy by

$$H(\mathbb{P}) := \begin{cases} \int_{\mathbb{R}^U} \frac{d\mathbb{P}}{d\mathrm{Leb}}(x) \log\left(\frac{d\mathbb{P}}{d\mathrm{Leb}}(x)\right) dx \text{ if } \mathbb{P} \ll \mathrm{Leb} \text{ and } \frac{d\mathbb{P}}{d\mathrm{Leb}} \log\left(\frac{d\mathbb{P}}{d\mathrm{Leb}}\right) \in L^1\left(\mathbb{R}^U\right), \\ +\infty \text{ otherwise.} \end{cases}$$

Then the surface tension $\nu(U,p)$ is the solution of the minimization problem

$$\nu(\square_n, p) = \inf_{\mathbb{P} \in \mathcal{P}(\mathbb{R}^U)} \left(\frac{1}{|U|} \mathbb{E}\left[\sum_{x,y \in U, |x-y|_1 = 1} V_e\left(p \cdot (x-y) + \phi(x) - \phi(y)\right) \right] + \frac{1}{|U|} H(\mathbb{P}) \right).$$

To derive a second variation estimate, the idea is to obtain uniform convexity of the functional above. This functional involves two terms: an energy term, for which convexity is provided by the assumptions made on the elastic energy V, and the entropy term. To obtain convexity for the latter, we appeal to the notion of displacement convexity introduced by McCann in [**115**] in the setting of optimal transport and provides convexity for the entropy: given two probability measures $\mathbb{P}_0$ and $\mathbb{P}_1$ on $\mathbb{R}^U$ and $t \in [0,1]$, we define $\mathbb{P}_t := ((1-t)I_d + tT)_* \mathbb{P}_0$, where T is the optimal transport map sending $\mathbb{P}_0$ to $\mathbb{P}_1$. The displacement convexity of McCann then asserts that the mapping $t \to H(\mathbb{P}_t)$ is convex in $[0,1]$, i.e.,

$$H(\mathbb{P}_t) \leq tH(\mathbb{P}_0) + (1-t)H(\mathbb{P}_1), \ \forall t \in [0,1].$$

Combining the uniform convexity of the energy and the convexity of the entropy provides a uniform convexity statement for the functional one wishes to minimize in the surface tension.

1.5. Perspectives

1.5.1. About percolation. An important motivation in the development of a quantitative theory of stochastic homogenization is its relation with the random walker on the infinite supercritical cluster: given a discrete environment $\mathbf{a}$ satisfying the assumptions of Section 1.2.3, one can consider the continuous-time random walk $(X_t)_{t\geq 0}$ whose generator is the discrete elliptic operator $-\nabla \cdot \mathbf{a}\nabla$, defined in (1.2.2), started from a point y in the infinite cluster $\mathscr{C}_\infty$ of edges with nonzero conductance. For $t \geq 0$, we denote by $p^{\mathbf{a}}(t,\cdot,y)$ the law of the random walker X_t at time t. Similarly to what is explained in Section 1.1.3, one can prove that the law $p^{\mathbf{a}}(\cdot,\cdot,y)$ is the solution of the parabolic equation

$$\begin{cases} \partial_t p^{\mathbf{a}}(\cdot,\cdot,y) - \nabla \cdot \left(\mathbf{a}\nabla p^{\mathbf{a}}(\cdot,\cdot,y)\right) = 0 & \text{in } (0,\infty) \times \mathscr{C}_\infty, \\ p^{\mathbf{a}}(0,\cdot,y) = \delta_y & \text{in } \mathscr{C}_\infty. \end{cases} \tag{1.5.1}$$

As a consequence, deriving precise information on the parabolic Green's function, i.e. the solution of (1.5.1), provides information on the law of the random walker $(X_t)_{t\geq 0}$. In [**18**, Chapters 8 and 9], a quantitative homogenization theorem is proved for the parabolic Green's function in the uniformly elliptic setting. The proof of this result makes use of the two-scale expansion, which relies on quantitative bounds on the corrector. An interesting question to investigate could be to use the optimal bounds obtained for the corrector on the percolation cluster in Chapter 3, to perform the two-scale expansion associated to the parabolic Green's function and to derive a quantitative homogenization theorem for the parabolic Green's function on the percolation cluster, i.e. prove that there exists a minimal scale random variable $\mathcal{X} \leq \mathcal{O}_s(C)$, for some integrability exponent $s > 0$, such that for every $t \geq \mathcal{X}$,

$$\left| p^{\mathbf{a}}(t,x,y) - p^{\bar{\mathbf{a}}}(t,x,y) \right| \leq C t^{-\delta} t^{-d/2} \exp\left(-\alpha \frac{|x-y|^2}{t} \right),$$

for some nonnegative exponents $\alpha, \delta > 0$, where $p^{\bar{\mathbf{a}}}(t,x,y)$ is the parabolic Green's function associated to the homogenized equation

$$\begin{cases} \partial_t p^{\bar{\mathbf{a}}}(t,x,y) + \nabla \cdot \left(\bar{\mathbf{a}} \nabla p^{\bar{\mathbf{a}}}(t,x,y)\right) = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ p^{\bar{\mathbf{a}}}(0,\cdot,y) = \delta_y & \text{in } \mathbb{R}^d, \end{cases}$$

which can be explicitly computed according to the formula

$$p^{\bar{\mathbf{a}}}(t,x,y) = (4\pi t)^{-d/2} (\det \bar{\mathbf{a}})^{-1/2} \exp\left(-\frac{(x-y)\cdot \bar{\mathbf{a}}^{-1}(x-y)}{4t}\right).$$

1.5.2. About the $\nabla\phi$ model. Using the new ideas introduced in Chapter 5, a few directions of research can be considered. First the results established in this chapter correspond to the beginning of the theory of stochastic homogenization: obtaining a quantitative rate of convergence for the energy quantity ν as stated in (1.1.37). The next step of this program is to prove that on a large scale the interface ϕ concentrates around a deterministic interface which can be characterized as a critical point of a Wulff functional involving the infinite volume surface tension $\overline{\nu}$. A result of this type would be a version of the homogenization Theorem 1.1.2 applied to the Ginzburg-Landau model. In the long run and similarly to what was achieved in stochastic homogenization, this program could hopefully provide an optimal rate of convergence for the finite volume surface tension as well as optimal rates for the convergence of the interface. Finally the analogous quantity of the corrector ϕ_p in the discrete Ginzburg-Landau model is the field ϕ whose law is distributed according the Funaki-Spohn state of slope p (see [**70, 72**] or [**68**, Chapter 4] for a precise definition of the Funaki-Spohn state). In this introduction, we mentioned the optimal bounds which can be derived on the corrector (Theorem 1.1.6) but one can prove a stronger result: following [**18**, Chapter 5], one knows that the properly rescaled version of the corrector converges to a Gaussian free field. Such a result was established by Naddaf and Spencer in [**133**] and requires to use the Helffer-Sjöstrand representation formula. This approach could hopefully provide a new proof of this result which does not rely on the Helffer-Sjöstrand formula.

Second, many models from statistical physics are dealing with elastic potential V which do not satisfy a uniform ellipticity assumption. The Helffer-Sjöstrand representation formula requires uniform convexity of the potential and going beyond the setting of uniform ellipticity is a challenging problem. Nevertheless, it has been proved that the theory of stochastic homogenization is robust enough to prove interesting results in some non uniformly elliptic setting: this is the subject of Chapters 2, 3 and 4 of this thesis and we refer to Section 1.1.4 for a review of some other works achieved in this direction. Hopefully the new ideas introduced in Chapter 5, which forgo any reference to the Helffer-Sjöstrand representation formula, can be used to study some non uniformly elliptic systems in statistical physics as well.

CHAPTER 2

Elliptic regularity and quantitative homogenization on percolation clusters

We establish quantitative homogenization, large-scale regularity and Liouville results for the random conductance model on a supercritical (Bernoulli bond) percolation cluster. The results are also new in the case that the conductivity is constant on the cluster. The argument passes through a series of renormalization steps: first, we use standard percolation results to find a large scale above which the geometry of the percolation cluster behaves (in a sense made precise) like that of Euclidean space. Then, following the work of Barlow [**24**], we find a succession of larger scales on which certain functional and elliptic estimates hold. This gives us the analytic tools to adapt the quantitative homogenization program of Armstrong and Smart [**21**] to estimate the yet larger scale on which solutions on the cluster can be well-approximated by harmonic functions on $\mathbb{R}^d$. This is the first quantitative homogenization result in a porous medium and the harmonic approximation allows us to estimate the scale on which a higher-order regularity theory holds. The size of each of these random scales is shown to have at least a stretched exponential moment. As a consequence of this regularity theory, we obtain a Liouville-type result that states that, for each $k \in \mathbb{N}$, the vector space of solutions growing at most like $o(|x|^{k+1})$ as $|x| \to \infty$ has the same dimension as the set of harmonic polynomials of degree at most k, generalizing a result of Benjamini, Duminil-Copin, Kozma, and Yadin [**29**] from $k \leq 1$ to $k \in \mathbb{N}$.

This chapter corresponds to the article [**13**] written in collaboration with S. Armstrong and published in Communications on Pure and Applied Mathematics.

Contents

2.1. Introduction

2.1.1. Motivation and informal summary of results. Consider the random conductance model on the infinite percolation cluster for supercritical bond percolation on the graph $(\mathbb{Z}^d, B_d)$ in dimension $d \geq 2$. Here B_d is the set of *bonds*, that is, unordered pairs $\{x, y\}$ with

$x, y \in \mathbb{Z}^d$ satisfying $|x - y| = 1$. We are given $\lambda \in (0,1)$ and a function

$$\mathbf{a} : B_d \longrightarrow \{0\} \cup [\lambda, 1].$$

We call $\mathbf{a}(\{x,y\})$ the *conductance* of the bond $\{x,y\} \in B_d$ and we assume that $\{a(e)\}_{e \in B_d}$ is an i.i.d. ensemble. We assume that the Bernoulli random variable $\mathbb{1}_{\{\mathbf{a}(e) \neq 0\}}$ has parameter $\mathfrak{p} > \mathfrak{p}_c(d)$, where $\mathfrak{p}_c(d)$ is the bond percolation threshold for the lattice $\mathbb{Z}^d$. It follows that the graph $(\mathbb{Z}^d, \mathcal{E}(\mathbf{a}))$, where $\mathcal{E}(\mathbf{a})$ is the set of edges $e \in B_d$ for which $\mathbf{a}(e) \neq 0$, has a unique infinite connected component, which we denote by $\mathscr{C}_\infty = \mathscr{C}_\infty(\mathbf{a})$.

Our interest in this paper is in the elliptic finite difference equation

$$-\nabla \cdot \mathbf{a} \nabla u = 0 \quad \text{in } \mathscr{C}_\infty. \tag{2.1.1}$$

Here the elliptic operator $-\nabla \cdot \mathbf{a} \nabla$ is defined on functions $u : \mathscr{C}_\infty \to \mathbb{R}$ by

$$(-\nabla \cdot \mathbf{a} \nabla u)(x) := \sum_{y \sim x} \mathbf{a}((x,y)) (u(x) - u(y)). \tag{2.1.2}$$

The operator $-\nabla \cdot \mathbf{a} \nabla$ is the generator of a continuous-time Markov chain $\{X(t)\}_{t \geq 0}$ which can be briefly described as follows. Each edge $e \in B_d$ is endowed with a clock which rings after exponential waiting times with expectation $\mathbf{a}^{-1}(e)$. The random walker begins at the origin, i.e., $X(0) = 0$. When $X(t) = x \in \mathbb{Z}^d$, the random walker waits until one of the clocks at an adjacent edge to x rings, and then instantly moves across the edge to the neighboring point. The reader may choose to focus on the special case that the conductance $\mathbf{a}$ takes only the values $\{0,1\}$ and the model reduces to the simple random walk on the supercritical percolation cluster, with the generator being the Laplacian. The results in this paper are new even in this simpler situation.

Of primary interest is the scaling limit of this random walk (conditioned on the event that $0 \in \mathscr{C}_\infty$), and more generally its long-time behavior. A *quenched invariance principle* for the random walk $X(t)$ in the case $\mathbf{a} \in \{0,1\}$ was first proved in dimensions $d \geq 4$ by Sidoravicius and Sznitman [**144**] and later, in every dimension $d \geq 2$ by Berger and Biskup [**30**] and, independently, Mathieu and Piatnitski [**113**]. It states that, $\mathbb{P}[\cdot \,|\, 0 \in \mathscr{C}_\infty]$–a.s., the process $\left\{\varepsilon X\left(\varepsilon^{-2} t\right)\right\}_{t \geq 0}$ converges in law, as $\varepsilon \to 0$, to a non-degenerate Brownian motion with covariance matrix σI_d. This result was extended to the setting considered here (and to even greater generality) by Biskup and Prescott [**33**] and Mathieu [**112**] (see also Andres, Barlow, Deuschel and Hambly [**7**]). We refer to the survey of Biskup [**32**] and the references therein for more on the many recent works on this problem.

The quenched invariance principle for the process $\{X_t\}_{t \geq 0}$ is closely related to questions of *homogenization*, that is, the study of the solutions of (2.1.1) on large length scales. The basic qualitative homogenization result states that, $\mathbb{P}$–a.s., a solution u_r of (2.1.1) in $\mathscr{C}_\infty \cap B_r$ converges, as $r \to \infty$, to solutions of the (continuum) partial differential equation

$$-\nabla \cdot \bar{\mathbf{a}} \nabla \overline{u}_r = 0 \quad \text{in } B_r$$

in the sense that

$$\limsup_{r \to \infty} \frac{1}{r^2} \sum_{x \in \mathscr{C}_\infty \cap B_r} |u_r(x) - \overline{u}_r(x)|^2 = 0, \tag{2.1.3}$$

where u_r and $\overline{u}_r$ are given the same Dirichlet boundary condition $f_r(x) = r f\left(\frac{x}{r}\right)$ for a fixed function $f : \partial B_1 \to \mathbb{R}$. The matrix $\bar{\mathbf{a}} = \frac{1}{2}\sigma^2 I_d$, where $\sigma > 0$ is the covariance of the limiting Brownian motion from the invariance principle. The study of (2.1.1) can be motivated independently, from the PDE perspective, by the desire to extend the theory of elliptic homogenization to random porous media (supercritical bond percolation being a very natural model of a random porous medium). However, from the probability point of view, the important point is that a homogenization result is essentially equivalent to an invariance principle. Certainly a quenched invariance principle implies a qualitative homogenization result, while *quantitative* homogenization results give *quantitative* invariance principles. Indeed, perhaps the main difficulty in proving a quenched invariance principle is establishing the sublinear growth of correctors (cf. [**30, 32**]), which is nothing but a quantitative homogenization estimate. (The definition of the correctors is given in the comments following Theorem 2.1.2, below.)

It is a well-known open problem to obtain quantitative information (for instance, rates of convergence) for this model, both in terms of the quenched invariance principle as well as homogenization. It is mentioned for example in [**32**, Section 4.4], [**26**, below Theorem 1.2] and [**108**]. The obstacle is that the various qualitative proofs of the quenched invariance principle rely on an appeal to the ergodic theorem which is difficult to quantify. On the other hand, in the uniformly elliptic setting (when all bonds are open and $\mathbf{a} \in [\lambda, 1]$), there is now a very precise quantitative theory due to Gloria and Otto [**82, 83**] (see also [**81**]) with implications to random walks explained in [**63**]. However, it is not obvious to see how to extend the methods of these papers to the case of percolation clusters since they rely heavily on uniform ellipticity and seem to require the geometry of the random environment to posses some homogeneity down to the smallest scales. In a recent work, Lamacz, Neukamm and Otto [**108**] adapt these methods to the case of Bernoulli bond percolation which is modified so that all bonds in a fixed unit direction are open. However, this model has the property that every lattice point of $\mathbb{Z}^d$ belongs to the infinite cluster and is still quite far from the setting of supercritical percolation clusters.

In this paper, we prove the first quantitative homogenization results for the random conductance model on supercritical percolation clusters (see Theorems 2.1.1 and 2.1.2 below for the precise statements). In particular, we give explicit bounds on the sublinear growth of correctors and rates of convergence for the limit (2.1.3). We also prove a higher-order *regularity theory*, extending recent results in the uniformly elliptic case [**21, 20, 81**] to this setting. In particular, we prove Liouville-type results of every order, which characterize the set of solutions on the infinite cluster which exhibit polynomial growth. Such a regularity theory also provides important gradient estimates which are an essential ingredient for obtaining an *optimal* quantitative theory and obtaining scaling limits for correctors. We expect that the results in this paper will open the way for the development of such a theory on percolation clusters and to resolve several open problems mentioned for example in [**30, 32**]. Indeed, in a forthcoming sequel [**51**] to this paper, we establish optimal bounds on the scaling of correctors as well as the decay of the gradient of the Green's function.

A main source of our inspiration comes from the work of the first author and Smart [**21**], who recently introduced an alternative approach to quantitative theory of homogenization in the uniformly elliptic setting. Their method is based on studying certain subadditive quantities related to the variational formulation of the equation (i.e., the Dirichlet form) and quantifying their convergence by an iteration argument. At each step of the iteration, one passes information from a certain (large) length scale to a multiple of the length scale, showing that the error contracts by a factor less than 1. In this way the method resembles a renormalization argument. Critically, information at the smallest scales can be "forgotten" and we only need to ensure that the model behaves like a uniformly elliptic equation, in some sense, on large scales. This adds some flexibility and robustness to the approach and, as we show here, it is well-suited to handling difficulties encountered in attempting a generalization to percolation clusters.

The ideas are therefore relatively straightforward, even if the details are many and the proofs are long. We begin in Sections 2.2 and 2.3 by finding a large (random) scale on which the percolation cluster has geometric properties which are close to those of $\mathbb{R}^d$. Since this random scale is not uniformly bounded (there will be some large regions where the cluster is badly behaved) we partition $\mathbb{Z}^d$ into triadic cubes of different sizes such that every cube is well-connected in the sense of Antal and Pisztora [**11**], using a Calderón-Zygmund-type stopping time argument. In regions where this partition is rather coarse, the geometry of the cluster is less well-behaved and where it is finer, the cluster is well-connected. Inspired by the work of Barlow [**24**] (which was itself inspired by the earlier work of Mathieu and Remy [**114**]), we continue to coarsen the graph in stages, obtaining functional and elliptic inequalities on larger and larger scales: first we pass to a larger scale on which a Sobolev-Poincaré holds, then obtain a scale on which solutions of (2.1.1) satisfy a reverse Hölder inequality, and again to get a scale on which the gradients of solutions satisfy a Meyers-type higher integrability estimate.

This provides us with the elliptic estimates needed to run the arguments of [**21**], which takes up the bulk of the analysis in the paper. In Section 2.4, we introduce analogues of the subadditive energy quantities and show that they possess similar properties to the ones in the uniformly elliptic setting, at least on large scales and with high probability. The main part of the analysis comes in Section 2.5, where we show convergence of the subadditive quantities to their deterministic limits. The main difficulty compared to the analysis of [**21**] is to deal with the possibility that the energy density of the solutions may be very large in regions of the cluster in which the connectivity is poor (i.e., where the cube partition mentioned above is quite coarse). The resolution comes by using the gain of integrability from Meyers estimate to show that *spatial averages* of the gradients of the solutions cannot concentrate in small regions, which gives us just what we need.

At this stage in the development of our theory, the difference in difficulty between the uniformly elliptic case and the percolation cluster has basically vanished. We conclude by showing first in Section 2.6, by a deterministic argument resembling a numerical analysis exercise, that the convergence of the subadditive quantities implies control of the error in homogenization for the Dirichlet problem. This concludes the proof of our first main result and gives us the harmonic approximation we need to run the arguments of [**21, 20, 81, 17**] to obtain the quantitative $C^{k,1}$ regularity theory and, in particular, the Liouville results. The latter is summarized in Section 2.7.

We continue in the next two subsections by giving the precise assumptions and then the statements of the main results.

2.1.2. Notation and assumptions. Let $\mathbb{Z}^d$ be the standard d-dimensional hypercube lattice and $B_d := \left\{\{x,y\} \,:\, x,y\in\mathbb{Z}^d, |x-y|=1\right\}$ denote the set of nearest neighbor, non-oriented edges. We denote the standard basis in $\mathbb{R}^d$ by $\{\mathfrak{e}_1,\ldots,\mathfrak{e}_d\}$. For $x,y\in\mathbb{Z}^d$, we write $x\sim y$ if x and y are nearest neighbors. We usually denote a generic edge by e. We fix a parameter $\lambda\in(0,1]$ and denote by Ω the set of all functions $\mathbf{a}: B_d\to\{0\}\cup[\lambda,1]$, in other words, $\Omega = (\{0\}\cup[\lambda,1])^{B_d}$ and we let $\mathbf{a}$ denote the canonical element of Ω. The Borel σ-algebra on Ω is denoted by $\mathcal{F}$. For each $U\subseteq\mathbb{Z}^d$, we let $\mathcal{F}(U)\subseteq\mathcal{F}$ denote the smallest σ-algebra such that each of the random variables $\mathbf{a}\mapsto\mathbf{a}(\{x,y\})$, for $x,y\in U$ with $x\sim y$, is $\mathcal{F}(U)$-measurable.

We fix an i.i.d. probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$, that is, a measure of the form $\mathbb{P}=\mathbb{P}_0^{B_d}$ where $\mathbb{P}_0$ is the law of a random variable $\mathbf{a}(0)$ taking values in $\{0\}\cup[\lambda,1]$ with the property that, for a fixed edge e,

$$\mathfrak{p} := \mathbb{P}_0\left[\mathbf{a}(e)\neq 0\right] > \mathfrak{p}_{\mathfrak{c}}(d)$$

and $\mathfrak{p}_{\mathfrak{c}}(d)$ is the bond percolation threshold for the lattice $\mathbb{Z}^d$. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$.

Given $\mathbf{a}\in\Omega$, we say that an edge $e\in B_d$ is *occupied* if $\mathbf{a}(e)>0$ and *vacant* if $\mathbf{a}(e)=0$. Given vertices $x,y\in\mathbb{Z}^d$, a *path connecting x and y* is a sequence of occupied edges of the form $\{x,z_1\},\{z_1,z_2\},\ldots,\{z_n,z_{n+1}\},\ldots,\{z_N,y\}$. We say that *$x$ and y are connected* and write $x\leftrightarrow_{\mathbf{a}} y$ if there exists a path connecting x and y. A *cluster* is a subset $\mathscr{C}\subseteq\mathbb{Z}^d$ with the property that, for every $x,y\in\mathscr{C}$, there exists a path connecting x and y consisting only of edges between elements of $\mathscr{C}$. The $\mathbf{a}$-*interior* of a subset $U\subseteq\mathbb{Z}^d$ is the subset

$$\operatorname{int}_{\mathbf{a}}(U) := \{x\in U \,:\, y\sim x \text{ and } \mathbf{a}(\{x,y\})\neq 0 \implies y\in U\}$$

The $\mathbf{a}$-*boundary of U* is $\partial_{\mathbf{a}}U := U\setminus\operatorname{int}_{\mathbf{a}}(U)$. The interior and boundary with respect to the nearest-neighbor lattice $(\mathbb{Z}^d,B_d)$ are denoted by

$$\operatorname{int}(U) := \{x\in U \,:\, y\sim x \implies y\in U\} \quad \text{and} \quad \partial U := U\setminus\operatorname{int}(U).$$

We write $x \leftrightarrow_{\mathbf{a}} \infty$ if x belongs to an unbounded cluster and we denote by

$$\mathscr{C}_\infty := \left\{x \in \mathbb{Z}^d \,:\, x \leftrightarrow_{\mathbf{a}} \infty\right\}$$

the maximal unbounded cluster, which $\mathbb{P}$–almost surely exists and is unique [**40**].

We next introduce our notation for vector fields and Dirichlet forms. We let

$$E_d := \left\{(x,y) \,:\, x,y \in \mathbb{Z}^d, x \sim y\right\}$$

denote the set of oriented nearest-neighbor pairs and $E_d(U) := \{(x,y) \,:\, x,y \in U, x \sim y\}$ denote the edges lying in a subset $U \subseteq \mathbb{Z}^d$. A *vector field* G on U is a function

$$G : E_d(U) \to \mathbb{R}$$

which is antisymmetric, that is, $G(x,y) = -G(y,x)$ for every $(x,y) \in E_d(U)$. If $u : U \to \mathbb{R}$, then ∇u is the vector field defined by

$$(\nabla u)(x,y) := u(x) - u(y)$$

and $\mathbf{a}\nabla u$ is the vector field defined by

$$(\mathbf{a}\nabla u)(x,y) := \mathbf{a}(\{x,y\})(u(x) - u(y)).$$

If $q \in \mathbb{R}^d$, we also let q denote the constant vector field given by

$$q(x,y) := q \cdot (x-y).$$

If F is a vector field on U, then we define, for each $x \in U$,

$$|F|(x) := \left(\frac{1}{2}\sum_{y\in U,\, y\sim x} |F(x,y)|^2\right)^{\frac{1}{2}}. \tag{2.1.4}$$

We note that the definition of $|F|$ depends on the underlying domain U, but we do not display this dependence explicitly since it is always clear from the context. We put an inner product $\langle \cdot, \cdot \rangle_U$ on the space of vector fields on U, defined by

$$\langle F, G\rangle_U := \sum_{x,y\in U,\, x\sim y} F(x,y)G(x,y).$$

We denote by $\langle F\rangle_U$ the unique vector in $\mathbb{R}^d$ such that, for every $p \in \mathbb{R}^d$,

$$p \cdot \langle F\rangle_U = \langle p, F\rangle_U.$$

Given $\mathbf{a} \in \Omega$ and two functions $u, v : U \to \mathbb{R}$, the (bilinear) Dirichlet form can be written in this notation as

$$\langle \nabla u, \mathbf{a}\nabla v\rangle_U = \frac{1}{2}\sum_{x,y\in U,\, x\sim y} (u(x)-u(y))\,\mathbf{a}(\{x,y\})(v(x)-v(y)).$$

The elliptic operator $-\nabla \cdot \mathbf{a}\nabla$ is defined for each $u : U \to \mathbb{R}$ and $x \in U$ by

$$(-\nabla \cdot \mathbf{a}\nabla u)(x) := \sum_{y\in U,\, x\sim y} (\mathbf{a}\nabla u)(x,y) = \sum_{y\in U,\, x\sim y} \mathbf{a}(\{x,y\})(u(x)-u(y)).$$

We denote set of solutions (i.e., $\mathbf{a}$-harmonic functions) on a subset $U \subseteq \mathbb{Z}^d$ by

$$\mathcal{A}(U) := \{u : U \to \mathbb{R} \,:\, -\nabla \cdot \mathbf{a}\nabla u(x) = 0 \ \text{ for every } x \in \mathrm{int}_{\mathbf{a}}(U)\}. \tag{2.1.5}$$

We denote by $\mathcal{C}_0^{\mathbf{a}}(U)$ the set of functions $w : U \to \mathbb{R}$ with compact support and satisfying $w = 0$ on $\partial_{\mathbf{a}} U$. Then it is easy to check that

$$u \in \mathcal{A}(U) \iff \langle \nabla w, \mathbf{a}\nabla u\rangle_U = 0 \quad \text{for every } w \in \mathcal{C}_0^{\mathbf{a}}(U). \tag{2.1.6}$$

We next introduce our notation for keeping track of the sizes and stochastic integrability of random variables. Given $s, \theta > 0$ and a random variable X on $(\Omega, \mathcal{F})$, we write

$$X \le \mathcal{O}_s(\theta) \iff \mathbb{E}\left[\exp\left(\left(\frac{X}{\theta}\right)^s\right)\right] \le 2.$$

Note that by Markov's inequality, $X \le \mathcal{O}_s(\theta)$ implies that, for every $t > 0$,

$$\mathbb{P}[X \ge \theta t] \le 2\exp(-t^s),$$

so roughly the notation means that X has characteristic size at most θ with the tails of the distribution of $\theta^{-1}X$ decaying at most like $\lesssim \exp(-t^s)$. If Y is another random variable and $a > 0$, we also write

$$X \le Y + a\mathcal{O}_s(\theta) \iff X - Y \le \mathcal{O}_s(a\theta)$$

and, if Y is nonnegative,

$$X \le \mathcal{O}_s(\theta) Y \iff \frac{X}{Y} \le \mathcal{O}_s(\theta).$$

This notation is transitive in the sense that (cf. [**17**, Lemma 2.3(i)]), for a universal constant C depending only on s, which may be taken to be 1 if $s \ge 1$,

$$X \le \mathcal{O}_s(\theta_1) \text{ and } Y \le \mathcal{O}_s(\theta_2) \implies X + Y \le \mathcal{O}_s(C(\theta_1 + \theta_2)) \tag{2.1.7}$$

Moreover, by [**17**, Lemma 2.3(ii)], for any $s > 0$ there exists $C(s) < \infty$ such that, for every measure space $(X, \mathcal{F}, \mu)$ and measurable function $f : E \to \mathbb{R}_+$ and jointly measurable family $\{X(z)\}_{z\in E}$ of nonnegative random variables,

$$\forall z \in E,\ X(z) \le \mathcal{O}_s(1) \implies \int_E X(z)\, d\mu(z) \le \mathcal{O}_s(C). \tag{2.1.8}$$

Young's and Hölder's inequalities imply (cf. [**17**, Remark 2.2])

$$|X| \le \mathcal{O}_{s_1}(\theta_1) \text{ and } |Y| \le \mathcal{O}_{s_2}(\theta_2) \implies |XY| \le \mathcal{O}_{\frac{s_1 s_2}{s_1+s_2}}(\theta_1\theta_2). \tag{2.1.9}$$

It is easy to check from the Young's inequality that, for every $s_1, s_2 \in (0,\infty)$, every $t \in [0,1]$ and random variable X, we have for $\alpha = \frac{ts_1}{ts_1+(1-t)s_2}$

$$X \le \mathcal{O}_{s_1}(\theta_1) \text{ and } X \le \mathcal{O}_{s_2}(\theta_2) \implies X \le \mathcal{O}_{ts_1+(1-t)s_2}(\theta_1^{\alpha}\theta_2^{1-\alpha}). \tag{2.1.10}$$

For a finite $U \subseteq \mathbb{Z}^d$ and $w : U \to \mathbb{R}$, we often denote sums by integrals; for example,

$$\text{we often write } \int_U w(x)\,dx \quad \text{in place of} \quad \sum_{x\in U} w(x). \tag{2.1.11}$$

If U is a finite set, we denote its cardinality by $|U|$. Sometimes we also use $|V|$ to denote the Lebesgue measure of a subset $V \subseteq \mathbb{R}^d$, but the meaning will always be clear from context. The normalized integral for a function $w : U \to \mathbb{R}$ for a finite subset $U \subseteq \mathbb{Z}^d$ is denoted

$$\fint_U w(x)\,dx = \frac{1}{|U|}\int_U w(x)\,dx = \frac{1}{|U|}\sum_{x\in U} w(x).$$

For $p \in [1,\infty)$, we denote the L^p and normalized L^p norms of w by

$$\|w\|_{L^p(U)} := \left(\int_U |w(x)|^p\,dx\right)^{\frac{1}{p}} \quad \text{and} \quad \|w\|_{\underline{L}^p(U)} := \left(\fint_U |w(x)|^p\,dx\right)^{\frac{1}{p}}$$

and $\|w\|_{L^\infty(U)} := \sup_{x\in U} |w(x)|$. We define the distance function dist with respect to the ℓ_∞ norm on the coordinates, i.e., $\operatorname{dist}(x,y) = \max_{i=1,\ldots,d} |x_i - y_i|$ and extend this to subsets of $\mathbb{R}^d$ by $\operatorname{dist}(U,V) = \inf_{x\in U, y\in V} \operatorname{dist}(x,y)$.

A *cube* is a subset of $\mathbb{Z}^d$ of the form

$$\mathbb{Z}^d \cap \left(z + [0,N)^d\right), \quad z \in \mathbb{Z}^d,\ N \in \mathbb{N}.$$

We define the *center* and *size* of the cube given in the previous display above to be z and N, respectively, and denote the size of a cube $\square$ by size$(\square)$. For a cube $\square$ and $r > 0$, we use the nonstandard convention of denoting by $r\square$ the cube with size $\lfloor r\,\mathrm{size}(\square)\rfloor$ and *having the same center as* $\square$. A *triadic cube* is a cube of the form

$$\square_m(z) := \mathbb{Z}^d \cap \left(z + \left(-\frac{1}{2}3^m, \frac{1}{2}3^m\right)^d\right), \quad z \in 3^m\mathbb{Z}^d,\ m \in \mathbb{N}.$$

We also write $\square_m = \square_m(0)$. Observe that $\operatorname{size}(\square_m) = 3^m$. For every $m,n \in \mathbb{N}$ with $n \le m$, each triadic cube $\square_m(z)$ may be uniquely partitioned into exactly $3^{d(m-n)}$ disjoint triadic cubes of the form $\square_n(y)$, $y \in 3^n\mathbb{Z}^d$. Moreover, any two triadic cubes (of possibly different sizes) are either disjoint or else one is a subset of the other. We denote the collection of triadic cubes by $\mathcal{T}$ and the set of triadic cubes of size 3^n by $\mathcal{T}_n$. Note that $\mathcal{T}_n \coloneqq \left\{ z + \square_n \,:\, z \in 3^n\mathbb{Z}^d \right\}$. For each $\square \in \mathcal{T}$, the *predecessor of* $\square$ is the unique triadic cube $\widetilde{\square} \in \mathcal{T}$ satisfying

$$\square \subseteq \widetilde{\square} \quad \text{and} \quad \frac{\operatorname{size}(\widetilde{\square})}{\operatorname{size}(\square)} = 3. \tag{2.1.12}$$

If $\widetilde{\square}$ is the predecessor of $\square$, then we also say that $\square$ is a *successor* of $\widetilde{\square}$. Note that, since we are working with subsets of the discrete lattice $\mathbb{Z}^d$, disjoint triadic cubes will be separated by a distance of at least 1. In fact, two disjoint cubes $\square$, $\square'$ are neighbors if and only if $\operatorname{dist}(\square,\square') = 1$.

2.1.3. Statement of the main results. The first main result gives an estimate of the length scale on which the homogenization approximation holds, up to an algebraic error threshold. In the statement, we use the notation $\mathscr{C}_*(\square_m)$ which is not defined until Section 2.2, but roughly denotes the largest connected component of $\mathscr{C}_\infty \cap \square_m$ (which is also the same as $\mathscr{C}_\infty \cap \square_m$, up to a small number of vertices near the boundary of $\square_m$).

Theorem 2.1.1 (Quantitative homogenization). *Fix an exponent $p > 2$. Then there exist $s(p,d,\mathfrak{p},\lambda) > 0$ and $\alpha(p,d,\mathfrak{p},\lambda) > 0$, a constant $C(p,d,\mathfrak{p},\lambda) < \infty$, a symmetric matrix $\bar{\mathbf{a}}$ such that*

$$\frac{1}{C} I_d \le \bar{\mathbf{a}} \le C I_d$$

and a nonnegative random variable $\mathcal{X}$ satisfying

$$\mathcal{X} \le \mathcal{O}_s(C)$$

such that the following holds: for every $m \in \mathbb{N}$ such that $3^m \ge \mathcal{X}$ and function $u : \mathscr{C}_*(\square_m) \to \mathbb{R}$ satisfying

$$-\nabla \cdot \mathbf{a} \nabla u = 0 \quad \text{in } \mathscr{C}_*(\square_m) \setminus \partial \square_m,$$

there exists an $\bar{\mathbf{a}}$-harmonic function u_{hom} on $\left[-\frac12(3^m-1), \frac12(3^m-1)\right]^d$ satisfying

$$u(x) = u_{\mathrm{hom}}(x) \quad \text{for every } x \in \mathscr{C}_*(\square_m) \cap \partial\left(-\frac12(3^m-1), \frac12(3^m-1)\right)^d, \tag{2.1.13}$$

$$\fint_{\left(-\frac12(3^m-1),\frac12(3^m-1)\right)^d} \left|\nabla u_{\mathrm{hom}}(x)\right|^2 dx \le \frac{1}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^p(x) \tag{2.1.14}$$

and

$$3^{-m}\left(\frac{1}{|\square_m|} \sum_{x\in\mathscr{C}_*(\square_m)} \left|u(x) - u_{\mathrm{hom}}(x)\right|^2\right)^{\frac12} \le C 3^{-m\alpha} \left(\frac{1}{|\square_m|}\sum_{x\in\mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^p(x)\right)^{\frac1p}. \tag{2.1.15}$$

In view of (2.1.13), Theorem 2.1.1 can be thought of as an error estimate for the Dirichlet problem. Indeed, as we will see in the proof in Section 2.6, the function u_{hom} is constructed by solving the Dirichlet problem with boundary data obtained by smoothing out u itself near $\partial\square_m$. The first estimate (2.1.14) just says that the gradient of u_{hom} in L^p is no larger than that of u itself. The second estimate (2.1.15) is the main part of the conclusion, which states that the (properly scaled) L^2 difference between u and u_{hom} is smaller than the size of ∇u in L^p by the factor $3^{-m\alpha}$, that is, some power of the length scale.

We turn to an important consequence of Theorem 2.1.1, namely the large-scale regularity theory for solutions on the infinite cluster, which we denote by $\mathcal{A}(\mathscr{C}_\infty)$. For each $k \in \mathbb{N}$, we also let $\mathcal{A}_k(\mathscr{C}_\infty)$ denote the subspace of $\mathcal{A}(\mathscr{C}_\infty)$ consisting of functions growing more slowly at infinity than a polynomial of degree $k+1$:

$$\mathcal{A}_k(\mathscr{C}_\infty) := \left\{ u \in \mathcal{A}(\mathscr{C}_\infty) \, : \, \limsup_{R\to\infty} R^{-(k+1)} \left\| u \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)} = 0 \right\}.$$

Our second main result (see Theorem 2.1.2 below) concerns the structure of $\mathcal{A}_k(\mathscr{C}_\infty)$.

THEOREM 2.1.2 (Regularity theory). *There exist $s(d,\mathfrak{p},\lambda) > 0$, $\delta(d,\mathfrak{p},\lambda) > 0$ and a nonnegative random variable $\mathcal{X}$ satisfying*

$$\mathcal{X} \le \mathcal{O}_s(C(d,\mathfrak{p},\lambda)) \tag{2.1.16}$$

such that the following hold:

(i) *For each $k \in \mathbb{N}$, there exists a constant $C(k,d,\mathfrak{p},\lambda) < \infty$ such that, for every $u \in \mathcal{A}_k(\mathscr{C}_\infty)$, there exists $p \in \overline{\mathcal{A}}_k$ such that, for every $r \ge \mathcal{X}$,*

$$\left\| u - p \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \le C r^{-\delta} \left\| p \right\|_{\underline{L}^2(B_r)}. \tag{2.1.17}$$

(ii) *For every $k \in \mathbb{N}$ and $p \in \overline{\mathcal{A}}_k$, there exists $u \in \mathcal{A}_k$ such that, for every $r \ge \mathcal{X}$, the inequality* (2.1.17) *holds.*

(iii) *For each $k \in \mathbb{N}$, there exists $C(k,d,\mathfrak{p},\lambda) < \infty$ such that, for every $R \ge 2X$ and $u \in \mathcal{A}(\mathscr{C}_\infty \cap B_R)$, there exists $\phi \in \mathcal{A}_k(\mathscr{C}_\infty)$ such that, for every $r \in \left[\mathcal{X}, \frac{1}{2}R\right]$, we have*

$$\left\| u - \phi \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \le C \left(\frac{r}{R} \right)^{k+1} \left\| u \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)}.$$

A consequence of statements (i) and (ii) of Theorem 2.1.2 is that the vector space $\mathcal{A}_k(\mathscr{C}_\infty)$ has the same dimension as $\overline{\mathcal{A}}_k$, that is, the same dimension as the space of harmonic polynomials of order at most k. This was previous proved in the case $k = 1$ and $\mathbf{a} \in \{0,1\}$ by Benjamini, Duminil-Copin, Kozma, and Yadin [**29**]. For $k > 1$, it was previously proved (in greater generality) that the subspace of $\mathcal{A}(\mathscr{C}_\infty)$ of functions growing at most like $O(|x|^k)$ had finite dimension: see Sapozhnikov [**140**].

The Liouville result summarized in (i) and (ii) imply, in the case $k = 1$, that every element u of $\mathcal{A}_1(\mathscr{C}_\infty)$ can be written as

$$u(x) = c + p \cdot x + \chi_p(x)$$

where $c \in \mathbb{R}$, $p \in \mathbb{R}^d$ and χ_p is a function satisfying, for every $r \ge \mathcal{X}$,

$$\left\| \chi_p \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \le C |p| r^{1-\delta}. \tag{2.1.18}$$

These functions $\{\chi_p : p \in \mathbb{R}^d\}$ are called the *correctors* and their sublinear growth is a very important property that was previously proved only qualitatively (cf. [**30, 32**]). Together with the bound on $\mathcal{X}$ in (2.1.16), this provides the first quantitative bound on the sublinearity of χ_p.

Moreover, the qualitative Liouville result is quantified by the third statement (iii), which tells us much more: any $\mathbf{a}$-harmonic function may be expanded to arbitrary order in terms of elements of $\mathcal{A}_k(\mathscr{C}_\infty)$ in the same way that analytic functions can be approximated by Taylor polynomials of degree k. Even this statement for $k = 0$ is new and, combining it with Caccioppoli inequality (see Lemma 2.3.5), gives the following gradient bound may be compared to the results of [**21, 20**]: for every $\mathcal{X} \le r \le R$,

$$\left\| \nabla u \mathbb{1}_{\{\mathbf{a} \ne 0\}} \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \le C \left\| \nabla u \mathbb{1}_{\{\mathbf{a} \ne 0\}} \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)}. \tag{2.1.19}$$

Gradient estimates like (2.1.19) play an important role in obtaining optimal quantitative homogenization estimates, in particular estimates for the sublinearity of the correctors in the uniformly elliptic setting; see [**80**]. In the forthcoming sequel [**13**] to this paper, we explore the consequences of (2.1.19) in the setting of supercritical percolation clusters and show that they allow us to prove optimal estimates on the decay of the Green's function, its gradient as well as optimal bounds on the scaling of the correctors. In particular, we improve (2.1.18) to the optimal sublinearity bound in all dimensions.

Remark 2.1.1. The gradient bound (2.1.19) allows us to immediately upgrade the bound in (2.1.18) from $\underline{L}^2(\mathscr{C}_\infty \cap B_r)$ to $L^\infty(\mathscr{C}_\infty \cap B_r)$. To see this, we use the following interpolation inequality for L^∞ between L^2 and $C^{0,1}$: there exists $C(d) < \infty$ such that, for every $w : \mathbb{Z}^d \cap B_r \to \mathbb{R}$ with zero mean on $\mathbb{Z}^d \cap B_r$, we have

$$\|w\|_{L^\infty(\mathbb{Z}^d\cap B_r)} \le C \|w\|^{\frac{2}{3}}_{\underline{L}^2(\mathbb{Z}^d\cap B_r)} \left(r \|\nabla w\|_{L^\infty(\mathbb{Z}^d\cap B_r)} \right)^{\frac{1}{3}}.$$

Next, we deduce from (2.1.16), (2.1.18) and (2.1.19) the bound

$$\left|\nabla \chi_p \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|(0) \le C\mathcal{X}^{\frac{d}{2}} \left\|\nabla \chi_p \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right\|_{\underline{L}^2(\mathscr{C}_\infty\cap B_{\mathcal{X}})} \le C|p|\mathcal{X}^{\frac{d}{2}} \le \mathcal{O}_{2s/d}\left(C|p|\right).$$

Therefore, by stationarity and (2.1.8), for every $m > 1$,

$$\left\|\nabla \chi_p \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right\|_{L^\infty(\mathscr{C}_\infty\cap B_r)} \le Cr^{\frac{d}{m}} \left\|\nabla \chi_p \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right\|_{\underline{L}^m(\mathscr{C}_\infty\cap B_r)} \le \mathcal{O}_{2s/d}\left(Cr^{\frac{d}{m}}|p|\right).$$

Here the C depends additionally on m. We deduce that, for every $\beta > 0$, there exists $s(d, \mathfrak{p}, \lambda) > 0$ and $C(\beta, d, \mathfrak{p}, \lambda) < \infty$ such that

$$\left\|\nabla \chi_p \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right\|_{L^\infty(\mathscr{C}_\infty\cap B_r)} \le \mathcal{O}_s\left(C|p|r^\beta\right). \tag{2.1.20}$$

We may now apply the interpolation inequality above to the coarsened function $[\chi_p]_{\mathcal{P}}$ (see Definition 2.3.1), using the previous estimates (2.1.18) and (2.1.20) and Lemmas 2.3.2 and 2.3.3 to obtain, for some $s(d, \mathfrak{p}, \lambda) > 0$ and $C(d, \mathfrak{p}, \lambda) < \infty$,

$$\left\|[\chi_p]_{\mathcal{P}}\right\|_{L^\infty(B_r)} \le \mathcal{O}_s\left(C|p|r^{1-\delta/2}\right). \tag{2.1.21}$$

Finally, we can then use Lemma 2.3.2 and the previous inequality to get

$$\|\chi_p\|_{L^\infty(\mathscr{C}_\infty\cap B_r)} \le \mathcal{O}_s\left(C|p|r^{1-\delta/2}\right), \tag{2.1.22}$$

as desired.

We conclude with some comments regarding some of the parameters appearing in the main theorems. First, we do not obtain the optimal exponent $s(d, \mathfrak{p}, \lambda) > 0$ for the stochastic integrability appearing in Theorems 2.1.1 and 2.1.2. In the uniformly elliptic setting, it is proved in [**21**] that we may take any $s \in (0, d)$, which is the optimal stochastic integrability (in the sense that the results are false for $s > d$). The arguments here do lead to an explicit estimate of s, although we expect that it is impossible to find the optimal exponent s (which should depend only on d) without a very deep understanding of the geometry of the percolation cluster which, at least for $\mathfrak{p}$ close to $\mathfrak{p}_c$ and $d > 2$, remains elusive. We therefore have not made any attempt to optimize or even keep track of the explicit s we obtain. Likewise, it would be very interesting to show that the constant $c(d, \mathfrak{p}, \lambda) > 0$ in the lower bound for the effective diffusivity we obtain in (2.5.1) depends on $\mathfrak{p}$ like a power of $\mathfrak{p} - \mathfrak{p}_c$ that is, for some $\beta > 0$,

$$c(d, \mathfrak{p}, \lambda) \ge c_0(d, \lambda) \cdot (\mathfrak{p} - \mathfrak{p}_c)^\beta.$$

Our arguments actually give such a bound provided we can quantify the constants $c(d, \mathfrak{p})$ and $C(d, \mathfrak{p})$ in Lemma 2.2.7 below, which is proved in Antal and Pisztora [**11**] and contains the basic geometric information about the supercritical percolation clusters that all of our renormalization argument rely on. We are not aware of any work which estimates these constants, even crudely, and such an estimate would obviously be of fundamental interest to the study of percolation clusters beyond its implications to random walks on the clusters.

The basic theme we wish to emphasize is that the bottleneck to getting estimates of these parameters, and improving our quantitative understanding in other ways, lies not in improving our quantitative homogenization methodology but rather in obtaining a better quantitative, geometric understanding of supercritical percolation clusters (especially near criticality).

As a final remark, we would like to mention that the arguments in this paper give similar results for other random graphs (besides the particular case of a supercritical percolation cluster), provided that we have some quantitative estimate, like the one in Lemma 2.2.7, which says that the graph behaves like Euclidean space above some random scale.

2.2. Triadic partitions of good cubes

The geometry of the percolation cluster $\mathscr{C}_\infty$ and, more generally, the behavior of solutions of (2.1.1), is highly irregular on small scales but becomes more regular as we look on larger scales. A large part of our effort in this paper is to quantify this vague assertion for various notions of "good behavior." To this end, we will find it very useful to partition $\mathbb{Z}^d$ and certain subsets of $\mathbb{Z}^d$ into "good" cubes (in which the percolation cluster and solutions of (2.1.1) are well-behaved in some sense). These partitions will be random and in particular the sizes of the cubes will necessarily be non-uniform, but we will prove quantitative estimates on the size of a typical cube (which will depend on what "good" means). The coarseness of the partition therefore provides a measure of the local scale above which the system is well-behaved.

In this section, we give a general scheme for creating such partitions. Then, as a first application, we partition $\mathbb{Z}^d$ into "well-connected" cubes which greatly simplifies the geometry of the cluster and will allow us in the next section to prove functional inequalities (for example, a Sobolev inequality) for functions on subsets of $\mathscr{C}_\infty$.

2.2.1. A general scheme for partitions of good cubes. The construction of the partition is accomplished by a stopping time argument reminiscent of a Calderón-Zygmund-type decomposition. We are given a notion of "good cube" represented by an $\mathcal{F}$-measurable function which maps Ω into the set of all subsets of $\mathcal{T}$. In order words, for each $\mathbf{a} \in \Omega$, we are given a subcollection $\mathcal{G}(\mathbf{a}) \subseteq \mathcal{T}$ of triadic cubes. We think of $\square \in \mathcal{T}$ as being a good cube if $\square \in \mathcal{G}(\mathbf{a})$. As usual, we typically drop the dependence on $\mathbf{a}$ and just write $\mathcal{G}$.

PROPOSITION 2.2.1. *Let $\mathcal{G} \subseteq \mathcal{T}$ be a random collection of triadic cubes, as above. Suppose that $\mathcal{G}$ satisfies, for every $\square = z + \square_n \in \mathcal{T}$,*

$$\text{the event } \{\square \notin \mathcal{G}\} \text{ is } \mathcal{F}(z + \square_{n+1})\text{–measurable,} \tag{2.2.1}$$

and, for some constants $K, s > 0$,

$$\sup_{z \in 3^n \mathbb{Z}^d} \mathbb{P}\left[z + \square_n \notin \mathcal{G}\right] \le K \exp\left(-K^{-1} 3^{ns}\right). \tag{2.2.2}$$

Then, $\mathbb{P}$–almost surely, there exists a partition $\mathcal{S} \subseteq \mathcal{T}$ of $\mathbb{Z}^d$ into triadic cubes with the following properties:

(i) *All predecessors of elements of $\mathcal{S}$ are good: for every $\square, \square' \in \mathcal{T}$,*
$$\square' \subseteq \square \text{ and } \square' \in \mathcal{S} \implies \square \in \mathcal{G}.$$

(ii) *Neighboring elements of $\mathcal{S}$ have comparable sizes: for every $\square, \square' \in \mathcal{S}$ such that $\operatorname{dist}(\square, \square') \le 1$, we have*
$$\frac{1}{3} \le \frac{\operatorname{size}(\square')}{\operatorname{size}(\square)} \le 3.$$

(iii) *Estimate for the coarseness of $\mathcal{S}$: if we denote $\square_{\mathcal{S}}(x)$ the unique element of $\mathcal{S}$ containing a point $x \in \mathbb{Z}^d$, then there exists $C(s,K,d) < \infty$ such that, for every $x \in \mathbb{Z}^d$,*
$$\operatorname{size}\left(\square_{\mathcal{S}}(x)\right) = \mathcal{O}_s(C).$$

(iv) *Approximate locality of $\mathcal{S}$: for each $m \in \mathbb{N}$ with $m \ge n$ and $z \in 3^m \mathbb{Z}^d$, there exists a constant $C(s,K,d) < \infty$ and a partition $\mathcal{S}_{\mathrm{loc}}(z + \square_m) \subseteq \mathcal{T}$ of $z + \square_m$ which is $\mathcal{F}(z + \square_m)$-measurable, finer than $\mathcal{S}$ and satisfies, for*
$$\square_m^{(n)} := \left\{ x \in \square_m \, : \, \operatorname{dist}(x, \partial \square_m) \ge 3^n \right\},$$

the estimate

$$\mathbb{P}\left[\exists x \in z + \square_m^{(n)},\, \square_{\mathcal{S}}(x) \neq \square_{\mathcal{S}_{\mathrm{loc}}(z+\square_m)}(x)\right] \leq C 3^{m-n} \exp\left(-C^{-1} 3^{ns}\right).$$

PROOF. We begin by giving an algorithmic construction of $\mathcal{S}$. First, we shrink $\mathcal{G}$ to make it closed under taking predecessors and neighbors of predecessors. This requires the following definition: for each $\square \in \mathcal{T}$, we set

$$\mathcal{K}(\square) := \Big\{\square' \in \mathcal{T} : \exists n \in \mathbb{N} \text{ and } \square^0, \ldots, \square^n \in \mathcal{T} \text{ such that } \square^0 = \square,\ \square^n = \square', \\ \forall m \in \{1, \cdots, n\},\ \mathrm{size}(\square^m) = 3^m \,\mathrm{size}(\square),\ \text{and } \mathrm{dist}\left(\widetilde{\square}^m, \square^{m-1}\right) \leq 1\Big\}.$$

In other words, $\mathcal{K}(\square)$ is the collection of triadic cubes we can obtain in finitely many steps starting from $\square$ and where, in the mth step, we move from a cube $\square^{m-1}$ of size $3^{m-1}\,\mathrm{size}(\square)$ to a cube $\square^m$ of size $3^m\,\mathrm{size}(\square)$ whose predecessor is the neighbor of, or contains, $\square^{m-1}$. Recall that the predecessor $\widetilde{\square}$ of $\square$ is defined in the sentence ending in (2.1.12).

It is clear from the definition of $\mathcal{K}(\square)$ that

$$\square' \in \mathcal{K}(\square) \implies \mathcal{K}(\square') \subseteq \mathcal{K}(\square). \tag{2.2.3}$$

We now define

$$\overline{\mathcal{G}} := \{\square \in \mathcal{G} : \mathcal{K}(\square) \subseteq \mathcal{G}\}.$$

By (2.2.3) we see immediately that

$$\square \in \overline{\mathcal{G}} \implies \mathcal{K}(\square) \subseteq \overline{\mathcal{G}}. \tag{2.2.4}$$

To estimate the probability that a cube belongs to $\overline{\mathcal{G}}$, we use a union bound, (2.2.2) and the fact that, for each $m \in \mathbb{N}$, there are at most C distinct cubes of size $3^m\,\mathrm{size}(\square)$ belonging to $\mathcal{K}(\square)$ (this is a consequence of (2.2.5), below). The union bound gives, for each $\square \in \mathcal{G}$,

$$\mathbb{P}\left[\square \notin \overline{\mathcal{G}}\right] \leq \sum_{m=0}^{\infty} CK \exp\left(-K^{-1} 3^{ms}\,\mathrm{size}(\square)^s\right) \leq C \exp\left(-C^{-1}\,\mathrm{size}(\square)^s\right).$$

It follows immediately that, $\mathbb{P}$–a.s., every element of $\mathbb{Z}^d$ belongs to infinitely many elements of $\overline{\mathcal{G}}$. In particular, $\overline{\mathcal{G}}$ covers $\mathbb{Z}^d$.

We then introduce the partition $\mathcal{S}$ by defining, for each $x \in \mathbb{Z}^d$, the cube $\square_{\mathcal{S}}(x)$ to be the largest element of $\overline{\mathcal{G}}$ containing x which has a successor which does not belong to $\overline{\mathcal{G}}$. If there is no such cube, we set $\square_{\mathcal{S}}(x) := x + \square_0 = \{x\}$, which must then ($\mathbb{P}$–a.s.) belong to $\overline{\mathcal{G}}$. It is easy to see from this construction that $\mathcal{S}$ is indeed a partition. Moreover, we can estimate, for each $n \in \mathbb{N}$,

$$\mathbb{P}\left[\mathrm{size}\left(\square_{\mathcal{S}}(x)\right) = 3^n\right] \leq 3^d \sup_{z \in 3^{n-1}\mathbb{Z}^d} \mathbb{P}\left[z + \square_{n-1} \notin \overline{\mathcal{G}}\right] \leq C \exp\left(-C^{-1} 3^{(n-1)s}\right).$$

It follows that

$$\mathrm{size}(\square_{\mathcal{S}}(x)) = \mathcal{O}_s(C),$$

which confirms property (iii).

To check property (i), we note simply that $\mathcal{S} \subseteq \overline{\mathcal{G}} \subseteq \mathcal{G}$ and that $\overline{\mathcal{G}}$ is closed under taking predecessors.

To check property (ii), consider an element $\square \in \mathcal{S}$ and another cube $\square' \in \mathcal{T}$ with $\mathrm{dist}(\square, \square') = 1$ and $\mathrm{size}(\square') \geq 9\,\mathrm{size}(\square)$. To see that $\square'$ cannot belong to $\mathcal{S}$, observe that, since $\mathrm{dist}(\square', \square) \leq 1$, each of the successors of $\square'$ belongs to $\mathcal{K}(\square)$. Therefore each of the successors of $\square'$ belongs $\overline{\mathcal{G}}$ by (2.2.4) and the fact that $\square \in \mathcal{S} \subseteq \overline{\mathcal{G}}$. Thus $\square' \notin \mathcal{S}$ by the definition of $\mathcal{S}$.

We have left to check property (iv), which is accomplished by localizing the construction above. First we observe that there exists $C(d) < \infty$ such that

$$\square \in \mathcal{T} \text{ and } \square' \in \mathcal{K}(\square) \implies \mathrm{dist}(\square, \square') \leq C\,\mathrm{size}(\square'). \tag{2.2.5}$$

To see this, suppose that $\square, \square^1 \in \mathcal{T}$ are such that

$$\mathrm{size}(\square_1) = 3\,\mathrm{size}(\square) \quad \text{and} \quad \mathrm{dist}\left(\widetilde{\square}_1, \square\right) \leq 1.$$

This implies that

$$\operatorname{dist}(\square^1, \square) \le \operatorname{diam}\left(\widetilde{\square}^1\right) + \operatorname{dist}\left(\widetilde{\square}^1, \square\right) \le 9 \operatorname{diam}(\square) + 1 \le C \operatorname{size}(\square).$$

Then if $\square^0, \ldots, \square^n$ are as in the definition of $\mathcal{K}(\square)$, we obtain

$$\operatorname{dist}(\square, \square') \le C \sum_{j=1}^{n} \operatorname{size}(\square^j) = C \operatorname{size}(\square) \sum_{j=1}^{n} 3^j \le C 3^n \operatorname{size}(\square) = C \operatorname{size}(\square').$$

This yields (2.2.5).

The implication (2.2.5) allows us to localize the previous construction of $\mathcal{S}$ by looking only at the triadic cubes for which we can evaluate membership in $\mathcal{G}$ by only looking at the edges in $z + \square_m$. All other triadic cubes will be considered to be "good" by default. This motivates the definition

$$\mathcal{G}_{\mathrm{loc}}(z + \square_m) := \mathcal{G} \cup \left(\cup \left\{ y + \square_n \,:\, n \in \mathbb{N},\, y \in 3^n \mathbb{Z}^d,\, y + \square_{n+1} \notin z + \square_m \right\} \right).$$

Using the property (2.2.1), we see that for every $\square' \in \mathcal{T}$,

$$\text{the event } \left\{ \mathbf{a} \in \Omega \,:\, \square' \in \mathcal{G}_{\mathrm{loc}}(z + \square_m) \right\} \text{ is } \mathcal{F}(z + \square_m)\text{–measurable.}$$

We now define $\mathcal{S}_{\mathrm{loc}}(z + \square_m)$ to be the partition obtained by applying the previous construction to $\mathcal{G}_{\mathrm{loc}}(z + \square_m)$. We write $\mathcal{S}_{\mathrm{loc}} = \mathcal{S}_{\mathrm{loc}}(z + \square_m)$ for short. It is clear from the construction that $\mathcal{S}_{\mathrm{loc}}$ is completely determined by the environment in $z + \square_m$, that is, $\mathcal{S}_{\mathrm{loc}}$ is $\mathcal{F}(z + \square_m)$–measurable. Moreover, we see immediately that $\square_{\mathcal{S}}(x)$ and $\square_{\mathcal{S}_{\mathrm{loc}}}(x)$ may differ only if there exists $\square' \in \mathcal{K}(\square_{\mathcal{S}_{\mathrm{loc}}}(x)) \setminus \mathcal{G}$ such that $\square' = y + \square_n$ and $y + \square_{n+1} \notin z + \square_m$. In this case, we have

$$\begin{aligned} \operatorname{dist}(x, \partial(z + \square_m)) &\le \operatorname{size}(\square_{\mathcal{S}_{\mathrm{loc}}}(x)) + \operatorname{dist}\left(\square_{\mathcal{S}_{\mathrm{loc}}}(x), \square'\right) + 3 \operatorname{diam}(\square') \\ &\le \operatorname{size}(\square_{\mathcal{S}_{\mathrm{loc}}}(x)) + C \operatorname{size}(\square') \\ &\le C \operatorname{size}(\square'). \end{aligned}$$

Thus there exists $k_0(d) \in \mathbb{N}$ such that, for every $x \in z + \square_m$,

$$\begin{aligned} &\square_S(x) \neq \square_{\mathcal{S}_{\mathrm{loc}}}(x) \\ &\qquad \Longrightarrow \exists \square, \square' \in \mathcal{T} \text{ s.t. } x \in \square,\ \square' \in \mathcal{K}(\square) \setminus \mathcal{G},\ \operatorname{dist}(x, \partial(z + \square_m)) \le 3^{k_0} \operatorname{size}(\square'). \end{aligned}$$

By (2.2.5), for each $j \in \mathbb{N}$ with $j \le m$, there are at most $C 3^{m-j}$ elements of the set

$$\left\{ \square' \,:\, \exists \square \in \mathcal{T},\ \square \cap z + \square_m \neq \varnothing,\ \square' \in \mathcal{K}(\square),\ \operatorname{size}(\square') = 3^j \right\}$$

while, for $j \ge m$, then there are at most C elements of this set. Thus by a union bound and (2.2.2),

$$\begin{aligned} \mathbb{P}\left[\exists x \in z + \square_m^{(n)},\ \square_S(x) \neq \square_{\mathcal{S}_{\mathrm{loc}}}(x) \right] &\le CK \sum_{j=n-k_0}^{\infty} (3^{m-j} + 1) \exp\left(-K^{-1} 3^{js}\right) \\ &\le C 3^{m-n} \exp\left(-c 3^{ns}\right). \end{aligned}$$

This completes the proof of (iv). □

Many times in this paper we will be required to estimate, with $\mathcal{S}$ as in the previous proposition and for a finite subset $U \subseteq \mathbb{Z}^d$ and exponent $t \ge 1$, the random variable

$$\Lambda_t(U, \mathcal{S}) := \frac{1}{|U|} \sum_{x \in \mathrm{cl}_{\mathcal{S}}(U)} \operatorname{size}(\square_{\mathcal{S}}(x))^t = \frac{1}{|U|} \sum_{\mathcal{S} \ni \square \subseteq \mathrm{cl}_{\mathcal{S}}(U)} \operatorname{size}(\square)^{d+t}, \tag{2.2.6}$$

where, given an arbitrary subset $U \subseteq \mathbb{Z}^d$, we define the *closure* $\mathrm{cl}_{\mathcal{S}}(U)$ *of* U *with respect to a partition* $\mathcal{S}$ by

$$\mathrm{cl}_{\mathcal{P}}(U) := \bigcup_{z \in U} \square_{\mathcal{S}}(z).$$

As in the statement of Proposition 2.2.1, we let $\square_{\mathcal{S}}(x)$ denote the unique element of $\mathcal{S}$ containing $x \in \mathbb{Z}^d$.

An immediate consequence of Proposition 2.2.1(iii) and (2.1.8) is the estimate

$$\Lambda_t(U,\mathcal{S}) \le \mathcal{O}_{\frac{s}{t}}(C), \tag{2.2.7}$$

for some constant $C := C(t,s,K,d) < +\infty$. While (2.2.7) is quite useful, some of our arguments require something slightly stronger: the existence of a *random scale* $\mathcal{M}_t(\mathcal{S}) \in \mathbb{N}$, with good quantitative bounds on the size of $\mathcal{M}_t(\mathcal{S})$, and a *deterministic* constant $C(t,K,s,d,\lambda,\mathfrak{p}) < \infty$ such that, for every $m \in \mathbb{N}$,

$$m \ge \mathcal{M}_t(\mathcal{S}) \implies \Lambda_t(\square_m,\mathcal{S}) \le C.$$

The precise result is stated below in Proposition 2.2.4. To prove it, we need to use independence and thus the localization provided by Proposition 2.2.1(iv). In the following lemma, we put the localization statement into a more convenient form, in terms of $\Lambda_t(U,\mathcal{S})$.

LEMMA 2.2.2. *Let $K,s>0$, $\mathcal{S}$ and $\mathcal{S}_{\mathrm{loc}}$ be as in the statement of Proposition 2.2.1. Fix $t\in[1,\infty)$. Then, for every $t'>1$, there exists $C(t',t,K,s,d)<\infty$ such that, for every $m\in\mathbb{N}$ and $z\in 3^m\mathbb{Z}^d$,*

$$\Lambda_t(z+\square_m,\mathcal{S}) \le \Lambda_t(z+\square_m,\mathcal{S}_{\mathrm{loc}}(z+\square_{m+1})) + \mathcal{O}_{\frac{s}{t+t'}}\left(C3^{-mt'}\right). \tag{2.2.8}$$

PROOF. Fix $m\in\mathbb{N}$ and $z\in 3^m\mathbb{Z}^d$ and write $\mathcal{S}_{\mathrm{loc}}$ in place of $\mathcal{S}_{\mathrm{loc}}(z+\square_{m+1})$ to lighten the notation and let D_m denote the event

$$D_m := \left\{\exists x\in z+\square_m \text{ such that } \square_{\mathcal{S}}(x) \ne \square_{\mathcal{S}_{\mathrm{loc}}}(x)\right\}.$$

According to Proposition 2.2.1, since $\square_m \subseteq \square_{m+1}^{(m)}$, we have

$$\mathbb{P}\left[D_m\right] \le C\exp\left(-c3^{ms}\right). \tag{2.2.9}$$

We estimate

$$\begin{aligned}\Lambda_t(z+\square_m,\mathcal{S})\mathbb{1}_{\Omega\setminus D_m} &= \left(\frac{1}{|\square_m|}\sum_{x\in z+\square_m}\mathrm{size}(\square_{\mathcal{S}}(x))^t\right)\mathbb{1}_{\Omega\setminus D_m}\\ &\le \Lambda_t(z+\square_m,\mathcal{S}_{\mathrm{loc}})\end{aligned}$$

On the other hand, (2.2.9) implies that, for every $t'>1$,

$$\mathbb{1}_{D_m} = \mathcal{O}_{\frac{s}{t'}}\left(C3^{-mt'}\right)$$

and, therefore by (2.1.9),

$$\Lambda_t(z+\square_m,\mathcal{S})\mathbb{1}_{D_m} \le \mathcal{O}_{\frac{s}{t}}(C)\cdot\mathcal{O}_{\frac{s}{t'}}\left(C3^{-mt'}\right) \le \mathcal{O}_{\frac{s}{t+t'}}\left(C3^{-mt'}\right).$$

Combining the above displays yields (2.2.8). □

We need the following technical lemma.

LEMMA 2.2.3. *Fix $K\ge1$, $s>0$ and $\beta>0$ and suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative random variables satisfying, for every $n\in\mathbb{N}$,*

$$X_n \le \mathcal{O}_s\left(K3^{-n\beta}\right). \tag{2.2.10}$$

Then there exists $C(s,\beta,K)<\infty$ such that the random scale

$$M := \sup\left\{3^n\in\mathbb{N} : X_n\ge1\right\}$$

satisfies the estimate

$$M \le \mathcal{O}_{s\beta}(C). \tag{2.2.11}$$

Proof. Chebyshev's inequality and (2.2.10) imply

$$\mathbb{P}[X_n \geq 1] \leq 2\exp\left(-\left(K^{-1}3^{-n\beta}\right)^s\right).$$

Fix $t > 0$ and $\delta > 0$ and compute

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\delta M^t\right)\right] &\leq 1 + \sum_{k=0}^{\infty} \delta t 3^{(k+1)t} \exp\left(\delta 3^{(k+1)t}\right) \mathbb{P}\left[M > 3^k\right] \\
&\leq 1 + \sum_{k=0}^{\infty} \delta t 3^{(k+1)t} \exp\left(\delta 3^{(k+1)t}\right) \sum_{n=k}^{\infty} \mathbb{P}[X_n \geq 1] \\
&\leq 1 + \sum_{k=0}^{\infty} 2\delta t 3^{(k+1)t} \exp\left(\delta 3^{(k+1)t}\right) \sum_{n=k}^{\infty} \exp\left(-\left(K^{-1}3^{-n\beta}\right)^s\right) \\
&\leq 1 + C\delta t \sum_{k=0}^{\infty} 3^{(k+1)t} \exp\left(\delta 3^{(k+1)t}\right) \exp\left(-\left(C^{-1}3^{-k\beta}\right)^s\right).
\end{aligned}$$

Taking $t := s\beta$ and imposing the condition $\delta < \frac{3^{-t}}{2} C^{-s}$, we get

$$\mathbb{E}\left[\exp\left(\delta M^t\right)\right] \leq 1 + C\delta s\beta \sum_{k=0}^{\infty} 3^{(k+1)t} \exp\left(-\frac{1}{2}\left(C^{-1}3^{-k\beta}\right)^s\right) \leq 1 + C\delta s\beta.$$

Taking $\delta < Cs\beta$ gives $\mathbb{E}\left[\exp\left(\delta M^t\right)\right] \leq 2$, thus

$$M \leq \mathcal{O}_{s\beta}\left(\delta^{-1}\right),$$

and the proof of (2.2.11) is complete. □

Proposition 2.2.4 (Minimal scales for $\mathcal{S}$). *Let $K, s > 0$ and $\mathcal{S}$ be as in Proposition 2.2.1. Fix $t \in [1,\infty)$. Then there exist $C(t, K, s, d, \mathfrak{p}) < \infty$, an $\mathbb{N}$-valued random variable $\mathcal{M}_t(\mathcal{S})$ and, for every exponent $r \in \left(0, \frac{sd}{d+t+s}\right)$, a constant $C'(r, t, K, s, d, \mathfrak{p}) < \infty$ such that*

$$\mathcal{M}_t(\mathcal{S}) = \mathcal{O}_r(C') \tag{2.2.12}$$

and

$$m \in \mathbb{N},\ 3^m \geq \mathcal{M}_t(\mathcal{S}) \implies \Lambda_t(\square_m, \mathcal{S}) \leq C \quad \textit{and} \quad \sup_{x \in \square_m} \operatorname{size}(\square_{\mathcal{S}}(x)) \leq 3^{\frac{dm}{d+t}}. \tag{2.2.13}$$

Proof. The proof is organized as follows. In steps 1 and 2, we prove that there exists a random variable $\mathcal{M}_t^0(\mathcal{S})$ satisfying (2.2.12) and

$$m \in \mathbb{N},\ 3^m \geq \mathcal{M}_t^0(\mathcal{S}) \implies \Lambda_t(\square_m, \mathcal{S}) \leq C.$$

In step 3 we prove that there exists andom variable $\mathcal{M}_t^1(\mathcal{S})$ satisfying (2.2.12) and

$$m \in \mathbb{N},\ 3^m \geq \mathcal{M}_t^1(\mathcal{S}) \implies \sup_{x \in \square_m} \operatorname{size}(\square_{\mathcal{S}}(x)) \leq 3^{\frac{dm}{d+t}}.$$

We then define $\mathcal{M}_t(\mathcal{S}) := \max(\mathcal{M}_t^0(\mathcal{S}), \mathcal{M}_t^1(\mathcal{S}))$ which, in view of the previous results, clearly satisfies (2.2.12) and (2.2.13).

Step 1. Fix $t \in [1,\infty)$ and $r \in \left(0, \frac{sd}{d+t+s}\right)$. Also fix $m, n \in \mathbb{N}$ with $m > n$. To shorten the notation, we write $\mathcal{S}_{\mathrm{loc}}(z') := \mathcal{S}_{\mathrm{loc}}(z' + \square_{n+1})$ for each $z' \in 3^n\mathbb{Z}^d$. Using Lemma 2.2.2, we have

$$\begin{aligned}
&\Lambda_t(z + \square_m, \mathcal{S}) \\
&\quad = \frac{|\square_n|}{|\square_m|} \sum_{z' \in 3^n\mathbb{Z}^d \cap (z + \square_m)} \Lambda_t(z' + \square_n, \mathcal{S}) \\
&\quad \leq \frac{|\square_n|}{|\square_m|} \sum_{z' \in 3^n\mathbb{Z}^d \cap (z + \square_m)} \Lambda_t\left(z' + \square_n, \mathcal{S}_{\mathrm{loc}}(z')\right) + \mathcal{O}_{\frac{s}{t+t'}}\left(C3^{-nt'}\right).
\end{aligned} \tag{2.2.14}$$

Denote

$$Z := \frac{|\square_n|}{|\square_m|} \sum_{z' \in 3^n\mathbb{Z}^d \cap (z+\square_m)} \Lambda_t\left(z' + \square_n, \mathcal{S}_{\text{loc}}(z')\right).$$

We estimate Z, using independence. The claim is that

$$Z \le C + \mathcal{O}_1\left(C3^{nt-d(m-n)}\right). \tag{2.2.15}$$

First we note that, by the properties of $\mathcal{S}_{\text{loc}}(z')$, we have

$$\Lambda_t(z' + \square_n, \mathcal{S}_{\text{loc}}(z')) \le 3^{nt}, \quad \mathbb{P}\text{–a.s.}$$

Thus $Z \le 3^{nt}$, $\mathbb{P}$–a.s.

We now take an enumeration $\{z^j : 1 \le j \le 3^{d(m-n-1)}\}$ of the elements of the set $3^{n+1}\mathbb{Z}^d \cap \square_m$. Next, for each $1 \le j \le 3^{md}$, we let $\{z^{i,j} : 1 \le i \le 3^d\}$ be an enumeration of the elements of the set $3^n\mathbb{Z}^d \cap \left(z^j + \square_{n+1}\right)$, such that, for every $1 \le j, j' \le 3^{d(m-n-1)}$ and $1 \le i \le 3^d$, $z^j - z^{j'} = z^{i,j} - z^{i,j'}$.The point of this is that, for every $1 \le i \le 3^d$ and $1 \le j < j' \le 3^{d(m-n-1)}$, we have $\operatorname{dist}\left(z^{i,j} + \square_n, z^{i,j'} + \square_n\right) \ge 3^n$ and therefore, $\Lambda_t\left(z^{i,j} + \square_n, \mathcal{S}_{\text{loc}}\left(z^{i,j}\right)\right)$ and $\Lambda_t\left(z^{i,j'} + \square_n, \mathcal{S}_{\text{loc}}\left(z^{i,j'}\right)\right)$ are $\mathbb{P}$–independent. Now fix $h > 0$ and compute, using the Hölder inequality and the independence

$$\begin{aligned}
&\log \mathbb{E}\left[\exp\left(h3^{-nt}Z\right)\right] \\
&\quad = \log \mathbb{E}\left[\prod_{i=1}^{3^d} \prod_{j=1}^{3^{d(m-n-1)}} \exp\left(h3^{-nt-d(m-n)}\Lambda_t\left(z^{i,j} + \square_n, \mathcal{S}_{\text{loc}}(z^{i,j})\right)\right)\right] \\
&\quad \le 3^{-d}\sum_{i=1}^{3^d} \log \mathbb{E}\left[\prod_{j=1}^{3^{d(m-n-1)}} \exp\left(h3^{-nt-d(m-n-1)}\Lambda_t\left(z^{i,j} + \square_n, \mathcal{S}_{\text{loc}}(z^{i,j})\right)\right)\right] \\
&\quad \le 3^{-d}\sum_{i=1}^{3^d} \sum_{j=1}^{3^{d(m-n-1)}} \log \mathbb{E}\left[\exp\left(h3^{-nt-d(m-n-1)}\Lambda_t\left(z^{i,j} + \square_n, \mathcal{S}_{\text{loc}}(z^{i,j})\right)\right)\right].
\end{aligned}$$

This inequality can be rewritten

$$\begin{aligned}
&\log \mathbb{E}\left[\exp\left(h3^{-nt}Z\right)\right] \\
&\qquad \le 3^{-d} \sum_{z' \in 3^n\mathbb{Z}^d \cap (z+\square_m)} \log \mathbb{E}\left[\exp\left(h3^{-nt-d(m-n-1)}\Lambda_t\left(z + \square_n, \mathcal{S}_{\text{loc}}(z')\right)\right)\right].
\end{aligned}$$

Next we use the elementary inequality

$$\forall y \in [0,1], \quad \exp(y) \le 1 + 2y$$

to get, for every $h \in [0, 3^{d(m-n-1)}]$,

$$\exp\left(h3^{-nt-d(m-n-1)}\Lambda_t\left(z' + \square_n, \mathcal{S}_{\text{loc}}(z')\right)\right) \le 1 + 2h3^{-nt-d(m-n-1)}\Lambda_t\left(z' + \square_n, \mathcal{S}_{\text{loc}}(z')\right).$$

Taking the expectation of this, applying the previous display and using the elementary inequality

$$\forall y \ge 0, \ \log(1 + y) \le y,$$

we get

$$\begin{aligned}
\log \mathbb{E}\left[\exp\left(h3^{-nt}Z\right)\right] &\le 3^{d(m-n)} \log\left(1 + 2h3^{-nt-d(m-n-1)}\mathbb{E}\left[\Lambda_t\left(z' + \square_n, \mathcal{S}_{\text{loc}}(z')\right)\right]\right) \\
&\le 2h3^{-nt+d}\mathbb{E}\left[\Lambda_t\left(z' + \square_n, \mathcal{S}_{\text{loc}}(z')\right)\right] \\
&\le Ch3^{-nt}.
\end{aligned}$$

Taking $h := 3^{d(m-n-1)}$ yields

$$\mathbb{E}\left[\exp\left(3^{d(m-n-1)-nt}Z\right)\right] \le \exp\left(C3^{d(m-n)-nt}\right).$$

From this and Chebyshev's inequality, we obtain a constant C such that

$$\mathbb{P}\left[Z \geq C + h\right] \leq \exp\left(-h 3^{d(m-n)-nt}\right)$$

This implies (2.2.15).

Step 2. We complete the proof by applying union bounds. Combining (2.2.14) and (2.2.15) yields

$$\Lambda_t(z + \square_m, \mathcal{S}) \leq C_0 + \mathcal{O}_1\left(C 3^{nt-d(m-n)}\right) + \mathcal{O}_{\frac{s}{t+t'}}\left(C 3^{-nt'}\right).$$

We now choose

$$n := \left\lceil \frac{dm}{d+t+s} \right\rceil$$

so that the previous line becomes

$$\Lambda_t(z + \square_m, \mathcal{S}) \leq C_0 + \mathcal{O}_1\left(C 3^{-\frac{sd}{d+t+s}m}\right) + \mathcal{O}_{\frac{s}{t+t'}}\left(C 3^{-\frac{dt'}{d+t+s}m}\right).$$

Define

$$\mathcal{M}_t^0(\mathcal{S}) := \sup\left\{3^m \,:\, \Lambda_t(z + \square_m, \mathcal{S}) \geq C_0 + 2\right\}$$

and apply Lemma 2.2.3 to find that

$$\mathcal{M}_t^0(\mathcal{S}) \leq \mathcal{O}_{\frac{sd}{d+t+s}}(C) + \mathcal{O}_{\frac{st'd}{(t+t')(d+t+s)}}(C) \leq \mathcal{O}_{\frac{st'd}{(t+t')(d+t+s)}}(C).$$

Taking t' sufficiently large, depending on (r, d, s, t), then

$$\frac{st'd}{(t+t')(d+t+s)} > r$$

and thus we obtain

$$\mathcal{M}_t^0(\mathcal{S}) \leq \mathcal{O}_r(C).$$

Step 3. By Proposition 2.2.1 (iii), we have, for every $m \in \mathbb{N}$

$$\mathbb{P}\left[\sup_{x \in \square_m} \operatorname{size}(\square_{\mathcal{S}}(x)) > 3^{\frac{dm}{d+t}}\right] \leq \sum_{x \in \square_m} \mathbb{P}\left[\operatorname{size}(\square_{\mathcal{S}}(x)) > 3^{\frac{dm}{d+t}}\right] \leq 2 \cdot 3^{dm} \exp\left(-C^{-1} 3^{\frac{dsm}{d+t}}\right).$$

From this we deduce that for every $m \in \mathbb{N}$

$$\mathbb{1}_{\left\{\sup_{x \in \square_m} \operatorname{size}(\square_{\mathcal{S}}(x)) > 3^{\frac{dm}{d+t}}\right\}} \leq \mathcal{O}_1\left(C 3^{-\frac{dsm}{d+t}}\right).$$

Applying Lemma 2.2.3 with $s = 1$, $\beta = \frac{ds}{d+t}$ and

$$X_m := 2 \cdot \mathbb{1}_{\left\{\sup_{x \in \square_m} \operatorname{size}(\square_{\mathcal{S}}(x)) > 3^{\frac{dm}{d+t}}\right\}}$$

shows that the random variable

$$\mathcal{M}_t^1(\mathcal{S}) := \sup\left\{3^m \in \mathbb{N} \;:\; X_m \geq 1\right\}$$

satisfies

$$\mathcal{M}_t^1(\mathcal{S}) \leq \mathcal{O}_{\frac{dm}{d+t}}(C).$$

Since $r \leq \frac{dm}{d+t+s} \leq \frac{dm}{d+t}$, we also have

$$\mathcal{M}_t^1(\mathcal{S}) \leq \mathcal{O}_r(C)$$

and by definition of $(X_m)_{m \in \mathbb{N}}$ and $\mathcal{M}_t^1(\mathcal{S})$, we have

$$m \in \mathbb{N},\, 3^m \geq \mathcal{M}_t^1(\mathcal{S}) \implies \sup_{x \in \square_m} \operatorname{size}(\square_{\mathcal{S}}(x)) \leq 3^{\frac{dm}{d+t}}.$$

The proof is complete. □

2.2.2. The partition $\mathcal{P}$ of well-connected cubes. We apply the construction of the previous subsection to obtain a random partition $\mathcal{P}$ of $\mathbb{Z}^d$ which simplifies the geometry of the percolation cluster. This partition plays an important role in the rest of the paper. For bounds on the "good event" which allows us to construct the partition, we use the important results of Pisztora [**138**], Penrose and Pisztora [**136**] and Antal and Pisztora [**11**]. We first recall some definitions introduced in those works.

DEFINITION 2.2.5 (Crossability and Crossing cluster). We say that a cube $\square$ is *crossable* (with respect to $\mathbf{a} \in \Omega$) if each of the d pairs of opposite $(d-1)$–dimensional faces of $\square$ are joined by an open path in $\square$. We say that a cluster $\mathscr{C} \subseteq \square$ is a *crossing cluster for* $\square$ if $\mathscr{C}$ intersects each of the $(d-1)$–dimensional faces of $\square$.

DEFINITION 2.2.6 (Good cube). We say that a triadic cube $\square \in \mathcal{T}$ is *well-connected* if there exists a crossing cluster $\mathscr{C}$ for the cube $\square$ such that:

(i) each cube $\square'$ with $\operatorname{size}(\square') \in \left[\frac{1}{10}\operatorname{size}(\square), \frac{1}{2}\operatorname{size}(\square)\right]$ and $\square' \cap \frac{3}{4}\square \neq \varnothing$ is crossable; and
(ii) for every cube $\square'$ as in (i) and every path $\gamma \subseteq \square'$ with $\operatorname{diam}(\gamma) \geq \frac{1}{10}\operatorname{size}(\square)$, we have that γ is connected to $\mathscr{C}$ within $\square'$. That is, there is another path γ' within $\square'$ which connects a point of γ to a point of $\mathscr{C}$.

We say that $\square \in \mathcal{T}$ is a *good cube* if $\operatorname{size}(\square) \geq 3$, $\square$ is well-connected and each of the 3^d successors of $\square$ are well-connected. We say that $\square \in \mathcal{T}$ is a *bad cube* if it is not a good cube.

The following estimate on the probability of the cube $\square_n$ being good is a consequence [**138**, Theorem 3.2] and [**136**, Theorem 5], as recalled in [**11**, (2.24)].

LEMMA 2.2.7 ([**11**, (2.24)]). *There exist $C(d,\mathfrak{p}) < \infty$ and $c(d,\mathfrak{p}) \in (0,\frac{1}{2}]$ such that, for every $m \in \mathbb{N}$,*

$$\mathbb{P}\left[\square_n \textit{ is good}\right] \geq 1 - C\exp\left(-c3^n\right). \tag{2.2.16}$$

It follows from Definition 2.2.6 that, for every good cube $\square$, there exists a unique maximal crossing cluster for $\square$ which is contained in $\square$. We denote this cluster by $\mathscr{C}_*(\square)$. In the next lemma, we record the observation that adjacent triadic cubes which have similar sizes and are both good have connected clusters.

LEMMA 2.2.8. *Let $n, n' \in \mathbb{N}$ with $|n - n'| \leq 1$ and $z, z' \in 3^n\mathbb{Z}^d$ such that*

$$\operatorname{dist}\left(\square_n(z), \square_{n'}(z')\right) \leq 1.$$

Suppose also that $\square_n(z)$ and $\square_{n'}(z')$ are good cubes. Then there exists a cluster $\mathscr{C}$ such that

$$\mathscr{C}_*(\square_n(z)) \cup \mathscr{C}_*(\square_{n'}(z')) \subseteq \mathscr{C} \subseteq \square_n(z) \cup \square_{n'}(z').$$

PROOF. We may suppose that $n \leq n'$. Let x be the center point on the face of $\square_n(z)$ which is adjacent to $\square_{n'}(z')$. If $n' = n+1$, then we let $\square_n'$ be the successor of $\square_{n'}(z')$ which is adjacent to $\square_n(z)$ and otherwise, if $n' = n$, we set $\square_n' := \square_{n'}(z')$. Consider the cube $\square$ of size $\frac{1}{2}3^n$ centered at x. Since $\square_n(z)$ is a good cube and therefore well-connected, $\square$ is crossable. Let $\gamma \subseteq \square$ be a path which connects the two faces of $\square$ which are parallel to the face of $\square_n(z)$ containing x. There are two subpaths of $\gamma \subseteq \square$ which, respectively, lie inside of $\square_n(z)$ and $\widetilde{\square}_n$ and have length at least $\frac{1}{4}3^n$. Therefore, since both of the cubes $\square_n(z)$ and $\square_n'$ are well-connected, we conclude that γ intersects both $\mathscr{C}_*(\square_n(z))$ and $\mathscr{C}_*(\square_n') \subseteq \mathscr{C}_*(\square_{n'}(z'))$. Taking $\mathscr{C}$ to be the cluster

$$\mathscr{C} := \gamma \cup \mathscr{C}_*(\square_n(z)) \cup \mathscr{C}_*(\square_{n'}(z'))$$

completes the proof. □

We next define our partition $\mathcal{P}$.

DEFINITION 2.2.9. We let $\mathcal{P} \subseteq \mathcal{T}$ be the partition $\mathcal{S}$ of $\mathbb{Z}^d$ obtained by applying Proposition 2.2.1 to the collection

$$\mathcal{G} := \{\square \in \mathcal{T} : \square \text{ is good}\}.$$

We also let $\mathcal{P}_{\mathrm{loc}}(z + \square_n)$ denote the local partitions $\mathcal{S}_{\mathrm{loc}}(z + \square_n)$ in the statement of Proposition 2.2.1(iv).

The (random) partition $\mathcal{P}$ plays an important role throughout the rest of the paper. We also denote by $\mathcal{P}_*$ the collection of triadic cubes which contain some element of $\mathcal{P}$, that is,

$$\mathcal{P}_* := \{\square : \square \text{ is a triadic cube and } \square \supseteq \square' \text{ for some } \square' \in \mathcal{P}\}.$$

Notice that every element of $\mathcal{P}_*$ can be written in a unique way as a disjoint union of elements of $\mathcal{P}$. According to Proposition 2.2.1(i), every triadic cube containing an element of $\mathcal{P}$ is good. By Propositions 2.2.1(iii) and Lemma 2.2.7, there exists $C(d,\mathfrak{p}) < \infty$ such that, for every $x \in \mathbb{Z}^d$,

$$\operatorname{size}(\square_{\mathcal{P}}(x)) \leq \mathcal{O}_1(C). \tag{2.2.17}$$

By the properties of $\mathcal{P}$ given in Proposition 2.2.1(i) and (ii) and Lemma 2.2.8, the maximal crossing cluster $\mathscr{C}_*(\square)$ of an element $\square \in \mathcal{P}_*$ must satisfy $\mathscr{C}_*(\square) \subseteq \mathscr{C}_\infty$, since the union of all crossing clusters of elements of $\mathcal{P}$ is unbounded and connected. Indeed, we have the stronger property that, for every $n \in \mathbb{N}$ and $z \in 3^n\mathbb{Z}^d$,

$$\square_{n+1}(z) \in \mathcal{P}_* \implies \varnothing \neq \mathscr{C}_*(\square_n(z)) \subseteq \mathscr{C}_*(\square_{n+1}(z)) \cap \square_n(z) = \mathscr{C}_\infty \cap \square_n(z). \tag{2.2.18}$$

Given $\square \in \mathcal{P}$ with $\square = \square_{n+1}(z)$ for $n \in \mathbb{N}$ and $z \in 3^{n+1}\mathbb{Z}^d$, we let $\overline{z}(\square)$ denote the element of $\mathscr{C}_*(\square) \cap \square_n(z)$ which is closest to z in the manhattan distance; if this is not unique, then we break ties by the lexicographical order.

COROLLARY 2.2.10. *For every $x, y \in \mathbb{Z}^d$ and path γ_0 connecting x and y, there exists a path $\gamma \subseteq \mathscr{C}_\infty$ connecting $\overline{z}(\square_{\mathcal{P}}(x))$ and $\overline{z}(\square_{\mathcal{P}}(y))$ such that*

$$\gamma \subseteq \bigcup_{z \in \gamma_0} \square_{\mathcal{P}}(z).$$

PROOF. This is immediate from Lemma 2.2.8 and the property of $\mathcal{P}$ from Proposition 2.2.1(ii). □

2.3. Elliptic and functional inequalities on clusters

In this section, we use the partition $\mathcal{P}$ constructed in the previous section to tame the large-scale geometry of the percolation cluster. In particular, we give direct arguments leading to a quantitative Sobolev-Poincaré inequality. This allows us to develop some basic elliptic estimates we will need later in the paper. Many of the results in this section are similar to, and overlap with, those of Barlow [**24**]. Our approach however is somewhat different and, we believe, can be pushed beyond the particular case of a Bernoulli bond percolation cluster considered in this paper. Also, some of the estimates we prove (e.g., the Meyers estimate in Proposition 2.3.8) are new and needed in the following sections.

2.3.1. Functional inequalities on clusters. As in the previous section, for each $\square \in \mathcal{P}_*$, we let $\mathscr{C}_*(\square)$ denote the unique maximal crossing cluster for $\square$. Recall that, given an arbitrary subset $U \subseteq \mathbb{Z}^d$, we define the *closure* $\mathrm{cl}_{\mathcal{P}}(U)$ *of U with respect to $\mathcal{P}$* by

$$\mathrm{cl}_{\mathcal{P}}(U) := \bigcup_{z \in U} \square_{\mathcal{P}}(z).$$

We then define $\mathscr{C}_*(U)$ to be the maximal cluster contained in $\mathrm{cl}_{\mathcal{P}}(U)$ which contains each of the clusters $\mathscr{C}_*(\square_{\mathcal{P}}(z))$ for every $z \in U$.

Definition 2.3.1. Given a function $w : \mathscr{C}_*(U) \to \mathbb{R}$, the *coarsening* $[w]_{\mathcal{P}}$ *of* w *with respect to* $\mathcal{P}$ is a function $\mathrm{cl}_{\mathcal{P}}(U) \to \mathbb{R}$ defined by

$$[w]_{\mathcal{P}}(x) := w\left(\overline{z}\left(\square_{\mathcal{P}}(x)\right)\right), \quad x \in \mathrm{cl}_{\mathcal{P}}(U).$$

where $\overline{z}\left(\square_{\mathcal{P}}(x)\right)$ is the point of $\mathscr{C}_*(\square_{\mathcal{P}}(x))$ which is the closest to the center of the cube for the infinite norm $|\cdot|_\infty$ (if there is more than one candidat, pick the one that comes first for the lexicographical order). In particular, $[w]_{\mathcal{P}}$ we note that is defined on the closure $\mathrm{cl}_{\mathcal{P}}(U)$ and is constant on the elements of $\mathcal{P}$.

The advantage of $[w]_{\mathcal{P}}$ is that it allows us to make use of the simpler and more favorable geometric structure of $\mathbb{Z}^d$ compared to the percolation clusters. The price to pay is the difference between w and $[w]_{\mathcal{P}}$, which depends of course on the coarseness of the partition $\mathcal{P}$ and the control one has on ∇u. Indeed, we show next that the difference $w - [w]_{\mathcal{P}}$ can be controlled in $L^s(\mathscr{C}_*(U))$ for $s \in [1,\infty)$ in terms of a weighted $L^s(\mathscr{C}_*(U))$ norm of ∇w. The weight function represents the coarseness of the partition $\mathcal{P}$ in $\mathrm{cl}_{\mathcal{P}}(U)$. In what follows, $\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}$ denotes the vector field of ∇w restricted to the open edges:

$$\left(\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right)(e) := \nabla w(e) \mathbb{1}_{\{\mathbf{a}(e)\neq 0\}}, \quad e \in E_d(U).$$

Recall that the notation $|F|(x)$ for a vector field F is defined in (2.1.4).

Lemma 2.3.2. *For every bounded* $U \subseteq \mathbb{Z}^d$*,* $1 \le s < \infty$ *and* $w : \mathscr{C}_*(U) \to \mathbb{R}$*,*

$$\int_{\mathscr{C}_*(U)} |w(x) - [w]_{\mathcal{P}}(x)|^s \, dx \le C^s \sum_{\square\in\mathcal{P},\, \square\subseteq \mathrm{cl}_{\mathcal{P}}(U)} \mathrm{size}(\square)^{sd} \int_{\square\cap\mathscr{C}_*(U)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^s(x)\, dx. \tag{2.3.1}$$

Proof. For each $x \in \mathscr{C}_*(U)$, there is a non-self intersecting path connecting x and $\overline{z}\left(\square_{\mathcal{P}}(x)\right)$ which belongs to the union of the elements $\square$ of $\mathcal{P}$ which are contained in $\mathrm{cl}_{\mathcal{P}}(U)$ and satisfy $\mathrm{dist}(\square, \square_{\mathcal{P}}(x)) \le 1$. It follows that

$$\begin{aligned} |w(x) - [w]_{\mathcal{P}}(x)| &= |w(x) - w(\overline{z}\left(\square_{\mathcal{P}}(x)\right))| \\ &\le \sum_{\square\in\mathcal{P},\ \mathrm{dist}(\square,\square_{\mathcal{P}}(x))\le 1} \int_{\square\cap\mathscr{C}_*(U)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|(y)\, dy. \end{aligned} \tag{2.3.2}$$

Summing over $x \in \square \cap \mathscr{C}_*(U)$ for a fixed $\square \in \mathcal{P}$ with $\square \subseteq \mathrm{cl}_{\mathcal{P}}(U)$ and using property (ii) from Proposition 2.2.1 for $\mathcal{P}$ yields

$$\begin{aligned} \int_{\square\cap\mathscr{C}_*(U)} |w(x) - [w]_{\mathcal{P}}(x)|^s \, dx &\le C^s |\square| \left(\int_{\square\cap\mathscr{C}_*(U)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|(x)\, dx \right)^s \\ &\le C^s |\square|^s \int_{\square\cap\mathscr{C}_*(U)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^s(x)\, dx. \end{aligned}$$

Summing over $\square \in \mathcal{P}$ with $\square \subseteq \mathrm{cl}_{\mathcal{P}}(U)$ yields the lemma. ∎

We next show that we can control L^s norms of $|\nabla [w]_{\mathcal{P}}|$ by those of $\left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|$ and the coarseness of the partition $\mathcal{P}$. The proof is very simple and similar to that of the previous lemma.

Lemma 2.3.3. *For every bounded* $U \subseteq \mathbb{Z}^d$*,* $1 \le s < \infty$ *and* $w : \mathscr{C}_*(U) \to \mathbb{R}$*,*

$$\int_{\mathrm{cl}_{\mathcal{P}}(U)} |\nabla [w]_{\mathcal{P}}|^s(x)\, dx \le C^s \sum_{\square\in\mathcal{P},\, \square\subseteq \mathrm{cl}_{\mathcal{P}}(U)} \mathrm{size}(\square)^{sd-1} \int_{\square\cap\mathscr{C}_*(U)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^s(x)\, dx. \tag{2.3.3}$$

Proof. The gradient $\nabla [w]_{\mathcal{P}}$ is supported on the edges $\{x,y\}$ such $x \in \square$ and $y \in \square'$ for two disjoint, neighboring elements $\square \sim \square'$ of $\mathcal{P}$. On such edges, we have

$$|\nabla [w]_{\mathcal{P}}(\{x,y\})| = \left| w(\overline{z}(\square(x))) - w(\overline{z}(\square'(y)))\right|$$

Recalling that there exists a path between $\overline{z}(\square(x))$ and $\overline{z}(\square'(y))$ which lies entirely in $\square\cup\square'$ and summing over the edges along this path, we find that

$$\left| w(\overline{z}(\square(x))) - w(\overline{z}(\square'(y)))\right| \le C \int_{(\square\cup\square')\cap\mathscr{C}_*(U)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|(z)\, dz.$$

Since two neighboring elements of $\mathcal{P}$ have sizes within a factor of three, a property given by Proposition 2.2.1(ii), the number of such edges between $\square$ and $\square'$ is at least

$$c \operatorname{size}(\square)^{d-1}.$$

Finally, we note that every $\square \in \mathcal{P}$ has at least 2^d neighboring elements of $\mathcal{P}$.

The above assertions imply that

$$\begin{aligned}
&\int_{\mathrm{cl}_{\mathcal{P}}(U)} \left|\nabla \left[w\right]_{\mathcal{P}}\right|^s(x)\,dx \\
&\le C^s \sum_{\square\in\mathcal{P},\,\square\subseteq \mathrm{cl}_{\mathcal{P}}(U)} \operatorname{size}(\square)^{d-1} \sum_{\square'\in\mathcal{P},\,\square\sim\square'} \left(\int_{(\square\cup\square')\cap\mathscr{C}_*(U)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|(z)\,dz \right)^s \\
&\le C^s \sum_{\square\in\mathcal{P},\,\square\subseteq \mathrm{cl}_{\mathcal{P}}(U)} \operatorname{size}(\square)^{d-1} \sum_{\square'\in\mathcal{P},\,\square\sim\square'} \left|\square\cup\square'\right|^{s-1} \int_{(\square\cup\square')\cap\mathscr{C}_*(U)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^s(z)\,dz \\
&\le C^s \sum_{\square\in\mathcal{P},\,\square\subseteq \mathrm{cl}_{\mathcal{P}}(U)} \operatorname{size}(\square)^{d-1+d(s-1)} \int_{\square\cap\mathscr{C}_*(U)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^s(z)\,dz.
\end{aligned}$$

This completes the proof. □

The previous two lemmas imply a Sobolev-Poincaré-type inequality on the clusters, borrowing the result from the classical inequalities on $\mathbb{R}^d$ by comparing w to $[w]_{\mathcal{P}}$. This is strong evidence of our informal assertion that "the geometry of $\mathscr{C}_\infty$ is quantitatively like that of $\mathbb{R}^d$ on scales larger than $\mathcal{P}$."

Before giving the statement, we recall that if $s \in [1,\infty)$ then the Sobolev conjugates s^* and s_* of s in dimension d are defined by

$$s^* := \begin{cases} \dfrac{sd}{d-s} & \text{if } s < d, \\ \infty & \text{if } s \ge d. \end{cases}$$

If $s \in \left[\frac{d}{d-1}, \infty\right)$, then we also define

$$s_* := \frac{sd}{s+d}$$

so that $(s_*)^* = s$.

Proposition 2.3.4 (Sobolev inequality for $\mathscr{C}_*(\square)$). *Suppose that $s \in \left[\frac{d}{d-1}, \infty\right)$ and $\square \in \mathcal{P}_*$. Let $w : \mathscr{C}_*(\square) \to \mathbb{R}$ satisfy one of the following conditions:*

$$\text{(2.3.4)}\qquad \begin{cases} \displaystyle\int_{\mathscr{C}_*(\square)} w(x)\,dx = 0 \quad \text{or} \quad w(x) = 0 \text{ for every } x \in \partial_{\mathbf{a}}\square, \\ \displaystyle\int_{\square} [w]_{\mathcal{P}}(x)\,dx = 0 \quad \text{or} \quad [w]_{\mathcal{P}}(x) = 0 \text{ for every } x \in \partial_{\mathbf{a}}\square. \end{cases}$$

Then there exists $C(s,d,\mathfrak{p}) < \infty$ such that

$$\text{(2.3.5)}\qquad \int_{\mathscr{C}_*(\square)} |w(x)|^s\,dx \le C \left(\sum_{\square'\in\mathcal{P},\,\square'\subseteq\square} \operatorname{size}(\square')^{sd} \int_{\square'\cap\mathscr{C}_*(\square)} \left|\nabla w \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{s_*}(x)\,dx \right)^{\frac{s}{s_*}}$$

Before giving the proof of Proposition 2.3.4, let us comment on the form of the right side in (2.3.5). The classical Sobolev inequality for functions $w \in W_0^{1,s_*}(\square)$ (or mean-zero functions $w \in W^{1,s_*}(\square)$) in a cube $\square \subseteq \mathbb{R}^d$ states that

$$\int_{\square} |w(x)|^s\,dx \le C \left(\int_{\square} |\nabla w(x)|^{s_*}\,dx \right)^{\frac{s}{s_*}}.$$

The term on the right side of this inequality is similar to the first term on the right side of (2.3.5), except there is the weight $\operatorname{size}(\square')^{sd}$ representing the size of the local cube in the partition $\mathcal{P}$. The size of the elements of $\mathcal{P}$ are of course not uniformly bounded, however they are typically of unit size, by (2.2.17), with exponential stochastic integrability. In particular, one may use Hölder's inequality to separate this weight from the function $|\nabla w|^{s_*}$ in the integrand at the cost of an arbitrarily small loss of the exponent s_* while the sums over the weights can be controlled above a minimal scale by Proposition 2.2.4.

Proof of Proposition 2.3.4. Rather than (2.3.4), we first prove the proposition under the assumption that

$$\int_{\square} [w]_{\mathcal{P}} (x) \, dx = 0 \quad \text{or} \quad [w]_{\mathcal{P}} = 0 \text{ on } \partial_{\mathbf{a}} \square \, . \tag{2.3.6}$$

In this case, the usual Sobolev inequality on $\mathbb{Z}^d$ (which follows easily from the one on $\mathbb{R}^d$ by affine interpolation, for example) applied to $[w]_{\mathcal{P}}$ gives us that

$$\left(\int_{\square} |[w]_{\mathcal{P}}(x)|^s \, dx \right)^{\frac{1}{s}} \le C \left(\int_{\square} |\nabla [w]_{\mathcal{P}}|^{s_*} (x) \, dx \right)^{\frac{1}{s_*}} .$$

We then apply Lemma 2.3.3 to estimate the right side, which gives

$$\left(\int_{\square} |[w]_{\mathcal{P}}(x)|^s \, dx \right)^{\frac{1}{s}} \le \left(\sum_{\square \in \mathcal{P}, \, \square' \subseteq \square} \operatorname{size}(\square')^{sd-1} \int_{\square' \cap \mathscr{C}_*(\square)} \left| \nabla w \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^{s_*} (x) \, dx \right)^{\frac{1}{s_*}} ,$$

and use Lemma 2.3.2 and the triangle inequality to estimate the left side and combine this with the previous inequality to get

$$\begin{aligned} \left(\int_{\mathscr{C}_*(\square)} |w(x)|^s \, dx \right)^{\frac{1}{s}} &\le C \left(\sum_{\square' \in \mathcal{P}, \, \square' \subseteq \square} \operatorname{size}(\square')^{sd-1} \int_{\square' \cap \mathscr{C}_*(\square)} \left| \nabla w \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^{s_*} (x) \, dx \right)^{\frac{1}{s_*}} \\ &\quad + C \left(\sum_{\square' \in \mathcal{P}, \, \square' \subseteq \square} \operatorname{size}(\square')^{sd} \int_{\square' \cap \mathscr{C}_*(\square)} \left| \nabla w \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^{s} (x) \, dx \right)^{\frac{1}{s}} . \end{aligned}$$

We may estimate the second term on the right side thanks to the following inequality: since $\frac{s}{s_*} > 1$, we have that, for every $n \in \mathbb{N}$ and finite sequence of nonnegative real numbers $\{a_i\}_{1 \le i \le n}$,

$$\sum_{i=1}^n a_i^{\frac{s}{s_*}} \le \left(\sum_{i=1}^n a_i \right)^{\frac{s}{s_*}} .$$

Applying this to the second term on the right side gives

$$\begin{aligned} &\left(\sum_{\square' \in \mathcal{P}, \, \square' \subseteq \square} \operatorname{size}(\square')^{sd} \int_{\square' \cap \mathscr{C}_*(\square)} \left| \nabla w \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^{s} (x) \, dx \right)^{\frac{1}{s}} \\ &\qquad \le \left(\sum_{\square' \in \mathcal{P}, \, \square' \subseteq \square} \operatorname{size}(\square')^{s_* d} \int_{\square' \cap \mathscr{C}_*(\square)} \left| \nabla w \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^{s_*} (x) \, dx \right)^{\frac{1}{s_*}} . \end{aligned}$$

Noticing that $s_* d \le sd$ and $sd - 1 \le sd$ completes the proof of the proposition under the assumption (2.3.6).

To prove the proposition under the assumption

$$\int_{\mathscr{C}_*(\square)} w(x) \, dx = 0, \tag{2.3.7}$$

we apply the result to the function $w - \frac{1}{|\mathscr{C}_*(\square)|}\int_{\mathscr{C}_*(U)} [w]_{\mathcal{P}}(x)\,dx$ which satisfies assumption (2.3.6). This yields

$$\left(\fint_{\mathscr{C}_*(\square)} \left| w(x) - \frac{1}{|\mathscr{C}_*(\square)|} \int_{\mathscr{C}_*(U)} [w]_{\mathcal{P}}(x)\,dx \right|^s dx \right)^{\frac{1}{s}} \\ \leq C \left(\sum_{\square' \in \mathcal{P},\, \square' \subseteq \square} \operatorname{size}(\square')^{sd-1} \int_{\square' \cap \mathscr{C}_*(\square)} \left| \nabla w \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^{s_*}(x)\,dx \right)^{\frac{1}{s_*}} \\ + C \left(\sum_{\square' \in \mathcal{P},\, \square' \subseteq \square} \operatorname{size}(\square')^{sd} \int_{\square' \cap \mathscr{C}_*(\square)} \left| \nabla w \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^{s}(x)\,dx \right)^{\frac{1}{s}}.$$

To complete the proof we use Lemma 2.3.2 to obtain

$$\left| \frac{1}{|\mathscr{C}_*(\square)|} \int_{\mathscr{C}_*(U)} [w]_{\mathcal{P}}(x)\,dx \right|^s \\ \leq \left| \frac{1}{|\mathscr{C}_*(\square)|} \int_{\mathscr{C}_*(U)} w(x)\,dx - \frac{1}{|\mathscr{C}_*(\square)|} \int_{\mathscr{C}_*(U)} [w]_{\mathcal{P}}(x)\,dx \right|^s \\ \leq \frac{1}{|\mathscr{C}_*(\square)|} \int_{\mathscr{C}_*(U)} |w(x) - [w]_{\mathcal{P}}(x)|^s\,dx \\ \leq \frac{C^s}{|\mathscr{C}_*(\square)|} \sum_{\square' \in \mathcal{P},\, \square' \subseteq \square} \operatorname{size}(\square')^{sd} \int_{\square' \cap \mathscr{C}_*(\square)} \left| \nabla w \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^s(x)\,dx.$$

Combining the two previous displays completes the proposition under the assumption (2.3.7).

We finally prove the proposition under the assumption

$$w(x) = 0 \text{ for every } x \in \partial_{\mathbf{a}} \square$$

The main idea is to apply the Sobolev inequality under the assumption $[w]_{\mathcal{P}} = 0$ on $\partial \square$. To do so we define the following function v on $\square$

$$v(x) := \begin{cases} [w]_{\mathcal{P}}(x) & \text{if } \square_{\mathcal{P}}(x) \notin \partial_{\mathcal{P}} \square, \\ 0 & \text{if } \square_{\mathcal{P}}(x) \in \partial_{\mathcal{P}} \square. \end{cases}$$

The function v is almost equal to $[w]_{\mathcal{P}}$ with a slight modification on the boundary cubes of the partition where we set $v = 0$. Since $u = 0$ on $\partial_{\mathbf{a}} \square$ and in view of Definition 2.3.1, one can observe that the results of Lemmas 2.3.2 and 2.3.3 hold with v instead of $[w]_{\mathcal{P}}$. We then complete the proof of the Sobolev inequality by adapting the argument as in the case $[w]_{\mathcal{P}} = 0$ on $\partial \square$ with v instead of $[w]_{\mathcal{P}}$. □

2.3.2. Basic elliptic estimates on clusters. In this subsection, we record some basic elliptic estimates and show how these allow us to improve some of the estimates from the previous subsection for **a**-harmonic functions. We remark that the estimates in this section do not use the independence of the ensemble $\{\mathbf{a}(e)\}_{e \in E_d}$, merely the independence of the ensemble $\{\mathbb{1}_{\{\mathbf{a}(x) \neq 0\}}\}_{e \in E_d}$, and so they work for general coefficient fields defined on the percolation clusters.

We begin with Caccioppoli's inequality, following the standard argument.

Lemma 2.3.5 (Caccioppoli inequality). *Assume $U \subseteq \mathbb{Z}^d$ is a cluster and $V \subseteq U$ such that* $\operatorname{dist}(V, \partial_{\mathbf{a}} U) \geq r \geq 1$. *Suppose that $u \in \mathcal{A}(U)$. Then there exists $C(\lambda) < \infty$ such that*

$$\int_V \left| \nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^2(x)\,dx \leq \frac{C}{r^2} \int_{U \setminus \operatorname{int}(V)} |u(x)|^2\,dx. \tag{2.3.8}$$

Proof. Select a function $\eta \in C^1(\mathbb{R}^d)$ satisfying

$$\mathbb{1}_V \leq \eta \leq 1, \quad \eta \equiv 0 \text{ on } \partial_{\mathbf{a}} U, \quad \text{and} \quad |\nabla \eta|^2 \leq \frac{C\eta}{r^2}. \tag{2.3.9}$$

Testing the equation for u with ηu (that is, applying (2.1.6) with $w = \eta u$) yields

$$\begin{aligned}0 &= \sum_{x,y\in U, x\sim y} (\eta(x)u(x) - \eta(y)u(y))\,\mathbf{a}(\{x,y\})\,(u(x)-u(y)) \\ &= \sum_{x,y\in U, x\sim y} \eta(x)\,(u(x)-u(y))\,\mathbf{a}(\{x,y\})\,(u(x)-u(y)) \\ &\quad + \sum_{x,y\in U, x\sim y} u(y)\,(\eta(x)-\eta(y))\,\mathbf{a}(\{x,y\})\,(u(x)-u(y))\,.\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\sum_{x,y\in U, x\sim y} \eta(x)\mathbf{a}(\{x,y\})\,(u(x)-u(y))^2 \\ &\quad\le \sum_{x,y\in U, x\sim y} |u(y)|\,|\eta(x)-\eta(y)|\,\mathbf{a}(\{x,y\})\,(u(x)-u(y)) \\ &\qquad + \sum_{x,y\in U, x\sim y} \zeta((x,y))\mathbf{a}(\{x,y\})\eta(x)\,(u(x)-u(y)) \\ &\quad\le C\sum_{x,y\in U, x\sim y} \frac{|\eta(x)-\eta(y)|^2}{\eta(x)+\eta(y)}\,|u(y)|^2 \\ &\qquad + \frac{1}{4}\sum_{x,y\in U, x\sim y} (\eta(x)+\eta(y))\,\mathbf{a}(\{x,y\})^2\,(u(x)-u(y))^2 \\ &\quad\le \frac{C}{r^2}\sum_{x,y\in U, x\sim y} \mathbb{1}_{\{\eta(x)\neq\eta(y)\}}(u(y))^2 + \frac{1}{2}\sum_{x,y\in U, x\sim y} \eta(x)\mathbf{a}(\{x,y\})^2\,(u(x)-u(y))^2\,.\end{aligned}$$

We obtain (2.3.8) after absorbing the last term on the right back on the left side and rewriting the expression, using $\mathbf{a} \ge \lambda\mathbb{1}_{\{\mathbf{a}\neq 0\}}$ and (2.3.9). □

An important tool for the arguments later in the paper is Meyers' improvement of integrability for the gradients of solutions, adapted to percolation clusters. In the classical setup (for uniformly elliptic equations in $\mathbb{R}^d$), this is a very simple consequence of the Caccioppoli and Sobolev inequalities which imply a reverse Hölder inequality and thus, after an application of the Gehring lemma, the desired estimate. The situation is more complicated in our setting, since the Sobolev inequality is not uniform and depends on the local coarseness of the partition $\mathcal{P}$, as we have seen. The first step is therefore to quantify the probability of a (deterministic) reverse Hölder inequality on large triadic cubes. This will give us another notion of "good cube" and thus another triadic partition $\mathcal{R}$ which we will use to prove our generalization of Meyers' estimate.

To simplify the statement, we define, for each cube $\square$ and exponent $s > \frac{1}{2}(d+2)$ (so the Hölder conjugate s' satisfies $s' < \frac{2}{2_*}$), the quantity

$$\mathrm{RH}_s(\square) := \begin{cases} \displaystyle\sup_{u\in\mathcal{A}(\mathscr{C}_{\max}(3\square))} \frac{\left(\frac{1}{|\square|}\int_{\mathscr{C}_*(\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right)^{\frac12}}{\left(\frac{1}{|3\square|}\int_{\mathscr{C}_{\max}(3\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{s'2_*}(x)\,dx\right)^{\frac{1}{s'2_*}}} & \text{if } \square \text{ is good,} \\ +\infty & \text{if } \square \text{ is bad,}\end{cases}$$

where $\mathscr{C}_{\max}(3\square)$ denote the maximal cluster of $3\square$ containing $\mathscr{C}_*(\square)$. Notice that if both $\square$ and $3\square$ are good cubes (which is in particular the case if $\square \in \mathcal{P}_*$) then $\mathscr{C}_{\max}(3\square) = \mathscr{C}_*(3\square)$ but thanks to this definition the random variable $\mathrm{RH}_s(\square)$ is $\mathcal{F}(3\square)$–measurable and thus hypothesis (2.2.1) will be satisfied when we apply Proposition 2.2.1 is Definition 2.3.7 below. Also we obviously mean the supremum to exclude constant functions. In other words, $\mathrm{RH}_s(\square)$ is the smallest constant C such that every $u \in \mathcal{A}(\mathscr{C}_{\max}(3\square))$ satisfies the reverse Hölder inequality

$$\left(\frac{1}{|\square|}\int_{\square\cap\mathscr{C}_*(\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right)^{\frac12} \le C\left(\frac{1}{|3\square|}\int_{\mathscr{C}_{\max}(3\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{s'2_*}(x)\,dx\right)^{\frac{1}{s'2_*}}. \tag{2.3.10}$$

We next estimate the probability that $\mathrm{RH}_s(\square_m)$ is larger than a fixed deterministic constant. Recall that the random variable $\mathcal{M}_t(\mathcal{P})$ is given in Proposition 2.2.4.

LEMMA 2.3.6 (Reverse Hölder inequality). *Fix an exponent $s > \frac{1}{2}(d+2)$. Then there exists a constant $C(s,d,\lambda,\mathfrak{p}) < \infty$ such that, for every $m \in \mathbb{N}$,*

$$3^m \geq \mathcal{M}_{2sd}(\mathcal{P}) + C \implies \mathrm{RH}_s(\square_m) \leq C. \tag{2.3.11}$$

In particular, for every $m \in \mathbb{N}$ and exponent $t \in \left(0, \frac{d}{d+1+2sd}\right)$, there exists a constant $C'(t,s,d,\lambda,\mathfrak{p}) < \infty$ such that

$$\mathbb{P}\left[\mathrm{RH}_s(\square_m) > C\right] \leq C' \exp\left(-3^{mt}\right). \tag{2.3.12}$$

PROOF. To setup the argument, fix an exponent

$$s > \frac{1}{2}(d+2).$$

We also fix an integer $m \in \mathbb{N}$ satisfying

$$3^m \geq \mathcal{M}_{2sd}(\mathcal{P})$$

and a function $u \in \mathcal{A}(\mathscr{C}_*(\square_m))$. Note that $m \geq \mathcal{M}_{2sd}(\mathcal{P})$ implies that

$$\frac{1}{|\square_m|} \sum_{x \in \square_m} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2sd} \leq C \tag{2.3.13}$$

and, in particular, that $\square_m$ is good. The goal is to prove (2.3.10) for a deterministic constant $C(s,d,\lambda,\mathfrak{p}) < \infty$.

We begin by applying (2.3.8) and then (2.3.5), which give us

$$\begin{aligned} &\int_{\mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2 (x)\,dx \\ &\quad \leq C 3^{-2m} \inf_{a\in\mathbb{R}} \int_{\mathscr{C}_*(\square_{m+1})} |u(x)-a|^2\,dx \\ &\quad \leq C 3^{-2m} \left(\sum_{\square\in\mathcal{P},\,\square\subseteq\square_{m+1}} \mathrm{size}(\square)^{2d} \int_{\square\cap\mathscr{C}_*(\square_{m+1})} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{2_*} (x)\,dx \right)^{\frac{2}{2_*}}. \end{aligned} \tag{2.3.14}$$

We turn our attention to the first term on the right of (2.3.14). Using the Hölder inequality with exponents s and its Hölder conjugate $s' := \frac{s}{s-1} < \frac{2}{2_*}$, we get

$$\begin{aligned} &\sum_{\square\in\mathcal{P},\,\square\subseteq\square_{m+1}} \mathrm{size}(\square')^{2d} \int_{\square\cap\mathscr{C}_*(\square_{m+1})} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{2_*} (x)\,dx \\ &\quad \leq \left(\sum_{\square\in\mathcal{P},\,\square\subseteq\square_{m+1}} \mathrm{size}(\square)^{2sd+d} \right)^{\frac{1}{s}} \\ &\qquad \times \left(\sum_{\square\in\mathcal{P},\,\square\subseteq\square_{m+1}} |\square| \left(\frac{1}{|\square|} \int_{\square\cap\mathscr{C}_*(\square_{m+1})} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{2_*} (x)\,dx \right)^{s'} \right)^{\frac{1}{s'}} \\ &\quad \leq \left(\sum_{x\in\square_{m+1}} \mathrm{size}(\square_{\mathcal{P}}(x))^{2sd} \right)^{\frac{1}{s}} \left(\int_{\mathscr{C}_*(\square_{m+1})} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{s'2_*} (x)\,dx \right)^{\frac{1}{s'}} \\ &\quad \leq C\,|\square_{m+1}|^{\frac{1}{s}} \left(\int_{\mathscr{C}_*(\square_{m+1})} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{s'2_*} (x)\,dx \right)^{\frac{1}{s'}}, \end{aligned}$$

where in the last line we used the first inequality of (2.3.13). Combining the above displays, we obtain

$$\begin{aligned}
&\frac{1}{|\,\square_m\,|}\int_{\mathscr{C}_*(\square_m)}\left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx \\
&\qquad \le C3^{-m\left(d+2-\frac{2d}{s2_*}\right)}\left(\int_{\mathscr{C}_*(\square_{m+1})}\left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{s'2_*}(x)\,dx\right)^{\frac{2}{s'2_*}} \\
&\qquad = C\left(\frac{1}{|\,\square_{m+1}\,|}\int_{\mathscr{C}_*(\square_{m+1})}\left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{s'2_*}(x)\,dx\right)^{\frac{2}{s'2_*}}.
\end{aligned}$$

This completes the proof of (2.3.10) and therefore of (2.3.11). The second statement is obtained from the first and (2.2.12). □

Definition 2.3.7 (The partition $\mathcal{R}$ and minimal scale $\mathcal{M}_t(\mathcal{R})$). We denote by $\mathcal{R}$ the partition obtained by applying Proposition 2.2.1 to the family of "good events" given by $\mathcal{G} := \{\square \in \mathcal{T} : \mathrm{RH}_{d+2}(\square) \le C\}$ in which a deterministic reverse Hölder inequality holds for gradients of elements of $\mathcal{A}(\mathscr{C}_*(\square))$ with exponent $s'2_* = \frac{2d}{d+1}$ and with $C(d,\lambda,\mathfrak{p}) < \infty$ as in the statement of Lemma 2.3.6. Given an exponent $t \ge 1$ and according to Proposition 2.2.4, we denote the minimal scale for this partition $\mathcal{M}_t(\mathcal{R})$, which we note has integrability

$$\mathcal{M}_t(\mathcal{R}) = \mathcal{O}_r(C'(r,t,d,\lambda,\mathfrak{p})) \quad \text{for every } r \in \left(0, \frac{d^2}{(d+t)(2d^2+5d+1)+d}\right). \tag{2.3.15}$$

We next obtain a version of the Meyers improvement of integrability estimate. For uniformly elliptic equations in Euclidean space, this estimate asserts the existence of an exponent $\varepsilon > 0$ (depending only on dimension and ellipticity) and a constant C such that, for every solution u in the ball B_R,

$$\left(\fint_{B_R}|\nabla u(x)|^{2+\varepsilon}\,dx\right)^{\frac{2}{2+\varepsilon}} \le C\fint_{B_R}|\nabla u(x)|^2\,dx.$$

This deterministic gain in integrability is an important ingredient in the theory developed in [**21**]. In our setting, the analogue presented below (which holds only above a random scale) plays an even more essential role in the developments in Section 2.5 because it allows us to "Hölder away" the sizes of random partitions from our estimates without giving up any exponent.

We define, for each triadic $\square \in \mathcal{T}$ and exponent $\varepsilon > 0$, the random variable

$$\mathrm{ME}_\varepsilon(\square) := \begin{cases} \displaystyle\sup_{u\in\mathcal{A}(\mathscr{C}_{\max}(3\square))} \frac{\left(\frac{1}{|\square|}\int_{\mathscr{C}_*(\square)}\left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{2+\varepsilon}(x)\,dx\right)^{\frac{1}{2+\varepsilon}}}{\left(\frac{1}{|3\square|}\int_{\mathscr{C}_{\max}(3\square)}\left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{2}(x)\,dx\right)^{\frac{1}{2}}} & \text{if } 3\,\square \text{ is good,} \\ +\infty & \text{if } 3\,\square \text{ is bad.}\end{cases}$$

We are interesting in showing that, for an exponent $\varepsilon(d,\lambda,\mathfrak{p}) > 0$, the quantity $\mathrm{ME}_\varepsilon(\square)$ is bounded by a deterministic constant above a random scale. The statement is given in the following proposition.

Proposition 2.3.8 (Meyers estimate). *There exist $\varepsilon(d,\lambda,\mathfrak{p}) > 0$ and $t(d,\lambda,\mathfrak{p}) < \infty$ and a deterministic constant $C(d,\lambda,\mathfrak{p}) < \infty$ such that, for every $m \in \mathbb{N}$,*

$$3^m \ge \mathcal{M}_t(\mathcal{R}) + C \implies \mathrm{ME}_\varepsilon(\square_m) \le C. \tag{2.3.16}$$

In particular, there exists $\delta(d,\lambda,\mathfrak{p}) > 0$ and $C(d,\lambda,\mathfrak{p}) < \infty$ such that, for every $m \in \mathbb{N}$,

$$\mathbb{P}\left[\mathrm{ME}_\varepsilon(\square_m) > C\right] \le C'\exp\left(-3^{m\delta}\right). \tag{2.3.17}$$

Proof. The classical proof of the Meyers estimate (cf. [**73**]) is a combination of the reverse Hölder inequality (given by the Caccioppoli and Sobolev inequalities, as in the previous lemma) and the Gehring lemma (cf. [**77**]). Here the situation is more complicated, because we only have the reverse Hölder inequality above a random scale, and therefore we must modify the strategy slightly by applying the Gehring lemma to a coarsening of $\left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2$ with respect to $\mathcal{R}$, using the reverse Hölder inequality holds on scales larger than $\mathcal{R}$, to obtain an improvement of integrability. Then we use the Hölder's inequality to get rid of the partition.

We fix $m \in \mathbb{N}$ with $m \geq \mathcal{M}_t(\mathcal{R}) + C$ and t is a fixed exponent to be selected at the end of the proof in such a way that it depends only on $(d, \lambda, \mathfrak{p})$. We fix a solution $u \in \mathcal{A}(\mathscr{C}_*(\square_{m+1}))$ and introduce the coarsened function

$$f(x) := \sum_{\square \in \mathcal{R}} \mathbb{1}_{\{[x] \in \square\}} \left(\frac{1}{|\square|} \int_{\square \cap \mathscr{C}_*(\square_{m+1})} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{\frac{2d}{d+1}} (y)\, dy \right), \quad x \in \mathbb{R}^d.$$

Step 1. We write the reverse Hölder estimate from the previous lemma in terms of f. The claim is that there exists a constant $C(d, \lambda, \mathfrak{p}) < \infty$ such that, for every cube $Q \subseteq \mathbb{R}^d$,

$$\fint_Q f(x)^{\frac{d+1}{d}}\, dx \leq C \left(\fint_{63Q} f(x)\, dx \right)^{\frac{d+1}{d}}. \tag{2.3.18}$$

First, observe that if the size of Q is smaller than 100, then since f is constant on all cubes of the form $z + [-\frac{1}{2}, \frac{1}{2}]^d$ with $z \in \mathbb{Z}^d$, it is easy to see that

$$\sup_{x \in Q} f(x) \leq 2^d \frac{1}{|Q| \wedge 1} \int_{2Q} f(x)\, dx \leq C \fint_{2Q} f(x)\, dx.$$

Therefore we need only to check (2.3.18) for cubes of size larger than 100. Note that every cube Q of size at least 100 contains some cube of the form $\frac{1}{9}\square$ for some triadic cube $\square \in \mathcal{T}$.

Let $\square$ be the largest triadic cube satisfying $\frac{1}{9}\square \subseteq Q$. It follows from simple geometric considerations that $Q \subseteq \square$. If $\square \notin \mathcal{R}_*$, then $Q \subseteq \square'$ for some $\square' \in \mathcal{R}$. In this case, f is constant on Q and therefore the bound (2.3.18) is obvious. We may therefore assume $\square \in \mathcal{R}_*$. We now compute, using (2.3.10),

$$\begin{aligned}
\fint_Q f(x)^{\frac{d+1}{d}}\, dx &\leq C \fint_{\square} f(x)^{\frac{d+1}{d}}\, dx \\
&\leq C \fint_{\square \cap \mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2 (y)\, dy \\
&\leq C \left(\frac{1}{|3\,\square|} \int_{\mathscr{C}_*(3\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{\frac{2d}{d+1}} (x)\, dx \right)^{\frac{d+1}{d}}.
\end{aligned}$$

Let W be the union of elements of $\mathcal{R}$ which have nonempty intersection with $3\square$. It is easy to check from the fact that $\mathcal{R}$ has property (ii) of Proposition 2.2.1 that $W \subseteq 7\square$. Thus

$$\begin{aligned}
&\left(\frac{1}{|3\,\square|} \int_{\mathscr{C}_*(3\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{\frac{2d}{d+1}} (x)\, dx \right)^{\frac{d+1}{d}} \\
&\qquad \leq C \left(\frac{1}{|W|} \int_{W \cap \mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{\frac{2d}{d+1}} (x)\, dx \right)^{\frac{d+1}{d}} \\
&\qquad = C \left(\fint_W f(x)\, dx \right)^{\frac{d+1}{d}} \leq C \left(\fint_{7\square} f(x)\, dx \right)^{\frac{d+1}{d}}.
\end{aligned}$$

Since $7\square \subseteq 63Q$, this completes the proof of (2.3.18).

Step 2. We apply the Gehring Lemma to the function f and show that the result implies (2.3.20). By an application of Theorem 6.6 & Corollary 6.1 from [**77**] (we again can get the result in the discrete case from the continuum case by using affine interpolation), we obtain the existence of an exponent $\varepsilon(d, \lambda, \mathfrak{p}) > 0$ and a constant $C(d, \lambda, \mathfrak{p}) < \infty$ such that

$$\fint_{\square_m} f(x)^{\frac{d+1}{d}(1+\varepsilon)}\, dx \leq C \left(\fint_{\square_{m+1}} f(x)^{\frac{d+1}{d}}\, dx \right)^{1+\varepsilon}. \tag{2.3.19}$$

It is obvious that

$$\int_{\square_{m+1}} f(x)^{\frac{d+1}{d}}\,dx \le \int_{\mathscr{C}_*(\square_{m+1})} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx.$$

To bound the left side of (2.3.19) from below, observe that for each $\square \in \mathcal{R}$ and $x \in \mathscr{C}_*(\square_{m+1})$, we have

$$f(x) \ge |\square_{\mathcal{R}}(\lceil x \rfloor)|^{-1} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{\frac{2d}{d+1}}(\lceil x \rfloor).$$

Therefore, by Hölder's inequality,

$$\begin{aligned}
&\int_{\mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{2+\varepsilon}(x)\,dx \\
&\quad\le \left(\int_{\mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{2+2\varepsilon}(x)\, |\square_{\mathcal{R}}(x)|^{-\frac{d+1}{d}(1+\varepsilon)}\,dx\right)^{\frac{2+\varepsilon}{2+2\varepsilon}} \\
&\qquad\times \left(\int_{\mathscr{C}_*(\square_m)} |\square_{\mathcal{R}}(x)|^{\frac{(2+\varepsilon)(1+\varepsilon)(d+1)}{2\varepsilon d}}\,dx\right)^{\frac{\varepsilon}{2+2\varepsilon}} \\
&\quad\le C\left(\int_{\square_m} f(x)^{\frac{d+1}{d}(1+\varepsilon)}\,dx\right)^{\frac{2+\varepsilon}{2+2\varepsilon}} \left(\int_{\mathscr{C}_*(\square_m)} |\square_{\mathcal{R}}(x)|^{\frac{(2+\varepsilon)(1+\varepsilon)(d+1)}{2\varepsilon d}}\,dx\right)^{\frac{\varepsilon}{2+2\varepsilon}} \\
&\quad\le C\,|\square_m| \left(\frac{1}{|\square_{m+1}|}\int_{\mathscr{C}_*(\square_{m+1})} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right)^{\frac{2+\varepsilon}{2}} \\
&\qquad\times \left(\frac{1}{|\square_m|}\int_{\mathscr{C}_*(\square_m)} |\square_{\mathcal{R}}(x)|^{\frac{(2+\varepsilon)(1+\varepsilon)(d+1)}{2\varepsilon d}}\,dx\right)^{\frac{\varepsilon}{2+2\varepsilon}}.
\end{aligned}$$

We now choose $t := \frac{(2+\varepsilon)(1+\varepsilon)(d+1)}{2\varepsilon}$ which, as required, depends only on $(d,\lambda,\mathfrak{p})$. Under the assumption that $m \ge \mathcal{M}_t(\mathcal{R})$, the second factor on the right is then bounded by $C(d,\lambda,\mathfrak{p})$ and we obtain, for every $u \in \mathcal{A}(\mathscr{C}_*(\square))$,

$$\frac{1}{|\square_m|}\int_{\mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^{2+\varepsilon}(x)\,dx \le C\left(\frac{1}{|\square_{m+1}|}\int_{\mathscr{C}_*(\square_{m+1})} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right)^{\frac{2+\varepsilon}{2}}. \tag{2.3.20}$$

Thus $\mathrm{ME}_\varepsilon(\square_m) \le C$, completing the proof of (2.3.16). The bound (2.3.17) is then a consequence of (2.3.16), (2.3.15) and the Chebyshev inequality. □

We finish this section with the definition of the partition $\mathcal{Q}$ quantifying the local scale on which the Meyers estimate holds.

Definition 2.3.9 (The partition $\mathcal{Q}$ and minimal scale $\mathcal{M}_t(\mathcal{Q})$). We denote by $\mathcal{Q}$ the partition obtained by applying Proposition 2.2.1 to the family of "good events" given by $\mathcal{G} := \{\square \in \mathcal{T} : \mathrm{ME}_\varepsilon(\square_m) \le C\}$ in which a deterministic Meyers estimates holds for gradients of elements of $\mathcal{A}(\mathscr{C}_*(\square))$ with exponent $\varepsilon(d,\lambda,\mathfrak{p}) > 0$ and with $C(d,\lambda,\mathfrak{p}) < \infty$ as in the statement of Proposition 2.3.8. We denote, for each exponent $t \ge 1$, the minimal scale for this partition (given by Proposition 2.2.4) by $\mathcal{M}_t(\mathcal{Q})$. Note that

$$\mathcal{M}_t(\mathcal{Q}) = \mathcal{O}_r(C'(r,t,d,\lambda,\mathfrak{p})) \quad \text{for every } r \in \left(0, \frac{\delta d}{d+t+\delta}\right). \tag{2.3.21}$$

where $\delta := \delta(d,\lambda,\mathfrak{p}) > 0$ is as in the statement of Proposition 2.3.8.

2.4. Subadditive energy quantities and basic properties

Here we introduce the subadditive energy quantities, which are modeled on the ones from [**21**], and record their basic properties. We also prove estimates on their uniform convexity, boundedness, the ordering relation between them and their subadditivity. These properties are mostly trivial in the uniformly elliptic case, but more technical in our setting since we must take into account the geometry of the percolation clusters (using the results from the previous two sections).

2.4.1. Definition of the subadditive energy quantities. We next introduce the subadditive quantities. These are based on similar quantities introduced in the continuum, uniformly elliptic setting in [**21**] and variants of the quantities which have been recently used to obtain optimal estimates and scaling limits in stochastic homogenization of uniformly elliptic equations [**16, 17**].

We first define, for each finite subset $U \subseteq \mathbb{Z}^d$, the set

$$\mathcal{A}_*(U) := \{ u : \mathscr{C}_*(U) \to \mathbb{R} \,:\, -\nabla \cdot \mathbf{a} \nabla u(x) = 0, \ \forall x \in \mathscr{C}_*(U) \setminus \partial \operatorname{cl}_{\mathcal{P}}(U) \} .$$

Note that $\mathcal{A}(\mathscr{C}_*(U)) \subseteq \mathcal{A}_*(U)$ since $\partial_{\mathbf{a}} \mathscr{C}_*(U) \subseteq \partial \operatorname{cl}_{\mathcal{P}}(U)$, but neither of these inclusions in necessarily an equality. As in (2.1.6), we have,

$$u \in \mathcal{A}_*(U) \iff \langle \nabla w, \mathbf{a} \nabla u \rangle_U = 0 \quad \text{for every } w \in \mathcal{C}_0(U), \tag{2.4.1}$$

where $\mathcal{C}_0(U)$ denotes the set of functions $w : \mathscr{C}_*(U) \to \mathbb{R}$ equal to 0 on $\mathscr{C}_*(U) \cap \operatorname{cl}_{\mathcal{P}}(U)$.

DEFINITION 2.4.1. For each $U \subseteq \mathbb{Z}^d$ and $p, q \in \mathbb{R}^d$, we define the random variables

$$\mu(U,q) := \inf_{u \in \mathcal{A}_*(U)} \frac{1}{|\operatorname{cl}_{\mathcal{P}}(U)|} \left(\frac{1}{2} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(U)} - \langle q, \nabla [u]_{\mathcal{P}} \rangle_{\operatorname{cl}_{\mathcal{P}}(U)} \right)$$

and

$$\nu(U,p) := \sup_{v \in \mathcal{A}_*(U)} \frac{1}{|\operatorname{cl}_{\mathcal{P}}(U)|} \left(-\frac{1}{2} \langle \nabla v, \mathbf{a} \nabla v \rangle_{\mathscr{C}_*(U)} + \langle p, \mathbf{a} \nabla v \rangle_{\mathscr{C}_*(U)} \right).$$

The optimization problems in the above definitions of $\mu(U,q)$ and $\nu(U,p)$ are strictly convex and concave, respectively, and therefore they have unique optimizers in $\mathcal{A}_*(U)$, up to additive constants, which we denote by

$$u(\cdot, U, q) := \text{minimizing element of } \mathcal{A}_*(U) \text{ in the definition of } \mu(U,q)$$

and

$$v(\cdot, U, p) := \text{maximizing element of } \mathcal{A}_*(U) \text{ in the definition of } \nu(U,q).$$

We choose the additive constants for $u(\cdot, U, q)$ and $v(\cdot, U, p)$ so that

$$\fint_{\operatorname{cl}_{\mathcal{P}}(U)} [u(\cdot, U, q)]_{\mathcal{P}}(x)\, dx = 0 \quad \text{and} \quad \fint_{\operatorname{cl}_{\mathcal{P}}(U)} [v(\cdot, U, p)]_{\mathcal{P}}(x)\, dx = 0. \tag{2.4.2}$$

Notice that, for each bounded $U \subseteq \mathbb{Z}^d$,

$$q \mapsto -\mu(U,q) \quad \text{and} \quad p \mapsto \nu(U,p) \quad \text{are nonnegative and quadratic.}$$

In particular, these maps are convex.

In Lemma 2.4.3, below, we will show that the function $v(\cdot, U, p)$ is the solution of the Dirichlet problem in $\mathscr{C}_*(U)$ with affine data $x \mapsto p \cdot x + c$ on $\mathscr{C}_*(U) \cap \partial \operatorname{cl}_{\mathcal{P}}(U)$, for some $c \in \mathbb{R}$. Therefore $\nu(U,p)$ is just (up to the normalization) the energy of the familiar cell problem solution in $\mathscr{C}_*(U)$. The quantity μ represents the energy of the "dual" cell problem introduced in [**21**]. It is important here that the linear term in the definition of μ is *not* $\langle q, \nabla u \rangle_{\mathscr{C}_*(U)}$, which is what one might naively guess when attempting to generalize from the uniformly elliptic case. This will not possess the correct convex dual relationship with ν: in particular, (2.4.29) would be false, rendering attempts at proving Proposition 2.5.2 hopeless. Indeed, if u is close to an affine function with slope p (for example, the function $v(\cdot, U, p)$), there is no reason to expect that $\langle q, \nabla u \rangle_{\mathscr{C}_*(U)}$ should be close to $q \cdot p$, because we are "missing" the contribution of ∇u in the closed edges. While the exact form of the linear term is not very important, we need something that will be close to $q \cdot p$ if (as expected on large scales) $u(\cdot, U, q)$ is close to an affine function with slope p. Using the spatial average of the gradient $\nabla [u]_{\mathcal{P}}$ of the coarsened function satisfies this property and turns out to be very convenient. One of the central ideas of [**21**] is that one should focus on the spatial averages of the gradients and energy densities of the solutions. We do the same in the generalization here, except that when it comes to gradient we consistently replace a solution u with its coarsening $[u]_{\mathcal{P}}$.

In the next section, we quantify the convergence of quantities $\mu(\square, q)$ and $\nu(\square, p)$ for $\square \in \mathcal{T}$ as $\operatorname{size}(\square) \to \infty$, see Proposition 2.5.2. In the rest of this section, we prepare for this analysis by presenting some basic properties of these quantities. The geometry of the percolation cluster forces us to give up some very nice properties possessed by μ and ν in the continuum uniformly elliptic setting [**21**]. For example, $-\mu$ and ν are not (strictly speaking) *uniformly* convex independently of U, because in general the partition can be quite coarse and the geometry of the percolation cluster very complicated. They are not *stationary* with respect to $\mathbb{Z}^d$-translations (because the partition $\mathcal{P}$ is not stationary), nor are they *local* quantities (since they depend on the coefficient field $\mathbf{a}(\cdot)$ on the whole of B_d, since $\mathcal{P}$ does), nor are they precisely subadditive. Most of this section is therefore consumed by the quite technical task of showing that each of these important properties does in fact hold in an approximate sense which is quantified with sufficiently strong stochastic integrability.

We conclude this subsection by computing the first and second variations of the optimization problems in the definitions of μ and ν and then checking that the function v is the solution of the Dirichlet problem with affine data as claimed above.

LEMMA 2.4.2 (First and second variations). *Fix a bounded $U \subseteq \mathbb{Z}^d$ and $p, q \in \mathbb{R}^d$. For every $w \in \mathcal{A}_*(U)$,*

$$\langle \nabla w, \mathbf{a}\nabla u(\cdot, U, q)\rangle_{\mathscr{C}_*(U)} = \langle q, \nabla [w]_{\mathcal{P}}\rangle_{\operatorname{cl}_{\mathcal{P}}(U)}, \tag{2.4.3}$$

$$\begin{aligned}\frac{1}{|\operatorname{cl}_{\mathcal{P}}(U)|}&\left(\frac{1}{2}\langle \nabla w, \mathbf{a}\nabla w\rangle_{\mathscr{C}_*(U)} - \langle q, \nabla [w]_{\mathcal{P}}\rangle_{\operatorname{cl}_{\mathcal{P}}(U)}\right) \\ &= \mu(U,q) + \frac{1}{|\operatorname{cl}_{\mathcal{P}}(U)|}\frac{1}{2}\langle \nabla (w - u(\cdot, U, q)), \mathbf{a}\nabla (w - u(\cdot, U, q))\rangle_{\mathscr{C}_*(U)},\end{aligned} \tag{2.4.4}$$

$$\langle \nabla w, \mathbf{a}\nabla v(\cdot, U, p)\rangle_{\mathscr{C}_*(U)} = \langle p, \mathbf{a}\nabla w\rangle_{\mathscr{C}_*(U)}, \tag{2.4.5}$$

and

$$\begin{aligned}\frac{1}{|\operatorname{cl}_{\mathcal{P}}(U)|}&\left(-\frac{1}{2}\langle \nabla w, \mathbf{a}\nabla w\rangle_{\mathscr{C}_*(U)} + \langle p, \mathbf{a}\nabla w\rangle_{\mathscr{C}_*(U)}\right) \\ &= \nu(U,p) - \frac{1}{|\operatorname{cl}_{\mathcal{P}}(U)|}\left\langle \frac{1}{2}\nabla (w - v(\cdot, U, p)), \mathbf{a}\nabla (w - v(\cdot, U, p))\right\rangle_{\mathscr{C}_*(U)}.\end{aligned} \tag{2.4.6}$$

PROOF. Let $w \in \mathcal{A}(\mathscr{C}_*(U))$. For each $h \in [0,1)$, define $u_h := u(\cdot, U, q) + hw$. By comparing u_h to $u := u_0 = u(\cdot, U, q)$ in the definition of $\mu(U,q)$, we obtain, for every $h > 0$,

$$\begin{aligned} 0 &\geq \frac{1}{2}\langle \nabla u, \mathbf{a}\nabla u\rangle_{\mathscr{C}_*(U)} - \frac{1}{2}\langle \nabla u_h, \mathbf{a}\nabla u_h\rangle_{\mathscr{C}_*(U)} - \langle q, \nabla [u - u_h]_{\mathcal{P}}\rangle_{\operatorname{cl}_{\mathcal{P}}(U)} \\ &= -\frac{1}{2}h^2\langle \nabla w, \mathbf{a}\nabla w\rangle_{\mathscr{C}_*(U)} - h\langle \nabla w, \mathbf{a}\nabla u\rangle_{\mathscr{C}_*(U)} + h\langle q, \nabla [w]_{\mathcal{P}}\rangle_{\operatorname{cl}_{\mathcal{P}}(U)}.\end{aligned}$$

Rearranging this and dividing by $h > 0$ gives

$$-\langle \nabla u, \mathbf{a}\nabla w\rangle_{\mathscr{C}_*(U)} + \langle q, \nabla [w]_{\mathcal{P}}\rangle_{\operatorname{cl}_{\mathcal{P}}(U)} \leq \frac{1}{2}h\langle \nabla w, \mathbf{a}\nabla w\rangle_{\mathscr{C}_*(U)}.$$

Sending $h \to 0$ yields, for every $w \in \mathcal{A}(\mathscr{C}_*(U))$,

$$\langle q, \nabla [w]_{\mathcal{P}}\rangle_{\operatorname{cl}_{\mathcal{P}}(U)} \leq \langle \nabla u, \mathbf{a}\nabla w\rangle_{\mathscr{C}_*(U)}.$$

The reverse of the previous inequality follows by replacing w by $-w$, which completes the proof of (2.4.3). Returning now to the first display and inserting (2.4.3), we obtain (2.4.4) for u_h in place of w.

The proofs of (2.4.5) and (2.4.6) are similar and thus omitted. □

For future reference, we record some identities which are consequences of those in Lemma 2.4.2. By combining (2.4.5) and (2.4.6) with $w = v(\cdot, U, p)$, we get

$$\nu(U,p) = \frac{1}{|\operatorname{cl}_{\mathcal{P}}(U)|} \frac{1}{2} \langle \nabla v(\cdot, U, p), \mathbf{a} \nabla v(\cdot, U, p) \rangle_{\mathscr{C}_*(U)} . \tag{2.4.7}$$

Next, inserting this into (2.4.4) with $w = v(\cdot, U, p)$, we get

$$\begin{aligned} \nu(U,p) - \mu(U,q) - \langle q, \nabla [v(\cdot, U, p)]_{\mathcal{P}} \rangle_{\operatorname{cl}_{\mathcal{P}}(U)} \\ = \frac{1}{|\operatorname{cl}_{\mathcal{P}}(U)|} \frac{1}{2} \langle \nabla (v(\cdot, U, p) - u(\cdot, U, q)), \mathbf{a} \nabla (v(\cdot, U, p) - u(\cdot, U, q)) \rangle_{\mathscr{C}_*(U)} . \end{aligned} \tag{2.4.8}$$

We next show that $v(\cdot, \square, p)$ is the solution of the Dirichlet problem with affine boundary data. Recall that the space $\mathcal{C}_0(U)$ is defined between (2.1.5) and (2.4.1), above.

LEMMA 2.4.3. *There exists $c \in \mathbb{R}$ such that*

$$v(x, U, p) = p \cdot x + c \quad \text{for every } x \in \mathscr{C}_*(U) \cap \partial \operatorname{cl}_{\mathcal{P}}(U) \tag{2.4.9}$$

and

$$\nu(U,p) = \inf_{w \in \mathcal{C}_0(\operatorname{cl}_{\mathcal{P}}(U))} \frac{1}{2|\operatorname{cl}_{\mathcal{P}}(U)|} \langle (p + \nabla w), \mathbf{a} (p + \nabla w) \rangle_{\mathscr{C}_*(U)} .$$

PROOF. From Lemma 2.4.2 we have that, for every $w \in \mathcal{A}_*(U)$,

$$\langle \nabla w, \mathbf{a} ((\nabla v(\cdot, U, p) - p) \rangle_{\mathscr{C}_*(U)} = 0. \tag{2.4.10}$$

Pick $w \in \mathcal{A}_*(U)$ such that $w(x) = v(x, U, p) - p \cdot x$ for every $x \in \mathscr{C}_*(U) \cap \partial \operatorname{cl}_{\mathcal{P}}(U)$. Then $w - v(\cdot, U, p) - p \cdot x \in \mathcal{C}_0(\operatorname{cl}_{\mathcal{P}}(U))$, since $w \in \mathcal{A}_*(U)$, we have

$$\langle \nabla w, \mathbf{a} (\nabla w - v(\cdot, U, p) + p) \rangle_{\mathscr{C}_*(U)} = 0. \tag{2.4.11}$$

Summing the equations (2.4.10) and (2.4.11) gives

$$\langle \nabla w, \mathbf{a} \nabla w \rangle_{\mathscr{C}_*(U)} = 0.$$

Therefore w is constant and so $x \mapsto v(x, U, p) - p \cdot x$ is constant on $\mathscr{C}_*(U) \cap \partial \operatorname{cl}_{\mathcal{P}}(U)$.

Since $v \in \mathcal{A}_*(U)$ and $v(x, U, p) = p \cdot x + c$ on $\mathscr{C}_*(U) \cap \partial \operatorname{cl}_{\mathcal{P}}(U)$, we have, by (2.4.7),

$$\begin{aligned} \nu(U,p) &= \langle \nabla v(\cdot, U, p), \mathbf{a} \nabla v(\cdot, U, p) \rangle_{\mathscr{C}_*(U)} \\ &= \inf_{w \in \mathcal{C}_0(\mathscr{C}_*(U))} \langle (p + \nabla w), \mathbf{a} (p + \nabla w) \rangle_{\mathscr{C}_*(U)} . \end{aligned}$$

The proof of (2.4.3) is complete. □

2.4.2. Subadditivity, boundedness, uniform convexity, and ordering. The purpose of this subsection is to prove that μ and ν retain most of their essential properties from the uniformly elliptic setting, with errors arising due to the coarseness of the random partition $\mathcal{P}$.

We begin by proving the upper bounds for $-\mu$ and ν. While the one for ν is obvious, the one for $-\mu$ uses the estimate (2.2.17).

LEMMA 2.4.4. *There exists a constant $C(d, \mathfrak{p}, \lambda) < \infty$ such that, for each $\square \in \mathcal{T}$ and $p, q \in \mathbb{R}^d$,*

$$-\mu(\square, q) \le C|q|^2 \fint_{\square} \operatorname{size}(\square_{\mathcal{P}}(x))^{2d-2} \, dx \tag{2.4.12}$$

and

$$\nu(\square, p) \le C|p|^2. \tag{2.4.13}$$

PROOF. We begin by noticing that the bound (2.4.13) is immediate from Young's inequality. We thus focus on the bound for $-\mu$. For convenience, we denote $u := u(\cdot, \square, q)$. Applying Lemma 2.3.3 with $s = 1$ gives

$$\int_{\mathrm{cl}_{\mathcal{P}}(\square)} |\nabla [u]_{\mathcal{P}}|(x)\,dx \le C \sum_{\square' \in \mathcal{P}, \square' \subseteq \mathrm{cl}_{\mathcal{P}}(\square)} \mathrm{size}(\square')^{d-1} \int_{\square' \cap \mathscr{C}_*(\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|(x)\,dx.$$

By the Hölder inequality, we deduce that

$$\begin{aligned}(2.4.14)\qquad &\int_{\mathrm{cl}_{\mathcal{P}}(\square)} |\nabla [u]_{\mathcal{P}}|(x)\,dx \\ &\le C \sum_{\square' \in \mathcal{P}, \square' \subseteq \mathrm{cl}_{\mathcal{P}}(\square)} \mathrm{size}(\square')^{\frac{3}{2}d-1} \left(\int_{\square' \cap \mathscr{C}_*(\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2 (x)\,dx \right)^{\frac12} \\ &\le C \left(\sum_{\square' \in \mathcal{P}, \square' \subseteq \mathrm{cl}_{\mathcal{P}}(\square)} \mathrm{size}(\square')^{3d-2} \right)^{\frac12} \left(\int_{\mathscr{C}_*(\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2 (x)\,dx \right)^{\frac12} \\ &\le C \left(\int_{\mathrm{cl}_{\mathcal{P}}(\square)} \mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\,dx \right)^{\frac12} \left(\int_{\mathscr{C}_*(\square)} \left|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2 (x)\,dx \right)^{\frac12}.\end{aligned}$$

Young's inequality then yields

$$\langle q, \nabla [u]_{\mathcal{P}} \rangle_{\mathrm{cl}_{\mathcal{P}}(\square)} \le C|q|^2 \int_{\mathrm{cl}_{\mathcal{P}}(\square)} \mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\,dx + \frac12 \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(\square)}.$$

Hence

$$\begin{aligned}-\mu(\square, q) &= \frac{1}{|\mathrm{cl}_{\mathcal{P}}(\square)|} \left(-\frac12 \langle \nabla u, \mathbf{a}\nabla u \rangle_{\mathscr{C}_*(\square)} + \langle q, \nabla [u]_{\mathcal{P}} \rangle_{\square} \right) \\ &\le C|q|^2 \fint_{\mathrm{cl}_{\mathcal{P}}(\square)} \mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\,dx.\end{aligned}$$

Notice that if $\mathrm{cl}_{\mathcal{P}}(\square) \neq \square$ then for each $x \in \mathrm{cl}_{\mathcal{P}}(\square)$, $\mathrm{size}(\square_{\mathcal{P}}(x)) = \mathrm{size}(\mathrm{cl}_{\mathcal{P}}(\square))$ and thus

$$(2.4.15)\qquad \fint_{\mathrm{cl}_{\mathcal{P}}(\square)} \mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\,dx = \mathrm{size}(\mathrm{cl}_{\mathcal{P}}(\square))^{2d-2} = \fint_{\square} \mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\,dx.$$

Combining the two previous displays gives (2.4.12). □

Observe that (2.2.17) and (2.4.16) imply that

$$(2.4.16)\qquad -\mu(\square, q) \le \mathcal{O}_{\frac{1}{2d-2}}\left(C|q|^2\right).$$

It is also useful to notice that we can bound the right side slightly differently (so that the random part is scaling better) to obtain

$$(2.4.17)\qquad -\mu(\square, q) \le C|q|^2 + \mathcal{O}_{\frac{1}{2d+1}}\left(C|q|^2\,\mathrm{size}^{-1}(\square)\right).$$

To see this, we combine (2.4.12) with the bounds for the minimal scale $\mathcal{M}_{2d-2}(\mathcal{P})$ given in Proposition 2.2.4 to obtain

$$\begin{aligned}(2.4.18)\qquad \fint_{\square_n} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2}\,dx &\le C + \fint_{\square_n} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2}\,dx\, \mathbb{1}_{\{\mathcal{M}_{2d-2}(\mathcal{P}) > 3^n\}} \\ &\le C + 3^{-n}\mathcal{M}_{2d-2}(\mathcal{P}) \fint_{\square_n} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2}\,dx \\ &\le C + \mathcal{O}_{\frac13}\left(C3^{-n}\right) \cdot \mathcal{O}_{\frac{1}{2d-2}}(C) \\ &\le C + \mathcal{O}_{\frac{1}{2d+1}}\left(C3^{-n}\right).\end{aligned}$$

For future reference, we also note that (2.4.16) and (2.4.13) imply upper bounds for the L^2 norm of the gradients of $u(\cdot,\square,q)$, $v(\cdot,\square,p)$ and their coarsenings. Indeed, by the first variations (2.4.3) and (2.4.5), we have

$$\begin{cases} \mu(U,q) = \dfrac{1}{|\mathrm{cl}_{\mathcal{P}}(U)|}\dfrac{1}{2}\left\langle \nabla u(\cdot,U,q), \mathbf{a}\nabla u(\cdot,U,q)\right\rangle_{\mathrm{cl}_{\mathcal{P}}(U)}, \quad \text{and} \\ \nu(U,p) = \dfrac{1}{|\mathrm{cl}_{\mathcal{P}}(U)|}\dfrac{1}{2}\left\langle \nabla v(\cdot,U,p), \mathbf{a}\nabla v(\cdot,U,p)\right\rangle_{\mathrm{cl}_{\mathcal{P}}(U)} \end{cases} \tag{2.4.19}$$

and thus (2.4.12) and (2.4.18) imply

$$\begin{aligned} \frac{1}{|\,\mathrm{cl}_{\mathcal{P}}(\square)|}\int_{\mathscr{C}_*(\square)} \left|\nabla u(\cdot,U,q)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx & \\ \leq C|q|^2 \fint_{\square} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2}\,dx & \\ \leq \mathcal{O}_{\frac{1}{2d-2}}(C|q|^2) \wedge \left(C|q|^2 + \mathcal{O}_{\frac{1}{2d+1}}\left(C|q|^2\,\mathrm{size}^{-1}(\square)\right)\right) & \end{aligned} \tag{2.4.20}$$

and (2.4.13) implies

$$\frac{1}{|\,\mathrm{cl}_{\mathcal{P}}(\square)|}\int_{\mathscr{C}_*(\square)} \left|\nabla v(\cdot,U,p)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx \leq C|p|^2. \tag{2.4.21}$$

Combining these with (2.4.14), (2.4.15) and the analogous bound for $v(\cdot,\square,p)$, we get the following bounds for the L^1 norm of the coarsened functions:

$$\fint_{\mathrm{cl}_{\mathcal{P}}(\square)} \left|\nabla\left[u(\cdot,\square,q)\right]_{\mathcal{P}}\right|(x)\,dx \leq C|q| \fint_{\square} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2}\,dx \leq \mathcal{O}_{\frac{1}{2d-2}}\left(C|q|\right) \tag{2.4.22}$$

and

$$\fint_{\mathrm{cl}_{\mathcal{P}}(\square)} \left|\nabla\left[v(\cdot,\square,p)\right]_{\mathcal{P}}\right|(x)\,dx \leq C|p| \left(\fint_{\square} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2}\,dx\right)^{\frac{1}{2}} \leq \mathcal{O}_{\frac{1}{d-1}}\left(C|p|\right). \tag{2.4.23}$$

By (2.2.17), we have that for every $t > 0$

$$\begin{aligned} \mathbb{P}\left[\sup_{x\in\square} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right) \geq t\right] & \leq \sum_{x\in\square} \mathbb{P}\left[\mathrm{size}\left(\square_{\mathcal{P}}(x)\right) \geq t\right] \\ & \leq C\,\mathrm{size}(\square)^d \exp\left(-C^{-1}t\right). \end{aligned}$$

From this we deduce that for every $\delta > 0$, there exists $C := C(d,\mathfrak{p},\delta) < \infty$ such that

$$\sup_{x\in\square} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right) \leq \mathcal{O}_1\left(C\,\mathrm{size}(\square)^\delta\right).$$

Combining this with Lemma 2.3.3 (with $s = 2$) gives

$$\begin{aligned} \fint_{\mathrm{cl}_{\mathcal{P}}(\square)} \left|\nabla\left[u(\cdot,\square,q)\right]_{\mathcal{P}}\right|^2(x)\,dx & \leq C|q|^2 \left(\sup_{x\in\square} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-1}\right) \fint_{\square} \mathrm{size}(\square)^{2d-2}\,dx \\ & \leq |q|^2 \mathcal{O}_{\frac{1}{2d-1}}\left(C\,\mathrm{size}(\square)^\delta\right) \mathcal{O}_{\frac{1}{2d-2}}(C) \\ & \leq \mathcal{O}_{\frac{1}{4d-3}}\left(C|q|^2\,\mathrm{size}(\square)^\delta\right). \end{aligned} \tag{2.4.24}$$

Similarly we obtain for each $\delta > 0$ and for some $C := C(d,\mathfrak{p},\delta) < \infty$

$$\begin{aligned} \fint_{\mathrm{cl}_{\mathcal{P}}(\square)} \left|\nabla\left[v(\cdot,\square,p)\right]_{\mathcal{P}}\right|^2(x)\,dx & \leq C|p|^2 \left(\sup_{x\in\square} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-1}\right) \left(\fint_{\square} \mathrm{size}(\square)^{2d-2}\,dx\right)^{\frac{1}{2}} \\ & \leq |p|^2 \mathcal{O}_{\frac{1}{2d-1}}\left(C\,\mathrm{size}(\square)^\delta\right) \mathcal{O}_{\frac{1}{d-1}}(C) \\ & \leq \mathcal{O}_{\frac{1}{3d-2}}\left(C|p|^2\,\mathrm{size}(\square)^\delta\right). \end{aligned} \tag{2.4.25}$$

We next prove the ordering relation between μ and ν. In the uniformly elliptic case, this is proved in one line and comes from testing the definition of μ with the minimizer of ν and using an integration by parts to see that the spatial average of ∇v is exactly p. In our situation, the latter computation is not exact but holds up to a coarsening error, as stated in the following lemma.

LEMMA 2.4.5. *There exists a constant $C(d,\lambda,\mathfrak{p})<\infty$ such that, for every $\square\in\mathcal{T}$ and $q,p\in\mathbb{R}^d$,*

$$\left|\frac{1}{|\mathrm{cl}_{\mathcal{P}}(\square)|}\left\langle q,\nabla[v(\cdot,\square,p)]_{\mathcal{P}}\right\rangle_{\mathrm{cl}_{\mathcal{P}}(\square)}-p\cdot q\right|\leq\mathcal{O}_{\frac{2}{2d-1}}\left(C|p||q|\,\mathrm{size}(\square)^{-\frac12}\right). \tag{2.4.26}$$

PROOF. Fix $p\in\mathbb{R}^d$ and denote $v:=v(\cdot,\square,p)$. The estimate (2.4.26) is a consequence of (2.2.17) and the following claimed inequality:

$$\left|\frac{1}{|\mathrm{cl}_{\mathcal{P}}(\square)|}\left\langle q,\nabla[v]_{\mathcal{P}}\right\rangle_{\mathrm{cl}_{\mathcal{P}}(\square)}-p\cdot q\right|\leq C|p||q|\left(\frac{|\partial_{\mathcal{P}}\,\square|}{|\,\square\,|}+\left(\frac{1}{|\,\square\,|}\sum_{x\in\partial\square}\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-1}\right)^{\frac12}\right). \tag{2.4.27}$$

Recall that

$$\partial_{\mathcal{P}}\square:=\bigcup\left\{\square'\in\mathcal{P}\,:\,\square'\subseteq\square,\ \mathrm{dist}\left(\square',\partial\square\right)=0\right\}.$$

First, we deal with the case $\mathrm{cl}_{\mathcal{P}}(\square)\neq\square$. In that case, we know, by definition of the partition $\mathcal{P}$, that $[v]_{\mathcal{P}}$ is constant, thus

$$\frac{1}{|\mathrm{cl}_{\mathcal{P}}(\square)|}\left\langle q,\nabla[v]_{\mathcal{P}}\right\rangle_{\mathrm{cl}_{\mathcal{P}}(\square)}=0.$$

We also have by definition of $\partial_{\mathcal{P}}\square$

$$\square\subseteq\mathrm{cl}_{\mathcal{P}}(\square)=\partial_{\mathcal{P}}\square$$

Thus

$$\begin{aligned}\left|\frac{1}{|\mathrm{cl}_{\mathcal{P}}(\square)|}\left\langle q,\nabla[v]_{\mathcal{P}}\right\rangle_{\mathrm{cl}_{\mathcal{P}}(\square)}-p\cdot q\right|&=|p\cdot q|\\&\leq|p||q|\\&\leq C|p||q|\frac{|\partial_{\mathcal{P}}\,\square|}{|\,\square\,|}.\end{aligned}$$

which shows (2.4.27). We now assume for the rest of the proof $\mathrm{cl}_{\mathcal{P}}(\square)=\square$.

For $x\in\partial\square$, denote the outer normal vector $\mathbf{n}(x)\in\mathbb{R}^d$ to $\partial\square$ at x by

$$\mathbf{n}(x):=\sum_{i=1}^{d}\left(e_i\mathbb{1}_{\{x-e_i\in\square\}}-e_i\mathbb{1}_{\{x+e_i\in\square\}}\right).$$

Note that $|\mathbf{n}(x)|\leq d$. Applying the discrete Stokes formula, we find that

$$\begin{aligned}\left|\frac{1}{|\,\square\,|}\left\langle q,\nabla[v]_{\mathcal{P}}\right\rangle_{\square}-p\cdot q\right|&=\left|\frac{1}{|\,\square\,|}\int_{\partial\square}\left([v]_{\mathcal{P}}(x)-p\cdot x\right)q\cdot\mathbf{n}(x)\,dx\right|\\&\leq\frac{d|q|}{|\,\square\,|}\int_{\partial\square}\left|[v]_{\mathcal{P}}(x)-p\cdot x\right|\,dx.\end{aligned}$$

For each $\square'\in\mathcal{P}$ with $\square'\subseteq\partial_{\mathcal{P}}\square$, we can find a point $\overline{y}(\square')\in\partial_{\mathbf{a}}\mathscr{C}_*(\square)$ and thus, by Lemma 2.4.3, for each $x\in\partial\square$,

$$\begin{aligned}\left|[v]_{\mathcal{P}}(x)-p\cdot x\right|&=\left|v\left(\overline{z}\left(\square_{\mathcal{P}}(x)\right)\right)-v\left(\overline{y}\left(\square_{\mathcal{P}}(x)\right)\right)\right|+\left|p\cdot\left(x-\overline{y}\left(\square_{\mathcal{P}}(x)\right)\right)\right|\\&\leq\int_{\square_{\mathcal{P}}(x)\cap\mathscr{C}_*(\square)}\left|\nabla v\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|(x')\,dx'+C|p|\,\mathrm{size}\left(\square_{\mathcal{P}}(x)\right).\end{aligned}$$

Integrating this over $x \in \partial\square$ and using the Hölder inequality gives

$$\begin{aligned}
&\int_{\partial\square} |[v]_{\mathcal{P}}(x) - p\cdot x| \, dx \\
&\le \int_{\partial\square} \left(\int_{\mathscr{C}_*(\square)\cap\square_{\mathcal{P}}(x)} \left|\nabla v \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|(y)\,dy + C|p|\,\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)\right) dx \\
&\le C|p|\,|\partial_{\mathcal{P}}\square| + \int_{\partial\square} |\square_{\mathcal{P}}(x)|^{\frac12} \left(\int_{\mathscr{C}_*(\square)\cap\square_{\mathcal{P}}(x)} \left|\nabla v \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(y)\,dy\right)^{\frac12} dx \\
&= C|p|\,|\partial_{\mathcal{P}}\square| + \left(\sum_{\mathcal{P}\ni\square'\subseteq\partial_{\mathcal{P}}\square} \mathrm{size}(\square')^{\frac32 d-1} \left(\int_{\mathscr{C}_*(\square)\cap\square'} \left|\nabla v \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(y)\,dy\right)^{\frac12}\right) \\
&\le C|p|\,|\partial_{\mathcal{P}}\square| + \left(\sum_{\mathcal{P}\ni\square'\subseteq\partial_{\mathcal{P}}\square} \mathrm{size}(\square')^{3d-2}\right)^{\frac12} \left(\sum_{\mathcal{P}\ni\square'\subseteq\partial_{\mathcal{P}}\square} \int_{\mathscr{C}_*(\square)\cap\square'} \left|\nabla v \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(y)\,dy\right)^{\frac12} \\
&\le C|p|\,|\partial_{\mathcal{P}}\square| + \left(\sum_{x\in\partial\square} \mathrm{size}(\square_{\mathcal{P}}(x))^{2d-1}\right)^{\frac12} \left(\int_{\mathscr{C}_*(\square)} \left|\nabla v \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(y)\,dy\right)^{\frac12}.
\end{aligned}$$

The first variation (2.4.5) for ν combined with (2.4.13) yields that

$$\frac{1}{|\mathrm{cl}_{\mathcal{P}}(\square)|}\frac12 \langle \nabla v, \mathbf{a}\nabla v\rangle_{\mathscr{C}_*(\square)} = \nu(\square,p) \le C|p|^2.$$

Thus

$$\left(\int_{\mathscr{C}_*(\square)} \left|\nabla v \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(y)\,dy\right)^{\frac12} \le C|p|\,|\mathrm{cl}_{\mathcal{P}}(\square)|.$$

Combining the above inequalities gives the desired bound (2.4.27).

To obtain (2.4.26), we observe that (2.2.17) implies that

$$\frac{|\partial_{\mathcal{P}}\square|}{|\square|} \le \mathcal{O}_1\left(\mathrm{size}(\square)^{-1}\right) \tag{2.4.28}$$

and

$$\left(\frac{1}{|\square|} \sum_{x\in\partial\square} \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-1}\right)^{\frac12} \le \mathcal{O}_{\frac{2}{2d-1}}\left(C\left(\frac{|\partial\square|}{|\square|}\right)^{-\frac12}\right) = \mathcal{O}_{\frac{2}{2d-1}}\left(C\,\mathrm{size}(\square)^{-\frac12}\right). \qquad \square$$

Corollary 2.4.6. *There exists a constant $C(d,\lambda,\mathfrak{p}) < \infty$ such that, for every $\square\in\mathcal{T}$ and $p,q\in\mathbb{R}^d$,*

$$\begin{aligned}
&\Bigg| \left(\nu(\square,p) - \mu(\square,q) - p\cdot q\right) \\
&\quad - \frac{1}{|\mathrm{cl}_{\mathcal{P}}(\square)|}\frac12 \langle \nabla(v(\cdot,\square,p) - u(\cdot,\square,q)), \mathbf{a}\nabla(v(\cdot,\square,p) - u(\cdot,\square,q))\rangle_{\mathscr{C}_*(\square)} \Bigg| \\
&\qquad\qquad \le \mathcal{O}_{\frac{2}{2d-1}}\left(C|p||q|\,\mathrm{size}(\square)^{-\frac12}\right).
\end{aligned} \tag{2.4.29}$$

Proof. Combine (2.4.8) and Lemma 2.4.5. $\square$

We next show that the combination of the upper bounds in Lemma 2.4.4 and the inequality in Corollary 2.4.6 give us the desired lower bounds.

Lemma 2.4.7. *There exists a constant $C := C(d,\mathfrak{p},\lambda) < \infty$ such that, for every $\square\in\mathcal{T}$ and $p,q\in\mathbb{R}^d$,*

$$-\mu(\square,q) \ge |q|^2\left(\frac1C - \mathcal{O}_{\frac{2}{2d-1}}\left(C\,\mathrm{size}(\square)^{-\frac12}\right)\right). \tag{2.4.30}$$

and

$$\nu(\square, p) \geq |p|^2 \left(\frac{1}{C} - \mathcal{O}_{\frac{2}{2d+1}} \left(C \operatorname{size}(\square)^{-\frac{1}{2}} \right) \right). \tag{2.4.31}$$

PROOF. These estimates are consequences of Lemma 2.4.4 and Lemma 2.4.6. We first give the lower bound for $-\mu$. Combining (2.4.13) and (2.4.29), we find that, for every $p, q \in \mathbb{R}^d$,

$$\begin{aligned}\mu(\square, q) &\leq \nu(\square, p) - p \cdot q + \mathcal{O}_{\frac{2}{2d-1}} \left(C|p||q| \operatorname{size}(\square)^{-\frac{1}{2}} \right) \\ &\leq C|p|^2 - p \cdot q + \mathcal{O}_{\frac{2}{2d-1}} \left(C|p||q| \operatorname{size}(\square)^{-\frac{1}{2}} \right).\end{aligned}$$

Choosing $p = q/2C$ to minimize the first two terms on the right, we get

$$\mu(\square, q) \leq -\frac{1}{C}|q|^2 + \mathcal{O}_{\frac{2}{2d-1}} \left(C|q|^2 \operatorname{size}(\square)^{-\frac{1}{2}} \right),$$

which is (2.4.30).

To prove the lower bound for ν, we argue similarly: by (2.4.16) and (2.4.29), for every $p, q \in \mathbb{R}^d$, we have

$$\begin{aligned}\nu(\square, p) &\geq \mu(\square, q) + p \cdot q - \mathcal{O}_{\frac{2}{2d-1}} \left(C|p||q| \operatorname{size}(\square)^{-\frac{1}{2}} \right) \\ &\geq -C|q|^2 \left(\fint_{\square} \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2} \, dx \right) + p \cdot q - \mathcal{O}_{\frac{2}{2d-1}} \left(C|p||q| \operatorname{size}(\square)^{-\frac{1}{2}} \right).\end{aligned}$$

Optimize by taking $q := C^{-1} \left(\fint_{\square} \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2} \, dx \right)^{-1} p$ in order to maximize the first two terms on the right side, we get

$$\nu(\square, p) \geq |p|^2 \left(\frac{1}{C} \left(\fint_{\square} \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2} \, dx \right)^{-1} - \mathcal{O}_{\frac{2}{2d-1}} \left(C \operatorname{size}(\square)^{-\frac{1}{2}} \right) \right).$$

Estimating the first term on the right side of the previous line by (2.4.18), we obtain

$$\nu(\square, p) \geq |p|^2 \left(\frac{1}{C} - \mathcal{O}_{\frac{1}{2d+1}} \left(C \operatorname{size}(\square)^{-1} \right) - \mathcal{O}_{\frac{2}{2d-1}} \left(C \operatorname{size}(\square)^{-\frac{1}{2}} \right) \right).$$

Since $\nu(\square, p)$ is nonnegative (and thus bounded below almost surely), the previous line and (2.1.10) implies

$$\nu(\square, p) \geq |p|^2 \left(\frac{1}{C} - \mathcal{O}_{\frac{2}{2d+1}} \left(C \operatorname{size}(\square)^{-\frac{1}{2}} \right) \right). \qquad \square$$

The final result of this subsection concerns the approximate subadditivity of $-\mu$ and ν.

LEMMA 2.4.8. *For every $p, q \in \mathbb{R}^d$ and $m, n \in \mathbb{N}$ with $n \leq m$ and $\square \in \mathcal{T}_m$,*

$$\mu(\square, q) \geq 3^{-d(m-n)} \sum_{z \in 3^n \mathbb{Z}^d \cap \square} \mu(\square_n(z), q) - |q|^2 \mathcal{O}_{\frac{1}{2d-1}} \left(C|q|^2 3^{-\frac{n}{4}} \right) \tag{2.4.32}$$

and

$$\nu(\square, p) \leq 3^{-d(m-n)} \sum_{z \in 3^n \mathbb{Z}^d \cap \square} \nu(\square_n(z), p) + |p|^2 \mathcal{O}_{\frac{2}{2d-1}} \left(C 3^{-n} \right). \tag{2.4.33}$$

PROOF. We first give the proof of (2.4.32). Denote $u := u(\cdot, \square, q)$. Testing the definition of $\mu(z + \square_n, q)$ for $z \in 3^n \mathbb{Z}^d \cap \square$ with u gives

$$\mu(z + \square_n, q) \mathbb{1}_{\{z + \square_n \in \mathcal{P}_*\}} \leq \frac{1}{|\square_n|} \left(\frac{1}{2} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(z + \square_n)} - \langle q, \nabla [u]_{\mathcal{P}} \rangle_{z + \square_n} \right) \mathbb{1}_{\{z + \square_n \in \mathcal{P}_*\}}. \tag{2.4.34}$$

If we sum the right side over $z \in 3^n \mathbb{Z}^d \cap \square$, the result is close to $\mu(\square, q)$. There are two sources of error: (i) $\mathscr{C}_*(\square)$ contains edges which do not belong to any of the clusters $\mathscr{C}_*(z + \square_n)$; (ii) when we sum the second term in parentheses we miss the edges between two adjacent subcubes and edges deleted because $z + \square_n \notin \mathcal{P}_*$. We treat each of these errors in turn. The claim is that

$$\frac{1}{|\square|} \sum_{z \in 3^n \mathbb{Z}^d \cap \square} \left(\frac{1}{2} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(z+\square_n)} - \langle q, \nabla [u]_{\mathcal{P}} \rangle_{z+\square_n} \right) \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}} \leq \mu(\square, q) + \mathcal{O}_{\frac{1}{2d-1}} \left(C|q|^2 3^{-\frac{n}{4}} \right). \tag{2.4.35}$$

First, it is clear that while $\mathscr{C}_*(\square) \cap (z + \square_n)$ and $\mathscr{C}_*(z + \square_n)$ may be different, every open edges in the latter cluster belongs to the former. Therefore, since the quadratic form is nonnegative on each edge, we have, for every $z \in 3^n \mathbb{Z}^d \cap \square$,

$$\sum_{z \in 3^n \mathbb{Z}^d \cap \square} \frac{1}{2} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(z+\square_n)} \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}} \leq \frac{1}{2} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(\square)}.$$

Next, let V be the set of vertices $x \in \square$ with an adjacent edge $\{x, y\}$ such that $y \notin \square_n(x)$ or such that $\square_n(x) \notin \mathcal{P}_*$. It is clear that

$$|V| \leq C|\square| 3^{-n} + C \sum_{x \in \square} \mathbb{1}_{\{\mathrm{size}(\square_{\mathcal{P}}(x)) > 3^n\}} \leq |\square| \left(C 3^{-n} + \mathcal{O}_1 \left(3^{-n} \right) \right) \leq \mathcal{O}_1 \left(C |\square| 3^{-n} \right).$$

From this, the Hölder inequality and (2.4.24) with $\delta = \frac{1}{4}$, we get

$$\begin{aligned} \frac{1}{|\square|} & \left| \langle q, \nabla [u]_{\mathcal{P}} \rangle_{\square} - \sum_{z \in 3^n \mathbb{Z}^d \cap \square} \langle q, \nabla [u]_{\mathcal{P}} \rangle_{z+\square_n} \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}} \right| \\ & \leq \frac{C|q|}{|\square|} \int_V |\nabla [u]_{\mathcal{P}}| (x) \, dx \\ & \leq C|q| \left(\frac{|V|}{|\square|} \right)^{\frac{1}{2}} \left(\fint_{\square} |\nabla [u]_{\mathcal{P}}|^2 (x) \, dx \right)^{\frac{1}{2}} \\ & \leq C|q| \cdot \mathcal{O}_2 \left(C 3^{-\frac{n}{2}} \right) \cdot \mathcal{O}_{\frac{2}{4d-3}} \left(C|q| 3^{\frac{n}{4}} \right) \\ & \leq \mathcal{O}_{\frac{1}{2d-1}} \left(C|q|^2 3^{-\frac{n}{4}} \right). \end{aligned}$$

Combining the above yields (2.4.35). To obtain (2.4.32) from (2.4.34) and (2.4.35), we just recall that $\mu(z + \square_n, q) \mathbb{1}_{\{z+\square_n \notin \mathcal{P}_*\}} = 0$.

We turn to the proof of (2.4.33), which is only slightly different. Testing the definition of $\nu(z + \square_n, p)$ with $v := v(\cdot, \square, p)$ gives

$$\nu(z + \square_n, p) \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}} \geq \frac{1}{|\square_n|} \left(-\frac{1}{2} \langle \nabla v, \mathbf{a} \nabla v \rangle_{\mathscr{C}_*(z+\square_n)} + \langle p, \mathbf{a} \nabla v \rangle_{\mathscr{C}_*(z+\square_n)} \right) \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}}. \tag{2.4.36}$$

As above, we have

$$\sum_{z \in 3^n \mathbb{Z}^d \cap \square} \frac{1}{2} \langle \nabla v, \mathbf{a} \nabla v \rangle_{\mathscr{C}_*(z+\square_n)} \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}} \leq \frac{1}{2} \langle \nabla v, \mathbf{a} \nabla v \rangle_{\mathscr{C}_*(\square)}.$$

Let W denote the set of vertices $x \in \square$ with an edge $\{x, y\}$ belonging to the cluster $\mathscr{C}_*(\square)$ but not any of the clusters $\mathscr{C}_*(z + \square_n)$ for $z \in 3^n \mathbb{Z}^d$ satisfying $z + \square_n \in \mathcal{P}_*$. It is clear that W must be contained in the union of elements of $\mathcal{P}$ which touch the boundaries of one of the cubes $z + \square_n$ and those cubes $z + \square_n$ which do not belong to $\mathcal{P}_*$. Therefore, in view of (2.4.28),

$$\begin{aligned} |W| & \leq \left| \bigcup_{z \in 3^n \mathbb{Z}^d \cap \square} \partial_{\mathcal{P}} (z + \square_n) \right| + \left| \bigcup_{z \in 3^n \mathbb{Z}^d \cap \square, \ z+\square_n \notin \mathcal{P}_*} (z + \square_n) \right| \\ & \leq C \sum_{z \in 3^n \mathbb{Z}^d \cap \square} |\partial_{\mathcal{P}} (z + \square_n)| + C \sum_{x \in \square} \mathbb{1}_{\{\mathrm{size}(\square_{\mathcal{P}}(x)) > 3^n\}} \\ & \leq \mathcal{O}_1 \left(C |\square| 3^{-n} \right). \end{aligned} \tag{2.4.37}$$

By the previous inequality, the Hölder inequality and (2.4.21), we get

$$\begin{aligned}\frac{1}{|\square|}&\left|\langle p, \mathbf{a}\nabla v\rangle_{\mathscr{C}_*(\square)} - \sum_{z\in 3^n\mathbb{Z}^d\cap\square} \langle p, \mathbf{a}\nabla v\rangle_{\mathscr{C}_*(z+\square_n)} \mathbb{1}_{\{z+\square_n\in\mathcal{P}_*\}}\right| \\ &\leq \frac{C|p|}{|\square|}\int_{W\cap\mathscr{C}_*(\square)} \left|\nabla v \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|(x)\,dx \\ &\leq C|p|\left(\frac{|W|}{|\square|}\right)^{\frac12}\left(\frac{1}{|\square|}\int_{\mathscr{C}_*(\square)} \left|\nabla v\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right)^{\frac12} \\ &\leq C|p|^2\cdot\mathcal{O}_2\left(C3^{-\frac{n}{2}}\right) \\ &\leq \mathcal{O}_2\left(C|p|^2 3^{-\frac{n}{2}}\right).\end{aligned}$$

Combining these yields

$$\frac{1}{|\square|}\sum_{z\in 3^n\mathbb{Z}^d\cap\square}\left(-\frac12\langle\nabla v, \mathbf{a}\nabla v\rangle_{\mathscr{C}_*(z+\square_n)} + \langle p, \mathbf{a}\nabla v\rangle_{\mathscr{C}_*(z+\square_n)}\right)\mathbb{1}_{\{z+\square_n\in\mathcal{P}_*\}} \geq \nu(\square, q) - \mathcal{O}_2\left(C|p|^2 3^{-\frac{n}{2}}\right). \tag{2.4.38}$$

Combined with (2.4.36), this yields

$$\nu(\square, p) \leq 3^{-d(m-n)}\sum_{z\in 3^n\mathbb{Z}^d\cap\square}\nu(z+\square_n, p)\mathbb{1}_{\{z+\square_n\in\mathcal{P}_*\}} + \mathcal{O}_2\left(C|p|^2 3^{-\frac{n}{2}}\right).$$

This implies (2.4.33) since ν is nonnegative. □

2.4.3. Localization and approximate stationarity. We next define local and stationary versions of μ and ν and show that they are the same as the original quantities, up to a small error. For each $m, n \in \mathbb{N}$ satisfying $m > n$, we denote

$$\mathcal{T}_m^{(n)} := \left\{z + \square_m \,:\, z \in 3^n\mathbb{Z}^d\right\}$$

and define the following random family of good cubes

$$\mathcal{G}^{(n)} := \mathcal{G} \cup \left(\cup\left\{\square' \in \mathcal{T} \,:\, \operatorname{size}(\square') \geq 3^n\right\}\right),$$

where $\mathcal{G}$ is the set of cube cubes from Definition 2.2.9. We then apply Proposition 2.2.1 which gives two partitions $\mathcal{P}^{(n)}$ and $\mathcal{P}^{(n)}_{\mathrm{loc}}(\square)$. Notice that thanks to this construction we obtain a partition $\mathcal{P}^{(n)}$ which is stationary for translations of vectors within $3^n\mathbb{Z}^d$ (in particular, this will ensure that (2.4.40) holds) and will be important in Section 6. Before introducing the local and stationary versions of μ and ν, we prove a quantitative result, showing that $\mathcal{P}$ and $\mathcal{P}^{(n)}$ are equals on □ on a set of large probability.

Proposition 2.4.9. *For each $m, n \in \mathbb{N}$ satisfying $m > n$, $3^n > Cm$ and $\square \in \mathcal{T}_m^{(n)}$, the following estimates holds*

$$\mathbb{P}\left[\forall x \in \square,\, \square_{\mathcal{P}}(x) = \square_{\mathcal{P}^{(n)}}(x)\right] \geq 1 - C\exp\left(-C^{-1}3^n\right).$$

Proof. For each $x \in \square$, we have

$$\square_{\mathcal{P}}(x) \neq \square_{\mathcal{P}^{(n)}}(x) \iff \exists \square' \in \mathcal{K}\left(\square_{\mathcal{P}^{(n)}}(x)\right) \setminus \mathcal{G}$$

Thus, with a similar argument as in the proof of (iv) of Proposition 2.2.1

$$\exists x \in \square,\, \square_{\mathcal{P}}(x) \neq \square_{\mathcal{P}^{(n)}}(x) \implies \exists \square' \notin \mathcal{G},\, \operatorname{size}(\square') \geq 3^n \text{ and } \operatorname{dist}(\square', \square) \leq C\operatorname{size}(\square')$$

and thus

$$\begin{aligned}\mathbb{P}\left[\forall x \in \square,\, \square_{\mathcal{P}}(x) \neq \square_{\mathcal{P}^{(n)}}(x)\right] &\leq C3^{m-n}\exp\left(-C^{-1}3^n\right) \\ &\leq C\exp\left(-C^{-1}3^n\right).\end{aligned}$$

The proof is complete. □

Now, for $\square \in \mathcal{T}_m^{(n)}$, consider the event

$$A_n(\square) := \left\{ \sup_{\square' \in \mathcal{P}_{\mathrm{loc}}^{(n)}(\square)} \operatorname{size}(\square') \le 3^{n-1} \right\}.$$

Observe that $A_n(\square) \in \mathcal{F}(\square)$. As in Section 2.2, we define, for $n \in \mathbb{N}$,

$$\square^{(n)} := \left\{ x \in \square \, : \, \operatorname{dist}(x, \partial \square) \ge 3^n \right\}.$$

Note that on the event $A_n(\square)$, by definition of the local partition $\mathcal{P}_{\mathrm{loc}}^{(n)}$ in Proposition 2.2.1, every cube belonging to $\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)$ contained in $\square^{(n)}$ is good (in the sense of Definition 2.2.6), so that the union of the clusters $\left(\mathscr{C}_* \left(\square_{\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)}(x) \right) \right)_{x \in \square^{(n)}}$ over $x \in \square^{(n)}$ is connected. We then define the local cluster $\mathscr{C}_*^{(n)}(\square)$ as the maximal cluster which is a subset of $\square$ and which contains every cluster of the form $\left(\mathscr{C}_* \left(\square_{\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)}(x) \right) \right)_{x \in \square^{(n)}}$ with $x \in \square^{(n)}$. If $A_n(\square)$ fails to hold, we define $\mathscr{C}_*^{(n)}(\square)$ to be empty.

From the local cluster we define the local versions of the energy quantities by

$$\mu_{\mathrm{loc}}^{(n)}(\square, q) := \mathbb{1}_{A_n(\square)} \inf_{u \in \mathcal{A}\left(\mathscr{C}_*^{(n)}(\square)\right)} \frac{1}{|\,\square\,|} \left(\frac{1}{2} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*^{(n)}(\square)} - \left\langle q, \nabla [u]_{\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)} \right\rangle_{\square^{(n)}} \right)$$

and

$$\nu_{\mathrm{loc}}^{(n)}(\square, q) := \mathbb{1}_{A_n(\square)} \sup_{v \in \mathcal{A}\left(\mathscr{C}_*^{(n)}(\square)\right)} \frac{1}{|\,\square\,|} \left(-\frac{1}{2} \left\langle \nabla v, \mathbf{a} \nabla v \right\rangle_{\mathscr{C}_*^{(n)}(\square)} + \left\langle p, \mathbf{a} \nabla v \right\rangle_{\mathscr{C}_*^{(n)}(\square)} \right).$$

In other words, $\mu_{\mathrm{loc}}^{(n)}(\square, q)$ is the same as $\mu(\square, q)$ except that we use the local partition $\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)$ instead of $\mathcal{P}$ and that we integrate the second term only on $\square^{(n)}$; meanwhile $\nu_{\mathrm{loc}}^{(n)}(\square, q)$ is the same as $\nu(\square, q)$, except in the event (which is unlikely for $\square$ large and n large) that $A_n(\square)$ does not hold.

It is immediate from Proposition 2.2.1(iv), which gives the locality of $\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)$, that for every $\square \in \mathcal{T}_m^{(n)}$ and $p, q \in \mathbb{R}^d$,

$$\mu_{\mathrm{loc}}^{(n)}(\square, q) \text{ and } \nu_{\mathrm{loc}}^{(n)}(\square, p) \quad \text{are } \mathcal{F}(\square)\text{–measurable.} \tag{2.4.39}$$

It is clear that the construction above yields that $\mu_{\mathrm{loc}}^{(n)}(\square_m, q)$ and $\nu_{\mathrm{loc}}^{(n)}(\square_m, p)$ are $3^m\mathbb{Z}^d$–stationary and thus

$$\text{the laws of } \mu_{\mathrm{loc}}^{(n)}(z + \square_m, q) \text{ and } \nu_{\mathrm{loc}}^{(n)}(z + \square_m, p) \text{ are independent of } z \in 3^n\mathbb{Z}^d. \tag{2.4.40}$$

In particular, $\left\{ \mu_{\mathrm{loc}}^{(n)}(z + \square_m, q) \right\}_{z \in 3^m\mathbb{Z}^d}$ and $\left\{ \nu_{\mathrm{loc}}^{(n)}(z + \square_m, p) \right\}_{z \in 3^m\mathbb{Z}^d}$ are i.i.d.

We denote by $u_{\mathrm{loc}}^{(n)}(\cdot, \square, q)$ and $v_{\mathrm{loc}}^{(n)}(\cdot, \square, q)$ the optimizers in the definitions of $\mu_{\mathrm{loc}}^{(n)}(\square, q)$ and $\nu_{\mathrm{loc}}^{(n)}(\square, p)$, respectively. We choose the additive constant in the same way as above, so that (2.4.2) holds. In the event that $A_n(\square)$ does not hold, we define $u_{\mathrm{loc}}^{(n)}(\cdot, \square, q) = v_{\mathrm{loc}}^{(n)}(\cdot, \square, p) = 0$.

Most of the estimates that we proved in the previous subsection continue to hold for the localized quantities. In particular, since $\mathcal{P}_{\mathrm{loc}}^{(n)}$ is finer than $\mathcal{P}^{(n)}$ which is finer than $\mathcal{P}$, we have, by the same proof as the one in Lemma 2.4.4, the bounds

$$\begin{cases} -\mu_{\mathrm{loc}}^{(n)}(\square, q) \le C|q|^2 \fint_{\square} \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2} \, dx, \\ \nu_{\mathrm{loc}}^{(n)}(\square, p) \le C|p|^2. \end{cases} \tag{2.4.41}$$

We also record the fact that, by the same argument as the one leading to (2.4.20), (2.4.21) and (2.4.24), we have the estimates

$$\begin{aligned} &\frac{1}{|\square|}\int_{\mathscr{C}_*(\square)} \left|\nabla u_{\mathrm{loc}}^{(n)}(\cdot,\square,q)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx \le C|q|^2 \fint_{\square} \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)^{2d-2}\,dx, \\ &\frac{1}{|\square|}\int_{\mathscr{C}_*^{(n)}(\square)} \left|\nabla v_{\mathrm{loc}}^{(n)}(\cdot,\square,p)\right|^2(x)\,dx \le C|p|^2, \end{aligned} \tag{2.4.42}$$

We next estimate the difference between μ and $\mu_{\mathrm{loc}}^{(n)}$ as well as ν and $\nu_{\mathrm{loc}}^{(n)}$ and their minimizers.

Proposition 2.4.10. *There exists $C(d,\lambda,\mathfrak{p})<\infty$ such that, for every $m,n\in\mathbb{N}$ with $m>n$ and $3^n\ge Cm$, every $\square\in\mathcal{T}_m^{(n)}$ and $p,q\in\mathbb{R}^d$, we have*

$$\begin{cases} \left|\mu(\square,q)-\mu_{\mathrm{loc}}^{(n)}(\square,q)\right| \le \mathcal{O}_{\frac{1}{2d-1}}\left(C|q|^2\left(3^{-\frac{m-n}{2}}+3^{-n}\right)\right), \\ \left|\nu(\square,p)-\nu_{\mathrm{loc}}^{(n)}(\square,p)\right| \le \mathcal{O}_1\left(C|p|^2 3^{-n}\right). \end{cases} \tag{2.4.43}$$

and

$$\begin{aligned} \frac{\mathbb{1}_{\left\{\mathscr{C}_*(\square)=\mathscr{C}_*^{(n)}(\square)\right\}}}{|\square|}\int_{\mathscr{C}_*(\square)} &\left|\left(\nabla u(\cdot,\square,q)-\nabla u_{\mathrm{loc}}^{(n)}(\cdot,\square,q)\right)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx \\ &\le \mathcal{O}_{\frac{1}{2d-1}}\left(C|q|^2\left(3^{-\left(\frac{m-n}{2}\right)}+3^{-n}\right)\right). \end{aligned} \tag{2.4.44}$$

Proof. First note that on the event $\{\sup_{x\in\square}\operatorname{size}(\square_{\mathcal{P}}(x))\le 3^n\}$, $\mathscr{C}_*^{(n)}(\square)=\mathscr{C}_*(\square)$. Moreover, since $\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)$ is finer than $\mathcal{P}$, we have

$$\left\{\sup_{x\in\square}\operatorname{size}(\square_{\mathcal{P}}(x))\le 3^n\right\}\subseteq A_n(\square).$$

By Proposition 2.2.1 (iii) and the assumption $3^n\ge Cm$, we can estimate the probability of this event by

$$\mathbb{P}\left[\sup_{x\in\square}\operatorname{size}(\square_{\mathcal{P}}(x))>3^n\right]\le 3^{dm}\exp(-C^{-1}3^n)\le C\exp\left(-C^{-1}3^n\right). \tag{2.4.45}$$

It is clear that $\nu_{\mathrm{loc}}^{(n)}(\square,p)=\nu(\square,p)$ if $\mathscr{C}_*^{(n)}(\square)=\mathscr{C}_*(\square)$ and $A_n(\square)$ holds. Thus on the event $\{\sup_{x\in\square}\operatorname{size}(\square_{\mathcal{P}}(x))\le 3^n\}$, $\nu_{\mathrm{loc}}^{(n)}(\square,p)=\nu(\square,p)$. Using the bounds (2.4.13) and (2.4.41), we therefore obtain

$$\begin{aligned} \left|\nu(\square,p)-\nu_{\mathrm{loc}}^{(n)}(\square,p)\right| &\le \left(|\nu(\square,p)|+\left|\nu_{\mathrm{loc}}^{(n)}(\square,p)\right|\right)\mathbb{1}_{\{\sup_{x\in\square}\operatorname{size}(\square_{\mathcal{P}}(x))>3^n\}} \\ &\le C|p|^2\mathbb{1}_{\{\sup_{x\in\square}\operatorname{size}(\square_{\mathcal{P}}(x))>3^n\}} \\ &\le \mathcal{O}_1\left(C|p|^2 3^{-n}\right). \end{aligned}$$

We turn to the bound for μ. Denote by $B_n(\square)$ the event

$$B_n(\square):=\left\{\sup_{x\in\square}\operatorname{size}(\square_{\mathcal{P}}(x))\le 3^n\right\}\bigcap\left\{\forall x\in\square^{(n)},\ \square_{\mathcal{P}}(x)=\square_{\mathcal{P}_{\mathrm{loc}}^{(n)}}(x)\right\}.$$

By (iv) of Proposition 2.2.1, Proposition 2.4.9, and (2.4.45), we can estimate

$$\begin{aligned} \mathbb{P}\left[B_n(\square)\right] &\ge 1-3^{dm}\exp\left(-C^{-1}3^n\right)-C3^{m-n}\exp\left(-C^{-1}3^n\right) \\ &\ge 1-C\exp\left(-C^{-1}3^n\right). \end{aligned}$$

On this event, we can use the function $u(\cdot,\square,q)$ as a minimizer candidate for $\mu_{\mathrm{loc}}^{(n)}(\square,q)$. This yields

$$\begin{aligned}\mu_{\mathrm{loc}}(\square,q) &\geq \frac{1}{|\square|}\left(\frac{1}{2}\langle\nabla u,\mathbf{a}\nabla u\rangle_{\mathscr{C}_*(\square)} - \left\langle q,\nabla[u]_{\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)}\right\rangle_{\square^{(n)}}\right)\\ &\geq \frac{1}{|\square|}\left(\frac{1}{2}\langle\nabla u,\mathbf{a}\nabla u\rangle_{\mathscr{C}_*(\square)} - \langle q,\nabla[u]_{\mathcal{P}}\rangle_{\square^{(n)}}\right)\\ &\geq \mu(\square,q) - \left|\frac{1}{|\square|}\langle q,\nabla[u]_{\mathcal{P}}\rangle_{\square\smallsetminus\square^{(n)}}\right|\\ &\geq \mu(\square,q) - |q|\frac{1}{|\square|}\int_{\square\smallsetminus\square^{(n)}}|\nabla[u(\cdot,\square,q)]_{\mathcal{P}}|(x)\,dx.\end{aligned}$$

To estimate the last term on the right-hand side, we can extract from the proof of Lemma 2.3.3 with $s=1$ the following inequality

$$\begin{aligned}&\int_{\square\smallsetminus\square^{(n)}}\left|\nabla\left[u_{\mathrm{loc}}^{(n)}(\cdot,\square,q)\right]_{\mathcal{P}}\right|(x)\,dx\\ &\quad\leq C\sum_{\square'\in\mathcal{P},\square'\subseteq\square\smallsetminus\square^{(n-1)}}\mathrm{size}(\square')^{d-1}\int_{\square'\cap\mathscr{C}_*(\square)}\left|\nabla u_{\mathrm{loc}}^{(n)}\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|(x)\,dx\\ &\quad\leq C\left(\sum_{\square'\in\mathcal{P},\square'\subseteq\square\smallsetminus\square^{(n-1)}}\mathrm{size}(\square)^{3d-2}\right)^{\frac12}\left(\int_{(\square\smallsetminus\square^{(n-1)})\cap\mathscr{C}_*(\square)}\left|\nabla u_{\mathrm{loc}}^{(n)}\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right)^{\frac12}\\ &\quad\leq C\left(\int_{\square\smallsetminus\square^{(n-1)}}\mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\,dx\right)^{\frac12}\left(\int_{\mathscr{C}_*(\square)}\left|\nabla u_{\mathrm{loc}}^{(n)}\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right)^{\frac12}\end{aligned}$$

This gives

$$\begin{aligned}&\frac{1}{|\square|}\int_{\square\smallsetminus\square^{(n)}}\left|\nabla\left[u_{\mathrm{loc}}^{(n)}(\cdot,\square,q)\right]_{\mathcal{P}}\right|(x)\,dx\\ &\qquad\leq C3^{\frac{n-m}{2}}\left(\frac{1}{\left|\square\smallsetminus\square^{(n-1)}\right|}\int_{\square\smallsetminus\square^{(n-1)}}\mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\,dx\right)^{\frac12}\\ &\qquad\qquad\times\left(\frac{1}{|\square|}\int_{\mathscr{C}_*(\square)}\left|\nabla u_{\mathrm{loc}}^{(n)}\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right)^{\frac12}.\end{aligned}$$

This yields by (2.1.8) and (2.4.42)

$$\begin{aligned}\mu_{\mathrm{loc}}(\square,q)\mathbb{1}_{B_n(\square)} &\geq \mu(\square,q)\mathbb{1}_{B_n(\square)} - C3^{\frac{n-m}{2}}\mathcal{O}_{\frac{1}{d-1}}(C)\mathcal{O}_{\frac{1}{d-1}}(C)\\ &\geq \mu(\square,q)\mathbb{1}_{B_n(\square)} - \mathcal{O}_{\frac{1}{2d-2}}\left(C|q|^2 3^{\frac{n-m}{2}}\right).\end{aligned}\tag{2.4.46}$$

Similarly and still on the event $B_n(\square)$, using $u_{\mathrm{loc}}^{(n)}$ as a minimizer candidate for $\mu(\square,q)$ shows

$$\begin{aligned}\mu(\square,q) &\geq \frac{1}{|\square|}\left(\frac12\langle\nabla u_{\mathrm{loc}},\mathbf{a}\nabla u_{\mathrm{loc}}\rangle_{\mathscr{C}_*(\square)} - \langle q,\nabla[u_{\mathrm{loc}}]_{\mathcal{P}}\rangle_{\square}\right)\\ &\geq \mu_{\mathrm{loc}}(\square,q) - \left|\frac{1}{|\square|}\langle q,\nabla[u_{\mathrm{loc}}]_{\mathcal{P}}\rangle_{\square\smallsetminus\square^{(n)}}\right|\\ &\geq \mu_{\mathrm{loc}}(\square,q) - \frac{|q|}{|\square|}\int_{\square\smallsetminus\square^{(n)}}|\nabla[u_{\mathrm{loc}}(\cdot,\square,q)]_{\mathcal{P}}|(x)\,dx.\end{aligned}$$

Thus

$$\mu(\square,q)\mathbb{1}_{B_n(\square)} \geq \mu_{\mathrm{loc}}^{(n)}(\square,q)\mathbb{1}_{B_n(\square)} - \mathcal{O}_{\frac{1}{2d-2}}(C|q|^2 3^{\frac{n-m}{2}}).\tag{2.4.47}$$

Combining (2.4.46) and (2.4.47) gives

$$\left|\mu(\square,q)-\mu_{\mathrm{loc}}^{(n)}(\square,q)\right|\mathbb{1}_{B_n(\square)}\le\mathcal{O}_{\frac{1}{2d-2}}\left(C|q|^2 3^{\frac{n-m}{2}}\right)$$

On the event $\Omega\smallsetminus B_n(\square)$, we have by (2.4.41),

$$\begin{aligned}\left|\mu(\square,q)-\mu_{\mathrm{loc}}^{(n)}(\square,q)\right|\mathbb{1}_{\{\Omega\smallsetminus B_n(\square)\}}&\le\left(|\mu(\square,q)|+\left|\mu_{\mathrm{loc}}^{(n)}(\square,q)\right|\right)\mathbb{1}_{\{\Omega\smallsetminus B_n(\square)\}}\\&\le\mathcal{O}_{\frac{1}{2d-2}}(C|q|^2)\mathcal{O}_1\left(C|q|^2 3^{-n}\right)\\&\le\mathcal{O}_{\frac{1}{2d-1}}\left(C|q|^2 3^{-n}\right).\end{aligned}$$

Summing the two previous displays completes the proof of (2.4.43).

We now turn to the proof of (2.4.44). We apply the second variations (2.4.4) to $w=u_{\mathrm{loc}}^{(n)}(\cdot,\square,q)$, on the event $B_n(\square)$, which yields

$$\begin{aligned}&\frac{1}{|\square|}\left(\frac{1}{2}\left\langle\nabla u_{\mathrm{loc}},\mathbf{a}\nabla u_{\mathrm{loc}}^{(n)}\right\rangle_{\mathscr{C}_*(\square)}-\left\langle q,\nabla\left[u_{\mathrm{loc}}^{(n)}\right]_{\mathcal{P}}\right\rangle_{\square}\right)\\&\qquad=\mu(\square,q)+\frac{1}{|\square|}\frac{1}{2}\left\langle\nabla u_{\mathrm{loc}}^{(n)}-\nabla u,\mathbf{a}\left(\nabla u_{\mathrm{loc}}^{(n)}-\nabla u\right)\right\rangle_{\mathscr{C}_*(\square)}.\end{aligned}$$

On the other hand, we have the estimate

$$\begin{aligned}&\frac{1}{|\square|}\left(\frac{1}{2}\left\langle\nabla u_{\mathrm{loc}}^{(n)},\mathbf{a}\nabla u_{\mathrm{loc}}^{(n)}\right\rangle_{\mathscr{C}_*(\square)}-\left\langle q,\nabla\left[u_{\mathrm{loc}}^{(n)}\right]_{\mathcal{P}}\right\rangle_{\square}\right)\\&\qquad\le\mu_{\mathrm{loc}}^{(n)}(\square,q)+\frac{|q|}{|\square|}\int_{\square\smallsetminus\square^{(n)}}\left|\nabla\left[u_{\mathrm{loc}}^{(n)}(\cdot,\square,q)\right]_{\mathcal{P}}\right|(x)\,dx.\end{aligned}$$

Combining the two previous displays with the same computation as the one leading to (2.4.47), we obtain

$$\begin{aligned}&\frac{1}{|\square|}\left\langle\nabla u_{\mathrm{loc}}^{(n)}-\nabla u,\mathbf{a}\left(\nabla u_{\mathrm{loc}}^{(n)}-\nabla u\right)\right\rangle_{\mathscr{C}_*(\square)}\mathbb{1}_{B_n(\square)}\\&\qquad\le 2\left|\mu_{\mathrm{loc}}^{(n)}(\square,q)-\mu(\square,q)\right|+\mathcal{O}_{\frac{1}{2d-2}}\left(C|q|^2 3^{\frac{n-m}{2}}\right)\le\mathcal{O}_{\frac{1}{2d-2}}\left(C|q|^2 3^{\frac{n-m}{2}}\right).\end{aligned}$$

On the event $\left\{\mathscr{C}_*(\square)=\mathscr{C}_*^{(n)}(\square)\right\}\smallsetminus B_n(\square)$, we have, by (2.4.20) and (2.4.42),

$$\begin{aligned}&\frac{\mathbb{1}_{\left\{\left\{\mathscr{C}_*(\square)=\mathscr{C}_*^{(n)}(\square)\right\}\smallsetminus B_n(\square)\right\}}}{|\square|}\left\langle\nabla u_{\mathrm{loc}}^{(n)}-\nabla u,\mathbf{a}\left(\nabla u_{\mathrm{loc}}^{(n)}-\nabla u\right)\right\rangle_{\mathscr{C}_*(\square)}\\&\qquad\le\mathcal{O}_{\frac{1}{2d-2}}(C|q|^2)\mathcal{O}_1\left(C3^{-n}\right)\le\mathcal{O}_{\frac{1}{2d-1}}\left(C|q|^2 3^{-n}\right).\end{aligned}$$

Combining the two previous displays and recalling that $\mathbf{a}\ge\lambda\mathbb{1}_{\{\mathbf{a}\neq0\}}$ completes the proof of (2.4.44). □

We conclude this section by recording some consequences of Proposition 2.4.10, for our reference. According to (2.4.40) and (2.4.43) with $n=\left\lceil\frac{m}{2}\right\rceil$, we have, for every $m\in\mathbb{N}$,

$$\sup_{z\in3^m\mathbb{Z}^d}\left|\mathbb{E}\left[\mu(z+\square_m,q)\right]-\mathbb{E}\left[\mu(\square_m,q)\right]\right|\le C|q|^2 3^{-\frac{m}{4}}.$$

Combining this with (2.4.32), we obtain, for each $m\in\mathbb{N}$,

$$\mathbb{E}\left[\mu(\square_{m+1},q)\right]\ge\mathbb{E}\left[\mu(\square_m,q)\right]-C|q|^2 3^{-\frac{m}{4}}.$$

Summing this from n to $m-1$ yields that, for every $m,n\in\mathbb{N}$ with $n<m$,

$$\mathbb{E}\left[\mu(\square_m,q)\right]\ge\mathbb{E}\left[\mu(\square_n,q)\right]-C|q|^2 3^{-\frac{n}{4}}.\tag{2.4.48}$$

By a similar argument, we have the bound

$$\mathbb{E}\left[\nu(\square_m,p)\right]\le\mathbb{E}\left[\nu(\square_n,p)\right]+C|p|^2 3^{-n}.\tag{2.4.49}$$

As a consequence of (2.4.44), we also get a localization estimate for the coarsened functions, summarized in the following lemma.

Lemma 2.4.11. *There exist* $C(d,\lambda,\mathfrak{p})<\infty$ *and* $s(d)>0$ *such that, for every* $m,n\in\mathbb{N}$ *with* $3^n\geq Cm$, *every* $\square\in\mathcal{T}_m^{(n)}$ *and* $p,q\in\mathbb{R}^d$, *we have*

$$\fint_{\square^{(n)}}\left|\nabla\left[u(\cdot,\square,q)\right]_{\mathcal{P}}-\nabla\left[u_{\mathrm{loc}}^{(n)}(\cdot,\square,q)\right]_{\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)}\right|(x)\,dx\leq\mathcal{O}_s\left(C|q|\left(3^{-\left(\frac{m-n}{4}\right)}+3^{-\frac{n}{4}}\right)\right). \tag{2.4.50}$$

Proof. Write $u:=u(\cdot,\square,q)$ and $u_{\mathrm{loc}}^{(n)}:=u_{\mathrm{loc}}^{(n)}(\cdot,\square,q)$ for short. We recall the definition of the event $B_n(\square)$:

$$B_n(\square):=\left\{\sup_{x\in\square}\mathrm{size}(\square_{\mathcal{P}}(x))\leq 3^n\right\}\bigcap\left\{\forall x\in\square^{(n)},\ \square_{\mathcal{P}}(x)=\square_{\mathcal{P}_{\mathrm{loc}}^{(n)}}(x)\right\}.$$

We also record that for some $C:=C(d,\mathfrak{p},\lambda)<+\infty$,

$$\mathbb{1}_{\Omega\setminus B_n(\square)}\leq\mathcal{O}_1\left(C3^{-n}\right).$$

We split the left side of (2.4.50):

$$\begin{aligned}\fint_{\square^{(n)}}&\left|\nabla\left[u\right]_{\mathcal{P}}-\nabla\left[u_{\mathrm{loc}}^{(n)}\right]_{\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)}\right|(x)\,dx\\&=\left(\fint_{\square^{(n)}}\left|\nabla\left[u\right]_{\mathcal{P}}-\nabla\left[u_{\mathrm{loc}}^{(n)}\right]_{\mathcal{P}}\right|(x)\,dx\right)\cdot\mathbb{1}_{B_n(\square)}\\&\quad+\left(\fint_{\square^{(n)}}\left|\nabla\left[u\right]_{\mathcal{P}}-\nabla\left[u_{\mathrm{loc}}^{(n)}\right]_{\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)}\right|(x)\,dx\right)\cdot\mathbb{1}_{\Omega\setminus B_n(\square)}.\end{aligned}\tag{2.4.51}$$

We estimate the first term on the right side of (2.4.51) using (2.3.3) and (2.4.44): for some $s(d)>0$, we have,

$$\begin{aligned}&\fint_{\square^{(n)}}\left|\nabla\left[u\right]_{\mathcal{P}}-\nabla\left[u_{\mathrm{loc}}^{(n)}\right]_{\mathcal{P}}\right|(x)\,dx\cdot\mathbb{1}_{B_n(\square)}\\&\leq\frac{C\mathbb{1}_{B_n(\square)}}{\left|\square^{(n)}\right|}\sum_{\mathcal{P}\ni\square'\subseteq\square^{(n)}}\mathrm{size}(\square')^{d-1}\int_{\square'\cap\mathscr{C}_*(\square^{(n)})}\left|\left(\nabla u-\nabla u_{\mathrm{loc}}^{(n)}\right)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|(x)\,dx\\&\leq\frac{C\mathbb{1}_{B_n(\square)}}{\left|\square^{(n)}\right|}\left(\sum_{\mathcal{P}\ni\square'\subseteq\square^{(n)}}\mathrm{size}(\square')^{3d-2}\right)^{\frac12}\left(\sum_{\mathcal{P}\ni\square'\subseteq\square^{(n)}}\int_{\square'\cap\mathscr{C}_*(\square^{(n)})}\left|\left(\nabla u-\nabla u_{\mathrm{loc}}^{(n)}\right)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\right)^{\frac12}\\&=C\mathbb{1}_{B_n(\square)}\left(\frac{1}{\left|\square^{(n)}\right|}\sum_{x\in\square^{(n)}}\mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\right)^{\frac12}\left(\frac{1}{\left|\square^{(n)}\right|}\int_{\mathscr{C}_*(\square^{(n)})}\left|\left(\nabla u-\nabla u_{\mathrm{loc}}^{(n)}\right)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\right)^{\frac12}\\&\leq C\mathbb{1}_{B_n(\square)}\left(\frac{1}{\left|\square^{(n)}\right|}\sum_{x\in\square^{(n)}}\mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\right)^{\frac12}\left(\frac{1}{|\square|}\int_{\mathscr{C}_*(\square)}\left|\left(\nabla u-\nabla u_{\mathrm{loc}}^{(n)}\right)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\right)^{\frac12}\\&\leq C\mathcal{O}_{\frac{2}{2d-1}}(C)\,\mathcal{O}_{\frac{2}{2d-1}}\left(C|q|\left(3^{-\left(\frac{m-n}{4}\right)}+3^{-\frac{n}{2}}\right)\right)\\&\leq\mathcal{O}_s\left(C|q|\left(3^{-\left(\frac{m-n}{4}\right)}+3^{-\frac{n}{2}}\right)\right),\end{aligned}$$

where we used that, for $m>n$, we have $|\square|\leq 3^d\left|\square^{(n)}\right|$, $\mathbb{1}_{B_n(\square)}\leq\mathbb{1}_{\left\{\mathscr{C}_*(\square)=\mathscr{C}_*^{(n)}(\square)\right\}}$ and $\mathscr{C}_*\left(\square^{(n)}\right)\subseteq\mathscr{C}_*(\square)$.

To estimate the second term on the right side of (2.4.51), notice that if $\mathrm{cl}_{\mathcal{P}}(\square)\neq\square$ then $\nabla\left[u\right]_{\mathcal{P}}$ is constant. This remark and (2.4.22) yields

$$\fint_{\square^{(n)}}\left|\nabla\left[u\right]_{\mathcal{P}}\right|(x)\,dx\leq C\fint_{\square}\left|\nabla\left[u\right]_{\mathcal{P}}\right|(x)\,dx\leq C|q|\fint_{\square}\mathrm{size}(\square_{\mathcal{P}}(x))^{2d-2}\,dx\leq\mathcal{O}_{\frac{1}{2d-2}}(C|q|).$$

A similar computation gives the same result with $\left[u_{\mathrm{loc}}^{(n)}\right]_{\mathcal{P}_{\mathrm{loc}}^{(n)}}$ instead of $[u]_{\mathcal{P}}$. Thus we have, for some (possibly smaller) exponent $s(d) > 0$,

$$\begin{aligned}
&\fint_{\square^{(n)}} \left| \nabla [u]_{\mathcal{P}} - \nabla \left[u_{\mathrm{loc}}^{(n)}\right]_{\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)} \right| (x)\, dx \cdot \mathbb{1}_{\Omega \setminus B_n(\square)} \\
&\quad \le \left(\fint_{\square^{(n)}} \left|\nabla [u]_{\mathcal{P}}\right| (x) + \left| \nabla \left[u_{\mathrm{loc}}^{(n)}\right]_{\mathcal{P}_{\mathrm{loc}}^{(n)}(\square)} \right| (x)\, dx \right) \cdot \mathbb{1}_{\Omega \setminus B_n(\square)} \\
&\quad \le \mathcal{O}_{\frac{1}{2d-2}}(C|q|) \mathcal{O}_1(C3^{-n}) \\
&\quad \le \mathcal{O}_s\left(C|q|3^{-n}\right).
\end{aligned}$$

This completes the proof of (2.4.50). □

2.5. Convergence of the subadditive quantities

An immediate consequence of the approximate subadditivity (2.4.33) and stationarity stated in (2.4.40), (2.4.43) is the approximate monotonicity of $n \mapsto \mathbb{E}\left[\nu(\square_n, p)\right]$: we have that, for every $p \in \mathbb{R}^d$ and $m, n \in \mathbb{N}$ with $n \le m$,

$$\mathbb{E}\left[\nu(\square_m, p)\right] \le \mathbb{E}\left[\nu(\square_n, p)\right] + C|p|^2 3^{-n}.$$

It follows that, for each $p \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \mathbb{E}\left[\nu(\square_m, p)\right] \quad \text{exists.}$$

Since $p \mapsto \mathbb{E}\left[\nu(\square_m, p)\right]$ is a quadratic form which, for sufficiently large n, is bounded above and below by multiples of $|p|^2$ by (2.4.13) and (2.4.31), the same is true of $\lim_{n\to\infty} \mathbb{E}\left[\nu(\square_m, p)\right]$. This allows us to make the following definition.

DEFINITION 2.5.1 (Homogenized diffusion matrix $\bar{\mathbf{a}}$). We define $\bar{\mathbf{a}}$ to be the unique (deterministic) positive matrix $\bar{\mathbf{a}}$

$$\frac{1}{2} p \cdot \bar{\mathbf{a}} p = \lim_{n\to\infty} \mathbb{E}\left[\nu(\square_m, p)\right].$$

By the bounds (2.4.13) and (2.4.31) on ν, there exist $0 < c(d, \lambda, \mathfrak{p}) \le C(d, \lambda) < \infty$ such that

$$cI_d \le \bar{\mathbf{a}} \le CI_d. \tag{2.5.1}$$

Arguing in a similar manner, we can show that $\mathbb{E}\left[-\mu(\square_n, q)\right]$ also has a limit as $n \to \infty$ to a quadratic function in q which is bounded above and below by $|q|^2$. As we will prove in this section, that quadratic form turns out to be $q \mapsto \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q$, the convex dual of the quadratic form $p \mapsto \frac{1}{2} p \cdot \bar{\mathbf{a}} p$. Moreover, by the approximate localization property (2.4.43), one can argue as in the proof of the subadditive ergodic theorem that these quantities converge $\mathbb{P}$–a.s. to these deterministic constants.

The main result of this section is a quantitative rate of convergence for the subadditive quantities to their limits, which is summarized in the following proposition.

PROPOSITION 2.5.2. *There exist $s(d) > 0$, $\alpha(d, \mathfrak{p}, \lambda) \in \left(0, \frac{1}{4}\right]$ and $C(d, \mathfrak{p}, \lambda) < \infty$ such that, for every $\square \in \mathcal{T}$ and $p, q \in \mathbb{R}^d$,*

$$\left| \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q + \mu(\square, q) \right| \le \mathcal{O}_s\left(C|q|^2 \operatorname{size}(\square)^{-\alpha}\right) \tag{2.5.2}$$

and

$$\left| \frac{1}{2} p \cdot \bar{\mathbf{a}} p - \nu(\square, p) \right| \le \mathcal{O}_s\left(C|p|^2 \operatorname{size}(\square)^{-\alpha}\right). \tag{2.5.3}$$

The proof of Proposition 2.5.2 is an adaptation of arguments in [**21**]. The main step is to control the *expectations* of the quantities under the absolute value signs in (2.5.2) and (2.5.3). That is, we want to show that there exists an exponent $\alpha(d,\mathfrak{p},\lambda) > 0$ such that, for every $n \in \mathbb{N}$ and $p, q \in \mathbb{R}^d$,

$$\left|\mathbb{E}\left[\mu(\square_n, q)\right] + \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q\right| + \left|\mathbb{E}\left[\nu(\square_n, p)\right] - \frac{1}{2} p \cdot \bar{\mathbf{a}} p\right| \le C\left(|p|^2 + |q|^2\right) 3^{-n\alpha}. \tag{2.5.4}$$

Once this is accomplished, we obtain the conclusion of Proposition 2.5.2 by gaining stochastic integrability via a straightforward use of subadditivity and independence.

It may appear from (2.5.4) that we have two estimates to prove, but one of the insights from [**21**] is that it is really just one estimate. Indeed, let us consider the quantity $\omega(\square, q)$, defined for each $\square \in \mathcal{T}$ and $q \in \mathbb{R}^d$ by

$$\omega(\square, q) := \nu\left(\square, \bar{\mathbf{a}}^{-1} q\right) - \mu(\square, q) - q \cdot \bar{\mathbf{a}}^{-1} q.$$

In order to prove (2.5.4), it is enough to show that, for every $n \in \mathbb{N}$ and $q \in \mathbb{R}^d$,

$$\mathbb{E}\left[\omega(\square_n, q)_+\right] \le C|q|^2 3^{-n\alpha}. \tag{2.5.5}$$

Since this fact motivates the rest of the analysis in this section, we pause now to prove it.

LEMMA 2.5.3. *There exists $C(d,\lambda,\mathfrak{p}) < \infty$ such that*

$$\sup_{q\in\mathbb{R}^d} \frac{1}{|q|^2}\left|\mu(\square, q) + \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q\right| + \sup_{p\in\mathbb{R}^d} \frac{1}{|p|^2}\left|\nu(\square, p) - \frac{1}{2} p \cdot \bar{\mathbf{a}} p\right| \le C \sup_{e\in\partial B_1} \omega(\square, e)_+^{\frac{1}{2}} + \mathcal{O}_{\frac{2}{2d+1}}\left(C \operatorname{size}^{-\frac{1}{4}}(\square)\right). \tag{2.5.6}$$

PROOF. According to Lemma 2.4.7, there exists $C < \infty$ such that

$$k(\square) := \sup_{q\in\mathbb{R}^d}\left(\frac{1}{C} + \frac{1}{|q|^2}\mu(\square, q)\right) \le \mathcal{O}_{\frac{2}{2d-1}}\left(C \operatorname{size}^{-\frac{1}{2}}(\square)\right). \tag{2.5.7}$$

Fix $p \in \mathbb{R}^d$. Define the function

$$f(q) := \nu(\square, p) - \mu(\square, q) - q \cdot p + k(\square)|q|^2, \quad q \in \mathbb{R}^d.$$

Observe that f is a quadratic function and

$$f(q) \ge \nu(\square, p) + \frac{1}{C}|q|^2 - q \cdot p.$$

It follows that f is uniformly convex. Thus there exists a unique point $q_0 \in \mathbb{R}^d$ at which f attains its minimum. From the inequality $f(q_0) \le f(0)$ we see that $|q_0| \le C|p|$, and we have

$$p = -\nabla\mu(\square, \cdot)(q_0) + 2k(\square) q_0.$$

In particular,

$$|p + \nabla\mu(\square, \cdot)(q_0)| \le C|q_0| k(\square) \le \mathcal{O}_{\frac{2}{2d-1}}\left(C|p| \operatorname{size}^{-\frac{1}{2}}(\square)\right). \tag{2.5.8}$$

By the uniform convexity of f and the fact it achieves its minimum at q,

$$f(\bar{\mathbf{a}} p) \ge f(q_0) + \frac{1}{C}\left|\bar{\mathbf{a}} p - q_0\right|^2.$$

By (2.4.29) and (2.5.7),

$$f(q_0) \ge k(\square)|q_0|^2 - \mathcal{O}_{\frac{2}{2d-1}}\left(C|p||q_0| \operatorname{size}(\square)^{-\frac{1}{2}}\right) \ge -\mathcal{O}_{\frac{2}{2d-1}}\left(C|p|^2 \operatorname{size}^{-\frac{1}{2}}(\square)\right).$$

Thus we obtain

$$\begin{aligned}|\bar{\mathbf{a}}p - q_0|^2 &\le C\left(f(\bar{\mathbf{a}}p) - f(q_0)\right)\\ &\le Cf(\bar{\mathbf{a}}p) + \mathcal{O}_{\frac{2}{2d-1}}\left(C|p|^2 \operatorname{size}^{-\frac12}(\square)\right)\\ &\le C\omega(\square,\bar{\mathbf{a}}p) + \mathcal{O}_{\frac{2}{2d-1}}\left(C|p|^2 \operatorname{size}^{-\frac12}(\square)\right).\end{aligned}$$

Since $-\mu(\square,\cdot)$ is quadratic, for every $q \in \mathbb{R}^d$,

$$-\mu(\square,q) = -\frac{1}{2} q\cdot\nabla\mu(\square,\cdot)(q).$$

The bound (2.4.17) gives us that, for every $q_1,q_2 \in \mathbb{R}^d$,

$$|\nabla\mu(\square,\cdot)(q_1) - \nabla\mu(\square,\cdot)(q_2)| \le C|q_1-q_2|\left(1 + \mathcal{O}_{\frac{2}{2d+1}}\left(C\operatorname{size}^{-\frac12}(\square)\right)\right).$$

We deduce from the previous three displays, $|\bar{\mathbf{a}}p - q_0| \le C|p|$ and (2.5.8) that

$$\begin{aligned}&\left|-\mu(\square,\bar{\mathbf{a}}p) - \frac12 p\cdot\bar{\mathbf{a}}p\right|\\ &\quad= \left|-\frac12\bar{\mathbf{a}}p\cdot\nabla\mu(\square,\cdot)(\bar{\mathbf{a}}p) - \frac12 p\cdot\bar{\mathbf{a}}p\right|\\ &\quad\le |\bar{\mathbf{a}}p|\,|\nabla\mu(\square,\cdot)(\bar{\mathbf{a}}p) + p|\\ &\quad\le C|p|\left(|\nabla\mu(\square,\cdot)(\bar{\mathbf{a}}p) - \nabla\mu(\square,\cdot)(q_0)| + \mathcal{O}_{\frac{2}{2d-1}}\left(C|p|\operatorname{size}^{-\frac12}(\square)\right)\right)\\ &\quad\le C|p|\left(|\bar{\mathbf{a}}p - q_0| + \mathcal{O}_{\frac{2}{2d+1}}\left(C|p|\operatorname{size}^{-\frac12}(\square)\right)\right)\\ &\quad\le C|p|\left(\omega(\square,\bar{\mathbf{a}}p) + \mathcal{O}_{\frac{2}{2d-1}}\left(C|p|^2\operatorname{size}^{-\frac12}(\square)\right)\right)^{\frac12} + \mathcal{O}_{\frac{2}{2d+1}}\left(C|p|^2\operatorname{size}^{-\frac12}(\square)\right).\end{aligned}$$

From this and (2.5.1), we deduce that

$$\left|-\mu(\square,q) - \frac12 q\cdot\bar{\mathbf{a}}^{-1}q\right| \le C|q|^2 \sup_{e\in\partial B_1}\omega(\square,e)_+^{\frac12} + \mathcal{O}_{\frac{2}{2d+1}}\left(C|q|^2\operatorname{size}^{-\frac14}(\square)\right).$$

Using the previous inequality and

$$|f(\bar{\mathbf{a}}p) - \omega(\square,\bar{\mathbf{a}}p)| = k(\square)|\bar{\mathbf{a}}p|^2 \le \mathcal{O}_{\frac{2}{2d-1}}\left(C|p|^2\operatorname{size}^{-\frac12}(\square)\right)$$

we also obtain

$$\left|\nu(\square,p) - \frac12 p\cdot\bar{\mathbf{a}}p\right| \le C|p|^2 \sup_{e\in\partial B_1}\omega(\square,e)_+^{\frac12} + \mathcal{O}_{\frac{2}{2d+1}}\left(C|p|^2\operatorname{size}^{-\frac14}(\square)\right).$$

This completes the proof. □

In view of the previous lemma, we are motivated to prove the bound (2.5.5), which would follow if we can show that the expectation of $\omega(\square_n,q)_+$ contracts by a factor $\theta(d,\mathfrak{p},\lambda) < 1$ as we pass from scale n to scale $n+1$ (so that an iteration produces the desired estimate). It is therefore natural to work with the change in the expectation of ω between triadic scales n and $n+1$. In fact, it is convenient to use the slightly different quantity

$$\tau_n := \sum_{i=1}^d \left(\mathbb{E}\left[\mu(\square_{n+1},\mathfrak{e}_i)\right] - \mathbb{E}\left[\mu(\square_n,\mathfrak{e}_i)\right]\right)_+ + \sum_{i=1}^d \left(\mathbb{E}\left[\nu(\square_n,\bar{\mathbf{a}}^{-1}\mathfrak{e}_i)\right] - \mathbb{E}\left[\nu(\square_{n+1},\bar{\mathbf{a}}^{-1}\mathfrak{e}_i)\right]\right)_+.$$

Recall that, by (2.4.48) and (2.4.49),

$$\begin{cases}\mathbb{E}\left[\mu(\square_{n+1},q)\right] - \mathbb{E}\left[\mu(\square_n,q)\right] + C|q|^2 3^{-\frac{n}{4}} \ge 0, \text{ and}\\ \mathbb{E}\left[\nu(\square_n,\bar{\mathbf{a}}^{-1}p)\right] - \mathbb{E}\left[\nu(\square_{n+1},\bar{\mathbf{a}}^{-1}p)\right] + C|p|^2 3^{-n} \ge 0.\end{cases}$$

Since these are quadratic functions of q and p, respectively, it follows that they are convex and therefore sums control supremums:

$$\sup_{q\in\mathbb{R}^d}\frac{1}{|q|^2}\left(\mathbb{E}\left[\mu(\square_{n+1},q)\right]-\mathbb{E}\left[\mu(\square_n,q)\right]+C|q|^2 3^{-\frac{n}{4}}\right)\le\sum_{i=1}^d\left(\mathbb{E}\left[\mu(\square_{n+1},\mathfrak{e}_i)\right]-\mathbb{E}\left[\mu(\square_n,\mathfrak{e}_i)\right]+C3^{-\frac{n}{4}}\right)$$

with a similar inequality for ν. Using this observation, we deduce that, for every $n\in\mathbb{N}$ and $p,q\in\mathbb{R}^d$,

$$\begin{aligned}(\mathbb{E}\left[\mu(\square_{n+1},q)\right]-\mathbb{E}\left[\mu(\square_n,q)\right])_+ + (\mathbb{E}\left[\nu(\square_n,p)\right]-\mathbb{E}\left[\nu(\square_{n+1},p)\right])_+ \\ \le C(|p|^2+|q|^2)\left(\tau_n+C3^{-\frac{n}{4}}\right).\end{aligned}\tag{2.5.9}$$

Since $\omega(\square,\cdot)$ is almost nonnegative and the quantities $\omega(\cdot,q)$, $-\mu(\cdot,q)$ and $\nu(\cdot,\bar{\mathbf{a}}^{-1}q)$ are almost subadditive, and therefore their expectations are (almost) monotone, τ_n is essentially the same (up to negligible errors) as

$$\sum_{i=1}^d\left(\mathbb{E}\left[\omega(\square_n,\mathfrak{e}_i)\right]-\mathbb{E}\left[\omega(\square_{n+1},\mathfrak{e}_i)\right]\right).$$

Thus to prove the inequality $\mathbb{E}\left[\omega(\square_{n+1},e)\right]\le\theta\mathbb{E}\left[\omega(\square_n,e)\right]$ for $\theta<1$, it suffices to show that $\mathbb{E}\left[\omega(\square_n,e)\right]\le C\tau_n$ for some $C<\infty$. We do not prove exactly this, but something close enough (see the statement of Lemma 2.5.9, below) which can still be iterated to obtain (2.5.5); see Lemma 2.5.10, below.

The proof of Proposition 2.5.2 begins with the simple observation that, by quadratic response, the expected difference in the gradients of $u(\cdot,\square,q)$ at two successive triadic scales is controlled by τ_n. This will aid us by localizing the functions $u(\cdot,\square,q)$. In the uniform elliptic setting, this argument is two lines (cf. [**21**, (2.25)]). In our situation, the idea is the same but the statement is necessarily weaker and the proof is more technical due to the discreteness and the non-uniformity of the geometry of the clusters.

LEMMA 2.5.4. *There exists $C(d,\lambda,\mathfrak{p})<\infty$ such that, for every $m,n\in\mathbb{N}$ with $n\in\left[\frac12 m,m\right)$, $\square\in\mathcal{T}_m$ and $p,q\in\mathbb{R}^d$,*

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square|}\sum_{z\in3^n\mathbb{Z}^d\cap\square}\int_{\mathscr{C}_*(z+\square_n)}\left|\nabla\left(u(\cdot,\square,q)-u(\cdot,z+\square_n,q)\right)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\right] \\ +\mathbb{E}\left[\frac{1}{|\square|}\sum_{z\in3^n\mathbb{Z}^d\cap\square}\int_{\mathscr{C}_*(z+\square_n)}\left|\nabla\left(v(\cdot,\square,p)-v(\cdot,z+\square_n,p)\right)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\right] \\ \le C\left(|p|^2+|q|^2\right)\left(\sum_{k=n}^{m-1}\tau_k+3^{-\frac{n}{4}}\right).\end{aligned}\tag{2.5.10}$$

PROOF. For convenience, we write $u:=u(\cdot,\square,q)$ and $u_z:=u(\cdot,z+\square_n,q)$ for $z\in3^n\mathbb{Z}^d\cap\square$. The second variation (2.4.4) gives, for every $z\in3^n\mathbb{Z}^d\cap\square$,

$$\begin{aligned}\frac{1}{|\square_n|}\int_{\mathscr{C}_*(z+\square_n)}\left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\,\mathbb{1}_{\{z+\square_n\in\mathcal{P}_*\}} \\ \le C\left(\frac{1}{|\square_n|}\left(\frac12\left\langle\nabla u,\mathbf{a}\nabla u\right\rangle_{\mathscr{C}_*(z+\square_n)}-\left\langle q,\nabla[u]_{\mathcal{P}}\right\rangle_{z+\square_n}\right)-\mu(z+\square_n,q)\right)\mathbb{1}_{\{z+\square_n\in\mathcal{P}_*\}}.\end{aligned}$$

Summing this inequality over $z \in 3^n\mathbb{Z}^d \cap \square$ yields

$$\begin{aligned}
&\frac{|\square_n|}{|\square|} \sum_{z \in 3^n\mathbb{Z}^d \cap \square} \frac{1}{|\square_n|} \int_{\mathscr{C}_*(z+\square_n)} \left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\, \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}} \\
&\quad \leq C\left(\frac{1}{|\square|} \sum_{z \in 3^n\mathbb{Z}^d \cap \square} \left(\frac{1}{2}\langle \nabla u, \mathbf{a}\nabla u\rangle_{\mathscr{C}_*(z+\square_n)} - \langle q, \nabla[u]_{\mathcal{P}}\rangle_{z+\square_n} \right) \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}} \right. \\
&\qquad\qquad \left. - \frac{1}{|3^n\mathbb{Z}^d \cap \square|} \sum_{z \in 3^n\mathbb{Z}^d \cap \square} \mu(z+\square_n, q)\, \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}} \right).
\end{aligned}$$

Next, we notice that if $z + \square_n \notin \mathcal{P}_*$, then $\mathrm{cl}_{\mathcal{P}}(z+\square_n)$ is an element of $\mathcal{P}$, thus all coarsened functions are constant on $\mathrm{cl}_{\mathcal{P}}(z+\square_n)$ and so $u_z \equiv 0$ and $\mu(z+\square_n, q) = 0$. Thus we may remove the indicator function in the last line of the previous display. Combining the above and using (2.4.35) gives

$$\begin{aligned}
(2.5.11)\quad &\frac{1}{|\square|} \sum_{z \in 3^n\mathbb{Z}^d \cap \square} \int_{\mathscr{C}_*(z+\square_n)} \left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\, \mathbb{1}_{\{z+\square_n \in \mathcal{P}_*\}} \\
&\quad \leq C\left(\mu(\square, q) - 3^{-d(m-n)} \sum_{z \in 3^n\mathbb{Z}^d \cap \square} \mu(z+\square_n, q) \right) + \mathcal{O}_{\frac{2}{4d-3}}\left(C|q|^2 3^{-\frac{n}{2}} \right).
\end{aligned}$$

Let Γ denote the event

$$\Gamma := \left\{ \exists z \in 3^n\mathbb{Z}^d \cap \square \text{ such that } z + \square_n \notin \mathcal{P}_* \right\}.$$

Observe that

$$\mathbb{P}[\Gamma] \leq \sum_{z \in 3^n\mathbb{Z}^d \cap \square} \mathbb{P}[z+\square_n \notin \mathcal{P}_*] \leq C3^{d(m-n)} \exp(-c3^n) \leq C\exp\left(-c3^{\frac{m}{2}}\right).$$

Thus $\mathbb{1}_\Gamma \leq O_1\left(C3^{-\frac{m}{2}}\right)$. Using this bound, (2.1.9) and (2.4.16) (and again the fact that $u_z = 0$ if $z + \square_n \notin \mathcal{P}_*$), we obtain

$$\begin{aligned}
&\frac{1}{|\square|} \sum_{z \in 3^n\mathbb{Z}^d} \int_{\mathscr{C}_*(z+\square_n)} \left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\, \mathbb{1}_{\{z+\square_n \notin \mathcal{P}_*\}} \\
&\quad = \frac{1}{|\square|} \sum_{z \in 3^n\mathbb{Z}^d} \int_{\mathscr{C}_*(z+\square_n)} \left|\nabla u\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\, \mathbb{1}_{\{z+\square_n \notin \mathcal{P}_*\}} \\
&\quad \leq \frac{1}{|\square|} \int_{\mathscr{C}_*(z+\square_n)} \left|\nabla u\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\, \mathbb{1}_\Gamma \\
&\quad \leq \mathcal{O}_{\frac{1}{2d-2}}\left(C|q|^2\right) \cdot \mathcal{O}_1\left(C3^{-\frac{m}{2}}\right) \\
&\quad \leq \mathcal{O}_{\frac{1}{2d-1}}\left(C3^{-\frac{m}{2}}\right).
\end{aligned}$$

Combining this with (2.5.11), taking the expectation of the result and applying (2.4.48) yields the estimate for the first term on the left side of (2.5.10). The estimate of the second term is similar and we omit the details, except for the remark that (2.4.38) should be used in place of (2.4.35). □

We next obtain a version of the previous lemma for the spatial averages of the gradients of the coarsened functions $[u]_{\mathcal{P}}$. This is tricky and somewhat technical, because in passing from u to $[u]_{\mathcal{P}}$, using (2.3.3), we make errors depending on the coarseness of the partition $\mathcal{P}$. If the energy density $|\nabla u|^2$ happens to be concentrated in the very largest cubes of $\mathcal{P}$, then this does not give us a good enough estimate. We deal with this issue by using the Meyers estimate, Proposition 2.3.8, which allows us to "Hölder away" the factors representing the coarseness of $\mathcal{P}$ on the right of (2.3.3).

LEMMA 2.5.5. *There exists* $C(d,\lambda,\mathfrak{p})<\infty$ *such that, for every* $m,n\in\mathbb{N}$ *with* $n\in\left[\left(\frac{4d+1}{4d+2}\right)m,m\right)$, $\square\in\mathcal{T}_m$ *and* $p,q\in\mathbb{R}^d$,

$$\mathbb{E}\left[\frac{3^{-d(m-n)}}{|\square_n|^2}\sum_{z\in3^n\mathbb{Z}^d\cap\square_m}\left|\langle[\nabla u(\cdot,\square,q)]_{\mathcal{P}}\rangle_{z+\square_n}-\langle[\nabla u(\cdot,z+\square_n,q)]_{\mathcal{P}}\rangle_{z+\square_n}\right|^2\right]$$
$$+\mathbb{E}\left[\frac{3^{-d(m-n)}}{|\square_n|^2}\sum_{z\in3^n\mathbb{Z}^d\cap\square}\left|\langle[\nabla v(\cdot,\square,p)]_{\mathcal{P}}\rangle_{z+\square_n}-\langle[\nabla v(\cdot,z+\square_n,p)]_{\mathcal{P}}\rangle_{z+\square_n}\right|^2\right]$$
$$\leq C\left(|p|^2+|q|^2\right)\left(\sum_{k=n}^{m+1}\tau_k+3^{-\frac{n}{4}}\right). \tag{2.5.12}$$

PROOF. The proof of the estimate for the second term on the left of (2.5.12) will be omitted, since it follows from a very similar argument as for the estimate of the first term. For convenience and simplicity, we give the proof only in the case that $\square=\square_m$. Throughout, we work with the partition $\mathcal{Q}$ defined in Definition 2.3.9 which gives us good cubes for the Meyers estimate. We estimate the left side of (2.5.12) by the left side of (2.5.10). For this we use (2.3.3), the Hölder inequality and the Meyers estimate.

We fix $q\in\mathbb{R}^d$ and denote, for $z\in3^n\mathbb{Z}^d\cap\square$, the functions $u:=u(\cdot,\square_m,q)$ and $u_z:=u(\cdot,z+\square_n,q)$.

Step 1. We reduce to a "good" event in which every element of the partition $\mathcal{Q}$ in $\square_{2n}\supseteq\square_m$ is not too large. Denote this event by

$$\Gamma:=\left\{\max_{x\in\square_{2n}}\operatorname{size}(\square_{\mathcal{Q}}(x))\leq3^{\frac{n}{2}-1}\right\}.$$

By (2.3.21), there exists an exponent $s(d,\lambda,\mathfrak{p})<\infty$ such that

$$\mathbb{1}_{\Omega\setminus\Gamma}\leq3^{-\frac{n}{2}+1}\sup_{x\in\square_{2n}}\operatorname{size}(\square_{\mathcal{Q}}(x))\leq3^{-\frac{n}{2}+1}\mathcal{O}_s\left(C3^{\frac{n}{8}}\right)\leq\mathcal{O}_s\left(C3^{-\frac{3n}{8}}\right).$$

Using (2.4.24) with $\delta=\frac18$, the fact that $u_z=0$ (resp. $u=0$) if $z+\square_n\notin\mathcal{P}_*$ (resp. $\square_m\notin\mathcal{P}_*$) and the Hölder inequality, we obtain

$$\begin{aligned}&\frac{3^{-d(m-n)}}{|\square_n|^2}\sum_{z\in3^n\mathbb{Z}^d\cap\square_m}\left|\langle\nabla[u]_{\mathcal{P}}\rangle_{z+\square_n}-\langle\nabla[u_z]_{\mathcal{P}}\rangle_{z+\square_n}\right|^2\mathbb{1}_{\Omega\setminus\Gamma}\\
&\leq C3^{-d(m-n)}\sum_{z\in3^n\mathbb{Z}^d\cap\square_m}\left(\fint_{z+\square_n}\left(|\nabla[u]_{\mathcal{P}}|^2(x)+|\nabla[u_z]_{\mathcal{P}}|^2(x)\right)dx\right)\mathbb{1}_{\Omega\setminus\Gamma}\\
&\leq C\left(\fint_{\square_m}|\nabla[u]_{\mathcal{P}}|^2(x)\,dx+3^{-d(m-n)}\sum_{z\in3^n\mathbb{Z}^d\cap\square_m}\fint_{z+\square_n}|\nabla[u_z]_{\mathcal{P}}|^2(x)\,dx\right)\mathbb{1}_{\Omega\setminus\Gamma}\\
&\leq\mathcal{O}_{\frac{1}{4d-3}}\left(C|q|^23^{\frac{n}{8}}\right)\cdot\mathcal{O}_s\left(C3^{-\frac{3n}{8}}\right).\end{aligned}$$

We deduce that

$$\mathbb{E}\left[\frac{3^{-d(m-n)}}{|\square_n|^2}\sum_{z\in3^n\mathbb{Z}^d\cap\square_m}\left|\langle\nabla[u]_{\mathcal{P}}\rangle_{z+\square_n}-\langle\nabla[u_z]_{\mathcal{P}}\rangle_{z+\square_n}\right|^2\mathbb{1}_{\Omega\setminus\Gamma}\right]\leq C|q|^23^{-\frac{n}{4}}. \tag{2.5.13}$$

Step 2. We further prepare for the use of the Meyers estimate application by removing a boundary layer around each of the subcubes $z+\square_n$. This is necessary because Proposition 2.3.8 is only an interior estimate and we do not have a good boundary condition anyway for minimizers of μ. Let K_z denote the union of the elements of $\mathcal{T}_{\lceil n/2\rceil}$ which are subsets of $z+\square_n$ and intersect the boundary of $z+\square_n$:

$$K_z:=\bigcup\left\{\square'\in\mathcal{T}:\operatorname{size}(\square')=3^{\lceil n/2\rceil},\ \square'\subseteq z+\square_n,\ \operatorname{dist}(\square',\partial(z+\square_n))=0\right\}.$$

Let E_z denote the edges (x,y) such that x or y belongs to K_z. Then by the Hölder inequality, the triangle inequality, and the fact that $|K_z| \le C3^{-\frac{n}{2}}|\square_n|$, we find that, for each $z \in 3^n\mathbb{Z}^d \cap \square_m$,

$$\mathbb{E}\left[\frac{1}{|\square_n|^2}\left|\sum_{e\in E_z} \nabla[u-u_z]_{\mathcal{P}}(e)\right|^2\right] \le C\mathbb{E}\left[\frac{|K_z|}{|\square_n|^2}\int_{K_z}\left(|\nabla[u]_{\mathcal{P}}|^2(x) + |\nabla[u_z]_{\mathcal{P}}|^2(x)\right)dx\right]$$
$$\le C3^{-\frac{n}{2}}\mathbb{E}\left[\fint_{z+\square_n}\left(|\nabla[u]_{\mathcal{P}}|^2(x) + |\nabla[u_z]_{\mathcal{P}}|^2(x)\right)dx\right].$$

Summing over $z \in 3^n\mathbb{Z}^d \cap \square_m$ and using (2.4.24) with $\delta = \frac{1}{4}$ gives

$$3^{-d(m-n)}\sum_{z\in 3^n\mathbb{Z}^d\cap\square_m}\mathbb{E}\left[\frac{1}{|\square_n|^2}\left|\sum_{e\in E_z}\nabla[u-u_z]_{\mathcal{P}}(e)\right|^2\right] \le C|q|^2 3^{-\frac{n}{4}}. \tag{2.5.14}$$

Step 3. We estimate the expected difference between $\nabla[u]_{\mathcal{P}}$ and $\nabla[u_z]_{\mathcal{P}}$ in the strong L^2 norm, after removing the K_z's and in the case that the good event Γ holds. The precise objective of this step is to prove (2.5.20), below.

We begin by using (2.3.3) to estimate, for each $z \in 3^n\mathbb{Z}^d \cap \square_m$,

$$\int_{(z+\square_n)\smallsetminus K_z}|\nabla[u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_\Gamma$$
$$\le C\sum_{\mathcal{P}\ni\square'\subseteq(z+\square_n)\smallsetminus K_z}\operatorname{size}(\square')^{2d-1}\int_{\mathscr{C}_*(\square')}\left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\,\mathbb{1}_\Gamma$$
$$\le C\sum_{\mathcal{Q}\ni\square'\subseteq(z+\square_n)\smallsetminus K_z}\operatorname{size}(\square')^{2d-1}\int_{\mathscr{C}_*(\square')}\left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\,\mathbb{1}_\Gamma.$$

Here we used that $\mathcal{Q}$ is coarser than $\mathcal{P}$ and that no element of $\mathcal{Q}$ in $z+\square_n$ is larger than $3^{\lceil n/2\rceil}$ on the event Γ. Applying the Hölder inequality to the previous sum yields, for every $s \in (2,\infty)$,

$$\frac{1}{|\square_n|}\int_{(z+\square_n)\smallsetminus K_z}|\nabla[u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_\Gamma$$
$$\le C\left(\frac{1}{|\square_n|}\sum_{\mathcal{Q}\ni\square'\subseteq(z+\square_n)\smallsetminus K_z}\operatorname{size}(\square')^{\frac{s(2d-1)}{s-2}}\right)^{\frac{s-2}{s}}$$
$$\times\left(\frac{1}{|\square_n|}\sum_{\mathcal{Q}\ni\square'\subseteq(z+\square_n)\smallsetminus K_z}\int_{\mathscr{C}_*(\square')}\left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^s(x)\,dx\right)^{\frac{2}{s}}\mathbb{1}_\Gamma.$$

Take $s = 2+\varepsilon$ with $\varepsilon(d,\lambda,\mathfrak{p}) > 0$ as in Proposition 2.3.8 and apply the proposition in each $\square' \in \mathcal{Q}$, $\square' \subseteq (z+\square_n)\smallsetminus K_z$, which we note that on the event Γ implies $3\square' \subseteq z+\square_n$, to get an estimate of the second factor on the right side:

$$\left(\frac{1}{|\square_n|}\sum_{\mathcal{Q}\ni\square'\subseteq(z+\square_n)\smallsetminus K_z}\int_{\mathscr{C}_*(\square')}\left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^s(x)\,dx\right)^{\frac{2}{s}}\mathbb{1}_\Gamma$$
$$\le\left(\frac{1}{|\square_n|}\sum_{\mathcal{Q}\ni\square'\subseteq(z+\square_n)\smallsetminus K_z}|\square'|\left(\frac{1}{|\square'|}\int_{\mathscr{C}_*(3\square')}\left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\right)^{\frac{s}{2}}\right)^{\frac{2}{s}}\mathbb{1}_\Gamma$$
$$\le\frac{1}{|\square_n|}\sum_{\mathcal{Q}\ni\square'\subseteq(z+\square_n)\smallsetminus K_z}|\square'|^{\frac{2}{s}-1}\int_{\mathscr{C}_*(3\square')}\left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx\,\mathbb{1}_\Gamma$$
$$\le\frac{1}{|\square_n|}\int_{\mathscr{C}_*(z+\square_n)}\left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq0\}}\right|^2(x)\,dx$$

To get the last line, we used that every point of $\mathscr{C}_*(z+\square_n)$ belongs to $\mathscr{C}_*(3\square')$ for at most C elements $\square'$ of the sum, since $\mathcal{Q}$ satisfies property (ii) of Proposition 2.2.1, and $|\square'|^{\frac{2}{s}-1} \le 1$. Putting the above inequalities together, we get

$$(2.5.15)\quad \frac{1}{|\square_n|}\int_{(z+\square_n)\setminus K_z} |\nabla [u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_\Gamma \le C\left(\frac{1}{|\square_n|}\sum_{\mathcal{Q}\ni\square'\subseteq z+\square_n} \operatorname{size}(\square')^{\frac{s(2d-1)}{s-2}}\right)^{\frac{s-2}{s}} \mathbb{1}_\Gamma\cdot\frac{1}{|\square_n|}\int_{\mathscr{C}_*(z+\square_n)} \left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x).$$

It follows that, for $r := \frac{s(2d-1)}{s-2} = \frac{s(2d-1)}{\varepsilon}$, which we note depends only on $(d,\lambda,\mathfrak{p})$,

$$C\left(\frac{1}{|\square_n|}\sum_{\mathcal{Q}\ni\square'\subseteq z+\square_n} \operatorname{size}(\square')^{\frac{s(2d-1)}{s-2}}\right)^{\frac{s-2}{s}} \mathbb{1}_\Gamma\,\mathbb{1}_{\{n\ge\mathcal{N}_r(z)\}} \le C.$$

and therefore

$$(2.5.16)\quad \frac{1}{|\square_n|}\int_{(z+\square_n)\setminus K_z} |\nabla [u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_\Gamma\,\mathbb{1}_{\{n\ge\mathcal{N}_r(z)\}} \le \frac{C}{|\square_n|}\int_{\mathscr{C}_*(z+\square_n)} \left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x).$$

To complete the proof of (2.5.12), we split the expectation of the left side of (2.5.15), using the minimal scale for the partition $\mathcal{Q}$:

$$(2.5.17)\quad \mathbb{E}\left[\fint_{(z+\square_n)\setminus K_z} |\nabla [u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_\Gamma\right] \le \mathbb{E}\left[\fint_{(z+\square_n)\setminus K_z} |\nabla [u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_\Gamma\,\mathbb{1}_{\{n\ge\mathcal{N}_r(z)\}}\right] + \mathbb{E}\left[\fint_{(z+\square_n)\setminus K_z} |\nabla [u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_\Gamma\,\mathbb{1}_{\{n<\mathcal{N}_r(z)\}}\right].$$

Taking the expectation of (2.5.16) gives an estimate for the first term on the right side:

$$(2.5.18)\quad \mathbb{E}\left[\fint_{(z+\square_n)\setminus K_z} |\nabla [u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_\Gamma\,\mathbb{1}_{\{n\ge\mathcal{N}_r(z)\}}\right] \le C\mathbb{E}\left[\frac{1}{|\square_n|}\int_{\mathscr{C}_*(z+\square_n)} \left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right].$$

We estimate the second term on the right of (2.5.15) rather crudely: we use (2.1.9) and combine this with (2.4.24) (with $\delta = \frac14$) to obtain, for some small exponent $s_0(d,\lambda,\mathfrak{p}) > 0$,

$$\begin{aligned}\fint_{z+\square_n} |\nabla [u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_{\{n<\mathcal{N}_r(z)\}} &\le 2\left(3^{d(m-n)}\fint_{\square_m} |\nabla [u]_{\mathcal{P}}|^2(x)\,dx + \fint_{z+\square_n} |\nabla [u_z]_{\mathcal{P}}|^2(x)\,dx\right)\mathbb{1}_{\{n<\mathcal{N}_r(z)\}}\\ &\le \mathcal{O}_{s_0}\left(C|q|^2 3^{d(m-n)+\frac{m}{4}-n}\right).\end{aligned}$$

Taking the expectation of this yields

$$(2.5.19)\quad \mathbb{E}\left[\fint_{z+\square_n} |\nabla [u-u_z]_{\mathcal{P}}|^2(x)\,dx\,\mathbb{1}_{\{n<\mathcal{N}_r\}}\right] \le C|q|^2 3^{d(m-n)+\frac{m}{4}-n}.$$

Notice that the assumptions $n \geq \left(\frac{4d+1}{4d+2}\right) m$ implies that $d(m-n) + \frac{m}{4} - n \leq -\frac{n}{2}$. Combining this observation with (2.5.17), (2.5.18) and (2.5.19) yields

$$\mathbb{E}\left[\frac{1}{|\,\square_n\,|}\int_{(z+\square_n)\smallsetminus K_z} \left|\nabla\left[u-u_z\right]_{\mathcal{P}}\right|^2(x)\,dx\,\mathbb{1}_{\Gamma}\right] \\ \leq C|q|^2 3^{-\frac{n}{2}} + C\mathbb{E}\left[\frac{1}{|\square_n|}\int_{\mathscr{C}_*(z+\square_n)} \left|\nabla(u-u_z)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right].$$

Summing over $z \in 3^n\mathbb{Z}^d \cap \square_m$ and using Lemma 2.5.4, we get

$$3^{-d(m-n)} \sum_{z\in 3^n\mathbb{Z}^d\cap\square_m} \mathbb{E}\left[\frac{1}{|\,\square_n\,|}\int_{(z+\square_n)\smallsetminus K_z} \left|\nabla\left[u-u_z\right]_{\mathcal{P}}\right|^2(x)\,dx\,\mathbb{1}_{\Gamma}\right] \\ \leq C|q|^2\left(\sum_{k=n}^{m-1}\tau_k + 3^{-\frac{n}{4}}\right). \tag{2.5.20}$$

Step 4. The conclusion. Combining (2.5.13), (2.5.14) and (2.5.20), we obtain

$$\frac{3^{-d(m-n)}}{|\,\square_n\,|^2} \sum_{z\in 3^n\mathbb{Z}^d\cap\square_m} \mathbb{E}\left[\left|\langle[\nabla u]_{\mathcal{P}}\rangle_{z+\square_n} - \langle[\nabla u_z]_{\mathcal{P}}\rangle_{z+\square_n}\right|^2\right] \leq C|q|^2\left(\sum_{k=n}^{m-1}\tau_k + 3^{-\frac{n}{4}}\right).$$

This completes the proof of the lemma. □

Definition 2.5.6 (The matrix $\bar{\mathbf{a}}_{\square_n}$). We define $\bar{\mathbf{a}}_{\square_n}^{-1}$ to be the matrix satisfying, for every $q \in \mathbb{R}^d$,

$$\frac{1}{2} q\cdot \bar{\mathbf{a}}_{\square_n}^{-1} q = \mathbb{E}\left[-\mu(\square_n, q)\right]. \tag{2.5.21}$$

Since the right side of (2.5.21) is a nonnegative and quadratic function of q, it can be written in terms of a matrix and thus $\bar{\mathbf{a}}_{\square_n}^{-1}$ is well-defined.

We continue with some observations regarding the matrices $\bar{\mathbf{a}}_{\square_n}$. Notice that (2.4.16) and (2.4.30) imply the existence of $C(d,\lambda,\mathfrak{p}) < \infty$ such that, for every $n \in \mathbb{N}$ with $n \geq C$,

$$\frac{1}{C} I_d \leq \bar{\mathbf{a}}_{\square_n}^{-1} \leq C I_d. \tag{2.5.22}$$

Since our model is invariant under permutations and reflections of the coordinate axes, the matrices $\bar{\mathbf{a}}_{\square_n}^{-1}$ (as well as $\bar{\mathbf{a}}$) are actually a multiple of the identity I_d. However, since we do not use this anywhere and our arguments which actually can handle more general models, we ignore this fact.

According to the first variation (2.4.3), we can also write $\bar{\mathbf{a}}_{\square_n}^{-1}$ in terms of the expected spatial average of the minimizers of $\mu(\square_n, q)$: for every $q, q' \in \mathbb{R}^d$,

$$q'\cdot\bar{\mathbf{a}}_{\square_n}^{-1} q = \mathbb{E}\left[\frac{1}{|\mathrm{cl}_{\mathcal{P}}(\square_n)|}\left\langle q', \nabla\left[u(\cdot,\square_n,q)\right]_{\mathcal{P}}\right\rangle_{\mathrm{cl}_{\mathcal{P}}(\square_n)}\right]. \tag{2.5.23}$$

In other words,

$$\bar{\mathbf{a}}_{\square_n}^{-1} q = \mathbb{E}\left[\frac{1}{|\mathrm{cl}_{\mathcal{P}}(\square_n)|}\left\langle \nabla[u(\cdot,\square_n,q)]_{\mathcal{P}}\right\rangle_{\mathrm{cl}_{\mathcal{P}}(\square_n)}\right].$$

For future reference, we record, for every $m, n \in \mathbb{N}$ with $m > n$, the estimate

$$\left|\bar{\mathbf{a}}_{\square_m}^{-1} - \bar{\mathbf{a}}_{\square_n}^{-1}\right| \leq C\left(3^{-\frac{n}{4}} + \sum_{k=n}^{m-1}\tau_k\right). \tag{2.5.24}$$

To prove (2.5.24), we use (2.4.48) and (2.5.21) to see that, for every $q \in \mathbb{R}^d$,

$$\begin{aligned}\left|\frac{1}{2} q \cdot \bar{\mathbf{a}}_{\square_m}^{-1} q - \frac{1}{2} q \cdot \bar{\mathbf{a}}_{\square_n}^{-1} q\right| &= \left|\mathbb{E}\left[\mu(\square_m, q)\right] - \mathbb{E}\left[\mu(\square_n, q)\right]\right| \\ &\le \sum_{k=n}^{m-1} \left|\mathbb{E}\left[\mu(\square_{k+1}, q)\right] - \mathbb{E}\left[\mu(\square_k, q)\right]\right| \\ &\le \sum_{k=n}^{m-1} \left(\left(\mathbb{E}\left[\mu(\square_{k+1}, q)\right] - \mathbb{E}\left[\mu(\square_k, q)\right]\right)_+ + C 3^{-\frac{k}{4}} \right) \\ &\le C|q|^2 \left(3^{-\frac{n}{4}} + \sum_{k=n}^{m-1} \sum_{i=1}^{d} \left(\mathbb{E}\left[\mu(\square_{k+1}, \mathfrak{e}_i)\right] - \mathbb{E}\left[\mu(\square_k, \mathfrak{e}_i)\right]\right)_+ \right) \\ &\le C|q|^2 \left(3^{-\frac{n}{4}} + \sum_{k=n}^{m-1} \tau_k \right).\end{aligned}$$

Taking the supremum of the previous inequality over $q \in \mathbb{R}^d \setminus \{0\}$, after dividing by $|q|^2$, yields (2.5.24).

We show in the next lemma that the variance of $\langle \nabla [u(\cdot, \square_n, q)]_{\mathcal{P}} \rangle_{\mathrm{cl}_{\mathcal{P}}(\square_n)}$ is controlled by τ_n. This is perhaps the main step in the proof of Proposition 2.5.2. It is a variation of [**21**, Lemma 3.2].

Lemma 2.5.7. *There exists $C(d, \mathfrak{p}, \lambda) < \infty$ such that, for every $n \in \mathbb{N}$ and $q \in \mathbb{R}^d$,*

$$\mathbb{E}\left[\left| \frac{1}{|\square_n|} \langle \nabla [u(\cdot, \square_n, q)]_{\mathcal{P}} \rangle_{\square_n} - \bar{\mathbf{a}}_{\square_n}^{-1} q \right|^2 \right] \le C|q|^2 \left(\tau_n + 3^{-\frac{n}{4}} \right). \tag{2.5.25}$$

Proof. Fix a unit direction $e \in \partial B_1$.

Step 1. We construct a (deterministic) compactly supported, bounded and solenoidal vector field $\mathbf{G}$ on $\square_{n+1}$ which is constant and equal to e on $\square_n$. Precisely, the claim is that $\mathbf{G}$ is a vector field on $\square_n$ satisfying

$$\sup_{(x,y) \in E_d(\square_{n+1})} |\mathbf{G}(x,y)| \le C, \tag{2.5.26}$$

is constant in the middle third subcube, i.e.,

$$\mathbf{G}(x,y) = e \cdot (x - y) \quad \text{for all } x, y \in \square_n \text{ with } x \sim y, \tag{2.5.27}$$

and is almost (discretely) divergence-free in the sense that, for every $w : \square_{n+1} \to \mathbb{R}$, we have

$$\left| \langle \nabla w, \mathbf{G} \rangle_{\square_{n+1}} \right| \le C 3^{-n} \sum_{x \in \square_{n+1}} |\nabla w|(x). \tag{2.5.28}$$

According to the proof of [**21**, Lemma 3.2], there exists a smooth, (continuum) vector field $\mathbf{g} \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ satisfying, for every $k \in \mathbb{N}$,

$$\operatorname{supp} \mathbf{g} \subseteq [-3,3]^d, \quad \mathbf{g}(x) = e \text{ in } [-1,1]^d, \quad \nabla \cdot \mathbf{g} = 0 \text{ in } \mathbb{R}^d, \quad \left|\nabla^k \mathbf{g}\right| \le C(k,d).$$

We define the (discrete) vector field $\mathbf{G}$ by

$$\mathbf{G}(x,y) = \frac{1}{2} \left(\mathbf{g}(3^{-n} x) + \mathbf{g}(3^{-n} y) \right) \cdot (x - y), \quad x, y \in \square_{n+1}.$$

To check (2.5.28), we fix $w : \square_{n+1} \to \mathbb{R}$, assume without loss of generality that $(w)_{\square_{n+1}} = 0$, and compute

$$\begin{aligned}\langle \nabla w, \mathbf{G} \rangle_{\square_{n+1}} &= \frac{1}{2} \sum_{x,y \in \square_{n+1},\, x \sim y} (w(x) - w(y)) \left(\mathbf{g}(3^{-n} x) + \mathbf{g}(3^{-n} y) \right) \cdot (x - y) \\ &= \sum_{x \in \square_{n+1}} w(x) \sum_{y \sim x} \left(\mathbf{g}(3^{-n} x) + \mathbf{g}(3^{-n} y) \right) \cdot (x - y).\end{aligned}$$

For each x, we have

$$\sum_{y\sim x} \mathbf{g}\left(3^{-n}x\right)\cdot(x-y) = \mathbf{g}\left(3^{-n}x\right)\sum_{y\sim x}\cdot(x-y) = 0,$$

and therefore

$$\sum_{y\sim x}\left(\mathbf{g}\left(3^{-n}x\right)+\mathbf{g}\left(3^{-n}y\right)\right)\cdot(x-y) = \sum_{y\sim x}\left(\mathbf{g}\left(3^{-n}y\right)-\mathbf{g}\left(3^{-n}x\right)\right)\cdot(x-y).$$

The divergence-free condition and $\left|\nabla^2\mathbf{g}\right|\le C$ yield, for each fixed x,

$$\begin{aligned}\left|\sum_{y\sim x}\left(\mathbf{g}\left(3^{-n}y\right)-\mathbf{g}\left(3^{-n}x\right)\right)\cdot(x-y)\right| &\le \left|\sum_{y\sim x}(x-y)\cdot\nabla\mathbf{g}\left(3^{-n}x\right)(y-x)\right| + C3^{-2n}\\ &= 2\left|\nabla\cdot\mathbf{g}\left(3^{-n}x\right)\right| + C3^{-2n} = C3^{-2n}.\end{aligned}$$

Combining the above yields

$$\begin{aligned}\left|\langle\nabla w,\mathbf{G}\rangle_{\square_{n+1}}\right| &\le \sum_{x\in\square_{n+1}}|w(x)|\left|\sum_{y\sim x}\left(\mathbf{g}\left(3^{-n}x\right)+\mathbf{g}\left(3^{-n}y\right)\right)\cdot(x-y)\right|\\ &\le C3^{-2n}\sum_{x\in\square_{n+1}}|w(x)|.\end{aligned}$$

The Poincaré inequality and $(w)_{\square_{n+1}}=0$ now yields (2.5.28).

Step 2. We next show that

$$\begin{aligned}\left|\langle\nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}},\mathbf{G}\rangle_{\square_{n+1}} - \sum_{z\in3^n\mathbb{Z}^d\cap\square_{n+1}}\langle\nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}},\mathbf{G}\rangle_{z+\square_n}\right|&\\ \le \mathcal{O}_{\frac{2}{4d-3}}\left(C|\square_{n+1}||q|3^{-\frac{n}{4}}\right).&\end{aligned}\tag{2.5.29}$$

For any vector field F,

$$\langle F,\mathbf{G}\rangle_{\square_{n+1}} - \sum_{z\in3^n\mathbb{Z}^d\cap\square_{n+1}}\langle F,\mathbf{G}\rangle_{z+\square_n} = \sum_{(x,y)\in D}\langle F,\mathbf{G}\rangle_{z+\square_n}$$

where D is the set of edges (x,y) such that $x,y\in\square_{n+1}$ and x and y belong to different subcubes of the form $z+\square_n$, $z\in\{-3^n,0,3^n\}^d$. Notice that $|D|\le C3^{-n}|\square_{n+1}|$. By the Hölder inequality and the bound $|\mathbf{G}|\le C$, we therefore obtain

$$\begin{aligned}\left|\langle F,\mathbf{G}\rangle_{\square_{n+1}} - \sum_{z\in3^n\mathbb{Z}^d\cap\square_{n+1}}\langle F,\mathbf{G}\rangle_{z+\square_n}\right| &\le \sum_{(x,y)\in D}\langle F,\mathbf{G}\rangle_{z+\square_n}\\ &\le C\int_D|F|(x)\,dx\\ &\le C|D|^{\frac12}\left(\int_D|F|^2(x)\,dx\right)^{\frac12}\\ &\le C3^{-\frac{n}{2}}|\square_{n+1}|^{\frac12}\left(\int_{\square_{n+1}}|F|^2(x)\,dx\right)^{\frac12}.\end{aligned}$$

Applying this inequality with $F=\nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}$ and then using (2.4.24) with $\delta=\frac12$ to bound the right side of the result, we obtain (2.5.29).

Step 3. The conclusion. We apply (2.5.28) with $w=\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}$ and combine this with (2.4.20), (2.5.12) and (2.5.29) to get

$$\mathbb{E}\left[\left|\sum_{z\in3^n\mathbb{Z}^d\cap\square_{n+1}}\frac{1}{|\square_n|}\langle\nabla\left[u(\cdot,z+\square_n,q)\right]_{\mathcal{P}},\mathbf{G}\rangle_{z+\square_n}\right|^2\right]\le C|q|^2\left(\tau_n+3^{-\frac{n}{4}}\right).\tag{2.5.30}$$

We conclude by noticing that, since the elements in the sum on the left side are almost $\mathbb{P}$–independent, the variance of each term in the sum should be controlled by the variance of the entire sum. To make this precise, we prove the following inequality:

$$\mathbb{E}\left[\left|\frac{1}{|\square_n|}\left\langle \nabla\left[u(\cdot,z+\square_n,q)\right]_{\mathcal{P}},\mathbf{G}(x,y)\right\rangle_{z+\square_n} - \frac{1}{|\square_n|}\left\langle \nabla\left[u_{\mathrm{loc}}^{(\lceil n/2\rceil)}(\cdot,z+\square_n,q)\right]_{\mathcal{P}_{\mathrm{loc}}(z+\square_n)},\mathbf{G}(x,y)\right\rangle_{z+\square_n}\right|^2\right] \le C3^{-\frac{n}{4}}. \tag{2.5.31}$$

To prove this inequality, we first appeal to (2.4.50), which gives

$$\mathbb{E}\left[\left|\frac{1}{|\square_n|}\left\langle \nabla\left[u(\cdot,z+\square_n,q)\right]_{\mathcal{P}},\mathbf{G}(x,y)\right\rangle_{(z+\square_n)^{(\lceil n/2\rceil)}} - \frac{1}{|\square_n|}\left\langle \nabla\left[u_{\mathrm{loc}}^{(\lceil n/2\rceil)}(\cdot,z+\square_n,q)\right]_{\mathcal{P}_{\mathrm{loc}}(z+\square_n)},\mathbf{G}(x,y)\right\rangle_{(z+\square_n)^{(\lceil n/2\rceil)}}\right|^2\right] \le C3^{-\frac{n}{4}}.$$

and combine it with the result of the following computation, which is an application of (2.4.24) (with $\delta = \frac{1}{4}$):

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{|\square_n|}\left\langle \nabla\left[u(\cdot,z+\square_n,q)\right]_{\mathcal{P}},\mathbf{G}(x,y)\right\rangle_{(z+\square_n)\setminus(z+\square_n)^{(\lceil n/2\rceil)}}\right|^2\right] \\
&\quad\le C\mathbb{E}\left[\left|\frac{1}{|\square_n|}\int_{(z+\square_n)\setminus(z+\square_n)^{(\lceil n/2\rceil)}}\left|\nabla\left[u(\cdot,z+\square_n,q)\right]_{\mathcal{P}}\right|(x)\,dx\right|^2\right] \\
&\quad\le C\mathbb{E}\left[\frac{\left|(z+\square_n)\setminus(z+\square_n)^{(\lceil n/2\rceil)}\right|}{|\square_n|}\fint_{z+\square_n}\left|\nabla\left[u(\cdot,z+\square_n,q)\right]_{\mathcal{P}}\right|^2(x)\,dx\right] \\
&\quad\le C3^{-\frac{n}{4}}.
\end{aligned}$$

Since

$$\left\langle \nabla\left[u_{\mathrm{loc}}^{(\lceil n/2\rceil)}(\cdot,z+\square_n,q)\right]_{\mathcal{P}_{\mathrm{loc}}(z+\square_n)},\mathbf{G}(x,y)\right\rangle_{(z+\square_n)^{(\lceil n/2\rceil)}} \quad \text{is } \mathcal{F}(z+\square_n)\text{–measurable,}$$

and $|\mathbf{G}| \le C$ by (2.5.26), we may use independence (in the second line in the display below), (2.5.31) and the triangle inequality (twice, in the first and third lines) and (2.5.30) (in the fourth line) to obtain

$$\begin{aligned}
&\mathrm{var}\left[\frac{1}{|\square_n|}\left\langle \nabla\left[u(\cdot,\square_n,q)\right]_{\mathcal{P}},\mathbf{G}(x,y)\right\rangle_{\square_n}\right] \\
&\le \sum_{z\in 3^n\mathbb{Z}^d\cap\square_{n+1}}\mathrm{var}\left[\frac{1}{|\square_n|}\left\langle \nabla\left[u_{\mathrm{loc}}^{(\lceil n/2\rceil)}(\cdot,z+\square_n,q)\right]_{\mathcal{P}_{\mathrm{loc}}(z+\square_n)},\mathbf{G}(x,y)\right\rangle_{(z+\square_n)^{(\lceil n/2\rceil)}}\right] + C3^{-\frac{n}{4}} \\
&= \mathrm{var}\left[\sum_{z\in 3^n\mathbb{Z}^d\cap\square_{n+1}}\frac{1}{|\square_n|}\left\langle \nabla\left[u_{\mathrm{loc}}^{(\lceil n/2\rceil)}(\cdot,z+\square_n,q)\right]_{\mathcal{P}_{\mathrm{loc}}(z+\square_n)},\mathbf{G}(x,y)\right\rangle_{(z+\square_n)^{(\lceil n/2\rceil)}}\right] + C3^{-\frac{n}{4}} \\
&\le \mathrm{var}\left[\sum_{z\in 3^n\mathbb{Z}^d\cap\square_{n+1}}\frac{1}{|\square_n|}\left\langle \nabla\left[u(\cdot,z+\square_n,q)\right]_{\mathcal{P}},\mathbf{G}(x,y)\right\rangle_{z+\square_n}\right] + C3^{-\frac{n}{4}} \\
&\le C|q|^2\left(\tau_n + 3^{-\frac{n}{4}}\right).
\end{aligned}$$

Using (2.5.27), we may rewrite this as

$$\operatorname{var}\left[\frac{1}{|\square_n|}\left\langle\nabla\left[u(\cdot,\square_n,q)\right]_{\mathcal{P}},e\right\rangle_{\square_n}\right]\le C\left(\tau_n+3^{-\frac{n}{4}}\right).$$

Summing this over $e\in\{\mathfrak{e}_1,\ldots,\mathfrak{e}_d\}$ and recalling (2.5.23) completes the proof. □

We next pass from control of the spatial averages of $\nabla\left[u\right]_{\mathcal{P}}$ given in the previous lemma to control over the function $[u]_{\mathcal{P}}$ itself. The ingredients for this are Lemmas 2.5.5 and 2.5.7 and the (discrete) multiscale Poincaré inequality (Proposition 2.A.2).

LEMMA 2.5.8. *There exists $C(d,\mathfrak{p},\lambda)<\infty$ such that, for $n\in\mathbb{N}$ and $p,q\in\mathbb{R}^d$,*

$$\mathbb{E}\left[\fint_{\square_{n+1}}\left|[u(\cdot,\square_{n+1},q)]_{\mathcal{P}}(x)-x\cdot\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2\,dx\right]\le C|q|^2 3^{2n}\left(3^{-\left(\frac{1}{4d+2}\right)n}+3^{-n}\sum_{k=0}^{n}3^k\tau_k\right). \tag{2.5.32}$$

and

$$\mathbb{E}\left[\fint_{\square_{n+1}}\left|[v(\cdot,\square_{n+1},p)]_{\mathcal{P}}(x)-x\cdot p\right|^2\,dx\right]\le C|p|^2 3^{2n}\left(3^{-\left(\frac{2}{2d+1}\right)n}+3^{-n}\sum_{k=0}^{n}3^k\tau_k\right). \tag{2.5.33}$$

PROOF. We first give the proof of (2.5.32). The main tool for passing from spatial averages of gradients to strong norms for the function itself is the multiscale Poincaré inequality (Proposition 2.A.2), which we apply to the function

$$x\mapsto[u(\cdot,\square_{n+1},q)]_{\mathcal{P}}(x)-x\cdot\bar{\mathbf{a}}_{\square_n}^{-1}q.$$

We note that this function has zero mean on $\square_{n+1}$ by the chosen normalization (2.4.2). Proposition 2.A.2 yields, for $n_0:=\left\lceil\left(\frac{4d+1}{4d+2}\right)(n+1)\right\rceil$,

(2.5.34)
$$\begin{aligned}\fint_{\square_{n+1}}\left|[u(\cdot,\square_{n+1},q)]_{\mathcal{P}}(x)-x\cdot\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2\,dx&\le C3^{2n_0}\fint_{\square_{n+1}}\left|\nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}(x)-\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2\,dx\\&\quad+C\left(\sum_{k=n_0}^{n}3^k\left(3^{-d(n-k)}\sum_{z\in3^k\mathbb{Z}^d\cap\square_{n+1}}\left|\frac{1}{|\square_k|}\left\langle\nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}\right\rangle_{z+\square_k}-\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2\right)^{\frac12}\right)^2.\end{aligned}$$

The first term on the right side is controlled by the triangle inequality, (2.4.24) (with $\delta=\frac{1}{4d+2}$) and (2.5.22), which give

$$\begin{aligned}&\fint_{\square_{n+1}}\left|\nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}(x)-\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2\,dx\\&\quad\le\fint_{\square_{n+1}}\left|\nabla\left[u\right]_{\mathcal{P}}(\cdot,\square_{n+1},q)\right|^2\,dx+\left|\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2\\&\quad\le\mathcal{O}_{\frac{1}{4d-3}}\left(C|q|^2 3^{\frac{n}{4d+2}}\right)+C|q|^2\le\mathcal{O}_{\frac{1}{4d-3}}\left(C|q|^2 3^{\frac{n}{4d+2}}\right).\end{aligned} \tag{2.5.35}$$

To bound the second term on the right side of (2.5.34), we have to estimate the expectation of the (square of the) difference between $\frac{1}{|\square_k|}\left\langle\nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}\right\rangle_{z+\square_k}$ and $\bar{\mathbf{a}}_{\square_n}^{-1}q$ in all successors $z+\square_k$ of $\square_{n+1}$, down to the mesoscale of size 3^{n_0}. The claim is that, for every $k\ge n_0$,

$$3^{-d(n-k)}\sum_{z\in3^k\mathbb{Z}^d\cap\square_{n+1}}\mathbb{E}\left[\left|\frac{1}{|\square_k|}\left\langle\nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}\right\rangle_{z+\square_k}-\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2\right]\le C|q|^2\left(\sum_{m=k}^{n}\tau_m+3^{-\frac{k}{4}}\right). \tag{2.5.36}$$

This inequality is a consequence of what we have shown above. Indeed, by the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, (2.5.12), (2.5.22), (2.5.24) and (2.5.25), we have

$$\begin{aligned}
&3^{-d(n-k)} \sum_{z\in 3^k\mathbb{Z}^d\cap\square_{n+1}} \mathbb{E}\left[\left|\frac{1}{|\square_k|}\left\langle \nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}\right\rangle_{z+\square_k} - \bar{\mathbf{a}}_{\square_n}^{-1} q\right|^2\right] \\
&\quad\le \frac{3^{1-d(n-k)}}{|\square_k|^2} \sum_{z\in 3^k\mathbb{Z}^d\cap\square_{n+1}} \mathbb{E}\left[\left|\left\langle \nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}\right\rangle_{z+\square_k} - \left\langle \nabla\left[u(\cdot,z+\square_k,q)\right]_{\mathcal{P}}\right\rangle_{z+\square_k}\right|^2\right] \\
&\qquad + 3^{1-d(n-k)} \sum_{z\in 3^k\mathbb{Z}^d\cap\square_{n+1}} \mathbb{E}\left[\left|\frac{1}{|\square_k|}\left\langle \nabla\left[u(\cdot,z+\square_k,q)\right]_{\mathcal{P}}\right\rangle_{z+\square_k} - \bar{\mathbf{a}}_{\square_k}^{-1} q\right|^2\right] \\
&\qquad + 3|q|^2\left|\bar{\mathbf{a}}_{\square_k}^{-1} - \bar{\mathbf{a}}_{\square_n}^{-1}\right|^2 \\
&\quad\le C|q|^2\left(\sum_{m=k}^{n}\tau_m + 3^{-\frac{k}{4}}\right).
\end{aligned}$$

This is (2.5.36).

We next combine (2.5.36) with (2.5.34) and (2.5.35), after taking the expectation of the latter two inequalities, to obtain the estimate

$$\mathbb{E}\left[\fint_{\square_{n+1}}\left|\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}(x) - x\cdot\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2\,dx\right] \le C\left(3^{2n_0+\frac{n}{4d+2}}|q|^2 + \mathbb{E}\left[\left(\sum_{k=n_0}^{n} 3^k X_k^{\frac12}\right)^2\right]\right)$$

where the random variable

$$X_k := 3^{-d(n-k)}\sum_{z\in 3^k\mathbb{Z}^d\cap\square_{n+1}}\left|\frac{1}{|\square_k|}\left\langle \nabla\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}\right\rangle_{z+\square_k} - \bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2$$

satisfies

$$\mathbb{E}\left[X_k\right] \le C|q|^2\left(\sum_{m=k}^{n}\tau_m + 3^{-\frac{k}{4}}\right).$$

Using the fact that

$$\left(\sum_{k=n_0}^{n} 3^k X_k^{\frac12}\right)^2 \le C3^n\sum_{k=n_0}^{n}3^kX_k$$

and taking expectations, we obtain

$$\begin{aligned}
\mathbb{E}\left[\fint_{\square_{n+1}}\left|\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}(x) - x\cdot\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2,dx\right] &\le C|q|^2\left(3^{2n_0+\frac{n}{4d+2}} + 3^n\sum_{k=n_0}^{n}3^k\left(\sum_{m=k}^{n}\tau_m + 3^{-\frac{k}{4}}\right)\right) \\
&\le C|q|^2\left(3^{2n_0+\frac{n}{4d+2}} + 3^{n+\frac{3n_0}{4}} + 3^n\sum_{k=0}^{n}3^k\tau_k\right).
\end{aligned}$$

Observe from the definition of n_0 that

$$3^{n+\frac{3n_0}{4}} \le C3^{2n_0+\frac{n}{4d+2}} \le C3^{2n-\frac{n}{4d+2}}.$$

We thus obtain

$$\mathbb{E}\left[\fint_{\square_{n+1}}\left|\left[u(\cdot,\square_{n+1},q)\right]_{\mathcal{P}}(x) - x\cdot\bar{\mathbf{a}}_{\square_n}^{-1}q\right|^2,dx\right] \le C|q|^2 3^{2n}\left(3^{-\left(\frac{1}{4d+2}\right)n} + 3^{-n}\sum_{k=0}^{n}3^k\tau_k\right).$$

This completes the proof of (2.5.32). The proof of (2.5.33) is so similar to that of (2.5.32) that we omit the details. The only difference is that we do not need Lemma 2.5.7 and, in place of (2.5.25), use the estimate (2.4.26), which implies

$$\mathbb{E}\left[\left|\frac{1}{|\square_k|}\left\langle \nabla\left[v(\cdot, z+\square_k, p)\right]_{\mathcal{P}}\right\rangle_{z+\square_k} - p\right|^2\right] \le C|p|^2 3^{-k}.$$

This completes the proof of the lemma. □

We now combine the previous lemma, the Caccioppoli inequality (Lemma 2.3.5) and quadratic response (2.4.6) to obtain an estimate on the expectation of a quantity very close to $\omega(\square_n, q)$ in terms of a weighted average of $\{\tau_k\}_{k\le n}$.

LEMMA 2.5.9. *There exists a constant $C(d,\mathfrak{p},\lambda) < \infty$ such that for every $n\in\mathbb{N}$ and $q\in\mathbb{R}^d$,*

$$\mathbb{E}\left[\left|\nu\left(\square_n, \bar{\mathbf{a}}_{\square_n}^{-1} q\right) - \mu(\square_n, q) - q\cdot \bar{\mathbf{a}}_{\square_n}^{-1} q\right|\right] \le C|q|^2\left(3^{-\left(\frac{1}{4d+2}\right)n} + \sum_{k=0}^{n} 3^{k-n}\tau_k\right). \tag{2.5.37}$$

PROOF. Denote $\bar{p}_n := \bar{\mathbf{a}}_{\square_n}^{-1} q$. According to Corollary 2.4.6 and $|p_n| \le C|q|$,

$$\begin{aligned} &|\nu(\square_n, \bar{p}_n) - \mu(\square_n, q) - q\cdot \bar{p}_n| \\ &\quad \le \frac{1}{|\square_n|}\int_{\mathscr{C}_*(\square_n)} \left|\left(\nabla u(\cdot,\square_n,q) - \nabla v(\cdot,\square_n,\bar{p}_n)\right)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx + \mathcal{O}_{\frac{2}{2d-1}}\left(C|q|^2 3^{-\frac{n}{2}}\right). \end{aligned} \tag{2.5.38}$$

We focus the rest of the argument on estimating the expectation of the first term on the right side of (2.5.38). Using (2.5.10) and the Caccioppoli inequality (Lemma 2.3.5), we have that

$$\begin{aligned} &\mathbb{E}\left[\frac{1}{|\square_n|}\int_{\mathscr{C}_*(\square_n)} \left|\left(\nabla u(\cdot,\square_n,q) - \nabla v(\cdot,\square_n,\bar{p}_n)\right)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right] \\ &\quad \le \mathbb{E}\left[\frac{1}{|\square_n|}\int_{\mathscr{C}_*(\square_n)} \left|\left(\nabla u(\cdot,\square_{n+1},q) - \nabla v(\cdot,\square_{n+1},\bar{p}_n)\right)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx\right] \\ &\qquad + C|q|^2\left(\sum_{k=n}^{m-1}\tau_k + 3^{-\frac{n}{2}}\right) \\ &\quad \le \mathbb{E}\left[\frac{C3^{-2n}}{|\square_n|}\int_{\mathscr{C}_*(\square_{n+1})} \left|u(x,\square_{n+1},q) - v(x,\square_{n+1},\bar{p}_n)\right|^2\,dx\right] \\ &\qquad + C|q|^2\left(\sum_{k=n}^{m-1}\tau_k + 3^{-\frac{n}{2}}\right). \end{aligned} \tag{2.5.39}$$

This reduces the lemma to an appropriate estimate of the last expectation in the previous display.

We continue by writing $u := u(x,\square_{n+1},q)$ and $v := v(x,\square_{n+1},\bar{p}_n)$ for short. We also allow s to be a positive exponent depending on d which may vary in each occurrence. According to Lemma 2.3.3,

$$\begin{aligned} &\int_{\mathscr{C}_*(\square_{n+1})} \left|(u-v)(x) - [u-v]_{\mathcal{P}}(x)\right|^2\,dx \\ &\qquad \le C \sup_{x\in\mathscr{C}_*(\square_{n+1})} |\square_{\mathcal{P}}(x)|^2 \int_{\mathscr{C}_*(\square_{n+1})} \left|\nabla(u-v)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x)\,dx. \end{aligned} \tag{2.5.40}$$

Let A_n be the event

$$A_n := \left\{\sup_{x\in\mathscr{C}_*(\square_{n+1})} |\square_{\mathcal{P}}(x)| > 3^{\frac{n}{2}}\right\}.$$

Then according to Proposition 2.2.4, there exists $s(d) > 0$ such that

$$\mathbb{1}_{A_n}\sup_{x\in\mathscr{C}_*(\square_{n+1})} |\square_{\mathcal{P}}(x)|^2 \le 3^{-\frac{n}{2}}\sup_{x\in\mathscr{C}_*(\square_{n+1})} |\square_{\mathcal{P}}(x)|^3 \le \mathcal{O}_s(C).$$

Hence, by (2.5.40) and the bounds (2.4.20) and (2.4.21), we get

$$\begin{aligned}
&\int_{\mathscr{C}_*(\square_{n+1})} \left|(u-v)(x) - [u-v]_{\mathcal{P}}(x)\right|^2 \, dx \\
&\leq C \sup_{x \in \mathscr{C}_*(\square_{n+1})} \left|\square_{\mathcal{P}}(x)\right|^2 \int_{\mathscr{C}_*(\square_{n+1})} \left(\left|\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}\right|^2 (x) + \left|\nabla v \mathbb{1}_{\{\mathbf{a} \neq 0\}}\right|^2 (x) \right) dx \\
&\leq C 3^n \int_{\mathscr{C}_*(\square_{n+1})} \left(\left|\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}\right|^2 (x) + \left|\nabla v \mathbb{1}_{\{\mathbf{a} \neq 0\}}\right|^2 (x) \right) dx \\
&\quad + C \mathbb{1}_{A_n} \sup_{x \in \mathscr{C}_*(\square_{n+1})} \left|\square_{\mathcal{P}}(x)\right|^2 \int_{\mathscr{C}_*(\square_{n+1})} \left(\left|\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}\right|^2 (x) + \left|\nabla v \mathbb{1}_{\{\mathbf{a} \neq 0\}}\right|^2 (x) \right) dx \\
&\leq \left(C 3^n + \mathcal{O}_s(C)\right) \mathcal{O}_{\frac{1}{2d+1}} \left(C \left|\square_n\right| |q|^2 \right) \\
&\leq \mathcal{O}_s \left(C \left|\square_n\right| |q|^2 3^n \right).
\end{aligned}$$

Thus

$$\mathbb{E}\left[\frac{1}{|\square_n|} \int_{\mathscr{C}_*(\square_{n+1})} \left|(u-v)(x) - [u-v]_{\mathcal{P}}(x)\right|^2 \, dx \right] \leq C |q|^2 3^n.$$

Taking expectations and using the triangle inequality, we get

$$\begin{aligned}
\mathbb{E}&\left[\frac{C 3^{-2n}}{|\square_n|} \int_{\mathscr{C}_*(\square_{n+1})} \left|u(x) - v(x)\right|^2 \, dx \right] \\
&\leq C 3^{-2n} \mathbb{E}\left[\frac{1}{|\square_n|} \int_{\mathscr{C}_*(\square_{n+1})} \left| [u]_{\mathcal{P}}(x) - [v]_{\mathcal{P}}(x) \right|^2 dx \right] + C |q|^2 3^{-n}.
\end{aligned} \tag{2.5.41}$$

Combining (2.5.39) and (2.5.41), we get

$$\begin{aligned}
\mathbb{E}&\left[\frac{1}{|\square_n|} \int_{\mathscr{C}_*(\square_n)} \left|(\nabla u - \nabla v) \mathbb{1}_{\{\mathbf{a} \neq 0\}}\right|^2 (x) \, dx \right] \\
&\leq C 3^{-2n} \mathbb{E}\left[\frac{1}{|\square_n|} \int_{\mathscr{C}_*(\square_{n+1})} \left| [u]_{\mathcal{P}}(x) - [v]_{\mathcal{P}}(x) \right|^2 dx \right] + C |q|^2 \left(\sum_{k=n}^{m-1} \tau_k + 3^{-\frac{n}{2}} \right).
\end{aligned}$$

An application of Lemma 2.5.8 and the triangle inequality yields

$$\mathbb{E}\left[\int_{\mathscr{C}_*(\square_{n+1})} \left| [u]_{\mathcal{P}}(x) - [v]_{\mathcal{P}}(x) \right|^2 \, dx \right] \leq C |q|^2 3^{2n} \left(3^{-\left(\frac{2}{2d+1}\right)n} + \sum_{k=0}^{n} 3^{k-n} \tau_k \right).$$

The previous two displays and (2.5.38) imply (2.5.37) and complete the proof. □

We next show that an iteration of the result of the previous lemma yields a rate of decay for the expectation of ω.

Lemma 2.5.10. *There exist an exponent $\alpha(d, \mathfrak{p}, \lambda) > 0$ and $C(d, \mathfrak{p}, \lambda) < \infty$ such that for every $n \in \mathbb{N}$,*

$$\mathbb{E}\left[\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left(\omega(\square_n, q)\right)_+ \right] \leq C 3^{-n\alpha}. \tag{2.5.42}$$

Proof. In view of the left side of (2.5.37), it is natural to consider the quantity D_n defined for each $n \in \mathbb{N}$ by

$$D_n := \sum_{i=1}^{d} \mathbb{E}\left[\nu\left(\square_n, \mathfrak{e}_i\right) - \mu(\square_n, \bar{\mathbf{a}}_{\square_n} \mathfrak{e}_i) - \mathfrak{e}_i \cdot \bar{\mathbf{a}}_{\square_n} \mathfrak{e}_i \right].$$

Notice that (2.4.13) and (2.4.16) imply that D_n is bounded and Corollary 2.4.6 implies that D_n has a small negative part: for every $n \in \mathbb{N}$,

$$-C 3^{-\frac{n}{2}} \leq D_n \leq C. \tag{2.5.43}$$

As we will see in Step 4, below, D_n controls $\mathbb{E}\left[\sup_{q\in\mathbb{R}^d}|q|^{-2}\left(\omega(\square_n,q)\right)_+\right]$ in the sense that (2.5.44) implies (2.5.42). Therefore our goal is to show that, for α and C as in the statement of the lemma,

$$D_n \le C3^{-n\alpha}. \tag{2.5.44}$$

Step 1. We show that the existence of $c(d,\lambda,\mathfrak{p})>0$, $C(d,\lambda,\mathfrak{p})<\infty$ and $N_0(d,\lambda,\mathfrak{p})\in\mathbb{N}$ such that, for every $n\in\mathbb{N}$ with $n\ge N_0$,

$$D_{n+1} \le D_n - c\tau_n + C3^{-\frac{n}{4}}. \tag{2.5.45}$$

By the definition of the matrix $\bar{\mathbf{a}}_{\square_{n+1}}$, we have that, for each $p\in\mathbb{R}^d$, the map

$$q\mapsto -\mathbb{E}\left[\mu(\square_{n+1},q)\right]-p\cdot q=\frac{1}{2}q\cdot\bar{\mathbf{a}}_{\square_{n+1}}^{-1}q-p\cdot q$$

achieves its minimum at the point $q=\bar{\mathbf{a}}_{\square_{n+1}}p$. By quadratic response and the bounds on $\mathbb{E}\left[\mu(\square_{n+1},q)\right]$ implied by (2.4.16), this implies the existence of $C(d,\lambda,\mathfrak{p})<\infty$ such that, for every $q\in\mathbb{R}^d$,

$$\begin{aligned}
&-\mathbb{E}\left[\mu(\square_{n+1},\bar{\mathbf{a}}_{\square_{n+1}}\mathfrak{e}_i)\right]-\mathfrak{e}_i\cdot\bar{\mathbf{a}}_{\square_{n+1}}\mathfrak{e}_i\\
&\quad\le -\mathbb{E}\left[\mu(\square_{n+1},q)\right]-\mathfrak{e}_i\cdot q\\
&\quad\le -\mathbb{E}\left[\mu(\square_{n+1},\bar{\mathbf{a}}_{\square_{n+1}}\mathfrak{e}_i)\right]-\mathfrak{e}_i\cdot\bar{\mathbf{a}}_{\square_{n+1}}\mathfrak{e}_i+C\left|q-\bar{\mathbf{a}}_{\square_{n+1}}\mathfrak{e}_i\right|^2.
\end{aligned} \tag{2.5.46}$$

Using the first line of (2.5.46) with $q=\bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i$ yields

$$\begin{aligned}
D_{n+1} &= \sum_{i=1}^d\left(\mathbb{E}\left[\left(\nu\left(\square_{n+1},\mathfrak{e}_i\right)-\mu(\square_{n+1},\bar{\mathbf{a}}_{\square_{n+1}}\mathfrak{e}_i)-\mathfrak{e}_i\cdot\bar{\mathbf{a}}_{\square_{n+1}}\mathfrak{e}_i\right)\right]\right)\\
&\le \sum_{i=1}^d\left(\mathbb{E}\left[\left(\nu\left(\square_{n+1},\mathfrak{e}_i\right)-\mu(\square_{n+1},\bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i)-\mathfrak{e}_i\cdot\bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i\right)\right]\right)\\
&= D_n+\sum_{i=1}^d\left(\mathbb{E}\left[\nu\left(\square_{n+1},\mathfrak{e}_i\right)\right]-\mathbb{E}\left[\nu\left(\square_n,\mathfrak{e}_i\right)\right]\right)\\
&\qquad -\sum_{i=1}^d\left(\mathbb{E}\left[\mu(\square_{n+1},\bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i)\right]-\mathbb{E}\left[\mu(\square_n,\bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i)\right]\right)\\
&\le D_n-c\tau_n+C3^{-\frac{n}{4}},
\end{aligned}$$

where in the last line we used the bounds (2.4.48), (2.4.49) and (2.5.22), which hold for all sufficiently large n depending only on $(d,\lambda,\mathfrak{p})$. This completes the proof of (2.5.45).

In view of the form of the right side of (2.5.37) as well as the result of the previous step, it is natural to modify D_n slightly by defining, for every $n\ge N_0$,

$$\widetilde{D}_n := 3^{-\frac{n}{2}}\sum_{k=N_0}^n 3^{\frac{k}{2}}D_k.$$

Notice that $\widetilde{D}_n$ is, up to a constant, a weighted average of $D_{N_0},\ldots,D_n$ and, in particular, by (2.5.43), we have that

$$\widetilde{D}_n = D_n+3^{-\frac{n}{2}}\sum_{k=N_0}^{n-1}3^{\frac{k}{2}}D_k \ge D_n-Cn3^{-\frac{n}{2}}\ge D_n-C3^{-\frac{n}{4}}. \tag{2.5.47}$$

Therefore, rather than (2.5.44), we may prove the stronger bound

$$\widetilde{D}_n \le C3^{-n\alpha}. \tag{2.5.48}$$

Step 2. We show that there exists $\theta(d,\lambda,\mathfrak{p})\in\left[\frac{1}{2},1\right)$ and $C(d,\lambda,\mathfrak{p})<\infty$ such that, for every $n\in\mathbb{N}$ with $n\ge N_0$,

$$\widetilde{D}_{n+1}\le\theta\widetilde{D}_n+C3^{-\left(\frac{2}{2d+1}\right)n}. \tag{2.5.49}$$

Using (2.5.43) and (2.5.45), we find that

$$\widetilde{D}_n - \widetilde{D}_{n+1} = 3^{-\frac{n}{2}} \sum_{k=N_0}^{n} 3^{\frac{k}{2}} \left(D_k - D_{k-1}\right) - 3^{\frac{1}{2}(N_0-(n+1))} D_{N_0}$$
$$\geq c3^{-\frac{n}{2}} \sum_{k=N_0}^{n} 3^{\frac{k}{2}} \left(\tau_k - C3^{-\frac{n}{4}}\right) - C3^{-\frac{n}{2}} \geq c3^{-\frac{n}{2}} \sum_{k=N_0}^{n} 3^{\frac{k}{2}} \tau_k - C3^{-\frac{n}{4}}. \tag{2.5.50}$$

Next, we apply Lemma 2.5.9, which tells us that

$$D_k \leq C\left(3^{-\left(\frac{2}{2d+1}\right)k} + \sum_{l=0}^{k} 3^{l-k}\tau_l\right) \leq C\left(3^{-\left(\frac{2}{2d+1}\right)k} + \sum_{l=N_0}^{n} 3^{l-k}\tau_l\right).$$

Summing this over $k \in \{N_0, \ldots, n\}$ gives

$$\begin{aligned}\widetilde{D}_n &\leq C3^{-\frac{n}{2}} \sum_{k=N_0}^{n} 3^{\frac{k}{2}} \left(3^{-\left(\frac{2}{2d+1}\right)k} + \sum_{l=N_0}^{k} 3^{l-k}\tau_l\right) \\ &= C3^{-\frac{n}{2}} \sum_{k=N_0}^{n} \sum_{l=N_0}^{k} 3^{-\frac{k}{2}} 3^l \tau_l + C3^{-\left(\frac{2}{2d+1}\right)n} \\ &= C3^{-\frac{n}{2}} \sum_{l=N_0}^{n} \sum_{k=l}^{n} 3^{-\frac{k}{2}} 3^l \tau_l + C3^{-\left(\frac{2}{2d+1}\right)n} \\ &\leq C3^{-\frac{n}{2}} \sum_{l=N_0}^{n} 3^{\frac{l}{2}} \tau_l + C3^{-\left(\frac{2}{2d+1}\right)n}.\end{aligned}$$

Combining the previous displays with (2.5.50) gives

$$\widetilde{D}_{n+1} \leq (1-c)\widetilde{D}_n + C3^{-\left(\frac{2}{2d+1}\right)n}.$$

This completes the proof of (2.5.49).

Step 3. We complete the proof of (2.5.48). By an interation of (2.5.49) we get, for every $n \geq N_0$,

$$\widetilde{D}_n \leq \left(\sum_{k=N_0}^{n} \theta^k 3^{\left(\frac{2}{2d-1}\right)(k-n)}\right) \widetilde{D}_{N_0}.$$

Taking θ closer to 1, if necessary, so that $\theta > \left(\frac{1}{3}\right)^{\frac{2}{2d-1}}$, we get that each term in the sum is at most θ^n. Using also $\widetilde{D}_{N_0} \leq C$, we therefore obtain

$$\widetilde{D}_n \leq Cn\theta^n \leq C\theta^{\frac{n}{2}}.$$

Taking $\alpha := \log 3/2\,|\log\theta|$ so that $\theta^{\frac{1}{2}} = 3^{-\alpha}$ yields the desired bound (2.5.48).

Step 4. We complete the proof of (2.5.42). First, we observe that, due to Corollary 2.4.29, we have the function

$$q \mapsto \omega(\square_n, q) + C3^{-\frac{n}{2}}|q|^2.$$

is nonnegative and quadratic and hence convex. It follows that

$$\sup_{q\in\mathbb{R}^d} \frac{1}{|q|^2} \left(\omega(\square_n, q)\right)_+ \leq \sup_{q\in\mathbb{R}^d} \frac{1}{|q|^2} \left(\omega(\square_n, q) + C3^{-\frac{n}{2}}|q|^2\right) \leq \sum_{i=1}^{d} \omega(\square_n, e_i) + C3^{-\frac{n}{2}}.$$

Next we observe that (2.4.29) and (2.5.44) (which we recall is a consequence of (2.5.47) and (2.5.48)) imply that

$$|D_n| \leq C3^{-n\alpha}.$$

Using this and (2.5.45), we find that

$$\tau_n \leq C3^{-\frac{n}{2}} + C(D_n - D_{n+1}) \leq C3^{-n\alpha}.$$

According to the previous line and (2.5.24), we deduce that

$$|\bar{\mathbf{a}}_{\square_n} - \bar{\mathbf{a}}| \le C3^{-n\alpha}.$$

Therefore, by the previous line, (2.4.29) and (2.5.46) applied with $q = \bar{\mathbf{a}}\mathfrak{e}_i$, we find

$$\begin{aligned}
&\left|\mathbb{E}\left[\omega(\square_n, \mathfrak{e}_i)_+\right] - \mathbb{E}\left[\nu\left(\square_n, \mathfrak{e}_i\right) - \mu(\square_n, \bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i) - \mathfrak{e}_i \cdot \bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i\right]\right| \\
&\qquad \le \left|\mathbb{E}\left[\omega(\square_n, \mathfrak{e}_i)\right] - \mathbb{E}\left[\nu\left(\square_n, \mathfrak{e}_i\right) - \mu(\square_n, \bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i) - \mathfrak{e}_i \cdot \bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i\right]\right| + C3^{-\frac{n}{2}} \\
&\qquad \le C\left|\bar{\mathbf{a}}_{\square_n} - \bar{\mathbf{a}}\right|^2 + C3^{-\frac{n}{2}} \\
&\qquad \le C3^{-n\alpha}.
\end{aligned}$$

Combining the above and using (2.5.44) again, we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{q\in\mathbb{R}^d} \frac{1}{|q|^2}(\omega(\square_n, q))_+\right] &\le \sum_{i=1}^d \mathbb{E}\left[(\omega(\square_n, \mathfrak{e}_i))_+\right] + C3^{-\frac{n}{2}} \\
&\le \sum_{i=1}^d \mathbb{E}\left[\nu\left(\square_n, \mathfrak{e}_i\right) - \mu(\square_n, \bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i) - \mathfrak{e}_i \cdot \bar{\mathbf{a}}_{\square_n}\mathfrak{e}_i\right] + C3^{-n\alpha} \\
&= D_n + C3^{-n\alpha} \\
&\le C3^{-n\alpha}.
\end{aligned}$$

This completes the argument. □

To complete the proof of Proposition 2.5.2, we need to show that the control over the expectation of ω given in the previous lemma can be enhanced, using independence, to control over exponential moments of ω. This is a consequence of the following lemma.

Lemma 2.5.11. *There exist exponents $s(d) > 0$ and $\alpha(s, \lambda, \mathfrak{p}) > 0$ and a constant $C(s, d, \lambda, \mathfrak{p}) < \infty$ such that, for every $\square \in \mathcal{T}$,*

$$\sup_{q\in\mathbb{R}^d} \frac{1}{|q|^2}|\omega(\square, q)| = \mathcal{O}_s\left(C\left(\operatorname{size}(\square)\right)^{-\alpha}\right). \tag{2.5.51}$$

Proof. The argument is an application of the exponential moment method and subadditivity, modified to take care of the fact that ω is not a bounded random variable. It is enough to prove the result for cubes of the form $\square = \square_m$ for $m \in \mathbb{N}$ by approximate stationarity, see (2.4.40) and (2.4.43). We fix $m \in \mathbb{N}$ and set $n := \left\lceil \frac{1}{2}m \right\rceil$ and $k := \left\lceil \frac{1}{4}m \right\rceil$. Throughout the argument, we let s denote a positive exponent depending only on d which may vary in each occurrence. Likewise α is a positive exponent depending only on $(d, \lambda, \mathfrak{p})$ which may vary.

We denote, for each $\square \in \mathcal{T}$,

$$\rho(\square) := \sup_{q\in\mathbb{R}^d} \frac{1}{|q|^2}\left(\omega(\square, q)\right)_+.$$

To prepare for the use of independence, we also let $\rho_{\mathrm{loc}}^{(k)}$ and $\omega_{\mathrm{loc}}^{(k)}$ denote the same quantities as ρ and ω except with the local quantities $\mu_{\mathrm{loc}}^{(k)}$ and $\nu_{\mathrm{loc}}^{(k)}$ in place of μ and ν in their definitions. That is,

$$\omega_{\mathrm{loc}}^{(k)}(\square, q) := \nu_{\mathrm{loc}}^{(k)}\left(\square, \bar{\mathbf{a}}^{-1}q\right) - \mu_{\mathrm{loc}}^{(k)}(\square, q) - q \cdot \bar{\mathbf{a}}^{-1}q$$

and

$$\rho_{\mathrm{loc}}^{(k)}(\square) := \sup_{q\in\mathbb{R}^d} \frac{1}{|q|^2}\left(\omega_{\mathrm{loc}}^{(k)}(\square, q)\right)_+.$$

By (2.4.39) and Proposition 2.4.10, we see that

$$\rho_{\mathrm{loc}}^{(k)}(\square) \text{ and } \omega_{\mathrm{loc}}^{(k)}(\square, q) \text{ is } \mathcal{F}(\square)\text{–measurable} \tag{2.5.52}$$

and

$$\left|\omega(\square, q) - \omega_{\mathrm{loc}}^{(k)}(\square, q)\right| + \left|\rho(\square) - \rho_{\mathrm{loc}}^{(k)}(\square)\right| \le \mathcal{O}_s\left(C3^{-\frac{k}{4}}\right). \tag{2.5.53}$$

Step 1. By removing a bad event of small probability, we essentially reduce to the case that $\rho(z+\square_n)$ is bounded on the subcubes $z+\square_n \subseteq \square_m$.

According to Lemma 2.4.4, its consequence (2.4.17) and the Markov inequality, there exists $C_1(d,\mathfrak{p},\lambda) < \infty$ such that, for every $z \in 3^n\mathbb{Z}^d$,

$$\mathbb{P}\left[\rho(z+\square_n) > C_1\right] \le C\exp\left(-\frac{1}{C}3^{\frac{n}{2d+1}}\right).$$

By (2.5.53), this implies that

$$\begin{aligned}\mathbb{P}\left[\rho_{\mathrm{loc}}^{(k)}(z+\square_n) > C_1\right] &\le C\exp\left(-\frac{1}{C}3^{\frac{n}{2d+1}}\right) + C\exp\left(-\frac{1}{C}3^{\frac{ks}{4}}\right)\\ &\le C\exp\left(-\frac{1}{C}3^{m\alpha}\right).\end{aligned}$$

Thus

$$\mathbb{P}\left[\sup_{z\in 3^n\mathbb{Z}^d\cap\square_m} \rho_{\mathrm{loc}}^{(k)}(z+\square_n) > C_1\right] \le C3^{d(m-n)}\exp\left(-\frac{1}{C}3^{m\alpha}\right) \le C\exp\left(-\frac{1}{C}3^{m\alpha}\right).$$

For each $z \in 3^n\mathbb{Z}^d$, denote the events

$$G_z := \left\{\rho_{\mathrm{loc}}^{(k)}(z+\square_n) \le C_1\right\}$$

and

$$H := \left\{\sup_{z\in 3^n\mathbb{Z}^d\cap\square_m} \rho_{\mathrm{loc}}^{(k)}(z+\square_n) > C_1\right\} = \Omega \setminus \left(\bigcap_{z\in 3^n\mathbb{Z}^d\cap\square_m} G_z\right).$$

Note that $G_z \in \mathcal{F}(z+\square_n)$. Note that the estimate above gives a bound for $\mathbb{1}_H$:

$$\mathbb{1}_H \le \mathcal{O}_1\left(C3^{-m\alpha}\right),$$

and thus we can bound ω from above on the "bad" event H:

$$\rho(\square_m)\,\mathbb{1}_H \le \mathcal{O}_s(C)\cdot\mathcal{O}_1\left(C3^{-m\alpha}\right) \le \mathcal{O}_s\left(C3^{-m\alpha}\right). \tag{2.5.54}$$

Step 2. The concentration argument. The claimed estimate is

$$\rho(\square_m)\mathbb{1}_{\Omega\setminus H} \le \mathcal{O}_s\left(C3^{-m\alpha}\right). \tag{2.5.55}$$

We begin by noticing that (2.4.32), (2.4.33) and (2.5.53) give us the approximate subadditivity bound

$$\begin{aligned}\rho(\square_m)\mathbb{1}_{\Omega\setminus H} &\le 3^{-d(m-n)}\sum_{z\in 3^n\mathbb{Z}^d\cap\square_m}\rho_{\mathrm{loc}}^{(k)}(z+\square_n)\mathbb{1}_{G_z} + \mathcal{O}_s\left(C3^{-\frac{k}{4}}\right)\\ &\le 3^{-d(m-n)}\sum_{z\in 3^n\mathbb{Z}^d\cap\square_m}\rho_{\mathrm{loc}}^{(k)}(z+\square_n)\wedge C_1 + \mathcal{O}_s\left(C3^{-\frac{k}{4}}\right).\end{aligned} \tag{2.5.56}$$

Note that for $z,z' \in 3^n\mathbb{Z}^d\cap\square_m$ with $z \ne z'$,

$$\rho_{\mathrm{loc}}^{(k)}\left(z+\square_n\right) \quad \text{and} \quad \rho_{\mathrm{loc}}^{(k)}\left(z'+\square_n\right) \quad \text{are } \mathbb{P}\text{–independent.} \tag{2.5.57}$$

We now fix $t>0$ and compute

$$\begin{aligned}&\log\mathbb{E}\left[\exp\left(t\sum_{z\in 3^n\mathbb{Z}^d\cap\square_m}\rho_{\mathrm{loc}}^{(k)}(z+\square_n)\wedge C_1\right)\right]\\ &\quad= \log\mathbb{E}\left[\prod_{z\in 3^n\mathbb{Z}^d\cap\square_m}\exp\left(t\rho_{\mathrm{loc}}^{(k)}\left(z+\square_n\right)\wedge C_1\right)\right]\\ &\quad\le \sum_{z\in 3^n\mathbb{Z}^d\cap\square_m}\log\mathbb{E}\left[\exp\left(t\rho_{\mathrm{loc}}^{(k)}\left(z+\square_n\right)\wedge C_1\right)\right] && \text{(by (2.5.57))}\\ &\quad= 3^{d(m-n)}\log\mathbb{E}\left[\exp\left(t\rho_{\mathrm{loc}}^{(k)}(\square_n)\wedge C_1\right)\right] && \text{(by (2.4.40)).}\end{aligned}$$

We take $t := 1/K$ and estimate the last term using the elementary inequalities

$$\begin{cases} \exp(s) \le 1 + Cs & \text{for all } 0 \le s \le C_1, \\ \log(1+s) \le s & \text{for all } s \ge 0, \end{cases}$$

to get

$$3^{-d(m-n)} \log \mathbb{E}\left[\exp\left(C^{-1} \sum_{z \in 3^n\mathbb{Z}^d \cap \square_m} \rho_{\mathrm{loc}}^{(k)}(z + \square_n) \wedge C_1\right)\right] \le C\mathbb{E}\left[\rho_{\mathrm{loc}}^{(k)}(\square_n)\right].$$

Applying (2.5.42) and (2.5.53), we find that

$$\mathbb{E}\left[\rho_{\mathrm{loc}}^{(k)}(\square_n)\right] \le C3^{-n\alpha} \le C3^{-\frac{m\alpha}{2}}.$$

The previous two lines and Chebyshev's inequality imply that

$$\mathbb{P}\left[3^{-d(m-n)} \sum_{z \in 3^n\mathbb{Z}^d \cap \square_m} \rho_{\mathrm{loc}}^{(k)}(z + \square_n) \wedge C_1 > t\right] \le \exp\left(-c3^{d(m-n)}\left(t - C3^{-\frac{m\alpha}{2}}\right)\right).$$

This implies that

$$3^{-d(m-n)} \sum_{z \in 3^n\mathbb{Z}^d \cap \square_m} \rho_{\mathrm{loc}}^{(k)}(z + \square_n) \wedge C_1 \le \mathcal{O}_1\left(C3^{-\frac{m\alpha}{2}}\right).$$

Combined with (2.5.56), we get

$$\rho(\square_m)\mathbb{1}_{\Omega \setminus H} \le \mathcal{O}_1\left(C3^{-\frac{m\alpha}{2}}\right) + \mathcal{O}_s\left(C3^{-\frac{k}{2}}\right) \le \mathcal{O}_s\left(C3^{-m\alpha}\right),$$

which is (2.5.55).

Step 3. We complete the argument. By combining the previous steps, we get

$$\rho(\square_m) = \rho(\square_m)\mathbb{1}_{\Omega \setminus H} + \rho(\square_m)\mathbb{1}_H \le \mathcal{O}_s\left(C3^{-\frac{m\alpha}{2}}\right).$$

We also recall that, by (2.4.29), we have

$$\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2}\omega(\square_m, q) \ge -\mathcal{O}_{\frac{2}{2d-1}}\left(C3^{-\frac{m}{2}}\right).$$

The previous two inequalities yield the lemma after we shrink α. □

Proof of Proposition 2.5.2. Applying Lemma 2.5.3 and then Lemma 2.5.11, we obtain

$$\begin{aligned} &\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^4}\left|\mu(\square, q) - \frac{1}{2}q \cdot \bar{\mathbf{a}}^{-1}q\right|^2 + \sup_{p \in \mathbb{R}^d} \frac{1}{|p|^4}\left|\nu(\square, p) - \frac{1}{2}p \cdot \bar{\mathbf{a}}p\right|^2 \\ &\qquad \le C\left(\sup_{e \in \partial B_1} \omega(\square, e)_+ + \mathcal{O}_{\frac{1}{2d+1}}\left(C\,\mathrm{size}(\square)^{-\frac{1}{2}}\right)\right) \le \mathcal{O}_s\left(C\,\mathrm{size}(\square)^{-\alpha}\right). \end{aligned}$$

Taking square roots and shrinking α gives the desired bound for some $s(d, \lambda, \mathfrak{p}) > 0$. Using (2.1.10) to interpolate this results with the bounds (2.4.16) and (2.4.13), we can allow the exponent s to depend only on d by further shrinking α. □

We conclude this section by introducing $\mathcal{N}$ which will be an important tool in the proofs of Section 6.

DEFINITION 2.5.12. For $m \in \mathbb{N}$, we define the random variable $\mathcal{N}$ by

$$\mathcal{N} := \sup \Bigg\{ 3^m \, : \, m \in \mathbb{N}, \, n = \left\lceil \frac{m}{4} \right\rceil, \; k = \left\lceil \frac{m}{8} \right\rceil, \\ \sup_{z \in 3^k\mathbb{Z}^d \cap \square_m} \left(\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left| \mu(z + \square_n, q) - \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q \right| + \sup_{p \in \mathbb{R}^d} \frac{1}{|p|^2} \left| \nu(z + \square_n, p) - \frac{1}{2} p \cdot \bar{\mathbf{a}} p \right| \right) \\ \geq C 3^{-m\alpha} \Bigg\}.$$

Here the exponent $\alpha = \alpha(d, \mathfrak{p}, \lambda) > 0$ is defined in the proof of the following proposition, and may be smaller than the one in Proposition 2.5.2.

PROPOSITION 2.5.13. *There exist $s(d, \mathfrak{p}, \lambda) > 0$ and $C(d, \mathfrak{p}, \lambda) < \infty$ such that*

$$\mathcal{N} \leq \mathcal{O}_s(C). \tag{2.5.58}$$

PROOF. First we prove, for each fixed $m \in \mathbb{N}$ and $z \in 3^k\mathbb{Z}^d$, an estimate of the form

$$\mathbb{P}\left[\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left| \mu(z + \square_n, q) - \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q \right| \geq C 3^{-m\alpha} \right] \leq C \exp\left(-C^{-1} 3^{-\alpha m} \right),$$

where $n := \left\lceil \frac{m}{4} \right\rceil$. To do this we apply Proposition 2.4.10 and use the stationarity property (2.4.40) and (2.4.43) to obtain

$$\begin{aligned} &\mathbb{P}\left[\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left| \mu(z + \square_n, q) - \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q \right| \geq C 3^{-m\alpha} \right] \\ &\quad \leq \mathbb{P}\left[\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left| \mu^{(k)}_{\mathrm{loc}}(z + \square_n, q) - \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q \right| \geq \frac{C}{2} 3^{-m\alpha} \right] + C \exp\left(-C^{-1} 3^{-\alpha m} \right) \\ &\quad \leq \mathbb{P}\left[\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left| \mu^{(k)}_{\mathrm{loc}}(\square_n, q) - \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q \right| \geq \frac{C}{2} 3^{-m\alpha} \right] + C \exp\left(-C^{-1} 3^{-\alpha m} \right) \\ &\quad \leq \mathbb{P}\left[\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left| \mu_{\mathrm{loc}}(\square_n, q) - \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q \right| \geq \frac{C}{4} 3^{-m\alpha} \right] + C \exp\left(-C^{-1} 3^{-\alpha m} \right) \\ &\quad \leq C \exp\left(-C^{-1} 3^{-\alpha m} \right). \end{aligned}$$

This results and the fact that $3^m \geq Cn$ gives to

$$\begin{aligned} &\mathbb{P}\left[\sup_{z \in 3^k\mathbb{Z}^d \cap \square_m} \sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left| \mu(z + \square_n, q) - \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q \right| \geq C 3^{-m\alpha} \right] \\ &\quad \leq \sum_{z \in 3^k\mathbb{Z}^d \cap \square_m} \mathbb{P}\left[\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left| \mu^{(k)}_{\mathrm{loc}}(z + \square_n, q) - \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q \right| \geq \frac{C}{2} 3^{-m\alpha} \right] \\ &\quad \leq 3^{d(m-k)} C \exp\left(-C^{-1} 3^{-\alpha m} \right) \\ &\quad \leq C \exp\left(-C^{-1} 3^{-\alpha m} \right). \end{aligned}$$

Similarly, we obtain

$$\mathbb{P}\left[\sup_{z \in 3^k\mathbb{Z}^d \cap \square_m} \sup_{p \in \mathbb{R}^d} \frac{1}{|p|^2} \left| \nu(z + \square_n, p) - \frac{1}{2} p \cdot \bar{\mathbf{a}} p \right| \geq C 3^{-m\alpha} \right] \leq C \exp\left(-C^{-1} 3^{-\alpha m} \right).$$

The estimate (2.5.58) is now a consequence of Lemma 2.2.3 with X_m defined as the indicator function of the event

$$\left\{ \sup_{z \in 3^k\mathbb{Z}^d \cap \square_m} \left(\sup_{q \in \mathbb{R}^d} \frac{1}{|q|^2} \left| \mu(z + \square_n, q) - \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q \right| + \sup_{p \in \mathbb{R}^d} \frac{1}{|p|^2} \left| \nu(z + \square_n, p) - \frac{1}{2} p \cdot \bar{\mathbf{a}} p \right| \right) \geq C 3^{-m\alpha} \right\}. \qquad \square$$

2.6. Homogenization error estimates for the Dirichlet problem

In this section, we pass from control on the subadditive quantities $-\mu$ and ν to quenched control on the error in homogenization for the Dirichlet problem. Combined with Proposition 2.5.2, this allows us to complete the proof of Theorem 2.1.1. The arguments here are entirely deterministic in the sense that they produce an estimate for the homogenization error in terms of the coarseness of the partition $\mathcal{P}$ and the convergence of $-\mu$ and ν in mesoscopic cubes. In particular, we are not using theory developed in the previous section. The arguments here are a variation of similar ones in [**21**].

In this section, we abuse notation by using the symbol $\square_m$ to also denote the *continuum* cube

$$\left[-\frac{1}{2}(3^m-1), \frac{1}{2}(3^m-1)\right]^d \subseteq \mathbb{R}^d.$$

It will be made clear from the context whether $\square_m$ refers to the continuum cube or the discrete one. Moreover in this section we will use

$$|\square_m| = \mathrm{Card}(\square_m) = 3^{dm}$$

which is slightly different from $\mathrm{Leb}(\square_m) = (3^m-1)^d$. We will also write $\fint_{\square_m} = \frac{1}{|\square_m|}\int_{\square_m} = 3^{-dm}\int_{\square_m}$. We further abuse notation by extending the coarsened function $[u]_{\mathcal{P}}$ to be defined on a continuum domain by taking it to be constant on each unit cube of the form $z + \left(-\frac{1}{2}, \frac{1}{2}\right]^d$ with $z \in \mathbb{Z}^d$. To avoid confusion, here we will use the symbols $\int$ and $\fint$ only to denote integration with respect to Lebesgue measure on $\mathbb{R}^d$ and write sums with $\sum$.

We fix, once and for all, a positive integer $m \in \mathbb{N}$ and a function $u \in \mathcal{A}_*(\square_m)$. We also fix an exponent $p > 2$ and set

$$M := \left(\frac{1}{|\square_m|}\sum_{x \in \mathscr{C}_*(\square_m)} |\nabla u \mathbb{1}_{\mathbf{a}\neq 0}|^p(x)\right)^{\frac{1}{p}}.$$

To define u_{hom} on the continuum cube $\square_m$, we first define

$$\widetilde{w}(x) := \begin{cases} u(x) & \text{if } x \in \mathscr{C}_*(\square_m)\cap\partial\square_m, \\ [u]_{\mathcal{P}}(x) & \text{otherwise.} \end{cases}$$

We then extend $\widetilde{w}$ to the continuum by taking it to be constant on unit cubes of the form $z + \left(-\frac{1}{2}, \frac{1}{2}\right]^d$ with $z \in \mathbb{Z}^d$. We then make it smooth by convolving it with a smooth bump function $\rho \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ which is supported in $\left(-\frac{1}{2}, \frac{1}{2}\right)^d$ and has unit mass. Call the function obtained in this way $w \in C^\infty(\square_m)$ and notice that,

$$\forall x \in \mathbb{Z}^d \cap \square_m,\ w(x) = \widetilde{w}(x).$$

We take $u_{\mathrm{hom}} \in H^1(\square_m)$ to be the solution of the homogenized Dirichlet problem

$$\begin{cases} -\nabla\cdot(\bar{\mathbf{a}}\nabla u_{\mathrm{hom}}) = 0 & \text{in } \square_m, \\ u_{\mathrm{hom}} = w & \text{on } \partial\square_m. \end{cases}$$

The purpose of this section is to prove the following proposition which, in view of our setup in this section, implies Theorem 2.1.1.

We recall that the random scales $\mathcal{M}_t$ and $\mathcal{N}$ are given in Proposition 2.2.4 and Definition 2.5.12, respectively, and the partition $\mathcal{Q}$ is the one for the Meyers estimate (see Definition 2.3.9) which is coarser than $\mathcal{P}$.

Proposition 2.6.1. *There exist $t := t(d,\mathfrak{p},\lambda,p) < +\infty$, $\alpha := \alpha(d,\mathfrak{p},\lambda,p) > 0$ and $C := C(d,\mathfrak{p},\lambda,p) < +\infty$ such that $3^m \geq \mathcal{N}\vee\mathcal{M}_t(\mathcal{Q})$ implies that*

$$\frac{1}{|\square_m|}\sum_{x\in\mathscr{C}_*(\square_m)} |u(x) - u_{\mathrm{hom}}(x)|^2 \leq CM^2\,\mathrm{size}(\square_m)^{(2-\alpha)}.$$

The main idea in the proof of the proposition is to construct a function $\widetilde{u} \in w + H_0^1(\square_m)$ which satisfies the property that its continuous homogenized energy is smaller that the discrete heterogeneous energy of u. This is explicited in Lemma 2.6.2. In the other direction, we will construct a function $\widetilde{u}_{\mathrm{hom}} : \mathscr{C}_*(\square_m) \to \mathbb{R}$ which statisfies the following three properties:

(1) $\widetilde{u}_{\mathrm{hom}}$ is equal to w on $\mathscr{C}_*(\square_m) \cap \partial \square_m$,
(2) The discrete heterogenous energy of $\widetilde{u}_{\mathrm{hom}}$ is smaller than the continuous homogenized energy of u_{hom},
(3) The discrete L^2-norm of $\widetilde{u}_{\mathrm{hom}} - u_{\mathrm{hom}}$ is small.

This is specified in Lemma 2.6.3. We will then deduce Proposition 2.6.1 from these results.

LEMMA 2.6.2. *There exists $t := t(d, \mathfrak{p}, \lambda, p) < +\infty$, $\alpha := \alpha(d, \mathfrak{p}, \lambda, p) > 0$ and $C := C(d, \mathfrak{p}, \lambda, p) < +\infty$ such that, in the case that $3^m \geq \mathcal{N} \vee \mathcal{M}_t(\mathcal{Q})$, there exists a function $\widetilde{u} \in w + H_0^1(\square_m)$ satisfying the energy bound*

$$\fint_{\square_m} \nabla \widetilde{u}(x) \cdot \bar{\mathbf{a}} \nabla \widetilde{u}(x)\, dx \leq \frac{1}{|\square_m|} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(\square_m)} + CM^2 3^{-m\alpha}. \tag{2.6.1}$$

LEMMA 2.6.3. *There exist $t := t(d, \mathfrak{p}, \lambda, p) < +\infty$, $\alpha := \alpha(d, \mathfrak{p}, \lambda, p) > 0$ and $C := C(d, \mathfrak{p}, \lambda, p) < +\infty$ such that, in the case that $3^m \geq \mathcal{N} \vee \mathcal{M}_t(\mathcal{Q})$, there exists a function $\widetilde{u}_{\mathrm{hom}} : \mathscr{C}_*(\square_m) \to \mathbb{R}$ satisfying the following three properties:*

(i) *Boundary condition:*

$$\forall x \in \mathscr{C}_*(\square_m) \cap \partial \square_m, \quad \widetilde{u}_{\mathrm{hom}}(x) = u(x).$$

(ii) *Energy estimate:*

$$\frac{1}{|\square_m|} \langle \nabla \widetilde{u}_{\mathrm{hom}}, \mathbf{a}(x) \nabla \widetilde{u}_{\mathrm{hom}} \rangle_{\mathscr{C}_*(\square_m)} \leq \fint_{\square_m} \left(\nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x) \right) dx + CM^2 3^{-m\alpha},$$

(iii) *L^2 estimate:*

$$\frac{3^{-2m}}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)} \left(\widetilde{u}_{\mathrm{hom}}(x) - u_{\mathrm{hom}}(x) \right)^2 \leq CM^2 3^{-m\alpha}.$$

We now prove Proposition 2.6.1 and postpone the proof of these Lemmas.

PROOF OF PROPOSITION 2.6.1. Using that $u \in \mathcal{A}_*(\square_m)$, the definition of u_{hom}, Lemma 2.6.2 and Property (ii) of Lemma 2.6.3, we have the following series of inequalities

$$\begin{aligned} \fint_{\square_m} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx &\leq \fint_{\square_m} \left(\nabla \widetilde{u} \cdot \bar{\mathbf{a}} \nabla \widetilde{u} \right) dx \\ &\leq \frac{1}{|\square_m|} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\square_m} + C 3^{-m\alpha} \\ &\leq \frac{1}{|\square_m|} \langle \nabla \widetilde{u}_{\mathrm{hom}}, \mathbf{a} \nabla \widetilde{u}_{\mathrm{hom}} \rangle_{\square_m} + C 3^{-m\alpha} \\ &\leq \fint_{\square_m} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx + C 3^{-m\alpha}. \end{aligned} \tag{2.6.2}$$

Using that for every $e \in B_d(\square_m)$, $\mathbf{a}(e) \in \{0\} \times [\lambda, 1]$, we obtain, for every $e \in B_d(\mathscr{C}_*(\square_m))$ such that $\mathbf{a}(e) \neq 0$

$$\begin{aligned} \lambda \left(\nabla \widetilde{u}_{\mathrm{hom}}(e) - \nabla u(e) \right)^2 &\leq \mathbf{a}(e) \left(\nabla \widetilde{u}_{\mathrm{hom}}(x) - \nabla u(e) \right)^2 \\ &\leq 2\mathbf{a}(e) \left(\nabla \widetilde{u}_{\mathrm{hom}}(e) \right)^2 + 2\mathbf{a}(e) \left(\nabla u(e) \right)^2 \\ &\quad - 4\mathbf{a}(e) \left(\frac{\nabla \widetilde{u}_{\mathrm{hom}}(e) + \nabla u(e)}{2} \right). \end{aligned}$$

Summing over $e \in B_d(\mathscr{C}_*(\square_m))$ and using (2.6.2) yields

$$\frac{1}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)} \left| \nabla \left(\widetilde{u}_{\mathrm{hom}} - u \right) \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^2 (x) \leq C 3^{-m\alpha}$$

$$+ C \left(4 \frac{1}{|\square_m|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\square_m} - 4 \frac{1}{|\square_m|} \left\langle \frac{\nabla \widetilde{u}_{\mathrm{hom}} + \nabla u}{2}, \mathbf{a} \frac{\nabla \widetilde{u}_{\mathrm{hom}} + \nabla u}{2} \right\rangle_{\square_m} \right).$$

By Property (i) of Lemma 2.6.3, $\widetilde{u}_{\mathrm{hom}} = u$ on $\mathscr{C}_*(\square_m) \cap \partial \square_m$. Thus, since $u \in \mathcal{A}_*(\square_m)$,

$$\left\langle \frac{\nabla \widetilde{u}_{\mathrm{hom}} + \nabla u}{2}, \mathbf{a} \frac{\nabla \widetilde{u}_{\mathrm{hom}} + \nabla u}{2} \right\rangle_{\square_m} \leq \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\square_m}.$$

This eventually gives

$$\frac{1}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)} \left| \nabla \left(\widetilde{u}_{\mathrm{hom}} - u \right) \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^2 (x) \leq C M^2 3^{-m\alpha}.$$

The Poincaré inequality (Proposition 2.3.4 with $s = 2$) then gives

$$\frac{3^{-2m}}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)} \left(\widetilde{u}_{\mathrm{hom}}(x) - u(x) \right)^2$$

$$\leq \left(\sup_{x \in \square_m} \operatorname{size}\left(\square_{\mathcal{P}}(x) \right)^{2d} \right) \frac{1}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)} \left| \nabla \left(\widetilde{u}_{\mathrm{hom}} - u \right) \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^2 (x)$$

Thus since $\mathcal{P}$ is finer than $\mathcal{Q}$, and $3^m \geq \mathcal{M}_t(\mathcal{Q})$, we have, for t large enough,

$$\frac{3^{-2m}}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)} \left(\widetilde{u}_{\mathrm{hom}}(x) - u(x) \right)^2 \leq C M^2 3^{-m\alpha}. \tag{2.6.3}$$

Combining this with Property (iii) of Lemma 2.6.3 shows

$$\frac{3^{-2m}}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)} \left(u_{\mathrm{hom}}(x) - u(x) \right)^2 \leq C M^2 3^{-m\alpha}$$

and the proof is complete. □

Before starting the proof of Lemmas 2.6.2 and 2.6.3, we need to introduce some definitions and vocabulary which will be useful for the proof of both lemmas. We keep the same notations as in Definition 2.5.12 and set

$$n := \left\lceil \frac{m}{4} \right\rceil \text{ and } k := \left\lceil \frac{m}{8} \right\rceil,$$

so that by definition of $\mathcal{N}$, for every $p, q \in \mathbb{R}^d$ and $z \in 3^k \mathbb{Z}^d \cap \square_m$,

$$\left| \frac{1}{2} q \cdot \bar{\mathbf{a}}^{-1} q + \mu(z + \square_n, q) \right| \leq C |q|^2 3^{-m\alpha} \tag{2.6.4}$$

and

$$\left| \frac{1}{2} p \cdot \bar{\mathbf{a}} p + \nu(z + \square_n, p) \right| \leq C |p|^2 3^{-m\alpha}. \tag{2.6.5}$$

We also pick t large enough such that

$$3^m \geq \mathcal{M}_t(\mathcal{Q}) \implies \sup_{x \in \square_m} \operatorname{size}\left(\square_{\mathcal{Q}}(x) \right) \leq 3^k,$$

in particular since $\mathcal{P}$ is finer than $\mathcal{Q}$,

$$3^m \geq \mathcal{M}_t(\mathcal{Q}) \implies \sup_{x \in \square_m} \operatorname{size}\left(\square_{\mathcal{P}}(x) \right) \leq 3^k.$$

We define l the size of a boundary layer we need to remove in our argument:

$$l := \left\lceil \frac{7m}{8} \right\rceil,$$

so that $3^l \simeq \mathrm{size}(\square_m)^{\frac{7}{8}}$ is the size of the boundary layer. Notice that $k \leq m \leq l$. From this we define the two interior cubes

$$\square^\circ := \left[-\frac{3^m - 3^l}{2}, \frac{3^m - 3^l}{2} \right]^d$$

and

$$\square^\circ_{\mathrm{int}} := \left[-\frac{3^m - 2\cdot 3^l}{2}, \frac{3^m - 2\cdot 3^l}{2} \right]^d.$$

They correspond to the cube $\square_m$ to which one has removed a boundary layer of size respectively 3^l and $2\cdot 3^l$. From this we define $\eta \in C_c^\infty(\square_m)$ which satisfies the following properties

$$\eta = 1 \text{ on } \square^\circ_{\mathrm{int}}, \ \eta = 0 \text{ on } \ \square_m \setminus \square^\circ \ \text{ and } \ |\nabla \eta| \leq C 3^{-l}. \tag{2.6.6}$$

In what follows, it is convenient to work with three different scales: the microscopic scale (of size 1), the macroscopic scale (of size $\mathrm{size}(\square_m) = 3^m$) and the mesoscopic scale (of size $\mathrm{size}(\square_m)^{\frac{1}{4}} = 3^{\frac{m}{4}}$). The last step before starting the proofs of Lemmas 2.6.2 and 2.6.3 is to prove some estimates pertaining to w.

LEMMA 2.6.4. *The following properties hold:*

(i) *For every $x \in \mathbb{Z}^d \cap \square_m$ such that* $\mathrm{dist}(x, \partial\square_m) > 1$ *and for every $y \in x + \left[-\frac{1}{2}, \frac{1}{2}\right]^d$,*

$$|w(y) - [u]_{\mathcal{P}}(x)| \leq \sum_{z\in\mathbb{Z}^d : |z-x|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(z)$$

and for every $x \in \mathbb{Z}^d \cap \square_m$ such that $\mathrm{dist}(x, \partial\square_m) \leq 1$, *every $y \in x + \left[-\frac{1}{2}, \frac{1}{2}\right]^d \cap \square_m$,*

$$|w(y) - [u]_{\mathcal{P}}(x)| \leq \sum_{z\in\mathbb{Z}^d : |z-x|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(z) + \sum_{z\in\mathscr{C}_*(\square_m)\cap\square' : \square'\in\mathcal{P}\, \mathrm{dist}(\square', \square_{\mathcal{P}}(x)) \leq 1} |\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}|(z).$$

(ii) *There exists a constant $C := \sup_{\mathbb{R}^d} |\nabla\rho| < +\infty$ such that, for every $x \in \mathbb{Z}^d \cap \square_m$ satisfying* $\mathrm{dist}(x, \partial\square_m) > 1$ *and for every $y \in x + \left[-\frac{1}{2}, \frac{1}{2}\right]^d$*

$$|\nabla w(y)| \leq C \sum_{z\in\mathbb{Z}^d : |z-x|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(z)$$

and for every $x \in \mathbb{Z}^d \cap \square_m$ such that $\mathrm{dist}(x, \partial\square_m) \leq 1$, *every $y \in x + \left[-\frac{1}{2}, \frac{1}{2}\right]^d \cap \square_m$,*

$$|\nabla w(y)| \leq C \sum_{z\in\mathbb{Z}^d : |z-x|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(z) + C \sum_{z\in\mathscr{C}_*(\square_m)\cap\square' : \square'\in\mathcal{P}\, \mathrm{dist}(\square', \square_{\mathcal{P}}(x)) \leq 1} |\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}|(z).$$

(iii) *For every $p' \in \left[2, \frac{p+2}{2}\right]$, there exists a constant $C := C(s, d, \mathfrak{p}, p) < +\infty$ such that*

$$\left(\fint_{\square_m} |\nabla w(x)|^{p'} \, dx \right)^{\frac{1}{p'}} \leq CM.$$

(iv) *For every $p' \in \left[2, \frac{p+2}{2}\right]$, there exists $C := C(s, d, \mathfrak{p}, p) < +\infty$ such that*

$$\left(\fint_{\square_m} |w(x)|^{p'} \right)^{\frac{1}{p'}} \leq CM3^m.$$

PROOF. We prove (i). For every $x \in \mathbb{Z}^d \cap \square_m$ and every $y \in \left(x + \left[-\frac{1}{2}, \frac{1}{2}\right]^d\right) \cap \square_m$,

$$|w(y) - [u]_{\mathcal{P}}(x)| \leq \int_{\mathbb{R}^d} |\widetilde{w}(z) - [u]_{\mathcal{P}}(x)| \, \rho(y - z) \, dz.$$

Since $\operatorname{supp}\rho \subseteq (-\frac{1}{2}, \frac{1}{2})^d$ we have

$$\begin{aligned} |w(y) - [u]_{\mathcal{P}}(x)| &\le \sup_{z \in \square_m \cap \left(x + [-1,1]^d\right)} |\widetilde{w}(z) - [u]_{\mathcal{P}}(x)| \\ &\le \sup_{z \in \mathbb{Z}^d \cap \square_m : |z-x|_\infty \le 1} |\widetilde{w}(z) - [u]_{\mathcal{P}}(x)| . \end{aligned}$$

If $\operatorname{dist}(x, \partial \square_m) > 1$, then for every $z \in \mathbb{Z}^d \cap \square_m$ such that $|z - x|_\infty \le 1$, $z \in \operatorname{int} \square_m$ and thus $\widetilde{w}(z) = [u]_{\mathcal{P}}(z)$. Hence

$$\begin{aligned} |w(y) - [u]_{\mathcal{P}}(x)| &\le \sup_{z \in \mathbb{Z}^d : |z-x|_\infty \le 1} |\widetilde{w}(z) - [u]_{\mathcal{P}}(x)| \\ &\le \sup_{z \in \mathbb{Z}^d : |z-x|_\infty \le 1} |[u]_{\mathcal{P}}(z) - [u]_{\mathcal{P}}(x)| \\ &\le \sum_{z \in \mathbb{Z}^d : |z-x|_\infty \le 1} |\nabla [u]_{\mathcal{P}}|(z). \end{aligned}$$

If $\operatorname{dist}(x, \partial \square_m) \le 1$, then for every $z \in \mathbb{Z}^d \cap \square_m$ such that $|z - x|_\infty \le 1$ we have either $\widetilde{w}(z) = u(z)$ or $\widetilde{w}(z) = [u]_{\mathcal{P}}(z)$. Thus

$$\begin{aligned} |w(y) - [u]_{\mathcal{P}}(x)| &\le \sup_{z \in \mathbb{Z}^d \cap \square_m : |z-x|_\infty \le 1} |\widetilde{w}(z) - [u]_{\mathcal{P}}(x)| \\ &\le \sup_{z \in \mathbb{Z}^d \cap \square_m : |z-x|_\infty \le 1} \left(|[u]_{\mathcal{P}}(z) - [u]_{\mathcal{P}}(x)| + |\widetilde{w}(z) - [u]_{\mathcal{P}}(z)|\right) \\ &\le \sum_{z \in \mathbb{Z}^d \cap \square_m : |z-x|_\infty \le 1} |\nabla [u]_{\mathcal{P}}|(z) + \sup_{z \in \mathbb{Z}^d \cap \square_m : |z-x|_\infty \le 1} |\widetilde{w}(z) - [u]_{\mathcal{P}}(z)| . \end{aligned}$$

Using the first inequality (2.3.2) in the proof of Lemma 2.3.2 yields

$$\sup_{z \in \mathbb{Z}^d \cap \square_m : |z-x|_\infty \le 1} |\widetilde{w}(z) - [u]_{\mathcal{P}}(z)| \le \sum_{z \in \mathscr{C}_*(\square_m) \cap \square' : \square' \in \mathcal{P} \operatorname{dist}(\square', \square_{\mathcal{P}}(x)) \le 1} |\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}|(z).$$

The proof of (i) is complete. To prove (ii), notice that for every $x \in \mathbb{Z}^d \cap \square_m$ and $y \in \left(x + \left[-\frac{1}{2}, \frac{1}{2}\right]^d\right) \cap \square_m$,

$$\begin{aligned} |\nabla w(y)| = |\nabla \left(w - [u]_{\mathcal{P}}(x)\right)(y)| &\le |\left(\left(w - [u]_{\mathcal{P}}(x)\right) * \nabla \rho\right)(y)| \\ &\le \int_{\mathbb{R}^d} |\widetilde{w}(z) - [u]_{\mathcal{P}}(x)| |\nabla \rho(y - z)| \, dz \\ &\le \left(\sup_{\mathbb{R}^d} |\nabla \rho|\right) \sup_{z \in \square_m \cap \left(x + [-1,1]^d\right)} |\widetilde{w}(z) - [u]_{\mathcal{P}}(x)| \\ &\le C \sup_{z \in \mathbb{Z}^d \cap \square_m : |z-x|_\infty \le 1} |\widetilde{w}(z) - [u]_{\mathcal{P}}(x)| . \end{aligned}$$

The end of the proof of (ii) is similar to the proof of (i) and thus omitted.

To prove (iii), we split the integral

$$\fint_{\square_m} |\nabla w(x)|^{p'} \, dx = \frac{1}{|\square_m|} \int_{\square_m \setminus \partial_{\mathcal{P}} \square_m} |\nabla w(x)|^{p'} \, dx + \frac{1}{|\square_m|} \int_{\partial_{\mathcal{P}} \square_m} |\nabla w(x)|^{p'} \, dx.$$

The first term of the right-hand side can be estimated by using the first inequality of (ii),

$$\begin{aligned} \int_{\square_m \setminus \partial_{\mathcal{P}} \square_m} |\nabla w(x)|^{p'} \, dx &\le C \sum_{x \in \square_m \setminus \partial_{\mathcal{P}} \square_m} \left| \sum_{z \in \mathbb{Z}^d : |z-x|_\infty \le 1} |\nabla [u]_{\mathcal{P}}|(z) \right|^{p'} \\ &\le C \sum_{x \in \square_m \setminus \partial_{\mathcal{P}} \square_m} \sum_{z \in \mathbb{Z}^d : |z-x|_\infty \le 1} |\nabla [u]_{\mathcal{P}}|^{p'}(z) \\ &\le C \sum_{x \in \square_m} |\nabla [u]_{\mathcal{P}}|^{p'}(x). \end{aligned}$$

Applying Lemma 2.3.3 yields

$$\begin{aligned}
&\int_{\square_m \setminus \partial_{\mathcal{P}} \square_m} |\nabla w(x)|^{p'} \, dx \\
&\quad \le C \sum_{\square \in \mathcal{P},\, \square \subseteq \mathrm{cl}_{\mathcal{P}}(\square_m)} \mathrm{size}(\square)^{p'd-1} \sum_{\square \cap \mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}\right|^{p'}(x) \\
&\quad \le C \left(\sum_{\square \in \mathcal{P},\, \square \subseteq \mathrm{cl}_{\mathcal{P}}(\square_m)} \mathrm{size}(\square)^{(p'd-1)\frac{p}{p-p'}} \right)^{1-\frac{p'}{p}} \left(\sum_{x \in \mathscr{C}_*(\square_m)} \left|\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}\right|^{p}(x) \right)^{\frac{p'}{p}}.
\end{aligned}$$

Taking the exponent t large enough and using the assumption $3^m \ge \mathcal{M}_t(\mathcal{Q})$, we obtain

$$\frac{1}{|\square_m|} \int_{\square_m \setminus \partial_{\mathcal{P}} \square_m} |\nabla w(x)|^{p'} \, dx \le C M^{p'}. \tag{2.6.7}$$

The second term of the right-hand side can be estimated by using the second inequality of (ii):

$$\begin{aligned}
&\int_{\partial_{\mathcal{P}} \square_m} |\nabla w(x)|^{p'} \, dx \\
&\quad \le C \sum_{x \in \partial_{\mathcal{P}} \square_m} \left(\sum_{z \in \mathbb{Z}^d \cap \square_m : |z-x|_\infty \le 1} |\nabla [u]_{\mathcal{P}}|(z) + \sum_{z \in \mathscr{C}_*(\square_m) \cap \square' : \square' \in \mathcal{P}\, \mathrm{dist}(\square', \square_{\mathcal{P}}(x)) \le 1} |\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}|(z) \right)^{p'} \\
&\quad \le C \sum_{x \in \square_m} |\nabla [u]_{\mathcal{P}}|^{p'}(x) + C \sum_{x \in \partial_{\mathcal{P}} \square_m} \mathrm{size}(\square_{\mathcal{P}}(x))^{d\left(1-\frac{1}{p'}\right)} \\
&\qquad \times \sum_{z \in \mathscr{C}_*(\square_m) \cap \square',\, \square' \in \mathcal{P},\, \mathrm{dist}(\square', \square_{\mathcal{P}}(x)) \le 1} |\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}|^{p'}(z) \\
&\quad \le C \sum_{x \in \square_m} |\nabla [u]_{\mathcal{P}}|^{p'}(x) + C \sum_{x \in \mathscr{C}_*(\square_m)} \mathrm{size}(\square_{\mathcal{P}}(x))^{d\left(2-\frac{1}{p'}\right)} |\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}|^{p'}(x) \\
&\quad \le C \sum_{x \in \square_m} |\nabla [u]_{\mathcal{P}}|^{p'}(x) + C \left(\sum_{x \in \square_m} \mathrm{size}(\square_{\mathcal{P}}(x))^{d\left(2-\frac{1}{p'}\right)\frac{p}{p-p'}} \right)^{1-\frac{p'}{p}} \\
&\qquad \times \left(\sum_{x \in \mathscr{C}_*(\square_m)} |\nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}}|^{p}(x) \right)^{\frac{p'}{p}}.
\end{aligned}$$

Taking the exponent t large enough and using $\mathrm{size}(\square_m) = 3^m \ge \mathcal{M}_t(\mathcal{Q})$ again, we get

$$\frac{1}{|\square_m|} \int_{\partial_{\mathcal{P}} \square_m} |\nabla w(x)|^{p'} \, dx \le C M^{p'}. \tag{2.6.8}$$

Combining (2.6.7) and (2.6.8) yields (iii).

The proof of (iv) is a consequence the usual Sobolev inequality on $\mathbb{Z}^d$ (2.3.1), the assumption (2.4.2) and the estimate which follows easily from Lemma 2.6.4[(i)] and the assumption $3^m \ge \mathcal{M}_t(\mathcal{Q})$,

$$\left(⨍_{\square_m} |w(x)|^{p'} \, dx \right)^{\frac{1}{p'}} \le C \left(\frac{1}{|\square_m|} \sum_{x \in \square_m} |[u]_{\mathcal{P}}|^{p'}(x) \right)^{\frac{1}{p'}} + \left(\frac{1}{|\square_m|} \sum_{x \in \square_m} |\nabla [u]_{\mathcal{P}}|^{p}(x) \right)^{\frac{1}{p}}.$$

□

We can now prove Lemma 2.6.2.

Proof of Lemma 2.6.2. We now construct the function $\widetilde{u} \in C^\infty(\square_m)$ by removing the microscopic oscillations from u. More precisely we remove the microscopic oscillations of w, which is close to u, by considering spatial averages of w on a mesoscopic scale. We thus define, for every $x \in \square_m$ such that $x + \square_n \subseteq \square_m$,

$$\xi(x) := \fint_{x+\square_n} w(z)\, dz.$$

We next modify ξ in order to get an element of $C^\infty(\square_m)$, equal to w on $\partial \square_m$, by setting

$$\widetilde{u}(x) := \eta(x)\xi(x) + (1-\eta(x))w(x).$$

It is clear that $\widetilde{u} \in w + H_0^1(\square_m)$, so we now focus on the proof of (2.6.1). We split the argument into several steps.

Step 1. Denote for each $y \in \square^\circ$,

$$p(y) := \nabla \xi(y) = \frac{1}{|\square_n|} \int_{y+\square_n} \nabla w(z)\, dz \in \mathbb{R}^d \text{ and } q(y) := \bar{\mathbf{a}} p(y) \in \mathbb{R}^d.$$

For $y \in 3^k \mathbb{Z}^d \cap \square^\circ$, testing u as a minimizer candidate in the definition of $\mu(y + \square_n, q(y))$, gives

$$\mu(y+\square_n, q(y)) \le \frac{1}{2|\square_n|} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(y+\square_n)} - \frac{1}{|\square_n|} \langle q(y), \nabla [u]_{\mathcal{P}} \rangle_{y+\square_n}. \tag{2.6.9}$$

Combining this result with (2.6.4),

$$\begin{aligned}
-\frac{1}{2} q(y) \cdot \bar{\mathbf{a}}^{-1} q(y) &\le \frac{1}{2|\square_n|} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(y+\square_n)} - \frac{1}{|\square_n|} \langle q(y), \nabla [u]_{\mathcal{P}} \rangle_{y+\square_n} + C|q(y)|^2 3^{-m\alpha} \\
&\le \frac{1}{2|\square_n|} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(y+\square_n)} - q(y) \cdot p(y) + C|q(y)|^2 3^{-m\alpha} \\
&\quad + |q(y)| \left| \frac{1}{|\square_n|} \langle \nabla [u]_{\mathcal{P}} \rangle_{y+\square_n} - p(y) \right|.
\end{aligned}$$

Thus by definition of $q(y)$

$$p(y) \cdot \bar{\mathbf{a}} p(y) \le \frac{1}{|\square_n|} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(y+\square_n)} + C|q(y)|^2 3^{-m\alpha} + 2|q(y)| \left| \frac{1}{|\square_n|} \langle \nabla [u]_{\mathcal{P}} \rangle_{y+\square_n} - p(y) \right|.$$

Summing over $y \in 3^k \mathbb{Z}^d \cap \square^\circ_{\mathrm{int}}$ and multiplying by $\frac{3^k}{|\square^\circ_{\mathrm{int}}|}$ yields

$$\begin{aligned}
&\frac{3^{dk}}{|\square^\circ_{\mathrm{int}}|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square^\circ_{\mathrm{int}}} p(y) \cdot \bar{\mathbf{a}} p(y) \\
&\quad \le \frac{3^k}{|\square^\circ_{\mathrm{int}}|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square^\circ_{\mathrm{int}}} \frac{1}{|\square_n|} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_*(y+\square_n)} + CM^2 3^{-m\alpha} \\
&\quad + \frac{3^{dk}}{|\square^\circ_{\mathrm{int}}|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square^\circ_{\mathrm{int}}} 2|q(y)| \left| \frac{1}{|\square_n|} \langle \nabla [u]_{\mathcal{P}} \rangle_{y+\square_n} - p(y) \right|.
\end{aligned} \tag{2.6.10}$$

Since by (iii) of Lemma 2.6.4 with $p' = 2$

$$\begin{aligned}
\frac{3^{dk}}{|\square^\circ_{\mathrm{int}}|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square^\circ_{\mathrm{int}}} |q(y)|^2 &\le C \frac{3^{dk}}{|\square^\circ_{\mathrm{int}}|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square^\circ_{\mathrm{int}}} |p(y)|^2 \\
&\le C \fint_{\square^\circ_{\mathrm{int}}} |\nabla w(z)|^2\, dz \\
&\le C \fint_{\square} |\nabla w(z)|^2\, dz \\
&\le CM^2.
\end{aligned}$$

The rest of the proof is organized as follows:

- In Step 2: we show

$$\frac{3^{dk}}{|\square_{\mathrm{int}}^{\circ}|}\sum_{y\in 3^k\mathbb{Z}^d\cap\square_{\mathrm{int}}^{\circ}}|q(y)|\left|\frac{1}{|\square_n|}\left\langle\nabla\left[u\right]_{\mathcal{P}}\right\rangle_{y+\square_n}-p(y)\right|\leq CM^2 3^{-m\alpha}.$$

- In Step 3: we show

$$\frac{3^{dk}}{|\square_{\mathrm{int}}^{\circ}|}\sum_{y\in 3^k\mathbb{Z}^d\cap\square_{\mathrm{int}}^{\circ}}\frac{1}{|\square_n|}\left\langle\nabla u,\mathbf{a}\nabla u\right\rangle_{\mathscr{C}_*(y+\square_n)}\,dy\leq\frac{1}{|\square_m|}\left\langle\nabla u,\mathbf{a}\nabla u\right\rangle_{\mathscr{C}_*(\square_m)}+CM^2 3^{-m\alpha}.$$

- In Step 4 and Step 5 : we show

$$\fint_{\square_m}\nabla\widetilde{u}(y)\cdot\bar{\mathbf{a}}\nabla\widetilde{u}(y)\,dy\leq\frac{3^{dk}}{|\square_{\mathrm{int}}^{\circ}|}\sum_{y\in 3^k\mathbb{Z}^d\cap\square_{\mathrm{int}}^{\circ}}p(y)\cdot\bar{\mathbf{a}}p(y)+CM^2 3^{-m\alpha}.$$

Combining these three results with (2.6.10) completes the proof of Lemma 2.6.2.

Step 2. We want to show

$$\frac{3^{dk}}{|\square_{\mathrm{int}}^{\circ}|}\sum_{y\in 3^k\mathbb{Z}^d\cap\square_{\mathrm{int}}^{\circ}}|q(y)|\left|\frac{1}{|\square_n|}\left\langle\nabla\left[u\right]_{\mathcal{P}}\right\rangle_{y+\square_n}-p(y)\right|\leq CM^2 3^{-m\alpha}. \tag{2.6.11}$$

We already saw that

$$\frac{3^{dk}}{|\square_{\mathrm{int}}^{\circ}|}\sum_{y\in 3^k\mathbb{Z}^d\cap\square_{\mathrm{int}}^{\circ}}|q(y)|^2\leq CM^2.$$

By the Cauchy-Schwarz inequality it is enough to obtain (2.6.11) to prove

$$\frac{3^{dk}}{|\square_{\mathrm{int}}^{\circ}|}\sum_{y\in 3^k\mathbb{Z}^d\cap\square_{\mathrm{int}}^{\circ}}\left|\frac{1}{|\square_n|}\left\langle\nabla\left[u\right]_{\mathcal{P}}\right\rangle_{y+\square_n}-p(y)\right|^2\leq CM^2 3^{-m\alpha}. \tag{2.6.12}$$

To prove this, we will prove, for every $y\in 3^k\mathbb{Z}^d\cap\square_{\mathrm{int}}^{\circ}$,

$$\left|\left\langle\nabla\left[u\right]_{\mathcal{P}}\right\rangle_{y+\square_n}-\int_{y+\square_n}\nabla w(z)\,dz\right|\leq C\sum_{x\in\partial\square_n}|\nabla\left[u\right]_{\mathcal{P}}|(x). \tag{2.6.13}$$

For the sake of simplicity we assume $y=0$ to prove (2.6.13). By the discrete Stokes formula,

$$\left\langle\nabla\left[u\right]_{\mathcal{P}}\right\rangle_{\square_n}=\sum_{x\in\partial\square_n}\left[u\right]_{\mathcal{P}}(x)\mathbf{n}(x).$$

For $i=\{-d,\ldots,-1,1,\ldots,d\}$, denote by $\partial_i\square_n$ the ith face of $\square_n$ given by

$$\partial_i\square_n:=\left\{x\in\mathbb{Z}^d\cap\square_n\ :\ x_{|i|}=\operatorname{sign}(i)\frac{1}{2}(3^n-1)\right\}.$$

Denote also by $\mathbf{n}_i$ the associated outer normal vector, i.e, $\mathbf{n}_i=\mathfrak{e}_i$ for i positive and $\mathbf{n}_i=-\mathfrak{e}_i$ for i negative. The previous identity can be rewritten

$$\left\langle\nabla\left[u\right]_{\mathcal{P}}\right\rangle_{\square_n}=\sum_{i=\pm1,\cdots,\pm d}\sum_{x\in\partial_i\square_n}\left[u\right]_{\mathcal{P}}(x)\mathbf{n}_i.$$

Thus

$$\left\langle\nabla\left[u\right]_{\mathcal{P}}\right\rangle_{\square_n}-\int_{\square_n}\nabla w(z)\,dz=\sum_{i=\pm1,\cdots,\pm d}\left(\sum_{x\in\partial_i\square_n}\left[u\right]_{\mathcal{P}}(x)-\int_{\partial_i\square_n}w(z)\,dz\right)\mathbf{n}_i.$$

Without loss of generality, it is sufficient to prove (2.6.13) to show

$$\left|\sum_{x\in\partial_1\square_n\cup\partial_{-1}\square_n}\left[u\right]_{\mathcal{P}}(x)-\int_{\partial_1\square_n\cup\partial_{-1}\square_n}w(z)\,dz\right|\leq C\sum_{x\in\partial\square_n}|\nabla\left[u\right]_{\mathcal{P}}|(x). \tag{2.6.14}$$

With a few modifications of the proof of (i) of Lemma 2.6.4, we can show: for every $x \in \partial_1 \square_n \cup \partial_{-1} \square_n$ and every $z \in \left(x + \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-1}\right) \bigcap \partial \square_n$

$$|w(z) - [u]_{\mathcal{P}}(x)| \leq \sum_{y \in \mathbb{Z}^d \cap \partial \square_n : |y-x|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(y). \tag{2.6.15}$$

The idea of the proof of (2.6.14) is to apply (2.6.15) with $x \in \partial_1 \square_n$ (resp. $\partial_{-1} \square_n$) and $z \in \left(x + \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-1}\right) \bigcap \partial \square_n$. Unfortunately, a technical difficulty appears if x lies on the boundary of $\partial_1 \square_n$ (resp. $\partial_{-1} \square_n$) which we denote by

$$\partial \partial_1 \square_n := \{x \in \partial_1 \square_n \ : \ \exists i \in \{2, \cdots, d\}, \ x_i \text{ is maximal or minimal}\}.$$

We define similarly $\partial \partial_{-1} \square_n$. Thus we distinguish two cases, whether $x \in \partial \partial_1 \square_n$ (resp. $\partial \partial_{-1} \square_n$) or not.

Case 1: $x \in \partial_1 \square_n \setminus \partial \partial_1 \square_n$, then by (2.6.15)

$$\left| [u]_{\mathcal{P}}(x) - \int_{x + \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-1}} w(z)\, dz \right| \leq \sum_{z \in \partial \square_n : |z-x|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(z).$$

Symmetrically for $x \in \partial_{-1} \square_n \setminus \partial \partial_{-1} \square_n$

$$\left| [u]_{\mathcal{P}}(x) - \int_{x + \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-1}} w(z)\, dz \right| \leq \sum_{z \in \partial \square_n : |z-x|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(z).$$

Case 2: $x \in \partial \partial_1 \square_n$ then denote by $\widehat{x} := x - (3^n - 1)\mathfrak{e}_1 \in \partial \partial_{-1} \square_n$. We have

$$|[u]_{\mathcal{P}}(x) - [u]_{\mathcal{P}}(\widehat{x})| \leq \sum_{i=0}^{3^n-1} |\nabla [u]_{\mathcal{P}}| (x - i\mathfrak{e}_1).$$

Summing over $x \in \partial \partial_1 \square_n$ yields

$$\begin{aligned} \sum_{x \in \partial \partial_1 \square_n} |[u]_{\mathcal{P}}(x) - [u]_{\mathcal{P}}(\widehat{x})| &\leq \sum_{x \in \partial \partial_1 \square_n} \sum_{i=0}^{3^n-1} |\nabla [u]_{\mathcal{P}}| (x - i\mathfrak{e}_1) \\ &\leq \sum_{x \in \partial \square_n} |\nabla [u]_{\mathcal{P}}| (x), \end{aligned}$$

since for every $x \in \partial \partial_1 \square_n$ and every $i \in \{0, \cdots, 3^n - 1\}$, $x - i\mathfrak{e}_1 \in \partial \square_n$ (and for every $y \in \partial \square_n$ there existsat most one $x \in \partial \partial_1 \square_n$ and one $i \in \{0, \cdots, 3^n - 1\}$ such that $y = x - i\mathfrak{e}_1$).

Moreover, for every $x \in \partial \partial_1 \square_n$ and every $z \in \left(x + \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-1}\right) \bigcap \partial \square_n$, we define $\widehat{z} := z - (3^n - 1)\mathfrak{e}_1$. By (2.6.15) and the previous computations

$$\begin{aligned} |w(z) - w(\widehat{z})| &\leq |w(z) - [u]_{\mathcal{P}}(x)| + |[u]_{\mathcal{P}}(x) - [u]_{\mathcal{P}}(\widehat{x})| + |[u]_{\mathcal{P}}(\widehat{x}) - w(\widehat{z})| \\ &\leq \sum_{y \in \square_m : |y-x|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(y) + \sum_{i=1}^{3^m} |\nabla [u]_{\mathcal{P}}| (x - i\mathfrak{e}_1) \\ &\quad + \sum_{y \in \square_m : |y-\widehat{x}|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(y). \end{aligned}$$

Thus, integrating over $\left(x + \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-1}\right) \bigcap \partial \square_n$,

$$\begin{aligned} &\left| \int_{\left(x + \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-1}\right) \cap \partial \square_n} w(z)\, dz - \int_{\left(\widehat{x} + \{0\} \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-1}\right) \cap \partial \square_n} w(z)\, dz \right| \\ &\qquad \leq \sum_{y \in \square_n : |y-x|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(y) + \sum_{i=0}^{3^n-1} |\nabla [u]_{\mathcal{P}}| (x - i\mathfrak{e}_1) + \sum_{y \in \square_n : |y-\widehat{x}|_\infty \leq 1} |\nabla [u]_{\mathcal{P}}|(y). \end{aligned}$$

Summing over $x \in \partial\partial_1 \square_n$ yields

$$\sum_{x \in \partial\partial_1 \square_n} \left| \int_{\left(x + \{0\} \times [-\frac{1}{2}, \frac{1}{2}]^{d-1}\right) \cap \partial \square_n} w(z)\, dz - \int_{\left(\widehat{x} + \{0\} \times [-\frac{1}{2}, \frac{1}{2}]^{d-1}\right) \cap \partial \square_n} w(z)\, dz \right| \leq C \sum_{y \in \partial \square_n} \left| \nabla \left[u\right]_{\mathcal{P}} \right|(y).$$

Combining the displays of cases 1 and 2 and using the triangle inequality shows

$$\left| \sum_{x \in \partial_1 \square_n \cup \partial_{-1} \square_n} \left[u\right]_{\mathcal{P}}(x) - \int_{\partial_1 \square_n \cup \partial_{-1} \square_n} w(z)\, dz \right| \leq C \sum_{x \in \partial \square_n} \left| \nabla \left[u\right]_{\mathcal{P}} \right|(x),$$

which is (2.6.13). We now turn to the proof of (2.6.12).

Applying the Cauchy-Schwarz inequality yields

$$\left| \frac{1}{|\square_n|} \left\langle \nabla \left[u\right]_{\mathcal{P}} \right\rangle_{y + \square_n} - \frac{1}{|\square_n|} \int_{y + \square_n} \nabla w(z)\, dz \right|^2 \leq C \left(\frac{|\partial \square_n|}{|\square_n|} \right) \left(\frac{1}{|\square_n|} \sum_{z \in (y + \square_n)} \left| \nabla \left[u\right]_{\mathcal{P}} \right|^2 (z) \right) \leq C 3^{-n} \left(\frac{1}{|\square_n|} \sum_{z \in (y + \square_n)} \left| \nabla \left[u\right]_{\mathcal{P}} \right|^2 (z) \right).$$

Summing this over $3^k \mathbb{Z}^d \cap \square_{\mathrm{int}}^\circ$ and applying (iii) of Lemma 2.6.4 with $p' = 2$ yields (2.6.12) and consequently the main result of this step (2.6.11).

Step 3. We want to show

$$\frac{3^{dk}}{\left|\square_{\mathrm{int}}^\circ\right|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square_{\mathrm{int}}^\circ} \frac{1}{|\square_n|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(y + \square_n)} \leq \frac{1}{|\square_m|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(\square_m)} + C M^2 3^{-m\alpha}. \tag{2.6.16}$$

Notice that, while $\mathscr{C}_*(\square_m) \cap (z + \square_n)$ and $\mathscr{C}_*(z + \square_n)$ may be different, every open edge in the latter cluster belongs to the former. This remark shows the following inequality, for each $y \in 3^k \mathbb{Z}^d \cap \square_{\mathrm{int}}^\circ$,

$$\left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(y + \square_n)} \leq \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(\square_m) \cap (y + \square_n)}.$$

This allows us to bound

$$\frac{3^{dk}}{\left|\square_{\mathrm{int}}^\circ\right|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square_{\mathrm{int}}^\circ} \frac{1}{|\square_n|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(y + \square_n)} \leq \frac{3^{dk}}{\left|\square_{\mathrm{int}}^\circ\right|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square_{\mathrm{int}}^\circ} \frac{1}{|\square_n|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(\square_m) \cap (y + \square_n)} \leq \frac{1}{\left|\square_{\mathrm{int}}^\circ\right|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(\square_m)}.$$

To complete the proof, we need to show the following estimate

$$\frac{1}{\left|\square_{\mathrm{int}}^\circ\right|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(\square_m)} \leq \frac{1}{|\square_m|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(\square_m)} + C M^2 3^{-m\alpha}.$$

The previous estimate follows from the following computation

$$\frac{1}{\left|\square_{\mathrm{int}}^\circ\right|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(\square_m)} - \frac{1}{|\square_m|} \left\langle \nabla u, \mathbf{a} \nabla u \right\rangle_{\mathscr{C}_*(\square_m)} \leq \frac{\left|\square_m \setminus \square_{\mathrm{int}}^\circ\right|}{|\square_m|^2} \left(\sum_{x \in \mathscr{C}_*(\square_m)} \left| \nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^2(x) \right) + \frac{1}{|\square_m|} \left(\sum_{x \in \mathscr{C}_*(\square_m) \setminus \square_{\mathrm{int}}^\circ} \left| \nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^2(x) \right).$$

We recall the definition of l which is the size of the boundary layer. We have

$$\frac{\left|\square_m \setminus \square_{\mathrm{int}}^\circ\right|}{|\square_m|} \leq C 3^{l-m} \leq C 3^{-m\alpha}.$$

This allows to bound the first term on the right-hand side

$$\frac{|\,\square_m \setminus \square_{\mathrm{int}}^\circ\,|}{|\,\square_m\,|^2}\left(\sum_{x\in\mathscr{C}_*(\square_m)} |\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}|^2(x)\right) \le C3^{-m\alpha} M^2.$$

The second term can be bounded by applying the Hölder inequality,

$$\begin{aligned}
&\frac{1}{|\,\square_m\,|}\sum_{x\in\mathscr{C}_*(\square_m)\setminus\square_{\mathrm{int}}^\circ} |\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}|^2(x) \\
&\quad\le C\left(\frac{|\mathscr{C}_*(\square_m)\setminus\square_{\mathrm{int}}^\circ|}{|\,\square_m\,|}\right)^{\frac{p-2}{p}}\left(\frac{1}{|\,\square_m\,|}\sum_{x\in\mathscr{C}_*(\square_m)\setminus\square_{\mathrm{int}}^\circ} |\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}|^p(x)\right)^{\frac{2}{p}} \\
&\quad\le C3^{(l-m)\frac{p-2}{p}}\left(\frac{1}{|\,\square_m\,|}\sum_{x\in\mathscr{C}_*(\square_m)} |\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}|^p(x)\right)^{\frac{2}{p}} \\
&\quad\le C3^{-m\alpha}M^2.
\end{aligned}$$

The proof of (2.6.16) is complete.

Step 4. We want to show

$$\fint_{\square_{\mathrm{int}}^\circ} p(x)\cdot\bar{\mathbf{a}}p(x)\,dx \le \frac{3^{dk}}{|\,\square_{\mathrm{int}}^\circ\,|}\sum_{x\in 3^k\mathbb{Z}^d\cap\square_{\mathrm{int}}^\circ} p(y)\cdot\bar{\mathbf{a}}p(y) + CM^2 3^{-m\alpha}. \tag{2.6.17}$$

For every $x\in\square_{\mathrm{int}}^\circ$ and $y\in x+\left[-\frac{3^k}{2},\frac{3^k}{2}\right]^d$,

$$\begin{aligned}
|p(x)-p(y)| &\le \frac{1}{|\,\square_n\,|}\int_{(x+\square_n)\Delta(y+\square_n)} |\nabla w(z)|\,dz \\
&\le \left(\frac{|(x+\square_n)\Delta(y+\square_n)|}{|\,\square_n\,|}\right)^{\frac12}\left(\frac{1}{|\,\square_n\,|}\int_{(x+\square_n)\Delta(y+\square_n)} |\nabla w(z)|^2\,dz\right)^{\frac12} \\
&\le C3^{\frac{k-n}{2}}\left(\fint_{x+\square_n+\square_k} |\nabla w(z)|^2\,dz\right)^{\frac12} \\
&\le C3^{-m\alpha}\left(\fint_{x+\square_n+\square_k} |\nabla w(z)|^2\,dz\right)^{\frac12}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|p(x)\cdot\bar{\mathbf{a}}p(x)-\fint_{x+\left[-\frac{3^k}{2},\frac{3^k}{2}\right]^d} p(y)\cdot\bar{\mathbf{a}}p(y)\,dy\right| \\
&\quad\le \sup_{y\in x+\left[-\frac{3^k}{2},\frac{3^k}{2}\right]^d} |p(x)-p(y)|\left(|p(x)|+|p(y)|\right) \\
&\quad\le C3^{-m\alpha}\left(\fint_{y+\square_m+\square_k} |\nabla w(z)|^2\,dz\right).
\end{aligned}$$

Summing over $x \in 3^k\mathbb{Z}^d \cap \square_{\mathrm{int}}^\circ$ and using (iii) of Lemma 2.6.4

$$\begin{aligned}
&\left| \frac{3^{dk}}{|\square_{\mathrm{int}}^\circ|} \sum_{x \in 3^k\mathbb{Z}^d \cap \square_{\mathrm{int}}^\circ} p(x)\cdot \bar{\mathbf{a}} p(x) - \fint_{\square_{\mathrm{int}}^\circ} p(x)\cdot \bar{\mathbf{a}} p(x)\, dx \right| \\
&\qquad \le 3^{-m\alpha} \frac{3^{dk}}{|\square_{\mathrm{int}}^\circ|} \sum_{x \in 3^k\mathbb{Z}^d \cap \square_{\mathrm{int}}^\circ} \left(\fint_{x+\square_n+\square_k} |\nabla w(z)|^2 \, dz \right) \\
&\qquad \le 3^{-m\alpha} \left(\frac{1}{|\square_{\mathrm{int}}^\circ|} \int_{\square_m} |\nabla w(z)|^2 \, dz \right) \\
&\qquad \le CM^2 3^{-m\alpha}.
\end{aligned}$$

The proof of (2.6.17) is complete.

Step 5. We want to show

$$\fint_{\square_m} \nabla \widetilde{u}(y)\cdot \bar{\mathbf{a}} \nabla \widetilde{u}(y)\, dy \le \fint_{\square_{\mathrm{int}}^\circ} p(y)\cdot \bar{\mathbf{a}} p(y)\, dx + CM^2 3^{-m\alpha}. \tag{2.6.18}$$

First we need to prove the following estimate: there exists $C := C(s,d,\mathfrak{p},\lambda,p) < +\infty$ such that for each $p' \in \left[2, \frac{p+2}{2}\right]$,

$$\fint_{\square_m} |\nabla \widetilde{u}(x)|^{p'} \, dx \le CM^{p'}. \tag{2.6.19}$$

Differentiating the expression for $\widetilde{u}$, we get

$$\nabla \widetilde{u}(x) = \nabla \eta(x)(\xi(x) - w(x)) + \eta(x)(\nabla \xi(x) - \nabla w(x)) + \nabla w(x).$$

By Lemma 2.6.4(iii), to prove (2.6.19) it suffices to prove the following estimate:

$$\fint_{\square^\circ} \left(3^{-l} |\xi(x) - w(x)| + |\nabla \xi(x)| \right)^{p'} dx \le CM^{p'}. \tag{2.6.20}$$

We first estimate the second term in the integrand:

$$\begin{aligned}
\fint_{\square^\circ} |\nabla \xi(x)|^{p'} \, dx &= \fint_{\square^\circ} \left| \fint_{x+\square_n} \nabla w(y)\, dy \right|^{p'} dx \\
&\le \fint_{\square^\circ} \fint_{x+\square_n} |\nabla w(y)|^{p'} \, dy\, dx \\
&\le \frac{|\square_m|}{|\square^\circ|} \fint_{\square_m} |\nabla w(x)|^{p'} \, dx \\
&\le CM^{p'}.
\end{aligned}$$

To estimate the first term in the integrand, we use that for every $y \in \square^\circ$,

$$\begin{aligned}
&\frac{1}{|\square_n|} \int_{y+\square_n} |\xi(x) - w(x)|^{p'} \, dx \\
&\quad = \frac{1}{|\square_n|} \int_{y+\square_n} \left| w(x) - \frac{1}{|\square_n|} \int_{x+\square_n} w(z)\, dz \right|^{p'} dx \\
&\quad \le C \frac{1}{|\square_n|} \int_{y+\square_n} \left| w(x) - \frac{1}{\mathrm{Leb}(\square_n)} \int_{y+\square_n} w(z)\, dz \right|^{p'} dx \\
&\qquad + C \frac{1}{|\square_n|} \int_{y+\square_n} \left| \frac{1}{|\square_n|} \int_{x+\square_n} w(z)\, dz - \frac{1}{|\square_n|} \int_{y+\square_n} w(z)\, dz \right|^{p'} dx \\
&\qquad + C \left((3^n - 1)^{-d} - 3^{-dn} \right)^{p'} \left| \int_{y+\square_n} w(z)\, dz \right|^{p'} dx.
\end{aligned}$$

Thanks to the Poincaré inequality, we can bound the first term on the right-hand side:

$$\fint_{y+\square_n} \left| w(x) - \frac{1}{\mathrm{Leb}(\square_n)} \int_{y+\square_n} w(z)\,dz \right|^{p'} dx \le C3^{p'n} \fint_{y+\square_n} |\nabla w(x)|^{p'}\,dx.$$

To compute the second term, we observe that for every $y \in \square^\circ$ and $x \in y + \square_n$,

$$\begin{aligned}\left| \fint_{x+\square_n} w(z)\,dz - \fint_{y+\square_n} w(z)\,dz \right| &= \left| \fint_{\square_n} \int_0^1 (x-y)\cdot \nabla w(tx + (1-t)y + z)\,dt\,dz \right| \\ &\le C3^n \fint_{y+\square_{n+1}} |\nabla w(z)|\,dz.\end{aligned}$$

Assembling these yields

$$\fint_{y+\square_n} |\xi(x) - w(x)|^{p'}\,dx \le C3^{p'n} \fint_{y+\square_{n+1}} |\nabla w(x)|^{p'}\,dx + 3^{-p'n} \fint_{y+\square_n} |w(x)|^{p'}\,dx$$

and then integrating over $y \in \square^\circ$ and applying Lemma 2.6.4[(iii) and (iv)] yields

$$\fint_{\square^\circ} |\xi(x) - w(x)|^{p'}\,dx \le C\left(3^{p'n} + 3^{-p'n+p'm}\right).$$

Inequality (2.6.19) is then a consequence of the two estimates $l \ge n$ and $l \ge m - n$.

To prove (2.6.18), notice that for $y \in \square^\circ$, $\nabla \widetilde{u}(y) = p(y)$. Hence we can write

$$\begin{aligned}&\fint_{\square_m} \nabla \widetilde{u}(y)\cdot \mathbf{\bar a} \nabla \widetilde{u}(y)\,dy - \fint_{\square^\circ} p(y)\cdot \mathbf{\bar a} p(y)\,dy \\ &\quad\le \frac{|\square_m \setminus \square^\circ|}{|\square_m|} \fint_{\square^\circ} |\nabla \widetilde{u}(x)|^2\,dx + \frac{1}{|\square_m|}\int_{\square_m\setminus\square^\circ} |\nabla \widetilde{u}(x)|^2\,dx \\ &\quad\le C3^{-m\alpha}M^2 + \left(\frac{|\square_m \setminus \square^\circ|}{|\square|}\right)^{\frac{p-2}{p+2}} \left(\frac{1}{|\square_m|}\int_{\square_m\setminus\square^\circ} |\nabla \widetilde{u}(x)|^{\frac{p+2}{2}}\,dx\right)^{\frac{4}{p+2}} \\ &\quad\le C3^{-m\alpha}M^2 + C3^{-m\alpha}\left(\fint_{\square_m} |\nabla \widetilde{u}(x)|^{\frac{p+2}{2}}\,dx\right)^{\frac{4}{p+2}} \\ &\quad\le C3^{-m\alpha}M^2.\end{aligned}$$

This completes the proof of (2.6.18) and thus the proof of Lemma 2.6.2. □

Before starting the proof of Lemma 2.6.3, we need to record two estimates from the regularity theory

PROPOSITION 2.6.5 (Meyers and H^2 estimates [**77**]). *Suppose $t \in (2,\infty)$, $f \in W^{1,t}(\square_m)$ and $v \in H^1(\square_m)$ satisfy*

$$\begin{cases} \nabla\cdot \mathbf{\bar a}\nabla v = 0 & \text{in } \square_m \\ v = f & \text{in } \partial\square_m\,. \end{cases}$$

Then there exist $r := r(d,\lambda,t) \in (2,t)$ and a constant $C := C(d,\lambda,t) < +\infty$ such that $v \in W^{1,r}(\square_m)$ and

$$\left(\fint_{\square_m} |\nabla v(x)|^r\,dx\right)^{\frac{1}{r}} \le C\left(\fint_{\square_m} |\nabla v(x)|^t\,dx\right)^{\frac{1}{t}}.$$

Moreover for every cube $\square' \subseteq \square_m$ with $\mathrm{dist}(\square', \partial\square_m) > 0$, $v \in H^2(\square_m)$ *and*

$$\mathrm{dist}(\square', \partial\square_m)\left(\fint_{\square'} |\nabla\nabla w(x)|^2\,dx\right)^{\frac{1}{2}} \le C\left(\fint_{\square_m} |\nabla v(x)|^t\,dx\right)^{\frac{1}{t}}.$$

Applying this result with $t = \frac{p+2}{2}$, $f = w$, $\square' = \square^\circ$ and using (iii) of Lemma 2.6.4 shows: there exists $r := r(d,\lambda,p) \in \left(2, \frac{p+2}{2}\right)$ such that

$$\left(\fint_{\square_m} |\nabla u_{\mathrm{hom}}(x)|^r\,dx\right)^{\frac{1}{r}} \le CM \tag{2.6.21}$$

and

$$3^l \left(\fint_{\square^\circ} |\nabla\nabla u_{\mathrm{hom}}(x)|^2 \, dx \right)^{\frac{1}{2}} \leq CM. \tag{2.6.22}$$

These two estimates will be useful in the proof of Lemma 2.6.3.

Proof of Lemma 2.6.3. We construct $\widetilde{u}_{\mathrm{hom}}$ from u_{hom} by patching together mesoscopic minimizers.

For every y in $\square^\circ$, we set

$$\zeta(y) := \frac{1}{|\square_n|} \int_{y+\square_n} u_{\mathrm{hom}}(x)\, dx, \;\; p(y) := \nabla \zeta(y) = \frac{1}{|\square_n|} \int_{y+\square_n} \nabla u_{\mathrm{hom}}(x)\, dx \text{ and } q(y) = \bar{\mathbf{a}} p(y).$$

We begin the construction by defining an affine approximation to u_{hom} in the mesoscopic cube $y + \square_n$ by setting, for each $y \in \square^\circ \cap 3^n \mathbb{Z}^d$,

$$l_y := p(y) \cdot (x - y) + \zeta(y).$$

For each $y \in \square^\circ \cap 3^n \mathbb{Z}^d$, denote by v_y the unique element of

$$\mathcal{A}(y + \square_n) \cap \left(l_y + C_0 \left(\mathscr{C}_*(y + \square_n) \right) \right),$$

that is, v_y is the unique element of $l_y + C_0 \left(\mathscr{C}_*(y + \square_n) \right)$ which satisfies:

$$\langle \nabla v_y, \mathbf{a} \nabla v_y \rangle_{y+\square_n} \leq \langle \nabla w, \mathbf{a} \nabla w \rangle_{y+\square_n} \text{ for every } w \in l_y + C_0 \left(\mathscr{C}_*(y + \square_n) \right).$$

The objective is then to patch these functions together to obtain a function defined on $\mathscr{C}_*(\square_m)$. A first technical issue has to be treated: in general we don't have $\mathscr{C}_*(\square_m) \cap (y + \square_n) = \mathscr{C}_*(y + \square_n)$. To deal with this technical point, we extend v_y to $\mathscr{C}_*(\square_m) \cap (y + \square_n)$ by setting

$$v_y(x) := l_y(x) \text{ for every } x \in \left(\mathscr{C}_*(\square_m) \cap (y + \square_n) \right) \setminus \mathscr{C}_*(y + \square_n).$$

In other words, v_y is the maximizer in the definition of $\nu(y + \square_n, p(y))$ except that we added a constant to it and extended its definition to the slightly larger set $\mathscr{C}_*(\square_m) \cap (y + \square_n)$. We patch these functions together by setting for each $x \in \mathscr{C}_*(\square_m)$

$$\widetilde{v}(x) := \sum_{y \in \square^\circ \cap 3^n \mathbb{Z}^d} v_y(x) \mathbb{1}_{\{x \in y + \square_n\}}. \tag{2.6.23}$$

Finally, we modify $\widetilde{v}$ to match the boundary condition. Take $\eta \in C_0^\infty(\mathbb{R}^d)$ to be the cutoff function satisfying (2.6.6) and define, for each $x \in \mathscr{C}_*(\square_m)$

$$\widetilde{u}_{\mathrm{hom}}(x) := \eta(x) \widetilde{v}(x) + (1 - \eta(x))\, u_{\mathrm{hom}}(x).$$

With this definition, it is clear that $\widetilde{u}_{\mathrm{hom}}$ satisfies the boundary condition (i):

$$\forall x \in \mathscr{C}_*(\square_m) \cap \partial \square_m, \; \widetilde{u}_{\mathrm{hom}}(x) = u(x).$$

We now prove the energy estimate (ii). We split the proof into three steps:

- In Steps 1 and 2, we prove the interior estimate

$$\frac{1}{|\square^\circ|} \langle \nabla \widetilde{v}, \mathbf{a} \nabla \widetilde{v} \rangle_{\mathscr{C}_*(\square_m) \cap \square^\circ} - \fint_{\square^\circ} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx \leq CM^2 3^{-m\alpha}.$$

- In Step 3 we prove the two boundary estimates

$$\frac{1}{|\square_m|} \langle \nabla \widetilde{u}_{\mathrm{hom}}, \mathbf{a} \nabla \widetilde{u}_{\mathrm{hom}} \rangle_{\mathscr{C}_*(\square_m)} \leq \frac{1}{|\square^\circ|} \langle \nabla \widetilde{v}, \mathbf{a} \nabla \widetilde{v} \rangle_{\mathscr{C}_*(\square_m) \cap \square^\circ} + CM^2 3^{-m\alpha}$$

  and

$$\fint_{\square^\circ} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx \leq \fint_{\square_m} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx + CM^2 3^{-m\alpha}.$$

Combining these three results gives the energy estimate (ii).

Step 1. In this step, we show the following interior estimate: for each $y \in 3^n\mathbb{Z}^d \cap \square^\circ$,

$$\frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a}\nabla v_y \rangle_{\mathscr{C}_*(\square_m)\cap(y+\square_n)} - \frac{1}{|\square_n|} \int_{y+\square_n} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx \le |p(y)|^2 3^{-m\alpha}. \tag{2.6.24}$$

For $y \in 3^n\mathbb{Z}^d \cap \square^\circ$, using v_y as a test function in the variational formulation associated to $\mu(y+\square_n, q(y))$ yields

$$\mu(y+\square_n, q(y)) \le \frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a}\nabla v_y \rangle_{\mathscr{C}_*(y+\square_n)} - \frac{1}{|\square_n|} \left\langle q(y), \nabla \left[v_y\right]_{\mathcal{P}} \right\rangle_{y+\square_n}.$$

As in the proof of Lemma 2.4.5, we have

$$\begin{aligned}
&\left| \frac{1}{|\square_n|} \left\langle q(y), \nabla \left[v_y\right]_{\mathcal{P}} \right\rangle_{y+\square_n} - q(y)\cdot p(y) \right| \\
&\qquad \le C|p(y)||q(y)| \frac{|\partial_{\mathcal{P}}(y+\square_n)|}{|\square_n|} + C|p(y)||q(y)| \left(\frac{1}{|\square_n|} \sum_{x\in\partial(y+\square_n)} \mathrm{size}(\square_{\mathcal{P}}(x))^{2d-1} \right)^{\frac12}.
\end{aligned}$$

Taking t large enough and using that $3^m \ge \mathcal{M}_t(\mathcal{Q})$, we have

$$\max_{x\in\square_m} \mathrm{size}(\square_{\mathcal{P}}(x)) \le C 3^{\frac{dm}{d+t}}$$

and thus

$$\frac{1}{|\square_n|} \sum_{x\in\partial(y+\square_n)} \mathrm{size}(\square_{\mathcal{P}}(x))^{2d-1} \le C \frac{3^{\frac{d(2d-1)m}{d+t}}}{3^n} \le C3^{-m\alpha}$$

and

$$\frac{|\partial_{\mathcal{P}}(y+\square_n)|}{|\square_n|} = \frac{1}{|\square_n|} \sum_{x\in\partial(y+\square_n)} \mathrm{size}(\square_{\mathcal{P}}(x)) \le C \frac{3^{\frac{dm}{d+t}}}{3^m} \le C3^{-m\alpha}.$$

Combining the two previous displays yields

$$\frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a}\nabla v_y \rangle_{\mathscr{C}_*(y+\square_n)} - q(y)\cdot p(y) \le \mu(y+\square_n, q(y)) + C|p(y)|^2 3^{-m\alpha}.$$

Then, by (2.6.4),

$$\begin{aligned}
|p(y)\cdot\bar{\mathbf{a}}p(y) - q(y)\cdot p(y) - \mu(y+\square_n, q(y))| &= |p(y)\cdot\bar{\mathbf{a}}p(y) + \mu(y+\square_n, q(y))| \\
&\le |p(y)|^2 3^{-m\alpha}.
\end{aligned}$$

This shows

$$\frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a}\nabla v_y \rangle_{\mathscr{C}_*(y+\square_n)} - p(y)\cdot\bar{\mathbf{a}}p(y) \le C|p(y)|^2 3^{-m\alpha}.$$

To complete the proof of (2.6.24), it is sufficient to show the two following inequalities

$$\fint_{y+\square_n} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx \ge p(y)\cdot\bar{\mathbf{a}}p(y) - C|p(y)|^2 3^{-m\alpha} \tag{2.6.25}$$

and

$$\frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a}\nabla v_y \rangle_{\mathscr{C}_*(\square_m)\cap(y+\square_n)} \le \frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a}\nabla v_y \rangle_{\mathscr{C}_*(y+\square_n)} + C|p(y)|^2 3^{-m\alpha}. \tag{2.6.26}$$

The proof of (2.6.25) relies on a convexity argument: we have, for every $p \in \mathbb{R}^d$,

$$p\cdot\bar{\mathbf{a}}p \ge p(y)\cdot\bar{\mathbf{a}}p(y) + 2p(y)\cdot\bar{\mathbf{a}}(p - p(y)).$$

Rewriting this inequality with $p = \nabla u_{\mathrm{hom}}(x)$ and integrating over $y + \square_n$ gives

$$\begin{aligned} &\fint_{y+\square_n} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx \\ &\qquad \geq \frac{(3^n-1)^d}{|\square_n|} p(y) \cdot \bar{\mathbf{a}} p(y) + 2 \fint_{y+\square_n} p(y) \cdot \bar{\mathbf{a}} (\nabla u_{\mathrm{hom}}(x) - p(y))\, dx \\ &\qquad \geq \frac{(3^n-1)^d}{|\square_n|} p(y) \cdot \bar{\mathbf{a}} p(y) \geq p(y) \cdot \bar{\mathbf{a}} p(y) - C|p(y)|^2 3^{-m\alpha}, \end{aligned}$$

by definition of $p(y)$. This gives (2.6.25). We now turn to the proof of (2.6.26). By the construction of the partition $\mathcal{P}$, it is clear that $(\mathscr{C}_*(\square_m) \cap (y+\square_n)) \setminus \mathscr{C}_*(y+\square_n)$ must be contained in the union of elements of $\mathcal{P}$ which intersect the boundary of the cube $y + \square_n$. Therefore,

$$|(\mathscr{C}_*(\square_m) \cap (y+\square_n)) \setminus \mathscr{C}_*(y+\square_n)| \leq |\partial_{\mathcal{P}}(y+\square_n)|$$

By definition of v_y,

$$v_y(x) := l_y(x) \text{ for every } x \in (\mathscr{C}_*(\square_m) \cap (y+\square_n)) \setminus \mathscr{C}_*(y+\square_n).$$

Combining the two previous displays yields

$$\begin{aligned} \frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a} \nabla v_y \rangle_{\mathscr{C}_*(\square_m)\cap(y+\square_n)} &\leq \frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a} \nabla v_y \rangle_{\mathscr{C}_*(y+\square_n)} + C|p(y)|^2 \frac{|\partial_{\mathcal{P}}(y+\square_n)|}{|\square_n|} \\ &\leq \frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a} \nabla v_y \rangle_{\mathscr{C}_*(y+\square_n)} + C|p(y)|^2 3^{-m\alpha}. \end{aligned}$$

This completes the proof of (2.6.26) and consequently the proof of (2.6.24).

Step 2. In this step we prove the following interior estimate,

$$\frac{1}{|\square^\circ_{\mathrm{int}}|} \langle \nabla \widetilde{v}, \mathbf{a} \nabla \widetilde{v} \rangle_{\square^\circ_{\mathrm{int}}} - \fint_{\square^\circ_{\mathrm{int}}} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx \leq CM^2 3^{-m\alpha}. \tag{2.6.27}$$

Summing (2.6.24) over all $y \in \square^\circ_{\mathrm{int}} \cap 3^m \mathbb{Z}^d$ and noticing that

$$\frac{3^{dn}}{|\square^\circ_{\mathrm{int}}|} \sum_{y \in \square^\circ_{\mathrm{int}} \cap 3^n \mathbb{Z}^d} |p(y)|^2 \leq C \fint_{\square_m} |\nabla u_{\mathrm{hom}}(x)|^2\, dx \leq CM^2$$

gives

$$\frac{1}{|\square^\circ_{\mathrm{int}}|} \sum_{y \in \square^\circ_{\mathrm{int}} \cap 3^n \mathbb{Z}^d} \langle \nabla \widetilde{v}, \mathbf{a} \nabla \widetilde{v} \rangle_{\mathscr{C}_*(\square_m)\cap(y+\square_n)} - \fint_{\square^\circ} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x)\, dx \leq CM^2 3^{-m\alpha}.$$

The technical point relies on the fact that $\mathscr{C}_*(\square_m)$ contains edges which do not belong to any of the clusters $(\mathscr{C}_*(\square_m) \cap (y+\square_m))_{y \in 3^n\mathbb{Z}^d \cap \square^\circ_{\mathrm{int}}}$. These edges are contained in the set V of edges connecting two vertices in different triadic cubes of size 3^n, i.e.

$$V := \left\{ \{x,z\} \ : \ x, z \in \mathbb{Z}^d,\ \exists y, y' \in 3^n \mathbb{Z}^d \cap \square^\circ_{\mathrm{int}},\ y \neq y',\ x \in y + \square_n \text{ and } z \in y' + \square_n \right\}.$$

Let $e = \{x,z\} \in \mathcal{B}_d(\mathscr{C}_*(\square_m)) \cap V$ be an edge connecting the two triadic cubes $y + \square_n$ and $y' + \square_n$. We have

$$\begin{aligned} |\nabla \widetilde{v}(e)| &= |l_y(x) - l_{y'}(z)| \\ &= |p(y) \cdot (x-y) + \zeta(y) - p(y') \cdot (z - y') - \zeta(y')| \\ &\leq |(p(y) - p(y')) \cdot (x-y)| + |p(y)||(x-z)| + |p(y) \cdot (y'-y) + \zeta(y) - \zeta(y')| \\ &\leq 3^n |p(y) - p(y')| + |p(y)| + |p(y)(y'-y) + \zeta(y) - \zeta(y')|. \end{aligned}$$

We estimate the first term on the right-hand side

$$\begin{aligned}|p(y)-p(y')| &= \left|\fint_{y+\square_n}\left(\nabla u_{\mathrm{hom}}(x)-\nabla u_{\mathrm{hom}}(x+y'-y)\right)dx\right|\\ &\le \left|\fint_{y+\square_n}\int_0^1 \nabla\nabla u_{\mathrm{hom}}(x+(1-t)(y'-y))\cdot(y-y')\,dt\,dx\right|\\ &\le |y-y'|\int_0^1\fint_{y+\square_n}\left|\nabla\nabla u_{\mathrm{hom}}(x+(1-t)(y'-y))\right|dx\,dt\\ &\le C3^n\int_0^1\fint_{y+\square_{n+1}}|\nabla\nabla u_{\mathrm{hom}}(x)|\,dx\,dt\\ &\le C3^n\fint_{y+\square_{n+1}}|\nabla\nabla u_{\mathrm{hom}}(x)|\,dx.\end{aligned}$$

A similar computation yields

$$|p(y)\cdot(y'-y)+\zeta(y)-\zeta(y')|\le C3^{2n}\fint_{y+\square_{n+1}}|\nabla\nabla u_{\mathrm{hom}}(x)|\,dx.$$

For $y\in 3^n\mathbb{Z}^d\cap\square^\circ$, denote by V_y the set of edges of V connecting $y+\square_n$ to another cube, i.e.

$$V_y := \{\{x,z\}\in V \ : \ x\in y+\square_n \text{ or } z\in y+\square_n\}.$$

The previous displays yield, for $y\in 3^n\mathbb{Z}^d\cap\square^\circ_{\mathrm{int}}$,

$$\begin{aligned}\frac{1}{|\square_m|}\sum_{e\in V_y\cap\mathscr{C}_*(\square_m)}|\nabla\widetilde{v}(e)|^2 &\le C\frac{|\partial\square_n|}{|\square_n|}\left(|p(y)|^2+(3^{2n}+3^{4n})\fint_{y+\square_{n+1}}|\nabla\nabla u_{\mathrm{hom}}(x)|^2\,dx\right)\\ &\le C3^{-n}\left(|p(y)|^2+3^{4n}\fint_{y+\square_{n+1}}|\nabla\nabla u_{\mathrm{hom}}(x)|\,dx\right).\end{aligned}$$

Thus

$$\begin{aligned}&\left|\frac{1}{|\square^\circ|}\sum_{y\in\square^\circ\cap 3^n\mathbb{Z}^d}\langle\nabla\widetilde{v},\mathbf{a}\nabla\widetilde{v}\rangle_{\mathscr{C}_*(\square_m)\cap(y+\square_n)}-\frac{1}{|\square^\circ|}\langle\nabla\widetilde{v},\mathbf{a}\nabla\widetilde{v}\rangle_{\mathscr{C}_*(\square_m)\cap\square^\circ}\right|\\ &\quad\le\frac{1}{|\square^\circ|}\sum_{y\in\square^\circ\cap 3^n\mathbb{Z}^d}\sum_{e\in V_y\cap\mathscr{C}_*(\square_m)}|\nabla\widetilde{v}(e)|^2\\ &\quad\le C\frac{|\square_m|}{|\square^\circ|}\sum_{y\in\square^\circ\cap 3^n\mathbb{Z}^d}\left(3^{-n}|p(y)|^2+3^{3n}\fint_{y+\square_{n+1}}|\nabla\nabla u_{\mathrm{hom}}(x)|^2\,dx\right)\\ &\quad\le C3^{-n}M^2+C3^{3n}\left(\fint_{\square^\circ}|\nabla\nabla u_{\mathrm{hom}}(x)|^2\,dx\right)\\ &\quad\le C3^{-n}M^2+C3^{3n-2l}M^2.\end{aligned}$$

Here we used (2.6.22) to derive the last line. Recall that we defined $m=\left\lceil\frac{m}{4}\right\rceil$ and $l=\left\lceil\frac{3m}{4}\right\rceil$ and so, for some $\alpha>0$,

$$3^{-n}\le C3^{-m\alpha} \text{ and } 3^{3n-2l}\le 3^{-m\alpha}.$$

This completes the proof of (2.6.27).

Step 3. In this final step, we estimate the contribution of u_{hom} and $\widetilde{u}_{\mathrm{hom}}$ in the boundary layer $\square_m\setminus\square^\circ_{\mathrm{int}}$. The claim is that

$$\fint_{\square^\circ_{\mathrm{int}}}\nabla u_{\mathrm{hom}}(x)\cdot\bar{\mathbf{a}}\nabla u_{\mathrm{hom}}(x)\,dx\le\fint_{\square_m}\nabla u_{\mathrm{hom}}(x)\cdot\bar{\mathbf{a}}\nabla u_{\mathrm{hom}}(x)\,dx+CM^2 3^{-m\alpha} \tag{2.6.28}$$

and

$$\frac{1}{|\square_m|}\langle\nabla\widetilde{u}_{\mathrm{hom}},\mathbf{a}\nabla\widetilde{u}_{\mathrm{hom}}\rangle_{\mathscr{C}_*(\square_m)}\le\frac{1}{|\square^\circ_{\mathrm{int}}|}\langle\nabla\widetilde{v},\mathbf{a}\nabla\widetilde{v}\rangle_{\mathscr{C}_*(\square_m)\cap\square^\circ_{\mathrm{int}}}+CM^2 3^{-m\alpha}. \tag{2.6.29}$$

To prove (2.6.28), we first recall the Meyers estimate (2.6.21) which gives, for some $r := (d, \lambda, p) \in \left(2, \frac{p+2}{2}\right)$,

$$\left(\fint_{\square_m} |\nabla u_{\mathrm{hom}}(x)|^r \, dx \right)^{\frac{1}{r}} \leq CM.$$

This allows to compute

$$\begin{aligned}
&\fint_{\square_{\mathrm{int}}^{\circ}} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x) \, dx - \fint_{\square_m} \nabla u_{\mathrm{hom}}(x) \cdot \bar{\mathbf{a}} \nabla u_{\mathrm{hom}}(x) \, dx \\
&\quad \leq \frac{|\square_m \setminus \square_{\mathrm{int}}^{\circ}|}{|\square|} \fint_{\square_m} |\nabla u_{\mathrm{hom}}(x)|^2 \, dx + \frac{1}{|\square_m|} \int_{\square_m \setminus \square_{\mathrm{int}}^{\circ}} |\nabla u_{\mathrm{hom}}(x)|^2 \, dx \\
&\quad \leq C 3^{l-n} M^2 + \left(\frac{|\square_m \setminus \square_{\mathrm{int}}^{\circ}|}{|\square_m|} \right)^{1-\frac{2}{r}} \left(\frac{1}{|\square_m|} \int_{\square_m \setminus \square_{\mathrm{int}}^{\circ}} |\nabla u_{\mathrm{hom}}(x)|^r \, dx \right)^{\frac{2}{r}} \\
&\quad \leq CM^2 3^{-m\alpha} + 3^{(l-n)\left(1-\frac{2}{r}\right)} \left(\fint_{\square_m} |\nabla u_{\mathrm{hom}}(x)|^r \, dx \right)^{\frac{2}{r}} \\
&\quad \leq CM^2 3^{-m\alpha},
\end{aligned}$$

and completes the proof of (2.6.28).

The proof of (2.6.29) follows from a similar computation but we need to prove the following discrete estimates

$$\frac{1}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m) \cap \left(\square_m \setminus \square_{\mathrm{int}}^{\circ}\right)} |\nabla \widetilde{u}_{\mathrm{hom}} \mathbb{1}_{\{\mathbf{a} \neq 0\}}|^2(x) \leq CM^2 3^{-m\alpha}. \tag{2.6.30}$$

and

$$\frac{1}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)} |\nabla \widetilde{u}_{\mathrm{hom}} \mathbb{1}_{\{\mathbf{a} \neq 0\}}|^2(x) \leq CM^2. \tag{2.6.31}$$

We will only prove (2.6.30). The proof of (2.6.31) is similar and can be easily deduced from the proof of (2.6.30).

For $x, y \in \mathscr{C}_*(\square_m)$ such that $x \sim y$ and $\mathbf{a}(\{x,y\}) \neq 0$, we compute

$$\nabla \widetilde{u}_{\mathrm{hom}}((x,y)) = \eta(y) \nabla \widetilde{v}((x,y)) + (1 - \eta(y)) \nabla u_{\mathrm{hom}}((x,y)) + \nabla \eta((x,y)) (\widetilde{v}(x) - u_{\mathrm{hom}}(x)).$$

Thus to prove (2.6.30) it is sufficient to prove the following three estimates:

(1) An estimate on $\nabla \widetilde{v}$:

$$\frac{1}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m) \cap \left(\square^{\circ} \setminus \square_{\mathrm{int}}^{\circ}\right)} |\nabla \widetilde{v} \mathbb{1}_{\{\mathbf{a} \neq 0\}}|^2(x) \leq CM^2 3^{-m\alpha}.$$

(2) An estimate on ∇u_{hom}:

$$\frac{1}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m) \cap \left(\square_m \setminus \square_{\mathrm{int}}^{\circ}\right)} |\nabla u_{\mathrm{hom}} \mathbb{1}_{\{\mathbf{a} \neq 0\}}|^2(x) \leq CM^2 3^{-m\alpha}.$$

(3) An estimate on $\widetilde{v} - u_{\mathrm{hom}}$:

$$\frac{3^{-2l}}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m) \cap \square^{\circ}} (\widetilde{v}(x) - u_{\mathrm{hom}}(x))^2 \leq CM^2 3^{-m\alpha}.$$

We prove the first estimate (1). For $y \in 3^n\mathbb{Z}^d \cap \square^\circ$, we have

$$\begin{aligned}
&\frac{1}{|\square_n|} \sum_{x \in \mathscr{C}_*(\square_m)\cap(y+\square_n)} |\nabla v_y \mathbb{1}_{\{\mathbf{a}\neq 0\}}|^2(x) \\
&\leq \frac{1}{|\square_n|} \sum_{x \in \mathscr{C}_*(y+\square_n)} |\nabla v_y \mathbb{1}_{\{\mathbf{a}\neq 0\}}|^2(x) + \frac{1}{|\square_n|} \sum_{x \in \mathscr{C}_*(\square_m)\cap(y+\square_n)\setminus\mathscr{C}_*(y+\square_n)} |\nabla l_y \mathbb{1}_{\{\mathbf{a}\neq 0\}}|^2(x) \\
&\leq \frac{1}{|\square_n|} \langle \nabla v_y, \mathbf{a}\nabla v_y\rangle_{\mathscr{C}_*(y+\square_n)} + \frac{C}{|\square_n|} \sum_{x \in \mathscr{C}_*(\square_m)\cap(y+\square_n)\setminus\mathscr{C}_*(y+\square_n)} |p(y)|^2 \\
&\leq \frac{1}{|\square_n|} \langle \nabla l_y, \mathbf{a}\nabla l_y\rangle_{\mathscr{C}_*(y+\square_n)} + C|p(y)|^2 \\
&\leq C|p(y)|^2.
\end{aligned}$$

As in the second step, we prove that for every edge e belonging to the cluster $\mathscr{C}_*(\square_m)$ and connecting the triadic cube $(y+\square_n)$ to another triadic cube of the same size,

$$|\nabla \widetilde{v}(e)|^2 \leq |p(y)|^2 + 3^{4n} \fint_{y+\square_{n+1}} |\nabla\nabla u_{\mathrm{hom}}(x)|^2 \, dx.$$

Notice that there are at most $C3^{(d-1)n}$ such edges since they must lie on the boundary of $(y+\square_n)$. Combining the two previous displays yields

$$\begin{aligned}
&\frac{1}{|\square_n|} \sum_{x \in \mathscr{C}_*(\square_m)\cap(y+\square_n)} |\nabla \widetilde{v}|^2(x) \\
&\leq C|p(y)|^2 + C3^{-n}|p(y)|^2 + C3^{3n} \fint_{y+\square_{n+1}} |\nabla\nabla u_{\mathrm{hom}}(x)|^2 \, dx \\
&\leq C|p(y)|^2 + C3^{3n} \fint_{y+\square_{n+1}} |\nabla\nabla u_{\mathrm{hom}}(x)|^2 \, dx \\
&\leq C \fint_{y+\square_n} |\nabla u_{\mathrm{hom}}(x)|^2 \, dx + C3^{3n} \fint_{y+\square_{n+1}} |\nabla\nabla u_{\mathrm{hom}}(x)|^2 \, dx.
\end{aligned}$$

Summing over $y \in 3^n\mathbb{Z}^d \cap (\square^\circ \setminus \square^\circ_{\mathrm{int}})$ gives

$$\begin{aligned}
\frac{1}{|\square_m|} \sum_{x \in \mathscr{C}_*(\square_m)\cap(\square^\circ\setminus\square^\circ_{\mathrm{int}})} |\nabla \widetilde{v}|^2(x) \leq \frac{C}{|\square_m|} \int_{\square^\circ\setminus\square^\circ_{\mathrm{int}}} |\nabla u_{\mathrm{hom}}(x)|^2 \, dx \\
+ C\frac{3^{3n}}{|\square_m|} \int_{\square^\circ\setminus\square^\circ_{\mathrm{int}}+\square_n} |\nabla\nabla u_{\mathrm{hom}}(x)|^2 \, dx.
\end{aligned}$$

The first term of the right-hand side can be estimated using the Meyers estimate as in the proof of (2.6.28). We obtain

$$\frac{1}{|\square_m|} \int_{\square^\circ\setminus\square^\circ_{\mathrm{int}}} |\nabla u_{\mathrm{hom}}(x)|^2 \, dx \leq CM^2 3^{-m\alpha}.$$

The second term of the right-hand side can be estimated using the interior H^2 estimate stated in Proposition 2.6.5,

$$\begin{aligned}
\frac{3^{3n}}{|\square_m|} \fint_{\square^\circ\setminus\square^\circ_{\mathrm{int}}+\square_n} |\nabla\nabla u_{\mathrm{hom}}(x)|^2 \, dx &\leq \frac{3^{3n}}{|\square_m|} \fint_{\square^\circ\setminus\square^\circ_{\mathrm{int}}+\square_n} |\nabla\nabla u_{\mathrm{hom}}(x)|^2 \, dx \\
&\leq \frac{3^{3n}}{|\square_m|} \fint_{\square^\circ+\square_n} |\nabla\nabla u_{\mathrm{hom}}(x)|^2 \, dx \\
&\leq C3^{3n-2l}M^2 \\
&\leq CM^2 3^{-m\alpha}.
\end{aligned}$$

The proof of (1) is complete.

To prove (2) we prove the stronger result

$$\frac{1}{|\square_m|}\sum_{x\in\square_m}|\nabla u_{\mathrm{hom}}|^{r\wedge\frac{p+2}{2}}(x)\le CM^{r\wedge\frac{p+2}{2}}3^{-m\alpha},$$

where $r>2$ is the exponent which appears in the Meyers estimate (2.6.22). This implies (2) by the same argument as in the proof of (2.6.28). Let $x,z\in\mathbb{Z}^d\cap\square_m$ with $x\sim z$. We need to distinguish three cases.

Case 1: $x,z\in\mathrm{int}(\square_m)$. If $x,z\in\mathrm{int}(\square_m)$

$$\nabla u_{\mathrm{hom}}(\{x,z\})=u_{\mathrm{hom}}(z)-u_{\mathrm{hom}}(x)=\int_0^1\nabla u_{\mathrm{hom}}(tx+(1-t)z)\cdot(x-z)\,dt.$$

Since u_{hom} is $\bar{\mathbf{a}}$-harmonic, ∇u is also $\bar{\mathbf{a}}$-harmonic hence it satisfies the mean value principle: for every $x\in\square_m$ and every $R>0$ such that $B(x,R)\subseteq\square_m$,

$$\nabla u(x)=\det(\bar{\mathbf{a}})^{-1}\fint_{\bar{\mathbf{a}}^{-1}B(x,R)}\nabla u(y)\,dy.$$

Denote for $x,z\in\mathrm{int}\,\square_m$,

$$W_{x,z}:=\left(x+\left(-\frac12,\frac12\right)^d\right)\bigcup\left(z+\left(-\frac12,\frac12\right)^d\right).$$

From this we deduce that, for every $x,z\in\mathrm{int}\,\square_m$ and every $t\in[0,1]$,

$$|\nabla u_{\mathrm{hom}}(tx+(1-t)z)|\le C\int_{W_{x,z}}|\nabla u(y)|\,dy,$$

thus

$$|\nabla u_{\mathrm{hom}}(\{x,z\})|\le C\int_{W_{x,z}}|\nabla u(y)|\,dy.$$

By the Jensen inequality, we obtain

$$|\nabla u_{\mathrm{hom}}(\{x,z\})|^{r\wedge\frac{p+2}{2}}\le C\int_{W'_{x,z}}|\nabla u_{\mathrm{hom}}(y)|^{r\wedge\frac{p+2}{2}}\,dy.$$

Case 2: $x,z\in\partial\square_m$.

$$\nabla u_{\mathrm{hom}}(\{x,z\})=\nabla w(\{x,z\}).$$

Case 3: $x\in\mathrm{int}(\square_m)$ *and* $z\in\partial\square_m$. Without loss of generality, we assume that $x-z=\mathfrak{e}_1$. Denote by S_0 the surface

$$S_0:=z+\{0\}\times\left(-\frac14,\frac14\right)^{d-1},$$

and S_1 its translation of vector $\mathfrak{e}_1$

$$S_1:=S_0+\mathfrak{e}_1=z+\{1\}\times\left(-\frac14,\frac14\right)^{d-1}.$$

By definition of w, we have for each $y\in S_0$,

$$w(y)=w(z).$$

Since $u_{\mathrm{hom}}=w$ on the boundary of $\partial\square_m$, for each $y\in S_0$,

$$u_{\mathrm{hom}}(z)=u_{\mathrm{hom}}(y).$$

With this in mind we have

$$\begin{aligned}
&|u_{\mathrm{hom}}(z) - u_{\mathrm{hom}}(x)| \\
&\quad = \left| \fint_{S_0} u_{\mathrm{hom}}(y)\,dy - u_{\mathrm{hom}}(x) \right| \\
&\quad \le C \left| \int_0^1 \fint_{S_0} \nabla u_{\mathrm{hom}}(y + t\mathfrak{e}_1)\,dy\,dt \right| + C \left| \fint_{S_1} u_{\mathrm{hom}}(y)\,dy - u_{\mathrm{hom}}(x) \right| \\
&\quad \le C \int_0^1 \fint_{S_0} |\nabla u_{\mathrm{hom}}(y + t\mathfrak{e}_1)|\,dy\,dt + C \fint_{S_1} |u_{\mathrm{hom}}(y) - u_{\mathrm{hom}}(x)|\,dy,
\end{aligned}$$

but for each $y \in S_1$,

$$\begin{aligned}
|u_{\mathrm{hom}}(y) - u_{\mathrm{hom}}(x)| &\le \left| \int_0^1 \nabla u_{\mathrm{hom}}(ty + (1-t)x)\,dt \right| \\
&\le \int_0^1 |\nabla u_{\mathrm{hom}}(ty + (1-t)x)|\,dt
\end{aligned}$$

by the mean value property

$$\begin{aligned}
&\le C \int_0^1 \int_{x + (-\frac{1}{2},\frac{1}{2})^d} |\nabla u_{\mathrm{hom}}(y)|\,dy\,dt \\
&\le C \int_{x + (-\frac{1}{2},\frac{1}{2})^d} |\nabla u_{\mathrm{hom}}(y)|\,dy.
\end{aligned}$$

Denote by

$$W'_{x,z} := \left(x + \left(-\frac{1}{2}, \frac{1}{2} \right)^d \right) \bigcup \left(\left(z + \left(-\frac{1}{2}, \frac{1}{2} \right)^d \right) \cap \square_m \right).$$

The previous computation yields

$$|\nabla u_{\mathrm{hom}}(\{x,z\})| \le C \int_{W'_{x,z}} |\nabla u_{\mathrm{hom}}(y)|\,dy.$$

We then apply the Jensen inequality to obtain

$$|\nabla u_{\mathrm{hom}}(\{x,z\})|^{r \wedge \frac{p+2}{2}} \le C \int_{W'_{x,z}} |\nabla u_{\mathrm{hom}}(y)|^{r \wedge \frac{p+2}{2}}\,dy.$$

Summing over all the edges of $\square_m$ gives

$$\frac{1}{|\square_m|} \sum_{x \in \square_m} |\nabla u_{\mathrm{hom}}|^{r \wedge \frac{p+2}{2}}(x) \le \fint_{\square_m} |\nabla u_{\mathrm{hom}}(z)|^{r \wedge \frac{p+2}{2}}\,dz + \frac{1}{|\square_m|} \sum_{x \in \partial \square_m} |\nabla w|^{r \wedge \frac{p+2}{2}}(x).$$

By Lemma 2.6.4 (ii) and a similar computation as in the proof of Lemma 2.6.4 (iii), we obtain

$$\frac{1}{|\square_m|} \sum_{x \in \partial \square_m} |\nabla w|^{r \wedge \frac{p+2}{2}}(x) \le C M^{r \wedge \frac{p+2}{2}}.$$

Combining the two previous displays yield

$$\frac{1}{|\square_m|} \sum_{x \in \square_m} |\nabla u_{\mathrm{hom}}|^{r \wedge \frac{p+2}{2}}(x) \le C M^{r \wedge \frac{p+2}{2}}.$$

We now prove (3). Since $3^m \ge \mathcal{M}_t(\mathcal{Q})$, picking t large enough, we obtain, by an application of the Poincaré inequality (which is a consequence of Proposition 2.3.4 with $s = 2$) combined with the bound on the L^2 norm of v_y given in (2.4.21): for every $y \in 3^n\mathbb{Z}^d \cap \square$,

$$\begin{aligned}\frac{3^{-2n}}{|\square_n|} &\sum_{x\in\mathscr{C}_*(\square_m)\cap(y+\square_n)} (v_y(x) - l_y(x))^2 \\ &\le \frac{C}{|\square_n|}\sum_{x\in\mathscr{C}_*(\square_m)\cap(y+\square_n)} \left|\nabla(v_y - l_y)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x) \\ &\le \frac{C}{|\square_n|}\sum_{x\in\mathscr{C}_*(\square_m)\cap(y+\square_n)} \left|\nabla v_y \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right|^2(x) + C|p(y)|^2 \\ &\le C|p(y)|^2.\end{aligned}$$

Since l_y is also $\bar{\mathbf{a}}$-harmonic, we can apply the mean value principle to the function $l_y(x) - u_{\mathrm{hom}}(x)$ as in the proof of (2),

$$\frac{3^{-2l}}{|\square_m|}\sum_{x\in\mathscr{C}_*(\square_m)\cap(y+\square_n)} (l_y(x) - u_{\mathrm{hom}}(x))^2 \le C3^{-2l} \fint_{y+\square_n} (l_y(x) - u_{\mathrm{hom}}(x))^2\,dx.$$

Applying the Poincaré inequality twice and taking into account that $\fint_{y+\square_n} = 3^{-dn}\int_{y+\square_n} \neq \frac{1}{\mathrm{Leb}(\square_n)}\int_{y+\square_n}$ by the conventions established at the beginning of this section then gives

$$\begin{aligned}\frac{3^{-2l}}{|\square_n|} &\sum_{x\in\mathscr{C}_*(\square_m)\cap(y+\square_n)} (l_y(x) - u_{\mathrm{hom}}(x))^2 \\ &\le C3^{-2l}\fint_{y+\square_n} (l_y(x) - u_{\mathrm{hom}}(x))^2\,dx \\ &\le C3^{2n-2l}\fint_{y+\square_n} |p(y) - \nabla u_{\mathrm{hom}}(x)|^2\,dx + C3^{-2n-2l}\fint_{y+\square_n} |u_{\mathrm{hom}}(x)|^2\,dx \\ &\le C3^{4n-2l}\fint_{y+\square_n} |\nabla\nabla u_{\mathrm{hom}}(x)|^2\,dx + C3^{-2l}|p(y)|^2 + C3^{-2n-2l}\fint_{y+\square_n} |u_{\mathrm{hom}}(x)|^2\,dx.\end{aligned}$$

Combining the two previous displays and using that $l - n \ge \alpha m$ yields, for each $y \in 3^n\mathbb{Z}^d \cap \square^\circ$

$$\begin{aligned}\frac{3^{-2l}}{|\square_n|} &\sum_{x\in\mathscr{C}_*(\square_m)\cap(y+\square_n)} (v_y(x) - u_{\mathrm{hom}}(x))^2 \\ &\le C3^{2n}\fint_{y+\square_n} |\nabla\nabla u_{\mathrm{hom}}(x)|^2\,dx + C|p(y)|^2 3^{-m\alpha} + C3^{-2n-2l}\fint_{y+\square_n} |u_{\mathrm{hom}}(x)|^2\,dx.\end{aligned}$$

Summing over $y \in 3^m\mathbb{Z}^d \cap \square^\circ$ gives

$$\begin{aligned}\frac{3^{-2l}}{|\square_m|}\sum_{x\in\mathscr{C}_*(\square_m)\cap\square^\circ} (v_y(x) - u_{\mathrm{hom}}(x))^2 &\le C\frac{3^{2n}}{|\square_m|}\int_{\square^\circ} |\nabla\nabla u_{\mathrm{hom}}(x)|^2\,dx \\ &\quad + \frac{C3^{-m\alpha}}{|\square_m|}\int_{\square^\circ} |\nabla u_{\mathrm{hom}}(x)|^2\,dx + C\frac{3^{-2n-2l}}{|\square_m|}\int_{\square^\circ} |u_{\mathrm{hom}}(x)|^2\,dx.\end{aligned}$$

To estimate the last term on the right-hand side, we recall that $u_{\mathrm{hom}} \in w + H^1_0(\square_m)$. By applying the Poincaré inequality

$$\begin{aligned}\frac{1}{|\square_m|}\int_{\square^\circ} |u_{\mathrm{hom}}(x)|^2\,dx &\le \fint_{\square_m} |u_{\mathrm{hom}}(x)|^2\,dx \\ &\le C\fint_{\square_m} |w(x)|^2\,dx + C3^{2m}\fint_{\square_m} |\nabla u_{\mathrm{hom}}(x) - \nabla w(x)|^2\,dx \\ &\le C\fint_{\square_m} |w(x)|^2\,dx + C3^{2m}\fint_{\square_m} |\nabla w(x)|^2\,dx \\ &\le CM^2 3^{2m},\end{aligned}$$

by (iii) and (iv) of Lemma 2.6.4. The estimate stated in (3) is then a consequence of the interior H^2 estimate of Proposition 2.6.5 and the fact that, by definition of l and n, we have $l+n-m \geq \alpha m$. This completes the proof of (3) and consequently the proof of Lemma 2.6.3(ii).

We now prove the L^2 estimate in (iii). By definition of u_{hom}, we have

$$\widetilde{u}_{\mathrm{hom}}(x) - u_{\mathrm{hom}}(x) = \eta(x)(\widetilde{v}(x) - u_{\mathrm{hom}}(x)).$$

Using the previous estimates as well as $m - l \geq \alpha m$ gives

$$\begin{aligned}\frac{3^{-2m}}{|\square_m|}\sum_{x\in\mathscr{C}_*(\square_m)}(\widetilde{u}_{\mathrm{hom}}(x) - u_{\mathrm{hom}}(x))^2 &\leq 3^{2(l-m)}\frac{3^{-2l}}{|\square_m|}\sum_{x\in\mathscr{C}_*(\square_m)\cap\square^\circ}(\widetilde{v}(x) - u_{\mathrm{hom}}(x))^2\\ &\leq CM^2 3^{-m\alpha}\end{aligned}$$

and the proof of (iii) is complete. □

2.7. Regularity theory

With Theorem 2.1.1 now proved, the second main result of the paper, Theorem 2.1.2, essentially follows from the arguments introduced in the uniformly elliptic case in [**21**] and elaborated in [**16, 17**]. The main idea is that an appropriate quantitative homogenization result, like Theorem 2.1.1, can be thought of as a result about harmonic approximation: it implies that an arbitrary solution of the heterogeneous equation can be well-approximated by an $\bar{\mathbf{a}}$-harmonic function. This allows us to transfer the regularity possessed by $\bar{\mathbf{a}}$–harmonic functions to $\mathbf{a}$-harmonic functions, following the classical ideas from elliptic regularity theory (as in, for instance, the proofs of the Schauder estimates). Of course, the regularity we obtain will only be valid on length scales on which the harmonic approximation is valid, which in our situation is all scales larger than a fixed (random) length scale of size $\mathcal{O}_s(C)$.

In this section, we abuse notation by letting $\square_m$ denote the *continuum* cube

$$\left(-\frac{1}{2}3^m, \frac{1}{2}3^m\right)^d \subseteq \mathbb{R}^d.$$

It will be made clear from the context whether $\square_m$ refers to the continuum cube or the discrete one. We further abuse notation by extending the coarsened function $[u]_{\mathcal{P}}$ to be defined on a continuum domain by taking it to be constant on each unit cube of the form $z + \square_0$ with $z \in \mathbb{Z}^d$. To avoid confusion, here we will use the symbols $\int$ and $\fint$ only to denote integration with respect to Lebesgue measure on $\mathbb{R}^d$ and write sums with $\sum$.

The first step in the proof of Theorem 2.1.2 is to post-process the error estimate proved in Theorem 2.1.1 by writing it in a form that is more convenient for the analysis in this section. We put it in terms of the coarsened functions $[u]_{\mathcal{P}}$ and emphasize harmonic approximation. The coarsening causes some technicalities to appear in the statement, so we emphasize that the second and third terms of the right side of (2.7.2) can be considered to be "small."

Lemma 2.7.1. *There exist $s(d,\lambda,\mathfrak{p}) > 0$, $\alpha(d,\lambda,\mathfrak{p}) > 0$, $C(d,\lambda,\mathfrak{p}) < \infty$ and a random variable $\mathcal{X}$ satisfying*

$$\mathcal{X} \leq \mathcal{O}_s(C) \tag{2.7.1}$$

such that, for every $m \in \mathbb{N}$ with $3^m \geq \mathcal{X}$ and every $u \in \mathcal{A}(\mathscr{C}_*(\square_{m+3}))$,

$$\begin{aligned}&\inf_{w\in\overline{\mathcal{A}}(\square_m)}\|[u]_{\mathcal{P}} - w\|_{\underline{L}^2(\square_m)}\\ &\leq C3^{-m\alpha}\inf_{a\in\mathbb{R}}\|[u]_{\mathcal{P}} - a\|_{\underline{L}^2(\square_m)} + C3^{-m\alpha}\sum_{k=1}^{n-1}3^{-k\left(m+\frac{3}{2}(k+1)\right)}\inf_{a\in\mathbb{R}}\|[u]_{\mathcal{P}} - a\|_{\underline{L}^2(\square_{m+3k})}\\ &\qquad + C3^{-m\alpha-n\left(m+\frac{3}{2}(n+1)\right)}\inf_{a\in\mathbb{R}}|\square_{m+3n}|^{-\frac{1}{2}}\|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3n}))}.\end{aligned} \tag{2.7.2}$$

PROOF. We take $p := 2+\varepsilon$, where $\varepsilon(d,\lambda,\mathfrak{p}) > 0$ is as in the statement of Proposition 2.3.8. We also take $\alpha(d,\lambda,\mathfrak{p}) > 0$ to be the exponent given in Theorem 2.1.1 with respect to the exponent p above (and which may be made smaller, if desired, in the course of the argument) and $\mathcal{X}$ to be the maximum of the random variable $\mathcal{X}$ appearing in the statement of Theorem 2.1.1 and $\mathcal{M}_t(\mathcal{Q}) + C'$ appearing in (2.3.16) with C' and t are large constants depending on $(d,\lambda,\mathfrak{p})$ to be selected in the course of the argument. It is clear then that $\mathcal{X}$ satisfies (2.7.1) for an exponent $s(d,\lambda,\mathfrak{p}) > 0$ and constant $C(d,\lambda,\mathfrak{p}) > 0$.

Step 1. We slightly tweak the statement of the error estimate. The claim is that there exists an exponent $p'(d,\lambda,\mathfrak{p}) > 2$ such that, for every $m \in \mathbb{N}$ with $3^m \geq \mathcal{X}$ and every $u \in \mathcal{A}(\mathscr{C}_*(\square_{m+3}))$,

$$\inf_{w\in\overline{\mathcal{A}}(\square_m)} |\square_m|^{-\frac{1}{p'}} \|u-w\|_{L^{p'}(\mathscr{C}_*(\square_m))} \leq C3^{-m\alpha} \inf_{a\in\mathbb{R}} |\square_m|^{-\frac{1}{2}} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3}))}. \tag{2.7.3}$$

Fix $u \in \mathcal{A}(\square_{m+2})$ and take $w = u_{\mathrm{hom}} \in \overline{\mathcal{A}}(\square_{m+1})$ to be the $\bar{\mathbf{a}}$-harmonic function given in the statement of Theorem 2.1.1 for the domain $\square = \square_{m+1}$. Then the conclusion of Theorem 2.1.1 gives the estimate

$$|\square_m|^{-\frac{1}{2}} \|u-w\|_{L^2(\mathscr{C}_*(\square_{m+1}))} \leq C3^{m(1-\alpha)} |\square_m|^{-\frac{1}{p}} \left\|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right\|_{L^p(\square_{m+1})}.$$

The Meyers estimate (Proposition 2.3.8) and the Caccioppoli inequality (Lemma 2.3.5) yield

$$\begin{aligned} |\square_m|^{-\frac{1}{p}} \left\|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right\|_{L^p(\square_{m+1})} &\leq C |\square_m|^{-\frac{1}{2}} \left\|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right\|_{L^2(\square_{m+2})} \\ &\leq C3^{-m} \inf_{a\in\mathbb{R}} |\square_m|^{-\frac{1}{2}} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3}))}. \end{aligned} \tag{2.7.4}$$

The last two displays give us

$$\|u-w\|_{L^2(\mathscr{C}_*(\square_{m+1}))} \leq C3^{-m\alpha} \inf_{a\in\mathbb{R}} |\square_m|^{-\frac{1}{2}} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3}))}. \tag{2.7.5}$$

It remains to improve the norm on the left side from L^2 to L^p. It is clear from the construction in Section 2.6, namely Lemma 2.6.4(iii), and the fact that w is harmonic that

$$\begin{aligned} \|\nabla w\|_{L^\infty(\square_m)} \leq \|\nabla w\|_{\underline{L}^2(\square_{m+1})} &\leq C \left\|\nabla u \mathbb{1}_{\{\mathbf{a}\neq 0\}}\right\|_{L^p(\square_{m+1})} \\ &\leq C3^{-m} \inf_{a\in\mathbb{R}} |\square_m|^{-\frac{1}{2}} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3}))}. \end{aligned} \tag{2.7.6}$$

We used 2.7.4 again in the last line above. Now take $p' := \frac{1}{2}(2+p)$ and apply the Sobolev-Poincaré inequality (Proposition 2.3.4) and then the Hölder inequality to deduce that, by taking $t(d,\lambda,\mathfrak{p}) < \infty$ sufficiently large,

$$\begin{aligned} &|\square_m|^{-\frac{1}{p'}} \|u-w\|_{L^{p'}(\mathscr{C}_*(\square_m))} \\ &\quad \leq C3^m \left(\frac{1}{|\square_m|} \sum_{\square\in\mathcal{P},\ \square\subseteq\square_m} \operatorname{size}(\square)^t \right)^{\frac{1}{p}-\frac{1}{p'}} |\square_m|^{-\frac{1}{p}} \left\|\nabla(u-w)\mathbb{1}_{\{\mathbf{a}\neq 0\}}\right\|_{L^p(\mathscr{C}_*(\square_m))} \\ &\quad \leq C \inf_{a\in\mathbb{R}} |\square_m|^{-\frac{1}{2}} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3}))}. \end{aligned}$$

The previous line and (2.7.5) give us, by interpolation between L^2 and $L^{p'}$ the bound, for the exponent $p'' := \left(\frac{1}{2}\left(\frac{1}{2}+\frac{1}{p'}\right)\right)^{-1} > 2$, of

$$|\square_m|^{-\frac{1}{p''}} \|u-w\|_{L^{p''}(\mathscr{C}_*(\square_m))} \leq C3^{-m\alpha/2} \inf_{a\in\mathbb{R}} |\square_m|^{-\frac{1}{2}} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3}))}.$$

After redefining α to be slightly smaller, this yields (2.7.3).

Step 2. We estimate the L^2 difference between $[u]_{\mathcal{P}}$ and w on the entire (continuum) cube $\square_m$, using the estimate from the previous step. The claim is that, for every $m \in \mathbb{N}$ with $3^m \geq \mathcal{X}$

and every $u \in \mathcal{A}(\mathscr{C}_*(\square_{m+3}))$,

$$\inf_{w \in \overline{\mathcal{A}}(\square_m)} \left\| [u]_{\mathcal{P}} - w \right\|_{\underline{L}^2(\square_m)} \le C 3^{-m\alpha} \inf_{a \in \mathbb{R}} |\square_m|^{-\frac12} \left\| u - a \right\|_{L^2(\mathscr{C}_*(\square_{m+3}))}. \tag{2.7.7}$$

Taking w as in the previous step and using that $[u]_{\mathcal{P}}$ is constant and equal to $u(\overline{z}(\square))$ on every element $\square$ of $\mathcal{P}$, we see that

$$\begin{aligned} \left\| [u]_{\mathcal{P}} - w \right\|^2_{\underline{L}^2(\square_m)} &= \frac{1}{|\,\square_m\,|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} \int_{\square} \left| w(x) - u(\overline{z}(\square)) \right|^2 \\ &\le \frac{2}{|\,\square_m\,|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} \left(|\,\square\,| \cdot \left| w(\overline{z}(\square)) - u(\overline{z}(\square)) \right|^2 + \int_{\square} \left| w(x) - w(\overline{z}(\square)) \right|^2 \right). \end{aligned}$$

The estimate for the second term inside the sum follows easily from (2.7.6):

$$\begin{aligned} &\frac{1}{|\,\square_m\,|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} \int_{\square} \left| w(x) - w(\overline{z}(\square)) \right|^2 \, dx \\ &\qquad \le \frac{1}{|\,\square_m\,|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} \operatorname{size}(\square)^2 \left\| \nabla w \right\|^2_{L^\infty(\square_m)} \le C 3^{-2m} \inf_{a \in \mathbb{R}} |\square_m|^{-1} \left\| u - a \right\|^2_{L^2(\mathscr{C}_*(\square_{m+3}))}. \end{aligned}$$

The estimate for the first term follows from (2.7.3) and the Hölder inequality, after making the parameter $t(d, \lambda, \mathfrak{p}) < \infty$ larger once again:

$$\begin{aligned} &\frac{1}{|\,\square_m\,|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} |\,\square\,| \cdot \left| w(\overline{z}(\square)) - u(\overline{z}(\square)) \right|^2 \\ &\le \left(\frac{1}{|\,\square_m\,|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} |\,\square\,|^{\frac{p'}{p'-2}} \right)^{\frac{p'-2}{p'}} \left(\frac{1}{|\,\square_m\,|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} \left| w(\overline{z}(\square)) - u(\overline{z}(\square)) \right|^{p'} \right)^{\frac{2}{p'}} \\ &\le C \left(\frac{1}{|\,\square_m\,|} \sum_{x \in \mathscr{C}_*(\square_m)} \left| w(x) - u(x) \right|^{p'} \right)^{\frac{2}{p'}} \\ &\le C 3^{-2m\alpha} \inf_{a \in \mathbb{R}} |\square_m|^{-1} \left\| u - a \right\|^2_{L^2(\mathscr{C}_*(\square_{m+3}))}. \end{aligned}$$

Combining the previous three displays yields (2.7.7).

Step 3. We compare u and $[u]_{\mathcal{P}}$. The claim is that there exists $p'(d, \lambda, \mathfrak{p}) > 2$ such that, for every $m \in \mathbb{N}$ with $3^m \ge \mathcal{X}$ and every $u \in \mathcal{A}(\mathscr{C}_*(\square_{m+3}))$,

$$|\square_m|^{-\frac{1}{p'}} \left\| u - [u]_{\mathcal{P}} \right\|_{L^{p'}(\mathscr{C}_*(\square_m))} \le C 3^{-m} \inf_{a \in \mathbb{R}} |\square_m|^{-\frac12} \left\| u - a \right\|_{L^2(\mathscr{C}_*(\square_{m+3}))}. \tag{2.7.8}$$

In fact, we can take $p' := \frac12(p+2)$. Then by (2.3.1), the Hölder inequality, and taking $t(d, \lambda, \mathfrak{p}) < \infty$ to be large enough, we get

$$\begin{aligned} &|\square_m|^{-1} \left\| u - [u]_{\mathcal{P}} \right\|^{p'}_{L^{p'}(\mathscr{C}_*(\square_{m+1}))} \\ &\quad \le \frac{C}{|\square_{m+1}|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_{m+1}} \operatorname{size}(\square)^{p'd} \int_{\square} \left| \nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right|^{p'}(x) \, dx \\ &\quad \le C \left(\frac{1}{|\square_{m+1}|} \sum_{\square \in \mathcal{P}, \square \subseteq \square_{m+1}} \operatorname{size}(\square)^{\frac{d(p'+1)p}{p-p'}} \right)^{\frac{p-p'}{p}} \left\| \nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right\|^{p'}_{\underline{L}^p(\square_{m+1})} \\ &\quad \le C \left\| \nabla u \mathbb{1}_{\{\mathbf{a} \neq 0\}} \right\|^{p'}_{\underline{L}^p(\square_{m+1})}. \end{aligned}$$

Combining this with (2.7.4), we get

$$|\square_m|^{-\frac{1}{p'}} \left\| u - [u]_{\mathcal{P}} \right\|_{L^{p'}(\mathscr{C}_*(\square_{m+1}))} \le C 3^{-m} \inf_{a \in \mathbb{R}} |\square_m|^{-\frac12} \left\| u - a \right\|_{L^2(\mathscr{C}_*(\square_{m+3}))}.$$

This yields (2.7.8).

Step 4. We complete the proof of the lemma by combining ingredients proved in the previous steps. According to (2.7.8) and the triangle inequality, for every $n \in \mathbb{N}$ with $3^n \geq \mathcal{X}$ and every $u \in \mathcal{A}(\mathscr{C}_*(\square_{n+3}))$,

$$\begin{aligned}
&\inf_{a\in\mathbb{R}} |\square_m|^{-\frac12} \|u-a\|_{L^2(\mathscr{C}_*(\square_m))} \\
&\quad \leq \inf_{a\in\mathbb{R}} |\square_m|^{-\frac12} \|[u]_{\mathcal{P}} - a\|_{L^2(\mathscr{C}_*(\square_m))} + |\square_m|^{-\frac12} \|u-[u]_{\mathcal{P}}\|_{L^2(\mathscr{C}_*(\square_m))} \\
&\quad \leq \inf_{a\in\mathbb{R}} \|[u]_{\mathcal{P}} - a\|_{\underline{L}^2(\square_m)} + C3^{-m} \inf_{a\in\mathbb{R}} |\square_m|^{-\frac12} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3}))}.
\end{aligned}$$

An iteration of this inequality yields, for every $m,n \in \mathbb{N}$ with $3^m \geq \mathcal{X}$ and every $u \in \mathcal{A}(\mathscr{C}_*(\square_{m+3n}))$, the bound

$$\begin{aligned}
&\inf_{a\in\mathbb{R}} |\square_m|^{-\frac12} \|u-a\|_{L^2(\mathscr{C}_*(\square_m))} \\
&\quad \leq \inf_{a\in\mathbb{R}} \|[u]_{\mathcal{P}} - a\|_{\underline{L}^2(\square_m)} + C\sum_{k=1}^{n-1} 3^{-\sum_{j=1}^{k}(m+3j)} \inf_{a\in\mathbb{R}} \|[u]_{\mathcal{P}} - a\|_{\underline{L}^2(\square_{m+3k})} \\
&\qquad + C3^{-\sum_{j=1}^{n}(m+3j)} \inf_{a\in\mathbb{R}} |\square_{m+3n}|^{-\frac12} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3n}))} \\
&\quad = \inf_{a\in\mathbb{R}} \|[u]_{\mathcal{P}} - a\|_{\underline{L}^2(\square_m)} + C\sum_{k=1}^{n-1} 3^{-k\left(m+\frac32(k+1)\right)} \inf_{a\in\mathbb{R}} \|[u]_{\mathcal{P}} - a\|_{\underline{L}^2(\square_{m+3k})} \\
&\qquad + C3^{-n\left(m+\frac32(n+1)\right)} \inf_{a\in\mathbb{R}} |\square_{m+3n}|^{-\frac12} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3n}))}.
\end{aligned}$$

Applying (2.7.7), we deduce that, for every $m,n \in \mathbb{N}$ with $3^m \geq \mathcal{X}$ and every $u \in \mathcal{A}(\mathscr{C}_*(\square_{m+3n}))$

$$\begin{aligned}
&\inf_{w\in\overline{\mathcal{A}}(\square_m)} \|[u]_{\mathcal{P}} - w\|_{\underline{L}^2(\square_m)} \\
&\quad \leq C3^{-m\alpha} \inf_{a\in\mathbb{R}} \|[u]_{\mathcal{P}} - a\|_{\underline{L}^2(\square_m)} + C3^{-m\alpha} \sum_{k=1}^{n-1} 3^{-k\left(m+\frac32(k+1)\right)} \inf_{a\in\mathbb{R}} \|[u]_{\mathcal{P}} - a\|_{\underline{L}^2(\square_{m+3k})} \\
&\qquad + C3^{-m\alpha-n\left(m+\frac32(n+1)\right)} \inf_{a\in\mathbb{R}} |\square_{m+3n}|^{-\frac12} \|u-a\|_{L^2(\mathscr{C}_*(\square_{m+3n}))}.
\end{aligned}$$

This is (2.7.2). □

With the result of the previous lemma in mind, we next give an elementary real analysis lemma which formalizes the transfer of regularity from harmonic functions to functions which are well-approximated by harmonic functions on large scales. It is a variation of [**16**, Lemma 2.4].

LEMMA 2.7.2. *Fix $A \geq 2$, $R \geq 2$, $\alpha > 0$ and $u \in L^2(B_R)$. For each $k \in \mathbb{N}$ and $s \in (0,R]$, denote*

$$D_k(s) := \inf_{w\in\overline{\mathcal{A}}_k} \|u-w\|_{\underline{L}^2(B_s)}.$$

Assume that $X \in [1, A^{-1}R]$ and $E > 0$ have the property that, for every $r \in [X, A^{-1}R]$,

$$\inf_{v\in\overline{\mathcal{A}}(B_r)} \|u-v\|_{\underline{L}^2(B_r)} \leq r^{1-\alpha}\left(\sup_{Ar\leq s\leq R} \frac{D_0(s)}{s} + \frac{E}{r}\right). \tag{2.7.9}$$

Then, for each $k \in \mathbb{N}$, there exists a constant $C(k,A,\alpha,d,\lambda,\mathfrak{p}) < \infty$ such that, for every $r \in \left[X \vee C, \frac12 R\right]$,

$$D_k(r) \leq C\left(\frac{r}{R}\right)^{k+1} D_k(R) + Cr^{1-\alpha}\left(\frac{D_0(R)}{R} + \frac{E}{r}\right). \tag{2.7.10}$$

PROOF. Fix $k \in \mathbb{N}$. Throughout we denote by C and c positive constants which depend only on $(k, A, \alpha, d, \lambda, \mathfrak{p})$ and may vary in each occurrence.

Step 1. We show that, for every $r \in \left[X, \frac{1}{2}R\right]$ and every $s \in \left(0, \frac{1}{2}r\right]$, we have

$$D_k(s) \le C\left(\frac{s}{r}\right)^{k+1} \inf_{w \in \overline{\mathcal{A}}_k} \|u - w\|_{\underline{L}^1(B_r)} + C\left(\frac{s}{r}\right)^{-\frac{d}{2}} \inf_{v \in \overline{\mathcal{A}}} \|u - v\|_{\underline{L}^2(B_r)}. \tag{2.7.11}$$

Select v so that

$$\|u - v\|_{\underline{L}^2(B_r)} = \inf_{v' \in \overline{\mathcal{A}}} \left\|u - v'\right\|_{\underline{L}^2(B_r)}.$$

Since v is an $\bar{\mathbf{a}}$-harmonic function, we have that

$$\inf_{w \in \mathcal{A}_k} \|v - w\|_{L^\infty(B_s)} \le C\left(\frac{s}{r}\right)^{k+1} \inf_{w \in \overline{\mathcal{A}}_k} \|v - w\|_{\underline{L}^1(B_r)}.$$

Using this and the triangle inequality twice, we find that

$$\begin{aligned}
D_k(s) &= \inf_{w \in \overline{\mathcal{A}}_k} \|u - w\|_{\underline{L}^2(B_s)} \\
&\le \inf_{w \in \overline{\mathcal{A}}_k} \|v - w\|_{\underline{L}^2(B_s)} + \|u - v\|_{\underline{L}^2(B_s)} \\
&\le C\left(\frac{s}{r}\right)^{k+1} \inf_{w \in \overline{\mathcal{A}}_k} \|v - w\|_{\underline{L}^1(B_r)} + C\left(\frac{|B_r|}{|B_s|}\right)^{\frac{1}{2}} \|u - v\|_{\underline{L}^2(B_r)} \\
&\le C\left(\frac{s}{r}\right)^{k+1} \inf_{w \in \overline{\mathcal{A}}_k} \|u - w\|_{\underline{L}^1(B_r)} + C\left(\left(\frac{s}{r}\right)^{k+1} + \left(\frac{r}{s}\right)^{\frac{d}{2}}\right) \|u - v\|_{\underline{L}^2(B_r)} \\
&\le C\left(\frac{s}{r}\right)^{k+1} \inf_{w \in \overline{\mathcal{A}}_k} \|u - w\|_{\underline{L}^1(B_r)} + C\left(\frac{s}{r}\right)^{-\frac{d}{2}} \|u - v\|_{\underline{L}^2(B_r)} \\
&= C\left(\frac{s}{r}\right)^{k+1} \inf_{w \in \overline{\mathcal{A}}_k} \|u - w\|_{\underline{L}^1(B_r)} + C\left(\frac{s}{r}\right)^{-\frac{d}{2}} \inf_{v' \in \overline{\mathcal{A}}} \left\|u - v'\right\|_{\underline{L}^2(B_r)}.
\end{aligned}$$

This is (2.7.11). Note that it also implies

$$D_k(s) \le C\left(\frac{s}{r}\right)^{k+1} D_k(r) + C\left(\frac{s}{r}\right)^{-\frac{d}{2}} \inf_{v \in \overline{\mathcal{A}}} \|u - v\|_{\underline{L}^2(B_r)}. \tag{2.7.12}$$

Step 2. We organize the rest of the argument. Denote

$$\widetilde{D}_k(r) := r^{-k} D_k(r) = r^{-k} \inf_{w \in \overline{\mathcal{A}}_k} \|u - w\|_{\underline{L}^2(B_r)}.$$

We also take $w_{k,r} \in \overline{\mathcal{A}}_k$ such that

$$\left\|u - w_{k,r}\right\|_{\underline{L}^2(B_r)} = \inf_{w \in \overline{\mathcal{A}}_k} \|u - w\|_{\underline{L}^2(B_r)} = D_k(r).$$

By (2.7.12), there exists $\theta(k, d, \lambda, \mathfrak{p}) \in \left(0, \frac{1}{2}\right]$ such that, for every $r \in \left[X, \frac{1}{2}R\right]$,

$$\widetilde{D}_k(\theta r) \le \frac{1}{2}\widetilde{D}_k(r) + C r^{-k} \inf_{v \in \overline{\mathcal{A}}} \|u - v\|_{\underline{L}^2(B_r)}.$$

Using the harmonic approximation hypothesis (2.7.9), we get

$$\widetilde{D}_k(\theta r) \le \frac{1}{2}\widetilde{D}_k(r) + C r^{1-k-\alpha}\left(\sup_{Ar \le s \le R} \frac{D_0(s)}{s} + \frac{E}{r}\right). \tag{2.7.13}$$

We will complete the proof of the lemma by iterating (2.7.13). We first must take care of the case $k = 0$ before handling general $k \in \mathbb{N}$.

Step 3. We prove (2.7.10) for $k = 0$. That is, we claim that, for every $r \in \left[X \vee C, \frac{1}{2}R\right]$,

$$\frac{D_0(r)}{r} \le C\left(\frac{D_0(R)}{R} + \frac{E}{r^{1+\alpha}}\right). \tag{2.7.14}$$

By adding a constant to u, we may suppose that $w_{0,R} = 0$. Then

$$\|w_{1,R}\|_{\underline{L}^2(B_R)} \le \|u\|_{\underline{L}^2(B_R)} + \|u - w_{1,R}\|_{\underline{L}^2(B_R)} = D_0(R) + D_1(R) \le 2D_0(R)$$

and therefore (note that $w_{1,r}$ is affine so that $\nabla w_{1,r}$ is a constant vector)

$$|\nabla w_{1,R}| \le \frac{C}{R} D_0(R). \tag{2.7.15}$$

Using the triangle inequality, we find that, for every $n \in \mathbb{N}$ with $\theta^n R \ge X \vee C$,

$$\begin{aligned}\left\|w_{1,\theta^n R} - w_{1,\theta^{n+1}R}\right\|_{\underline{L}^2(B_{\theta^{n+1}R})} &\le \left\|w_{1,\theta^n R} - u\right\|_{\underline{L}^2(B_{\theta^{n+1}R})} + \left\|u - w_{1,\theta^{n+1}R}\right\|_{\underline{L}^2(B_{\theta^{n+1}R})} \\ &\le \left(\frac{|B_{\theta^n R}|}{|B_{\theta^{n+1}R}|}\right)^{\frac{1}{2}} D_1\left(\theta^n R\right) + D_1\left(\theta^{n+1}R\right) \\ &= \theta^{-\frac{d}{2}} D_1(\theta^n R) + D_1\left(\theta^{n+1}R\right).\end{aligned}$$

Hence

$$\begin{aligned}\left|\nabla w_{1,\theta^n R} - \nabla w_{1,\theta^{n+1}R}\right| &\le \frac{1}{\theta^{n+1}R}\left\|w_{1,\theta^n R} - w_{1,\theta^{n+1}R}\right\|_{\underline{L}^2(B_{\theta^{n+1}R})} \\ &\le C\left(\widetilde{D}_1(\theta^n R) + \widetilde{D}_1\left(\theta^{n+1}R\right)\right).\end{aligned}$$

Summing this and using (2.7.15), we deduce that, for every $n \in \mathbb{N}$ with $\theta^n R \ge X \vee C$,

$$\begin{aligned}\left|\nabla w_{1,\theta^{n+1}R}\right| &\le |\nabla w_{1,R}| + \sum_{k=0}^{n}\left|\nabla w_{1,\theta^k R} - \nabla w_{1,\theta^{k+1}R}\right| \\ &\le \frac{C}{R} D_0(R) + C\sum_{k=0}^{n+1} \widetilde{D}_1(\theta^k R).\end{aligned}$$

Since the triangle inequality gives us

$$D_0(\theta^{n+1}R) \le \theta^{n+1}R\left|\nabla w_{1,\theta^{n+1}R}\right| + D_1(\theta^{n+1}R),$$

we obtain

$$\frac{D_0(\theta^{n+1}R)}{\theta^{n+1}R} \le C\left(\frac{D_0(R)}{R} + \sum_{k=0}^{n+1}\widetilde{D}_1(\theta^k R)\right). \tag{2.7.16}$$

By an iteration of (2.7.13), we get

$$\widetilde{D}_1(\theta^k R) \le 2^{-k}\widetilde{D}_1(R) + C\left(\theta^k R\right)^{-\alpha}\left(\sup_{A\theta^k \le s \le R}\frac{D_0(s)}{s} + \left(\theta^k R\right)^{-1} E\right)$$

and thus

$$\sum_{k=0}^{n+1}\widetilde{D}_1(\theta^k R) \le C\widetilde{D}_1(R) + C\left(\theta^n R\right)^{-\alpha}\left(\sup_{A\theta^{n+1} \le s \le R}\frac{D_0(s)}{s} + \left(\theta^n R\right)^{-1} E\right).$$

Combining the above with (2.7.16) and using $\widetilde{D}_1(R) \le R^{-1}D_0(R)$ yields

$$\frac{D_0(\theta^{n+1}R)}{\theta^{n+1}R} \le C\left(\frac{D_0(R)}{R} + \left(\theta^{n+1}R\right)^{-\alpha}\left(\sup_{\theta^{n+1} \le s \le R}\frac{D_0(s)}{s} + \left(\theta^{n+1}R\right)^{-1} E\right)\right).$$

If we take C sufficiently large, we obtain that $\theta^n R \ge X \vee C$ implies $C(\theta^n R)^{-\alpha} \le C X^{-\alpha} \le \frac{1}{4}\theta^{1+\frac{d}{2}}$ and we get from this and the previous display that

$$\frac{D_0(\theta^{n+1}R)}{\theta^{n+1}R} \le \frac{CD_0(R)}{R} + \frac{1}{2}\sup_{\theta^n \le s \le R}\frac{D_0(s)}{s} + \frac{E}{(\theta^{n+1}R)^{1+\alpha}}.$$

This and an easy induction argument gives us, for all such $n \in \mathbb{N}$,

$$\frac{D_0(\theta^n R)}{\theta^n R} \le \frac{C D_0(R)}{R} + \frac{E}{(\theta^{n+1}R)^{1+\alpha}}.$$

This implies (2.7.14).

Step 3. The proof of (2.7.10) for general $k \in \mathbb{N}$. In view of (2.7.14), we can improve (2.7.13) to the bound

$$\widetilde{D}_k(\theta r) \le \frac{1}{2}\widetilde{D}_k(r) + C r^{1-k-\alpha}\left(\frac{D_0(R)}{R} + \frac{E}{r}\right). \tag{2.7.17}$$

The details of the proof of (2.7.10) for general $k \in \mathbb{N}$ now follow very closely from the argument in Step 3 in the proof of [**16**, Lemma 2.4]. We omit the details. □

Following [**16, 17**], we next show that Theorem 2.1.1 and Lemma 2.7.2 imply a form of higher regularity for coarsenings of **a**-harmonic functions on mesoscopic scales. The following lemma can be compared to [**16**, Theorem 2.1].

LEMMA 2.7.3. *There exist exponents $s(d) > 0$, $\delta(d, \mathfrak{p}, \lambda) > 0$ and a random variable $\mathcal{X}$ satisfying*

$$\mathcal{X} \le \mathcal{O}_s\left(C(d, \mathfrak{p}, \lambda)\right)$$

and, for each $k \in \mathbb{N}$, a constant $C(k, d, \mathfrak{p}, \lambda) < \infty$ such that, for every $R \ge 2\mathcal{X}$, $u \in \mathcal{A}(\mathscr{C}_\infty \cap B_R)$ and $r \in \left[\mathcal{X}, \frac{1}{2}R\right]$,

$$\begin{aligned} &\inf_{w \in \overline{\mathcal{A}}_k} |B_r|^{-\frac12} \left\| u - w \right\|_{L^2(\mathscr{C}_\infty \cap B_r)} \\ &\qquad \le C\left(\frac{r}{R}\right)^{k+1} \inf_{w \in \overline{\mathcal{A}}_k} |B_R|^{-\frac12} \left\| u - w \right\|_{L^2(\mathscr{C}_\infty \cap B_R)} + C r^{-\delta}\left(\frac{r}{R}\right) |B_R|^{-\frac12} \left\| u \right\|_{L^2(\mathscr{C}_\infty \cap B_R)}. \end{aligned} \tag{2.7.18}$$

PROOF. We take $\mathcal{X}$ as in the proof of Lemma 2.7.1 and fix $R \ge 2\mathcal{X}$ and $u \in \mathcal{A}(\mathscr{C}_\infty \cap B_R)$. Note that, due to the definition of $\mathcal{X}$, for every $r \ge \mathcal{X}$ and $m \in \mathbb{N}$ such that $B_r \subseteq \square_m$, we have that $\mathscr{C}_\infty \cap B_r \subseteq \mathscr{C}_*(\square_{m+1})$. As in the statement of Lemma 2.7.2, we denote

$$D_k(s) := \inf_{w \in \overline{\mathcal{A}}_k} \left\| [u]_{\mathcal{P}} - w \right\|_{\underline{L}^2(B_s)}.$$

Also define

$$E := \frac{1}{R} \inf_{a \in \mathbb{R}} R^{-\frac{d}{2}} \left\| u - a \right\|_{L^2(\mathscr{C}_\infty \cap B_R)}.$$

Note that if $r \le \frac{1}{27} d^{-\frac12} R$, then $\square_{m+3} \subseteq B_R$. Thus (2.7.2) implies that, for every $r \in [\mathcal{X}, cR]$,

$$\inf_{w \in \overline{\mathcal{A}}(B_r)} \left\| [u]_{\mathcal{P}} - w \right\|_{\underline{L}^2(B_r)} \le C r^{1-\alpha} \sup_{r \le s \le R} \left(\frac{D_0(s)}{s} + \frac{E}{r}\right)$$

An application of Lemma 2.7.2 therefore gives us that, for every $r \in [\mathcal{X}, cR]$,

$$D_k(r) \le C\left(\frac{r}{R}\right)^{k+1} D_k(cR) + C r^{1-\alpha}\left(\frac{D_0(cR)}{cR} + \frac{E}{r}\right). \tag{2.7.19}$$

Using (2.7.11), we also have that

$$D_k(cR) \le C \inf_{w \in \overline{\mathcal{A}}_k} \left\| [u]_{\mathcal{P}} - w \right\|_{\underline{L}^1(B_{cR})}$$

(the second c is larger) and therefore we deduce that

$$D_k(r) \le C\left(\frac{r}{R}\right)^{k+1} \inf_{w \in \overline{\mathcal{A}}_k} \left\| [u]_{\mathcal{P}} - w \right\|_{\underline{L}^1(B_{cR})} + C r^{1-\alpha}\left(\frac{D_0(cR)}{cR} + \frac{E}{r}\right). \tag{2.7.20}$$

The bound (2.7.20) is close to the desired result. What is left to do is to rewrite it in terms of u rather than $[u]_{\mathcal{P}}$. To this end, it is useful to define

$$\widetilde{D}_k(s) := \inf_{w \in \overline{\mathcal{A}}_k} |B_s|^{-\frac{1}{2}} \|u - w\|_{L^2(\mathscr{C}_\infty \cap B_s)} .$$

The required approximations are presented in the following two steps and then the conclusion in the third and final step. As usual, C denotes a positive constant depending only on $(k, d, \mathfrak{p}, \lambda)$ which may vary.

Step 1. We claim there exists $C < \infty$ such that, for every $r \in [\mathcal{X}, C^{-1}R]$,

$$\widetilde{D}_k(r) \le C \left(D_k(Cr) + \frac{1}{r} \widetilde{D}_0(Cr) \right). \tag{2.7.21}$$

Take $m \in \mathbb{N}$ such that $3^{m-1} \le r \le 3^m$. Observe that the condition on r, if C is large enough, gives us that $3^m \ge \mathcal{X}$ and that $\square_{m+4} \subseteq B_R$. We may therefore apply (2.7.8) to deduce

$$\begin{aligned}
\widetilde{D}_k(r) &\le C \inf_{w \in \overline{\mathcal{A}}_k} |\square_m|^{-\frac{1}{2}} \|u - w\|_{L^2(\mathscr{C}_\infty \cap \square_m)} \\
&\le C \inf_{w \in \overline{\mathcal{A}}_k} |\square_m|^{-\frac{1}{2}} \|u - w\|_{L^2(\mathscr{C}_*(\square_{m+1}))} \\
&\le C \inf_{w \in \overline{\mathcal{A}}_k} |\square_m|^{-\frac{1}{2}} \|[u]_{\mathcal{P}} - w\|_{L^2(\mathscr{C}_*(\square_{m+1}))} + C 3^{-m} \inf_{a \in \mathbb{R}} |\square_m|^{-\frac{1}{2}} \|u - a\|_{L^2(\mathscr{C}_*(\square_{m+4}))} \\
&\le C \left(D_k(Cr) + \frac{1}{r} \widetilde{D}_0(Cr) \right),
\end{aligned}$$

which confirms (2.7.21).

Step 2. We claim that there exists $C < \infty$ such that

$$\inf_{w \in \overline{\mathcal{A}}_k} \|[u]_{\mathcal{P}} - w\|_{\underline{L}^1(B_{cR})} \le C \left(\widetilde{D}_k(R) + \frac{D_0(R)}{R} \right). \tag{2.7.22}$$

With m as in Step 1 for $r = cR$, we compute, for any $w \in \overline{\mathcal{A}}_k$,

$$\begin{aligned}
&\|[u]_{\mathcal{P}} - w\|_{\underline{L}^1(B_{cR})} \\
&\quad \le C |\square_m|^{-\frac{1}{2}} \|[u]_{\mathcal{P}} - w\|_{L^1(\square_m)} \\
&\quad = \frac{C}{|\square_m|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} \int_{\square} |u(\overline{z}(\square)) - w(x)| \, dx \\
&\quad \le \frac{C}{|\square_m|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} \left(|\square| \cdot |u(\overline{z}(\square)) - w(\overline{z}(\square))| + \int_{\square} |w(x) - w(\overline{z}(\square))| \, dx \right) \\
&\quad \le C |\square_m|^{-\frac{1}{2}} \|u - w\|_{L^2(\mathscr{C}_\infty \cap \square_{m+1})} + \|\nabla w\|_{L^\infty(\square_m)} .
\end{aligned}$$

To get the last line, we use Hölder's inequality and make the exponent t in the definition of $\mathcal{X}$ larger, if necessary. We now take $w \in \overline{\mathcal{A}}_k$ to achieve the infimum in the definition of $\widetilde{D}_k(R)$. This yields that

$$\inf_{w \in \overline{\mathcal{A}}_k} \|[u]_{\mathcal{P}} - w\|_{\underline{L}^1(B_{cR})} \le C \widetilde{D}_k(R) + C \|\nabla w\|_{L^\infty(\square_m)} .$$

It remains to show that

$$\|\nabla w\|_{L^\infty(\square_m)} \le C R^{-1} \widetilde{D}_0(R). \tag{2.7.23}$$

To see this, we note that since w is a harmonic polynomial, we have

$$\|\nabla w\|_{L^\infty(\square_m)} \le C 3^{-m} \inf_{a \in \mathbb{R}} \|w - a\|_{\underline{L}^1(\square_m)} .$$

By the Hölder inequality, we have, for any $a \in \mathbb{R}$,

$$\begin{aligned}\|w - a\|_{\underline{L}^1(\square_m)} &\le \frac{C}{|\square_m|} \sum_{\square \in \mathcal{P},\, \square \subseteq \square_m} \|w - a\|_{L^1(\square)} \\ &\le \frac{C}{|\square_m|} \sum_{x \in \mathscr{C}_\infty \cap \square_m} \Big(|w(x) - a| + \operatorname{size}(\square_{\mathcal{P}}(x))^{d+2} \|\nabla w\|_{L^\infty(\square_m)} \Big) \\ &\le C|\square_m|^{-1} \|w - a\|_{L^1(\mathscr{C}_\infty \cap \square_m)} + C \|\nabla w\|_{L^\infty(\square_m)}\end{aligned}$$

Combining these after optimizing over $a \in \mathbb{R}$, we get

$$\|\nabla w\|_{L^\infty(\square_m)} \le C3^{-m} \inf_{a \in \mathbb{R}} |\square_m|^{-1} \|w - a\|_{L^1(\mathscr{C}_\infty \cap \square_m)} + C3^{-m} \|\nabla w\|_{L^\infty(\square_m)}.$$

Since $3^m \ge cR \ge C$, we can absorb the second term on the right side to get

$$\|\nabla w\|_{L^\infty(\square_m)} \le \frac{C}{R} \inf_{a \in \mathbb{R}} |\square_m|^{-1} \|w - a\|_{L^1(\mathscr{C}_\infty \cap \square_m)}.$$

Since w achieves the infimum in the definition of $\widetilde{D}_k(R)$, we have

$$\begin{aligned}\inf_{a \in \mathbb{R}} |\square_m|^{-1} \|w - a\|_{L^1(\mathscr{C}_\infty \cap \square_m)} &\le C \inf_{a \in \mathbb{R}} |B_R|^{-\frac{1}{2}} \|w - a\|_{\underline{L}^2(B_R)} \\ &\le \widetilde{D}_0(R) + \widetilde{D}_k(R) \le 2\widetilde{D}_0(R).\end{aligned}$$

This completes the proof of (2.7.23) and thus of (2.7.22).

Step 3. The conclusion. Combining (2.7.20), (2.7.21) and (2.7.22), we obtain that, for every $r \in [\mathcal{X}, C^{-1}R]$,

$$\widetilde{D}_k(r) \le C \left(\frac{r}{R} \right)^{k+1} \widetilde{D}_k(R) + Cr^{1-\delta} \frac{\widetilde{D}_0(R)}{R}. \tag{2.7.24}$$

By adjusting the constants C, we obtain the same inequality for all $r \in [\mathcal{X}, \frac{1}{2}R]$. This yields (2.7.18) and completes the proof. □

Notice that Lemma 2.7.3 already gives the statement of Theorem 2.1.2 in the case $k = 0$ and gives us the Lipschitz estimate (2.1.19). To complete the proof of Theorem 2.1.2, we need to do an induction on k.

Proof of Theorem 2.1.2. Now that we have proved Lemma 2.7.3, the proof of Theorem 2.1.2 closely follows the argument of [**17**, Proposition 3.1] with only very minor (mostly notational) modifications, using Theorem 2.1.1 and Lemma 2.7.3 in place of [**17**, Proposition 3.2] and [**17**, Proposition 3.3], respectively. We therefore refer the reader to [**17**] and do not repeat the argument here. □

2.A. Multiscale Poincaré inequality

The purpose of this appendix is to recall a useful inequality introduced in [**16**], modified here for the discrete lattice, which allows for controlling the L^2 norm of a function by the spatial averages of its gradient.

In this appendix, we will deviate from the notation for cubes introduced in Section 2.2 and used in the rest of the paper by denoting

$$\square_m := \left(-\frac{1}{2}3^m, \frac{1}{2}3^m \right)^d \subseteq \mathbb{R}^d.$$

Thus $\square_m$ is an open subset of $\mathbb{R}^d$ and not just a collection of integer lattice points. We will also reserve the symbols $\int$ and $\fint$ to denote integration with respect to Lebesgue measure on subsets of $\mathbb{R}^d$, with discrete sums denoted by $\sum$.

The inequality we consider here is a refinement of the usual Poincaré inequality which asserts that, for a constant $C(d) < \infty$ and every $u \in H^1(\square_m)$,

$$\fint_{\square_m} \left| u(x) - (u)_{\square_m} \right|^2 dx \le C 3^{2m} \fint_{\square_m} \left| \nabla u(x) \right|^2 dx. \tag{2.1.1}$$

The discrete version of this inequality can be written

$$\sum_{x \in \mathbb{Z}^d \cap \square_m} \left| u(x) - (u)_{\square_m} \right|^2 \le C 3^{2m} \sum_{x \in \mathbb{Z}^d \cap \square_m} \left| \nabla u \right|^2 (x). \tag{2.1.2}$$

Here we denote averages by $(u)_U := \fint_U u(x)\, dx$ or $(u)_U := |\mathbb{Z}^d \cap U|^{-1} \sum_{x \in \mathbb{Z}^d \cap U} u(x)$, depending on whether u denotes a continuum or discrete function (which will always be clear from the context). It is not possible to improve the scaling of the constant $C3^{2m}$ on the right side of these inequalities for general functions, as we see by considering the case that u is affine. However, if the gradient ∇u of u has spatial averages on large scales which are very small compared to the (normalized) L^2 norm of $|\nabla u|$, then it is possible to improve the scaling of the constant. In other words, we can improve the scaling in the Poincaré inequality if we use the weaker H^{-1} norm on the right side rather than the L^2 norm. This is, of course, the situation we often find ourselves in when considering highly oscillating functions. The following result, which we call the *multiscale Poincaré inequality*, was proved in [**16**].

Proposition 2.A.1 ([**16**, Proposition 6.1]). *Fix $n, m \in \mathbb{N}$ with $n < m$. Then there exists a constant $C(d) < \infty$ such that, for every $u \in H^1(\square_m)$,*

$$\left\| u - (u)_{\square_m} \right\|_{\underline{L}^2(\square_m)} \le C 3^n \left\| \nabla u \right\|_{\underline{L}^2(\square_m)} + C \sum_{k=n}^{m-1} 3^k \left(\frac{1}{|3^k \mathbb{Z}^d \cap \square_m|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square_m} \left| (\nabla u)_{y + \square_k} \right|^2 \right)^{\frac12}. \tag{2.1.3}$$

The purpose of this appendix is to explain how to use affine interpolation to derive from Proposition 2.A.1 the following discrete version of it.

Proposition 2.A.2. *Fix $n, m \in \mathbb{N}$ with $n \in \left[\frac{m}{2}, m\right]$. Then there exists a constant $C(d) < \infty$ such that, for every $u : \mathbb{Z}^d \cap \square_m \to \mathbb{R}$,*

$$\begin{aligned} &\left\| u - (u)_{\square_m} \right\|_{\underline{L}^2(\mathbb{Z}^d \cap \square_m)} \\ &\quad \le C 3^n \left\| \nabla u \right\|_{\underline{L}^2(\mathbb{Z}^d \cap \square_m)} + C \sum_{k=n}^{m-1} 3^k \left(\frac{1}{|3^k \mathbb{Z}^d \cap \square_m|} \sum_{y \in 3^k \mathbb{Z}^d \cap \square_m} \left| \frac{1}{|\square_k|} \left\langle \nabla u \right\rangle_{\mathbb{Z}^d \cap (y + \square_k)} \right|^2 \right)^{\frac12}. \end{aligned} \tag{2.1.4}$$

Proof. We construct a smooth $\widetilde{u} \in C^\infty(\square_m)$ which is close to the discrete function u by first extending u to be constant on each cube of the form $z + \square_0$ with $z \in \mathbb{Z}^d \cap \square_m$ and then taking the convolution of it against a smooth approximation of the identity with support contained in $B_{1/2}$. It follows that $\widetilde{u}(z) = u(z)$ for each $z \in \mathbb{Z}^d \cap \square_m$ and, for each $z \in \mathbb{Z}^d \cap \square_m$,

$$\sup_{x \in z + \square_0} |\nabla \widetilde{u}(x)| \le C \sum_{y \in \mathbb{Z}^d, |y - z|_\infty \le 1} |\nabla u|(z).$$

We then check from these facts, the discrete and continuum Stokes formulas and a similar calculation as in (2.6.14) that, for each $z \in \mathbb{Z}^d \cap \square_m$ and $k \in \mathbb{N}$ with $k < m$,

$$\begin{aligned} \left| (\nabla \widetilde{u})_{z + \square_k} - \frac{1}{|\square_k|} \left\langle \nabla u \right\rangle_{\mathbb{Z}^d \cap (z + \square_k)} \right| &\le C \frac{1}{|\square_k|} \sum_{y \in \mathbb{Z}^d \cap \partial (z + \square_k)} |\nabla u|(y) \\ &\le C 3^{-\frac{k}{2}} \left(\frac{1}{|\square_k|} \sum_{y \in \mathbb{Z}^d \cap \partial(z + \square_k)} |\nabla u|^2 (y) \right)^{\frac12}. \end{aligned}$$

Applying Proposition 2.A.1 to $\widetilde{u}$ and using the above inequalities to rewrite the result in terms of u, we get (2.1.4), as desired. □

CHAPTER 3

Optimal corrector estimates on percolation clusters

We prove optimal quantitative estimates on the first-order correctors on supercritical percolation clusters: we show that they are bounded in $d \geq 3$ and have logarithmic growth in $d = 2$, in the sense of stretched exponential moments. The main ingredients are a renormalization scheme of the supercritical percolation cluster, following the works of Pisztora [**138**]; large-scale regularity estimates developed in the previous paper [**13**]; and a nonlinear concentration inequality of Efron-Stein type which is used to transfer quantitative information from the environment to the correctors.

This chapter corresponds to the article [**51**].

Contents

3.1. Introduction

3.1.1. Motivation and informal summary of results. We consider the random conductance model on the supercritical percolation cluster defined as follows. We let $\mathbb{Z}^d$ be the standard hypercubic lattice and $\mathcal{B}_d$ be the set of *bonds* of $\mathbb{Z}^d$. We fix a parameter $\lambda \in (0,1)$ and we are given a function

$$\mathbf{a} : \mathcal{B}_d \to \{0\} \cup [\lambda, 1], \tag{3.1.1}$$

the value $\mathbf{a}(e)$ is called the conductance of the bond e and we assume that the collection $(\mathbf{a}(e))_{e \in \mathcal{B}_d}$ is an i.i.d collection of random variables. We assume that the probability $\mathfrak{p} := \mathbb{P}(\mathbf{a}(e) \neq 0) > \mathfrak{p}_\mathfrak{c}(d)$, where $\mathfrak{p}_\mathfrak{c}(d)$ is the bond percolation threshold for the lattice $\mathbb{Z}^d$. It follows that, almost surely, there exists a unique maximal connected component of bonds with nonzero conductance which we denote by $\mathscr{C}_\infty = \mathscr{C}_\infty(\mathbf{a})$. One then wishes to study the continuous time *random walk* X_t in the random environment $\mathbf{a}$ defined as follows. We select an environment $\mathbf{a}$ such that $0 \in \mathscr{C}_\infty$ and start a random walker at the origin, $X(0) = 0$. Each edge e is equipped with a random clock and rings after exponential waiting time with expectation $\mathbf{a}(e)^{-1}$. When $X(t) = x$, the random walker waits until a clock of an edge adjacent to x rings and then moves across that edge. Note that the random walker is confined to the infinite cluster $\mathscr{C}_\infty$. This random walk is a Markov process and a common strategy to study it is to look at its generator, which is given by the random discrete elliptic PDE

$$-\nabla \cdot \mathbf{a} \nabla u = 0 \text{ in } \mathscr{C}_\infty,$$

where the operator $-\nabla\cdot\mathbf{a}\nabla u$ is defined on functions $u:\mathscr{C}_\infty\to\mathbb{R}$ by, for each $x\in\mathscr{C}_\infty$,

$$\nabla\cdot\mathbf{a}\nabla u(x)=\sum_{y\sim x}\mathbf{a}(\{x,y\})(u(y)-u(x)).$$

In this article, we wish to study this random elliptic PDE by studying the (random) set of *harmonic functions* for this operator. In [**29**], it was proved, in the case when $\mathbf{a}$ takes only the two values 0 and 1, that every harmonic function h with prescribed linear growth is close to a linear function: the random vector space of harmonic functions with growth at most linear is finite dimensional, and its dimension is $(d+1)$ almost surely. Moreover for each harmonic function in this space, there exists a unique vector $p\in\mathbb{R}^d$ such that the difference $\chi_p(x):=h(x)-p\cdot x$ is $o(|x|)$ as $x\to\infty$. This result was quantified and extended to the generality presented in this introduction by Armstrong and the author in [**13**], where it is shown that the corrector is $o\left(|x|^{1-\delta}\right)$ for some small but strictly positive δ.

The map χ_p is called the corrector and is the central object of this article: our goal is to prove optimal bounds in terms of spatial scaling (and suboptimal with respect to stochastic integrability) on the first-order correctors. We show, in the sense of stretched exponential moments, that the correctors are bounded in dimensions $d\geq 3$, and have increments which grow like the square root of the logarithm of the distance in dimension 2. This is summarized in the following theorem.

Theorem 3.1.1 (Optimal L^∞ estimates for first-order correctors). *There exist an exponent $s:=s(d,\mathfrak{p},\lambda)>0$ and a constant $C:=C(d,\mathfrak{p},\lambda)<\infty$ such that for each $x,y\in\mathbb{Z}^d$ and each $p\in\mathbb{R}^d$,*

$$|\chi_p(x)-\chi_p(y)|\,\mathbb{1}_{\{x,y\in\mathscr{C}_\infty\}}\leq\begin{cases}\mathcal{O}_s\left(C|p|\log^{\frac12}|x-y|\right) & \text{if } d=2,\\ \mathcal{O}_s\left(C|p|\right) & \text{if } d\geq 3,\end{cases}$$

where, for a random variable X, we write $X\leq\mathcal{O}_s(K)$, to mean

$$\mathbb{E}\left[\exp\left(\left(\frac{X}{K}\right)^s\right)\right]\leq 2.$$

Obtaining information on the corrector is important and has proved to be useful. For instance, qualitative sublinarity of the corrector can be used to prove invariance principles for the random walker X_t following the general principle described below: if one denotes by $\chi:=(\chi_1,\ldots,\chi_d)$ the vector-valued corrector, where χ_i is the corrector such that $e_i\cdot x+\chi_i(x)$ is harmonic, then the process

$X_t+\chi(X_t)$ is a martingale, almost surely with respect to the environment.

The strategy is to apply a standard martingale convergence theorem and then to derive a quenched invariance principle for the rescaled process $\varepsilon X_{t/\varepsilon^2}+\varepsilon\chi(X_{t/\varepsilon^2})$. Using the sublinearity of the corrector χ allows to prove an invariance principle for the diffusion process X itself. This was carried out on the infinite supercritical cluster (in the case when $\mathbf{a}$ takes only the values 0 and 1) first by Sidoravicius and Sznitman in [**144**] in dimension larger than 4, and a few years later by Mathieu, Piatnitski [**11**] and Berger, Biskup in [**30**] in all dimensions $d\geq 2$. Prior to these results, the generator of the random walk was studied by Barlow in [**24**] and by Mathieu, Remy in [**114**], who proved heat-kernel type bounds for the transition probability.

In the more general setting of i.i.d random conductances, when $\mathbf{a}$ can a priori take values in $[0,\infty)$, a quenched functional central limit theorem was established by Andres, Barlow, Deuschel and Hambly in [**7**], provided that there exists an infinite cluster of nonzero conductances, based on the previous works of Mathieu [**112**], Biskup and Prescott [**33**], Barlow and Deuschel [**25**]. More general model of random walks on percolation clusters with long range correlation, including random interlacements and level sets of the Gaussian free field, are studied by Procaccia, Rosenthal and Sapozhnikov in [**139**].

Tight bounds on the corrector are useful to derive invariance principles but they are also the crucial ingredient for the derivation of optimal error and two-scale expansion estimates for the homogenization of general boundary value problems. They can be used to obtain *a Berry-Essen theorem*, in the spirit of Mourrat [**128**] in the uniformly elliptic setting and are also important to obtain precise information on the Green's function for the Laplacian on the infinite cluster as well as on the transition probability for the random walk, as is explained in [**18**, Chapters 8 and 9]. They can also inform the performance of numerical algorithms for the computation of the homogenized diffusivity [**129**] and of solutions to the heterogeneous equation [**15**].

The tools developed in this article come from the theory of stochastic homogenization which studies the solutions of the elliptic equations

$$-\nabla \cdot \mathbf{a} \nabla u = 0 \text{ in } \mathbb{R}^d,$$

where the environment $\mathbf{a}$ is a random map from $\mathbb{R}^d$ to the set of symmetric matrices, satisfying some assumptions of ellipticity, stationarity and ergodicity. There have been recent developments in the quantitative homogenization of uniformly elliptic random envionments, which started with the work of Gloria and Otto in [**82**]. In this article they were able to obtain moments bounds on the corrector with an optimal spatial scaling, by using a Spectral Gap inequality, which was first introduced into stochastic homogenization by Naddaf and Spencer in [**133**], to quantify the ergodicity. This program was then continued by Gloria and Otto in [**83, 85, 86**] and by Neukamm Gloria and Otto in [**79, 81, 80**] and has implications to random walks as explained in [**63**].

Another approach was later initiated by Armstrong and Smart in [**21**], who extended the techniques of Avellaneda and Lin [**22, 23**] and the ones of Dal Maso and Modica [**49, 50**], and were able to obtain a large scale $C^{0,1}$ regularity theory under an assumption of finite range dependence on the environment. This was then generalized by Armstrong, Kuusi and Mourrat to general mixing conditions and to other types of equation [**20**] and improved to obtain *optimal* rates of convergence [**17, 18**].

The theory is now well-understood in the uniformly elliptic setting. Going beyond this setting has been the subject of much research recently in different directions. In [**108**], Lamacz, Neukamm and Otto were able to extend these results to a model of Bernoulli bond percolation, where the standard model is modified such that all the bonds in a fixed unit direction are always open. Another way of removing the ellipticity assumption can be the following: we define some (scalar) random variables $0 < \lambda \leq \mu < \infty$ according to the formulas

$$\lambda := \inf_{\xi \in \mathbb{R}^d \setminus \{0\}} \frac{\xi \cdot \mathbf{a} \xi}{|\xi|^2} \text{ and } \mu := \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} \frac{\xi \cdot \mathbf{a} \xi}{|\xi|^2},$$

and add an assumption on the integrability of λ and μ of the form : there exist $p, q \in [1, \infty]$ such that

$$\mathbb{E}\left[\lambda^{-p}\right] + \mathbb{E}\left[\mu^{q}\right] < \infty. \tag{3.1.2}$$

This setting was first considered by Andres, Deuschel, Slowik in [**9**] (see also [**10**]), and then by Chiarini and Deuschel in [**44**]. They are able to derive an invariance principle for the diffusion process under the assumption $1/p + 1/q < 2/d$, which allowed them to perform a Moser iteration. In [**28**], Bella, Fehrman and Otto, still working under the assumption $1/p + 1/q < 2/d$, were able to obtain a first-order Liouville theorem and a large scale $C^{1,\alpha}$ estimate for $\mathbf{a}$-harmonic functions. An extension of these results to the case of time-dependent coefficients has been carried out by [**8**]. The condition (3.1.2) requires the value of the conductances to be non-zero almost surely, an extension of this model in a case when the conductance is allowed to be zero and to be small (under some moments condition) was investigated by Deuschel, Nguyen and Slowik in [**57**].

The setting considered in this article is different from the models satisfying condition (3.1.2): we are working with the i.i.d. random conductance model, and we assume the value of the conductances to be either 0 or larger than some deterministic constant $\lambda > 0$ (see (3.1.1)), with the property that $\mathbb{P}(\mathbf{a}(e) \neq 0) > \mathfrak{p}_c(d)$. Despite this difference, the main challenge is essentially the same: adapting the various tools and proofs, available in the uniformly elliptic setting, to the degenerate elliptic environment. To this end, we follow the strategy initiated in the previous paper [**13**] and appeal to a renormalization structure for the supercritical percolation cluster. The construction is recalled in Section 3.2, where $\mathbb{Z}^d$ is partitioned into triadic cubes of different random sizes, well-connected in the sense of Antal an Pisztora [**11**]. This partition allows to distinguish regions of $\mathbb{Z}^d$ where the infinite cluster is well-behaved, its geometry looks like the geometry of the lattice $\mathbb{Z}^d$, from regions where the infinite cluster is badly-behaved. In the first case, it is rather straightforward to adapt the theory developed in the uniformly elliptic setting. Problems arise where the infinite cluster is badly-behaved. In this situation the theory cannot be adapted. Fortunately there are few regions were the cluster is badly-behaved, and the theory of stochastic homogenization in the uniformly elliptic setting is robust enough to be adapted to the supercritical cluster.

Our strategy to prove the optimal scaling of the corrector relies on a concentration inequality (cf. Proposition 3.2.16), which gives us a convenient way to transfer quantitative information from the coefficients to the correctors. This idea originates in an unpublished paper from Naddaf and Spencer [**133**], and was then developed by Gloria and Otto [**82, 83**] and Gloria, Neukamm and Otto [**81**] (see also Mourrat [**127**]) to study stochastic homogenization. More precisely, thanks to this inequality we are able to obtain quantitative estimates on the spatial average of the gradient of the corrector.

We then need one last ingredient to transfer bounds on the spatial average of the gradient of the corrector to the oscillation of the correctors. This will be achieved by the multiscale Poincaré inequality, Proposition 3.2.17. This inequality is a refinement of the Poincaré inequality, more suited to the study of rapidly oscillating functions such as the corrector.

3.1.2. Notation and assumptions.

3.1.2.1. *General notation for the probabilistic model.* We denote by $\mathbb{Z}^d$ the standard d-dimensional hypercubic lattice. A point $x \in \mathbb{Z}^d$ will often be called a *vertex*. The set of edges of $\mathbb{Z}^d$, that is the set of unoriented pairs of nearest neighbors, is denoted by $\mathcal{B}_d := \left\{\{x,y\} : x,y \in \mathbb{Z}^d, |x-y|_1 = 1\right\}$. More specifically, given a subset $U \subseteq \mathbb{Z}^d$, we denote by $\mathcal{B}_d(U)$ the set of the edges of U, i.e., $\mathcal{B}_d(U) := \{\{x,y\} : x,y \in U, |x-y|_1 = 1\}$. The canonical basis of $\mathbb{R}^d$ is denoted by $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$. For $x,y \in \mathbb{Z}^d$, we write $x \sim y$ if $\{x,y\} \in \mathcal{B}_d$. For some fixed parameter $\lambda \in (0,1]$, we define the probability space $\Omega := (\{0\} \cup [\lambda,1])^{\mathcal{B}_d}$ and we equip this probability space with the Borel σ-algebra $\mathcal{F} := \mathcal{B}(\{0\} \cup [\lambda,1])^{\otimes \mathcal{B}_d}$. Given an edge $e \in \mathcal{B}_d$, we denote by $\mathbf{a}(e)$ the projection

$$\mathbf{a}(e) : \begin{cases} \Omega & \to \quad \{0\} \cup [\lambda,1], \\ (\omega_{e'})_{e' \in \mathcal{B}_d} & \mapsto \quad \omega_e. \end{cases}$$

We denote by $\mathbf{a}$ the collection $(\mathbf{a}(e))_{e \in \mathcal{B}_d}$ and we refer to this mapping as the *environment*. For every $U \subseteq \mathbb{Z}^d$, we denote by $\mathcal{F}(U) \subseteq \mathcal{F}$ the σ-algebra generated by the mappings $(\mathbf{a}(e))_{e \in \mathcal{B}_d(U)}$.

We fix a probability measure $\mathbb{P}_0$ supported in $\{0\} \cup [\lambda,1]$ satisfying the property

$$\mathfrak{p} := \mathbb{P}_0([\lambda,1]) > \mathfrak{p}_\mathfrak{c}(d). \tag{3.1.3}$$

where $\mathfrak{p}_\mathfrak{c}(d)$ is the bond percolation threshold for the lattice $\mathbb{Z}^d$. We then equip the measurable space $(\Omega,\mathcal{F})$ with the i.i.d. probability measure $\mathbb{P} = \mathbb{P}_0^{\otimes \mathcal{B}_d}$, so that the sequence of random variables $(\mathbf{a}(e))_{e \in \mathcal{B}_d}$ is an i.i.d. family of random variables of law $\mathbb{P}_0$. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

Given an environment $\mathbf{a}$, we say that an edge $e \in \mathcal{B}_d$ is *open* if $\mathbf{a}(e) > 0$ and *closed* if $\mathbf{a}(e) = 0$. Given two vertices $x,y \in \mathbb{Z}^d$, we say that there is a *path connecting x and y* if there exists a sequence of open edges of the form $\{x,z_1\}, \ldots, \{z_n,z_{n+1}\}, \ldots, \{z_N,y\}$. The two vertices x and y are then said to be *connected*, which we denote by $x \leftrightarrow_{\mathbf{a}} y$, if there exists a path connecting x and y. A *cluster* is a connected subset $\mathscr{C} \subseteq \mathbb{Z}^d$. Thanks to (3.1.3), we know that, $\mathbb{P}$–almost surely, there exists a unique maximal infinite cluster [**40**]. This cluster is denoted by $\mathscr{C}_\infty := \mathscr{C}_\infty(\mathbf{a})$.

We also denote by $E_d := \{(x,y) : x,y \in \mathbb{Z}^d, x \sim y\}$ the set of oriented edges. More generally, we define, for a subset $U \subseteq \mathbb{Z}^d$, $E_d(U) := \{(x,y) : x,y \in U, x \sim y\}$.

For $x \in \mathbb{Z}^d$, we define the translation τ_x on Ω to be the application

$$\tau_x : \left\{ \begin{array}{lll} \Omega & \to & \Omega, \\ (\omega_e)_{e \in \mathcal{B}_d} & \mapsto & (\omega_{e+x})_{e \in \mathcal{B}_d}. \end{array} \right.$$

Note that the measure $\mathbb{P}$ is stationary with respect to the $\mathbb{Z}^d$-translations: for each $x \in \mathbb{Z}^d$,

$$(\tau_x)_* \mathbb{P} = \mathbb{P}, \tag{3.1.4}$$

where $(\tau_x)_* \mathbb{P}$ is the pushforward measure defined by, for each $A \in \mathcal{F}$, $(\tau_x)_* \mathbb{P}(A) = \mathbb{P}\left(\tau_x^{-1}(A)\right)$.

3.1.2.2. *Notation for functions.* We define a *vector field* to be a function $G : E_d \to \mathbb{R}$ satisfying the following antisymmetry property: for each $(x,y) \in E_d$,

$$G(x,y) = -G(y,x).$$

For a given a function $u : \mathbb{Z}^d \to \mathbb{R}$, we define its gradient ∇u to be the vector field

$$(\nabla u)(x,y) := u(x) - u(y).$$

For a random function defined on a cluster $\mathscr{C}$, $u : \mathscr{C} \to \mathbb{R}$, we define ∇u to be the vector field defined by

$$(\nabla u)(x,y) := \left\{ \begin{array}{ll} u(x) - u(y) & \text{if } x,y \in \mathscr{C} \text{ and } \mathbf{a}(\{x,y\}) \neq 0, \\ 0 & \text{otherwise.} \end{array} \right. \tag{3.1.5}$$

and $\mathbf{a}\nabla u$ to be the vector field defined by

$$(\mathbf{a}\nabla u)(x,y) := \mathbf{a}(\{x,y\})(\nabla u)(x,y).$$

We typically think of $\mathscr{C}$ as being the infinite cluster $\mathscr{C}_\infty$.
For $q \in \mathbb{R}^d$, we denote by q the constant vector field, defined according to the formula

$$q(x,y) := q \cdot (x-y).$$

For a given vector field G, we define, for every $x \in \mathbb{Z}^d$,

$$|G|(x) := \left(\frac{1}{2} \sum_{(x,y) \in E_d} |G(x,y)|^2 \right)^{\frac{1}{2}}. \tag{3.1.6}$$

For a given a subset $U \subseteq \mathbb{Z}^d$, we equip the space of vector fields with a scalar product, defined by

$$\langle F, G \rangle_U := \sum_{(x,y) \in E_d(U)} F(x,y) G(x,y).$$

We will also frequently make use of the following notation, given a vector field G, we define

$$\langle G \rangle_U = \sum_{(x,y) \in E_d(U)} G(x,y)(x-y).$$

Given an environment $\mathbf{a}$, two functions $u, v : \mathbb{Z}^d \to \mathbb{R}$, and a subset $U \subseteq \mathbb{Z}^d$, the Dirichlet form can be written with the previous notation as

$$\langle \nabla u, \mathbf{a}\nabla v \rangle_U = \sum_{(x,y) \in E_d(U)} (u(x) - u(y))\, \mathbf{a}(\{x,y\})\, (v(x) - v(y)).$$

We then define the elliptic operator $-\nabla \cdot \mathbf{a}\nabla$ by, for each $u : \mathbb{Z}^d \to \mathbb{R}$ and $x \in \mathbb{Z}^d$,

$$(-\nabla \cdot \mathbf{a}\nabla u)(x) := \sum_{x \sim y} \mathbf{a}(\{x,y\})(u(x) - u(y)).$$

For a given a subset $U \subseteq \mathbb{Z}^d$, we define the random set of $\mathbf{a}$-harmonic functions in U by,

$$\mathcal{A}(U) := \{ u : U \mapsto \mathbb{R} : (-\nabla \cdot \mathbf{a}\nabla u)(x) = 0,\, x \in \mathrm{int}_{\mathbf{a}} U \},$$

where $\mathrm{int}_{\mathbf{a}}U$ is the interior of U with respect to the environment $\mathbf{a}$, defined according to the formula

$$\mathrm{int}_{\mathbf{a}}U := \left\{ x \in U \;:\; \forall y \in \mathbb{Z}^d,\ (y \sim x \text{ and } \mathbf{a}(\{x,y\}) \neq 0) \implies y \in U \right\}.$$

Given a subset $U \subseteq \mathbb{Z}^d$ and a function $w : U \to \mathbb{R}$, we generally denote sums by integrals; for instance,

$$\text{we write } \int_U w(x)\,dx \text{ instead of } \sum_{x \in U} w(x). \tag{3.1.7}$$

If U is a finite (resp. a continuous) set, we denote its cardinality (resp. its Lebesgue measure) by $|U|$. It will always be clear from context whether we are referring to the continuous integral (resp. to the Lebesgue measure) or to the discrete integral (resp. the cardinality). The normalized integral for a discrete (resp. continuous) function $w : U \to \mathbb{R}$ defined on a discrete (resp. continous) subset $U \subseteq \mathbb{Z}^d$ (resp. $U \subseteq \mathbb{R}^d$) is denoted by

$$\fint_U w(x)\,dx = \frac{1}{|U|} \int_U w(x)\,dx.$$

To shorten the notation, we sometimes write

$$(w)_U := \fint_U w(x)\,dx.$$

We denote by $C_c^\infty(\mathbb{R}^d, \mathbb{R})$ (resp. $C^\infty(\mathbb{R}^d, \mathbb{R})$) the set of smooth compactly supported (resp. smooth) functions in $\mathbb{R}^d$ and by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space, i.e.,

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d, \mathbb{R}) : \forall\,(k, \alpha_1, \cdots, \alpha_d) \in \mathbb{N}^{d+1}, \quad \sup_{x \in \mathbb{R}^d} |x|^k \left|\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f(x)\right| < \infty \right\}$$

and by $\mathcal{S}'(\mathbb{R}^d)$ (or $\mathcal{S}'$ for short) its topological dual, the space of tempered distribution. Given $U \subseteq \mathbb{R}^d$ a domain, we denote by $C_c^\infty(U, \mathbb{R})$ (resp. $C^\infty(U, \mathbb{R})$) the set of smooth compactly supported (resp. smooth) functions in U.

For $q \in [1, \infty)$, we denote the L^q and normalized L^q norms by

$$\|w\|_{L^q(U)} := \left(\int_U |w(x)|^q \,dx \right)^{\frac{1}{q}} \text{ and } \|w\|_{\underline{L}^q(U)} := \left(\fint_U |w(x)|^q \,dx \right)^{\frac{1}{q}}.$$

Moreover, we write $\|w\|_{L^\infty(U)} := \sup_{x \in U} |w(x)|$. For $k \in \mathbb{N}$, we denote by $W^{k,q}(U)$ the Sobolev space, by $W_0^{k,q}(U)$ the closure of $C_c^\infty(U, \mathbb{R})$ in $W^{k,q}(U)$, and by $W_{\mathrm{loc}}^{k,q}(U)$ the space of local Sobolev functions. For $k \in \mathbb{Z}$ with $k < 0$, we denote by $W^{k,q}(U)$ the topological dual of $W_0^{-k,p}(U)$, with $p = \frac{q}{q-1}$.

For vectors of $\mathbb{R}^d$, we denote by $|\cdot|$ the standard infinite norm given by $|x| = \max_{i=1,\ldots,d} |x_i|$. This distance can then be extended to a pseudometric on the subsets of $\mathbb{Z}^d$ by $\mathrm{dist}(U, V) = \inf_{x \in U, y \in V} |x - y|$.

We also use the notation $B_R(x)$ to denote the ball centered in $x \in \mathbb{Z}^d$ with radius $R > 0$ with respect to the infinite norm. The ball $B_R(0)$ is simply denoted B_R.

3.1.2.3. *Notation for cubes.* A *cube* is a subset of $\mathbb{Z}^d$ of the form

$$\left(z + (-N, N)^d \right) \cap \mathbb{Z}^d,\ N \in \mathbb{N}, z \in \mathbb{Z}^d.$$

For the cube given in the previous display, which we denote by $\square$, we define its *center* and its *size* to be the point $z \in \mathbb{Z}^d$ and the integer $2N - 1$. We denote its size by $\mathrm{size}(\square)$. In particular, with this convention, we have $|\square| = (\mathrm{size}(\square))^d$. For a non-negative real number $r > 0$ and a cube $\square$, of center $z \in \mathbb{Z}^d$ and size $N \in \mathbb{N}$, we denote by $r\square$ the cube

$$r\square := \left(z + (-rN, rN)^d \right) \cap \mathbb{Z}^d.$$

This notation is non-standard because the multiplication by r only affects the size of the cube, indeed the cube $r\square$ has size $\lfloor r\,\mathrm{size}(\square)\rfloor$, but the center of the cube remains unchanged. We now introduce a specific category of cubes, namely the *triadic cubes*. A triadic cube is a cube of the form

$$\square_n(z) := \left(z + \left(-\frac{1}{2}3^n, \frac{1}{2}3^n\right)^d\right) \cap \mathbb{Z}^d,\ n \in \mathbb{N},\ z \in 3^n\mathbb{Z}^d. \tag{3.1.8}$$

To simplify the notation, we also write $\square_n = \square_n(0)$. This collection of cubes enjoys a number of very convenient properties. First, any two triadic cubes (of possibly different sizes) are either disjoint or else one is included in the other. Moreover, for every $m, n \in \mathbb{N}$ with $n \le m$, the triadic cube $\square_m$ can be uniquely partitioned into $3^{d(m-n)}$ disjoint triadic cubes of size 3^n, i.e., cubes of the form $\square_n(z)$ with $z \in 3^n\mathbb{Z}^d$. We denote by $\mathcal{T}$ the collection of triadic cubes and by $\mathcal{T}_n$ the collection of triadic cubes of size 3^n, i.e. $\mathcal{T}_n := \left\{z + \square_n : z \in 3^n\mathbb{Z}^d\right\}$.

For each $n \in \mathbb{N}$ and each $\square \in \mathcal{T}_n$, we define the predecessor of $\square$, to be the unique triadic cube $\widetilde{\square} \in \mathcal{T}_{n+1}$ such that $\square \subseteq \widetilde{\square}$. If $\widetilde{\square}$ is the predecessor of $\square$, then we also say that $\square$ is a successor $\widetilde{\square}$. In particular, a cube of $\mathcal{T}_0$ does not have any successor, while each cube of $\mathcal{T} \smallsetminus \mathcal{T}_0$ has exactly 3^d successors.

3.1.2.4. *The $\mathcal{O}_s$ notation.* We next introduce a series of notation and properties which will be useful to measure the stochastic integrability and sizes of random variables. Given two parameters $s, \theta > 0$ and a non-negative random variable X, we denote by

$$X \le \mathcal{O}_s(\theta) \text{ if and only if } \mathbb{E}\left[\exp\left(\left(\frac{X}{\theta}\right)^s\right)\right] \le 2.$$

Note that by Markov's inequality, the tail of a random variable X satisfying $X \le \mathcal{O}_s(\theta)$ decreases exponentially fast: for every $t > 0$,

$$\mathbb{P}\left[X \ge \theta t\right] \le 2\exp\left(-t^s\right).$$

For a given sequence $(Y_i)_{i\in\mathbb{N}}$ of non-negative random variables and a sequence $(\theta_i)_{i\in\mathbb{N}}$ of non-negative real numbers, we write

$$X \le \sum_{i\in\mathbb{N}} Y_i \mathcal{O}_s(\theta_i),$$

if there exists a sequence of non-negative random variables $(Z_i)_{i\in\mathbb{N}}$ such that for each $i \in \mathbb{N}$, $Z_i \le \mathcal{O}_s(\theta_i)$ and

$$X \le \sum_{i\in\mathbb{N}} Y_i Z_i.$$

We now record some properties pertaining to this notation. All these properties are proved in [**18**, Appendix A] and we refer to this reference for the proofs. This notation is compatible with the addition, meaning that, for any $s > 0$, there exists a constant C depending only on s, which may be taken to be 1 if $s \ge 1$, such that

$$X_1 \le \mathcal{O}_s(\theta_1) \text{ and } X_2 \le \mathcal{O}_s(\theta_2) \implies X_1 + X_2 \le \mathcal{O}_s(C(\theta_1 + \theta_2)). \tag{3.1.9}$$

More generally, for any $s > 0$, there exists a constant $C(s) < \infty$ such that, for every measure space $(X, \mathcal{F}, \mu)$, every jointly measurable family $\{X(x)\}_{x\in E}$ of non-negative random variables and every measurable function $\theta : E \to \mathbb{R}_+$, we have

$$\forall x \in E,\ X(x) \le \mathcal{O}_s(\theta(x)) \implies \int_E X(x)\,d\mu(x) \le \mathcal{O}_s\left(C\int_E \theta(x)\,d\mu(x)\right). \tag{3.1.10}$$

Moreover the constant can be chosen to be

$$\begin{cases} C(s) = \left(\frac{1}{s\ln 2}\right)^{\frac{1}{s}} & \text{if } s < 1 \\ C(s) = 1 & \text{if } s \ge 1. \end{cases} \tag{3.1.11}$$

From the definition, we have, for each $\lambda \in \mathbb{R}_+$,

$$X \le \mathcal{O}_s(\theta) \implies \lambda X \le \mathcal{O}_s(\lambda\theta).$$

This notation is also compatible with the multiplication in the sense that

$$|X_1| \le \mathcal{O}_{s_1}(\theta_1) \text{ and } |X_2| \le \mathcal{O}_{s_2}(\theta_2) \implies |XY| \le \mathcal{O}_{\frac{s_1 s_2}{s_1+s_2}}(\theta_1\theta_2)\,. \tag{3.1.12}$$

Moreover, it is easy to check that one can decrease the integrability exponent s, i.e., for each $0 < s' < s$, there exists a constant $C := C(s') < \infty$ such that

$$X \le \mathcal{O}_s(\theta_1) \implies X \le \mathcal{O}_{s'}(C\theta_1). \tag{3.1.13}$$

3.1.2.5. *Convention for constants and exponents.* In this article, the symbols c and C denote positive constants which may vary from line to line. These constants depend mainly on three parameters which are fixed through the proofs: the dimension of the space d, the ellipticity λ and the probability $\mathfrak{p} = \mathbb{P}[\mathbf{a}(e) \neq 0]$. Usually, we use C for large constants (whose value is expected to belong to $[1,\infty)$) and c for small constants (whose value is expected to be small $(0,1]$).

For the stochastic integrability, we use the letter s and will typically have inequalities of the form $X \le \mathcal{O}_s(C)$. This exponent s depends on the parameters d, λ and $\mathfrak{p}$. Its value can also vary from line to line and is expected to be small.

In Sections 3.4 and 3.5, another parameter will be involved in the dependence of the constants and exponents, the spatial integrability $q \in (2,\infty)$ (see Theorem 3.1.2 below), the dependence in this additional parameter will be displayed thanks to the following convention : we write $C := C(d,\lambda,\mathfrak{p}) < \infty$ (resp. $C := C(d,\lambda,\mathfrak{p},q) < \infty$) to mean that the constant C depends only on the parameters $d,\lambda,\mathfrak{p}$ (resp. $d,\lambda,\mathfrak{p},q$) and that its value is expected to be large. For small constants or exponents we use the notations $c := c(d,\lambda,\mathfrak{p}) > 0$, $s := s(d,\lambda,\mathfrak{p}) > 0$ (resp. $c := c(d,\lambda,\mathfrak{p},q) > 0$, $s := s(d,\lambda,\mathfrak{p},q) > 0$).

3.1.3. Outline of the paper. The rest of the paper is organized as follows. In Section 3.2, we recall (mostly without proof) some properties of the infinite cluster which were stated and proved in [**13**] to develop a quantitative homogenization theory on the infinite percolation cluster. In Subsections 3.2.5 and 3.2.6, we state the concentration inequality and the multiscale Poincaré inequality, which are the two key ideas in the proof of Theorem 3.1.2. In Section 3.3, we use the concentration inequality and the properties of the infinite cluster recorded in Section 3.2 to obtain an estimate on the spatial averages of the corrector. In Section 3.4, we use the result established in Section 3.3 combined with the multiscale Poincaré inequality to prove the optimal L^q-bound on the gradient of the corrector, stated in the following theorem.

Theorem 3.1.2 (Optimal L^q estimates for first-order corrector). *For each $q \ge 2$, there exist an exponent $s := s(d,\mathfrak{p},\lambda) > 0$ and a constant $C(d,\mathfrak{p},\lambda,q) < \infty$ such that for each $R \ge 1$ and each $p \in \mathbb{R}^d$,*

$$\left(R^{-d}\int_{\mathscr{C}_\infty\cap B_R}\left|\chi_p - (\chi_p)_{\mathscr{C}_\infty\cap B_R}\right|^q\right)^{\frac{1}{q}} \le \begin{cases} \mathcal{O}_s\left(C|p|\log^{\frac12} R\right) & \text{if } d=2, \\ \mathcal{O}_s\left(C|p|\right) & \text{if } d\ge 3. \end{cases} \tag{3.1.14}$$

This theorem is strictly weaker than Theorem 3.1.1 but is an important step in its proof. In Section 3.5 we upgrade the previous L^q bound into the L^∞ bound stated in Theorem 3.1.1. In Appendix 3.A, we give a proof of the multiscale Poincaré inequality stated in subsection 3.2.6. In Appendix 3.B, we give the proof of a technical lemma used in Section 3.3.

Acknowledgement. I would like to thank Scott Armstrong, Jean-Christophe Mourrat and Chenlin Gu for helpful discussions and comments.

3.2. Preliminaries

In this section we record some properties about the infinite percolation cluster in the supercritical regime. Most of these properties were established in [**13**].

3.2.1. The corrector : Existence and first properties. Denote by $\mathcal{A}_1$ the (random) vector space of $\mathbf{a}$-harmonic functions with at most linear growth, i.e.

$$\mathcal{A}_1 := \left\{ u : \mathscr{C}_\infty \to \mathbb{R} \;\middle|\; \nabla \cdot (\mathbf{a} \nabla u) = 0 \text{ in } \mathscr{C}_\infty \text{ and } \lim_{R \to \infty} \frac{1}{R^2} \| u \|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)} = 0 \right\}$$

By [**13**, Theorem 2], we know that, $\mathbb{P}$-almost surely, the space $\mathcal{A}_1$ has dimension $(d+1)$ and that every function $u \in \mathcal{A}_1$ is close to a linear function : there exists $p \in \mathbb{R}^d$ and $c \in \mathbb{R}$ such that u can be written $u = c + p \cdot x + \chi_p(x)$. The functions $\{\chi_p\}_{p \in \mathbb{R}^d}$ are called the correctors. They are defined up to a constant and are unique. In particular, to work with these quantities, one has to be careful to only consider quantities which are invariant by adding a constant, such as the oscillation, the gradient, the difference $\chi_p(x) - \chi_p(y)$ etc.

The sublinear growth of the corrector is already known, indeed by [**13**, (1.22)], there exist two exponents $\delta := \delta(d, \mathfrak{p}, \lambda) > 0$, $s := s(d, \mathfrak{p}, \lambda) > 0$ and a constant $C := C(d, \mathfrak{p}, \lambda)$ such that, for each $R \geq 1$,

$$\underset{\mathscr{C}_\infty \cap B_R}{\operatorname{osc}} \chi_p \leq \mathcal{O}_s \left(C |p| R^{1-\delta} \right). \tag{3.2.1}$$

where the oscillation, on a subset $A \subseteq \mathbb{Z}^d$, of a function $f : A \to \mathbb{R}$ is defined by $\operatorname{osc}_A f := \sup_A f - \inf_A f$. The sublinear growth of the corrector is a very important property which was proven quantitatively in [**13**] and qualitatively in [**55**]. It can also be expressed with a minimal scale, indeed by [**13**, (1.18)], there exists a random variable $\mathcal{X}$ satisfying

$$\mathcal{X} \leq \mathcal{O}_s(C),$$

such that for each $R \geq \mathcal{X}$,

$$\left\| \chi_p - (\chi_p)_{\mathscr{C}_\infty \cap B_R} \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)} \leq C |p| R^{1-\delta}. \tag{3.2.2}$$

Moreover, the corrector satisfies the following stationarity property, for each $x, y \in \mathbb{Z}^d$, each $p \in \mathbb{R}^d$ and each $z \in \mathbb{Z}^d$,

$$(\chi_p(x) - \chi_p(y)) \mathbb{1}_{\{x, y \in \mathscr{C}_\infty\}}(\mathbf{a}) = (\chi_p(x+z) - \chi_p(y+z)) \mathbb{1}_{\{z+x, z+y \in \mathscr{C}_\infty\}}(\tau_z \mathbf{a}). \tag{3.2.3}$$

3.2.2. Triadic partitions of good cubes. This second section shows how to use the tools developed by Antal and Pisztora [**11**] to obtain a renormalization structure of the infinite cluster of supercritical percolation.

3.2.2.1. *A general scheme for partition of good cubes.* The construction of the partition is accomplished by a stopping time argument reminiscent of a Calderón-Zygmund-type decomposition. We are given a notion of "good cube" represented by an $\mathcal{F}$-measurable function which maps Ω into the set of all subsets of $\mathcal{T}$. In order words, for each $\mathbf{a} \in \Omega$, we are given a subcollection $\mathcal{G}(\mathbf{a}) \subseteq \mathcal{T}$ of triadic cubes. We think of $\square \in \mathcal{T}$ as being a good cube if $\square \in \mathcal{G}(\mathbf{a})$. We frequently drop the dependence in $\mathbf{a}$ and write $\mathcal{G}$.

Proposition 3.2.1 (Proposition 2.1 of [**13**]). *Let $\mathcal{G} \subseteq \mathcal{T}$ be a random collection of triadic cubes, as above. Suppose that $\mathcal{G}$ satisfies, for every $\square = z + \square_n \in \mathcal{T}$,*

$$\text{the event } \{\square \notin \mathcal{G}\} \text{ is } \mathcal{F}(z + \square_{n+1})\text{-measurable}, \tag{3.2.4}$$

and, for some constants $K, s > 0$,

$$\sup_{z \in 3^n \mathbb{Z}^d} \mathbb{P}\left[z + \square_n \notin \mathcal{G} \right] \leq K \exp\left(-K^{-1} 3^{ns} \right).$$

Then, $\mathbb{P}$–almost surely, there exists a partition $\mathcal{S} \subseteq \mathcal{T}$ of $\mathbb{Z}^d$ into triadic cubes with the following properties:

(i) *All predecessors of elements of $\mathcal{S}$ are good: for every $\square, \square' \in \mathcal{T}$,*

$$\square' \subseteq \square \text{ and } \square' \in \mathcal{S} \implies \square \in \mathcal{G}.$$

(ii) *Neighboring elements of $\mathcal{S}$ have comparable sizes: for every $\square, \square' \in \mathcal{S}$ such that* $\mathrm{dist}(\square, \square') \le 1$*, we have*

$$\frac{1}{3} \le \frac{\mathrm{size}(\square')}{\mathrm{size}(\square)} \le 3.$$

(iii) *Estimate for the coarseness of $\mathcal{S}$: if we denote by $\square_{\mathcal{S}}(x)$ the unique element of $\mathcal{S}$ containing a point $x \in \mathbb{Z}^d$, then there exists $C(s,K,d) < \infty$ such that, for every $x \in \mathbb{Z}^d$,*

$$\mathrm{size}\left(\square_{\mathcal{S}}(x)\right) \le \mathcal{O}_s(C).$$

In addition, if one has the following independence property, for every $\square = z + \square_n \in \mathcal{T}$,

$$\text{the event } \{\square \notin \mathcal{G}\} \text{ is } \mathcal{F}(z + \square_{n+1})\text{-measurable,} \tag{3.2.5}$$

then one has the following minimal scale property

(iv) *Minimal scale for $\mathcal{S}$. For each $t \in [1,\infty)$, there exists $C := C(t,s,K,d) < \infty$, an $\mathbb{N}$-valued random variable $\mathcal{M}_t(\mathcal{S})$ and exponent $r := r(t,s,K,d) > 0$ such that*

$$\mathcal{M}_t(\mathcal{S}) \le \mathcal{O}_r(C)$$

and for each $m \in \mathbb{N}$ satisfying $3^m \ge \mathcal{M}_t(\mathcal{S})$,

$$\frac{1}{|\square_m|} \sum_{x \in \square_m} \mathrm{size}\left(\square_{\mathcal{S}}(x)\right)^t \le C \quad \textit{and} \quad \sup_{x \in \square_m} \mathrm{size}\left(\square_{\mathcal{S}}(x)\right) \le 3^{\frac{dm}{d+t}}.$$

3.2.2.2. *The partition $\mathcal{P}$ of well-connected cubes.* We apply the construction of the previous subsection to obtain a random partition $\mathcal{P}$ of $\mathbb{Z}^d$ which simplifies the geometry of the percolation cluster. This partition plays an important role in the rest of the paper. To obtain bounds on the "good event" which allows us to construct the partition, we use the important results of Pisztora [**138**], Penrose and Pisztora [**136**] and Antal and Pisztora [**11**]. We first recall some definitions introduced in those works.

DEFINITION 3.2.2 (Crossability and crossing cluster). We say that a cube $\square$ is *crossable* (with respect to an environment $\mathbf{a} \in \Omega$) if each of the d pairs of opposite $(d-1)$–dimensional faces of $\square$ is joined by an open path in $\square$. We say that a cluster $\mathscr{C} \subseteq \square$ is a *crossing cluster for* $\square$ if $\mathscr{C}$ intersects each of the $(d-1)$–dimensional faces of $\square$.

DEFINITION 3.2.3 (Good cube). We say that a triadic cube $\square \in \mathcal{T}$ is *well-connected* if there exists a crossing cluster $\mathscr{C}$ for the cube $\square$ such that:

(i) each cube $\square'$ with $\mathrm{size}(\square') \in \left[\frac{1}{10}\mathrm{size}(\square), \frac{1}{2}\mathrm{size}(\square)\right]$ and $\square' \cap \frac{3}{4}\square \neq \varnothing$ is crossable; and
(ii) every path $\gamma \subseteq \square'$ with $\mathrm{diam}(\gamma) \ge \frac{1}{10}\mathrm{size}(\square)$ is connected to $\mathscr{C}$ within $\square'$.

We say that $\square \in \mathcal{T}$ is a *good cube* if $\mathrm{size}(\square) \ge 3$, $\square$ is well-connected and each of the 3^d successors of $\square$ are well-connected. We say that $\square \in \mathcal{T}$ is a *bad cube* if it is not a good cube.

The following estimate on the probability of the cube $\square_n$ being good is a consequence [**138**, Theorem 3.2] and [**136**, Theorem 5], as recalled in [**11**, (2.24)].

LEMMA 3.2.4 ([**11**, (2.24)]). *For each $\mathfrak{p} \in (\mathfrak{p}_c, 1]$, there exists $C(d,\mathfrak{p}) < \infty$ such that, for every $n \in \mathbb{N}$,*

$$\mathbb{P}\left[\square_n \textit{ is good}\right] \ge 1 - C\exp\left(-C^{-1}3^n\right). \tag{3.2.6}$$

It follows from Definition 3.2.3 that, for every good cube $\square$, there exists a unique maximal crossing cluster for $\square$ which is contained in $\square$. We denote this cluster by $\mathscr{C}_*(\square)$. In the next lemma, we record the observation that adjacent triadic cubes which have similar sizes and are both good have connected clusters.

LEMMA 3.2.5 (Lemma 2.8 of [**13**]). *Let $n, n' \in \mathbb{N}$ with $|n-n'| \le 1$ and $z, z' \in 3^n\mathbb{Z}^d$ such that*

$$\operatorname{dist}\left(\square_n(z), \square_{n'}(z')\right) \le 1.$$

Suppose also that $\square_n(z)$ and $\square_{n'}(z')$ are good cubes. Then there exists a cluster $\mathscr{C}$ such that

$$\mathscr{C}_*(\square_n(z)) \cup \mathscr{C}_*(\square_{n'}(z')) \subseteq \mathscr{C} \subseteq \square_n(z) \cup \square_{n'}(z').$$

We next define our partition $\mathcal{P}$.

DEFINITION 3.2.6. We let $\mathcal{P} \subseteq \mathcal{T}$ be the partition $\mathcal{S}$ of $\mathbb{Z}^d$ obtained by applying Proposition 3.2.1 to the collection

$$\mathcal{G} := \{\square \in \mathcal{T} : \square \ \text{ is good}\}.$$

More generally, for each $y \in \mathbb{Z}^d$, we let be $\mathcal{P}_y \subseteq \mathcal{T}$ be the partition $\mathcal{S}$ of $\mathbb{Z}^d$ obtained by applying Proposition 3.2.1 to the collection

$$\mathcal{G} := \{y + \square : \square \in \mathcal{T} \,\text{and}\, y + \square \text{ is good}\}.$$

From the construction of $\mathcal{P}$ and $\mathcal{P}_y$, we also have

$$\mathcal{P}_y = y + \mathcal{P}(\tau_{-y}\mathbf{a}) = \{y + \square : \square \in \mathcal{P}(\tau_{-y}\mathbf{a})\}.$$

The (random) partition $\mathcal{P}$ plays an important role throughout the rest of the paper. We also denote by $\mathcal{P}_*$ the collection of triadic cubes which contains some elements of $\mathcal{P}$, that is,

$$\mathcal{P}_* := \{\square : \square \text{ is a triadic cube and } \square \supseteq \square' \text{ for some } \square' \in \mathcal{P}\}.$$

Notice that every element of $\mathcal{P}_*$ can be written in a unique way as a disjoint union of elements of $\mathcal{P}$. According to Proposition 3.2.1(i), every triadic cube containing an element of $\mathcal{P}$ is good. By Proposition 3.2.1(iii) and Lemma 3.2.4, there exists $C(d,\mathfrak{p}) < \infty$ such that, for every $x \in \mathbb{Z}^d$,

$$\operatorname{size}(\square_{\mathcal{P}}(x)) \le \mathcal{O}_1(C). \tag{3.2.7}$$

By the properties of $\mathcal{P}$ given in Proposition 3.2.1(i) and (ii) and Lemma 3.2.5, the maximal crossing cluster $\mathscr{C}_*(\square)$ of an element $\square \in \mathcal{P}_*$ must satisfy $\mathscr{C}_*(\square) \subseteq \mathscr{C}_\infty$, since the union of all crossing clusters of elements of $\mathcal{P}$ is unbounded and connected. Notice also that, although we may not have $\mathscr{C}_*(\square) = \mathscr{C}_\infty \cap \square$, by definition of the partition $\mathcal{P}$ and (ii) of Definition 3.2.3, we have that, for every cube $\square \in \mathcal{P}$, there exists a cluster $\mathscr{C}$ such that

$$\mathscr{C}_\infty \cap \square \subseteq \mathscr{C} \subseteq \bigcup_{\square' \in \mathcal{P},\ \operatorname{dist}(\square,\square') \le 1} \square'. \tag{3.2.8}$$

In other words, for any cube $\square \in \mathcal{P}$ and every $x, y \in \mathscr{C}_\infty \cap \square$, there exists a path connecting x to y which stays in $\square$ or in its neighbors.

It is also interesting to note that, for $m \in \mathbb{N}$ such that $3^m \ge \mathcal{M}_{2d}(\mathcal{P})$, $\mathscr{C}_*(\square_m)$, $\mathscr{C}_\infty \cap \square_m$ and $\square_m$ are of comparable size, precisely, there exists a constant $C := C(d,\mathfrak{p}) < \infty$ such that

$$C^{-1} |\square_m| \le |\mathscr{C}_*(\square_m)| \le |\mathscr{C}_\infty \cap \square_m| \le |\square_m|. \tag{3.2.9}$$

This result is a consequence of the Cauchy-Schwarz inequality and the three relations, under the assumption $3^m \ge \mathcal{M}_{2d}(\mathcal{P})$, which implies in particular that $\square_m$ is good,

$$\sum_{\square \in \mathcal{P}, \square \subseteq \square_m} 1 \le \mathscr{C}_*(\square_m), \qquad \sum_{\square \in \mathcal{P}, \square \subseteq \square_m} \operatorname{size}(\square_{\mathcal{P}})^d = |\square_m| \quad \text{and} \quad \sum_{\square \in \mathcal{P}, \square \subseteq \square_m} \operatorname{size}(\square_{\mathcal{P}})^{2d} \le C\,|\square_m|.$$

The first inequality comes from the fact that each cube of $\mathcal{P}$ contained in $\square_m$ must have non-empty intersection with $\mathscr{C}_*(\square_m)$, the second is the preservation of the volume and the third is where we use the assumption $3^m \ge \mathcal{M}_{2d}(\mathcal{P})$.

Given $\square \in \mathcal{P}$, we let $\overline{z}(\square)$ denote the element of $\mathscr{C}_*(\square)$ which is closest to z in the Manhattan distance; if this is not unique, we break ties by the lexicographical order.

DEFINITION 3.2.7. Given a function $u : \mathscr{C}_\infty \to \mathbb{R}$, we define *the coarsened function with respect to $\mathcal{P}$* to be

$$\begin{array}{rccc} [u]_{\mathcal{P}} & : & \mathbb{Z}^d & \to & \mathbb{R} \\ & & x & \mapsto & u\left(\overline{z}\left(\square_{\mathcal{P}}(x)\right)\right). \end{array}$$

The reason we use the coarsened function is that it is defined on the entire lattice $\mathbb{Z}^d$ and not on the infinite cluster. This allows to make use of the simpler and more favorable geometric structure of $\mathbb{Z}^d$. The price to pay is the difference between u and $[u]_{\mathcal{P}}$. This depends on the coarseness of the partition $\mathcal{P}$ and the control one has on ∇u in a way that is made precise in the following proposition. The dependence on the coarseness of $\mathcal{P}$ is present via the size of the cubes of the partition. Recall that the notation $|F|(x)$ for a vector field F is defined in (3.1.6).

PROPOSITION 3.2.8 (Lemma 3.2 of [**13**]). *For every triadic cube $\square \in \mathcal{P}_*$, $1 \le s < \infty$ and $w : \mathscr{C}_\infty \to \mathbb{R}$,*

$$\int_{\mathscr{C}_*(\square)} |w(x) - [w]_{\mathcal{P}}(x)|^s \, dx \le C^s \int_{\mathscr{C}_*(\square)} \operatorname{size}(\square_{\mathcal{P}}(x))^{sd} \, |\nabla w|^s (x) \, dx. \tag{3.2.10}$$

More generally, for any family of disjoint cubes $\{\square_i\}_{i\in I} \in (\mathcal{P}_*)^I$, we have

$$\int_{\mathscr{C}_*(\cup_{i\in I}\square_i)} |w(x) - [w]_{\mathcal{P}}(x)|^s \, dx \le C^s \int_{\mathscr{C}_*(\cup_{i\in I}\square_i)} \operatorname{size}(\square_{\mathcal{P}}(x))^{sd} \, |\nabla w|^s (x) \, dx, \tag{3.2.11}$$

where $\mathscr{C}_*(\cup_{i\in I}\square_i)$ denotes the union of the maximal clusters of each connected component of $\cup_{i\in I}\square_i$.

REMARK 3.2.9. Unfortunately, we do not have $\mathscr{C}_*(\cup_{i\in I}\square_i) = \bigcup_{i\in I} \mathscr{C}_*(\square_i)$. The problem is the same than the one we had in (3.2.8) and thus (3.2.11) can not be directly obtained from (3.2.10). Nevertheless, thanks to this equation, we do have the inclusion

$$\mathscr{C}_\infty \cap \square \subseteq \mathscr{C}\left(\bigcup_{\square' \in \mathcal{P},\ \operatorname{dist}(\square,\square') \le 1} \square' \right). \tag{3.2.12}$$

Moreover we can control the L^s norm of the vector field $\nabla [w]_{\mathcal{P}}$ depending on the L^s norm of ∇w and the coarseness of the partition $\mathcal{P}$ thanks to the following proposition.

PROPOSITION 3.2.10 (Lemma 3.3 of [**13**]). *For every triadic cube $\square \in \mathcal{P}_*$, $1 \le s < \infty$ and $w : \mathscr{C}_\infty \to \mathbb{R}$,*

$$\int_{\mathscr{C}_*(\square)} |\nabla [w]_{\mathcal{P}}|^s (x) \, dx \le C^s \int_{\mathscr{C}_*(\square)} \operatorname{size}(\square_{\mathcal{P}}(x))^{sd-1} \, |\nabla w|^s (x) \, dx. \tag{3.2.13}$$

More generally, for any family of disjoint cubes $\{\square_i\}_{i\in I} \in (\mathcal{P}_*)^I$, we have

$$\int_{\mathscr{C}_*(\cup_{i\in I}\square_i)} |\nabla [w]_{\mathcal{P}}|^s (x) \, dx \le C^s \int_{\mathscr{C}_*(\cup_{i\in I}\square_i)} \operatorname{size}(\square_{\mathcal{P}}(x))^{sd-1} \, |\nabla w|^s (x) \, dx. \tag{3.2.14}$$

3.2.3. Solving the Poisson equation with divergence form source term. In this section we study the existence and uniqueness of the equation $-\nabla \cdot \mathbf{a} \nabla u = -\nabla \cdot \xi$ on the infinite cluster $\mathscr{C}_\infty$. We denote by $\sum_{e \subseteq \mathscr{C}_\infty}$ the sum over all the edges of the infinite cluster of nonzero conductances.

The results of this section can be summarized in the two following propositions.

PROPOSITION 3.2.11 (Gradient of Green's function). *For $a \in \Omega$, let $e = (x, y)$ be an edge of $E_d^{\mathbf{a}}$. There exist a constant $C := C(d, \lambda) < \infty$ and a function $\nabla G(e, \cdot) : \mathscr{C}_\infty \to \mathbb{R}$, whose gradient with respect to the second variable, denoted by $\nabla\nabla G$ satisfies,*

$$\langle \nabla\nabla G(e, \cdot), \nabla\nabla G(e, \cdot) \rangle_{\mathscr{C}_\infty} \le C, \tag{3.2.15}$$

and is a solution to the equation

$$-\nabla \cdot \mathbf{a} \nabla \left(\nabla G(e, \cdot) \right) = \delta_x - \delta_y \ \text{in}\ \mathscr{C}_\infty,$$

Moreover, we have, for each $e, e' \in E_d^{\mathbf{a}}$,

$$\nabla\nabla G(e, e') = \nabla\nabla G(e', e). \tag{3.2.16}$$

We then deduce how to solve the general equation $-\nabla \cdot \mathbf{a} \nabla w_\xi = -\nabla \cdot \xi$ from the previous proposition.

PROPOSITION 3.2.12. *Let $\xi : E_d \to \mathbb{R}$ be a vector field satisfying*

$$\xi(x,y) = 0 \text{ if } \mathbf{a}(x,y) = 0 \text{ or } x,y \notin \mathscr{C}_\infty. \tag{3.2.17}$$

If ξ satisfies $\langle \xi, \xi \rangle_{\mathscr{C}_\infty} < \infty$ then there exists a unique (a.s in the environment and up to a constant) solution w_ξ of

$$-\nabla \cdot \mathbf{a} \nabla w_\xi = -\nabla \cdot \xi \text{ in } \mathscr{C}_\infty.$$

Moreover, we have the following representation

$$\nabla w_\xi(\cdot) = \sum_{e \subseteq \mathscr{C}_\infty} \xi(e) \nabla \nabla G(e, \cdot). \tag{3.2.18}$$

PROOF OF PROPOSITION 3.2.11 AND 3.2.12. Let ξ be a vector field satisfying (3.2.17) and $\langle \xi, \xi \rangle_{\mathscr{C}_\infty} < \infty$. We denote by $\dot{H}^1$ the space of functions defined on the infinite cluster whose gradient is L^2, i.e. $\dot{H}^1 := \left\{ u : \mathscr{C}_\infty \to \mathbb{R} \ : \ \langle \nabla u, \nabla u \rangle_{\mathscr{C}_\infty} < \infty \right\}$, and look at the minimization problem

$$\inf_{u \in \dot{H}^1} \frac{1}{2} \langle \nabla u, \mathbf{a} \nabla u \rangle_{\mathscr{C}_\infty} - \langle \xi, \nabla u \rangle_{\mathscr{C}_\infty}.$$

By the standard techniques of the calculus of variations, there exists a unique solution (up to a constant) to this problem denoted by w_ξ. In particular, when ξ is the indicator of an edge e, we obtain the function $\nabla G(e, \cdot)$. To prove (3.2.16), we note that

$$\nabla \nabla G(e', e) = \left\langle \nabla \nabla G(e, \cdot), \mathbf{a} \nabla \nabla G(e', e) \right\rangle_{\mathscr{C}_\infty} = \nabla \nabla G(e, e').$$

The representation formula (3.2.18) follows from standard arguments. □

3.2.4. Regularity theory. In this subsection, we record a result from the regularity theory established in [**13**] giving a Lipschitz bound for the gradient of **a**-harmonic functions. This result is only a small part of the regularity theory established in [**13**, Theorem 2], but is the only result needed in the proofs of Theorems 3.1.1 and 3.1.2.

PROPOSITION 3.2.13 (Regularity theory). *There exist a constant $C < \infty$, an exponent $s > 0$ and a random variable $\mathcal{X}$ satisfying*

$$\mathcal{X} \le \mathcal{O}_s(C), \tag{3.2.19}$$

such that for each $u : \mathscr{C}_\infty \mapsto \mathbb{R}$ solution of the equation

$$-\nabla \cdot \mathbf{a} \nabla u = 0 \tag{3.2.20}$$

and each $R \ge r \ge \mathcal{X}$, we have

$$\|\nabla u\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \le C \frac{r}{R} \left\| u - (u)_{\mathscr{C}_\infty \cap B_R} \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R)}.$$

We also introduce the notation, for each $x \in \mathbb{Z}^d$

$$\mathcal{X}(x) := \mathcal{X} \circ \tau_x.$$

This proposition is much weaker than Theorem 2 of [**13**], it is indeed a consequence of the Caccioppoli inequality and Theorem 2 (iii) of [**13**] for $k = 0$. As a consequence, we obtain the following Lipschitz bound on the corrector.

PROPOSITION 3.2.14 (Lipschitz bound on the corrector). *There exists a constant $C < \infty$ and an exponent $s > 0$ such that, for each edge $e = (x,y) \in E_d$ and each $p \in \mathbb{R}^d$,*

$$|\nabla \chi_p(e)| \mathbb{1}_{\{e \in \mathscr{C}_\infty\}} \le C |p| \mathcal{X}^{d/2}(x). \tag{3.2.21}$$

which implies by (3.2.19),

$$|\nabla \chi_p(e)| \mathbb{1}_{\{e \in \mathscr{C}_\infty\}} \le \mathcal{O}_s(C|p|), \tag{3.2.22}$$

for some smaller exponent s (c.f. Subsection 3.1.2.5). Moreover the same estimate holds for the coarsened corrector

$$\left| \nabla \left[\chi_p \right]_{\mathcal{P}} (e) \right| \le \mathcal{O}_s(C|p|). \tag{3.2.23}$$

REMARK 3.2.15. The same estimate than (3.2.21) would hold with $\mathcal{X}^{d/2}(y)$ instead of $\mathcal{X}^{d/2}(x)$ in the right-hand side.

PROOF. By the stationarity of the law of the corrector, we can assume that the edge e touches 0, i.e. that $x = 0$. First note that, for each $r \geq 1$

$$|\nabla \chi_p(e)| \mathbb{1}_{\{e \in \mathscr{C}_\infty\}} \leq r^d \|\nabla \chi_p\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)}.$$

By applying Proposition 3.2.13 with $r = \mathcal{X}$, and taking the limit $R \to \infty$, we obtain

$$\|p + \nabla \chi_p\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_{\mathcal{X}})} \leq C \mathcal{X}^{d/2} \liminf_{R \to \infty} \frac{1}{R} \|l_p + \chi_p\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_R(x'))} \leq C \mathcal{X}^{d/2} |p|.$$

A combination of the two previous displays with the integrability estimate (3.2.19) yields (3.2.22). To prove (3.2.23), we combine (3.2.22) with Proposition 3.2.10 and use the integrability estimate $\mathrm{size}(\square_{\mathcal{P}}(x)) \leq \mathcal{O}_s(C)$, for each $x \in \mathbb{Z}^d$. This is performed in the following computation: for each edge $e = (x, y) \in \mathbb{R}^d$, we have

$$\begin{aligned} \left|\nabla [\chi_p]_{\mathcal{P}}(e)\right| &\leq \int_{\mathscr{C}_*(\square_{\mathcal{P}}(x) \cup \square_{\mathcal{P}}(y))} \left|\nabla [\chi_p]_{\mathcal{P}}\right|(x')\, dx' \\ &\leq C \int_{\mathscr{C}_\infty \cap B(x, C\,\mathrm{size}(\square_{\mathcal{P}}(x)))} \mathrm{size}\left(\square_{\mathcal{P}}(x')\right)^{d-1} |\nabla \chi_p|(x')\, dx' \\ &\leq C \sum_{x' \in \mathbb{Z}^d} \mathbb{1}_{\{x' \in \mathscr{C}_\infty \cap B(x, C\,\mathrm{size}(\square_{\mathcal{P}}(x)))\}} \mathrm{size}\left(\square_{\mathcal{P}}(x')\right)^{d-1} |\nabla \chi_p|(x'). \end{aligned} \tag{3.2.24}$$

Moreover for each $x \in \mathbb{Z}^d$, $\mathrm{size}(\square_{\mathcal{P}}(x)) \leq \mathcal{O}_s(C)$. As a consequence

$$\mathbb{1}_{\{x' \in B(x, C\,\mathrm{size}(\square_{\mathcal{P}}(x)))\}} \leq C \frac{\mathrm{size}(\square_{\mathcal{P}}(x))^{d+1}}{|x - x'|^{d+1} \vee 1} \leq \frac{\mathcal{O}_s(C)}{|x - x'|^{d+1} \vee 1}, \tag{3.2.25}$$

where we used the notation $a \vee b = \max(a, b)$. Using the summability of the map $x \to |x|^{-d-1}$, the properties (3.1.10) and (3.1.12) on the $\mathcal{O}_s$ notation and the Lipschitz bounds (3.2.22) on the corrector, we obtain the result. □

We now present the two main tools to prove Theorem 3.1.2. The first one is a concentration inequality, thanks to which we can obtain some quantitative control on the spatial averages of the gradient at scale R (see Proposition 3.3.3). We then deduce Theorem 3.1.2 from Proposition 3.3.3 thanks to the multiscale Poincaré inequality.

3.2.5. Concentration inequality for stretched exponential moments. The following concentration inequality is the key point in the proof of Proposition 3.3.3 in the next section. A proof of this inequality can be found in [**19**, Proposition 2.2].

PROPOSITION 3.2.16 (Proposition 2.2 of [**19**]). *Fix $\beta \in (0, 2)$. Let X be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and set for each $e \in \mathcal{B}_d(\mathbb{Z}^d)$,*

$$X'_e = \mathbb{E}[X | \mathcal{F}(\mathcal{B}_d \setminus \{e\})] \quad \text{and} \quad \mathbb{V}[X] = \sum_{e \in \mathcal{B}_d} \left(X - X'_e\right)^2,$$

then there exists $C := C(d, \beta) < \infty$ such that

$$\mathbb{E}\left[\exp\left(|X - \mathbb{E}[X]|^\beta\right)\right] \leq C \mathbb{E}\left[\exp\left((C \mathbb{V}[X])^{\frac{\beta}{2-\beta}}\right)\right]^{\frac{2-\beta}{\beta}}.$$

3.2.6. Multiscale Poincaré inequality. The next proposition is a version of the multiscale Poincaré inequality. It controls the oscillations of a function in the L^q norm (left-hand side of (3.2.26)) by the spatial average of the gradient of the function (right-hand side of (3.2.26)).

PROPOSITION 3.2.17 (Multiscale Poincaré inequality, heat kernel version). *For each $r > 0$, we define*

$$\Phi_r := \left\{ \begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R} \\ x & \mapsto & r^{-d} \exp\left(-\frac{|x|^2}{r^2}\right). \end{array} \right.$$

For each $q \geq 1$, there exists a constant $C := C(d,q) < \infty$ such that for each tempered distribution $u \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ and each $R > 0$,

$$\|u - (u)_{B_R}\|_{\underline{L}^q(B_R)} \leq C \left(\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r \left|\Phi_r * \nabla u(x)\right|^2 dr \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}. \tag{3.2.26}$$

Moreover the dependence on the variable q of the constant C can be estimated as follows, for each $q \geq 2$

$$C(d,q) \leq A q^{\frac{3}{2}}$$

for some constant $A := A(d) < \infty$.

The proof of this proposition heavily relies on [**18**, Proposition D.1 and Remark D.6] and is presented in Appendix A.

3.3. Estimates of the spatial averages of the first-order correctors

We now have all the necessary tools to prove the optimal L^q bounds of the corrector, stated in Theorem 3.1.2. The idea is to first prove Proposition 3.3.3 thanks to the concentration inequality, Proposition 3.2.16. We then deduce the bound on the coarsened corrector thanks to the multiscale Poincaré inequality, Proposition 3.2.17 and remove the coarsening thanks to Proposition 3.2.8. This eventually yields Theorem 3.1.2.

DEFINITION 3.3.1. Fix a function $\eta \in C_c^\infty\left(B_{\frac{1}{2}}\right)$ satisfying

$$\forall x \in \mathbb{R}^d, \eta(x) \geq 0 \text{ and } \int_{\mathbb{R}^d} \eta(x)\, dx = 1.$$

Given a function $w : \mathscr{C}_\infty \mapsto \mathbb{R}$, we consider the function $[w]_{\mathcal{P}}$ defined on the entire lattice $\mathbb{Z}^d$. We then extend this function to a function piecewise constant on $\mathbb{R}^d$ by setting, for each $x \in \mathbb{Z}^d$ and each $y \in x + B_{\frac{1}{2}}$, $[w]_{\mathcal{P}}(y) = [w]_{\mathcal{P}}(x)$. We then smoothen this function by taking the convolution against η and define

$$[w]_{\mathcal{P}}^\eta : \left\{ \begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R} \\ x & \mapsto & \left([w]_{\mathcal{P}} * \eta\right)(x). \end{array} \right.$$

This creates a smooth function defined on $\mathbb{R}^d$. This property will be convenient when we apply the multiscale Poincaré inequality, to obtain Theorem 3.1.2. This is the only reason we need to go from a discrete function defined on $\mathbb{Z}^d$ to a continuous function defined on $\mathbb{R}^d$. Additionally, this function satisfies a number of convenient properties, recorded in the following proposition.

PROPOSITION 3.3.2. *Given a function $w : \mathscr{C}_\infty \mapsto \mathbb{R}$, the function $[w]_{\mathcal{P}}^\eta$ defined in Definition 3.3.1 satisfies*

(i) *For each $x \in \mathbb{Z}^d$,*

$$[w]_{\mathcal{P}}^\eta(x) = [w]_{\mathcal{P}}(x).$$

(ii) *For each $x \in \mathbb{Z}^d$ and each $y \in \mathbb{Z}^d + B_{\frac{1}{2}}$, we have,*

$$\left|\nabla [w]_{\mathcal{P}}^\eta(y)\right| \leq C \left|\nabla [w]_{\mathcal{P}}\right|(x),$$

for some constant C depending only on d and η.

We now state the main technical proposition of this article.

PROPOSITION 3.3.3. *For each $R \geq 1$, and each $x \in \mathbb{R}^d$, the quantity $\nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}}^{\eta}\right)(x)$ is well-defined and there exist an exponent $s > 0$ and a constant $C < \infty$ such that it satisfies*

$$\left|\nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}}^{\eta}\right)(x)\right| \leq \mathcal{O}_s\left(C|p|R^{-\frac{d}{2}}\right), \tag{3.3.1}$$

where we used the notation introduced earlier for the gaussian function

$$\Phi_R := \left\{ \begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R} \\ x & \mapsto & R^{-d}\exp\left(-\frac{|x|^2}{R^2}\right). \end{array} \right.$$

We now turn to the proof of (3.3.1). By stationarity of the gradient of the corrector, it is enough to prove the result when $x = 0$. By linearity of the mapping $p \to \chi_p$, we may and do assume $|p| = 1$. We denote by $X = \nabla \left(\Phi_R * [\chi_p]_{\mathcal{P}}^{\eta}\right)(0)$. We are going to prove

$$|X| \leq \mathcal{O}_s\left(CR^{-\frac{d}{2}}\right),$$

The main idea of the proof is to apply Proposition 3.2.16 to X. To do so, we need to prove the two following lemmas. The first one focuses on the expectation of X.

LEMMA 3.3.4. *There exists a constant $C < \infty$ such that*

$$|\mathbb{E}[X]| \leq CR^{-\frac{d}{2}}.$$

The second one estimates the quantity $\mathbb{V}[X]$.

LEMMA 3.3.5. *There exists a constant $C < \infty$ and an exponent $s > 0$, such that*

$$\mathbb{V}[X] \leq \mathcal{O}_s\left(CR^{-d}\right).$$

These lemmas are proved in the following two subsections.

3.3.1. Estimating the expectation of the spatial averages. The main objective of this step is to show Lemma 3.3.4.

Proof of Lemma 3.3.4. The idea of the proof is to use the stationarity and the sublinearity of the corrector to prove that the expectation of its gradient is 0. The technical difficulty which arises is that the partition $\mathcal{P}$ is not stationary and consequently we lose the stationarity of the random variable $\nabla [\chi_p]_{\mathcal{P}}^{\eta}(0)$. To fix this issue we introduce a stationary partition $\mathcal{P}_{\mathrm{stat}}$ which is stationary and equal to $\mathcal{P}$ on a set of large probability. We finally show that the error we make by considering $\mathcal{P}_{\mathrm{stat}}$ instead of $\mathcal{P}$ is small.

For each $x, y, z \in \mathbb{Z}^d$ with $x \sim y$, denote by $\tau_z \mathbf{a}$ the translated environment defined by

$$\tau_z \mathbf{a}(\{x, y\}) = \mathbf{a}(\{x - z, y - z\}).$$

For $k \in \mathbb{N}$, we construct the partition $\mathcal{P}_{\mathrm{stat}}^k$ by applying Proposition 3.2.1 to the collection of triadic cubes

$$\mathcal{G}_{\mathrm{stat}}^k := \mathcal{G} \bigcup \left(\bigcup_{n=k}^{\infty} \mathcal{T}_n\right).$$

Note that this collection is not a set of good cubes in the sense of Definition 3.2.3 but it is $3^k\mathbb{Z}^d$-translation invariant. A straightforward consequence is that $\mathcal{P}_{\mathrm{stat}}^k$ is $3^k\mathbb{Z}^d$-stationary: for every environment $\mathbf{a}$, every $x \in \mathbb{Z}^d$, $z \in 3^k\mathbb{Z}^d$,

$$\mathrm{size}\left(\square_{\mathcal{P}_{\mathrm{stat}}^k}(x + z)\right)(\tau_z \mathbf{a}) = \mathrm{size}\left(\square_{\mathcal{P}_{\mathrm{stat}}^k}(x)\right)(\mathbf{a}). \tag{3.3.2}$$

With a proof similar to the proof of [**13**, Proposition 2.1 (iv)], we derive

$$\mathbb{P}\left[\exists x \in \square_k, \, \square_{\mathcal{P}}(x) \neq \square_{\mathcal{P}_{\mathrm{stat}}^k}(x)\right] \leq C\exp\left(-C^{-1}3^k\right). \tag{3.3.3}$$

For a function $u : \mathscr{C}_\infty \to \mathbb{R}$, we define the coarsened function with respect to the partition $\mathcal{P}^k_{\mathrm{stat}}$ by the formula

$$[u]_{\mathcal{P}^k_{\mathrm{stat}}} := u\left(\overline{z}_{\mathrm{stat}}\left(\square_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right)\right)$$

with the notation, for $\square \in \mathcal{T}$,

$$\overline{z}_{\mathrm{stat}}(\square) := \begin{cases} \overline{z}(\square) & \text{if } \overline{z}(\square) \in \mathscr{C}_\infty \text{ and } \square \text{ is a good cube,} \\ \operatorname*{argmin}_{z \in \mathscr{C}_\infty} \operatorname{dist}(z, \square) & \text{otherwise.} \end{cases} \tag{3.3.4}$$

If there is more than one choice in the argument of the minima, we select the one which is minimal for the lexicographical order. In particular, by the statinarity of the gradient of the corrector and (3.3.2), we have

$$\nabla [\chi_p]^\eta_{\mathcal{P}^k_{\mathrm{stat}}} \quad \text{is } 3^k\mathbb{Z}^d - \text{stationnary}, \tag{3.3.5}$$

where $[\chi_p]^\eta_{\mathcal{P}^k_{\mathrm{stat}}}$ is defined from $[\chi_p]_{\mathcal{P}^k_{\mathrm{stat}}}$ by a convolution with η, as in Definition 3.3.1. We fix $k \in \mathbb{Z}^d$ such that $3^k \le R^{\frac{1}{2}} \le 3^{k+1}$ and split the proof of Lemma 3.3.4 into three steps.

(i) In Step 1, we prove

$$\mathbb{E}\left[\left|\nabla\left(\Phi_R * [\chi_p]^\eta_{\mathcal{P}}\right)(0) - \nabla\left(\Phi_R * [\chi_p]^\eta_{\mathcal{P}^k_{\mathrm{stat}}}\right)(0)\right|\right] \le CR^{-\frac{d}{2}}.$$

(ii) In Step 2, we prove

$$\mathbb{E}\left[\int_{\left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \nabla [\chi_p]^\eta_{\mathcal{P}^k_{\mathrm{stat}}}(x)\,dx\right] = 0.$$

Note that we wrote $\left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d$ and not $\square_k$ because we are referring to the continuous cube and not the discrete one as it was defined in (3.1.8).

(iii) In Step 3, we use the result obtained in Step 2 to show

$$\left|\mathbb{E}\left[\nabla\left(\Phi_R * [\chi_p]^\eta_{\mathcal{P}^k_{\mathrm{stat}}}\right)(0)\right]\right| \le CR^{-\frac{d}{2}}.$$

Lemma 3.3.4 is then a consequence of the main results of Steps 1 and 3.

Step 1. The main result of this step is a consequence of the following computation, by (3.1.5), Proposition 3.3.2 and Proposition 3.2.10,

$$\begin{aligned} &\mathbb{E}\left[\left|\nabla\left(\Phi_R * [\chi_p]^\eta_{\mathcal{P}}\right)(0) - \nabla\left(\Phi_R * [\chi_p]^\eta_{\mathcal{P}^k_{\mathrm{stat}}}\right)(0)\right|\right] \\ &\quad \le \mathbb{E}\left[\left|\int_{B_{R^2}} \left(\nabla [\chi_p]^\eta_{\mathcal{P}}(x) - \nabla [\chi_p]^\eta_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right)\Phi_R(x)\,dx\right| \mathbb{1}_{\left\{\exists x \in B_{R^2} : \square_{\mathcal{P}^k_{\mathrm{stat}}}(x) \neq \square_{\mathcal{P}}(x)\right\}}\right] \\ &\quad\quad + \mathbb{E}\left[\left|\int_{\mathbb{R}^d \setminus B_{R^2}} \left(\nabla [\chi_p]^\eta_{\mathcal{P}}(x) - \nabla [\chi_p]^\eta_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right)\Phi_R(x)\,dx\right|\right]. \end{aligned} \tag{3.3.6}$$

The first term on the right-hand side can be estimated crudely the following way. We denote by U_0 the set

$$U_0 := \bigcup_{x \in B_{R^2}} \square_{\mathcal{P}}(x),$$

we then enlarge this set by adding two additional layers of cubes and define

$$U_1 := \bigcup_{\square \in \mathcal{P}, \operatorname{dist}(\square, U_0) \le 1} \square \quad \text{and} \quad U := \bigcup_{\square \in \mathcal{P}, \operatorname{dist}(\square, U_1) \le 1} \square.$$

Note that, by the properties of the partition $\mathcal{P}$, and (3.1.10),

$$|U| = C|U_1| \le C|U_0| \le C \sum_{x \in B_{R^2}} \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)^d \le \mathcal{O}_s\left(CR^{2d}\right). \tag{3.3.7}$$

Also with these definitions, the definition of $\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}$, which essentially amounts to (3.3.4), and (3.2.8), we have, for each $x \in B_{R^2}$,

$$\left|\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right| \leq \int_{\mathscr{C}_\infty \cap U} \left|\nabla \chi_p\right|(y)\,dy.$$

Similarly, by definition of $\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}}$ and the properties of the partition $\mathcal{P}$, we have, for each $x \in B_{R^2}$,

$$\left|\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}}(x)\right| \leq \int_{\mathscr{C}_\infty \cap U} \left|\nabla \chi_p\right|(y)\,dy.$$

This leads to the estimate

$$\begin{aligned}(3.3.8)\qquad &\left|\int_{B_{R^2}} \left(\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}}(x) - \nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right)\Phi_R(x)\,dx\right| \\ &\quad \leq C\left(\int_{\mathscr{C}_\infty \cap U} \left|\nabla \chi_p\right|(y)\,dy\right)\int_{B_{R^2}} \Phi_R(x)\,dx \\ &\quad \leq C\int_{\mathscr{C}_\infty \cap U} \left|\nabla \chi_p\right|(y)\,dy.\end{aligned}$$

Using Proposition 3.2.22, the estimate on the volume of U given in (3.3.7) and a computation similar to the one performed in (3.2.24), we obtain

$$\int_{\mathscr{C}_\infty \cap U} \left|\nabla \chi_p\right|(y)\,dy \leq \mathcal{O}_s\left(CR^{2d}\right).$$

Then by (3.3.3), we also have

$$\begin{aligned}\mathbb{P}\left[\exists x \in B_{R^2} \,:\, \square_{\mathcal{P}^k_{\mathrm{stat}}}(x) \neq \square_{\mathcal{P}}(x)\right] &\leq \sum_{z \in 3^k\mathbb{Z}^d \cap B_{R^2}} \mathbb{P}\left[\exists x \in z + \square_k \,:\, \square_{\mathcal{P}^k_{\mathrm{stat}}}(x) \neq \square_{\mathcal{P}}(x)\right] \\ &\leq \frac{R^{2d}}{3^{dk}}\mathbb{P}\left[\exists x \in \square_k \,:\, \square_{\mathcal{P}^k_{\mathrm{stat}}}(x) \neq \square_{\mathcal{P}}(x)\right] \\ &\leq \frac{CR^{2d}}{3^{dk}}\exp\left(-C^{-1}3^k\right).\end{aligned}$$

In particular, since k has been chosen such that $3^k \leq R^{\frac{1}{2}} < 3^{k+1}$, for each $q > 0$, there exist a constant $C := C(d, \mathfrak{p}, \lambda, q) < \infty$ and an exponent $s := s(d, \mathfrak{p}, \lambda, q) > 0$ such that

$$\mathbb{1}_{\left\{\exists x \in B_{R^2}(x') : \square_{\mathcal{P}^k_{\mathrm{stat}}}(x) \neq \square_{\mathcal{P}}(x)\right\}} \leq \mathcal{O}_s(CR^{-q}).$$

Combining the three previous displays with q chosen large enough, the Cauchy-Schwarz inequality and (3.1.12), we obtain

$$\left|\int_{B_{R^2}} \left(\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}}(x) - \nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right)\Phi_R(x)\,dx\right| \mathbb{1}_{\left\{\exists x \in B_{R^2} : \square_{\mathcal{P}^k_{\mathrm{stat}}}(x) \neq \square_{\mathcal{P}}(x)\right\}} \leq \mathcal{O}_s\left(CR^{-\frac{d}{2}}\right),$$

which yields in particular

$$\mathbb{E}\left[\left|\int_{B_{R^2}} \left(\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}}(x) - \nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right)\Phi_R(x)\,dx\right| \mathbb{1}_{\left\{\exists x \in B_{R^2} : \square_{\mathcal{P}^k_{\mathrm{stat}}}(x) \neq \square_{\mathcal{P}}(x)\right\}}\right] \leq CR^{-\frac{d}{2}}.$$

We now focus on estimating the second term on the right-hand side of (3.3.6). With the same computation as the one we just wrote, we obtain,

$$\int_{B_{R^2}} \left|\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}}(x) - \nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right| dx \leq \mathcal{O}_s\left(CR^{4d}\right),$$

indeed the proof is the same, we only need to replace $\int_{B_{R^2}} \Phi_R(x)\,dx$ by CR^{2d} in (3.3.8). Since this result is true for any $R \geq 1$, we obtain, for any $n \in \mathbb{N}$,

$$\int_{\mathscr{C}_\infty \cap (\square_{n+1} \setminus \square_n)} \left| \nabla \left[\chi_p\right]_{\mathcal{P}}^{\eta}(x) - \nabla \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x) \right| dx \leq \int_{\mathscr{C}_\infty \cap B_{3^n}} \left| \nabla \left[\chi_p\right]_{\mathcal{P}}^{\eta}(x) - \nabla \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x) \right| dx \leq \mathcal{O}_s\left(C 3^{4dn}\right).$$

This allows the computation

$$\begin{aligned}
&\mathbb{E}\left[\left| \int_{\mathbb{R}^d \setminus B_{R^2}} \left(\nabla \left[\chi_p\right]_{\mathcal{P}}^{\eta}(x) - \nabla \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x) \right) \Phi_R(x)\,dx \right|\right] \\
&\quad \leq \sum_{n=\lfloor 2\log_3(R) \rfloor}^{+\infty} \mathbb{E}\left[\exp\left(-\frac{3^{2n}}{R^2}\right) R^{-d} \int_{\mathscr{C}_\infty \cap (\square_{n+1} \setminus \square_n)} \left| \nabla \left[\chi_p\right]_{\mathcal{P}}^{\eta}(x) - \nabla \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x) \right| dx \right] \\
&\quad \leq \sum_{n=2\log_3(R)}^{+\infty} \exp\left(-\frac{3^{2n}}{R^2}\right) R^{-d} 3^{4dn} \\
&\quad \leq C \exp\left(-C^{-1} R^2\right).
\end{aligned}$$

Combining the estimates of the first and the second terms of the right-hand side completes the proof of Step 1.

Remark 3.3.6. Most of the estimates of this proof are very crude and precise estimates are not needed. Indeed the same proof shows the following stronger result: for each $q > 0$, there exists $C := C(d, \mathfrak{p}, \lambda, q) < \infty$ such that for each $R \geq 1$ and $k \in \mathbb{N}$ such that $3^k \leq R^{\frac{1}{2}} < 3^{k+1}$,

$$\mathbb{E}\left[\left| \nabla\left(\Phi_R * \left[\chi_p\right]_{\mathcal{P}}^{\eta}\right) - \nabla\left(\Phi_R * \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}\right) \right|\right] \leq C R^{-q},$$

but the proof of Lemma 3.3.4 only requires the result with $q = \frac{d}{2}$.

Step 2. We prove the main result of this step by combining the stationarity property (3.3.5) with the sublinear growth of the corrector. First notice that by (3.2.1), we have, for each $R > 1$,

$$\operatorname*{osc}_{\mathscr{C}_\infty \cap B_R} \chi_p \leq \mathcal{O}_s\left(C R^{1-\delta}\right).$$

By the Stokes formula and the definition of $\left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}$ we have, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\left| \int_{\left(-\frac{3^{nk}}{2}, \frac{3^{nk}}{2}\right)^d} \nabla \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x)\,dx \right| &= \left| \int_{\partial\left(\left(-\frac{3^{nk}}{2}, \frac{3^{nk}}{2}\right)^d\right)} \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x) \mathbf{n}(x)\,dx \right| \\
&\leq 3^{kn(d-1)} \operatorname*{osc}_{\mathscr{C}_\infty \cap B_{3^{kn}}} \chi_p \\
&\leq \mathcal{O}_s\left(C 3^{kn(d-\delta)}\right).
\end{aligned}$$

This yields in particular

$$\left| \mathbb{E}\left[\int_{\left(-\frac{3^{nk}}{2}, \frac{3^{nk}}{2}\right)^d} \nabla \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x)\,dx \right] \right| \leq C 3^{kn(d-\delta)}$$

Or we also have, by the stationarity property (3.3.5).

$$\begin{aligned}
\mathbb{E}\left[\int_{\left(-\frac{3^{nk}}{2}, \frac{3^{nk}}{2}\right)^d} \nabla \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x)\,dx \right] &= \sum_{z \in (3^k \mathbb{Z}^d \cap \square_{kn})} \mathbb{E}\left[\int_{z + \left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \nabla \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x)\,dx \right] \\
&= \frac{3^{dkn}}{3^{dk}} \mathbb{E}\left[\int_{\left(-\frac{3^k}{2}, \frac{3^k}{2}\right)^d} \nabla \left[\chi_p\right]_{\mathcal{P}^k_{\mathrm{stat}}}^{\eta}(x)\,dx \right].
\end{aligned}$$

Combining the two previous results shows

$$\left|\mathbb{E}\left[\int_{\left(-\frac{3^k}{2},\frac{3^k}{2}\right)^d} \nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\,dx\right]\right| \leq C 3^{dk} 3^{-kn\delta}.$$

Sending $n \to \infty$ shows

$$\left|\mathbb{E}\left[\int_{\left(-\frac{3^k}{2},\frac{3^k}{2}\right)^d} \nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\,dx\right]\right| = 0.$$

Step 3. First notice that

$$\mathbb{E}\left[\left(\Phi_R * \nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}\right)(0)\right] = \left(\Phi_R * \mathbb{E}\left[\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}\right]\right)(0).$$

By (3.3.5), the function

$$f := \begin{cases} \mathbb{R}^d & \to & \mathbb{R}^d \\ x & \mapsto & \mathbb{E}\left[\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right] \end{cases}$$

is $3^k\mathbb{Z}^d$-periodic, consequently there exists $(a_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^d} \in \left(\mathbb{C}^d\right)^{\mathbb{Z}^d}$ such that

$$f(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}} \exp\left(\frac{2i\pi\mathbf{n}\cdot x}{3^k}\right).$$

By an explicit computation, we obtain

$$\left(\Phi_R * \mathbb{E}\left[\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}\right]\right)(0) = \sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}} \pi^{\frac{d}{2}} \exp\left(-\frac{|\pi R\mathbf{n}|^2}{3^{2k}}\right).$$

Notice that the main result of Step 2 is equivalent to the following equality

$$a_{(0,\cdots,0)} = 0.$$

Using this relation and the Cauchy-Schwarz inequality, we obtain

$$\left|\left(\Phi_R * \mathbb{E}\left[\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}\right]\right)(0)\right|^2 \leq C\left(\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus(0,\cdots,0)} |a_{\mathbf{n}}|^2\right)\left(\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus(0,\cdots,0)} \exp\left(-2\frac{|\pi R\mathbf{n}|^2}{3^{2k}}\right)\right). \tag{3.3.9}$$

In particular, since k was chosen such that $3^k \leq R^{\frac{1}{2}} < 3^{k+1}$, we have

$$\sum_{\mathbf{n}\in\mathbb{Z}^d\setminus(0,\cdots,0)} \exp\left(-2\frac{|\pi R\mathbf{n}|^2}{3^{2k}}\right) \leq C\exp\left(-C^{-1}R\right). \tag{3.3.10}$$

Moreover, we have

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} |a_{\mathbf{n}}|^2 \leq \mathbb{E}\left[\int_{\left(-\frac{3^k}{2},\frac{3^k}{2}\right)^d} \left|\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right|^2 dx\right].$$

With the same computation as the one performed in Step 1, we obtain

$$\int_{\left(-\frac{3^k}{2},\frac{3^k}{2}\right)^d} \left|\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}(x)\right|^2 dx \leq \mathcal{O}_s\left(C|p|^2 3^{4kd}\right).$$

Taking the expectation yields

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} |a_{\mathbf{n}}|^2 \leq C 3^{4kd}.$$

Combining this with (3.3.9) and (3.3.10), we obtain

$$\left|\left(\Phi_R * \mathbb{E}\left[\nabla\left[\chi_p\right]^{\eta}_{\mathcal{P}^k_{\mathrm{stat}}}\right]\right)(x')\right|^2 \leq CR^{2d}\exp\left(-C^{-1}R\right) \leq C\exp\left(-C^{-1}R\right),$$

where we increased the value of the constant C in the second inequality to absorb the term R^{2d}. This implies in particular the main result of Step 3 and completes the proof of Lemma 3.3.4.

3.3.2. Estimating the resampling of the spatial averages. In this section, we prove Lemma 3.3.5, which we recall below.

LEMMA 3.3.5. *There exists a constant $C < \infty$ and an exponent $s > 0$, such that*

$$\mathbb{V}[X] \leq \mathcal{O}_s\left(CR^{-d}\right).$$

Proof of Lemma 3.3.5. We recall Proposition 3.2.16 and the notation $X = \nabla\left(\Phi_R * [\chi_p]_{\mathcal{P}}^{\eta}\right)(0)$. Given an environment $\mathbf{a} \in \Omega$ and an edge $e = (x,y) \in E_d$, we want to estimate $\left(X - X_e'\right)^2$. To do so, we need to understand how changing the value of the edge e can impact the infinite cluster $\mathscr{C}_\infty$ and the partition $\mathcal{P}$. This is studied in the following lemma.

LEMMA 3.3.7. *There exist two constants $C_0 := C_0(d) < \infty$ and $C := C(d) < \infty$ such that for each edge $e = (x,y) \in E_d$, environments $\mathbf{a}, \widetilde{\mathbf{a}} \in \Omega$ satisfying $\mathbf{a}(e') = \widetilde{\mathbf{a}}(e')$ for each edge $e' \in E_d \setminus \{e\}$ and for every $z \in \mathbb{Z}^d \setminus B\left(x, C_0 \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)\right)$,*

$$\operatorname{size}\left(\square_{\mathcal{P}(\widetilde{\mathbf{a}})}(z)\right) \leq C \operatorname{size}\left(\square_{\mathcal{P}(\mathbf{a})}(x)\right).$$

Moreover, if $z \in \mathbb{Z}^d \setminus B\left(x, C_0 \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)\right)$ then

$$\operatorname{size}\left(\square_{\mathcal{P}(\widetilde{\mathbf{a}})}(z)\right) = \operatorname{size}\left(\square_{\mathcal{P}(\mathbf{a})}(z)\right).$$

PROOF OF LEMMA 3.3.7. The main ingredients of the proof are the following:

1. If a good cube $\square \in \mathcal{P}_*$ is such that $3\square \cap \{x,y\} = \emptyset$ then the cube $\square$ is a good cube under the environment $\widetilde{\mathbf{a}}$.
2. By the properties of the partition $\mathcal{P}$, every cube $\square \in \mathcal{P}$ which does not contain x nor y is crossable under the environment $\widetilde{\mathbf{a}}$. The predecessors of $\square_{\mathcal{P}}(x)$ and $\square_{\mathcal{P}}(y)$ are also crossable under the environment $\widetilde{\mathbf{a}}$.
3. Notice that by resampling one edge we cannot create an isolated cluster which is not connected to $\mathscr{C}_\infty$ of size larger than $C\operatorname{size}(\square_{\mathcal{P}}(x))$, for some $C_0 := C_0(d) < \infty$. In particular, there exists a constant $C := C(d) < \infty$ such that every good cube of size larger than $C\operatorname{size}\left(\square_{\mathcal{P}}(x)\right)$ under the environment $\mathbf{a}$ satisfies (ii) of Definition 3.2.3 under the environment $\widetilde{\mathbf{a}}$.
4. There exists a constant $C := C(d) < \infty$ such that every cube of size larger than $C\operatorname{size}\left(\square_{\mathcal{P}}(x)\right)$ intersecting $\square_{\mathcal{P}}(x)$ is crossable by a cluster which does not intersect $\square_{\mathcal{P}}(x)$.
5. If, for $y \in B\left(x, C_0 \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)\right)$, the cube $\square_{\mathcal{P}}(y)$ is larger than the cube $C\operatorname{size}\left(\square_{\mathcal{P}}(x)\right)$, then the point x belongs to the cube $\square_{\mathcal{P}}(y)$ or one of its neighbors and thus $\operatorname{size}\left(\square_{\mathcal{P}}(y)\right) \leq C\operatorname{size}\left(\square_{\mathcal{P}}(x)\right)$.

Combining these properties shows that every good cube $\square$ under the environment $\mathbf{a}$ satisfying $\operatorname{size}(\square) \geq C\operatorname{size}\left(\square_{\mathcal{P}}(x)\right)$ is a good cube under the environment $\widetilde{\mathbf{a}}$. It is then straightforward to see from the previous remarks and the construction of the partition $\mathcal{P}$ in the proof of Proposition 3.2.1 that the conclusion of the lemma is valid. □

To estimate $\left(X - X_e'\right)^2$, we introduce an extended probability space by doubling the variables $(\Omega', \mathcal{F}', \mathbb{P}') = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \mathbb{P} \otimes \mathbb{P})$. For a given environment $(\mathbf{a}(e), \widetilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d} \in \Omega'$, we denote by pr_1 (resp. pr_2) the first (resp. second) projection, i.e., $\mathrm{pr}_1\left((\mathbf{a}(e), \widetilde{\mathbf{a}}(e))\right) = (\mathbf{a}(e))_{e \in \mathcal{B}_d}$ (resp. $\mathrm{pr}_2((\mathbf{a}(e), \widetilde{\mathbf{a}}(e))) = (\widetilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d}$). Any random variable Z defined on $(\Omega, \mathcal{F}, \mathbb{P})$ can be seen as a random variable defined on the extended space $(\Omega', \mathcal{F}', \mathbb{P}')$ by the formula $Z = Z \circ \mathrm{pr}_1$, i.e.

$$Z\left((\mathbf{a}(e), \widetilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d}\right) = Z\left((\mathbf{a}(e))_{e \in \mathcal{B}_d}\right).$$

All the random variables in this proof must be considered as random variable on $(\Omega', \mathcal{F}', \mathbb{P}')$, unless explicitly stated.

We will denote $\mathbb{E}'$ the expectation of a random variable $Z : \Omega' \to \mathbb{R}$ with respect to the measure $\mathbb{P}'$. For a constant $C \in (0,\infty)$ and an exponent $s > 0$, we denote

$$Z \leq \mathcal{O}_s'(C) \text{ if and only if } \mathbb{E}'\left[\exp\left(\left(\frac{Z}{C}\right)^s\right)\right] \leq 2.$$

In particular any random variable Z defined on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying $Z \leq \mathcal{O}_s(C)$ satisfies, as a random variable defined on $(\Omega', \mathcal{F}', \mathbb{P}')$, $Z \leq \mathcal{O}'_s(C)$.

Given an enlarged environment given the enlarged environment $(\mathbf{a}(e), \widetilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d}$, we denote by $\mathbf{a}$ the environment $(\mathbf{a}(e))_{e \in \mathcal{B}_d}$ and by $\mathbf{a}^{e'}$ the environment $\left((\mathbf{a}(e))_{e \in \mathcal{B}_d \smallsetminus \{e'\}}, \widetilde{\mathbf{a}}(e')\right)$. Similarly, given Z a random variable defined on Ω and $e' \in \mathcal{B}_d$ an edge, we denote by Z^e the random variable defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ by the formula, for each $(\mathbf{a}(e), \widetilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d} \in \Omega'$

$$Z^{e'}\left((\mathbf{a}(e), \widetilde{\mathbf{a}}(e))_{e \in \mathcal{B}_d}\right) := Z\left(\mathbf{a}^{e'}\right).$$

We give a similar definition for partitions, $\mathcal{P}^{e'}$ will denote the random partition of $\mathbb{Z}^d$ under the environment $\mathbf{a}^{e'}$, and for the infinite cluster ($\mathbb{P}'$ almost surely there exists a unique infinite cluster under the environment $\mathbf{a}^{e'}$ which will be denoted by $\mathscr{C}_\infty^{e'}$).

To prove Lemma 3.3.5, we will prove the following estimate

$$\sum_{e \in \mathcal{B}_d} \left| \nabla \left(\Phi_R * \left([\chi_p]_{\mathcal{P}}^{\eta} - \left[\chi_p^e\right]_{\mathcal{P}^e}^{\eta} \right) \right)(0) \right|^2 \leq \mathcal{O}'_s\left(\frac{C}{R^d}\right). \tag{3.3.11}$$

This is enough to prove Result 2 as we now prove. With the same argument as in [**18**, Lemma 2.3], we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\left(\frac{\sum_{e \in \mathcal{B}_d} (X - X'_e)^2}{CR^{-d}}\right)^s\right)\right] \\
&= \int_\Omega \exp\left(\left(\frac{\sum_{e \in \mathcal{B}_d} \left|\int_\Omega \nabla \left(\Phi_R * \left([\chi_p]_{\mathcal{P}}^{\eta} - \left[\chi_p^e\right]_{\mathcal{P}^e}^{\eta}\right)\right)(x')\, d\mathbb{P}(\widetilde{\mathbf{a}})\right|^2}{CR^{-d}}\right)^s\right) d\mathbb{P}(\mathbf{a}) \\
&\leq \int_\Omega \exp\left(\left(\int_\Omega \frac{\sum_{e \in \mathcal{B}_d} \left|\nabla \left(\Phi_R * \left([\chi_p]_{\mathcal{P}}^{\eta} - \left[\chi_p^e\right]_{\mathcal{P}^e}^{\eta}\right)\right)(x')\right|^2}{CR^{-d}}\, d\mathbb{P}(\widetilde{\mathbf{a}})\right)^s\right) d\mathbb{P}(\mathbf{a}) \\
&\leq C \int_\Omega \int_\Omega \exp\left(\left(\frac{\sum_{e \in \mathcal{B}_d} \left|\nabla \left(\Phi_R * \left([\chi_p]_{\mathcal{P}}^{\eta} - \left[\chi_p^e\right]_{\mathcal{P}^e}^{\eta}\right)\right)(x')\right|^2}{CR^{-d}}\right)^s\right) d\mathbb{P}(\mathbf{a})\, d\mathbb{P}(\widetilde{\mathbf{a}}) \\
&\leq C\mathbb{E}'\left[\exp\left(\left(\frac{\sum_{e \in \mathcal{B}_d} \left|\nabla \left(\Phi_R * \left([\chi_p]_{\mathcal{P}}^{\eta} - \left[\chi_p^e\right]_{\mathcal{P}^e}^{\eta}\right)\right)(x')\right|^2}{CR^{-d}}\right)^s\right)\right] \\
&\leq 2C.
\end{aligned}$$

This yields, after redefinition of the constant C,

$$\sum_{e \in \mathcal{B}_d} \left(X - X'_e\right)^2 \leq \mathcal{O}_s\left(\frac{C}{R^d}\right).$$

Before starting the proof of (3.3.11), we select one of the correctors χ_p^e arbitrarily (we recall that they are a priori defined up to a constant) and wish to give a meaning to the function, for $e = \{x, y\} \in \mathcal{B}_d$,

$$\left[\chi_p^e\right]_{\mathcal{P}}^{\eta}$$

as a random variable defined on the extended probability space Ω'.

Since we do not necessarily have $\mathscr{C}_\infty = \mathscr{C}_\infty^e$, we cannot simply write $\left[\chi_p^e\right]_{\mathcal{P}}(z) = \chi_p^e\left(z\left(\square_{\mathcal{P}}(z)\right)\right)$. Nevertheless, since the two environments $\left((\mathbf{a}(e'))_{e' \in \mathcal{B}_d \smallsetminus \{e\}}, \widetilde{\mathbf{a}}(e)\right)$ and $(\mathbf{a}(e'))_{e' \in \mathcal{B}_d}$ are only different by one edge, we have either $\mathscr{C}_\infty \subseteq \mathscr{C}_\infty^e$ or $\mathscr{C}_\infty^e \subseteq \mathscr{C}_\infty$. In the former case, we can define $\left[\chi_p^e\right]_{\mathcal{P}}(z) = \chi_p^e\left(z\left(\square_{\mathcal{P}}(z)\right)\right)$. In the latter case, $\mathscr{C}_\infty \smallsetminus \mathscr{C}_\infty^e$ is connected to $\mathscr{C}_\infty$ by the edge e. Without loss of generality, we denote by $e = (x, y)$ and assume that $x \in \mathscr{C}_\infty^e$. One can then check that the function

$$w := \begin{cases} \mathscr{C}_\infty & \to & \mathbb{R} \\ z & \mapsto & \chi_p^e(z) \mathbb{1}_{\{z \in \mathscr{C}_\infty^e\}} + \left(p \cdot (x - z) + \chi_p^e(z)\right) \mathbb{1}_{\{z \notin \mathscr{C}_\infty^e\}} \end{cases} \tag{3.3.12}$$

is a solution of

$$-\nabla \cdot (\mathbf{a} \nabla (p \cdot x + w)) = 0$$

and more precisely that $x \to p \cdot x + w(x) \in \mathcal{A}_1(\mathscr{C}_\infty)$. In particular, this gives

$$w = \chi_p.$$

This leads to the definition,

$$\left[\chi_p^e\right]_{\mathcal{P}} = [w]_{\mathcal{P}}.$$

We then extend $\left[\chi_p^e\right]_{\mathcal{P}}$ to a piecewise constant function on $\mathbb{R}^d$ and perform the convolution with η, as in Definition 3.3.1, to obtain a smooth function $\left[\chi_p^e\right]_{\mathcal{P}}^\eta$. It satisfies the equality

$$\nabla \left[\chi_p^e\right]_{\mathcal{P}}^\eta = \nabla \left[\chi_p\right]_{\mathcal{P}}^\eta. \tag{3.3.13}$$

To prove the estimate (3.3.11), we split the sum into two terms

$$\begin{aligned} &\left|\nabla \left(\Phi_R * \left([\chi_p]_{\mathcal{P}}^\eta - \left[\chi_p^e\right]_{\mathcal{P}^e}^\eta\right)\right)(0)\right|^2 \\ &\qquad \le 2\left|\nabla \left(\Phi_R * \left(\left[\chi_p^e\right]_{\mathcal{P}}^\eta - \left[\chi_p^e\right]_{\mathcal{P}^e}^\eta\right)(0)\right)\right|^2 + 2\left|\nabla \left(\Phi_R * \left([\chi_p]_{\mathcal{P}}^\eta - \left[\chi_p^e\right]_{\mathcal{P}}^\eta\right)\right)(0)\right|^2. \end{aligned} \tag{3.3.14}$$

Step 1. We estimate the first term on the right-hand side of (3.3.14) the following way. Using Proposition 3.3.2, Lemma 3.3.7 and Proposition 3.2.8 with $s = 1$, we have

$$\begin{aligned} &\left|\left(\Phi_R * \left(\nabla \left[\chi_p^e\right]_{\mathcal{P}}^\eta - \nabla \left[\chi_p^e\right]_{\mathcal{P}^e}^\eta\right)\right)(0)\right|^2 \\ &\quad \le \left(\int_{\mathbb{Z}^d \cap B(x, C \operatorname{size}(\square_{\mathcal{P}}(x)))} \left|\nabla \left[\chi_p^e\right]_{\mathcal{P}}(z)\right| + \left|\nabla \left[\chi_p^e\right]_{\mathcal{P}^e}(z)\right| dz\right)^2 \sup_{z \in B(x, C \operatorname{size}(\square_{\mathcal{P}}(x)))} \Phi_R^2(z) \\ &\quad \le C\left(\int_{\mathscr{C}_\infty^e \cap B(x, C \operatorname{size}(\square_{\mathcal{P}}(x)))} \operatorname{size}(\square_{\mathcal{P}}(x))^{d-1} \left(\left|\nabla \chi_p^e\right|(z) + 1\right) dy\right)^2 \sup_{z \in B(x, C \operatorname{size}(\square_{\mathcal{P}}(x)))} \Phi_R^2(z). \end{aligned}$$

The "+1" term on the right-hand side comes from the fact that we assumed $|p| = 1$ combined with the definition of w in (3.3.12), in the case when $\mathscr{C}_\infty^e \subseteq \mathscr{C}_\infty$. This gives

$$\begin{aligned} &\left|\left(\Phi_R * \left(\nabla \left[\chi_p^e\right]_{\mathcal{P}}^\eta - \nabla \left[\chi_p^e\right]_{\mathcal{P}^e}^\eta\right)\right)(0)\right|^2 \\ &\quad \le C \operatorname{size}(\square_{\mathcal{P}}(x))^{3d-2} \int_{\mathscr{C}_\infty^e \cap B(x, C \operatorname{size}(\square_{\mathcal{P}}(x)))} \left(\left|\nabla \chi_p^e\right|^2(z) + 1\right) dy \sup_{z \in B(x, C \operatorname{size}(\square_{\mathcal{P}}(x)))} \Phi_R^2(z). \end{aligned}$$

Moreover, there exists a constant $C(d) < \infty$ such that, for each $z \in \mathbb{Z}^d$,

$$\exp(-|z|^2) \le C \frac{1}{|z|^{\frac{d+1}{2}}} \wedge 1. \tag{3.3.15}$$

We denote by ζ the function on the right-hand side, i.e., $\zeta(z) := C \frac{1}{|z|^{\frac{d+1}{2}}} \wedge 1$. We similarly denote $\zeta_R(z) = \frac{1}{R^d} \zeta\left(\frac{z}{R}\right)$. We will use this function instead of Φ_R to complete the estimate of the term on the right-hand side because, since it is decreasing slower than Φ_R, it satisfies the following properties

$$\sup_{z \in B(x, C \operatorname{size}(\square_{\mathcal{P}}(x)))} \zeta_R^2(z) \le C \operatorname{size}(\square_{\mathcal{P}}(x))^{d+1} \inf_{z \in B(x, C \operatorname{size}(\square_{\mathcal{P}}(x)))} \zeta_R^2(z) \qquad \text{and} \qquad \sum_{z \in \mathbb{Z}^d} \zeta(z)^2 < \infty.$$

In particular, the previous estimate can be rewritten

$$\left|\left(\Phi_R * \left(\nabla\left[\chi_p^e\right]^\eta_{\mathcal{P}} - \nabla\left[\chi_p^e\right]^\eta_{\mathcal{P}^e}\right)\right)(0)\right|^2 \leq C\,\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{4d-1}\int_{\mathscr{C}^e_\infty\cap B(x,C\,\mathrm{size}(\square_{\mathcal{P}}(x)))}\zeta_R(z)^2\left(\left|\nabla\chi_p^e\right|^2(z)+1\right)dz.$$

Summing over all the edges $e\in\mathcal{B}_d$ gives

$$\begin{aligned}(3.3.16)\quad \sum_{e\in\mathcal{B}_d}&\left|\left(\Phi_R * \left(\nabla\left[\chi_p^e\right]^\eta_{\mathcal{P}} - \nabla\left[\chi_p^e\right]^\eta_{\mathcal{P}^e}\right)\right)(0)\right|^2\\ &\leq C\sum_{x\in\mathbb{Z}^d}\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{4d-1}\int_{\mathscr{C}^e_\infty\cap B(x,C\,\mathrm{size}(\square_{\mathcal{P}}(x)))}\zeta_R(z)^2\left(\left|\nabla\chi_p^e\right|^2(z)+1\right)dz\\ &\leq C\sum_{z\in\mathbb{Z}^d}\zeta_R(z)^2\left(\sum_{x\in\mathbb{Z}^d}\left(\left|\nabla\chi_p^e\right|^2(z)+1\right)\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{4d-1}\mathbb{1}_{\{z\in\mathscr{C}^e_\infty\cap B(x,C\,\mathrm{size}(\square_{\mathcal{P}}(x)))\}}\right).\end{aligned}$$

But, since for each $x\in\mathbb{Z}^d$, $\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)\leq\mathcal{O}'_s(C)$, we have

$$\mathbb{1}_{\{z\in B(x,C\,\mathrm{size}(\square_{\mathcal{P}}(x)))\}}\leq C\frac{\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{d+1}}{|x-z|^{d+1}}\leq\frac{\mathcal{O}'_s(C)}{|x-z|^{d+1}}.$$

Additionally, by Proposition 3.2.22, we have the Lipschitz bound on the corrector, for each $z\in\mathbb{Z}^d$,

$$(3.3.17)\qquad \left|\nabla\chi_p^e(z)\right|\mathbb{1}_{\{z\in\mathscr{C}^e_\infty\}}\leq\mathcal{O}'_s(C).$$

Since $\sum_{x\in\mathbb{Z}^d\setminus\{z\}}\frac{1}{|x-z|^{d+1}}<\infty$, we can use (3.1.10) to obtain

$$(3.3.18)\qquad \sum_{x\in\mathbb{Z}^d}\left(\left|\nabla\chi_p^e\right|^2(z)+1\right)\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{4d-1}\mathbb{1}_{\{z\in\mathscr{C}^e_\infty\cap B(x,C\,\mathrm{size}(\square_{\mathcal{P}}(x)))\}}\leq\mathcal{O}'_s(C).$$

which in turn gives

$$\sum_{e\in\mathcal{B}_d}\left|\left(\Phi_R * \left(\nabla\left[\chi_p^e\right]^\eta_{\mathcal{P}} - \nabla\left[\chi_p^e\right]^\eta_{\mathcal{P}^e}\right)\right)(0)\right|^2\leq C\sum_{y\in\mathbb{Z}^d}\zeta_R(y)^2\mathcal{O}'_s(C).$$

Since $\sum_{y\in\mathbb{Z}^d}\zeta_R(y)^2\leq\frac{C}{R^d}$, we can use the estimate (3.1.10) to obtain

$$\sum_{e\in\mathcal{B}_d}\left|\left(\Phi_R * \left(\nabla\left[\chi_p^e\right]^\eta_{\mathcal{P}} - \nabla\left[\chi_p^e\right]^\eta_{\mathcal{P}^e}\right)\right)(0)\right|^2\leq\mathcal{O}'_s\left(\frac{C}{R^d}\right).$$

This completes the proof of the estimate of the first term on the right-hand side of (3.3.14).

Step 2. We now estimate the second term on the right-hand side of (3.3.14)

$$\left|\left(\Phi_R * \left(\nabla\left[\chi_p-\chi_p^e\right]^\eta_{\mathcal{P}}\right)\right)(0)\right|^2\leq\mathcal{O}'_s\left(\frac{C}{R^d}\right).$$

To prove this estimate, we need to distinguish three cases, we recall that we denoted $e=(x,y)$.

Case 1. ($x\notin\mathscr{C}_\infty$ and $y\notin\mathscr{C}_\infty$) or $\mathbf{a}=\mathbf{a}^e$. In that case, $\mathscr{C}_\infty=\mathscr{C}^e_\infty$, and the two correctors χ_p and χ_p^e are equal up to a constant. In particular, this yields

$$\left|\left(\Phi_R * \left(\nabla\left[\chi_p-\chi_p^e\right]^\eta_{\mathcal{P}}\right)\right)(0)\right|^2=0.$$

Case 2. $\mathscr{C}_\infty\neq\mathscr{C}^e_\infty$. In that case, (3.3.13) is true. It implies

$$\left|\left(\Phi_R * \left(\nabla\left[\chi_p-\chi_p^e\right]^\eta_{\mathcal{P}}\right)\right)(0)\right|^2=0.$$

Case 3. $x,y\in\mathscr{C}_\infty$ and $\mathscr{C}_\infty=\mathscr{C}^e_\infty$ and $\mathbf{a}\neq\mathbf{a}^e$. We compute

$$-\nabla\cdot\left(\mathbf{a}\nabla\left(\chi_p-\chi_p^e\right)\right)=\nabla\cdot\left((\mathbf{a}-\mathbf{a}^e)\nabla\left(p\cdot z+\chi_p^e\right)\right),$$

which can be rewritten

$$(3.3.19)\qquad -\nabla\cdot\left(\mathbf{a}\nabla\left(\chi_p-\chi_p^e\right)\right)=\left(\mathbf{a}-\mathbf{a}^e\right)(x,y)\left(p\cdot(x-y)+\chi_p^e(x)-\chi_p^e(y)\right)\left(\delta_x-\delta_y\right).$$

Recall the notation $\nabla G(e,\cdot)$ introduced in Proposition 3.2.11. If the edge $e = (x,y)$ does not belong to the infinite cluster, i.e., if $\mathbf{a}(e) = 0$, then denote by $e_1, \ldots, e_n$ a path of edges of the infinite cluster connecting x to y and denote by

$$\nabla G(e,\cdot) := \sum_{i=1}^{n} \nabla G(e_i,\cdot). \tag{3.3.20}$$

This function is the unique solution (up to a constant) of the equation

$$-\nabla \cdot \mathbf{a}\nabla\left(\nabla G(e,\cdot)\right) = \delta_x - \delta_y.$$

We can solve (3.3.19) by using Proposition 3.2.11. Indeed the function

$$\chi_p - \chi_p^e - \left(\mathbf{a} - \mathbf{a}^e\right)(x,y)\left(p\cdot(x-y) + \chi_p^e(x) - \chi_p^e(y)\right)\nabla G(e,\cdot) \text{ is } \mathbf{a}-\text{harmonic}.$$

Moreover, by the sublinear growth of the corrector (3.2.2), the L^2-bound on the gradient of the function $\nabla G(e,\cdot)$ stated in (3.2.15) and a version of the Poincaré inequality on the percolation cluster (see for instance the proof of Proposition 3.2.13), one can show that the function

$$\chi_p - \chi_p^e - \left(\mathbf{a} - \mathbf{a}^e\right)(x,y)\left(p\cdot(x-y) + \chi_p^e(x) - \chi_p^e(y)\right)\nabla G(e,\cdot) \text{ has a sublinear growth.}$$

This implies, by [**13**, Theorem 2] that this function is constant. In particular, this shows

$$\nabla\chi_p - \nabla\chi_p^e = \left(\mathbf{a} - \mathbf{a}^e\right)(x,y)\left(p\cdot(x-y) + \chi_p^e(x) - \chi_p^e(y)\right)\nabla\left(\nabla G(e,\cdot)\right).$$

But, if $\mathbf{a}^e(e) = \widetilde{\mathbf{a}}(e) \neq 0$, we have the estimate, by (3.3.17),

$$\left|\chi_p^e(x) - \chi_p^e(y)\right| \mathbb{1}_{\{y\in\mathscr{C}_\infty^e, \widetilde{\mathbf{a}}(e)\neq 0\}} \leq \left|\nabla\chi_p^e(y)\right| \mathbb{1}_{\{y\in\mathscr{C}_\infty^e\}} \leq C\left(\mathcal{X}^e(x)\right)^{d/2}.$$

If $\mathbf{a}^e(e) = \widetilde{\mathbf{a}}(e) = 0$, then there exists a path going from x to y which lies in the cube $\square_{\mathcal{P}^e}(x)$ and its neighbors (its neighbors because we may not have $\square_{\mathcal{P}^e}(x) = \square_{\mathcal{P}^e}(y)$ or we may have $x,y\in\mathscr{C}_\infty \setminus \mathscr{C}_*\left(\square_{\mathcal{P}^e}(x)\right)$). Combining this remark with Lemma 3.3.7, we obtain

$$\begin{aligned}\left|\chi_p^e(x) - \chi_p^e(y)\right| &\leq C\int_{\mathscr{C}_\infty^e\cap B(x, C\,\mathrm{size}(\square_{\mathcal{P}}(x)))} \left|\nabla\chi_p^e\right|(z)\,dz \\ &\leq C\,\mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^d \left\|\nabla\chi_p^e\right\|_{\underline{L}^2(\mathscr{C}_\infty^e\cap B(x,C\,\mathrm{size}(\square_{\mathcal{P}}(x))))}.\end{aligned}$$

Using the Lipschitz bounds on the corrector again, we have

$$\left|\chi_p^e(x) - \chi_p^e(y)\right| \mathbb{1}_{\{x,y\in\mathscr{C}_\infty^e, \widetilde{\mathbf{a}}(e)=0\}} \leq \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^d \left(\mathcal{X}^e(x)\right)^{d/2}.$$

Thus

$$\left|\nabla\left(\Phi_R * \left(\left[\chi_p - \chi_p^e\right]_{\mathcal{P}}^{\eta}\right)\right)(0)\right|^2 \leq \left|\nabla\left(\Phi_R * \left(\left[\nabla G(e,\cdot)\right]_{\mathcal{P}}^{\eta}\right)\right)(0)\right|^2 \mathrm{size}\left(\square_{\mathcal{P}}(x)\right)^{2d}\left(\mathcal{X}^e(x)\right)^d. \tag{3.3.21}$$

The next step of the proof consists in removing the coarsening in the right-hand side of (3.3.21). To this end, we prove that there exist a constant $C := C(d) < \infty$ and a (random) vector field $\gamma_R : E_d \to \mathbb{R}$ satisfying, for each $e' = (x', y') \in E_d$,

$$\left|\gamma_R(e')\right| \leq C\,\mathrm{size}(\square_{\mathcal{P}}(x'))^{2d}\zeta_R\left(x'\right)$$

such that for each function $u : \mathscr{C}_\infty \to \mathbb{R}$ satisfying $\langle\nabla u, \nabla u\rangle_{\mathscr{C}_\infty} < \infty$,

$$\left(\Phi_R * \nabla\left[u\right]_{\mathcal{P}}^{\eta}\right)(0) = \langle\gamma_R, \nabla u\rangle_{\mathscr{C}_\infty}. \tag{3.3.22}$$

We first compute

$$\begin{aligned}\left(\Phi_R * \nabla\left[u\right]_{\mathcal{P}}^{\eta}\right)(0) &= \int_{\mathbb{R}^d} \Phi_R(z)\,\nabla\left[u\right]_{\mathcal{P}}^{\eta}(z)\,dz \\ &= \int_{\mathbb{R}^d} \Phi_R(z)\int_{B_{\frac12}} \left[u\right]_{\mathcal{P}}(z-s)\nabla\eta(s)\,ds\,dz \\ &= \int_{\mathbb{R}^d} \Phi_R(z)\int_{B_{\frac12}} \left(\left[u\right]_{\mathcal{P}}(z-s) - \left[u\right]_{\mathcal{P}}(z)\right)\nabla\eta(s)\,ds\,dz.\end{aligned}$$

But $[u]_{\mathcal{P}}(z)$ and $[u]_{\mathcal{P}}(z-s)$ are only different if the points z and $(z-s)$ belong to two different cubes of the partition $\mathcal{P}$, in that case, we have

$$[u]_{\mathcal{P}}(z-s) - [u]_{\mathcal{P}}(z) = u(\overline{z}(\square_{\mathcal{P}}(z-s))) - u(\overline{z}(\square_{\mathcal{P}}(z))).$$

Recall that there exists a path between $\overline{z}(\square_{\mathcal{P}}(z))$ and $\overline{z}(\square_{\mathcal{P}}(z-s))$ which lies entirely in $\square_{\mathcal{P}}(z) \cup \square_{\mathcal{P}}(z-s)$, which will be denoted $p_{z,z-s} \subseteq E_d$. Summing over the edges along this path, we find that

$$u(\overline{z}(\square_{\mathcal{P}}(z))) - u(\overline{z}(\square_{\mathcal{P}}(z-s))) = \sum_{e' \in p_{z,z-s}} \nabla u(e') = \sum_{e' \in E_d} \nabla u(e') \mathbb{1}_{\{e' \in p_{z,z-s}\}}.$$

If z and $z-s$ belongs to the same cube of the partition $\mathcal{P}$, we keep the same notation with the convention $p_{z,z-s} = \emptyset$. Consequently, we have for each $(z,s) \in \mathbb{R}^d \times B_{1/2}$,

$$[u]_{\mathcal{P}}(z-s) - [u]_{\mathcal{P}}(z) = \sum_{e' \in E_d} \nabla u(e') \mathbb{1}_{\{e' \in p_{z,z-s}\}}.$$

With this formula, we can rewrite

$$\begin{aligned}
\left(\Phi_R * \nabla [u]^{\eta}_{\mathcal{P}}\right)(0) &= \int_{\mathbb{R}^d} \int_{B_{1/2}} \Phi_R(z) \left([u]_{\mathcal{P}}(z-s) - [u]_{\mathcal{P}}(z)\right) \nabla \eta(s) \, ds \, dz \\
&= \sum_{e' \in E_d} \nabla u(e') \int_{\mathbb{R}^d} \int_{B_{1/2}} \Phi_R(z) \mathbb{1}_{\{e' \in p_{z,z-s}\}} \nabla \eta(s) \, ds \, dz \\
&= \langle \gamma_R, \nabla u \rangle_{\mathscr{C}_\infty}
\end{aligned}$$

with for each $e' \in E_d$

$$\gamma_R(e') = \int_{\mathbb{R}^d} \int_{B_{1/2}} \Phi_R(z) \mathbb{1}_{\{e' \in p_{z,z-s}\}} \nabla \eta(s) \, ds \, dz.$$

But, for each pair of points $(z,s) \in \mathbb{R}^d \times B_{1/2}$ such that the cube $\square_{\mathcal{P}}(z-s)$ is not equal to the cube $\square_{\mathcal{P}}(z)$, the path connecting the points $\overline{z}(\square_{\mathcal{P}}(z-s))$ and $\overline{z}(\square_{\mathcal{P}}(z))$ lies entirely in the set $\square_{\mathcal{P}}(z-s) \cup \square_{\mathcal{P}}(z)$. In particular an edge $e' = (x', y')$ belongs to the set $p_{z,z-s}$ only if the point z belongs to the set $\partial \square_{\mathcal{P}}(x') + B_{1/2}$. This shows

$$\gamma_R(e') = \int_{\partial \square_{\mathcal{P}}(x') + B_{1/2}} \int_{B_{1/2}} \Phi_R(t) \mathbb{1}_{\{e' \in p_{z,z-s}\}} \nabla \eta(s) \, ds \, dz$$

and thus, we have the estimate,

$$\begin{aligned}
\left|\gamma_R(e')\right| &\leq C \int_{\partial \square_{\mathcal{P}}(x') + B_{1/2}} \int_{B_{1/2}} \Phi_R(z) \, ds \, dz \\
&\leq \int_{\partial \square_{\mathcal{P}}(x') + B_{1/2}} \Phi_R(z) \, dz.
\end{aligned}$$

As in (3.3.15), we appeal to the inequality, for each $x \in \mathbb{R}^d$, $\exp(-|x|^2) \leq \zeta(x)$. The function ζ_R satisfies the inequality, for each triadic cube $\square \in \mathcal{T}$,

$$\sup_{\square + B_{1/2}} \zeta_R \leq C \operatorname{size}(\square)^{\frac{d+1}{2}} \inf_{\square} \zeta_R.$$

As a consequence of the two previous displays, we can rewrite the previous estimate

$$\begin{aligned}
\left|\gamma_R(e')\right| &\leq \int_{\partial \square_{\mathcal{P}}(x') + B_{1/2}} \Phi_R(z) \, dz \\
&\leq C \operatorname{size}(\square_{\mathcal{P}}(x'))^{d-1} \sup_{\partial \square_{\mathcal{P}}(x') + B_{1/2}} \zeta_R \\
&\leq C \operatorname{size}(\square_{\mathcal{P}}(x'))^{2d} \zeta_R(x'),
\end{aligned} \tag{3.3.23}$$

which is the desired estimate. This completes the proof of (3.3.22).

Applying this property with $u = \nabla G(e,\cdot)$, the inequality (3.3.21) becomes

$$\left|\left(\Phi_R * \left(\nabla\left[\chi_p - \chi_p^e\right]_{\mathcal{P}}^\eta\right)\right)(0)\right|^2 \le \left|\langle \gamma_R, \nabla\nabla G(e,\cdot)\rangle_{\mathscr{C}_\infty}\right|^2 \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)^{2d}\left(\mathcal{X}^e(x)\right)^d.$$

Applying Proposition 3.2.12, we denote by $w_{\gamma_R} : \mathscr{C}_\infty \to \mathbb{R}$ the solution of the elliptic equation

$$-\nabla\cdot\left(\mathbf{a}\nabla w_{\gamma_R}\right) = -\nabla\cdot\gamma_R \text{ in } \mathscr{C}_\infty,$$

so that, for each edge e' in the infinite cluster $\mathscr{C}_\infty$,

$$\nabla w_{\gamma_R}(e') = \sum_{e\subseteq\mathscr{C}_\infty}\gamma_R(e)\nabla\nabla G\left(e,e'\right) = \sum_{e\subseteq\mathscr{C}_\infty}\gamma_R(e)\nabla\nabla G(e',e) = \left\langle\gamma_R, \nabla\nabla G(e',\cdot)\right\rangle_{\mathscr{C}_\infty}.$$

This implies in particular, in both cases $\mathbf{a}(e) = 0$ and $\mathbf{a}(e) \neq 0$,

$$w_{\gamma_R}(x) - w_{\gamma_R}(y) = \langle\gamma_R, \nabla\nabla G(e,\cdot)\rangle_{\mathscr{C}_\infty}.$$

This gives consequently

$$\left|\left(\Phi_R * \left(\nabla\left[\chi_p - \chi_p^e\right]_{\mathcal{P}}^\eta\right)\right)(0)\right|^2 \le \left|w_{\gamma_R}(x) - w_{\gamma_R}(y)\right|^2 \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)^{2d}\left(\mathcal{X}^e(x)\right)^d.$$

We now combine Cases 1, 2 and 3 to obtain the following estimate, using the new notation, for each $x\in\mathbb{Z}^d$, $\mathcal{B}_d^x := \left\{\{x,y\} \,:\, y\in\mathbb{Z}^d, y\sim x\right\}$ the set of bonds connecting x to another vertex of $\mathbb{Z}^d$. One has

$$\begin{aligned}\sum_{e\in\mathcal{B}_d}&\left|\left(\Phi_R * \left(\nabla\left[\chi_p - \chi_p^e\right]_{\mathcal{P}}^\eta\right)\right)(0)\right|^2 \\ &\le C\sum_{x,y\in\mathscr{C}_\infty,|x-y|_1=1}\left|w_{\gamma_R}(x) - w_{\gamma_R}(y)\right|^2 \operatorname{size}\left(\square_{\mathcal{P}}(x)\right)^{2d}\sum_{e\in\mathcal{B}_d^x}\left(\mathcal{X}^e(x)\right)^d.\end{aligned}$$

Using that for each $x,y\in\mathscr{C}_\infty$ with $|x-y|_1 = 1$, there exists a path connecting x to y which is contained in $\mathscr{C}_\infty$, the cube $\square_{\mathcal{P}}(x)$ and its neighbors (the path is simply (x,y) if $\mathbf{a}(\{x,y\})\neq 0$), we obtain

$$\begin{aligned}\sum_{e\in\mathcal{B}_d}&\left|\left(\Phi_R * \left(\nabla\left[\chi_p - \chi_p^e\right]_{\mathcal{P}}^\eta\right)\right)(0)\right|^2 \\ &\le C\int_{\mathscr{C}_\infty}\left|\nabla w_{\gamma_R}\right|^2(z)\operatorname{size}\left(\square_{\mathcal{P}}(z)\right)^{3d} \\ &\qquad\times\left(\sum_{x\in\mathbb{Z}^d,\,\operatorname{dist}(\square_{\mathcal{P}}(x),\square_{\mathcal{P}}(z))\le 1, e\in\mathcal{B}_d^x}\left(\mathcal{X}^e(x)\right)^d\right)dz.\end{aligned} \tag{3.3.24}$$

To estimate the term on the right-hand side, we note, by definition of w_{γ_R} and (3.1.10),

$$\begin{aligned}\int_{\mathscr{C}_\infty}\left|\nabla w_{\gamma_R}\right|^2(z)\,dz &\le C\int_{\mathscr{C}_\infty}\gamma_R^2(z)\,dz \\ &\le C\int_{\mathbb{Z}^d}C\operatorname{size}(\square_{\mathcal{P}}(x))^{4d}\zeta_R(x)\,dx \\ &\le \mathcal{O}_s'\left(\frac{C}{R^d}\right).\end{aligned}$$

There remains to estimate the term $\operatorname{size}\left(\square_{\mathcal{P}}(z)\right)^{3d}\sum_{x\in\mathbb{Z}^d,\,\operatorname{dist}(\square_{\mathcal{P}}(x),\square_{\mathcal{P}}(z))\le 1, e\in\mathcal{B}_d^x}\left(\mathcal{X}^e(x)\right)^d$ on the right-hand side of (3.3.24). To this end, we need to prove a minimal scale statement and a Meyers estimate as stated below.

Lemma 3.3.8 (Minimal scale). *There exist a constant $C := C(d,\mathfrak{p},\lambda) < \infty$, an exponent $s := s(d,\mathfrak{p},\lambda) > 0$ and a random variable $\mathcal{M}_1 \le \mathcal{O}_s'(C)$ such that for each integer $m\in\mathbb{N}$ satisfying the inequality $3^m \ge \mathcal{M}_1$,*

$$3^{-dm} \sum_{z \in \square_m} \operatorname{size}(\square_{\mathcal{P}}(z))^{\frac{3d(2+\varepsilon)}{\varepsilon}} \left(\sum_{x \in \mathbb{Z}^d,\, \operatorname{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1,\, e \in \mathcal{B}_d^x} (\mathcal{X}^e(x))^d \right)^{\frac{2+\varepsilon}{\varepsilon}} \leq C \tag{3.3.25}$$

where $\varepsilon := \varepsilon(d, \mathfrak{p}, \lambda) > 0$ *is the exponent which appears in Proposition 3.C.2.*

Definition 3.3.9 (The partition $\mathcal{U}$). We define the following family of good cubes

$$\mathcal{G} := \{\square \in \mathcal{T} : (3.3.4) \text{ and } (3.3.25) \text{ hold}\}$$

in which a deterministic Meyers estimate and a minimal scale inequality hold. By Lemma 3.3.8 and Proposition 3.C.2, this family satisfies the assumptions of Proposition 3.2.1 (but not the assumption (3.2.5)). We denote by $\mathcal{U}$ the partition thus obtained. By (iii) of Proposition 3.2.1, one has the inequality

$$\operatorname{size}(\square_{\mathcal{U}}(x)) \leq \mathcal{O}_s(C),$$

for some exponent $s := s(d, \mathfrak{p}, \lambda) > 0$ and some constant $C := C(d, \mathfrak{p}, \lambda) < \infty$.

We postpone the proof of Lemma 3.3.8 and complete the proof of Lemma 3.3.5. Using the partition $\mathcal{U}$, we have

$$\begin{aligned}
&\int_{\mathscr{C}_\infty} |\nabla w_{\gamma_R}|^2(z) \operatorname{size}(\square_{\mathcal{P}}(z))^{3d} \left(\sum_{x \in \mathbb{Z}^d,\, \operatorname{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1,\, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right) dz \\
&\quad = \sum_{\square \in \mathcal{U}} \int_{\square \cap \mathscr{C}_\infty} |\nabla w_{\gamma_R}|^2(z) \\
&\qquad \times \operatorname{size}(\square_{\mathcal{P}}(z))^{3d} \left(\sum_{x \in \mathbb{Z}^d,\, \operatorname{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1,\, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right) dz \\
&\quad \leq \sum_{\square \in \mathcal{U}} |\square| \left(\frac{1}{|\square|} \int_{\square \cap \mathscr{C}_\infty} |\nabla w_{\gamma_R}|^{2+\varepsilon}(z)\, dz \right)^{\frac{2}{2+\varepsilon}} \\
&\qquad \times \left(\frac{1}{|\square|} \sum_{z \in \square} (\operatorname{size}(\square_{\mathcal{P}}(z)))^{\frac{3d(2+\varepsilon)}{\varepsilon}} \left(\sum_{x \in \mathbb{Z}^d,\, \operatorname{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1,\, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right)^{\frac{2+\varepsilon}{\varepsilon}} \right)^{\frac{\varepsilon}{2+\varepsilon}} \\
&\quad \leq C \sum_{\square \in \mathcal{U}} \left(\int_{\frac{4}{3}\square \cap \mathscr{C}_\infty} |\nabla w_{\gamma_R}|^2(x)\, dx + |\square| \left(\frac{1}{\left|\frac{4}{3}\square\right|} \int_{\frac{4}{3}\square \cap \mathscr{C}_\infty} \gamma_R^{2+\varepsilon}(x)\, dx \right)^{\frac{2}{2+\varepsilon}} \right).
\end{aligned}$$

To estimate the term on the right-hand side, we note that the cube $\frac{4}{3}\square$ is included in the set $\bigcup_{\square' \in \mathcal{U},\, \operatorname{dist}(\square', \square) \leq 1} \square'$ and the cardinality of the set $\{\square' \in \mathcal{U} : \operatorname{dist}(\square', \square) \leq 1\}$ is bounded by a constant depending only on the dimension d. This leads to

$$\begin{aligned}
\sum_{\square \in \mathcal{U}} \int_{\frac{4}{3}\square \cap \mathscr{C}_\infty} |\nabla w_{\gamma_R}|^2(x)\, dx &\leq C \int_{\mathscr{C}_\infty} |\nabla w_{\gamma_R}|^2(x)\, dx \\
&\leq \mathcal{O}_s'\left(\frac{C}{R^d} \right).
\end{aligned}$$

To estimate the second term on the right-hand side, we recall the discrete $l^1 - l^t$-estimate: for any finite sequence of positive numbers $(b_i)_{0 \leq i \leq n} \in \mathbb{R}_+^{n+1}$ and any $t \geq 1$, $\sum_{i=0}^n b_i^t \leq \left(\sum_{i=0}^n b_i\right)^t$. Using this inequality gives

$$\begin{aligned}\sum_{\square \in \mathcal{U}} |\square| \left(\frac{1}{\left|\frac{4}{3}\square\right|} \int_{\frac{4}{3}\square \cap \mathscr{C}_\infty} \gamma_R^{2+\varepsilon}(x)\, dx \right)^{\frac{2}{2+\varepsilon}} & \leq C \sum_{\square \in \mathcal{U}} |\square|^{1-\frac{2}{2+\varepsilon}} \int_{\frac{4}{3}\square \cap \mathscr{C}_\infty} \gamma_R^2(x)\, dx \\ & \leq C \sum_{x \in \mathscr{C}_\infty} \gamma_R(x)^2 \operatorname{size}(\square_{\mathcal{U}}(x))^{d\left(1-\frac{2}{2+\varepsilon}\right)} \\ & \leq C \sum_{x \in \mathbb{Z}^d} \zeta_R(x)^2 \operatorname{size}(\square_{\mathcal{P}}(x))^{4d} \operatorname{size}(\square_{\mathcal{U}}(x))^{d\left(1-\frac{2}{2+\varepsilon}\right)}.\end{aligned}$$

Using the inequalities $\operatorname{size}(\square_{\mathcal{U}}(x)) \leq \mathcal{O}_s'(C)$, $\operatorname{size}(\square_{\mathcal{P}}(x)) \leq \mathcal{O}_s'(C)$ and (3.1.10) we obtain

$$\sum_{\square \in \mathcal{U}} |\square| \left(\frac{1}{\left|\frac{4}{3}\square\right|} \int_{\frac{4}{3}\square \cap \mathscr{C}_\infty} \gamma_R^{2+\varepsilon}(x)\, dx \right)^{\frac{2}{2+\varepsilon}} \leq \mathcal{O}_s'\left(\frac{C}{R^d}\right).$$

The proof of Result 2, and thus of Proposition 3.3.3, is complete.

3.4. Optimal L^q estimates for first order corrector

We now show how to obtain the L^q optimal scaling bounds on the corrector, Theorem 3.1.2, from Proposition 3.3.3. Theorem 3.1.2, is restated below and proved in this section.

THEOREM 3.1.2 (Optimal L^q estimates for first order corrector). *There exist two exponents $s := s(d,\mathfrak{p},\lambda) > 0$, $k := k(d,\mathfrak{p},\lambda) < \infty$ and a constant $C(d,\mathfrak{p},\lambda) < \infty$ such that for each $R \geq 1$, each $q \geq 1$ and each $p \in \mathbb{R}^d$,*

$$\left(R^{-d} \int_{\mathscr{C}_\infty \cap B_R} \left|\chi_p - (\chi_p)_{\mathscr{C}_\infty \cap B_R}\right|^q\right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s\left(C|p|q^k \log^{\frac{1}{2}} R\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(C|p|q^k\right) & \text{if } d \geq 3. \end{cases} \tag{3.4.1}$$

Before starting the proof, we mention an important caveat. In this section we need to keep track of the dependence on the parameter q of the constants. We will thus be careful to track every dependence in the q variable. This will be useful in the next section to obtain the L^∞ bounds on the corrector. In particular in this section the exponent k may vary from line to line but will always remain finite and will depend solely on the variables $d, \mathfrak{p}, \lambda$.

PROOF OF THEOREM 3.1.2. As in the proof of Proposition 3.3.3, we assume that $|p| = 1$ to ease the notations. Additionally, note that by the Jensen inequality, it is enough to prove Theorem 3.1.2 in the case $q \geq 2$. We consequently make this assumption for the rest of the proof. The proof of this theorem is split into two steps.

- In Step 1, we use Proposition 3.3.3 and the multiscale Poincaré inequality, Proposition 3.2.17, to show, for each $R \geq 1$,

$$\left(\fint_{B_R} \left|[\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}}\right)_{B_R}\right|^q\right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s\left(Cq^k \log^{\frac{1}{2}} R\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(Cq^k\right) & \text{if } d \geq 3, \end{cases}$$

  with C, k and s depending only on $s, \mathfrak{p}, \lambda$.
- In Step 2, we remove the coarsening, thanks to Proposition 3.2.8, to eventually obtain

$$\left(R^{-d} \int_{\mathscr{C}_\infty \cap B_R} \left|\chi_p - (\chi_p)_{\mathscr{C}_\infty \cap B_R}\right|^q\right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s\left(Cq^k \log^{\frac{1}{2}} R\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(Cq^k\right) & \text{if } d \geq 3. \end{cases}$$

Step 1. Fix some $R \geq 1$. The main idea of this step is to apply Proposition 3.2.17 to the function $u = [\chi_p]_{\mathcal{P}}^{\eta}$. The assumption of Proposition 3.2.17 is satisfied (it is a consequence of the construction of $[\chi_p]_{\mathcal{P}}^{\eta}$ and of the sublinearity property (3.2.2)). Consequently, we have, for each $R \geq 1$,

$$\left\| [\chi_p]_{\mathcal{P}}^{\eta} - \left([\chi_p]_{\mathcal{P}}^{\eta} \right)_{B_R} \right\|_{L^q(B_R)} \leq C \left(\int_{\mathbb{R}^d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr \right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}}.$$

To study the term on the right-hand side, we split the interior integral into two terms

$$\int_0^{2R} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr = \int_0^{1} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr + \int_1^{2R} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr. \tag{3.4.2}$$

But, by Proposition 3.3.3, we know that for each $r \geq 1$ and each $x \in \mathbb{R}^d$,

$$\left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right| \leq \mathcal{O}_s \left(C r^{-\frac{d}{2}} \right).$$

This implies,

$$\left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 \leq \mathcal{O}_s \left(C r^{-d} \right).$$

The second term on the right-hand side can be estimated by using Proposition 3.3.3 and the inequality (3.1.10), this yields

$$\int_1^{2R} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr \leq \begin{cases} \mathcal{O}_s (C \log R) & \text{if } d = 2, \\ \mathcal{O}_s (C) & \text{if } d \geq 3. \end{cases}$$

To estimate the first term on the right-hand side of (3.4.2), we use Proposition 3.2.14 which reads, for each $x \in \mathbb{R}^d$,

$$\left| \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right| \leq \mathcal{O}_s (C).$$

By this and (3.1.10), we obtain

$$\int_0^{1} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr \leq \mathcal{O}_s (C).$$

Combining the previous displays shows

$$\int_0^{2R} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr \leq \begin{cases} \mathcal{O}_s (C \log R) & \text{if } d = 2, \\ \mathcal{O}_s (C) & \text{if } d \geq 3. \end{cases}$$

We then obtain

$$\left(\int_0^{2R} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr \right)^{\frac{q}{2}} \leq \begin{cases} \mathcal{O}_{\frac{2s}{q}} \left(C^{\frac{q}{2}} (\log R)^{\frac{q}{2}} \right) & \text{if } d = 2, \\ \mathcal{O}_{\frac{2s}{q}} \left(C^{\frac{q}{2}} \right) & \text{if } d \geq 3. \end{cases}$$

We then apply (3.1.10) and keep track of the constants thanks to (3.1.11), we obtain

$$\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr \right)^{\frac{q}{2}} dx \leq \begin{cases} \mathcal{O}_{\frac{2s}{q}} \left(\left(\frac{q}{s \ln(2)} \right)^{\frac{q}{s}} C^{\frac{q}{2}} (\log R)^{\frac{q}{2}} \right) & \text{if } d = 2, \\ \mathcal{O}_{\frac{2s}{q}} \left(\left(\frac{q}{s \ln(2)} \right)^{\frac{q}{s}} C^{\frac{q}{2}} \right) & \text{if } d \geq 3. \end{cases}$$

This eventually yields

$$\left(\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r \left| \Phi_r * \nabla [\chi_p]_{\mathcal{P}}^{\eta}(x) \right|^2 dr \right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(q^{\frac{1}{s}} C (\log R)^{\frac{1}{2}} \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(q^{\frac{1}{s}} C \right) & \text{if } d \geq 3. \end{cases}$$

We now set $k := \frac{1}{s} + \frac{3}{4}$. This exponent depends only on the parameters $d, \mathfrak{p}, \lambda$. By applying Proposition 3.2.17, we obtain

$$\left(\fint_{B_R} \left| [\chi_p]_{\mathcal{P}}^{\eta} - \left([\chi_p]_{\mathcal{P}}^{\eta} \right)_{B_R} \right|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(C q^k \log^{\frac{1}{2}} R \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(C q^k \right) & \text{if } d \geq 3. \end{cases} \tag{3.4.3}$$

The next goal is to remove the regularization by convolution by η. We first apply (3.3.2) to obtain

$$\left(\fint_{B_R} \left| [\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}}^{\eta} \right)_{B_R} \right|^q \right)^{\frac{1}{q}} \le C \left(\fint_{B_R} \left| [\chi_p]_{\mathcal{P}}^{\eta} - \left([\chi_p]_{\mathcal{P}}^{\eta} \right)_{B_R} \right|^q \right)^{\frac{1}{q}}.$$

Note that by the triangle inequality and Jensen inequality, we have

$$\begin{aligned} \left(\fint_{B_R} \left| [\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}} \right)_{B_R} \right|^q \right)^{\frac{1}{q}} &\le 2 \inf_{a \in \mathbb{R}} \left(\fint_{B_R} \left| [\chi_p]_{\mathcal{P}} - a \right|^q (x) \, dx \right)^{\frac{1}{q}} \\ &\le 2 \left(\fint_{B_R} \left| [\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}}^{\eta} \right)_{B_R} \right|^q \right)^{\frac{1}{q}} \\ &\le C \left(\fint_{B_R} \left| [\chi_p]_{\mathcal{P}}^{\eta} - \left([\chi_p]_{\mathcal{P}}^{\eta} \right)_{B_R} \right|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Combining the previous estimates completes the proof of Step 1.

Step 2. We remove the coarsening, thanks to Proposition 3.2.8. We split the L^q norm of the corrector into two terms,

$$\begin{aligned} \left(\int_{\mathscr{C}_\infty \cap \square_m} \left| \chi_p - (\chi_p)_{\mathscr{C}_\infty \cap \square_m} \right|^q (x) \, dx \right)^{\frac{1}{q}} &\le \left(\int_{\mathscr{C}_\infty \cap \square_m} \left| \chi_p - (\chi_p)_{\mathscr{C}_\infty \cap \square_m} \right|^q (x) \, dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \ge \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\quad + \left(\int_{\mathscr{C}_\infty \cap \square_m} \left| \chi_p - (\chi_p)_{\mathscr{C}_\infty \cap \square_m} \right|^q (x) \, dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \le \mathcal{M}_{2d}(\mathcal{P})\}}. \end{aligned}$$

The reason we use the indicator $\mathbb{1}_{\{3^m \le \mathcal{M}_{2d}(\mathcal{P})\}}$ is to be able to apply (3.2.9) in the computation below. But first, we estimate the first term on the right-hand side, to do so we can use the L^∞ bound (3.2.1) (applied with $\delta = 0$ to simplify the computation), this gives:

$$\begin{aligned} \left(\fint_{\mathscr{C}_\infty \cap \square_m} \left| \chi_p - (\chi_p)_{\mathscr{C}_\infty \cap \square_m} \right|^q (x) \, dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \le \mathcal{M}_{2d}(\mathcal{P})\}} &\le \left\| \chi_p - (\chi_p)_{\mathscr{C}_\infty \cap \square_m} \right\|_{L^\infty(\mathscr{C}_\infty \cap \square_m)} \mathbb{1}_{\{3^m \le \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\le \mathcal{O}_s(C 3^m) \mathbb{1}_{\{3^m \le \mathcal{M}_{2d}(\mathcal{P})\}}. \end{aligned}$$

Since $\mathbb{1}_{\{3^m \le \mathcal{M}_{2d}(\mathcal{P})\}} \le \mathcal{O}_s\left(3^{-m}\right)$, we obtain,

$$\left(\fint_{\mathscr{C}_\infty \cap \square_m} \left| \chi_p - (\chi_p)_{\mathscr{C}_\infty \cap \square_m} \right|^q (x) \, dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \le \mathcal{M}_{2d}(\mathcal{P})\}} \le \mathcal{O}_s(C). \tag{3.4.4}$$

To estimate the second term in the right-hand side, we compute

$$\begin{aligned} &\left(\fint_{\mathscr{C}_\infty \cap \square_m} \left| \chi_p - (\chi_p)_{\mathscr{C}_\infty \cap \square_m} \right|^q (x) \, dx \right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \ge \mathcal{M}_{2d}(\mathcal{P})\}} \\ &\qquad \le C \mathbb{1}_{\{3^m \ge \mathcal{M}_{2d}(\mathcal{P})\}} \left(\fint_{\mathscr{C}_\infty \cap \square_m} \left| \chi_p - [\chi_p]_{\mathcal{P}} \right|^q (x) \, dx \right)^{\frac{1}{q}} \\ &\qquad\quad + C \mathbb{1}_{\{3^m \ge \mathcal{M}_{2d}(\mathcal{P})\}} \left(\fint_{\mathscr{C}_\infty \cap \square_m} \left| [\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}} \right)_{\mathscr{C}_\infty \cap \square_m} \right|^q (x) \, dx \right)^{\frac{1}{q}}. \end{aligned} \tag{3.4.5}$$

To estimate the first term on the right-hand side, we first use (3.2.8) and Proposition 3.2.8, to obtain for each $m \in \mathbb{N}$ such that $\square_m \in \mathcal{P}_*$

$$\begin{aligned} \int_{\mathscr{C}_\infty \cap \square_m} \left| \chi_p - [\chi_p]_{\mathcal{P}} \right|^q (x) \, dx &\le \int_{\mathscr{C}_*(\square_{m+1})} \left| \chi_p - [\chi_p]_{\mathcal{P}} \right|^q (x) \, dx \\ &\le C \int_{\mathscr{C}_*(\square_{m+1})} \operatorname{size}(\square_{\mathcal{P}}(x))^{qd} \left| \nabla \chi_p \right|^q (x) \, dx \\ &\le C \int_{\mathscr{C}_\infty \cap \square_{m+1}} \operatorname{size}(\square_{\mathcal{P}}(x))^{qd} \left| \nabla \chi_p \right|^q (x) \, dx. \end{aligned}$$

By the Lipschitz bounds on the gradient of the corrector and the property of the partition $\mathcal{P}$, we have, for each $x \in \mathbb{Z}^d$

$$\operatorname{size}(\square_{\mathcal{P}}(x))^{qd} \left|\nabla \chi_p\right|^q (x) \mathbb{1}_{\{x \in \mathscr{C}_\infty\}} \leq \mathcal{O}_{\frac{s}{q}}\left(C^q\right).$$

Consequently, by (3.1.10) and using (3.1.11) to keep track of the dependence of the constants in the q variable

$$\begin{aligned}
\int_{\mathscr{C}_\infty \cap \square_{m+1}} \operatorname{size}(\square_{\mathcal{P}}(x))^{qd} \left|\nabla \chi_p\right|^q (x)\, dx &= \sum_{x \in \square_{m+1}} \operatorname{size}(\square_{\mathcal{P}}(x))^{qd} \left|\nabla \chi_p\right|^q (x) \mathbb{1}_{\{x \in \mathscr{C}_\infty\}} \\
&\leq \mathcal{O}_{\frac{s}{q}}\left(3^{d(m+1)} \left(\frac{q}{s \ln(2)}\right)^{\frac{q}{s}} C^q\right) \\
&\leq \mathcal{O}_{\frac{s}{q}}\left(3^{d(m+1)} q^{\frac{q}{s}} C^q\right).
\end{aligned}$$

In particular, if 3^m is larger than $\mathcal{M}_{2d}(\mathcal{P})$, then the cube $\square_m$ belongs to $\mathcal{P}_*$, the previous computations consequently show

$$\mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \int_{\mathscr{C}_\infty \cap \square_m} \left|\chi_p - [\chi_p]_{\mathcal{P}}\right|^q (x)\, dx \leq \mathcal{O}_{\frac{s}{q}}\left(3^{d(m+1)} q^{\frac{q}{s}} C^q\right).$$

Then by (3.2.9), we obtain

$$\mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \left(\fint_{\mathscr{C}_\infty \cap \square_m} \left|\chi_p - [\chi_p]_{\mathcal{P}}\right|^q (x)\, dx\right)^{\frac{1}{q}} \leq \mathcal{O}_s\left(q^{\frac{1}{s}} C\right).$$

To estimate the second term on the right-hand side of (3.4.5), we compute, by (3.2.9)

$$\begin{aligned}
&\left(\fint_{\mathscr{C}_\infty \cap \square_m} \left|[\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}}\right)_{\mathscr{C}_\infty \cap \square_m}\right|^q (x)\, dx\right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \\
&\qquad \leq 2 \inf_{a \in \mathbb{R}} \left(\fint_{\mathscr{C}_\infty \cap \square_m} \left|[\chi_p]_{\mathcal{P}} - a\right|^q (x)\, dx\right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \\
&\qquad \leq 2 \left(\fint_{\mathscr{C}_\infty \cap \square_m} \left|[\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}}\right)_{\square_m}\right|^q (x)\, dx\right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \\
&\qquad \leq C \left(\fint_{\square_m} \left|[\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}}\right)_{\square_m}\right|^q (x)\, dx\right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}}.
\end{aligned}$$

We then apply (3.4.3) and obtain

$$\left(\fint_{\mathscr{C}_\infty \cap \square_m} \left|\chi_p - (\chi_p)_{\mathscr{C}_\infty \cap \square_m}\right|^q (x)\, dx\right)^{\frac{1}{q}} \mathbb{1}_{\{3^m \geq \mathcal{M}_{2d}(\mathcal{P})\}} \leq \begin{cases} \mathcal{O}_s\left(q^k C m^{\frac{1}{2}}\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(q^k C\right) & \text{if } d \geq 3, \end{cases}$$

for some exponents $k := k(d, \mathfrak{p}, \lambda)$, $s := s(d, \mathfrak{p}, \lambda) > 0$ and some constant $C := C(d, \mathfrak{p}, \lambda) < \infty$. Combining this with (3.4.4), we obtain

$$\left(\fint_{\mathscr{C}_\infty \cap \square_m} \left|\chi_p - (\chi_p)_{\mathscr{C}_\infty \cap \square_m}\right|^q (x)\, dx\right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s\left(q^k C m^{\frac{1}{2}}\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(q^k C\right) & \text{if } d \geq 3, \end{cases}$$

The result of Theorem 3.1.2 requires to prove the previous inequality for a general ball B_R and not a cube $\square_m$. This result is obtained by selecting, for each radius $R \geq 1$, the integer l such that $3^m < R \leq 3^{m+1}$ and by performing a similar analysis. □

3.5. Optimal L^∞ estimates for the first order corrector

In this section, we prove the L^∞ bound on the corrector, Theorem 3.1.1.

THEOREM 3.1.1 (Optimal L^∞ estimates for first order correctors). *There exist an exponent $s := s(d,\mathfrak{p},\lambda) > 0$ and a constant $C := C(d,\mathfrak{p},\lambda) < \infty$ such that for each $x, y \in \mathbb{Z}^d$ and each $p \in \mathbb{R}^d$,*

$$|\chi_p(x) - \chi_p(y)| \mathbb{1}_{\{x,y\in\mathscr{C}_\infty\}} \leq \begin{cases} \mathcal{O}_s\left(C|p|\log^{\frac{1}{2}}|x-y|\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(C|p|\right) & \text{if } d \geq 3. \end{cases}$$

PROOF OF THEOREM 3.1.1. First by the stationarity of the gradient of the corrector, we can assume without loss of generality that $y = 0$. Without loss of generality, we can also assume $|p| = 1$, as it was done in the proofs of Proposition 3.3.3 and of Theorem 3.1.2. We thus want to prove, for each $x \in \mathbb{Z}^d$,

$$|\chi_p(x) - \chi_p(0)| \mathbb{1}_{\{0,x\in\mathscr{C}_\infty\}} \leq \begin{cases} \mathcal{O}_s\left(C\log^{\frac{1}{2}}|x|\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(C\right) & \text{if } d \geq 3. \end{cases}$$

Before starting the proof, note that, for every $q > 0$ and every $x \in \mathbb{R}_+$

$$\exp(x) \geq \frac{x^q}{q^q \exp(-q)}.$$

This implies, for each $s, q, \theta > 0$,

$$X \leq \mathcal{O}_s(\theta) \implies \mathbb{E}[X^q] \leq 2\theta^q \left(\frac{q}{s}\right)^{\frac{q}{s}} \exp\left(\frac{q}{s}\right). \tag{3.5.1}$$

We split the proof into six steps.

- In Step 1, we prove that for each $q \geq 1$ and each $m \in \mathbb{N}$,

$$\mathbb{E}\left[\left|[\chi_p]_{\mathcal{P}}(0) - 3^{-2dm}\sum_{y\in\square_m}\sum_{z\in y+\square_m}[\chi_p]_{\mathcal{P}}(z)\right|^q\right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

- In Step 2, we use the result of Step 1 to prove that for each $q \geq 1$ and each $m \in \mathbb{N}$,

$$\mathbb{E}\left[\left|[\chi_p]_{\mathcal{P}}(x) - 3^{-2dm}\sum_{y\in\square_m}\sum_{z\in x+y+\square_m}[\chi_p]_{\mathcal{P}}(z)\right|^q\right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

  Note that this statement is not just a consequence of Step 1 and the stationarity of the corrector since the partition $\mathcal{P}$ is not stationary. One additional argument is needed to conclude.

- In Step 3, we prove that for each $q \geq 1$ and $m \in \mathbb{N}$, chosen such that $3^m \leq |x| < 3^{m+1}$,

$$\mathbb{E}\left[\left|[\chi_p]_{\mathcal{P}}(0) - 3^{-2dm}\sum_{y\in\square_m}\sum_{z\in x+y+\square_m}[\chi_p]_{\mathcal{P}}(z)\right|^q\right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

- In Step 4, we combine Steps 2 and 3 to obtain, for each $q \geq 1$

$$\mathbb{E}\left[\left|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)\right|^q\right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases} \tag{3.5.2}$$

- In Step 5, we prove that there exist an exponent $s := s(d,\mathfrak{p},\lambda) > 0$ and a constant $C := C(d,\mathfrak{p},\lambda) < \infty$ such that

$$\left|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)\right| \leq \begin{cases} \mathcal{O}_s\left(C\log^{\frac{1}{2}}|x|\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(C\right) & \text{if } d \geq 3. \end{cases} \tag{3.5.3}$$

- In Step 6, we remove the coarsening and eventually show that

$$|\chi_p(x) - \chi_p(0)| \mathbb{1}_{\{0,x\in\mathscr{C}_\infty\}} \leq \begin{cases} \mathcal{O}_s\left(C\log^{\frac{1}{2}}|x|\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(C\right) & \text{if } d \geq 3. \end{cases}$$

Step 1. The main tool of this step is the following inequality which was proved in Step 1 of the proof of Theorem 3.1.2, for each $m \in \mathbb{N}$, and each $q \geq 1$,

$$\left(\fint_{\square_m} \left| [\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}} \right)_{\square_m} \right|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s \left(C q^k \sqrt{m} \right) & \text{if } d = 2, \\ \mathcal{O}_s \left(C q^k \right) & \text{if } d \geq 3. \end{cases} \tag{3.5.4}$$

Note that this implies, by increasing the values of C and k,

$$\mathbb{E} \left[\fint_{\square_m} \left| [\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}} \right)_{\square_m} \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases} \tag{3.5.5}$$

For some fixed $y \in \mathbb{Z}^d$, note that by stationarity of the corrector (3.2.3), for almost every $\mathbf{a} \in \Omega$, one has

$$\left([\chi_p]_{\mathcal{P}} (-y) - \left([\chi_p]_{\mathcal{P}} \right)_{\square_m} \right) (\mathbf{a}) = \left([\chi_p]_{\mathcal{P}_y} (0) - \left([\chi_p]_{\mathcal{P}_y} \right)_{y + \square_m} \right) (\tau_y \mathbf{a}),$$

where we recall the notation $\mathcal{P}_y = y + \mathcal{P}(\tau_{-y} \mathbf{a})$. Using the stationarity property (3.1.4), we obtain, for each $q \geq 1$,

$$\mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}_y} (0) - \left([\chi_p]_{\mathcal{P}_y} \right)_{y + \square_m} \right|^q \right] = \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}} (-y) - \left([\chi_p]_{\mathcal{P}} \right)_{\square_m} \right|^q \right].$$

Since this is true for each $y \in \mathbb{Z}^d$, we can integrate over y to obtain

$$\fint_{\square_m} \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}_y} (0) - \left([\chi_p]_{\mathcal{P}_y} \right)_{y + \square_m} \right|^q \right] dy = \fint_{\square_m} \mathbb{E} \left[\left| [\chi_p]_{\mathcal{P}} (-y) - \left([\chi_p]_{\mathcal{P}} \right)_{\square_m} \right|^q \right] dy.$$

Thus, by (3.5.5),

$$\mathbb{E} \left[\fint_{\square_m} \left| [\chi_p]_{\mathcal{P}_y} (0) - \left([\chi_p]_{\mathcal{P}_y} \right)_{y + \square_m} \right|^q dy \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases} \tag{3.5.6}$$

We now remove the translation of the partition and prove, for each $z \in \mathbb{Z}^d$

$$\left| [\chi_p]_{\mathcal{P}_y} (z) - [\chi_p]_{\mathcal{P}} (z) \right| \leq \mathcal{O}_s(C). \tag{3.5.7}$$

To prove this, note that, by definition of the coarsening (3.2.7), we have

$$[\chi_p]_{\mathcal{P}_y} (z) - [\chi_p]_{\mathcal{P}} (z) = \chi_p \left(\overline{z}(\square_{\mathcal{P}_y}(z)) \right) - \chi_p \left(\overline{z}(\square_{\mathcal{P}}(z)) \right),$$

and by definition of the two partitions $\mathcal{P}$ and $\mathcal{P}_y$, there exists a path connecting $\square_{\mathcal{P}_y}(z)$ to $\square_{\mathcal{P}}(z)$ which lies in $B\left(z, C \max\left(\operatorname{size}\left(\square_{\mathcal{P}_y}(z)\right), \operatorname{size}\left(\square_{\mathcal{P}}(z)\right)\right)\right)$. To simplify the notation in the following computation, we denote by $R' = C \max\left(\operatorname{size}\left(\square_{\mathcal{P}_y}(z)\right), \operatorname{size}\left(\square_{\mathcal{P}}(z)\right)\right)$. As a consequence, we have the estimate

$$\left| [\chi_p]_{\mathcal{P}_y} (z) - [\chi_p]_{\mathcal{P}} (z) \right| \leq \int_{\mathscr{C}_\infty \cap B_{R'}(z)} |\nabla \chi_p| (x) \, dx.$$

By Propositions 3.2.13, the bounds $R' \leq \mathcal{O}_s(C)$ and $\mathcal{X}(z) \leq \mathcal{O}_s(C)$ and the assumption $|p| = 1$, we have

$$\int_{\mathscr{C}_\infty \cap B'_R(z)} |\nabla \chi_p| (y) \, dy \leq \mathcal{O}_s(C).$$

Combining the previous displays completes the proof of (3.5.7). To remove the parameter y in (3.5.6), we compute

$$\begin{aligned} & \mathbb{E} \left[\fint_{\square_m} \left| [\chi_p]_{\mathcal{P}} (0) - \left([\chi_p]_{\mathcal{P}} \right)_{y + \square_m} \right|^q dy \right] \\ & \quad \leq 2^q \mathbb{E} \left[\fint_{\square_m} \left| [\chi_p]_{\mathcal{P}_y} (0) - \left([\chi_p]_{\mathcal{P}_y} \right)_{y + \square_m} \right|^q dy \right] \\ & \qquad + 2^q \mathbb{E} \left[\fint_{\square_m} \left| [\chi_p]_{\mathcal{P}_y} (0) - \left([\chi_p]_{\mathcal{P}_y} \right)_{y + \square_m} - [\chi_p]_{\mathcal{P}} (0) - \left([\chi_p]_{\mathcal{P}} \right)_{y + \square_m} \right|^q dy \right]. \end{aligned} \tag{3.5.8}$$

By (3.5.7) and (3.1.10), we have, for each $y \in \square_m$,

$$\left| [\chi_p]_{\mathcal{P}_y}(0) - \left([\chi_p]_{\mathcal{P}_y}\right)_{y+\square_m} - [\chi_p]_{\mathcal{P}}(0) - \left([\chi_p]_{\mathcal{P}}\right)_{y+\square_m} \right| \le \mathcal{O}_s(C),$$

and thus

$$\mathbb{E}\left[\left| [\chi_p]_{\mathcal{P}_y}(0) - \left([\chi_p]_{\mathcal{P}_y}\right)_{y+\square_m} - [\chi_p]_{\mathcal{P}}(0) - \left([\chi_p]_{\mathcal{P}}\right)_{y+\square_m} \right|^q\right] \le C^q q^{qk}.$$

Integrating over $y \in \square_m$ yields

$$\mathbb{E}\left[\fint_{\square_m}\left| [\chi_p]_{\mathcal{P}_y}(0) - \left([\chi_p]_{\mathcal{P}_y}\right)_{y+\square_m} - [\chi_p]_{\mathcal{P}}(0) - \left([\chi_p]_{\mathcal{P}}\right)_{y+\square_m} \right|^q dy\right] \le C^q q^{qk}.$$

By the previous display and (3.5.6), we have

$$\mathbb{E}\left[\fint_{\square_m}\left| [\chi_p]_{\mathcal{P}}(0) - \left([\chi_p]_{\mathcal{P}}\right)_{y+\square_m} \right|^q dy\right] \le \left\{ \begin{array}{ll} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \ge 3. \end{array}\right.$$

By the Jensen inequality, we obtain

$$\mathbb{E}\left[\left|\fint_{\square_m} [\chi_p]_{\mathcal{P}}(0) - \left([\chi_p]_{\mathcal{P}}\right)_{y+\square_m} dy\right|^q\right] \le \left\{ \begin{array}{ll} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \ge 3. \end{array}\right.$$

but notice that

$$\fint_{\square_m} [\chi_p]_{\mathcal{P}}(0) - \left([\chi_p]_{\mathcal{P}}\right)_{y+\square_m} dy = [\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in y+\square_m} [\chi_p]_{\mathcal{P}}(z).$$

Combining the two previous displays completes the proof of Step 1.

Step 2. By the stationarity of the corrector (3.2.3), for almost every $\mathbf{a} \in \Omega$, every $y, z \in \mathbb{Z}^d$,

$$[\chi_p]_{\mathcal{P}}(z)(\mathbf{a}) = [\chi_p]_{\mathcal{P}_y}(z+y)(\tau_y \mathbf{a}).$$

Using this property, we have

$$\begin{aligned}\mathbb{E}\left[\left|[\chi_p]_{\mathcal{P}_x}(x) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}_x}(z)\right|^q\right] &= \mathbb{E}\left[\left|[\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in y+\square_m} [\chi_p]_{\mathcal{P}}(z)\right|^q\right] \\ &\le \left\{ \begin{array}{ll} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \ge 3. \end{array}\right.\end{aligned}$$

Doing the same computation as in (3.5.8), we can replace $\mathcal{P}_x$ by $\mathcal{P}$ in the previous display, this yields

$$\mathbb{E}\left[\left|[\chi_p]_{\mathcal{P}}(x) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z)\right|^q\right] \le \left\{ \begin{array}{ll} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d = 2, \\ C^q q^{qk} & \text{if } d \ge 3. \end{array}\right.$$

This completes the proof of the main estimate of Step 2.

Step 3. This step is similar to Step 1, but the main tool of this step is slightly different and presented below. For $m \in \mathbb{N}$ such that $3^m \le |x| < 3^{m+1}$, and for each $q \ge 1$,

$$\left(\fint_{\square_m}\left|[\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}}\right)_{x+\square_m}\right|^q\right)^{\frac{1}{q}} \le \left\{ \begin{array}{ll} \mathcal{O}_s\left(Cq^k \sqrt{m}\right) & \text{if } d = 2, \\ \mathcal{O}_s\left(Cq^k\right) & \text{if } d \ge 3. \end{array}\right.$$

To prove this result, we note that $x + \square_m \subseteq \square_{m+2}$. With this in mind, we can compute

$$\begin{aligned}&\left(\fint_{\square_m}\left|[\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}}\right)_{x+\square_m}\right|^q\right)^{\frac{1}{q}} \\ &\qquad \le C\left(\fint_{\square_{m+2}}\left|[\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}}\right)_{\square_{m+2}}\right|^q\right)^{\frac{1}{q}} + C\left|\left([\chi_p]_{\mathcal{P}}\right)_{\square_{m+2}} - \left([\chi_p]_{\mathcal{P}}\right)_{x+\square_m}\right|.\end{aligned}$$

We estimate the second term on the right-hand side as follows

$$\left| \left([\chi_p]_{\mathcal{P}} \right)_{\square_{m+2}} - \left([\chi_p]_{\mathcal{P}} \right)_{x+\square_m} \right| \leq \fint_{x+\square_m} \left| [\chi_p]_{\mathcal{P}}(x) - \left([\chi_p]_{\mathcal{P}} \right)_{\square_{m+2}} \right| dx \leq C \left(\fint_{\square_{m+2}} \left| [\chi_p]_{\mathcal{P}}(x) - \left([\chi_p]_{\mathcal{P}} \right)_{\square_{m+2}} \right|^q dx \right)^{\frac{1}{q}}.$$

Combining the two previous displays with (3.5.4) shows

$$\left(\fint_{\square_m} \left| [\chi_p]_{\mathcal{P}} - \left([\chi_p]_{\mathcal{P}} \right)_{x+\square_m} \right|^q \right)^{\frac{1}{q}} \leq \begin{cases} \mathcal{O}_s\left(Cq^k\sqrt{m}\right) & \text{if } d=2, \\ \mathcal{O}_s\left(Cq^k\right) & \text{if } d \geq 3. \end{cases}$$

With the same proof as in Step 1, we obtain, for each $q \geq 1$

$$\mathbb{E}\left[\left| \fint_{\square_m} [\chi_p]_{\mathcal{P}}(0) - \left([\chi_p]_{\mathcal{P}} \right)_{y+x+\square_m} dy \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d=2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

But note that

$$[\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z) = \fint_{\square_m} [\chi_p]_{\mathcal{P}}(0) - \left([\chi_p]_{\mathcal{P}} \right)_{y+x+\square_m} dy.$$

Combining the two previous displays completes the proof of Step 3.

Step 4. In this step, we first split the integral,

$$\mathbb{E}\left[\left| [\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0) \right|^q \right] \leq 2^q \mathbb{E}\left[\left| [\chi_p]_{\mathcal{P}}(0) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z) \right|^q \right] + 2^q \mathbb{E}\left[\left| [\chi_p]_{\mathcal{P}}(x) - 3^{-2dm} \sum_{y \in \square_m} \sum_{z \in x+y+\square_m} [\chi_p]_{\mathcal{P}}(z) \right|^q \right].$$

Combining the results of Step 2 and Step 3, we have, for $m \in \mathbb{N}$ chosen such that $3^m \leq |x| \leq 3^{m+1}$ and for each $q \geq 1$,

$$\mathbb{E}\left[\left| [\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0) \right|^q \right] \leq \begin{cases} C^q q^{qk} m^{\frac{q}{2}} & \text{if } d=2, \\ C^q q^{qk} & \text{if } d \geq 3. \end{cases}$$

Since $m \leq \frac{\log |x|}{\log 3}$, the proof of Step 3 is complete.

Step 5. First we extend the result of Step 4 to the case $0 < q < 1$. By the Jensen inequality, we have, for each $0 < q \leq 1$

$$\mathbb{E}\left[\left| [\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0) \right|^q \right] \leq \mathbb{E}\left[\left| [\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0) \right|^2 \right]^{\frac{q}{2}} \leq \begin{cases} C^q m^{\frac{q}{2}} & \text{if } d=2, \\ C^q & \text{if } d \geq 3. \end{cases}$$

We now prove the main result of this step. We first deal with the case $d = 2$. Select an exponent $s > 0$ depending only on $d, \mathfrak{p}, \lambda$ such that $s < \frac{1}{k}$, where k is the exponent (depending only on $d, \mathfrak{p}, \lambda$) which appears in (3.5.2).

We then compute

$$\mathbb{E}\left[\exp\left(\left(\frac{\left| [\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0) \right|}{\log^{\frac{1}{2}} |x|} \right)^s \right) \right] = \sum_{l=0}^{\infty} \frac{1}{l!} \mathbb{E}\left[\frac{\left| [\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0) \right|^{sl}}{\log^{\frac{sl}{2}} |x|} \right] \leq \sum_{l=0}^{\lfloor \frac{1}{s} \rfloor} \frac{C^{sl}}{l!} + \sum_{l=\lceil \frac{1}{s} \rceil}^{\infty} \frac{C^{sl} (sl)^{skl}}{l!} < \infty,$$

by the Stirling formula. We now set $\sigma := \max\left(\frac{\log 2}{\log\left(\sum_{l=0}^{\lfloor \frac{1}{s}\rfloor} \frac{C^{sl}}{l!} + \sum_{l=\lceil \frac{1}{s}\rceil}^{\infty} \frac{C^{sl}(sl)^{skl}}{l!}\right)}, 1\right) > 0$. Note that σ depends only on $d, \mathfrak{p}, \lambda$. With this value of σ, we have

$$\mathbb{E}\left[\exp\left(\sigma\left(\frac{\left|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)\right|}{\log^{\frac{1}{2}}|x|}\right)^s\right)\right] \le 2.$$

From this computation, we obtain

$$\left|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)\right| \le \mathcal{O}_s\left(\sigma^{-\frac{1}{s}}\log^{\frac{1}{2}}|x|\right).$$

Setting $C := \sigma^{-\frac{1}{s}}$, we obtain (3.5.3). The proof in dimension $d \ge 3$ follows the same lines and is even simpler since we do not have the square root of the logarithm.

Step 6. In this step, we remove the coarsening. To do so, we prove, for each $y \in \mathbb{Z}^d$,

$$\left|\chi_p(y) - [\chi_p]_{\mathcal{P}}(y)\right| \mathbb{1}_{\{y \in \mathscr{C}_\infty\}} \le \mathcal{O}_s(C).$$

To prove this note that if $y \in \mathscr{C}_\infty$ then there exists a path connecting y to $\overline{z}(\square_{\mathcal{P}}(y))$ which lies in $\square_{\mathcal{P}}(y)$ and its neighbors. Consequently we have the estimate

$$\left|\chi_p(y) - [\chi_p]_{\mathcal{P}}(y)\right| \mathbb{1}_{\{y \in \mathscr{C}_\infty\}} \le \int_{\mathscr{C}_\infty \cap B_{C\,\mathrm{size}(\square_{\mathcal{P}}(y))}(y)} |\nabla \chi_p|(x)\,dx.$$

By Proposition 3.2.13, this gives

$$\left|\chi_p(y) - [\chi_p]_{\mathcal{P}}(y)\right| \mathbb{1}_{\{y \in \mathscr{C}_\infty\}} \le \mathcal{O}_s(C).$$

From this we deduce

$$\begin{aligned} &|\chi_p(x) - \chi_p(0)| \mathbb{1}_{\{0, x \in \mathscr{C}_\infty\}} \\ &\qquad \le \left|\chi_p(0) - [\chi_p]_{\mathcal{P}}(0)\right| \mathbb{1}_{\{0 \in \mathscr{C}_\infty\}} + \left|\chi_p(x) - [\chi_p]_{\mathcal{P}}(x)\right| \mathbb{1}_{\{x \in \mathscr{C}_\infty\}} + \left|[\chi_p]_{\mathcal{P}}(x) - [\chi_p]_{\mathcal{P}}(0)\right|. \end{aligned}$$

Combining the result of Step 5 with the previous displays shows

$$|\chi_p(x) - \chi_p(0)| \mathbb{1}_{\{0, x \in \mathscr{C}_\infty\}} \le \begin{cases} \mathcal{O}_s\left(C\log^{\frac{1}{2}}|x|\right) & \text{if } d = 2, \\ \mathcal{O}_s(C) & \text{if } d \ge 3. \end{cases}$$

The proof of Step 6 is complete. □

3.A. Proof of the L^q multiscale Poincaré inequality

In this appendix, we prove the L^q multiscale Poincaré inequality.

PROPOSITION 3.2.17 (Multiscale Poincaré inequality, heat kernel version). *For each $r > 0$, we define*

$$\Phi_r : \begin{cases} \mathbb{R}^d & \to & \mathbb{R} \\ x & \mapsto & r^{-d}\exp\left(-\frac{|x|^2}{r^2}\right). \end{cases} \tag{3.1.1}$$

for each $q \ge 2$, there exists a constant $C := C(d, q) < \infty$ such that for each tempered distribution $u \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ and each $R > 0$,

$$\|u - (u)_{B_R}\|_{\underline{L}^q(B_R)} \le C\left(\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|}{R}}\left(\int_0^R r\left|\Phi_r * \nabla u(x)\right|^2 dr\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}. \tag{3.1.2}$$

Moreover the dependence on the q variable of the constant C can be estimated as follows, for each $q \ge 2$,

$$C(d, q) \le A q^{\frac{1}{2}},$$

for some constant $A := A(d) < \infty$.

Before starting the proof, we need to state the following proposition from [**18**, Proposition D.1 and Remark D.6] and to record a result from the elliptic regularity theory.

Proposition 3.A.1 (Proposition D.1 and Remark D.6 of [**18**]). *For each $q \geq 2$, there exists a constant $C := C(d,q) < \infty$ such that for every tempered distribution $w \in \mathcal{S}'(\mathbb{R}^d)$,*

$$\|w\|_{W^{-1,q}(B_1)} \leq C \left(\int_{\mathbb{R}^d} e^{-|x|} \left(\int_0^1 r \left|\Phi_r * w(x)\right|^2 dr \right)^{\frac{q}{2}} \right)^{\frac{1}{q}}.$$

Moreover the constant C satisfies, for each $q \geq 2$

$$C(d,q) \leq A\sqrt{q},$$

for some constant $A := A(d) < \infty$.

Remark 3.A.2. (1) The Proposition D.1 and Remark D.6 of [**18**] are written with the standard heat kernel defined by $\Phi(t,x) := t^{-d/2} \exp\left(-\frac{|x|^2}{t}\right)$, which is related to our definition of Φ_r in (3.1.1) through the identity

$$\Phi_r = \Phi(r^2, \cdot).$$

This explains why the result stated in [**18**] is slightly different from the one presented here: one has to perform a change of variable to go from one to the other.

(2) The dependence on the q variable of the constant C is not explicit in [**18**]. It can nevertheless be recovered by a careful investigation of the proof.

We then record a result from the theory of elliptic regularity, it can be found in [**74**, Lemma 7.12 and Proposition 9.9].

Proposition 3.A.3 (Lemma 7.12 and Proposition 9.9 of [**74**]). *Let $V \subseteq \mathbb{R}^d$ be a bounded domain of $\mathbb{R}^d$. Let $f \in L^p(V)$, $1 < p < \infty$, and let w be the Newtonian potential of f, i.e.,*

$$w(x) := \int_V \Gamma(x-y) f(y)\, dy,$$

where Γ is the fundamental solution of the Laplace equation, i.e.,

$$\Gamma(x) := \begin{cases} \frac{1}{2\pi} \log|x| & \text{if } d = 2, \\ \frac{1}{d(2-d)\omega_d} |x|^{2-d} & \text{if } d \geq 3, \end{cases}$$

where ω_d is the volume of the unit sphere in $\mathbb{R}^d$. Then $w \in W^{2,p}(V), \Delta w = f$ a.e,

$$\left\|\nabla^2 w\right\|_{L^p(\Omega)} \leq C_0 \|f\|_{L^p(V)}$$

and

$$\|w\|_{L^p(V)} + \|\nabla w\|_{L^p(V)} \leq C_1 \|f\|_{L^p(V)},$$

for some constants $C_1 := C_1(d,V) < \infty$ and $C_0 := C_0(d,p,V) < \infty$. Moreover, the dependence on p of the constant C_0 can be explicited:

$$C_0(d,p,V) \leq Ap, \text{ if } p \geq 2 \qquad \text{and} \qquad C_0(d,p,V) \leq A\frac{1}{p-1} \text{ if } 1 < p \leq 2,$$

for some $A := A(d,V) < \infty$.

Before starting the proof, we mention that the dependence on the p variable is not explicit in [**74**, Proposition 9.9], but can be recovered by keeping track of the constant p in the application of the Marcinkiewicz interpolation theorem. We also mention that the case of the logarithmic potential is not considered in [**74**, Lemma 7.12] (it is useful to obtain the estimate of the L^p norm of w in dimension 2). Nevertheless their proof is general enough to be applied in this setting.

Proof of Proposition 3.2.17. Let $\psi \in C_c^\infty\left(B_{\frac{1}{4}}, \mathbb{R}\right)$ and $2 \leq q < \infty$. We denote by p the conjugate exponent of q, i.e. $p := \frac{q}{q-1} \in (1,2]$. We split the proof into 5 steps.

- In Step 1, we show that there exists a constant $C := C(d,\psi) < \infty$ such that, for each $u \in W^{1,q}(B_1)$,
$$\|u - \psi * u\|_{W^{-1,q}\left(B_{\frac34}\right)} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}.$$
- In Step 2, we prove that there exists a constant $C := C(d,\psi) < \infty$ such that, for each $u \in W^{1,q}(B_1)$,
$$\|u - \psi * u(0)\|_{W^{-1,q}\left(B_{\frac34}\right)} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}. \tag{3.1.3}$$
- In Step 3, we prove that there exists a constant $C := C(d,q,\psi) < \infty$ such that, for each $u \in W^{1,q}(B_1)$,
$$\|u\|_{L^q\left(B_{\frac12}\right)} \leq C \|\nabla u\|_{W^{-1,q}(B_1)} + C\|u\|_{W^{-1,q}\left(B_{\frac34}\right)}$$
and that the constant C satisfies $C(d,\psi,q) \leq Aq$ for some $A := A(d,\psi) < \infty$.
- In Step 4, we show that there exists a constant $C := C(d,q,\psi) < \infty$ such that, for each $u \in W^{1,q}(B_1)$,
$$\left\|u - (u)_{B_{\frac12}}\right\|_{L^q\left(B_{\frac12}\right)} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}$$
and that the constant C satisfies $C(d,\psi,q) \leq Aq$ for some $A := A(d,\psi) < \infty$.
- In Step 5, we show that for each tempered distribution $u \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ and each $R > 0$,
$$\|u - (u)_{B_R}\|_{\underline{L}^q(B_R)} \leq C \left(\int_{\mathbb{R}^d} R^{-d} e^{-\frac{|x|}{2R}} \left(\int_0^{2R} r \left|\Phi_r * \nabla u(x)\right|^2 dr \right)^{\frac q2} \right)^{\frac1q}.$$

Step 1. We prove that there exists a constant $C := C(d) < \infty$ such that
$$\|u - u * \psi\|_{W^{-1,q}\left(B_{\frac34}\right)} \leq C \|\nabla u\|_{W^{-1,q}(B_1)}.$$
Let $\psi \in C_c^\infty\left(B_{\frac14}, \mathbb{R}\right)$ and define, for $n \in \mathbb{N}$,
$$\psi_n := 2^{-dn}\psi\left(\frac{\cdot}{2^n}\right).$$
Since $\psi_n * u \to u$ in $L^q\left(B_{\frac34}\right)$, we can use the triangle inequality to bound
$$\|u - \psi * u\|_{W^{-1,q}\left(B_{\frac34}\right)} \leq \sum_{n=0}^{\infty} \|\psi_{n+1} * u - \psi_n * u\|_{W^{-1,q}\left(B_{\frac34}\right)}. \tag{3.1.4}$$
Since the function $\psi_1 - \psi_0$ is compactly supported in $B_{\frac14}$ and of mean 0, we can apply [**18**, Lemma 5.7], to show that there exists a function $\Psi \in C_c^\infty\left(B_{\frac14}, \mathbb{R}\right)$ satisfying
$$\nabla \cdot \Psi = \psi_1 - \psi_0.$$
For each $n \in \mathbb{N}$, we denote
$$\Psi_n := 2^{-dn}\Psi\left(\frac{\cdot}{2^n}\right),$$
by scaling invariance we also have
$$2^{-n}\nabla \cdot \Psi_n = \psi_{n+1} - \psi_n.$$

For each function $g \in W_0^{1,p}\left(B_{\frac{3}{4}}\right)$, we have

$$\int_{\left(B_{\frac{3}{4}}\right)} (\psi_{n+1} - \psi_n) * u(x) g(x)\, dx = 2^{-n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla \cdot \Psi_n(x-y) u(y) g(x)\, dx dy$$
$$= 2^{-n} \int_{\mathbb{R}^d} \nabla u(y) \cdot \left(\int_{\mathbb{R}^d} \Psi_n(x-y) g(x)\, dx \right) dy.$$

By construction, the function $y \to \left(\int_{\mathbb{R}^d} \Psi_n(x-y) g(x)\, dx \right)$ is supported in B_1, we can thus estimate

$$\left| \int_{\left(B_{\frac{3}{4}}\right)} (\psi_{n+1} - \psi_n) * u(x) g(x)\, dx \right| \le 2^{-n} \left\| \left(\int_{\mathbb{R}^d} \Psi_n(x - \cdot) g(x)\, dx \right) \right\|_{W_0^{1,p}(B_1)} \|\nabla u\|_{W^{-1,q}(B_1)}.$$

Moreover, one can check that there exists a constant $C := C(d,\psi) < \infty$ such that

$$\left\| \left(\int_{\mathbb{R}^d} \Psi_n(x - \cdot) g(x)\, dx \right) \right\|_{W_0^{1,p}(B_1)} \le C \|g\|_{W_0^{1,p}(B_1)} = C \|g\|_{W_0^{1,p}\left(B_{\frac{3}{4}}\right)}.$$

Taking the supremum over $g \in W_0^{1,p}\left(B_{\frac{3}{4}}\right)$ of norm 1 and combining this with (3.1.4), we obtain

$$\|u - \psi * u\|_{W^{-1,q}\left(B_{\frac{3}{4}}\right)} \le C \|\nabla u\|_{W^{-1,q}(B_1)},$$

for some constant $C := C(d) < \infty$. The proof of Step 1 is complete.

Step 2. We split the norm

$$\|u - \psi * u(0)\|_{W^{-1,q}\left(B_{\frac{3}{4}}\right)} \le \|u - \psi * u\|_{W^{-1,q}\left(B_{\frac{3}{4}}\right)} + \|\psi * u - \psi * u(0)\|_{W^{-1,q}\left(B_{\frac{3}{4}}\right)}. \tag{3.1.5}$$

But note that, for each $x \in B_{\frac{3}{4}}$,

$$|\psi * u(x) - \psi * u(0)| \le C \|\nabla u\|_{W^{-1,q}(B_1)}. \tag{3.1.6}$$

The proof of this estimate is very similar to the previous step, only simpler: by [**18**, Lemma 5.7], we represent $\psi(\cdot - x) - \psi$ in the form

$$\nabla \Psi_x = \psi(\cdot - x) - \psi$$

with $\Psi_x \in C_c^\infty(B_1, \mathbb{R})$ and then prove (3.1.6) thanks to an integration by parts. From this we deduce

$$\|\psi * u - \psi * u(0)\|_{W^{-1,q}\left(B_{\frac{3}{4}}\right)} \le C \|\nabla u\|_{W^{-1,q}(B_1)}.$$

Combining this estimate with (3.1.5) and the estimate proved in the previous step completes the proof of Step 2.

Step 3. Let $\eta \in C_c^\infty(B_1)$ be a cutoff function satisfying

$$\mathbb{1}_{B_{\frac{1}{2}}} \le \eta \le \mathbb{1}_{B_{\frac{3}{4}}}, \qquad \text{and} \qquad \left|\nabla^2 \eta\right| + |\nabla \eta| \le C.$$

For any function $f \in L^p(B_1)$, we denote by w_f the Newtonian potential of f introduced in Proposition 3.A.3 with $\Omega = B_1$. We then compute

$$\int_{B_1} \eta(x) u(x) f(x)\, dx = \int_{B_1} \eta(x) u(x) \Delta w_f(x)\, dx$$
$$= \int_{B_1} \nabla \eta(x) u(x) \nabla w_f(x) + \eta(x) \nabla u(x) \nabla w_f(x)\, dx$$
$$\le \|u\|_{W^{-1,q}\left(B_{\frac{3}{4}}\right)} \left\|\nabla \eta \nabla w_f\right\|_{W_0^{1,p}\left(B_{\frac{3}{4}}\right)} + \|\nabla u\|_{W^{-1,q}\left(B_{\frac{3}{4}}\right)} \left\|\eta \nabla w_f\right\|_{W_0^{1,p}\left(B_{\frac{3}{4}}\right)}.$$

By the properties of η and by Proposition 3.A.3, we have

$$\left\|\nabla \eta \nabla w_f\right\|_{W_0^{1,p}\left(B_{\frac{3}{4}}\right)} + \left\|\eta \nabla w_f\right\|_{W_0^{1,p}\left(B_{\frac{3}{4}}\right)} = \left\|\nabla \eta \nabla w_f\right\|_{W_0^{1,p}(B_1)} + \left\|\eta \nabla w_f\right\|_{W_0^{1,p}(B_1)} \le C \|f\|_{L^p(B_1)},$$

for some constant $C := C(d,p,\eta) < \infty$ satisfying

$$C(d,p,\eta) \le A\frac{1}{p-1},$$

with $A := A(d,\eta) < \infty$. Consequently

$$\|u\|_{L^q\left(B_{\frac12}\right)} \le \|\eta u\|_{L^q(B_1)} = \sup_{f\in L^p(B_1), \|f\|_{L^p(B_1)}=1} \int_{B_1} \eta(x)u(x)f(x)\,dx$$

$$\le C\left(\|u\|_{W^{-1,q}\left(B_{\frac34}\right)} + \|\nabla u\|_{W^{-1,q}(B_1)}\right).$$

The proof of Step 3 is complete.

Step 4. Applying the main result of the previous step to the function $u - \psi * u(0)$, we have

$$\|u-\psi * u(0)\|_{L^q\left(B_{\frac12}\right)} \le C\left(\|u-\psi * u(0)\|_{W^{-1,q}\left(B_{\frac34}\right)} + \|\nabla u\|_{W^{-1,q}(B_1)}\right).$$

Then by Step 2, we obtain

$$\|u-\psi * u(0)\|_{L^q\left(B_{\frac12}\right)} \le C\,\|\nabla u\|_{W^{-1,q}(B_1)}.$$

But we have, for each $a\in\mathbb{R}$

$$\left\|u-(u)_{B_{\frac12}}\right\|_{L^q\left(B_{\frac12}\right)} \le 2\,\|u-a\|_{L^q\left(B_{\frac12}\right)}.$$

Thus

$$\left\|u-(u)_{B_{\frac12}}\right\|_{L^q\left(B_{\frac12}\right)} \le 2\inf_{a\in\mathbb{R}}\|u-a\|_{L^q\left(B_{\frac12}\right)} \le 2\,\|u-\psi * u(0)\|_{L^q\left(B_{\frac12}\right)}.$$

Combining the previous displays completes the proof of Step 4.

Step 5. Applying the result of Step 4 and Proposition 3.A.1, we obtain, for each $q\ge 2$ and each $u\in\mathcal{S}'\left(\mathbb{R}^d\right)\cap W^{1,q}_{\mathrm{loc}}\left(\mathbb{R}^d\right)$,

$$\left\|u-(u)_{B_{\frac12}}\right\|_{L^q\left(B_{\frac12}\right)} \le C\left(\int_{\mathbb{R}^d} e^{-|x|}\left(\int_0^1 r\left|\Phi_r * \nabla u(x)\right|^2 dr\right)^{\frac q2}\right)^{\frac1q}$$

For some constant $C := C(d,q)$ satisfying $C(d,q)\le Aq^{\frac32}$. Rescaling the previous estimates eventually shows

$$\|u-(u)_{B_R}\|_{\underline{L}^q(B_R)} \le C\left(\int_{\mathbb{R}^d} R^{-d}e^{-\frac{|x|}{2R}}\left(\int_0^{2R} r\left|\Phi_r * \nabla u(x)\right|^2 dr\right)^{\frac q2}\right)^{\frac1q}.$$

and the proof of Proposition 3.2.17 is complete. □

3.B. Proof of Lemma 3.3.8

In this appendix, we prove Lemma 3.3.8. We first restate the lemma.

LEMMA 3.3.8 (Minimal scale). *There exist a constant $C := C(d,\mathfrak{p},\lambda) < \infty$, an exponent $s := s(d,\mathfrak{p},\lambda) > 0$ and a random variable $\mathcal{M}_1 \le \mathcal{O}_s'(C)$ such that for each $m\in\mathbb{N}$ satisfying $3^m \ge \mathcal{M}_1$,*

$$3^{-dm}\sum_{z\in\square_m}\mathrm{size}\left(\square_{\mathcal{P}}(z)\right)^{\frac{3d(2+\varepsilon)}{\varepsilon}}\left(\sum_{x\in\mathbb{Z}^d,\,\mathrm{dist}(\square_{\mathcal{P}}(x),\square_{\mathcal{P}}(z))\le 1,\, e\in\mathcal{B}_d^x}\left(1+\mathcal{X}^e(x)\right)^d\right)^{\frac{2+\varepsilon}{\varepsilon}} \le C$$

where $\varepsilon := \varepsilon(d, \mathfrak{p}, \lambda) > 0$ *is the exponent which appears in Proposition 3.C.2.*

Proof of Lemma 3.3.8. First, notice that one can rewrite

$$\begin{aligned} & 3^{-dm} \sum_{z \in \square_m} \operatorname{size}(\square_{\mathcal{P}}(z))^{\frac{3d(2+\varepsilon)}{\varepsilon}} \left(\sum_{x \in \mathbb{Z}^d,\, \operatorname{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1,\, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^d \right)^{\frac{2+\varepsilon}{\varepsilon}} \\ & \quad \leq C 3^{-dm} \sum_{z \in \square_m} \operatorname{size}(\square_{\mathcal{P}}(z))^{\frac{3d(2+\varepsilon)+2}{\varepsilon}} \sum_{x \in \mathbb{Z}^d,\, \operatorname{dist}(\square_{\mathcal{P}}(x), \square_{\mathcal{P}}(z)) \leq 1,\, e \in \mathcal{B}_d^x} (1 + \mathcal{X}^e(x))^{d \frac{2+\varepsilon}{\varepsilon}} \\ & \quad \leq C 3^{-dm} \sum_{x \in \mathbb{Z}^d,\, \operatorname{dist}(\square_{\mathcal{P}}(x), \square_m) \leq 1,\, e \in \mathcal{B}_d^x} \operatorname{size}(\square_{\mathcal{P}}(x))^{\frac{3d(2+\varepsilon)+2}{\varepsilon}} (1 + \mathcal{X}^e(x))^{d \frac{2+\varepsilon}{\varepsilon}} \end{aligned}$$

By (iv) of Proposition 3.2.1 applied with $t = \frac{6d(2+\varepsilon)+4}{\varepsilon}$, it is clear that for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}_t(\mathcal{P})$, we have

(1) $\sup_{x \in \square_{m+1}} \operatorname{size}(\square_{\mathcal{P}}(x)) \leq 3^{\frac{dm}{d+t}}$, this implies in particular

$$\left\{ x \in \mathbb{Z}^d,\, \operatorname{dist}(\square_{\mathcal{P}}(x), \square_m) \leq 1 \right\} \subseteq \square_{m+1}.$$

(2) the following estimate

$$\begin{aligned} \left(3^{-dm} \sum_{x \in \mathbb{Z}^d,\, \operatorname{dist}(\square_{\mathcal{P}}(x), \square_m) \leq 1} \operatorname{size}(\square_{\mathcal{P}}(x))^{\frac{6d(2+\varepsilon)+4}{\varepsilon}} \right)^{\frac{1}{2}} & \leq C \left(3^{-d(m+1)} \sum_{x \in \square_{m+1}} \operatorname{size}(\square_{\mathcal{P}}(x))^{\frac{6d(2+\varepsilon)+4}{\varepsilon}} \right)^{\frac{1}{2}} \\ & \leq C. \end{aligned}$$

Thus by the Cauchy-Schwarz inequality, it is enough to prove that there exists a random variable $\mathcal{M}$ satisfying $\mathcal{M} \leq \mathcal{O}_s'(C)$, such that for each $m \in \mathbb{N}$ satisfying $3^m \geq \mathcal{M}$,

$$3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} (\mathcal{X}^e(x))^{\frac{d(4+2\varepsilon)}{\varepsilon}} \leq C. \tag{3.2.1}$$

Unfortunately, we cannot prove this exact statement but we will prove a slightly weaker estimate, Lemma 3.B.1, which is still strong enough to prove Proposition 3.3.3. Define, for each $C > 0$, the random variable

$$\begin{aligned} \mathcal{X}_C := \inf \Big\{ r \in [1, \infty) \,:\, \forall r', R' \in [r, \infty),\ \text{with } r' \leq R',\ \forall u \in \mathcal{A}(\mathscr{C}_\infty \cap B_{R'}) \\ \left\| \nabla u \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_{r'})} \leq C \frac{r'}{R'} \left\| \nabla u \right\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_{R'})} \Big\}, \end{aligned}$$

and we similarly define, for each $x \in \mathbb{Z}^d$,

$$\mathcal{X}_C(x) := \mathcal{X}_C \circ \tau_x.$$

Denote by $C_0 := C_0(d, \mathfrak{p}\lambda) < \infty$ the constant appearing in Proposition 3.2.13. By definition we have

$$\mathcal{X}_{C_0} = \mathcal{X}.$$

Note also that $\mathcal{X}_C$ is decreasing in C. With this new notation in mind, we have the following lemma.

Lemma 3.B.1. *For every integrability parameter* $t > 0$*, there exist a constant* $C(d, \mathfrak{p}, \lambda, t) < \infty$*, an exponent* $s(d, \mathfrak{p}, \lambda, t) > 0$ *and a random variable* $\mathcal{M}_t^{\mathcal{X}}$ *satisfying*

$$\mathcal{M}_t^{\mathcal{X}} \leq \mathcal{O}_s'(C)$$

such that for every $m \in \mathbb{N}$ *satisfying*

$$3^m \geq \mathcal{M}_t^{\mathcal{X}}$$

the following inequality holds

$$3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{C_0^2}(x) \right|^t \leq C.$$

REMARK 3.B.2. (1) This statement is weaker than (3.2.1) since, for each $x \in \mathbb{Z}^d$ and $e \in \mathcal{B}_d^x$,

$$\mathcal{X}^e_{C_0^2}(x) \leq \mathcal{X}^e_{C_0}(x) = \mathcal{X}^e(x).$$

Nevertheless it is enough to prove Result 2, since we only need to replace C_0 by C_0^2 in every computation involving the estimates on the random variables $\chi_p^e(x)$ and the result remain the same, only the value of the constants will be increased.

(2) Applying this result with $t = \frac{d(4+2\varepsilon)}{\varepsilon}$ completes the proof of Lemma 3.3.8.

□

There remains to prove Lemma 3.B.1 but before starting the proof, we need to introduce a few ingredients and preliminary results. First define, for $R, C \in [1, \infty)$, the random variable $\mathcal{X}_{R,C}$ by the formula,

$$\mathcal{X}_{R,C} := \inf \Big\{ r \in [1, R] \,:\, \forall r', R' \in [r, R], \text{ with } r' \leq R',\, \forall u \in \mathcal{A}(\mathscr{C}_{\max}(B_R) \cap B_{R'}) \\ \left\| \nabla u \right\|_{\underline{L}^2(\mathscr{C}_{\max}(B_R) \cap B_{r'})} \leq C \frac{r'}{R'} \left\| \nabla u \right\|_{\underline{L}^2(\mathscr{C}_{\max}(B_R) \cap B_{R'})} \Big\}, \tag{3.2.2}$$

Where $\mathscr{C}_{\max}(B_R)$ denotes the largest cluster contained in B_R. Similarly we define, for each $x \in \mathbb{Z}^d$,

$$\mathcal{X}_C(x) := \mathcal{X}_C \circ \tau_x.$$

Note that this random variable is defined on the enlarged probability space $\Omega \times \Omega$ and is measurable with respect to $\mathcal{F}(x + B_R) \otimes \{\varnothing, \Omega\}$ (it depends on the edges in the ball $x + B_R$ of the first variable and does not depend on the edges of the second variable).

The reason why we were careful to write $\mathscr{C}_{\max}(B_R)$ in (3.2.2) and not $\mathscr{C}_*(B_R)$ or $\mathscr{C}_\infty \cap B_R$ (these three clusters are morally equal for large R), is to constrain the random variable $\mathcal{X}_{R,C}$ be measurable with respect to $\mathcal{F}(B_R) \otimes \{\varnothing, \Omega\}$.

The only incentive of this quantity is that the random variable $X_{R,C}$ is local (or is measurable with respect to $\mathcal{F}(B_R) \otimes \{\varnothing, \Omega\}$) and in particular the random variables $\mathcal{X}_{R,C}(x)$ and $\mathcal{X}_{R,C}(y)$ are independent as soon as $|x - y| > 2R$.

Note also that $\mathcal{X}_{R,C}$ is decreasing in the C variable and, for $R \geq \mathcal{M}_t(\mathcal{P})$, it is increasing in the R variable. We thus denote by, for each $C \geq 1$

$$\mathcal{X}_C := \lim_{R \to \infty} \mathcal{X}_{R,C} = \limsup_{R \geq 1} \mathcal{X}_{R,C} \in [1, \infty].$$

By Proposition 3.2.13, we know that there exists a constant $C_0 := C_0(d, \mathfrak{p}, \lambda) < \infty$ such that

$$\mathcal{X}_{C_0} = \mathcal{X} \leq \mathcal{O}'_s(C). \tag{3.2.3}$$

thus, for each $R > \mathcal{M}_t(\mathcal{P})$,

$$\mathcal{X}_{R,C_0} \leq \mathcal{X}_{C_0} \leq \mathcal{O}'_s(C).$$

Moreover, for each $R \in [1, \mathcal{M}_t(\mathcal{P})]$, we have

$$\mathcal{X}_{R,C_0} \leq \mathcal{M}_t(\mathcal{P}) \leq \mathcal{O}'_s(C).$$

Combining the two previous displays yields, for each $R \geq 1$,

$$\mathcal{X}_{R,C_0} \leq \mathcal{O}'_s(C).$$

We now prove the following inequality, for each $R, C > 1$,

$$\mathcal{X}_{C^2} \leq \mathcal{X}_{R,C} + R \mathbb{1}_{\{R \leq \mathcal{M}_t(\mathcal{P})\}} + \mathcal{X}_C \mathbb{1}_{\{\mathcal{X}_C > R\}}. \tag{3.2.4}$$

We split the proof of this inequality into two cases.

Case 1. If $\mathcal{X}_C > R$, then since $C \geq 1$ and $\mathcal{X}_C$ is decreasing in C, the inequality (3.2.4) follows from the computation

$$\mathcal{X}_{C^2} \leq \mathcal{X}_C \leq \mathcal{X}_{R,C} + R\mathbb{1}_{\{R \leq \mathcal{M}_t(\mathcal{P})\}} + \mathcal{X}_C \mathbb{1}_{\{\mathcal{X}_C > R\}}.$$

Case 2. If $\mathcal{X}_C \leq R$ and $R \leq \mathcal{M}_t(\mathcal{P})$, then

$$\mathcal{X}_{C^2} \leq R\mathbb{1}_{\{R \leq \mathcal{M}_t(\mathcal{P})\}} \leq \mathcal{X}_{R,C} + R\mathbb{1}_{\{R \leq \mathcal{M}_t(\mathcal{P})\}} + \mathcal{X}_C \mathbb{1}_{\{\mathcal{X}_C > R\}}.$$

Case 3. If $\mathcal{X}_C \leq R$ and $R \geq \mathcal{M}_t(\mathcal{P})$ then $\mathscr{C}_{\max}(B_R)$ is equal to the maximal connected component of $\mathscr{C}_\infty \cap B_R$ and we have, for each $r, R' > R$ with $R' \geq r$

$$\|\nabla u\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \leq C \frac{r}{R'} \|\nabla u\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_{R'})}.$$

Moreover, for each $r, R' \in [\mathcal{X}_{R,C}, R]$ with $R' \geq r$, we have

$$\|\nabla u\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \leq C \frac{r}{R'} \|\nabla u\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_{R'})}.$$

Recall that we picked C under the assumption $C \geq 1$ so that $C^2 \geq C$. Combining the two previous displays yields for each $r, R' \geq \mathcal{X}_{R,C}$ with $R' \geq r$,

$$\|\nabla u\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_r)} \leq C^2 \frac{r}{R'} \|\nabla u\|_{\underline{L}^2(\mathscr{C}_\infty \cap B_{R'})}$$

and thus by definition of $\mathcal{X}_{C^2}$,

$$\mathcal{X}_{C^2} \leq \mathcal{X}_{R,C}$$

and the proof of the inequality (3.2.4) is complete.

For $x \in \mathbb{Z}^d, e = \{x,y\} \in \mathcal{B}_d, C, R \in [1,\infty)$, denote by $\mathcal{X}^e_{R,C}(x)$ the translated and resampled random variable

$$\mathcal{X}^e_{R,C}(x) := \inf \Big\{ r \in [1,R] : \text{such that } \forall 1 \leq r' \leq R' \leq R,\ u \in \mathcal{A}^e(\mathscr{C}^e_{\max}(B_R(x)) \cap B_{R'}(x)) \\ \|\nabla u\|_{\underline{L}^2(\mathscr{C}^e_{\max}(B_R) \cap B_{r'}(x))} \leq C \frac{r'}{R'} \|\nabla u\|_{\underline{L}^2(\mathscr{C}^e_{\max}(B_R(x)) \cap B_{R'}(x))} \Big\}.$$

We also define, for each $x \in \mathbb{Z}^d$

$$\mathcal{X}^e_C(x) := \lim_{R \to \infty} \mathcal{X}^e_{R,C}(x) = \limsup_{R \geq 1} \mathcal{X}^e_{R,C}(x) \in [1,\infty].$$

The second ingredient in the proof of Lemma 3.B.1 is the following minimal scale lemma. It is a slight modification of [**13**, Lemma 2.3] and will be used at the very end of the proof of Lemma 3.B.1.

Lemma 3.B.3. *Fix $K \geq 1$, $s > 0$ and $\beta > 0$ and suppose that $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of random variables satisfying, for every $n \in \mathbb{N}$,*

$$X_n \leq K + \mathcal{O}_s\left(K 3^{-n\beta}\right).$$

Then there exists $C(s,\beta,K) < \infty$ such that the random scale

$$M := \sup\{3^n \in \mathbb{N} : X_n \geq K + 1\}$$

satisfies the estimate

$$M \leq \mathcal{O}_{s\beta}(C).$$

Proof. This result can be deduced by applying [**13**, Lemma 2.3] to the sequence of random variables $X'_n = \max(X_n - K, 0)$. □

We now turn to the proof of Lemma 3.B.1.

Proof of Lemma 3.B.1. Fix $t \in (0, \infty)$ and $m, n \in \mathbb{N}$ with $m > n$. Using (3.2.4), we have

$$
\begin{aligned}
(3.2.5) \qquad & 3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{C_0^2}(x) \right|^t \\
& \leq C 3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{3^n, C_0}(x) \right|^t + C 3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{C_0}(x) \right|^t \mathbb{1}_{\left\{ \mathcal{X}^e_{C_0}(x) > 3^n \right\}} \\
& \quad + C 3^{-dm} \sum_{x \in \square_m} 3^{tn} \mathbb{1}_{\{3^n \leq \mathcal{M}_t(\mathcal{P})\}} \circ \tau_x .
\end{aligned}
$$

Since $\mathcal{X}^e_{C_0}(x) \leq \mathcal{O}_s'(C)$, for every $t, t' > 0$, there exist an exponent $s'(d, \mathfrak{p}, \lambda, t, t') > 0$ and a constant $C'(d, \mathfrak{p}, \lambda, t, t') < \infty$ such that

$$
3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{C_0}(x) \right|^t \mathbb{1}_{\left\{ \mathcal{X}^e_{C_0}(x) > 3^n \right\}} \leq \mathcal{O}_{s'}'(C' 3^{-nt'})
$$

and

$$
3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} 3^{nt} \mathbb{1}_{\{3^n \geq \mathcal{M}_t(\mathcal{P})\}} \circ \tau_x \leq \mathcal{O}_{s'}'(C' 3^{-nt'}).
$$

Combining the previous displays yields

$$
3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{C_0^2}(x) \right|^t \leq C 3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{3^n, C_0}(x) \right|^t + \mathcal{O}_{s'}'(C' 3^{-nt'}).
$$

Moreover, notice that by definition of the localized random variable $\mathcal{X}^e_{3^n, C_0}(x)$, we have for each $x \in \mathbb{Z}^d$

$$
\sum_{e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{3^n, C_0}(x) \right|^t \leq 2d \times 3^{nt}.
$$

The proof of the lemma is then the same as the proof of Steps 1 and 2 of [**13**, Proposition 2.1] with $3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{C_0^2}(x) \right|^t dx$ instead of $\Lambda_t(z + \square_m, \mathcal{S})$ and $3^{-dm} \sum_{x \in z + \square_n,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{3^n, C_0^2}(x) \right|^t dx$ instead of $\Lambda_t(z' + \square_n, \mathcal{S}_{\mathrm{loc}}(z'))$. We rewrite it for completeness.

We denote

$$
Z := 3^{-dm} \sum_{x \in \square_m,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{3^n, C_0}(x) \right|^t = \frac{|\square_n|}{|\square_m|} \sum_{z \in 3^n \mathbb{Z}^d \cap \square_m} 3^{-dm} \sum_{x \in z + \square_n,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{3^n, C_0}(x) \right|^t .
$$

We first prove that there exists a constant $C := C(d, \mathfrak{p}, \lambda, t) < \infty$ such that

$$
(3.2.6) \qquad Z \leq C + \mathcal{O}_1'\left(3^{nt - d(m-n)} \right).
$$

To do so, choose an enumeration $\{z^j : 1 \leq j \leq 3^{d(m-n-2)}\}$ of the elements of the set $3^{n+2}\mathbb{Z}^d \cap \square_m$. Next, for each $1 \leq j \leq 3^{d(m-n-2)}$, we let $\{z^{i,j} : 1 \leq i \leq 3^{2d}\}$ be an enumeration of the elements of the set $3^n \mathbb{Z}^d \cap \left(z^j + \square_{n+2} \right)$, such that, for every $1 \leq j, j' \leq 3^{d(m-n-2)}$ and $1 \leq i \leq 3^{2d}$, $z^j - z^{j'} = z^{i,j} - z^{i,j'}$. The point of this is that, for every $1 \leq i \leq 3^{2d}$ and $1 \leq j < j' \leq 3^{d(m-n-2)}$, we have $\mathrm{dist}\left(z^{i,j} + \square_n, z^{i,j'} + \square_n \right) \geq 3^{n+1}$ and therefore, $3^{-dm} \sum_{x \in z^{i,j} + \square_n,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{3^n, C_0^2}(x) \right|^t$ and $3^{-dm} \sum_{x \in z^{i,j'} + \square_n,\, e \in \mathcal{B}_d^x} \left| \mathcal{X}^e_{3^n, C_0^2}(x) \right|^t$ are independent. Now fix $h > 0$ and compute, using the Hölder inequality and the independence

$$\begin{aligned}
&\log \mathbb{E}\left[\exp\left(h3^{-nt}Z\right)\right]\\
&\quad = \log \mathbb{E}\left[\prod_{i=1}^{3^{2d}}\prod_{j=1}^{3^{d(m-n-2)}} \exp\left(h3^{-nt-d(m-n)}3^{-dm}\sum_{x\in z^{i,j}+\square_n,\, e\in\mathcal{B}_d^x}\left|\mathcal{X}^e_{3^n,C_0^2}(x)\right|^t\right)\right]\\
&\quad \le 3^{-2d}\sum_{i=1}^{3^{2d}}\log \mathbb{E}\left[\prod_{j=1}^{3^{d(m-n-2)}} \exp\left(h3^{-nt-d(m-n-2)}3^{-dm}\sum_{x\in z^{i,j}+\square_n,\, e\in\mathcal{B}_d^x}\left|\mathcal{X}^e_{3^n,C_0^2}(x)\right|^t\right)\right]\\
&\quad \le 3^{-2d}\sum_{i=1}^{3^{2d}}\sum_{j=1}^{3^{d(m-n-2)}}\log \mathbb{E}\left[\exp\left(h3^{-nt-d(m-n-2)}3^{-dm}\sum_{x\in z^{i,j}+\square_n,\, e\in\mathcal{B}_d^x}\left|\mathcal{X}^e_{3^n,C_0^2}(x)\right|^t\right)\right].
\end{aligned}$$

This inequality can be rewritten

$$\begin{aligned}
&\log \mathbb{E}\left[\exp\left(h3^{-nt}Z\right)\right]\\
&\qquad \le 3^{-2d}\sum_{z'\in 3^n\mathbb{Z}^d\cap(z+\square_m)}\log \mathbb{E}\left[\exp\left(h3^{-nt-d(m-n-2)}3^{-dm}\sum_{x\in z'+\square_n,\, e\in\mathcal{B}_d^x}\left|\mathcal{X}^e_{3^n,C_0^2}(x)\right|^t\right)\right].
\end{aligned}$$

Next we use the elementary inequality

$$\forall y\in[0,1],\quad \exp(y)\le 1+2y$$

to get, for every $h\in[0,(2d)^{-t}3^{d(m-n-2)}]$,

$$\exp\left(h3^{-nt-d(m-n-2)}\sum_{x\in z'+\square_n,\, e\in\mathcal{B}_d^x}\left|\mathcal{X}^e_{3^n,C_0^2}(x)\right|^t\right)\le 1+2h3^{-nt-d(m-n-2)}\sum_{x\in z'+\square_n,\, e\in\mathcal{B}_d^x}\left|\mathcal{X}^e_{3^n,C_0^2}(x)\right|^t.$$

Taking the expectation in the previous display and using the elementary inequality

$$\forall y\ge 0,\ \log(1+y)\le y,$$

we get

$$\begin{aligned}
\log \mathbb{E}\left[\exp\left(h3^{-nt}Z\right)\right] &\le 3^{d(m-n)}\log\left(1+2h3^{-nt-d(m-n-1)}\mathbb{E}\left[\sum_{x\in z'+\square_n,\, e\in\mathcal{B}_d^x}\left|\mathcal{X}^e_{3^n,C_0^2}(x)\right|^t\right]\right)\\
&\le 2h3^{-nt+d}\mathbb{E}\left[\sum_{x\in z'+\square_n,\, e\in\mathcal{B}_d^x}\left|\mathcal{X}^e_{3^n,C_0^2}(x)\right|^t\right]\\
&\le Ch3^{-nt}.
\end{aligned}$$

Taking $h:=(2d)^{-t}3^{d(m-n-2)}$ yields

$$\mathbb{E}\left[\exp\left((2d)^{-t}3^{d(m-n-2)-nt}Z\right)\right]\le \exp\left(C3^{d(m-n)-nt}\right).$$

From this and Chebyshev's inequality, we obtain a constant C such that

$$\mathbb{P}\left[Z\ge C+h\right]\le \exp\left(-hC^{-1}3^{d(m-n)-nt}\right)$$

This implies (3.2.6).

Step 2. We complete the proof by applying a union bound. Combining (3.2.5) and (3.2.6) yields

$$\sum_{x\in\square_m,\, e\in\mathcal{B}_d^x}\left|\mathcal{X}^e_{3^n,C_0}(x)\right|^t\le C+\mathcal{O}_1\left(C3^{nt-d(m-n)}\right)+\mathcal{O}_{s'}\left(C3^{-nt'}\right).$$

We set

$$n:=\left\lceil\frac{dm}{d+t+1}\right\rceil \text{ and } t'=1$$

so that the previous line becomes

$$\sum_{x\in\square_m,\, e\in\mathcal{B}_d^x} \left|\mathcal{X}^e_{3^n,C_0}(x)\right|^t dx \le C + \mathcal{O}_1'\left(C3^{-\frac{d}{d+t+1}m}\right) + \mathcal{O}_{s'}'\left(C3^{-\frac{d}{d+t+1}m}\right).$$

Thus, by (3.1.13) and (3.1.9), we obtain the existence of two exponents $s := s(d,\mathfrak{p},\lambda,t) > 0$, $\beta := \beta(d,\mathfrak{p},\lambda,t) > 0$ and of a constant $C_0 := C_0(d,\mathfrak{p},\lambda,t) < \infty$ such that

$$\sum_{x\in\square_m,\, e\in\mathcal{B}_d^x} \left|\mathcal{X}^e_{3^n,C_0}(x)\right|^t dx \le C_0 + \mathcal{O}_s'\left(C_0 3^{-\beta m}\right).$$

Define

$$\mathcal{M}_t^{\mathcal{X}} := \sup\left\{3^m \,:\, \sum_{x\in\square_m,\, e\in\mathcal{B}_d^x} \left|\mathcal{X}^e_{3^n,C_0}(x)\right|^t dx \ge C_0 + 1\right\}$$

We want to prove

$$\mathcal{M}_t^{\mathcal{X}} \le \mathcal{O}_{s\beta}(C)$$

This is exactly Lemma 3.B.3 with $X_n = \sum_{x\in\square_m,\, e\in\mathcal{B}_d^x} \left|\mathcal{X}^e_{3^n,C_0}(x)\right|^t dx$ and $K = C_0$. □

3.C. Elliptic inequalities on the supercritical percolation cluster

In this section, we record some simple elliptic inequalities, the Caccioppoli inequality and the Meyers estimate. These inequalities were written in [**13**] for harmonic functions. In our context, we need to apply these results when the right-hand term is not 0 but the divergence of a vector field.

Proposition 3.C.1 (Caccioppoli inequality). *Assume that we are given a function* $u : \mathscr{C}_\infty \to \mathbb{R}$ *and a vector field* $\xi : E_d \to \mathbb{R}$ *satisfying the following condition*

$$\xi(x,y) = 0 \text{ if } \mathbf{a}(x,y) = 0 \text{ or } x,y \notin \mathscr{C}_\infty. \tag{3.3.1}$$

In particular, gradients of functions defined on the infinite cluster satisfy this condition by (3.1.5)*. Assume additionally that* u *and* ξ *satisfy the following equation,*

$$-\nabla\cdot(\mathbf{a}\nabla u) = -\nabla\cdot\xi \text{ in } \mathscr{C}_\infty.$$

Select two bounded sets $U, V \subseteq \mathbb{Z}^d$ *such that* $V \subseteq U$ *and* $\operatorname{dist}(V,\partial U) \ge r \ge 1$*. Then there exists* $C(\lambda) < \infty$ *such that*

$$\int_{\mathscr{C}_\infty\cap V} |\nabla u|^2(x)\,dx \le \frac{C}{r^2}\int_{\mathscr{C}_\infty\cap(U\setminus V)} |u(x)|^2\,dx + C\int_{\mathscr{C}_\infty\cap U}|\xi|^2(x)\,dx. \tag{3.3.2}$$

Proof. The strategy of the proof follows the standard technique to prove the Caccioppoli inequality, we select a cutoff function $\eta : \mathbb{Z}^d \to \mathbb{R}$ satisfying

$$\mathbb{1}_V \le \eta \le 1,\ \eta \equiv 0 \text{ on } \mathbb{R}^d \setminus U, \text{ and } \forall x,y \in \mathbb{Z}^d \text{ such that } x \sim y,\ |\eta(x)-\eta(y)|^2 \le \frac{C(\eta(x)+\eta(y))}{r^2}, \tag{3.3.3}$$

test the equation satisfied by u with ηu and perform a straightforward computation as well as the usual simplifications. □

The second important elliptic estimate needed in this article is the Meyers estimate. This estimate was also proved in [**13**] in the case of **a**-harmonic functions.

Proposition 3.C.2 (Meyers estimate). *There exist a constant* $C := C(d,\lambda,\mathfrak{p}) < \infty$*, two exponents* $s := s(d,\lambda,\mathfrak{p}) > 0$ *and* $\varepsilon := \varepsilon(d,\lambda,\mathfrak{p}) > 0$ *and a random variable* $\mathcal{M}_{\mathrm{Meyers}} \le \mathcal{O}_s(C)$ *such that for each* $m \in \mathbb{N}$ *with* $3^m \ge \mathcal{M}_{\mathrm{Meyers}}$*, and each function* $v : \mathscr{C}_\infty \to \mathbb{R}$ *satisfying*

$$-\nabla\cdot(\mathbf{a}\nabla v) = -\nabla\cdot\xi \text{ in } \mathscr{C}_\infty,$$

for some vector field $\xi : E_d \to \mathbb{R}$ *satisfying* (3.3.1), *the following estimate holds,*

$$\left(\frac{1}{|\square_m|} \int_{\square_m \cap \mathscr{C}_\infty} |\nabla v|^{2+\varepsilon} (x) \, dx \right)^{\frac{1}{2+\varepsilon}} \leq C \left(\frac{1}{\left|\frac{4}{3}\square_m\right|} \int_{\frac{4}{3}\square_m \cap \mathscr{C}_\infty} |\nabla v|^2 (x) \, dx \right)^{\frac{1}{2}} + C \left(\frac{1}{\left|\frac{4}{3}\square_m\right|} \int_{\frac{4}{3}\square_m \cap \mathscr{C}_\infty} |\xi|^{2+\varepsilon} (x) \, dx \right)^{\frac{1}{2+\varepsilon}} . \tag{3.3.4}$$

Proof of Proposition 3.C.2. The results of Proposition 3.8 and Definition 3.9 of [**13**] can be adapted in our context to prove (3.3.4). The Meyers estimate is indeed a consequence of the three following ingredients: the Caccioppoli inequality, the Sobolev inequality and the Gehring's lemma. But Proposition 3.C.1 provides a version of the Caccioppoli inequality well-suited to deal with a divergence form right-hand side. The Sobolev inequality is valid for any functions. The usual version of the Gehring's Lemma, see for instance Theorem 6.6 & Corollary 6.1 of [**77**], is general enough to be applied in our context.

□

CHAPTER 4

Quantitative Homogenization of Differential Forms

We develop a quantitative theory of stochastic homogenization in the more general framework of differential forms. Inspired by recent progress in the uniformly elliptic setting, the analysis relies on the study of certain subadditive quantities. We establish an algebraic rate of convergence from these quantities and deduce from this an algebraic error estimate for the homogenization of the Dirichlet problem. Most of the ideas needed in this article comes from two distinct theories, the theory of quantitative stochastic homogenization, and the generalization of the main results of functional analysis and of the regularity theory of second-order elliptic equations to the setting of differential forms.

This chapter corresponds to the article [**52**].

Contents

4.1. Introduction

The classical theory of stochastic homogenization focuses on the study of the second-order elliptic equation

$$\nabla \cdot (\mathbf{a}(x) \nabla u) = 0, \tag{4.1.1}$$

where $\mathbf{a}$ is a random, rapidly oscillating, uniformly elliptic coefficient field with law $\mathbb{P}$. The basic qualitative result roughly states that, under appropriate assumptions on $\mathbb{P}$, a solution u_r of (4.1.1) in $B(0,r)$, the ball of center 0 and radius r, converges as $r \to \infty$, $\mathbb{P}$-a.s, to a solution $\overline{u}_r$ of the equation

$$\nabla \cdot (\bar{\mathbf{a}} \nabla \overline{u}_r) = 0, \tag{4.1.2}$$

where $\bar{\mathbf{a}}$ is a constant, symmetric, definite-positive matrix, in the sense that

$$\frac{1}{r^d} \int_{B(0,r)} |u_r(x) - \overline{u}_r(x)|^2 \, dx \underset{r \to \infty}{\longrightarrow} 0. \tag{4.1.3}$$

This second equation (4.1.2) is frequently called the homogenized equation. Obtaining quantitative information, for instance rates of convergence in (4.1.3), drew a lot of attention in the recent years, and there has been some notable progress, in particular by the works of Armstrong, Kuusi, Mourrat and Smart [**21, 20, 17, 18**] and the works of Gloria, Neukamm and Otto [**81, 82, 83, 85**]. Quantitative rates of convergence are also interesting in particular because they can provide information on the performance of numerical algorithms for the computation of the homogenized coefficients [**129**].

The purpose of this article is to develop a theory of quantitative stochastic homogenization for the more general equation

$$\mathrm{d}\left(\mathbf{a}(x)\mathrm{d}u\right) = 0, \tag{4.1.4}$$

where u is an r-form, d is the exterior derivative and $\mathbf{a}$ is a random, rapidly oscillating tensor which maps the space of r-forms into the space of $(d-r)$-forms, satisfying some suitable properties which will be described below. When $r = 0$, u is a 0-form, that is to say a function, the differential equation (4.1.4) reduces to (4.1.1) and we recover the classical theory of stochastic homogenization.

The main result of this article, Theorem 4.1.2 below, is to prove a quantitative homogenization theorem for differential forms, i.e a quantitative version of (4.1.3) for differential forms. In our last main result, stated in Theorem 4.1.3 below, we prove that homogenization commutes with the natural duality structure of differential forms. This duality structure is behind certain exact formulas for the homogenized matrix which are known to hold in dimension $d = 2$ (see for instance [**98**, Chapter 1]). We note that similar results were obtained independently by Serre [**143**] in the case of periodic coefficients.

Note that the system (4.1.4), under natural assumptions on the coefficient field $\mathbf{a}$, is elliptic but not uniformly elliptic (since the operator vanishes on every closed form). To our knowledge, the results in this paper are the first quantitative stochastic homogenization estimates for such degenerate elliptic systems. The proof of our main results are based on an adaptation of the theory of quantitative stochastic homogenization developed in [**18**].

4.1.1. Notations and assumptions. In this section, we introduce the main notation and assumptions needed in this paper as well as a statement of the main theorems, Theorems 4.1.1 and 4.1.2.

4.1.1.1. *General Notations and Definitions.* We begin by recalling some definitions and recording some properties about differential forms which will be useful in this article. We consider the space $\mathbb{R}^d$ for some positive integer d, equipped with the standard $|\cdot|$. Denote by $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$ the canonical basis of $\mathbb{R}^d$. A cube of $\mathbb{R}^d$, generally denoted by $\square$, is a set of the form

$$z + R(-1,1)^d. \tag{4.1.5}$$

Given a cube $\square := z + R(-1,1)^d$, we also denote by $\mathrm{size}(\square)$ the size of the cube, in this case $\mathrm{size} = R$. A triadic cube of $\mathbb{R}^d$ is a cube of the specific form

$$z + \left(-\frac{3^m}{2}, \frac{3^m}{2}\right)^d, \, m \in \mathbb{N}, z \in 3^m\mathbb{Z}^d.$$

We use the notation, for $m \in \mathbb{N}$,

$$\square_m := \left(-\frac{3^m}{2}, \frac{3^m}{2}\right)^d.$$

If U is a measurable subset of $\mathbb{R}^d$, we denote its Lebesgue measure by $|U|$. The normalized integral for a function $u : U \to \mathbb{R}$ for a measurable subset $U \subseteq \mathbb{R}^d$ is denoted by

$$\fint_U u(x)\,\mathrm{d}x := \frac{1}{|U|}\int_U u(x)\,\mathrm{d}x.$$

Given two sets $U, V \subseteq \mathbb{R}^d$, we denote by $\mathrm{dist}(U,V) := \inf_{x\in U, y\in V}|x-y|$.

For $0 \le r \le d$, we denote by $\Lambda^r(\mathbb{R}^d)$ the space of r-linear forms. This is a vector space of dimension $\binom{d}{r}$, a canonical basis is given by

$$\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_r}, \; 1 \le i_1 < \ldots < i_r \le d.$$

We will denote by

$$\mathrm{d}x_I := \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_r}, \text{ for } I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, d\}.$$

Given U an open subset of $\mathbb{R}^d$, a differential form is a map

$$u : \begin{cases} U \to \Lambda^r(\mathbb{R}^d), \\ x \to \sum_{|I|=r} u_I(x) \mathrm{d}x_I. \end{cases}$$

Given $\xi := \xi_1 \mathfrak{e}_1 + \cdots + \xi_d \mathfrak{e}_d \in \mathbb{R}^d$, we denote by $\mathrm{d}\xi := \xi_1 \mathrm{d}x_1 + \cdots + \xi_d \mathrm{d}x_d \in \Lambda^1\left(\mathbb{R}^d\right)$.

In practice, we need to assume some regularity on u, so we introduce the following spaces.

- The space of smooth differential forms on U up to the boundary, denoted by $C^\infty \Lambda^r(U)$, i.e,
$$C^\infty \Lambda^r(U) := \left\{ u = \sum_{|I|=r} u_I(x) \mathrm{d}x_I \ : \ \forall I,\ u_I \in C^\infty(U) \right\}.$$
- The space of compactly supported smooth differential forms on U, denoted by $C_c^\infty \Lambda^r(U)$, i.e,
$$C_c^\infty \Lambda^r(U) := \left\{ u = \sum_{|I|=r} u_I(x) \mathrm{d}x_I \ : \ \forall I,\ u_I \in C_c^\infty(U) \right\}.$$
With this definition in mind, we denote by $\mathcal{D}_r(U)$ the space of r-currents, i.e, the space of formal sums
$$\sum_{|I|=r} u_I \mathrm{d}x_I$$
where the u_I are distributions on Ω. It is equivalently defined as the topological dual of $C_c^\infty \Lambda^r(U)$.
- For $1 \le p \le \infty$ the set of L^p differential forms on U, denoted by $L^p \Lambda^r(U)$ i.e,
$$L^p \Lambda^r(U) := \left\{ u = \sum_{|I|=r} u_I(x) \mathrm{d}x_I \ : \ \forall I,\ u_I \in L^p(U) \right\}$$
equipped with the norm
$$\|u\|_{L^p \Lambda^r(U)} := \sum_{|I|=r} \|u_I\|_{L^p \Lambda^r(U)},$$
and, for $1 \le p < \infty$, the normalized L^p-norm
$$\|u\|^p_{\underline{L}^p \Lambda^r(U)} := \sum_{|I|=r} \fint_U |u_I(x)|^p \, \mathrm{d}x = \frac{1}{|U|} \sum_{|I|=r} \int_U |u_I(x)|^p \, \mathrm{d}x.$$
We also equip the space $L^2 \Lambda^r(U)$ with the scalar product $\langle u, v \rangle_U := \sum_{|I|=r} \langle u_I, v_I \rangle_{L^2(U)}$.
- For $s \in \mathbb{R}$, the set of H^s differential forms on U, denoted by $H^s \Lambda^r(U)$, i.e,
$$H^s \Lambda^r(U) := \left\{ u = \sum_{|I|=r} u_I(x) \mathrm{d}x_I \ : \ \forall I,\ u_I \in H^s(U) \right\}$$
equipped with the scalar product $\langle u, v \rangle_{H^s \Lambda^r(U)} := \sum_{|I|=r} \langle u_I, v_I \rangle_{H^s(U)}$.

If $U \subseteq \mathbb{R}$ and $u : U \to \Lambda^r(\mathbb{R}^d)$, we denote the ith-partial derivative of u by $\partial_i u$, it is understood in the sense of currents according to the formula

$$\partial_i u = \sum_{|I|=r} \partial_i u_I \mathrm{d}x_I,$$

where $\partial_i u_I$ is understood in the sense of distribution. The gradient of u, denoted by $\nabla u := (\partial_1 u, \ldots, \partial_d u)$, is a vector-valued differential form. Higher derivatives, which are also vector-valued forms, are denoted by, for $l \ge 1$,

$$\nabla^l u := \left(\partial_{i_1} \ldots \partial_{i_l} u\right)_{i_1, \ldots, i_l \in \{1, \ldots, d\}}.$$

Given an m-form α and an r-form ω, we consider the exterior product $\alpha \wedge \omega$ which is an $(m+r)$-form and satisfies the following property

$$\alpha \wedge \omega = (-1)^{mr} \omega \wedge \alpha.$$

If $m + r > d$, we set $\omega \wedge \alpha = 0$.

We then define the exterior derivative which maps $C^\infty \Lambda^r(U)$ to $C^\infty \Lambda^{r+1}(U)$ according to the formula,

$$\mathrm{d}u = \sum_{|I|=r} \sum_{k \notin I} \frac{\partial u_I}{\partial x_k} \mathrm{d}x_k \wedge \mathrm{d}x_I,$$

and can then be extended to currents. In particular, if u is a differential form of degree d, then $\mathrm{d}u = 0$. This operator satisfies the following properties

$$\mathrm{d} \circ \mathrm{d} = 0 \text{ and } \mathrm{d}(u \wedge v) = (\mathrm{d}u) \wedge v + (-1)^r u \wedge (\mathrm{d}v). \tag{4.1.6}$$

Given a form $u := \sum_{|I|=r} u_I \mathrm{d}x_I \in C^\infty \Lambda^r(U)$, an open set $V \subseteq \mathbb{R}^d$ and a smooth map $\Phi = (\Phi_1, \ldots, \Phi_d) : V \to U$, we define the pullback u by Φ to be the smooth form

$$\Phi^* u := \begin{cases} V \to \Lambda^r(\mathbb{R}^d), \\ x \to \displaystyle\sum_{I = \{i_1, \ldots, i_r\}} u_I(\Phi(x)) \, \mathrm{d}\Phi_{i_1}(x) \wedge \cdots \wedge \mathrm{d}\Phi_{i_r}(x). \end{cases}$$

where $\mathrm{d}\Phi(x)$ denotes the differential of Φ evaluated at x. The pullback satisfies the following properties, given an r-form u and an m-form v,

$$\Phi^* \mathrm{d}u = \mathrm{d}\Phi^* u \text{ and } \Phi^* (u \wedge v) = \Phi^* u \wedge \Phi^* v. \tag{4.1.7}$$

Given another open set $W \subseteq \mathbb{R}^d$ and another smooth map $\Psi : W \to V$, we have the composition rule

$$\Psi^* (\Phi^* u) = (\Phi \circ \Psi)^* v.$$

Moreover, if we assume that Φ is a smooth diffeomorphism from V to U such that all the derivatives of Φ are bounded then, for $k \in \mathbb{N}$, Φ^* maps $H^k \Lambda^r(U)$ into $H^k \Lambda^r(V)$ and we have the estimate

$$\|\Phi^* u\|_{H^k \Lambda^r(V)} \le C \|u\|_{H^k \Lambda^r(U)}, \tag{4.1.8}$$

for some $C := C(d, k, \Phi) < \infty$.

We can also define a scalar product on $\Lambda^r(\mathbb{R}^d)$ such that $(\mathrm{d}x_I)_{|I|=r}$ is an orthonormal basis, i.e,

$$\left(\sum_{|I|=r} \alpha_I \mathrm{d}x_I, \sum_{|I|=r} \beta_I \mathrm{d}x_I \right) = \sum_{|I|=r} \alpha_I \beta_I. \tag{4.1.9}$$

We will use the notation, for $\alpha \in \Lambda^r(\mathbb{R}^d)$

$$|\alpha| = \sqrt{(\alpha, \alpha)}.$$

We denote by $B_1 \Lambda^r(\mathbb{R}^d)$ the unit ball of $\Lambda^r(\mathbb{R}^d)$, i.e,

$$B_1 \Lambda^r(\mathbb{R}^d) := \left\{ \alpha \in \Lambda^r(\mathbb{R}^d) \, : \, |\alpha| \le 1 \right\}.$$

Moreover for each r, notice that

$$\dim \Lambda^r(\mathbb{R}^d) = \dim \Lambda^{(d-r)}(\mathbb{R}^d) = \binom{d}{r}.$$

There is a canonical bijection between these spaces, the Hodge star operator, denoted by $\star$, which sends $\Lambda^r(\mathbb{R}^d)$ to $\Lambda^{(d-r)}(\mathbb{R}^d)$ and satisfies the property, for each $\alpha, \beta \in \Lambda^r(\mathbb{R}^d)$

$$\alpha \wedge (\star \beta) = (\alpha, \beta) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d.$$

It is defined on the canonical basis by

$$\star\left(\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_r}\right):=\mathrm{d}x_{i_{r+1}}\wedge\cdots\wedge\mathrm{d}x_{i_d}$$

where $(i_1,\ldots,i_d)$ is an even permutation of $\{1,\ldots,d\}$. An important property of this operator is the following, for each $\alpha\in\Lambda^r(\mathbb{R}^d)$,

$$\star\star\alpha=(-1)^{r(d-r)}\alpha. \tag{4.1.10}$$

We then define the integral of a d–form over a domain U. Let $u=u_{\{1,\ldots,d\}}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_d$ be a d–form over U. If $u_{\{1,\ldots,d\}}\in L^1(U)$, we say that u is integrable and define

$$\int_U u:=\int_U u_{\{1,\ldots,d\}}(x)\,\mathrm{d}x. \tag{4.1.11}$$

In particular, the scalar product on $L^2\Lambda^r(U)$ can be rewritten, for each $\alpha,\beta\in L^2\Lambda^r(U)$,

$$\langle u,v\rangle_U=\int_U u\wedge(\star v).$$

Additionally, if Φ is a smooth diffeomorphism mapping V to U positively oriented, i.e if $\det\mathrm{d}\Phi>0$, then the change of variables formula reads, for each integrable d-form u,

$$\int_V\Phi^*u=\int_U u. \tag{4.1.12}$$

We then want to define the normal and tangential components of a form u on the boundary of a smooth bounded domain U. To achieve this, consider $U\subseteq\mathbb{R}^d$ a smooth bounded domain of $\mathbb{R}^d$, denote by ν the outward normal of ∂U and fix $u\in C^\infty\Lambda^r(\mathbb{R}^d)$ a smooth r-form. For each $x\in\partial U$, we define $\mathbf{n}u(x)\in\Lambda^r\left(\mathbb{R}^d\right)$, the normal component of $u(x)$, to be the orthogonal projection of $u(x)$ with respect to the scalar product $(\cdot,\cdot)$ defined in (4.1.9) on the kernel of the mapping

$$\mathrm{d}\nu(x)\wedge\cdot\;:\begin{cases}\Lambda^r(\mathbb{R}^d)\to\Lambda^{r+1}(\mathbb{R}^d),\\ \qquad v\to\mathrm{d}\nu(x)\wedge v.\end{cases} \tag{4.1.13}$$

The tangential component of $u(x)$, denoted by $\mathbf{t}u(x)$, is given by the formula

$$\mathbf{t}u(x)=u(x)-\mathbf{n}u(x). \tag{4.1.14}$$

Let now $u\in C^\infty\Lambda^{d-1}(U)$, using the previous notation there exists a smooth function $v:\partial U\to\mathbb{R}$ such that, for each $x\in\partial U$,

$$\mathbf{t}u(x)=v(x)\mathrm{d}e_1^x\wedge\cdots\wedge\mathrm{d}e_{d-1}^x,$$

where $e_1^x,\ldots,e_{d-1}^x\in\mathbb{R}^d$ are such that $(e_1^x,\ldots,e_{d-1}^x,\nu(x))$ is an orthonormal basis positively oriented of $\mathbb{R}^d$. With this notation, we define the integral of u on ∂U by the formula

$$\int_{\partial U}u=\int_{\partial U}v(x)\mathrm{d}\mathcal{H}^{d-1}(x), \tag{4.1.15}$$

where $\mathcal{H}^{d-1}$ is the Hausdorff measure of dimension $(d-1)$ on $\mathbb{R}^d$.

The two definition of integrals (4.1.11) and (4.1.15) are linked together by the Stokes' formula: for each smooth bounded domain $U\subseteq\mathbb{R}^d$ and each $u\in C^\infty\Lambda^{d-1}(U)$,

$$\int_{\partial U}u=\int_U\mathrm{d}u. \tag{4.1.16}$$

We can now define δ, the formal adjoint of d with respect to the scalar product $\langle\cdot,\cdot\rangle_{L^2\Lambda^r(U)}$, i.e, the operator which satisfies for each $(u,v)\in C_c^\infty\Lambda^{r-1}(U)\times C_c^\infty\Lambda^r(U)$,

$$\langle\mathrm{d}u,v\rangle_{L^2\Lambda^r(U)}=\langle u,\delta v\rangle_{L^2\Lambda^{r-1}(U)}.$$

This operator can be explicitly computed using the second equality in (4.1.6), the equality (4.1.10), and the Stokes' formula (4.1.16). Indeed we have

$$\begin{aligned} 0 = \int_{\partial U} u \wedge (\star v) &= \int_U \mathrm{d}u \wedge (\star v) + (-1)^{r-1} \int_U u \wedge \mathrm{d}(\star v) \\ &= \int_U \mathrm{d}u \wedge \star v + (-1)^{r-1+(r-1)(d-r+1)} \int_U u \wedge \star(\star \mathrm{d} \star v). \end{aligned}$$

Consequently,

$$\delta = (-1)^{(r-1)d+1} \star \mathrm{d} \star . \tag{4.1.17}$$

We now define the set of L^2 forms u such that $\mathrm{d}u$ is also L^2. This will play a crucial role in this article. Note that this space is different from the Sobolev space $H^1\Lambda^r(U)$ introduced earlier.

Definition 4.1.1. For each open subset $U \subseteq \mathbb{R}^d$, and each $0 \le r < d$, we define the space $H^1_{\mathrm{d}}\Lambda^r(U)$ to be the set of forms in $L^2\Lambda^r(U)$ such that $\mathrm{d}u \in L^2\Lambda^{r+1}(U)$, i.e,

$$H^1_{\mathrm{d}}\Lambda^r(U) := \left\{ u \in L^2\Lambda^r(U) \ : \ \exists f \in L^2\Lambda^{r+1}(U), \forall v \in C^\infty_c\Lambda^{d-r-1}(U), \int_U (u \wedge \mathrm{d}v + (-1)^r \mathrm{d}u \wedge v) = 0 \right\}.$$

If $u \in H^1_{\mathrm{d}}\Lambda^r(U)$, we denote by $\mathrm{d}u$ the unique form in $L^2\Lambda^{r+1}(U)$ which satisfies, for every $v \in C^\infty_c\Lambda^{d-r-1}(U)$,

$$\int_U (u \wedge \mathrm{d}v + (-1)^r \mathrm{d}u \wedge v) = 0. \tag{4.1.18}$$

This space is a Hilbert space equipped with the norm

$$\|u\|_{H^1_{\mathrm{d}}\Lambda^r(U)} = \langle u, u \rangle_U + \langle \mathrm{d}u, \mathrm{d}u \rangle_U .$$

In the case $r = d$, we have $\mathrm{d}u = 0$ for each $u \in L^2\Lambda^d(U)$ and $H^1_{\mathrm{d}}\Lambda^d(U) = L^2\Lambda^d(U)$. We also denote by $H^1_{\mathrm{d},0}\Lambda^r(U)$ the closure of $C^\infty_c\Lambda^r(U)$ in $H^1_{\mathrm{d}}\Lambda^r(U)$, i.e,

$$H^1_{\mathrm{d},0}\Lambda^r(U) := \overline{C^\infty_c\Lambda^r(U)}^{H^1_{\mathrm{d}}\Lambda^r(U)}.$$

Symmetrically, for each $0 < r \le d$, we define $H^1_\delta\Lambda^r(U)$ to be the set of forms in $L^2\Lambda^r(U)$ such that $\delta u \in L^2\Lambda^{r-1}(U)$, i.e,

$$H^1_\delta\Lambda^r(U) := \left\{ u \in L^2\Lambda^r(U) \ : \ \exists f \in L^2\Lambda^{r-1}(U), \forall v \in C^\infty_c\Lambda^{d-r+1}(U), \int_U \left(u \wedge \delta v + (-1)^{d-r} f \wedge v\right) = 0 \right\}.$$

and in that case, we denote by $\delta u = f$. In the case $r = 0$, we have $\delta u = 0$ for each $u \in L^2(U)$ and $H^1_\delta\Lambda^0(U) = L^2(U)$. We also denote by $H^1_{\delta,0}\Lambda^r(U)$ the closure of $C^\infty_c\Lambda^r(U)$ in $H^1_\delta\Lambda^r(U)$, i.e,

$$H^1_{\delta,0}\Lambda^r(U) := \overline{C^\infty_c\Lambda^r(U)}^{H^1_\delta\Lambda^r(U)}.$$

We then introduce the subspaces of closed (resp. co-closed) forms of $H^1_{\mathrm{d}}\Lambda^r(U)$ (resp. $H^1_\delta\Lambda^r(U)$).

Definition 4.1.2. For each open $U \subseteq \mathbb{R}^d$ and each $0 \le r \le d$, we say that a form $u \in H^1_{\mathrm{d}}\Lambda^r(U)$ is closed (resp. co-closed) if and only if $\mathrm{d}u = 0$ (resp. $\delta u = 0$). We denote by $C^r_{\mathrm{d}}(U)$ the subset of closed r forms, i.e,

$$C^r_{\mathrm{d}}(U) := \left\{ u \in H^1_{\mathrm{d}}\Lambda^r(U) \ : \ \mathrm{d}u = 0 \right\}.$$

We also define

$$C^r_{\mathrm{d},0}(U) := C^r_{\mathrm{d}}(U) \cap H^1_{\mathrm{d},0}\Lambda^r(U).$$

Symetrically, we denote by $C^r_\delta(U)$ the subset of co-closed r forms, i.e,

$$C^r_\delta(U) := \left\{ u \in H^1_{\mathrm{d}}\Lambda^r(U) \ : \ \delta u = 0 \right\}.$$

We also define

$$C^r_{\delta,0}(U) := C^r_\delta(U) \cap H^1_{\delta,0}\Lambda^r(U).$$

4.1.1.2. *Notation related to the probability space.* For a random variable X, an exponent $s \in (0, +\infty)$ and a constant $C \in (0, \infty)$, we write

$$X \le \mathcal{O}_s(C)$$

to mean that

$$\mathbb{E}\left[\exp\left(\left(\frac{X_+}{C}\right)^s\right)\right] \le 2,$$

where $X_+ := \max(X, 0)$. The notation is clearly homogeneous:

$$X \le \mathcal{O}_s(C) \iff \frac{X}{C} \le \mathcal{O}_s(1).$$

More generally, for $\theta_0, \theta_1, \ldots, \theta_n \in \mathbb{R}^+$ and $C_1, \ldots, C_n \in \mathbb{R}^+_*$, we write

$$X \le \theta_0 + \theta_1 \mathcal{O}_s(C_1) + \cdots + \theta_n \mathcal{O}_s(C_n)$$

to mean that there exist nonnegative random variables $X_1, \ldots, X_n$ satisfying $X_i \le \mathcal{O}_s(C_n)$ such that

$$X \le \theta_0 + \theta_1 X_1 + \cdots + \theta_n X_n.$$

We now record an important property about this notation, the proof of which can be found in [**18**, Lemma A.4].

PROPOSITION 4.1.3. *For each $s \in (0, \infty)$, there exists a constant $C_s < \infty$ such that the following holds. Let μ be a measure over an arbitrary measurable space E, let $\theta : E \to (0, \infty)$ be a measurable function and $(X(x))_{x \in E}$ be a jointly measurable family of nonnegative random variables such that, for every $x \in E, X(x) \le \mathcal{O}_s(C(x))$. We have*

$$\int_E X(x)\mu(dx) \le \mathcal{O}_s\left(C_s \int_E C(x)\mu(dx)\right). \tag{4.1.19}$$

We then record a corollary which will be useful in Section 4.4.

COROLLARY 4.1.4. (i) *Given positive random variables $X_1, \ldots, X_n$ such that, for each $i \in \{1, \ldots, n\}$, $X_i \le \mathcal{O}_s(C_i)$, then*

$$\sum_{i=1}^n X_i \le \mathcal{O}_s\left(C_s \sum_{i=1}^n C_i\right),$$

where C_s is the constant in Proposition 4.1.3.

(ii) *Given a real number $r > 1$ and $X_1, \ldots, X_n$ such that for each $i \in \{1, \ldots, n\}, X_i \le \mathcal{O}_s(C)$, then*

$$\sum_{i=1}^n r^i X_i \le \mathcal{O}_s\left(C_s C \frac{r^{n+1}}{r-1}\right),$$

where C_s is the constant in Proposition 4.1.3.

4.1.1.3. *Notation and assumptions related to homogenization.* Given $\lambda \in (0, 1]$ and $1 \le r \le d$, we consider the space of measurable functions from $\mathbb{R}^d$ to $\mathcal{L}\left(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d)\right)$ satisfying the symmetry assumption, for each $x \in \mathbb{R}^d$,

$$p \wedge \mathbf{a}(x) q = q \wedge \mathbf{a}(x) p, \ \forall p, q \in \Lambda^r(\mathbb{R}^d), \tag{4.1.20}$$

and the ellipticity assumption, for each $x \in \mathbb{R}^d$,

$$\lambda |p|^2 \le \star(p \wedge \mathbf{a}(x) p) \le \frac{1}{\lambda}|p|^2, \ \forall p \in \Lambda^r(\mathbb{R}^d). \tag{4.1.21}$$

We denote by Ω_r the collection of all such measurable functions,

$$\Omega_r := \Big\{\mathbf{a}(\cdot) \, : \, \mathbf{a} : \mathbb{R}^d \to \mathcal{L}\left(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}\left(\mathbb{R}^d\right)\right) \text{ is Lebesgue measurable and satisfies (4.1.20) and (4.1.21)}\Big\}. \tag{4.1.22}$$

We endow Ω_r with the translation group $(\tau_y)_{y\in\mathbb{R}^d}$, acting on Ω_r via

$$(\tau_y \mathbf{a})(x) := \mathbf{a}(x+y)$$

and with the family $\{\mathcal{F}_r(U)\}$ of σ-algebras on Ω_r, with $\mathcal{F}_r(U)$ defined for each Borel subset $U \subseteq \mathbb{R}^d$ by

$$\mathcal{F}_r(U) := \Big\{ \sigma\text{-algebra on } \Omega_r \text{ generated by the family of maps} \\ \mathbf{a} \to \int_U p \wedge \mathbf{a}(x) q \phi(x),\ p,q \in \Lambda^r(\mathbb{R}^d),\ \phi \in C_c^\infty(U) \Big\}.$$

The largest of these σ-algebras is $\mathcal{F}_r(\mathbb{R}^d)$, simply denoted by $\mathcal{F}_r$. The translation group may be naturally extended to $\mathcal{F}_r$ itself by defining, for $A \in \mathcal{F}_r$,

$$\tau_y A := \{\tau_y \mathbf{a} : \mathbf{a} \in A\}. \tag{4.1.23}$$

We then endow the measurable space $(\Omega_r, \mathcal{F}_r)$ with a probability measure $\mathbb{P}_r$ satisfying the two following conditions:

- $\mathbb{P}_r$ is invariant under $\mathbb{Z}^d$-translations: for every $z \in \mathbb{Z}^d$, $A \in \mathcal{F}_r$,

$$\mathbb{P}[\tau_z A] = \mathbb{P}[A]. \tag{4.1.24}$$

- $\mathbb{P}_r$ has a unit range dependence: for every pair of Borel subsets $U, V \subseteq \mathbb{R}^d$ with $\operatorname{dist}(U,V) \geq 1$,

$$\mathcal{F}_r(U) \text{ and } \mathcal{F}_r(V) \text{ are independent.} \tag{4.1.25}$$

The expectation of an $\mathcal{F}_r$-measurable random variable X with respect to $\mathbb{P}_r$ is denoted by $\mathbb{E}_r[X]$ or simply $\mathbb{E}[X]$ when there is no confusion about the value of r.

DEFINITION 4.1.5. Given an integer $1 \leq r \leq d$, an environment $\mathbf{a} \in \Omega_r$ and an open subset $U \subseteq \mathbb{R}^d$, we say that $u \in H^1_{\mathrm{d}}\Lambda^{r-1}(U)$ is a solution of the equation

$$\mathrm{d}(\mathbf{a}\mathrm{d}u) = 0,$$

if for every smooth compactly supported form $v \in C_c^\infty \Lambda^r(U)$,

$$\int_U \mathrm{d}u \wedge \mathbf{a}\mathrm{d}v = 0.$$

We denote by $\mathcal{A}_r^{\mathbf{a}}(U)$ the set of solutions, i.e,

$$\mathcal{A}_r^{\mathbf{a}}(U) := \left\{ u \in H^1_{\mathrm{d}}\Lambda^r(U) : \forall v \in C_c^\infty \Lambda^r(U), \int_U \mathrm{d}u \wedge \mathbf{a}\mathrm{d}v = 0 \right\}. \tag{4.1.26}$$

When there is no confusion, we omit the subscripts r and $\mathbf{a}$ and only write $\mathcal{A}(U)$.

4.1.2. Statement of the main results.

DEFINITION 4.1.6. For every convex bounded domain $U \subseteq \mathbb{R}^d$, we define, for $(p,q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{(d-r)}(\mathbb{R}^d)$,

$$J(U,p,q) := \sup_{v \in \mathcal{A}(U)} \fint_U \left(-\frac{1}{2}\mathrm{d}v \wedge \mathbf{a}\mathrm{d}v - p \wedge \mathbf{a}\mathrm{d}v + \mathrm{d}v \wedge q \right). \tag{4.1.27}$$

The quantity J is nonnegative and satisfies a subadditivity property with respect to the domain U: see [**18**, Chapter 2] or Proposition 4.4.1 below. In particular the mapping

$$n \mapsto \mathbb{E}[J(\square_n, p, q)]$$

is decreasing and nonnegative, thus it converges as $n \to \infty$. The idea is then to show that there exists a linear mapping $\bar{\mathbf{a}} \in \mathcal{L}\left(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d)\right)$ such that for each r-form p, $J(\square_n, p, \bar{\mathbf{a}}p)$ tends to 0 and to quantify this statement. Precisely, we prove the following result.

THEOREM 4.1.1 (Quantitative homogenization). *Given $1 \le r \le d$, there exist an exponent $\alpha(d,\lambda) > 0$, a constant $C(d,\lambda) < \infty$ and a unique linear mapping $\bar{\mathbf{a}} \in \mathcal{L}\left(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d)\right)$, which is symmetric and satisfies the ellipticity condition* (4.1.21)*, such that for every $n \in \mathbb{N}$,*

$$\sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} J(\square_n, p, \bar{\mathbf{a}}p) \le \mathcal{O}_1\left(C 3^{-n\alpha}\right). \tag{4.1.28}$$

This is the subject of Section 4.4. In Section 4.5, we study the solvability of the equation $\mathrm{d}\mathbf{a}\mathrm{d}u = 0$ on a smooth bounded domain U. The first main proposition is the following, which establishes the well-posedness of the Dirichlet problem for differential forms.

PROPOSITION 4.1.7. *Let U be a bounded smooth domain of $\mathbb{R}^d$ and $1 \le r \le d$. Let $f \in H^1_{\mathrm{d}}\Lambda^{r-1}(U)$, then for any measurable map $\mathbf{a} : \mathbb{R}^d \to \mathcal{L}\left(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}\left(\mathbb{R}^d\right)\right)$ satisfying* (4.1.21) *and* (4.1.20)*, there exists a unique solution in $f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \cap \left(C^{r-1}_{\mathrm{d},0}(U)\right)^{\perp}$ of the equation*

$$\begin{cases} \mathrm{d}\left(\mathbf{a}\mathrm{d}u\right) = 0 & \text{in } U \\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U, \end{cases} \tag{4.1.29}$$

in the sense that, for each $v \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$,

$$\int_U \mathrm{d}u \wedge \mathbf{a}\mathrm{d}v = 0.$$

Moreover if we enlarge the space of admissible solutions to the space $f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$, we loose the uniqueness property, but if $v, w \in f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$ are two solutions of (4.1.29)*, then*

$$v - w \in C^{r-1}_{\mathrm{d},0}.$$

Before stating the homogenization theorem, there are two things to note about this proposition. First the suitable notion to replace the trace of a function when the degree of the form is not 0 is the tangential part of the form. This is the only information which is available when one has access to the form u and its differential derivative $\mathrm{d}u$. It will become clear in the next section when Propositions 4.2.2 and 4.2.3 are stated. Also note that for functions, or 0-forms, the notion of trace and tangential trace are the same.

Second, note that if $v \in C^{r-1}_{\mathrm{d},0}(U)$ and u is a solution of (4.1.29), then $u + v$ is also a solution of (4.1.29). This problem does not appear when one works with functions (or 0-forms) because in that case $C^0_{\mathrm{d},0}(U) = \{0\}$. This explains why we need to be careful when solving (4.1.29).

We then deduce from the previous proposition and Theorem 4.1.1 the homogenization theorem.

THEOREM 4.1.2 (Homogenization Theorem). *Let U be a bounded smooth domain of $\mathbb{R}^d$ and $1 \le r \le d$, fix $\varepsilon \in (0,1]$ and $f \in H^2\Lambda^{r-1}(U)$. Let $u^\varepsilon, u \in f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \cap \left(C^r_{\mathrm{d},0}(U)\right)^{\perp}$ respectively denote the solutions of the Dirichlet problems*

$$\begin{cases} \mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathrm{d}u^\varepsilon\right) = 0 & \text{in } U \\ \mathbf{t}u^\varepsilon = \mathbf{t}f & \text{on } \partial U. \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{d}\left(\bar{\mathbf{a}}\mathrm{d}u\right) = 0 & \text{in } U \\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U. \end{cases}$$

Then there exist an exponent $\alpha := \alpha(d,\lambda,U) > 0$ and a constant $C := C(d,\lambda,U) < \infty$ such that

$$\|u^\varepsilon - u\|_{L^2\Lambda^r(U)} + \|\mathrm{d}u^\varepsilon - \mathrm{d}u\|_{H^{-1}\Lambda^r(U)} \le \mathcal{O}_1\left(C\|\mathrm{d}f\|_{H^1\Lambda^r(U)}\,\varepsilon^\alpha\right).$$

The previous theorem is often stated, when one is dealing with functions (or 0-forms) in the case that U is a bounded Lipschitz domain and with a boundary condition $f \in W^{1,2+\delta}(U)$ for some $\delta > 0$: see for instance [**18**, Theorem 2.16]. This is convenient since this assumption ensures that the energy of the solution does not concentrate in a region of small Lebesgue measure near ∂U. Indeed, the global Meyers estimate gives some additional regularity on the function u,

$$\left(\fint_U |\nabla u|^{2+\varepsilon}(x)dx \right)^{\frac{1}{2+\varepsilon}} \leq C \left(\fint_U |\nabla f|^{2+\varepsilon}(x)dx \right)^{\frac{1}{2+\varepsilon}},$$

for some tiny $\varepsilon > 0$. On the other hand, this assumption is natural in view of the interior Meyers estimate, which ensures that the restriction of any solution to the heterogeneous equation to a smaller domain will possess such regularity.

Unfortunately, we were not able to prove a global Meyers-type estimate for the solutions of

$$\begin{cases} \mathrm{d}\left(\bar{\mathbf{a}}\mathrm{d}u\right) = 0 & \text{in } U, \\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U. \end{cases}$$

To bypass this difficulty, we made the additional assumptions U smooth and $\mathrm{d}f \in H^1\Lambda^r(U)$, this implies, by Proposition 4.A.4 that $\mathrm{d}u \in H^1\Lambda^r(U)$ with the estimate

$$\|\mathrm{d}u\|_{H^1\Lambda^r(U)} \leq C \|\mathrm{d}f\|_{H^1\Lambda^r(U)}.$$

Then, via the Sobolev embedding Theorem, we obtain that $\mathrm{d}u$ belongs to some L^p, for some $p := p(d) > 2$. This allows to control the L^2 norm of $\mathrm{d}u$ in a boundary layer of small volume, as it used to be done with the Meyers' estimate.

The last section is devoted to the study of the following dual problem. If $\mathbf{a} \in \Omega_r$, then for each $x \in \mathbb{R}^d$, $\mathbf{a}(x)$ is invertible and $\mathbf{a}^{-1} \in \mathcal{L}(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r(\mathbb{R}^d))$ satisfies the symmetry assumption (4.1.20) and the following ellipticity condition

$$\frac{1}{\lambda}|p|^2 \leq \mathbf{a}(x)^{-1}p \wedge p \leq \lambda |p|^2, \ \forall p \in \Lambda^{(d-r)}(\mathbb{R}^d).$$

We can thus define, for each $(p,q) \in \Lambda^{(d-r)}(\mathbb{R}^d) \times \Lambda^r(\mathbb{R}^d)$ and each $m \in \mathbb{N}$, the random variable

$$J_{\mathrm{inv}}(\square_m, p, q) := \sup_{u \in \mathcal{A}^{\mathrm{inv}}(\square_m)} \fint_{\square_m} \left(-\frac{1}{2}\mathbf{a}^{-1}\mathrm{d}u \wedge \mathrm{d}u - \mathbf{a}^{-1}\mathrm{d}u \wedge p + q \wedge \mathrm{d}u \right),$$

where $\mathcal{A}^{\mathrm{inv}}(\square_m)$ is the set of solution under the environment $\mathbf{a}^{-1}$, i.e,

$$\mathcal{A}_{\mathrm{inv}}(\square_m) := \left\{ u \in H^1_{\mathrm{d}}\Lambda^{(d-r-1)}(\square_m) \, : \, \forall v \in C_c^\infty \Lambda^{r-1}(\square_m), \int_{\square_m} \mathrm{d}u \wedge \mathbf{a}^{-1}\mathrm{d}v = 0 \right\}.$$

In Section 4.6, we prove that there exist a constant $C(d,\lambda) < \infty$, an exponent $\alpha(d,\lambda) > 0$ a linear operator $\overline{\mathrm{inv}\,\mathbf{a}} \in \mathcal{L}\left(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r(\mathbb{R}^d)\right)$ such that, for each $m \in \mathbb{N}$,

$$\sup_{p \in B_1\Lambda^{(d-r)}(\mathbb{R}^d)} J_{\mathrm{inv}}(\square_m, p, \overline{\mathrm{inv}\,\mathbf{a}}\,p) \leq \mathcal{O}_1\left(C3^{-m\alpha}\right).$$

We also prove that $\overline{\mathrm{inv}\,\mathbf{a}}$ is linked to $\bar{\mathbf{a}}$ according to the following theorem.

Theorem 4.1.3 (Duality). *The homogenized linear maps* $\bar{\mathbf{a}}$ *and* $\overline{\mathrm{inv}\,\mathbf{a}}$ *satisfy*

$$\overline{\mathrm{inv}\,\mathbf{a}} = (\bar{\mathbf{a}})^{-1}.$$

Outline of the paper. The rest of this article is organized as follows. In Section 4.2, we state without proof some important properties of differential forms, in particular we give a trace theorem for differential forms, study the solvability of the equation $\mathrm{d}f = u$ and state the Hodge-Morrey decomposition theorem. In Section 4.3, we generalize some inequalities known for functions to the setting of differential forms, in particular the Caccioppoli inequality and the multiscale Poincaré inequality. In Section 4.4, we combine all the ingredients established in the previous sections and prove the first main theorem of this article, Theorem 4.1.1. In Section 4.5, we use the results from Section 4.4 and the regularity estimates (pointwise interior estimate and boundary H^2-regularity) proved in the Appendix 4.A, to show the second main theorem of this article, Theorem 4.1.2. In Section 4.6, we study a duality structure between r-forms and $(d-r)$-forms and we deduce from that some results about the homogenized matrix in the case $d = 2$ and $r = 1$. Finally Appendix 4.A is devoted to the proof of some regularity estimates (more specifically pointwise interior estimate and H^2 boundary estimate) for the solution of the elliptic degenerate system $\mathrm{d}\bar{\mathbf{a}}\mathrm{d}u = 0$, where $\bar{\mathbf{a}}$ is a linear mapping sending r-forms to $(d-r)$-forms satisfying some suitable properties of symmetry and ellipticity, more formally explained in Section 4.1.1.

Acknowledgement. I would like to thank Scott Armstrong and Jean-Christophe Mourrat for helpful discussions and comments.

4.2. Some results pertaining to forms

In this section, we record some properties related to the spaces $H^1_{\mathrm{d}}\Lambda^r(U)$, $H^1_\delta\Lambda^r(U)$ and $C^r_{\mathrm{d}}(U)$. Most of these results and their proofs can be found in [**121**] and [**123**].

4.2.1. Tangential and normal trace of a differential form. Given $U \subseteq \mathbb{R}^d$ Lipschitz and bounded, we define the Sobolev space $H^{1/2}(\partial U)$ as the set of functions of $L^2(\partial U)$ which satisfy

$$[g]_{H^{1/2}(\partial U)} := \left(\int_{\partial U}\int_{\partial U} \frac{|g(x)-g(y)|^2}{|x-y|^{d+1}} \mathrm{d}\mathcal{H}^{d-1}(x)\mathrm{d}\mathcal{H}^{d-1}(y)\right)^{\frac{1}{2}} < \infty.$$

It is a Hilbert space equipped with the norm

$$\|g\|_{H^{1/2}(\partial U)} := \|g\|_{L^2(\partial U)} + [g]_{H^{1/2}(\partial U)}.$$

Define $H^{-1/2}(\partial U)$ to be the dual of $H^{1/2}(\partial U)$, i.e,

$$H^{-1/2}(\partial U) := \left(H^{1/2}(\partial U)\right)^*.$$

We can then extend this definition to differential forms by defining, for each $0 \le r \le d$,

$$H^{1/2}\Lambda^r(\partial U) := \left\{ u \in L^2\Lambda^r(\partial U) \text{ s.t } u = \sum_{|I|=r} u_I \mathrm{d}x_I \text{ and } \forall I, [u_I]_{H^{1/2}(\partial U)} < \infty \right\}.$$

This is also a Hilbert space, equipped with the norm,

$$\|u\|_{H^{1/2}\Lambda^r(\partial U)} := \|u\|_{L^2\Lambda^r(\partial U)} + \sum_{|I|=r} [u_I]_{H^{1/2}(\partial U)}.$$

We can also define $H^{-1/2}\Lambda^r(\partial U)$ by duality, according to the formula,

$$H^{-1/2}\Lambda^r(\partial U) := \left(H^{1/2}\Lambda^{d-r}(\partial U)\right)^*.$$

We then recall the classical Sobolev Trace Theorem for Lipschitz domains, it is a special case of [**99**, Chapter VII, Theorem 1] (see also [**111**]). The second half of this result is a consequence of the solvability of the Dirichlet problem for the Poisson equation in Lipschitz domains, which was proved in [**97**] or [**64**, Theorem 10.1].

Proposition 4.2.1 (Sobolev Trace Theorem). *Let U be a bounded Lipschitz domain. The linear operator $C^\infty\left(\overline{U}\right) \to \mathrm{Lip}(\partial U)$ that restricts a smooth function on $\overline{U}$ to ∂U has an extension to a bounded linear mapping $H^1(U) \to H^{1/2}(\partial U)$. That is, there exists a linear operator*

$$\mathrm{Tr} : H^1(U) \to H^{1/2}(\partial U),$$

and a constant $C(d,U) < \infty$ such that for each $u \in H^1(U)$,

$$\|\mathrm{Tr}\, u\|_{H^{1/2}(\partial U)} \le C\|u\|_{H^1(U)}$$

and for each $u \in C^\infty\left(\overline{U}\right)$,

$$\mathrm{Tr}\, u = u \text{ on } \partial U.$$

Moreover this map has a bounded right-inverse

$$E : H^{1/2}(\partial U) \to H^1(U).$$

In particular, the map Tr *is surjective.*

The trace can then be extended to differential forms by setting, for $u = \sum_{|I|=r} u_I \mathrm{d}x_I \in H^1\Lambda^r(U)$,

$$\operatorname{Tr} u = \sum_{|I|=r} \operatorname{Tr} u_I \, \mathrm{d}x_I \in H^{1/2}\Lambda^r(\partial U).$$

In the case when u does not belong to the space $H^1\Lambda^r(U)$ but only belongs to the larger space $H^1_{\mathrm{d}}\Lambda^r(U)$, one still has a Sobolev trace theorem, but one can only get information about the tangential component of the trace of u. The following proposition is a specific case of [**123**, Proposition 4.1 and Proposition 4.3] .

PROPOSITION 4.2.2 ([**123**], Proposition 4.1 and Proposition 4.3). *For each $u \in H^1_{\mathrm{d}}\Lambda^{r-1}(U)$, the map*

$$\langle \mathbf{t}u, \cdot \rangle : \begin{cases} H^{1/2}\Lambda^{(d-r)}(\partial U) \to \mathbb{R}, \\ \psi \to \displaystyle\int_U \left(\mathrm{d}u \wedge \Psi + (-1)^r u \wedge \mathrm{d}\Psi \right), \end{cases}$$

where $\Psi \in H^1\Lambda^{d-r}(U)$ is chosen such that $\operatorname{Tr}\Psi = \psi$, is well-defined, linear and bounded. The tangential trace

$$\mathbf{t} : \begin{cases} H^1_{\mathrm{d}}\Lambda^r(U) \to H^{-1/2}\Lambda^r(\partial U), \\ u \to \langle \mathbf{t}u, \cdot \rangle . \end{cases}$$

is linear and continuous. Moreover this notation is consistent with the tangential component introduced in (4.1.14). *Similarly, one can define the normal trace for $H^1_\delta\Lambda^{r+1}(U)$ according to the formula, for each $v \in H^1_\delta\Lambda^r(U)$,*

$$\langle \mathbf{n}v, \cdot \rangle : \begin{cases} H^{1/2}\Lambda^{(d-r)}(\partial U) \to \mathbb{R}, \\ \psi \to \displaystyle\int_U \left(\delta v \wedge \Psi + (-1)^{d-r} v \wedge \delta\Psi \right), \end{cases}$$

where $\Psi \in H^1\Lambda^{d-r}(U)$ is chosen such that $\operatorname{Tr}\Psi = \psi$. The linear operator $v \to \mathbf{n}v$ sends $H^1_\delta\Lambda^r(U)$ to $H^{-1/2}\Lambda^r(\partial U)$, is continuous and the notation is consistent with the normal component introduced in (4.1.13).

The following property shows that, when U is Lipschitz, the space $H^1_{\mathrm{d},0}\Lambda^r(U)$ (resp. $H^1_{\delta,0}\Lambda^r(U)$) is also the space of differential forms in $H^1_{\mathrm{d}}\Lambda^r(U)$ (resp. $H^1_\delta\Lambda^r(U)$) with tangential (resp. normal) trace equal to 0. A proof for these results can be found in [**121**, Lemma 2.13].

PROPOSITION 4.2.3 ([**121**], Lemma 2.13). *Let U be an open bounded Lipschitz subset of $\mathbb{R}^d$. For each $0 \le r \le d$, the following results hold:*

- *The space of smooth differential forms $C^\infty\Lambda^r(U)$ is dense in $H^1_{\mathrm{d}}\Lambda^r(U)$ (resp. $H^1_\delta\Lambda^r(U)$).*
- *The space $C^\infty_c\Lambda^r(U)$ of smooth and compactly supported differential forms is dense in $\left\{u \in H^1_{\mathrm{d}}\Lambda^r(U) \,:\, \mathbf{t}u = 0\right\}$ and in $\left\{u \in H^1_\delta\Lambda^r(U) \,:\, \mathbf{n}u = 0\right\}$). In particular, one has*

$$H^1_{\mathrm{d},0}\Lambda^r(U) = \left\{u \in H^1_{\mathrm{d}}\Lambda^r(U) : \mathbf{t}u = 0\right\} \quad \text{and} \quad H^1_{\delta,0}\Lambda^r(U) = \left\{u \in H^1_\delta\Lambda^r(U) : \mathbf{n}u = 0\right\}.$$

An interesting corollary of this proposition is that the space of solutions $\mathcal{A}(U)$, defined by (4.1.26), can be equivalently defined by the formula

$$\mathcal{A}(U) := \left\{ u \in H^1_{\mathrm{d}}\Lambda^r(U) : \forall v \in H^1_{\mathrm{d},0}\Lambda^{d-r}(U), \int_U \mathrm{d}u \wedge \mathbf{a}\mathrm{d}v = 0 \right\}. \tag{4.2.1}$$

4.2.2. Solvability of the equation $\mathrm{d}u = f$**.** We then record one important result concerning the solvability of the equation $\mathrm{d}u = f$ on bounded Lipschitz star-shaped domains.

PROPOSITION 4.2.4 ([**121**], Theorem 1.5 and Theorem 4.1). *Let $U \subseteq \mathbb{R}^d$ be a bounded Lipschitz star-shaped domain. The following statements hold.*

- *For $1 \le r \le d$ (resp. $0 \le r \le d-1$), given $f \in L^2\Lambda^r(U)$, the problem*

$$\left\{ \begin{array}{l} \mathrm{d}u = f \text{ in } U, \\ u \in H^1_{\mathrm{d}}\Lambda^{r-1}(U), \end{array} \right. \qquad \textit{resp.} \qquad \left\{ \begin{array}{l} \delta u = f \text{ in } U, \\ u \in H^1_{\delta}\Lambda^{r+1}(U), \end{array} \right. \tag{4.2.2}$$

has a solution if and only if f satisfies $\mathrm{d}f = 0$ (resp. $\delta f = 0$). In this case, there exist a constant $C(d,U) < \infty$ and a solution u of (4.2.2) which belongs to $H^1\Lambda^{r-1}(U)$ (resp. $u \in H^1\Lambda^{r+1}(U)$) and satisfies

$$\|u\|_{H^1\Lambda^{r-1}(U)} \le C\|f\|_{L^2\Lambda^r(U)} \quad \textit{resp.} \quad \|u\|_{H^1\Lambda^{r+1}(U)} \le C\|f\|_{L^2\Lambda^r(U)}.$$

- *For $1 \le r \le d-1$, given $f \in L^2\Lambda^r(U)$, the problem*

$$\left\{ \begin{array}{l} \mathrm{d}u = f, \\ u \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U), \end{array} \right. \qquad \textit{resp.} \qquad \left\{ \begin{array}{l} \delta u = f \text{ in } U, \\ u \in H^1_{\delta,0}\Lambda^{r+1}(U), \end{array} \right. \tag{4.2.3}$$

has a solution if and only if f satisfies

$$\left\{ \begin{array}{l} \mathrm{d}f = 0, \\ \mathbf{t}f = 0. \end{array} \right. \qquad \textit{resp.} \qquad \left\{ \begin{array}{l} \delta f = 0, \\ \mathbf{n}f = 0. \end{array} \right.$$

In this case, there exist a constant $C(d,U) < \infty$ and a solution u of (4.2.3) which belongs to $H^1\Lambda^{r-1}(U)$ (resp. $u \in H^1\Lambda^{r+1}(U)$) and satisfies

$$\|u\|_{H^1\Lambda^{r-1}(U)} \le C\|f\|_{L^2\Lambda^r(U)} \quad \textit{resp.} \quad \|u\|_{H^1\Lambda^{r+1}(U)} \le C\|f\|_{L^2\Lambda^r(U)}. \tag{4.2.4}$$

- *For $r = d$ (resp. $r = 0$), given $f \in L^2\Lambda^r(U)$, the problem*

$$\left\{ \begin{array}{l} \mathrm{d}u = f, \\ u \in H^1_{\mathrm{d},0}\Lambda^{d-1}(U), \end{array} \right. \qquad \textit{resp.} \qquad \left\{ \begin{array}{l} \delta u = f \text{ in } U, \\ u \in H^1_{\delta,0}\Lambda^{1}(U), \end{array} \right.$$

has a solution if and only if f satisfies

$$\int_U f = 0 \qquad \textit{resp.} \qquad \int_U \star f = 0.$$

Moreover there exists a solution $u \in H^1\Lambda^{d-1}(U)$ (resp. $u \in H^1\Lambda^1(U)$) which satisfies (4.2.4).

4.2.3. The Hodge-Morrey Decomposition Theorem. In this section, we record the Hodge-Morrey Decomposition Theorem. This requires to introduce the subspaces of exact, co-exact and harmonic forms.

DEFINITION 4.2.5. For each open $U \subseteq \mathbb{R}^d$ and each $1 \le r \le d$, we say that a form $u \in H^1_{\mathrm{d}}\Lambda^r(U)$ is exact if and only if there exists $\alpha \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$ such that $\mathrm{d}\alpha = u$. We denote by $\mathcal{E}^r(U)$ the subset of exact r forms with null tangential trace, i.e,

$$\mathcal{E}^r(U) := \left\{ u \in H^1_{\mathrm{d}}\Lambda^r(U) \ : \ \exists \alpha \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \text{ such that } \mathrm{d}\alpha = u \right\} \subseteq C^r_{\mathrm{d}}(U),$$

the subset of co-exact r forms with null normal trace $\mathcal{C}^r(U)$, i.e,

$$\mathcal{C}^r(U) := \left\{ v \in H^1_{\mathrm{d}}\Lambda^r(U) \ : \ \exists \beta \in H^1_{\delta,0}\Lambda^{r+1}(U) \text{ such that } \delta\beta = v \right\} \subseteq C^r_{\delta}(U),$$

and the subset of r harmonic forms, i.e,

$$\mathcal{H}^r(U) := \left\{ w \in L^2\Lambda^r(U) \ : \ \mathrm{d}w = 0 \text{ and } \delta w = 0 \right\}.$$

We now state the Hodge decomposition Theorem. This theorem is stated for two kinds of bounded domains, the convex domains in which case the situation is simple and the result can be deduced from Proposition 4.2.4, and the smooth domains. In the latter case the proof is more complicated and we refer to [**142**, Theorem 2.4.2] for the demonstration.

PROPOSITION 4.2.6 (Hodge-Morrey Decomposition, Theorem 2.4.2 of [**142**]). *Let $U \subseteq \mathbb{R}^d$ be an open, bounded domain. We assume that this domain is either convex or smooth, then for each $0 \le r \le d$,*

(i) the spaces $\mathcal{E}^r(U)$, $\mathcal{C}^r(U)$ and $\mathcal{H}^r(U)$ are closed in the $L^2\Lambda^r(U)$ topology.
(ii) the following orthogonal decomposition holds

$$L^2\Lambda^r(U) = \mathcal{E}^r(U) \overset{\perp}{\oplus} \mathcal{C}^r(U) \overset{\perp}{\oplus} \mathcal{H}^r(U).$$

4.3. Functional inequalities and differential forms

The goal of this section is to prove some functional inequalities which will be important in the proof of Theorem 4.1.1 in Section 4.5. To do so, we first deduce from the results of the previous section the Poincaré inequality for differential forms on convex or smooth bounded domains of $\mathbb{R}^d$, Proposition 4.3.2 and Proposition 4.3.1. We then state, without proof, the Gaffney-Friedrichs inequality for convex or smooth bounded domains of $\mathbb{R}^d$. We deduce from these propositions the multiscale Poincaré inequality, Proposition 4.3.6. We finally conclude this section by stating and proving the Caccioppoli inequality for differential forms.

4.3.1. The Poincaré inequality. The goal of this section is to generalize the standard Poincaré inequalities to the setting of differential forms.

PROPOSITION 4.3.1 (Poincaré). *Let U be a bounded domain of $\mathbb{R}^d$. We assume that U is either smooth or convex. There exists a constant $C := C(U) < \infty$, such that for all $0 \le r \le d$, for all $v \in H^1_{\mathrm{d},0}\Lambda^r(U)$,*

$$\inf_{\alpha \in C^r_{\mathrm{d},0}(U)} \|v - \alpha\|_{L^2\Lambda^r(U)} \le C\|\mathrm{d}v\|_{L^2\Lambda^{r+1}(U)}. \tag{4.3.1}$$

Moreover, the contant C has the following scaling property, for each $\lambda > 0$,

$$C(U) = \lambda C(\lambda^{-1}U).$$

The Poincaré Wirtinger inequality can also be generalized according to the following proposition.

PROPOSITION 4.3.2 (Poincaré-Wirtinger). *Let U be a bounded domain of $\mathbb{R}^d$. We assume that U is either smooth or convex. There exists a constant $C := C(U) < \infty$, such that for all $v \in H^1_{\mathrm{d}}\Lambda^r(U)$,*

$$\inf_{\alpha \in C^r_{\mathrm{d}}(U)} \|v - \alpha\|_{L^2\Lambda^r(U)} \le C\|\mathrm{d}v\|_{L^2\Lambda^{r+1}(U)}. \tag{4.3.2}$$

Moreover, the constant C has the following scaling property, for each $\lambda > 0$,

$$C(U) = \lambda C(\lambda^{-1}U).$$

PROOF OF PROPOSITIONS 4.3.1 AND 4.3.2. First notice that both estimates are easy when $r = d$ since in that case $C^r_{\mathrm{d}}(U) = H^1_{\mathrm{d}}\Lambda^d(U)$. From now on, we assume $0 \le r \le d-1$. In the case U convex, both inequalities (4.3.1) and (4.3.2) are a consequence of Proposition 4.2.4. We thus assume that U is smooth. The proof can be split into two steps.

- In Step 1, we prove that the space
$$\left\{u \in L^2\Lambda^{r+1}(U) \ : \ \exists \alpha \in H^1_{\mathrm{d}}\Lambda^r(U) \text{ such that } u = \mathrm{d}\alpha\right\}$$
is closed in the $L^2\Lambda^{r+1}$ topology.
- In Step 2, we deduce, from Step 1 and Proposition 4.2.6, the estimates (4.3.1) and (4.3.2).

Step 1. The argument relies on a decomposition of the space $\mathcal{H}^{r+1}(U)$ of harmonic forms, called the Friedrichs decomposition. By [**142**, Theorem 2.4.8], we have the following orthogonal decomposition,

$$\mathcal{H}^{r+1}(U) = \left(\mathcal{H}^{r+1}(U) \cap H^1_{\delta,0}\Lambda^{r+1}(U)\right) \overset{\perp}{\oplus} \left\{u \in \mathcal{H}^{r+1}(U) \mid \exists \alpha \in H^1_{\mathrm{d}}\Lambda^r(U) \text{ such that } u = \mathrm{d}\alpha\right\}.$$

Combining this result with Proposition 4.2.6 shows that

$$\begin{aligned}&\left\{u \in L^2\Lambda^{r+1}(U) \ : \ \exists \alpha \in H^1_{\mathrm{d}}\Lambda^r(U) \text{ such that } u = \mathrm{d}\alpha\right\}\\ &\qquad = \mathcal{E}^r(U) \overset{\perp}{\oplus} \left\{u \in \mathcal{H}^{r+1}(U) \mid \exists \alpha \in H^1_{\mathrm{d}}\Lambda^r(U) \text{ such that } u = \mathrm{d}\alpha\right\}\end{aligned}$$

is closed for the $L^2\Lambda^{r+1}$ topology.

Step 2. We first prove (4.3.1). By Proposition 4.2.6, we know that the space $\mathcal{E}^r$ is closed in $L^2\Lambda^{r+1}(U)$. This yields that the range of the linear operator

$$\mathrm{d} : \begin{cases} H^1_{\mathrm{d},0}\Lambda^r(U) \to L^2\Lambda^{r+1}(U), \\ \qquad\qquad u \to \mathrm{d}u. \end{cases}$$

is closed. Thus, by the Open Mapping Theorem, see [**38**, Theorem 2.6 and Corollary 2.7], there exists a constant $C(d,U) < \infty$ such that for each $v \in H^1_{\mathrm{d},0}\Lambda^r(U)$,

$$\inf_{\alpha \in \ker \mathrm{d}} \|v - \alpha\|_{L^2\Lambda^r(U)} \le C\|\mathrm{d}v\|.$$

But one has $\ker \mathrm{d} = C^r_{\mathrm{d}}(U) \cap H^1_{\mathrm{d},0}\Lambda^r(U)$. This completes the proof of (4.3.1).

The proof of (4.3.2) is similar, the only difference is that we use Step 1, instead of Proposition 4.2.6, to obtain that

$$\left\{u \in L^2\Lambda^{r+1}(U) \ : \ \exists \alpha \in H^1_{\mathrm{d}}\Lambda^r(U) \text{ such that } u = \mathrm{d}\alpha\right\}$$

is closed in the $L^2\Lambda^{r+1}$ topology.

The scaling of the constant comes from the change of variable $x \to \lambda x$. □

4.3.2. The Gaffney-Friedrichs inequality. We now state the Gaffney-Friedrichs inequality. The idea behind this inequality is to measure the global smoothness of a form u satisfying

$$\mathrm{d}u \in L^2\Lambda^{r+1}(U), \ \delta u \in L^2\Lambda^{r-1}(U) \text{ and } \mathbf{t}u = 0 \text{ on } \partial U. \tag{4.3.3}$$

According to a result from Gaffney [**71**] and Friedrichs [**67**], provided that U is smooth, the former assumption (4.3.3) implies that u is $H^1\Lambda^r(U)$ with the estimate

$$\|u\|_{H^1\Lambda^r(U)} \le C\left(\|\mathrm{d}u\|_{L^2\Lambda^{r+1}(U)} + \|\delta u\|_{L^2\Lambda^{r-1}(U)} + \|u\|_{L^2\Lambda^r(U)}\right), \tag{4.3.4}$$

for some $C := C(d,U) < \infty$. Conversely, one clearly has

$$\left(\|\mathrm{d}u\|_{L^2\Lambda^{r+1}(U)} + \|\delta u\|_{L^2\Lambda^{r-1}(U)}\right) \le C\,\|\nabla u\|_{L^2\Lambda^r(U)}.$$

Thus one can wonder whether the former inequality (4.3.4) can be refined into

$$\|\nabla u\|_{L^2\Lambda^r(U)} \le C\left(\|\mathrm{d}u\|_{L^2\Lambda^{r+1}(U)} + \|\delta u\|_{L^2\Lambda^{r-1}(U)}\right). \tag{4.3.5}$$

This inequality is false in general, indeed the set of harmonic forms with Dirichlet boundary condition

$$\mathcal{H}^r_D := \left\{u \in L^2\Lambda^r(U) \ : \ \mathrm{d}u = 0, \ \delta u = 0 \text{ and } \mathbf{t}u = 0 \text{ on } \partial U\right\}$$

is known to be finite dimensional and of dimension $\beta^{d-r}(U)$, the Betti number of the set U, cf [**142**, Theorem 2.2.2]. In particular, as soon as $\dim \mathcal{H}^r_D > 0$, the inequality (4.3.5) cannot hold. Nevertheless it is the only obstruction and we have the following result, which is a consequence of [**142**, Proposition 2.2.3].

PROPOSITION 4.3.3 (Gaffney-Friedrichs inequality for smooth domains). *Let U be a bounded smooth domain of $\mathbb{R}^d$, then there exists a constant $C := C(d,U) < \infty$ such that if $\omega \in L^2\Lambda^r(U)$ satisfies $d\omega \in L^2\Lambda^{r+1}(U)$, $\delta\omega \in L^2\Lambda^{r-1}(U)$, $\mathbf{t}\omega = 0$ on ∂U and $\omega \in (\mathcal{H}_D^r)^\perp$, then $\omega \in H^1\Lambda^r(U)$ and*

$$\|\nabla\omega\|_{L^2\Lambda^r(U)} \le C\left(\|\mathrm{d}\omega\|_{L^2\Lambda^{r+1}(U)} + \|\delta\omega\|_{L^2\Lambda^{r-1}(U)}\right).$$

One can also expect the inequality (4.3.5) to be true on convex domains, which are not necessarily smooth but satisfy $\beta^r(U) = 0$ for each $0 \le r \le d$. This result is stated in the following proposition and can be found in [**124**, Theorem 5.5].

PROPOSITION 4.3.4 (Gaffney-Friedrichs inequality for convex domains). *Let U be a convex bounded domain of $\mathbb{R}^d$. Then there exists a constant $C := C(d,U) < \infty$ such that if $\omega \in L^2\Lambda^r(U)$ satisfies $d\omega \in L^2\Lambda^{r+1}(U)$, $\delta\omega \in L^2\Lambda^{r-1}(U)$ and either $\mathbf{t}\omega = 0$ or $\mathbf{n}\omega = 0$ on ∂U, then $\omega \in H^1\Lambda^r(U)$ and*

$$\|\nabla\omega\|_{L^2\Lambda^r(U)} \le C\left(\|\mathrm{d}\omega\|_{L^2\Lambda^{r+1}(U)} + \|\delta\omega\|_{L^2\Lambda^{r-1}(U)}\right).$$

These inequalities are a key ingredient in the proofs of Theorem 4.1.1 and Theorem 4.1.2.

4.3.3. The multiscale Poincaré inequality. Another important ingredient needed in the proof of Theorem 4.1.1 is the multiscale Poincaré inequality stated below (Proposition 4.3.6). This inequality is valid for cubes and the statement and the proofs of Theorems 4.1.1 and 4.1.2 only require to apply the following results to cubes of $\mathbb{R}^d$. Thus, from now on and until the end of Section 4.3, we will only be dealing with cubes of $\mathbb{R}^d$, denoted by $\square$, instead of convex bounded domains. Recall that a cube of $\mathbb{R}^d$ is a set of the form

$$z + R(-1,1)^d \text{ with } z \in \mathbb{R}^d, R \in \mathbb{R}_+$$

and a triadic cube, denoted by $\square_m$, for $m \in \mathbb{N}$, is defined according to the formula

$$\square_m := \left(-\frac{3^m}{2}, \frac{3^m}{2}\right)^d.$$

We then define the mean value of a form on a cube according to the following proposition.

DEFINITION 4.3.5. Given $\square$ a cube of $\mathbb{R}^d$ and $0 \le r \le d$ and a form $\alpha = \sum_{|I|=r} \alpha_I \mathrm{d}x_I \in L^2\Lambda^r(\square)$. We denote by

$$(\alpha)_\square := \sum_{|I|=r} \left(\fint_\square \alpha_I(x)\,dx\right)\mathrm{d}x_I \in \Lambda^r(\mathbb{R}^d).$$

The multiscale Poincaré inequality then reads.

PROPOSITION 4.3.6 (Multiscale Poincaré). *Fix $m \in \mathbb{N}$ and, for each $0 \le r \le d$, each $n \in \mathbb{N}$, $n \le m$, define $\mathcal{Z}_{m,n} = 3^n\mathbb{Z}^d \cap \square_m$. There exists a constant $C(d) < \infty$ such that, for every $u \in C^r_{\mathrm{d}}(U)^\perp$,*

$$\|u\|_{\underline{L}^2(\square_m)} \le C\|\mathrm{d}u\|_{\underline{L}^2(\square_m)} + C\sum_{n=0}^{m-1} 3^n \left(|\mathcal{Z}_{m,n}|^{-1} \sum_{z\in\mathcal{Z}_{m,n}} |(\mathrm{d}u)_{z+\square_n}|^2\right)^{\frac{1}{2}}.$$

To prove this estimate, we first need to introduce the following H^{-1} norm for cubes.

DEFINITION 4.3.7. For each cube $\square$ of $\mathbb{R}^d$ and each $\omega \in L^2\Lambda^r(\square)$, we define the following H^{-1} norm

$$\|\omega\|_{\underline{H}^{-1}\Lambda^r(\square)} := \sup\left\{\frac{1}{|\square|}\langle\omega,\alpha\rangle_\square \,:\, \alpha \in H^1\Lambda^r(\square),\ \mathrm{size}(\square)^{-1}\left|(\alpha)_\square\right| + \|\nabla\alpha\|_{\underline{L}^2\Lambda^r(\square)} \le 1\right\}.$$

By the Poincaré-Wirtinger inequality, there exists a constant $C(d) < \infty$ such that,

$$\|\omega\|_{\underline{H}^{-1}\Lambda^r(\square)} \le C\,\mathrm{size}(\square)\|\omega\|_{\underline{L}^2\Lambda^r(\square)}.$$

The Multiscale Poincaré inequality is a consequence of this improved version of the Poincaré-Wirtinger inequality. The particular case $r = 0$ of this statement can be found in [**18**, Lemma 1.9].

PROPOSITION 4.3.8. *There exists a constant $C := C(d) < \infty$ such that for every cube $\square \in \mathbb{R}^d$, every $0 \le r \le d$ and every $u \in H^1_{\mathrm{d}}\Lambda^r(\square)$,*

$$\inf_{\alpha \in C^r_{\mathrm{d}}(\square)} \|u - \alpha\|_{\underline{L}^2\Lambda^r(\square)} \le C \|\mathrm{d}u\|_{\underline{H}^{-1}\Lambda^{r+1}(\square)}.$$

Before starting the proof, we need to state and prove the following lemma.

LEMMA 4.3.9. *There exists $C := C(d) < \infty$ such that for each cube $\square \in \mathbb{R}^d$, each $0 \le r \le (d-1)$ and each $u \in C^r_{\mathrm{d}}(\square)^\perp$, there exists a unique $w \in H^1_{\mathrm{d}}\Lambda^r(\square) \cap C^r_{\mathrm{d}}(\square)^\perp$ solution of the Neumann problem*

$$\begin{cases} \delta \mathrm{d}w = u \ \text{in } \square, \\ \mathbf{n}\mathrm{d}w = 0 \ \text{on } \partial\square, \end{cases} \tag{4.3.6}$$

in the sense that, for each $v \in H^1_{\mathrm{d}}\Lambda^r(U)$,

$$\langle \mathrm{d}w, \mathrm{d}v \rangle_\square = \langle u, v \rangle_\square .$$

Moreover, $\mathrm{d}w \in H^1\Lambda^{r+1}(\square)$ and

$$\|\nabla \mathrm{d}w\|_{L^2\Lambda^{r+1}(\square)} \le C\|u\|_{L^2\Lambda^r(\square)}. \tag{4.3.7}$$

PROOF. The proof can be split in two steps, first we need to prove that there exists a function w in $H^1_{\mathrm{d}}\Lambda^r(\square)$ solution of the Neumann problem (4.3.6) and then that the function w satisfies $\mathrm{d}w \in H^1\Lambda^{r+1}(\square)$ with the regularity estimate (4.3.7).

Step 1. To solve (4.3.6), denote for $v \in H^1_{\mathrm{d}}\Lambda^r(\square)$ by

$$\mathcal{J}(v) := \langle \mathrm{d}v, \mathrm{d}v \rangle_\square - \langle u, v \rangle_\square$$

and look at the variational problem

$$\inf_{v \in H^1_{\mathrm{d}}\Lambda^r(\square) \cap C^r_{\mathrm{d}}(\square)^\perp} \mathcal{J}(v).$$

By the standard minimization techniques of the calculus of variations and the Poincaré-Wirtinger inequality (Proposition 4.3.2), it is straightforward to prove that there exists a unique minimizer w of this problem. By the first variation, w solves (4.3.6).

Step 2. This proof is an adaptation from [**124**, Corollary 6.6]. The main ingredient of this step is the Gaffney-Friedrichs inequality (Proposition 4.3.4) applied with $U = \square$ and $\omega = \mathrm{d}w$. This form satisfies $w \in L^2\Lambda^{r+1}(\square)$, $\mathrm{d}w = \mathrm{dd}u = 0 \in L^2\Lambda^{r+2}(\square)$, $\delta w = u \in L^2\Lambda^r(\square)$ and $\mathbf{n}\omega = 0$. Thus, by the Gaffney-Friedrichs inequality, $\omega \in H^1\Lambda^{r+1}(\square)$, and for some $C := C(\square) < \infty$,

$$\|\nabla \omega\|_{L^2\Lambda^{r+1}(\square)} \le C\|u\|_{L^2\Lambda^r(\square)}.$$

By translation and scaling invariance, one obtains the existence of a constant $C := C(d) < \infty$ such that

$$\|\nabla \omega\|_{L^2\Lambda^{r+1}(\square)} \le C\|u\|_{L^2\Lambda^r(\square)}.$$

This is exactly (4.3.7). □

We now apply Lemma 4.3.9 to prove Proposition 4.3.8.

PROOF OF PROPOSITION 4.3.8. First notice that is enough to prove the result when $u \in H^1_{\mathrm{d}}\Lambda^r(\square) \cap \left(C^r_{\mathrm{d}}(\square)\right)^\perp$. Using the function $w \in H^1_{\mathrm{d}}\Lambda^r(\square)$ solution of the Neumann problem (4.3.6) in the cube $\square$, one has

$$\begin{aligned} \|u\|^2_{\underline{L}^2\Lambda^r(\square)} &= \frac{1}{|\square|} \langle u, u \rangle_\square \\ &= \frac{1}{|\square|} \langle \mathrm{d}u, \mathrm{d}w \rangle_\square \\ &\le \|\mathrm{d}u\|_{\underline{H}^{-1}\Lambda^{r+1}(\square)} \left(\operatorname{size}(\square)^{-1} \left|(\mathrm{d}w)_\square\right| + \|\nabla \mathrm{d}w\|_{\underline{L}^2\Lambda^{r+1}(\square)} \right). \end{aligned}$$

By Lemma 4.3.9,

$$\|\nabla \mathrm{d}w\|_{\underline{L}^2\Lambda^{r+1}(\square)} \le C\|u\|_{\underline{L}^2\Lambda^r(\square)}.$$

To complete the proof, there remains to estimate $\left|(\mathrm{d}w)_\square\right|$, to do so denote by

$$p = \sum_{i_1<\cdots<i_p} p_{i_1,\cdots,i_p}\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_1} := \frac{(\mathrm{d}w)_\square}{\left|(\mathrm{d}w)_\square\right|}.$$

and

$$l_p : \begin{cases} \mathbb{R}^d \to \Lambda^p(\mathbb{R}^d), \\ x \to \sum_{i_1<\cdots<i_p} p_{i_1,\cdots,i_p} x_{i_1}\mathrm{d}x_{i_2}\wedge\cdots\wedge\mathrm{d}x_{i_p}, \end{cases} \tag{4.3.8}$$

such that $\mathrm{d}l_p = p$. Testing the equation (4.3.6) with $\alpha = l_p$, one obtains

$$\begin{aligned}\left|(\mathrm{d}w)_\square\right| = \frac{1}{|\square|}\left|\langle p, \mathrm{d}w\rangle_\square\right| &= \frac{1}{|\square|}\left|\langle \mathrm{d}l_p, \mathrm{d}w\rangle_\square\right| \\ &= \frac{1}{|\square|}\left|\langle l_p, u\rangle_\square\right| \\ &\le C\,\mathrm{size}(\square)\|u\|_{\underline{L}^2\Lambda^r(\square)}.\end{aligned}$$

Combining the previous results completes the proof of the proposition. □

We then apply the Multiscale Poincaré inequality stated below. A proof of this inequality can be found in [**18**, Proposition 1.8].

PROPOSITION 4.3.10 (Multiscale Poincaré, Proposition 1.8 of [**18**]). *Fix $m \in \mathbb{N}$ and, for each $n \in \mathbb{N}$, such that $n \le m$, define $\mathcal{Z}_{m,n} = 3^n\mathbb{Z}^d \cap \square_m$. There exists a constant $C(d) < \infty$ such that, for every $f \in L^2(\square_m)$,*

$$\|f\|_{\underline{H}^{-1}(\square_m)} \le C\|f\|_{\underline{L}^2(\square_m)} + C\sum_{n=0}^{m-1} 3^n \left(|\mathcal{Z}_{m,n}|^{-1}\sum_{z\in\mathcal{Z}_{m,n}} |(f)_{z+\square_n}|^2\right)^{\frac{1}{2}}.$$

PROOF OF PROPOSITION 4.3.6. The result is then a consequence of Proposition 4.3.8 and Proposition 4.3.10 applied with $f = \mathrm{d}u$. □

4.3.4. The Caccioppoli inequality. We complete Section 4.3 by proving a version of the Caccioppoli inequality for differential forms. Recall the definitions of the space Ω_r in (4.1.22) and, given an environment $\mathbf{a} \in \Omega_r$, the definition of the space of solutions $\mathcal{A}(U)$ in (4.1.26).

PROPOSITION 4.3.11 (Caccioppoli inequality). *There exists a constant $C := C(d,\lambda) < \infty$ such that, for every $1 \le r \le d$, every open subsets $V, U \subseteq \mathbb{R}^d$ satisfying $\overline{V} \subseteq U$, and every $u \in \mathcal{A}(U)$,*

$$\|\mathrm{d}u\|_{L^2\Lambda^{r+1}(V)} \le \frac{C}{\mathrm{dist}(V,\partial U)}\|u\|_{L^2\Lambda^r(U\setminus V)}.$$

PROOF. Let $\eta \in C_c^\infty(U)$ be such that

$$\mathbb{1}_V \le \eta \le 1,\ |\nabla\eta| \le \frac{C}{\mathrm{dist}(V,\partial U)}.$$

The r-form ηu belongs to $H^1_{\mathrm{d},0}\Lambda^r(U)$ from this we deduce that

$$\int_U \mathrm{d}u \wedge \mathbf{a}\mathrm{d}(\eta u) = 0,$$

which gives

$$\begin{aligned}0 &= \int_U \mathrm{d}u \wedge \mathbf{a}\mathrm{d}(\eta^2 u)\\ &= \int_U \left(\mathrm{d}u \wedge \mathbf{a}\mathrm{d}\eta^2 \wedge u + \mathrm{d}u \wedge \mathbf{a}\eta^2 \mathrm{d}u\right)\\ &= \int_U \left(\mathrm{d}u \wedge \mathbf{a}2\eta\mathrm{d}\eta \wedge u + \mathrm{d}u \wedge \mathbf{a}\eta^2\mathrm{d}u\right).\end{aligned}$$

Thus, since by the ellipticity assumption (4.1.21) and the symmetry assumption (4.1.20), for each $x \in \mathbb{R}^d$, the bilinear form $(p,p') \to p \wedge \mathbf{a}(x)p'$ is a scalar product on $\Lambda^r(\mathbb{R}^d)$. In particular one can apply the Cauchy-Schwarz inequality in the following computation.

$$\begin{aligned}\int_U \mathrm{d}u \wedge \mathbf{a}\eta^2 du &= \int_U \mathrm{d}u \wedge \mathbf{a}(2\eta d\eta \wedge u)\\ &\leq 2\left(\int_U \mathrm{d}u \wedge \mathbf{a}(\eta^2 \mathrm{d}u)\right)^{\frac{1}{2}} \left(\int_U \mathrm{d}\eta \wedge u \wedge (\mathbf{a}\mathrm{d}\eta \wedge u)\right)^{\frac{1}{2}}.\end{aligned}$$

Using the ellipticity condition (4.1.21), one obtains

$$\begin{aligned}\|\mathrm{d}u\|_{L^2\Lambda^{r+1}(V)} &\leq C\|\mathrm{d}\eta \wedge u\|_{L^2\Lambda^r(U\smallsetminus V)}\\ &\leq \frac{C}{\mathrm{dist}(V,\partial U)}\|u\|_{L^2\Lambda^r(U\smallsetminus V)}.\end{aligned}$$

The proof is complete. □

4.4. Quantitative Homogenization

4.4.1. the subadditive quantity J and some of its properties. The goal of this section is to study the quantity J defined, according to Definition 4.1.6, by the formula, for $(p,q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$

$$J(U,p,q) := \sup_{v \in \mathcal{A}(U)} \fint_U \left(-\frac{1}{2}\mathrm{d}v \wedge \mathbf{a}\mathrm{d}v - p \wedge \mathbf{a}\mathrm{d}v + \mathrm{d}v \wedge q\right).$$

Thanks to the Poincaré-Wirtinger inequality, Proposition 4.3.2, one can prove that there exists a unique maximizer in $\mathcal{A}(U) \cap C^{r-1}_{\mathrm{d}}(U)^{\perp}$, denoted by $v(\cdot,U,p,q)$. The proof is very similar to Step 1 of the proof of Lemma 4.3.9 and the details are omitted.

We first record some useful properties about J, Proposition 4.4.1. We then establish a series of Lemmas, Lemmas 4.4.2 to 4.4.8, before proving the main result of this section, namely Theorem 4.1.1. We eventually deduce from Theorem 4.1.1 a corollary pertaining to the maximizer $v(\cdot,U,p,q)$, Proposition 4.4.10.

Proposition 4.4.1 (Basic propeties of J). *Fix a bounded Lipschitz domain $U \subseteq \mathbb{R}^d$. For each $1 \leq r \leq d$, the quantity $J(U,p,q)$ and its maximizer $v(\cdot,U,p,q)$ satisfy the following properties:*

(1) Decomposition of the maximizer $v(\cdot,U,p,q)$. *The map*

$$\begin{cases}\Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d) \to \mathcal{A}(U) \cap C^{r-1}_{\mathrm{d}}(U)^{\perp},\\ \qquad (p,q) \to v(\cdot,U,p,q),\end{cases} \tag{4.4.1}$$

is linear. Moreover, $v(\cdot,U,p,0)$ is, up to a closed form, equal to a solution of the Dirichlet problem

$$\begin{cases}\mathrm{d}\,(\mathbf{a}\mathrm{d}u) = 0 \in U,\\ \mathbf{t}u = \mathbf{t}l_{-p} \text{ on } \partial U,\end{cases} \tag{4.4.2}$$

where l_p is defined by (4.3.8). The precise interpretation of (4.4.2) is:

$$u \text{ solves } (4.4.2) \Leftrightarrow u \in l_{-p} + H^1_{\mathrm{d},0}\Lambda^{(r-1)}(U) \text{ and } \forall w \in H^1_{\mathrm{d},0}\Lambda^{(r-1)}(U), \int_U \mathrm{d}u \wedge \mathbf{a}\mathrm{d}w = 0.$$

Similarly $v(\cdot, U, 0, q)$ *is a solution of the Neumann problem*

$$\begin{cases} \mathrm{d}\left(\mathbf{a}\mathrm{d}u\right) = 0 \text{ in } U, \\ \mathbf{t}\left(\mathbf{a}\mathrm{d}u\right) = \mathbf{t}q \text{ on } \partial U. \end{cases} \tag{4.4.3}$$

the precise interpretation of (4.4.3) *is:*

$$u \text{ solves } (4.4.3) \Leftrightarrow u \in H^1_{\mathrm{d}}\Lambda^{r-1}(U) \text{ and } \forall w \in H^1_{\mathrm{d}}\Lambda^{r-1}(U), \int_U \mathrm{d}u \wedge \mathbf{a}\mathrm{d}w - \mathrm{d}w \wedge q = 0.$$

(2) Decomposition of $J(U,p,q)$. *For each* $(p,q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$*, the quantity* $J(U,p,q)$ *can be decomposed*

$$J(U,p,q) = \nu(U,p) + \nu^*(U,q) - \star(p \wedge q), \tag{4.4.4}$$

where $p \to \nu(U,p)$ *and* $q \to \nu^*(U,q)$ *are quadratic forms given by the formulas*

$$\nu(U,p) = \inf_{u \in l_{-p} + H^1_{\mathrm{d},0}\Lambda^{r-1}(U)} \fint_U \mathrm{d}u \wedge \mathbf{a}\mathrm{d}u \tag{4.4.5}$$

and

$$\nu^*(U,q) = \sup_{u \in H^1_{\mathrm{d}}\Lambda^{r-1}(U)} \fint_U \left(-\frac{1}{2}\mathrm{d}u \wedge \mathbf{a}\mathrm{d}u + \mathrm{d}u \wedge q \right). \tag{4.4.6}$$

As a remark note that there is a star before $q \wedge p$ *in* (4.4.4) *because* $q \wedge p$ *is a* d*-form and all the other terms are real numbers.*

(3) Upper and lower bound on $\nu(U,p)$ and $\nu^*(U,q)$. *There exists a constant* $C(d,\lambda) < \infty$ *such that for every* $p \in \Lambda^r(\mathbb{R}^d)$, $q \in \Lambda^{d-r}(\mathbb{R}^d)$,

$$\frac{1}{C}|p|^2 \le \nu(U,p) \le C|p|^2 \tag{4.4.7}$$

and

$$\frac{1}{C}|q|^2 \le \nu^*(U,q) \le C|q|^2. \tag{4.4.8}$$

This implies, according to (4.4.4)*, for some* $C := C(d,\lambda) < \infty$,

$$J(U,p,q) \le C(|p|^2 + |q|^2) \tag{4.4.9}$$

and

$$\|\mathrm{d}v(\cdot, U, p, q)\|_{\underline{L}^2\Lambda^r(U)} \le C(|p|^2 + |q|^2). \tag{4.4.10}$$

(4) Uniform convexity and $C^{1,1}$ regularity in the p and q variables separately. *There exists* $C(d,\lambda) < \infty$ *such that for every* $p_1, p_2 \in \Lambda^r(\mathbb{R}^d)$ *and* $q \in \Lambda^{d-r}(\mathbb{R}^d)$,

$$\frac{1}{C}|p_1 - p_2|^2 \le \frac{1}{2}J(U,p_1,q) + \frac{1}{2}J(U,p_2,q) - J\left(U, \frac{p_1+p_2}{2}, q\right) \le C|p_1 - p_2|^2. \tag{4.4.11}$$

For every $q_1, q_2 \in \Lambda^{d-r}(\mathbb{R}^d)$ *and* $p \in \Lambda^r(\mathbb{R}^d)$,

$$\frac{1}{C}|q_1 - q_2|^2 \le \frac{1}{2}J(U,p,q_1) + \frac{1}{2}J(U,p,q_2) - J\left(U, p, \frac{q_1+q_2}{2}\right) \le C|q_1 - q_2|^2. \tag{4.4.12}$$

(5) Subadditivity. *Let* $U_1, \ldots, U_n \subseteq U$ *be bounded Lipschitz domains that form a partition of* U*, in the sense that*

$$U_i \cap U_j = \varnothing \quad \text{if} \quad i \neq j \quad \text{and} \quad \left| U \setminus \bigcup_{i=1}^N U_i \right| = 0, \tag{4.4.13}$$

then, for every $(p,q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$,

$$J(U,p,q) \le \sum_{i=1}^N J(U_i,p,q). \tag{4.4.14}$$

(6) First variation for J. *For each* $(p,q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$, *the function* $v(\cdot,U,p,q)$ *is characterized as the unique element of* $\mathcal{A}(U) \cap C_{\mathrm{d}}^{r-1}(U)^{\perp}$ *which satisfies, for each* $u \in \mathcal{A}(U)$,

$$\int_U \mathrm{d}v \wedge \mathbf{a}\mathrm{d}u = \int_U (-p \wedge \mathbf{a}\mathrm{d}u + \mathrm{d}u \wedge q) \tag{4.4.15}$$

(7) Quadratic response *For every* $(p,q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$ *and* $w \in \mathcal{A}(U)$,

$$\begin{aligned} \frac{1}{C} \left\| \mathrm{d}w - \mathrm{d}v(\cdot,U,p,q) \right\|_{\underline{L}^2\Lambda^r(U)}^2 & \\ \leq J(U,p,q) - \fint_U \left(-\frac{1}{2}\mathrm{d}w \wedge \mathbf{a}\mathrm{d}w - p \wedge \mathbf{a}\mathrm{d}w + \mathrm{d}w \wedge q \right) & \\ \leq C \left\| \mathrm{d}w - \mathrm{d}v(\cdot,U,p,q) \right\|_{\underline{L}^2\Lambda^r(U)}^2 . & \end{aligned} \tag{4.4.16}$$

(8) Control of the difference of the optimizers by the subadditivity. *Let* $U_1,\ldots,U_n \subseteq U$ *be bounded Lipschitz domains that form a partition of* U, *in the sense of* (4.4.13). *Then for each* $(p,q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$,

$$\sum_{i=1}^n \frac{|U_i|}{|U|} \left\| \mathrm{d}v(\cdot,U,p,q) - \mathrm{d}v(\cdot,U_i,p,q) \right\|_{\underline{L}^2\Lambda^r(U_i)}^2 \leq C \sum_{i=1}^n \frac{|U_i|}{|U|} \left(J(U_i,p,q) - J(U,p,q) \right). \tag{4.4.17}$$

Proof. These properties are easy to check and their proofs are almost the same of those of [**18**, Lemma 2.2], so we omit the details. □

We now turn to the proof of a series of lemmas, which will be then used in the proof of Theorem 4.1.1. In the following lemma, we denote by $\mathcal{Z}_{m,n} := 3^n\mathbb{Z}^d \cap \square_m$. It is a finite set of cardinality $3^{d(m-n)}$.

Lemma 4.4.2. *Fix* $m,n \in \mathbb{N}$ *with* $n < m$, $(p,q) \in \Lambda^r(\mathbb{R}^d) \times \Lambda^{d-r}(\mathbb{R}^d)$ *and* $\{q'_z\}_{z\in\mathcal{Z}_{m,n}} \in \Lambda^{d-r}(\mathbb{R}^d)$, *then*

$$\frac{1}{|\square_m|} \sum_{z\in\mathcal{Z}_{n,m}} \left| \int_{z+\square_n} (\mathrm{d}v - \mathrm{d}v_z) \wedge q'_z \right| \leq \left(\sum_{z\in\mathcal{Z}_{n,m}} |q'_z|^2 \right)^{\frac{1}{2}} \left(\sum_{z\in\mathcal{Z}_{n,m}} J(z+\square_n,p,q) - J(\square_m,p,q) \right)^{\frac{1}{2}}. \tag{4.4.18}$$

Proof. We shorten the notations by setting, for each $z \in \mathcal{Z}_{m,n}$,

$$v := v(\cdot,\square_m,p,q), \quad v_z := v(\cdot,z+\square_n,p,q).$$

We compute, using Hölder inequality,

$$\begin{aligned} &\frac{1}{|\square_m|} \sum_{z\in\mathcal{Z}_{n,m}} \left| \int_{z+\square_n} q'_z \wedge (\mathrm{d}v - \mathrm{d}v_z) \right| \\ &\quad \leq \frac{C}{|\square_m|} \sum_{z\in\mathcal{Z}_{n,m}} |q'_z| \left\| (\mathrm{d}v - \mathrm{d}v_z) \right\|_{L^2(z+\square_n)} \\ &\quad \leq C \left(\frac{1}{|\square_m|} \sum_{z\in\mathcal{Z}_{n,m}} |q'_z|^2 \right) \left(\frac{1}{|\square_m|} \sum_{z\in\mathcal{Z}_{n,m}} \left\| (\mathrm{d}v - \mathrm{d}v_z) \right\|_{L^2(z+\square_n)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

by (4.4.17),

$$\leq C \left(\sum_{z\in\mathcal{Z}_{n,m}} |q'_z|^2 \right)^{\frac{1}{2}} \left(\sum_{z\in\mathcal{Z}_{n,m}} J(z+\square_n,p,q) - J(\square_m,p,q) \right)^{\frac{1}{2}}.$$

□

4.4.2. Estimate on the variance of the slope of the maximizer v. Given a differential form u, by analogy to the case of functions, we refer to the slope of u over a bounded domain $U \subseteq \mathbb{R}^d$ as the mean value of its exterior derivative, $(\mathrm{d}v(\cdot, \square_m, p, q))_U$.

LEMMA 4.4.3. *Let $m, n \in \mathbb{N}$ with $0 \le n \le m-2$. Then there exists $C(d,\lambda) < \infty$ such that, for every $(p,q) \in B_1\Lambda^r(\mathbb{R}^d) \times B_1\Lambda^{d-r}(\mathbb{R}^d)$,*

$$\operatorname{var}\left[(\mathrm{d}v(\cdot, \square_m, p, q))_{\square_m}\right] \le C3^{-d(m-n)} \operatorname{var}\left[(\mathrm{d}v(\cdot, \square_n, p, q))_{\square_n}\right] + C\mathbb{E}\left[J(\square_n, p, q) - J(\square_m, p, q)\right]. \tag{4.4.19}$$

PROOF. We first fix $n \in \mathbb{N}$ with $n \le m-2$, $q' \in B_1\Lambda^{d-r}(\mathbb{R}^d)$ and apply Lemma 4.4.2 with $q'_z := q'$ to derive

$$\frac{1}{|\square_m|}\left|\int_{\square_m} \mathrm{d}v(\cdot, \square_m, p, q) \wedge q' - \sum_{z \in \mathcal{Z}_{n,m}} \int_{z+\square_n} \mathrm{d}v(\cdot, z+\square_n, p, q) \wedge q'\right| \le C\left(\sum_{z \in \mathcal{Z}_{n,m}} J(z+\square_n, p, q) - J(\square_m, p, q)\right)^{\frac{1}{2}}. \tag{4.4.20}$$

From this we obtain

$$\operatorname{var}\left[\fint_{\square_m} \mathrm{d}v(\cdot, \square_m, p, q) \wedge q'\right] \le 2\operatorname{var}\left[\frac{1}{|\square_m|}\sum_{z \in \mathcal{Z}_{n,m}} \int_{z+\square_n} \mathrm{d}v(\cdot, z+\square_n, p, q) \wedge q'\right] + 2C\mathbb{E}\left[\left(\sum_{z \in \mathcal{Z}_{n,m}} J(z+\square_n, p, q) - J(\square_m, p, q)\right)\right].$$

We take an enumeration $\{z_{i,j} \,:\, 1 \le i \le 3^d, 1 \le j \le 3^{d(m-n-1)}\}$ of $\mathcal{Z}_{m,n}$ such that for each $1 \le i \le 3^d$ and each $1 \le j, j' \le 3^{d(m-n-1)}$,

$$|z_{i,j} - z_{i,j'}| \ge 2 \cdot 3^n.$$

This gives in particular

$$\operatorname{dist}(z_{i,j} + \square_n, z_{i,j'} + \square_n) \ge 3^n,$$

and, according to the finite range dependence assumption (4.1.25),

$$\mathcal{F}_r(z_{i,j} + \square_n) \text{ and } \mathcal{F}_r(z_{i,j'} + \square_n) \text{ are independent.}$$

We can thus estimate the first term on the right-hand side of (4.4.18), using the previous display and the stationarity (4.1.24) to get

$$\begin{aligned}
&\operatorname{var}\left[\frac{1}{|\square_m|}\sum_{z \in \mathcal{Z}_{n,m}} \int_{z+\square_n} \mathrm{d}v(\cdot, z+\square_n, p, q) \wedge q'\right] \\
&= 3^{-2dm} \operatorname{var}\left[\sum_{i=1}^{3^d} \sum_{j=1}^{3^{d(m-n-1)}} \int_{z_{i,j}+\square_n} \mathrm{d}v(\cdot, z_{i,j}+\square_n, p, q) \wedge q'\right] \\
&\le 3^{-2dm+d} \sum_{i=1}^{3^d} \operatorname{var}\left[\sum_{j=1}^{3^{d(m-n-1)}} \int_{z_{i,j}+\square_n} \mathrm{d}v(\cdot, z_{i,j}+\square_n, p, q) \wedge q'\right] \\
&\le 3^{-2dm+d} \sum_{i=1}^{3^d} \sum_{j=1}^{3^{d(m-n-1)}} \operatorname{var}\left[\int_{z_{i,j}+\square_n} \mathrm{d}v(\cdot, z_{i,j}+\square_n, p, q) \wedge q'\right] \\
&\le 3^{d(-m+1-n)} \operatorname{var}\left[\int_{\square_n} \mathrm{d}v(\cdot, \square_n, p, q) \wedge q'\right] \\
&\le C3^{-d(m-n)} \operatorname{var}\left[\fint_{\square_n} \mathrm{d}v(\cdot, \square_n, p, q) \wedge q'\right].
\end{aligned}$$

Combining the previous display with (4.4.20) and taking the supremum over $q' \in B_1\Lambda^{d-r}(\mathbb{R}^d)$ completes the proof of the lemma. □

The previous lemma controls the fluctuations of the slope of the maximizer v in the L^2 norm by the variation of the energy between two different scales. This latter quantity is an essential ingredients in the proofs of Theorems 4.1.1 and 4.1.2. This prompts the introduction of the following definition.

DEFINITION 4.4.4. For $n \in \mathbb{N}$, we define by

$$\begin{aligned}\tau_n &:= \sup_{(p,q)\in B_1\Lambda^r(\mathbb{R}^d)\times B_1\Lambda^{d-r}(\mathbb{R}^d)} \mathbb{E}\left[J(\square_n,p,q) - J(\square_{n+1},p,q)\right] \\ &= \sup_{p\in B_1\Lambda^r(\mathbb{R}^d)} \mathbb{E}\left[\nu(\square_n,p) - \nu(\square_{n+1},p)\right] + \sup_{q\in B_1\Lambda^{d-r}(\mathbb{R}^d)} \mathbb{E}\left[\nu^*(\square_n,q) - \nu^*(\square_{n+1},q)\right]\end{aligned}$$

With this definition, one can prove the following lemma.

LEMMA 4.4.5. *For each $n \in \mathbb{N}$, there exists a constant $C(d,\lambda) < \infty$ and an exponent $\beta := \beta(d,\lambda) > 0$ such that for every $(p,q) \in B_1\Lambda^r(\mathbb{R}^d) \times B_1\Lambda^{d-r}(\mathbb{R}^d)$,*

$$\operatorname{var}\left[\left(\mathrm{d}v(\cdot,\square_m,p,q)\right)_{\square_m}\right] \le C\sum_{n=0}^{m} 3^{\beta(n-m)}\tau_n + C3^{-\beta m}. \tag{4.4.21}$$

PROOF. Denote by $C := C(d,\lambda) < \infty$ the constant of Lemma 4.4.3 and select $l := l(d,\lambda) \in \mathbb{N}$ such that

$$\frac{1}{9} < C3^{-dl} \le \frac{1}{3}.$$

The inequality (4.4.19) applied with $n = m - l$ yields

$$\operatorname{var}\left[\left(\mathrm{d}v(\cdot,\square_m,p,q)\right)_{\square_{m-l}}\right] \le \frac{1}{3}\operatorname{var}\left[\left(\mathrm{d}v(\cdot,\square_m,p,q)\right)_{\square_{m-l}}\right] + C\sum_{k=n-l}^{n}\tau_k.$$

Iterating this estimate and using the bound on the L^2 norm of $\mathrm{d}v$ (4.4.10) gives, for some $C := C(d,\lambda) < \infty$,

$$\operatorname{var}\left[\left(\mathrm{d}v(\cdot,\square_m,p,q)\right)_{\square_m}\right] \le C3^{-\frac{n}{l}} + C\sum_{k=0}^{n} 3^{\frac{n-k}{l}}\tau_k.$$

This completes the proof of the lemma with $\beta = \frac{1}{l}$. □

4.4.3. Flatness of the maximizers and control of the energy. We begin this section by defining, for a bounded domain $U \subseteq \mathbb{R}^d$, an approximation of the homogenized matrix $\bar{\mathbf{a}}$ obtained by considering the environment only in the domain U.

DEFINITION 4.4.6. Consider U a bounded open subset of $\mathbb{R}^d$. Since $q \to J(U,0,q)$ is quadratic and bounded from above and below according to (4.4.8), there exists a linear mapping, denoted by $\bar{\mathbf{a}}_U$, from $\Lambda^r(\mathbb{R}^d)$ to $\Lambda^{d-r}(\mathbb{R}^d)$, satisfying the symmetry assumption (4.1.20) and such that, for every $q \in \Lambda^{d-r}(\mathbb{R}^d)$,

$$\mathbb{E}\left[J(U,0,q)\right] = \frac{1}{2} \star \left(\bar{\mathbf{a}}_U^{-1} q \wedge q\right). \tag{4.4.22}$$

We also write $\bar{\mathbf{a}}_n = \bar{\mathbf{a}}_{\square_n}$ for short.

There are two properties to notice about this quantity. First since J satisfies the subadditivity property (4.4.14), and by the stationarity assumption (4.1.24), the sequence $\left(\mathbb{E}\left[J(\square_n,0,q)\right]\right)_{n\in\mathbb{N}}$ is decreasing. Consequently it converges for each $q \in \Lambda^{d-r}(\mathbb{R}^d)$. From this, we deduce that there exists a linear symmetric map $\bar{\mathbf{a}} \in \mathcal{L}\left(\Lambda^r(\mathbb{R}^d), \Lambda^{d-r}(\mathbb{R}^d)\right)$ such that, for each $q \in \Lambda^{d-r}(\mathbb{R}^d)$

$$\mathbb{E}\left[J(\square_n,0,q)\right] \underset{n\to\infty}{\to} \frac{1}{2} \star \left(\bar{\mathbf{a}}^{-1} q \wedge q\right)$$

which also implies

$$\bar{\mathbf{a}}_n \to \bar{\mathbf{a}} \text{ in } \mathcal{L}\left(\Lambda^r(\mathbb{R}^d), \Lambda^{d-r}(\mathbb{R}^d)\right).$$

Moreover, by (4.4.8), one can check that there exists a constant $C(d,\lambda) < \infty$ such that, for each $p \in \Lambda^r(\mathbb{R}^d)$ and each $n \in \mathbb{N}$,

$$\frac{1}{C}|p|^2 \le p \wedge \bar{\mathbf{a}}_n p \le C|p|^2.$$

Sending $n \to \infty$ shows that the same estimate holds for $\bar{\mathbf{a}}$.

Second, one has the formula, for $q \in \Lambda^{d-r}(\mathbb{R}^d)$,

$$\bar{\mathbf{a}}_n^{-1} q = \mathbb{E}\left[(\mathrm{d}v(\cdot,\square_n,0,q))_{\square_n}\right]. \tag{4.4.23}$$

To prove this formula, one has, according to the first variation (4.4.15), for each $q \in \Lambda^{d-r}(\mathbb{R}^d)$,

$$J(\square_n,0,q) = \fint_{\square_n} \mathrm{d}v(\cdot,\square_n,0,q)\wedge q$$

Taking the expectation proves

$$\bar{\mathbf{a}}_n^{-1} q \wedge q = \mathbb{E}\left[(\mathrm{d}v(\cdot,\square_n,0,q))_{\square_n}\right]\wedge q.$$

To prove (4.4.23), it is thus sufficient to prove that $q \to \mathbb{E}\left[(\mathrm{d}v(\cdot,\square_n,0,q))_{\square_n}\right]$ satisfies the following symmetry property, for each $q, q' \in \Lambda^{d-r}(\mathbb{R}^d)$,

$$\mathbb{E}\left[(\mathrm{d}v(\cdot,\square_n,0,q))_{\square_n}\right]\wedge q' = \mathbb{E}\left[\left(\mathrm{d}v(\cdot,\square_n,0,q')\right)_{\square_n}\right]\wedge q.$$

It is a consequence of the following computation

$$\begin{aligned}
\mathbb{E}\left[(\mathrm{d}v(\cdot,\square_n,0,q))_{\square_n}\right]\wedge q' &= \mathbb{E}\left[\fint_{\square_n} \mathrm{d}v(\cdot,\square_n,0,q)\wedge q'\right]\\
&= \mathbb{E}\left[\fint_{\square_n} \mathrm{d}v(\cdot,\square_n,0,q)\wedge \mathbf{a}\mathrm{d}v(\cdot,\square_n,0,q')\right]\\
&= \mathbb{E}\left[\fint_{\square_n} \mathrm{d}v(\cdot,\square_n,0,q')\wedge \mathbf{a}\mathrm{d}v(\cdot,\square_n,0,q)\right]\\
&= \mathbb{E}\left[\fint_{\square_n} \mathrm{d}v(\cdot,\square_n,0,q')\wedge q\right]\\
&= \mathbb{E}\left[\left(\mathrm{d}v(\cdot,\square_n,0,q')\right)_{\square_n}\right]\wedge q.
\end{aligned}$$

We then note that, for every $q \in B_1\Lambda^{d-r}(\mathbb{R}^d)$, $m,n \in \mathbb{N}$ such that $n < m$, we have

$$\begin{aligned}
&|\bar{\mathbf{a}}_m^{-1} q - \bar{\mathbf{a}}_n^{-1} q|^2 \\
&\quad = \left|\mathbb{E}\left[(\mathrm{d}v(\cdot,\square_m,0,q))_{\square_m} - 3^{d(n-m)}\sum_{z\in 3^n\mathbb{Z}^d\cap\square_m}(\mathrm{d}v(\cdot,z+\square_n,0,q))_{z+\square_n}\right]\right|^2\\
&\quad \le \mathbb{E}\left[3^{d(n-m)}\sum_{z\in 3^n\mathbb{Z}^d\cap\square_m}\|\mathrm{d}v(\cdot,\square_m,0,q)-\mathrm{d}v(\cdot,z+\square_n,0,q)\|^2_{\underline{L}^2\Lambda^r(U)}\right]\\
&\quad \le C\mathbb{E}\left[J(\square_n,0,q)-J(\square_m,0,q)\right]\\
&\quad \le C\sum_{k=n}^{m-1}\tau_k.
\end{aligned} \tag{4.4.24}$$

For $p \in \Lambda^r(\mathbb{R}^d)$ and $m \in \mathbb{N}$, we denote by l_p^m the unique element of $C_{\mathrm{d}}^{r-1}(\square_m)^\perp$ such that $\mathrm{d}l_p^m = p$. It is the projection of the function l_p defined in (4.3.8) on $C_{\mathrm{d}}^{r-1}(\square_m)^\perp$.

LEMMA 4.4.7. *There exists $C := C(d,\lambda) < \infty$ such that, for every $m \in \mathbb{N}$, $(p,q) \in B_1\Lambda^r(\mathbb{R}^d) \times B_1\Lambda^{d-r}(\mathbb{R}^d)$,*

$$\mathbb{E}\left[\left\|v(\cdot,\square_{m+1},p,q) - l^{m+1}_{\bar{\mathbf{a}}_m^{-1}q-p}\right\|^2_{\underline{L}^2\Lambda^{r-1}(\square_{m+1})}\right] \le C3^{(2-\beta)m} + C3^{(2-\beta)m}\sum_{k=0}^m 3^{\beta k}\tau_k.$$

PROOF. Fix $(p,q) \in B_1\Lambda^r(\mathbb{R}^d) \times B_1\Lambda^{d-r}(\mathbb{R}^d)$ and denote by $\mathcal{Z}_{m,n} := 3^n\mathbb{Z}^d \cap \square_{m+1}$. We split the proof into two steps.

Step 1. Since, by definition, both $v(\cdot,\square_{m+1},p,q)$ and $l^{m+1}_{\bar{\mathbf{a}}_m^{-1}q-p}$ are in $C^{r-1}_{\mathrm{d}}(\square_{m+1})^\perp$, the difference belongs to $C^{r-1}_{\mathrm{d}}(\square_{m+1})^\perp$, thus we can apply the Multiscale Poincaré inequality (4.3.6),

$$\begin{aligned}
&\left\| v(\cdot,\square_{m+1},p,q) - l^{m+1}_{\bar{\mathbf{a}}_m^{-1}q-p} \right\|^2_{\underline{L}^2\Lambda^{r-1}(\square_{m+1})} \\
&\quad \le C \left\| \mathrm{d}v(\cdot,\square_{m+1},p,q) - \bar{\mathbf{a}}_m^{-1}q + p \right\|^2_{\underline{L}^2\Lambda^r(\square_{m+1})} \\
&\quad + C\left(\sum_{n=0}^m 3^n \left(|\mathcal{Z}_{m,n}|^{-1} \sum_{y\in\mathcal{Z}_{m,n}} \left| \left(\mathrm{d}v(x,\square_{m+1},p,q)\,dx - \bar{\mathbf{a}}_m^{-1}q + p \right)_{z+\square_n} \right|^2 \right)^{\frac12} \right)^2 .
\end{aligned} \tag{4.4.25}$$

We first bound the first term on the right-hand side

$$\left\| \mathrm{d}v(\cdot,\square_{m+1},p,q) - \bar{\mathbf{a}}_m^{-1}q + p \right\|^2_{\underline{L}^2\Lambda^r(\square_{m+1})} \le 2\left| -\bar{\mathbf{a}}_m^{-1}q + p \right|^2 + 2\left\| \mathrm{d}v(\cdot,\square_{m+1},p,q) \right\|^2_{\underline{L}^2\Lambda^r(\square_{m+1})} \le C.$$

Step 2. We prove the estimate, for every $0 \le n \le m$,

$$\mathbb{E}\left[|\mathcal{Z}_{m,n}|^{-1} \sum_{y\in\mathcal{Z}_{m,n}} \left| \left(\mathrm{d}v(\cdot,\square_{m+1},p,q) - \bar{\mathbf{a}}_m^{-1}q + p \right)_{y+\square_n} \right|^2 \right] \le C\left(3^{-n} + \sum_{k=0}^n 3^{k-n}\tau_k + \sum_{k=n}^m \tau_k \right).$$

By (4.4.17), we have, for every $(p,q) \in B_1\Lambda^r(\mathbb{R}^d) \times B_1\Lambda^{d-r}(\mathbb{R}^d)$,

$$\begin{aligned}
&|\mathcal{Z}_{m,n}|^{-1} \sum_{y\in\mathcal{Z}_{m,n}} \left\| \mathrm{d}v(\cdot,\square_{m+1},p,q) - \mathrm{d}v(\cdot,z+\square_n,p,q) \right\|^2_{\underline{L}^2\Lambda^r(y+\square_n)} \\
&\qquad \le C|\mathcal{Z}_{m,n}|^{-1} \sum_{z\in\mathcal{Z}_{m,n}} \left(J(z+\square_n,p,q) - J(\square_m,p,q) \right).
\end{aligned}$$

Taking expectations and using the stationarity yields

$$\begin{aligned}
&|\mathcal{Z}_{m,n}|^{-1}\mathbb{E}\left[\sum_{y\in\mathcal{Z}_{m,n}} \left\| \mathrm{d}v(\cdot,\square_{m+1},p,q) - \mathrm{d}v(\cdot,z+\square_n,p,q) \right\|^2_{\underline{L}^2\Lambda^r(y+\square_n)} \right] \\
&\qquad \le C\mathbb{E}\left[J(\square_n,p,q) - J(\square_m,p,q) \right] \le C\sum_{k=n}^m \tau_k .
\end{aligned}$$

The triangle inequality, the previous display and Lemma 4.4.5 then yield,

$$\begin{aligned}
&|\mathcal{Z}_{m,n}|^{-1} \sum_{y\in\mathcal{Z}_{m,n}} \mathbb{E}\left[\left| \left(\mathrm{d}v(\cdot,\square_{m+1},p,q) - \bar{\mathbf{a}}_n^{-1}q + p \right)_{y+\square_n} \right|^2 \right] \\
&\quad \le 3|\mathcal{Z}_{m,n}|^{-1} \sum_{y\in\mathcal{Z}_{m,n}} \mathbb{E}\left[\left| \left(\mathrm{d}v(\cdot,\square_{m+1},p,q) - \mathrm{d}v(x,y+\square_n,p,q) \right)_{y+\square_n} \right|^2 \right] \\
&\quad + 3|\mathcal{Z}_{m,n}|^{-1} \sum_{y\in\mathcal{Z}_{m,n}} \mathbb{E}\left[\left| \left(\mathrm{d}v(\cdot,y+\square_n,p,q) - \bar{\mathbf{a}}_n^{-1}q + p \right)_{y+\square_n} \right|^2 \right] \\
&\quad + 3|\bar{\mathbf{a}}_m^{-1}q - \bar{\mathbf{a}}_n^{-1}q|^2 \\
&\quad \le C\sum_{k=n}^m \tau_k + C\sum_{k=0}^n 3^{\beta(k-n)} + C3^{-\beta n}.
\end{aligned}$$

Combining this estimate and inequality (4.4.25) shows

$$\left\| v(\cdot,\square_{m+1},p,q) - l^{m+1}_{\bar{\mathbf{a}}_m^{-1}q-p} \right\|^2_{\underline{L}^2\Lambda^{r-1}(\square_{m+1})} \le C\left(1 + \left(\sum_{n=0}^m 3^n X_n^{\frac12} \right)^2 \right) \tag{4.4.26}$$

where the random variable

$$X_n := |\mathcal{Z}_{m,n}|^{-1} \sum_{y \in \mathcal{Z}_{m,n}} \left| \left(\mathrm{d}v(x, \square_{m+1}, p, q)\, dx - \bar{\mathbf{a}}_m^{-1} q + p \right)_{y+\square_n} \right|^2$$

satisfies

$$\mathbb{E}\left[X_n\right] \le C \sum_{k=n}^{m} \tau_k + C \sum_{k=0}^{n} 3^{\beta(k-n)} \tau_k + C 3^{-\beta n}.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{n=0}^{m} 3^n X_n^{\frac{1}{2}} \right)^2 \le \left(\sum_{n=0}^{m} 3^n \right) \left(\sum_{n=0}^{m} 3^n X_n \right) \le C 3^m \sum_{n=0}^{m} 3^n X_n.$$

Taking the expectation thus yields

$$\mathbb{E}\left[\left(\sum_{n=0}^{m} 3^n X_n^{\frac{1}{2}} \right)^2 \right] \le C 3^m \left(\sum_{n=0}^{m} \sum_{k=n}^{m} 3^n \tau_k + C \sum_{n=0}^{m} \sum_{k=0}^{n} 3^{(1-\beta)n} 3^{\beta k} \tau_k + C \sum_{n=0}^{m} 3^{(1-\beta)n} \right).$$

We then compute the term on the right-hand side

$$\sum_{n=0}^{m} \sum_{k=n}^{m} 3^n \tau_k = \sum_{k=0}^{m} \sum_{n=0}^{k} 3^n \tau_k \le C \sum_{k=0}^{m} 3^k \tau_k$$

and

$$\sum_{n=0}^{m} \sum_{k=0}^{n} 3^{(1-\beta)n} 3^{\beta k} \tau_k \le \sum_{k=0}^{m} \sum_{n=k}^{n} 3^{(1-\beta)n} 3^{\beta k} \tau_k \le C 3^{(1-\beta)m} \sum_{k=0}^{m} 3^{\beta k} \tau_k.$$

Combining the three previous displays shows

$$\mathbb{E}\left[\left(\sum_{n=0}^{m} 3^n X_n^{\frac{1}{2}} \right)^2 \right] \le C 3^{(2-\beta)m} + C 3^{(2-\beta)m} \sum_{k=0}^{m} 3^{\beta k} \tau_k + C 3^m \sum_{k=0}^{m} 3^k \tau_k. \tag{4.4.27}$$

Moreover, since $0 < \beta \le 1$, we notice that for each $k, m \in \mathbb{N}$ with $k \le m$, $3^{(k-m)} \le 3^{\beta(k-m)}$. In particular the third term on the right-hand side of (4.4.27) is smaller than the second term on the right-hand side. Consequently, estimate (4.4.27) can be simplified to obtain

$$\mathbb{E}\left[\left(\sum_{n=0}^{m} 3^n X_n^{\frac{1}{2}} \right)^2 \right] \le C 3^{(2-\beta)m} + C 3^{(2-\beta)m} \sum_{k=0}^{m} 3^{\beta k} \tau_k.$$

Thus the estimate (4.4.26) becomes

$$\mathbb{E}\left[\left\| v(\cdot, \square_{m+1}, p, q) - l^{m+1}_{\bar{\mathbf{a}}_m^{-1} q - p} \right\|^2_{\underline{L}^2 \Lambda^{r-1}(\square_{m+1})} \right] \le C 3^{(2-\beta)m} + C 3^{(2-\beta)m} \sum_{k=0}^{m} 3^{\beta k} \tau_k.$$

□

Now that we have some control on the flatness of the maximizers of $J(\square_m, p, q)$, we can estimate $J(\square_m, p, \bar{\mathbf{a}}_m p)$ thanks to the Caccioppoli inequality.

Lemma 4.4.8. *There exists a constant $C(d, \lambda) < \infty$ such that, for every $m \in \mathbb{N}$ and $p \in B_1 \Lambda^r(\mathbb{R}^d)$,*

$$\mathbb{E}\left[J(\square_m, p, \bar{\mathbf{a}}_m p) \right] \le C 3^{-\beta m} + C 3^{-\beta m} \sum_{k=0}^{m} 3^{\beta k} \tau_k.$$

Proof. Fix $p \in B_1 \Lambda^r(\mathbb{R}^d)$, by Lemma 4.4.7,

$$\mathbb{E}\left[\left\| v(\cdot, \square_{m+1}, p, \bar{\mathbf{a}}_m p) \right\|^2_{\underline{L}^2 \Lambda^{r-1}(\square_{m+1})} \right] \le C 3^{(2-\beta)m} + C 3^{(2-\beta)m} \sum_{k=0}^{m} 3^{\beta k} \tau_k.$$

Applying the Caccioppoli inequality, Proposition 4.3.11, one obtains

$$\mathbb{E}\left[\|\mathrm{d}v(\cdot,\square_{m+1},p,\bar{\mathbf{a}}_m p)\|^2_{\underline{L}^2\Lambda^{r-1}(\square_m)}\right] \leq C3^{-\beta m} + C3^{-\beta m}\sum_{k=0}^{m} 3^{\beta k}\tau_k. \tag{4.4.28}$$

By (4.4.17), we have

$$3^{-d}\sum_{y\in 3^m\mathbb{Z}^d\cap\square_{m+1}} \mathbb{E}\left[\|\mathrm{d}v(\cdot,\square_{m+1},p,\bar{\mathbf{a}}_m p) - \mathrm{d}v(\cdot,y+\square_m,p,\bar{\mathbf{a}}_m p)\|^2_{\underline{L}^2\Lambda^r(y+\square_m)}\right] \leq C\tau_m.$$

In particular, this yields

$$\mathbb{E}\left[\|\mathrm{d}v(\cdot,\square_{m+1},p,\bar{\mathbf{a}}_m p) - \mathrm{d}v(\cdot,\square_m,p,\bar{\mathbf{a}}_m p)\|^2_{\underline{L}^2\Lambda^r(\square_m)}\right] \leq C\tau_m.$$

Combining the previous display with (4.4.28) gives

$$\begin{aligned}\mathbb{E}\left[\|\mathrm{d}v(\cdot,\square_m,p,\bar{\mathbf{a}}_m p)\|^2_{\underline{L}^2\Lambda^r(\square_m)}\right] &\leq C\tau_m + C3^{-\beta m} + C3^{-\beta m}\sum_{k=0}^{m} 3^{\beta k}\tau_k \\ &\leq C3^{-\beta m} + C3^{-\beta m}\sum_{k=0}^{m} 3^{\beta k}\tau_k.\end{aligned}$$

By (4.4.16) with $w = 0$, we deduce

$$\mathbb{E}\left[J(\square_m,p,\bar{\mathbf{a}}_m p)\right] \leq C3^{-\beta m} + C3^{-\beta m}\sum_{k=0}^{m} 3^{\beta k}\tau_k.$$

The proof of the lemma is complete. □

4.4.4. Quantitative convergence of the subadditive quantity J. We are now able to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. First note that since, for each $m \in \mathbb{N}$, the mapping $p \to \mathbb{E}\left[J(\square_m,p,\bar{\mathbf{a}}_m p)\right]$ is a positive definite quadratic form, we have

$$\frac{1}{d}\sum_{i=1}^{d}\mathbb{E}\left[J(\square_m,e_i,\bar{\mathbf{a}}_m e_i)\right] \leq \sup_{p\in B_1\Lambda^r(\mathbb{R}^d)}\mathbb{E}\left[J(\square_m,p,\bar{\mathbf{a}}_m p)\right] \leq \sum_{i=1}^{d}\mathbb{E}\left[J(\square_m,e_i,\bar{\mathbf{a}}_m e_i)\right].$$

Thus if we denote by

$$D_m = \sum_{i=1}^{d}\mathbb{E}\left[J(\square_m,e_i,\bar{\mathbf{a}}_m e_i)\right],$$

we get from the previous remark that the estimate (4.1.28) is equivalent to

$$D_m \leq C3^{-\alpha m}. \tag{4.4.29}$$

The reason we consider this particular quantity is because of the bound, for some $c := c(d,\lambda) > 0$,

$$D_m - D_{m+1} \geq c\tau_m. \tag{4.4.30}$$

Moreover notice that using the definition of $\bar{\mathbf{a}}_{m+1}$ (4.4.22) and the decomposition of J (4.4.4), for each $p \in \Lambda^r(\mathbb{R}^d)$, the quadratic form

$$q \to \mathbb{E}\left[J(\square_{m+1},p,q)\right] = \mathbb{E}\left[\nu(\square_{m+1},p)\right] + \star\left(\frac{1}{2}\bar{\mathbf{a}}^{-1}_{m+1}q\wedge q - p\wedge q\right)$$

attains it minimum at $q = \bar{\mathbf{a}}_{m+1}p$. Consequently

$$D_{m+1} = \sum_{i=1}^{d}\mathbb{E}\left[J(\square_{m+1},e_i,\bar{\mathbf{a}}_{m+1}e_i)\right] \leq \sum_{i=1}^{d}\mathbb{E}\left[J(\square_{m+1},e_i,\bar{\mathbf{a}}_m e_i)\right].$$

Thus we can compute

$$\begin{aligned} D_m - D_{m+1} &= \sum_{i=1}^{d} \left(\mathbb{E}\left[J(\square_m, e_i, \bar{\mathbf{a}}_m e_i) \right] - \mathbb{E}\left[J(\square_{m+1}, e_i, \bar{\mathbf{a}}_{m+1} e_i) \right] \right) \\ &\geq \sum_{i=1}^{d} \left(\mathbb{E}\left[J(\square_m, e_i, \bar{\mathbf{a}}_m e_i) \right] - \mathbb{E}\left[J(\square_{m+1}, e_i, \bar{\mathbf{a}}_m e_i) \right] \right) \\ &\geq \sum_{i=1}^{d} \left(\mathbb{E}\left[\nu(\square_m, e_i) \right] - \mathbb{E}\left[\nu(\square_{m+1}, e_i) \right] \right) \\ &\quad + \sum_{i=1}^{d} \left(\mathbb{E}\left[\nu^*(\square_m, \bar{\mathbf{a}}_m e_i) \right] - \mathbb{E}\left[\nu^*(\square_{m+1}, \bar{\mathbf{a}}_m e_i) \right] \right) \\ &\geq c \sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} \mathbb{E}\left[\nu^*(\square_m, e_i) \right] - \mathbb{E}\left[\nu^*(\square_{m+1}, e_i) \right] \\ &\quad + c \sup_{p \in B_1 \Lambda^r(\mathbb{R}^d)} \mathbb{E}\left[\nu^*(\square_m, e_i) \right] - \mathbb{E}\left[\nu^*(\square_{m+1}, e_i) \right] \\ &\geq c\tau_m. \end{aligned}$$

The main ingredient in the proof of Theorem 4.1.1 is to define the alternative quantity

$$\widetilde{D}_m := 3^{-\frac{\beta m}{2}} \sum_{n=0}^{m} 3^{\frac{\beta n}{2}} D_n,$$

where $\beta := \beta(d, \lambda)$ is the exponent which appears in Lemmas 4.4.7 and 4.4.8, and to use Lemma 4.4.8 to prove the estimate

$$\widetilde{D}_m \leq C 3^{-\alpha m}. \tag{4.4.31}$$

for some $\alpha := \alpha(d, \lambda) > 0$. The estimate (4.4.29) follows since, for each $m \in \mathbb{N}$

$$D_m \leq \widetilde{D}_m.$$

The proof of (4.4.31) can be split in 5 steps

Step 1. We show that there exist $\theta(d, \lambda) \in (0,1)$ and $C(d, \lambda) < \infty$ such that, for every $m \in \mathbb{N}$,

$$\widetilde{D}_{m+1} \leq \theta \widetilde{D}_m + C 3^{-\frac{\beta m}{2}}. \tag{4.4.32}$$

By (4.4.30) and $D_0 \leq C$, we have

$$\widetilde{D}_m - \widetilde{D}_{m+1} = 3^{-\frac{\beta m}{2}} \sum_{n=0}^{m} 3^{\frac{\beta n}{2}} (D_n - D_{n+1}) - C 3^{-\frac{\beta m}{2}} \geq 3^{-\frac{\beta m}{2}} \sum_{n=0}^{m} 3^{\frac{\beta n}{2}} \tau_n - C 3^{-\frac{\beta m}{2}}.$$

In particular, the previous estimate gives

$$\widetilde{D}_{m+1} \leq \widetilde{D}_m + C 3^{-\frac{\beta m}{2}}$$

From this and Lemma 4.4.8, we compute

$$\begin{aligned} \widetilde{D}_{m+1} \leq \widetilde{D}_m + D_0 3^{-\frac{\beta m}{2}} &= 3^{-\frac{\beta m}{2}} \sum_{n=0}^{m} 3^{\frac{\beta n}{2}} D_n + D_0 3^{-\frac{\beta m}{2}} \\ &\leq C 3^{-\frac{\beta m}{2}} \sum_{n=0}^{m} 3^{\frac{\beta n}{2}} \left(3^{-\beta n} + 3^{-\beta n} \sum_{k=0}^{n} 3^{\beta k} \tau_k \right) + C 3^{-\frac{\beta m}{2}} \\ &\leq C 3^{-\frac{\beta m}{2}} \sum_{n=0}^{m} \sum_{k=0}^{n} 3^{-\frac{\beta n}{2}} 3^{\beta k} \tau_k + C 3^{-\frac{\beta m}{2}} \\ &\leq C 3^{-\frac{\beta m}{2}} \sum_{k=0}^{m} \sum_{n=k}^{m} 3^{-\frac{\beta n}{2}} 3^{\beta k} \tau_k + C 3^{-\frac{\beta m}{2}} \\ &\leq C 3^{-\frac{\beta m}{2}} \sum_{k=0}^{m} 3^{\frac{\beta k}{2}} \tau_k + C 3^{-\frac{\beta m}{2}}. \end{aligned}$$

Combining the two previous displays gives

$$\widetilde{D}_{m+1} \le C\left(\widetilde{D}_m - \widetilde{D}_{m+1}\right) + C3^{-\frac{\beta m}{2}}.$$

A rearrangement of this estimates yields (4.4.32).

Step 2. Iterating (4.4.32) gives

$$\widetilde{D}_m \le \theta^m D_0 + C\sum_{k=0}^{n} \theta^k 3^{-\frac{\beta(m-k)}{2}}.$$

Without loss of generality, we can assume $\theta > 3^{-\frac{\beta}{2}}$ (since we can make θ closer to 1 if necessary). With this assumption, the second term on the right-hand side can be estimated,

$$\sum_{k=0}^{n} \theta^k 3^{-\frac{\beta(m-k)}{2}} \le C\theta^m.$$

Combining this with the fact that $\widetilde{D}_0 = D_0 \le C$, we obtain

$$\widetilde{D}_m \le C\theta^m,$$

which can be rewritten, with $\alpha = -\frac{\ln\theta}{\ln 3} > 0$,

$$\widetilde{D}_m \le C3^{-\alpha m}.$$

Step 3. We need to get the same estimate as (4.4.29) but with $\bar{\mathbf{a}}$ instead of $\bar{\mathbf{a}}_m$. First notice that by (4.4.29) and (4.4.30),

$$c\tau_m \le D_m - D_{m+1} \le D_m \le C3^{-\alpha m}.$$

Thus by (4.4.24), for every $q \in B_1\Lambda^{d-r}(\mathbb{R}^d)$, every $m \in \mathbb{N}$,

$$|\bar{\mathbf{a}}^{-1}q - \bar{\mathbf{a}}_m^{-1}q|^2 = \lim_{l\to\infty} |\bar{\mathbf{a}}_l^{-1}q - \bar{\mathbf{a}}_m^{-1}q|^2 \le \sum_{k=m}^{\infty} \tau_k \le C\sum_{k=m}^{\infty} 3^{-\alpha k} \le C3^{-\alpha m}.$$

Using the ellipticity assumption (4.1.21), we deduce, for each $p \in B_1\Lambda^{d-r}(\mathbb{R}^d)$,

$$|\bar{\mathbf{a}}p - \bar{\mathbf{a}}_m p|^2 \le C3^{-\alpha m}.$$

Using that J is a quadratic form according to (4.4.9), one obtains that there exists a constant $C(d,\lambda) < \infty$ such that, for each $m \in \mathbb{N}$, each $p, p' \in \Lambda^r(\mathbb{R}^d)$ and each $q, q' \in \Lambda^{d-r}(\mathbb{R}^d)$,

$$\left|J(\square_m, p, q) - J(\square_m, p', q')\right| \le C(|p-p'| + |q-q'|)(|p| + |p'| + |q| + |q'|).$$

Consequently, for each $p \in B_1\Lambda^r(\mathbb{R}^d)$ and each $m \in \mathbb{N}$

$$\begin{aligned} |J(\square_m, p, \bar{\mathbf{a}}p) - J(\square_m, p, \bar{\mathbf{a}}_m p)| &\le C|\bar{\mathbf{a}}p - \bar{\mathbf{a}}_m p|\left(1 + |\bar{\mathbf{a}}p| + |\bar{\mathbf{a}}_n p|\right) \\ &\le C3^{-\frac{\alpha}{2}m}. \end{aligned}$$

Redefining $\alpha = \frac{\alpha}{2}$ completes the proof of the quantitative homogenization estimate (4.1.28).

Step 4. We need to show that the mapping $\bar{\mathbf{a}}$ is unique. Given two maps $\bar{\mathbf{a}}, \bar{\mathbf{a}}' \in \mathcal{L}\left(\Lambda^r(\mathbb{R}^d), \Lambda^{d-r}(\mathbb{R}^d)\right)$ such that the estimate (4.1.28) is satisfied, one has, by (4.4.12), for each $m \in \mathbb{N}$, and each $p \in B_1\Lambda^r(\mathbb{R}^d)$,

$$\begin{aligned} \frac{1}{C}|\bar{\mathbf{a}}p - \bar{\mathbf{a}}'p|^2 &\le \mathbb{E}\left[\frac{1}{2}J(\square_m, p, \bar{\mathbf{a}}p) + \frac{1}{2}J(\square_m, p, \bar{\mathbf{a}}'p)\right] \\ &\le C3^{-\alpha m}. \end{aligned}$$

Sending $m \to \infty$ gives, for each $p \in B_1\Lambda^r(\mathbb{R}^d)$, $\bar{\mathbf{a}}p = \bar{\mathbf{a}}'p$. Consequently $\bar{\mathbf{a}} = \bar{\mathbf{a}}'$ and the proof of the first part of Theorem 4.1.1 is complete.

Step 5. We can now complete the proof of Theorem 4.1.1 by upgrading the stochastic integrability. This is a consequence of the following Lemma, the proof of which can be found in [**18**, Lemma 2.14].

LEMMA 4.4.9. *Suppose that $U \to \rho(U)$ is a (random) map from the set of bounded Lipschitz domains to $[0, +\infty)$ and satisfies, for a fixed $K \geq 0$:*

$$\rho(U) \text{ is } \mathcal{F}(U) - \text{measurable}$$

$$\rho(U) \leq K.$$

and, whenever U is the disjoint union of $U_1, \ldots, U_k$ up to a set of zero Lebesgue measure, one has

$$\rho(U) \leq \sum_{i=1}^{k} \frac{|U_i|}{|U|} \rho(U_i)$$

Then there exists a universal constant $C < \infty$ such that, for every $m, n \in \mathbb{N}$,

$$\rho(\square_{n+m+1}) \leq 2\mathbb{E}[\rho(\square_n)] + \mathcal{O}_1\left(CK3^{-md}\right).$$

Applying this result to

$$\rho(U) := \sup_{p \in B_1} J(U, p, \bar{\mathbf{a}}p),$$

gives, for each $m, n \in \mathbb{N}$,

$$\rho(\square_{n+m+1}) \leq 2\mathbb{E}[\rho(\square_n)] + \mathcal{O}_1\left(C3^{-md}\right) \leq C3^{-n\alpha} + \mathcal{O}_1\left(C3^{-md}\right).$$

Taking $n = m$ yields, for every $n \in \mathbb{N}$,

$$\rho(\square_{2n+1}) \leq C3^{-n\alpha} + \mathcal{O}_1\left(C3^{-nd}\right).$$

By redefining $\alpha := \min\left(\frac{\alpha}{2}, \frac{d}{2}\right)$, we obtain, for each $n \in \mathbb{N}$,

$$\rho(\square_n) \leq C3^{-n\alpha} + \mathcal{O}_1\left(C3^{-n\alpha}\right) \leq \mathcal{O}_1\left(C3^{-n\alpha}\right).$$

The proof of Theorem 4.1.1 is now complete. □

4.4.5. Quantitative convergence of the exterior derivative of the maximizer v. Before turning to the proof of Theorem 4.1.2 in the next section, we state and prove the following proposition, which is a consequence of Theorem 4.1.1 and gives some information about the flatness of the minimizers.

PROPOSITION 4.4.10. *There exist $\alpha := \alpha(d, \lambda) > 0$ and $C := C(d, \lambda) < \infty$ such that for each $1 \leq r \leq d$, each $(p, q) \in B_1\Lambda^r(\mathbb{R}^d) \times B_1\Lambda^{d-r}(\mathbb{R}^d)$ and each $m \in \mathbb{N}$,*

$$3^{-m} \left\| \mathrm{d}v(\cdot, \square_m, p, q) - \left(\bar{\mathbf{a}}^{-1}q - p\right) \right\|_{\underline{H}^{-1}\Lambda^r(\square)} + 3^{-m} \left\| \mathbf{a}\mathrm{d}v(\cdot, \square_m, p, q) - (q - \bar{\mathbf{a}}p) \right\|_{\underline{H}^{-1}\Lambda^{d-r}(\square)} \leq \mathcal{O}_1\left(C3^{-m\alpha}\right). \tag{4.4.33}$$

PROOF. The proof is split into 2 steps.

- *Step 1.* We prove that, for each $q \in B_1\Lambda^{d-r}(\mathbb{R}^d)$ and every $m, n \in \mathbb{N}$ such that $m \geq n$

$$3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| \left(\mathrm{d}v(\cdot, \square_m, 0, q) - \bar{\mathbf{a}}^{-1}q\right)_{y+\square_n} \right|^2 \leq \mathcal{O}_1\left(C3^{-\alpha n}\right). \tag{4.4.34}$$

  Similarly, for each $p \in B_1\Lambda^r(\mathbb{R}^d)$ and every $m, n \in \mathbb{N}$ such that $m \geq n$

$$3^{d(n-m)} \sum_{y \in \mathcal{Z}_{n,m}} \left| \left(\mathbf{a}\mathrm{d}v(\cdot, \square_m, p, 0) - \bar{\mathbf{a}}p\right)_{y+\square_n} \right|^2 \leq \mathcal{O}_1\left(C3^{-\alpha n}\right). \tag{4.4.35}$$

- *Step 2.* We deduce from the previous step and the multiscale Poincaré inequality, Proposition 4.3.10, the estimate (4.4.33).

Step 1. We first deal with the case $m = n$. In this specific case, the estimate (4.4.34) reads

$$\left| \left(\mathrm{d}v(\cdot, \square_n, 0, q) - \bar{\mathbf{a}}^{-1}q\right)_{\square_n} \right|^2 \leq \mathcal{O}_1\left(C3^{-\alpha n}\right).$$

To argue this, note that, by the first variation for J,

$$J(\square_n, 0, q) = \frac{1}{2} \left(\mathrm{d}v(\cdot, \square_m, 0, q)\right)_{\square_m} \wedge q.$$

Moreover, the map $q \to (\mathrm{d}v(\cdot,\square_m,0,q))_{\square_m}$ is bounded by (4.4.10) and symmetric since, for each $q,q' \in \Lambda^{d-r}(\mathbb{R}^d)$,

$$\begin{aligned}\left(\mathrm{d}v\left(\cdot,\square_m,0,q'\right)\right)_{\square_m} \wedge q &= \fint_{\square_m} \mathrm{d}v\left(\cdot,\square_m,0,q'\right) \wedge \mathbf{a}\mathrm{d}v\left(\cdot,\square_m,0,q\right) \\ &= \fint_{\square_m} \mathrm{d}v\left(\cdot,\square_m,0,q\right) \wedge \mathbf{a}\mathrm{d}v\left(\cdot,\square_m,0,q'\right) \\ &= \left(\mathrm{d}v\left(\cdot,\square_m,0,q\right)\right)_{\square_m} \wedge q'.\end{aligned}$$

A combination of the two previous ideas and Theorem 4.1.1 gives

$$\begin{aligned}\sup_{q\in B_1\Lambda^{d-r}} \left|\left(\mathrm{d}v\left(\cdot,\square_n,0,q\right) - \bar{\mathbf{a}}^{-1}q\right)_{\square_n}\right|^2 &\leq C \sup_{q\in B_1\Lambda^{d-r}} \left|J(\square_n,0,q) - \bar{\mathbf{a}}^{-1}q\wedge q\right| \\ &\leq \mathcal{O}_1\left(C3^{-n\alpha}\right).\end{aligned} \tag{4.4.36}$$

To obtain the general case $m \geq n$ from the specific case $m = n$, we compute

$$\begin{aligned}&3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} \left|\left(\mathrm{d}v\left(\cdot,\square_m,0,q\right) - \bar{\mathbf{a}}^{-1}q\right)_{y+\square_n}\right|^2 \\ &\quad\leq C3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} \left|\left(\mathrm{d}v\left(\cdot,\square_m,0,q\right) - \mathrm{d}v\left(\cdot,y+\square_n,0,q\right)\right)_{y+\square_n}\right|^2 \\ &\quad\quad+ C3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} \left|\left(\mathrm{d}v\left(\cdot,y+\square_n,0,q\right) - \bar{\mathbf{a}}^{-1}q\right)_{y+\square_n}\right|^2 \\ &\quad\leq C3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} \left\|\mathrm{d}v\left(\cdot,\square_m,0,q\right) - \mathrm{d}v\left(\cdot,y+\square_n,0,q\right)\right\|^2_{\underline{L}^2(y+\square_n)} \\ &\quad\quad+ C3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} \left|\left(\mathrm{d}v\left(\cdot,y+\square_n,0,q\right) - \bar{\mathbf{a}}^{-1}q\right)_{y+\square_n}\right|^2 \\ &\quad\leq C3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} J(y+\square_n,0,q) - J(\square_m,0,q) \\ &\quad\quad+ C3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} \left|\left(\mathrm{d}v\left(\cdot,y+\square_n,0,q\right) - \bar{\mathbf{a}}^{-1}q\right)_{y+\square_n}\right|^2\end{aligned}$$

To deal with the first term on the right-hand side, we note that, for each $y \in \mathcal{Z}_{m,n}$,

$$\begin{aligned}J(y+\square_n,0,q) - J(\square_m,0,q) &\leq \left|J(y+\square_n,0,q) - \bar{\mathbf{a}}^{-1}q\wedge q\right| + \left|J(\square_m,0,q) - \bar{\mathbf{a}}^{-1}q\wedge q\right| \\ &\leq \mathcal{O}_1(C3^{-\alpha n}) + \mathcal{O}_1(C3^{-\alpha m}) \\ &\leq \mathcal{O}_1(C3^{-\alpha n}),\end{aligned}$$

by the stationarity assumption (4.1.24). Using the inequality (4.1.19), we eventually obtain

$$3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} J(y+\square_n,0,q) - J(\square_m,0,q) \leq \mathcal{O}_1(C3^{-n\alpha}).$$

To deal with the second term on the right-hand side, we have by the stationarity assumption (4.1.24) and (4.4.36), for each $y \in \mathcal{Z}_{m,n}$,

$$\left|\left(\mathrm{d}v\left(\cdot,y+\square_n,0,q\right) - \bar{\mathbf{a}}^{-1}q\right)_{y+\square_n}\right|^2 \leq \mathcal{O}_1\left(C3^{-\alpha n}\right).$$

Using the inequality (4.1.19), we obtain

$$3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} \left|\left(\mathrm{d}v\left(\cdot,y+\square_n,0,q\right) - \bar{\mathbf{a}}^{-1}q\right)_{y+\square_n}\right|^2 \leq \mathcal{O}_1\left(C3^{-\alpha n}\right).$$

The proof of (4.4.34) is thus complete. The proof of (4.4.35) is similar, the details are left to the reader.

Step 2. From Step 1 and (ii) of Corollary 4.1.4, one has

$$\sum_{n=0}^{m-1} 3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} \left|\left(\mathrm{d}v\left(\cdot, y+\square_n, 0, q\right) - \bar{\mathbf{a}}^{-1}q\right)_{y+\square_n}\right|^2 \leq \mathcal{O}_1\left(C3^{(1-\alpha)m}\right)$$

and

$$\sum_{n=0}^{m-1} 3^{d(n-m)} \sum_{y\in\mathcal{Z}_{n,m}} \left|\left(\mathbf{a}\mathrm{d}v\left(\cdot, y+\square_n, p, 0\right) - \bar{\mathbf{a}}p\right)_{y+\square_n}\right|^2 \leq \mathcal{O}_1\left(C3^{(1-\alpha)m}\right).$$

By the multiscale Poincaré inequality, Proposition 4.3.6, the bound on the L^2 norm of $\mathrm{d}v$, estimate (4.4.10), and the previous estimates, one obtains for each $(p,q)\in B_1\Lambda^r(\mathbb{R}^d)\times B_1\Lambda^{d-r}(\mathbb{R}^d)$,

$$\left\|\mathrm{d}v\left(\cdot,\square_m,p,q\right) - \left(\bar{\mathbf{a}}^{-1}q - p\right)\right\|_{\underline{H}^{-1}\Lambda^r(\square)} + \left\|\mathbf{a}\mathrm{d}v\left(\cdot,\square_m,p,q\right) - \left(q - \bar{\mathbf{a}}p\right)\right\|_{\underline{H}^{-1}\Lambda^{d-r}(\square)} \leq \mathcal{O}_1\left(C3^{(1-\alpha)m}\right).$$

Dividing both sides of the previous inequality by 3^m yields (4.4.33) and completes the proof of Proposition 4.4.10. □

4.5. Homogenization of the Dirichlet problem

The goal of this section is to study the Dirichlet problem for the equation $\mathrm{d}\mathbf{a}\mathrm{d}u = 0$ and to establish Theorem 4.1.2. We first prove existence and uniqueness of solution for this equation.

PROPOSITION 4.1.7. *Let U be a bounded smooth domain of $\mathbb{R}^d$ and $1\leq r\leq d$. Let $f\in H^1_{\mathrm{d}}\Lambda^{r-1}(U)$, then for any measurable map $\mathbf{a}:\mathbb{R}^d\to\mathcal{L}\left(\Lambda^r(\mathbb{R}^d),\Lambda^{(d-r)}\left(\mathbb{R}^d\right)\right)$ satisfying* (4.1.21) *and* (4.1.20)*, there exists a unique solution in $f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U)\cap\left(C^{r-1}_{\mathrm{d},0}(U)\right)^\perp$ of the equation*

$$\begin{cases}\mathrm{d}\left(\mathbf{a}\mathrm{d}u\right) = 0 & \text{in } U\\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U,\end{cases} \tag{4.5.1}$$

in the sense that, for each $v\in H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$,

$$\int_U \mathrm{d}u\wedge\mathbf{a}\mathrm{d}v = 0.$$

The solution satisfies the estimate, for some $C := C(d,\lambda,U)<\infty$,

$$\|u\|_{H^1_{\mathrm{d}}\Lambda^{r-1}(U)} \leq C\,\|\mathrm{d}f\|_{L^2\Lambda^r(U)}.$$

Moreover if we enlarge the set of admissible solutions to the space $f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$, one looses the uniqueness property, but if $v,w\in f+H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$ are two solutions of (4.5.1)*, then*

$$v - w\in C^{r-1}_{\mathrm{d},0}.$$

PROOF. The existence and uniqueness of such a solution are obtained by minimizing the quantity

$$\mathcal{J}(v) := \langle\mathrm{d}f + \mathrm{d}v, \mathrm{d}f + \mathrm{d}v\rangle_U$$

on the space $H^1_{\mathrm{d},0}\Lambda^{r-1}(U)\cap\left(C^r_{\mathrm{d},0}\right)^\perp$ and requires to use the Poincaré inequality, Proposition 4.3.1. The techniques are standard, we thus omit the details. □

We now turn to the statement and the proof of the main theorem of this section, Theorem 4.1.2.

THEOREM 4.1.2 (Homogenization Theorem). *Let U be a bounded smooth domain of $\mathbb{R}^d$ and $1\leq r\leq d$. Fix $\varepsilon\in(0,1]$ and $f\in H^1_{\mathrm{d}}\Lambda^{r-1}(U)$ such that $\mathrm{d}f\in H^1_{\mathrm{d}}\Lambda^r(U)$. Let $u^\varepsilon, u\in f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U)\cap\left(C^r_{\mathrm{d},0}(U)\right)^\perp$ respectively denote the solutions of the Dirichlet problems*

$$\begin{cases}\mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathrm{d}u^\varepsilon\right) = 0 & \text{in } U\\ \mathbf{t}u^\varepsilon = \mathbf{t}f & \text{on } \partial U.\end{cases} \quad \text{and} \quad \begin{cases}\mathrm{d}\left(\bar{\mathbf{a}}\mathrm{d}u\right) = 0 & \text{in } U\\ \mathbf{t}u = \mathbf{t}f & \text{on } \partial U.\end{cases}$$

Then there exist an exponent $\alpha := \alpha(d,\lambda,U) > 0$ *and a constant* $C := C(d,\lambda,U) < \infty$ *such that*

$$\|u^\varepsilon - u\|_{L^2\Lambda^r(U)} + \|\mathrm{d}u^\varepsilon - \mathrm{d}u\|_{\underline{H}^{-1}\Lambda^r(U)} \leq \mathcal{O}_1\left(C\,\|\mathrm{d}f\|_{H^1\Lambda^r(U)}\,\varepsilon^\alpha\right).$$

Proof. Without loss of generality, one can assume that $|U| = 1$. Fix $l > 0$, this parameter represents the thickness of a boundary layer we need to remove in the argument, it will be chosen at the end of the proof and shall solely depend on ϵ. For $R > 0$, denote by $U_R := \{x \in U : \operatorname{dist}(x,\partial U) > R\}$. For $I \subseteq \{1,\ldots,d\}$ of cardinality r and $m \in \mathbb{N}$, we denote by $\phi_{m,I}$ the unique solution in $l_{\mathrm{d}x_I} + H^1_{\mathrm{d},0}\Lambda^{r-1}(\square_m) \cap \left(C^r_{\mathrm{d},0}(\square_m)\right)^\perp$ of

$$\begin{cases} \mathrm{d}(\mathbf{a}\mathrm{d}u) = 0 & \text{in } \square_m \\ \mathbf{t}u = \mathbf{t}l_{\mathrm{d}x_I} & \text{on } \partial\square_m, \end{cases}$$

where $l_{\mathrm{d}x_I}$ is defined in (4.3.8) satisfies $\mathrm{d}l_{\mathrm{d}x_I} = \mathrm{d}x_I$. In particular, one has

$$\mathrm{d}\phi_{m,I} = \mathrm{d}v(\cdot,\square_m,p).$$

Let m be the smallest integer such that

$$U \subseteq \varepsilon\square_m,$$

and define the two-scale expansion, with the convention $\mathrm{d}u := \sum_{|I|=r}(\mathrm{d}u)_I \mathrm{d}x_I$,

$$w_0^\varepsilon(x) := u(x) + \varepsilon\zeta_l(x)\sum_{|I|=r}(\mathrm{d}u)_I(x)\phi_{m,I}\left(\frac{x}{\varepsilon}\right), \tag{4.5.2}$$

where $\zeta_l \in C_c^\infty(U)$ is a smooth cutoff function satisfying, for every $k \in \mathbb{N}$:

$$0 \leq \zeta_l \leq 1, \quad \zeta_l = 1 \text{ in } U_{2l}, \quad \zeta_l = 0 \text{ in } U \smallsetminus U_l, \quad \left|\nabla^k \zeta_l\right| \leq C(k,d,U) l^{-k}. \tag{4.5.3}$$

Note that $w_0^\varepsilon \in f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$. Since it is more convenient to work with an element of $f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \cap \left(C^r_{\mathrm{d},0}(U)\right)^\perp$ (to have the Poincaré inequality), we further define

$$w^\varepsilon := f + \operatorname{Proj}_{\left(C^r_{\mathrm{d},0}(U)\right)^\perp}(w_0^\varepsilon - f).$$

where $\operatorname{Proj}_{\left(C^r_{\mathrm{d},0}(U)\right)^\perp}$ denotes the L^2-orthogonal projection on the space $\left(C^r_{\mathrm{d},0}(U)\right)^\perp$. Note that $w^\varepsilon \in f + H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \cap \left(C^r_{\mathrm{d},0}(U)\right)^\perp$ by construction and that it satisfies

$$\mathrm{d}w_0^\varepsilon = \mathrm{d}w^\varepsilon \text{ in } U. \tag{4.5.4}$$

We then consider the map

$$\mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathrm{d}w^\varepsilon\right) : \begin{cases} H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \to \mathbb{R}, \\ v \to \int_U \mathrm{d}w^\varepsilon \wedge \mathbf{a}\left(\frac{x}{\varepsilon}\right)\mathrm{d}v. \end{cases}$$

and denote by

$$\left\|\mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathrm{d}w^\varepsilon\right)\right\|_{H^{-1}_{\mathrm{d}}\Lambda^{r-1}(U)} := \sup\left\{\int_U \mathrm{d}w^\varepsilon \wedge \mathbf{a}\left(\frac{x}{\varepsilon}\right)\mathrm{d}v : v \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \text{ s.t } \|v\|_{H^1_{\mathrm{d}}\Lambda^r(U)} \leq 1\right\}. \tag{4.5.5}$$

The idea of the proof is to compare u^ε to the function w^ε. The proof is split into 7 steps.

Step 1. In this step, we show that the norm $H^{-1}_{\mathrm{d}}\Lambda^{d-r+1}(U)$ defined in (4.5.5) is equivalent to the norm

$$\left\|\mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathrm{d}w^\varepsilon\right)\right\|_{H^{-1}\Lambda^{d-r+1}(U)} := \sup\left\{\int_U \mathrm{d}w^\varepsilon \wedge \mathbf{a}\left(\frac{x}{\varepsilon}\right)\mathrm{d}v : v \in H^1_0\Lambda^{r-1}(U) \text{ s.t } \|v\|_{H^1\Lambda^r(U)} \leq 1\right\}.$$

This result is a consequence of the following property, for some $C := C(d,\lambda,U) < \infty$,

$$\forall v \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U), \exists w \in H^1_0\Lambda^{r-1}(U) \text{ such that } \mathrm{d}w = \mathrm{d}v \text{ and } \|w\|_{H^1\Lambda^r(U)} \leq C\,\|v\|_{H^1_{\mathrm{d}}\Lambda^r(U)}.$$

To prove this, we follow the arguments of the proof of [**121**, Theorem 1.1]. Let $(O_j)_{1\le j\le N}$ be a finite, open covering of $\overline{U}$ such that $O_j \cap U$ is a Lipschitz bounded star-shaped domain. Then let $(\phi_j)_{1\le j\le N}$ be a smooth partition of unity such that $\operatorname{supp}\phi_j \subseteq O_j$ for $1 \le j \le N$. Note that the form $\phi_j v$ belongs to $H^1_{\mathrm{d},0}\Lambda^{r-1}(O_j\cap U)$. Thus by Proposition 4.2.4, there exists a function $w_j \in H^1_0\Lambda^{r-1}(O_j \cap U)$ satisfying

$$\|w_j\|_{H^1\Lambda^{r-1}(O_j\cap U)} \le C\,\|\phi_j v\|_{H^1_{\mathrm{d}}\Lambda^{r-1}(O_j\cap U)}.$$

We then extend the forms $\phi_j v$ and w_j by 0 to $\mathbb{R}^d$, so that $\phi_j v \in H^1_{\mathrm{d},0}\Lambda^{r-1}\left(\mathbb{R}^d\right)$ and $w_j \in H^1\Lambda^{r-1}\left(\mathbb{R}^d\right)$ satisfy

$$\mathrm{d}w_j = \mathrm{d}\left(\phi_j v\right) \text{ in } \mathbb{R}^d.$$

We then define

$$w := \sum_{j=1}^N w_j,$$

so that

$$w \in H^1_0\Lambda^{r-1}(U) \text{ and } \mathrm{d}w = \sum_{j=1}^N \mathrm{d}\left(\phi_j v\right) = \mathrm{d}v.$$

We also have the estimate

$$\|w\|_{H^1\Lambda^{r-1}(U)} \le C\,\|v\|_{H^1_{\mathrm{d}}\Lambda^{r-1}(U)}.$$

This completes the proof of Step 1.

Step 2. We show the $H^{-1}\Lambda^{r-1}(U)$ estimate

$$\left\|\mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathrm{d}w^\varepsilon\right)\right\|_{H^{-1}\Lambda^{r-1}(U)} \le \begin{cases} C\,\|\mathrm{d}f\|_{H^1\Lambda^r(U)}\left(l^{\frac{1}{d-2}} + \mathcal{O}_1\left(\dfrac{\varepsilon^\alpha}{l^{3+d/2}}\right)\right) & \text{if } d\ge 3,\\ C\,\|\mathrm{d}f\|_{H^1\Lambda^r(U)}\left(l^{\frac14} + \mathcal{O}_1\left(\dfrac{\varepsilon^\alpha}{l^{3+d/2}}\right)\right) & \text{if } d=2.\end{cases} \tag{4.5.6}$$

We first compute the exterior derivative of w^ε, by (4.5.2) and (4.5.4),

$$\begin{aligned}\mathrm{d}w^\varepsilon &= \mathrm{d}u + \zeta_l\sum_{|I|=r}(\mathrm{d}u)_I\mathrm{d}\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right) + \varepsilon\sum_{|I|=r}\mathrm{d}\left(\zeta_l(\mathrm{d}u)_I\right)\wedge\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\\ &= (1-\zeta_l)\mathrm{d}u + \sum_{|I|=r}\zeta_l(\mathrm{d}u)_I\left(\mathrm{d}x_I + \mathrm{d}\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\right) + \varepsilon\sum_{|I|=r}\mathrm{d}\left(\zeta_l(\mathrm{d}u)_I\right)\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right).\end{aligned}$$

From this we deduce, in the weak sense

$$\begin{aligned}\mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathrm{d}w^\varepsilon\right) &= \mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\left((1-\zeta_l)\mathrm{d}u + \varepsilon\sum_{|I|=r}\mathrm{d}\left(\zeta_l(\mathrm{d}u)_I\right)\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\right)\right)\\ &\quad + \sum_{|I|=r}\mathrm{d}\left(\zeta_l(\mathrm{d}u)_I\right)\wedge\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\left(\mathrm{d}x_I + \mathrm{d}\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\right).\end{aligned}$$

On the other hand, since u satisfies $\mathrm{d}(\bar{\mathbf{a}}\mathrm{d}u) = 0$, one sees that

$$\sum_{|I|=r}\mathrm{d}\left(\zeta_l\left(\mathrm{d}u\right)_I\right)\wedge\bar{\mathbf{a}}\mathrm{d}x_I = \mathrm{d}\left(\zeta_l\bar{\mathbf{a}}\mathrm{d}u\right) = -\mathrm{d}\left((1-\zeta_l)\bar{\mathbf{a}}\mathrm{d}u\right).$$

Consequently, in the weak sense

$$\begin{aligned}\mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathrm{d}w^\varepsilon\right) &= \mathrm{d}\left(\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) - \bar{\mathbf{a}}\right)(1-\zeta_l)\mathrm{d}u\right) + \varepsilon\sum_{|I|=r}\mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\mathrm{d}\left(\zeta_l(\mathrm{d}u)_I\right)\wedge\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\right)\\ &\quad + \sum_{|I|=r}\mathrm{d}\left(\zeta_l(\mathrm{d}u)_I\right)\wedge\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right)\left(\mathrm{d}x_I + \mathrm{d}\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\right) - \bar{\mathbf{a}}\mathrm{d}x_I\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\left\| \mathrm{d} \left(\mathbf{a} \left(\frac{\cdot}{\varepsilon} \right) \mathrm{d} w^{\varepsilon} \right) \right\|_{H^{-1}\Lambda^{r-1}(U)} \\ &\leq \sum_{|I|=r} \left\| \mathrm{d} \left(\zeta_l (\mathrm{d}u)_I \right) \right\|_{W^{1,\infty}(U)} \left\| \mathbf{a} \left(\frac{\cdot}{\varepsilon} \right) \left(\mathrm{d}x_I + \mathrm{d}\phi_{m,I} \left(\frac{\cdot}{\varepsilon} \right) \right) - \bar{\mathbf{a}} \mathrm{d}x_I \right\|_{H^{-1}\Lambda^r(U)} \\ &+ \left\| \left(\mathbf{a} \left(\frac{\cdot}{\varepsilon} \right) - \bar{\mathbf{a}} \right) (1 - \zeta_l) \mathrm{d}u \right\|_{L^2\Lambda^r(U)} \\ &+ \varepsilon \sum_{|I|=r} \left\| \mathbf{a} \left(\frac{\cdot}{\varepsilon} \right) \mathrm{d} \left(\zeta_l (\mathrm{d}u)_I \right) \wedge \phi_{m,I} \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2\Lambda^r(U)} \\ &=: T_1 + T_2 + T_3\end{aligned}$$

To bound the term on the right-hand side, we appeal to the interior regularity estimate, Proposition 4.A.3 and the assumption (4.5.3) on ζ_l, so that one has

$$\left\| \mathrm{d} \left(\zeta_l (\mathrm{d}u)_I \right) \right\|_{W^{1,\infty}(U)} \leq \frac{C}{l^{3+d/2}} \| \mathrm{d}f \|_{L^2\Lambda^r(U)}, \tag{4.5.7}$$

hence by Proposition 4.4.10,

$$T_1 \leq \frac{C}{l^{3+d/2}} \| \mathrm{d}f \|_{H^1(U)} \mathcal{O}_1 \left(\varepsilon^{\alpha} \right).$$

The bound for T_3 is similar, by Proposition 4.4.10 and the Poincaré inequality, Proposition 4.3.1, one has

$$\varepsilon \left\| \phi_{m,I} \left(\frac{\cdot}{\varepsilon} \right) \right\|_{L^2\Lambda^r(U)} \leq \mathcal{O}_1 \left(C\varepsilon^{\alpha} \right).$$

So by (4.5.7), one has

$$T_3 \leq \frac{C}{l^{3+d/2}} \| \mathrm{d}f \|_{H^1(U)} \mathcal{O}_1 \left(\varepsilon^{\alpha} \right).$$

To estimate the second term T_2, the idea is to apply the boundary regularity result proved in the appendix, Proposition 4.A.4. Since $\mathrm{d}f$ is assumed to be in $H^1\Lambda^r(U)$ and U is assumed to be smooth, one has

$$\| \mathrm{d}u \|_{H^1\Lambda^r(U)} \leq C \| \mathrm{d}\bar{\mathbf{a}}\mathrm{d}f \|_{L^2\Lambda^{d-r+1}(U)} \leq C \| \mathrm{d}f \|_{H^1\Lambda^r(U)}.$$

This implies, via the Sobolev imbedding Theorem, that $\mathrm{d}u$ is in $L^{\frac{2d}{d-2}}\Lambda^r(U)$ if $d \geq 3$ and any $L^p\Lambda^r(U)$ if $d = 2$, with the estimate

$$\begin{cases} \| \mathrm{d}u \|_{L^{\frac{2d}{d-2}}\Lambda^r(U)} \leq C & \| \mathrm{d}f \|_{H^1\Lambda^r(U)} \quad \text{if } d \geq 3, \\ \| \mathrm{d}u \|_{L^p\Lambda^r(U)} \leq C_p & \| \mathrm{d}f \|_{H^1\Lambda^r(U)} \quad \text{if } d = 2, \end{cases}$$

for some $C := C(d, U) < \infty$ and $C_p := C(p, U) < \infty$. We now set $p = 4$ (but any $p > 2$ would work). Using this estimate and the fact that $(1 - \zeta_l)$ is supported in $U \setminus U_{2l}$, gives, by Hölder inequality,

$$T_2 \leq C \| \mathrm{d}u \|_{L^2\Lambda^r(U \setminus U_{2r})} \leq \begin{cases} C |U \setminus U_{2l}|^{\frac{1}{d-2}} \| \mathrm{d}u \|_{L^{\frac{2d}{d-2}}\Lambda^r(U)} & \text{if } d \geq 3, \\ C |U \setminus U_{2l}|^{\frac{1}{4}} \| \mathrm{d}u \|_{L^4\Lambda^r(U)} & \text{if } d = 2, \end{cases}$$

Combining the few previous results completes the proof of (4.5.6).

Step 3. We deduce from the previous step the H^1 estimate

$$\| u^{\varepsilon} - w^{\varepsilon} \|_{H^1_{\mathrm{d}}\Lambda^{r-1}(U)} \leq \begin{cases} C \| \mathrm{d}f \|_{H^1\Lambda^r(U)} \left(l^{\frac{1}{d-2}} + \mathcal{O}_1 \left(\frac{\varepsilon^{\alpha}}{l^{3+d/2}} \right) \right) & \text{if } d \geq 3, \\ C \| \mathrm{d}f \|_{H^1\Lambda^r(U)} \left(l^{\frac{1}{4}} + \mathcal{O}_1 \left(\frac{\varepsilon^{\alpha}}{l^{3+d/2}} \right) \right) & \text{if } d = 2. \end{cases} \tag{4.5.8}$$

Indeed, testing (4.5.6) with $u^\varepsilon - w^\varepsilon \in H^1_{\mathrm{d},0}(U)$, and using Step 1, one obtains

$$\left| \int_U \mathrm{d}\left(u^\varepsilon - w^\varepsilon\right)(x) \wedge \mathbf{a}\left(\frac{x}{\varepsilon}\right) \mathrm{d}w^\varepsilon(x) \right| \leq \left\| u^\varepsilon - w^\varepsilon \right\|_{H^1_{\mathrm{d}}\Lambda^{r-1}(U)} \left\| \mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \mathrm{d}w^\varepsilon\right) \right\|_{H^{-1}_{\mathrm{d}}\Lambda^{r-1}(U)}$$
$$\leq \left\| u^\varepsilon - w^\varepsilon \right\|_{H^1_{\mathrm{d}}\Lambda^{r-1}(U)} \left\| \mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \mathrm{d}w^\varepsilon\right) \right\|_{H^{-1}\Lambda^{r-1}(U)}.$$

Meanwhile, testing this equation with u^ε shows

$$\int_U \mathrm{d}\left(u^\varepsilon - w^\varepsilon\right)(x) \wedge \mathbf{a}\left(\frac{x}{\varepsilon}\right) \mathrm{d}u^\varepsilon(x) = 0.$$

Combining the two previous displays with the Poincaré inequality yields,

$$\left\| \mathrm{d}u^\varepsilon - \mathrm{d}w^\varepsilon \right\|^2_{L^2\Lambda^r(U)} \leq C \int_U \mathrm{d}\left(u^\varepsilon - w^\varepsilon\right)(x) \wedge \mathbf{a}\left(\frac{x}{\varepsilon}\right) \mathrm{d}\left(u^\varepsilon - v^\varepsilon\right)(x)$$
$$\leq C \left\| u^\varepsilon - w^\varepsilon \right\|_{H^1_{\mathrm{d}}\Lambda^r(U)} \left\| \mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \mathrm{d}w^\varepsilon\right) \right\|_{H^{-1}_{\mathrm{d}}\Lambda^{r-1}(U)}$$
$$\leq C \left\| \mathrm{d}u^\varepsilon - \mathrm{d}w^\varepsilon \right\|_{L^2\Lambda^r(U)} \left\| \mathrm{d}\left(\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \mathrm{d}w^\varepsilon\right) \right\|_{H^{-1}\Lambda^{r-1}(U)}.$$

Thus

$$\left\| \mathrm{d}u^\varepsilon - \mathrm{d}w^\varepsilon \right\|_{L^2\Lambda^r(U)} \leq C \left\| \mathrm{d}\mathbf{a}\left(\frac{\cdot}{\varepsilon}\right) \mathrm{d}w^\varepsilon \right\|_{H^{-1}\Lambda^{r-1}(U)}.$$

Using the estimate (4.5.6) and another application of the Poincaré inequality completes the proof of (4.5.8)

Step 4. Recall that at the beginning of the proof, we assumed $|U| = 1$. We extend Definition 4.3.7 to the set U by setting, for each $\omega \in L^2\Lambda^r(U)$,

$$\|\omega\|_{\underline{H}^{-1}\Lambda^r(U)} := \sup\left\{ \langle \omega, \alpha \rangle_U \,:\, \alpha \in H^1\Lambda^r(U),\, |(\alpha)_U| + \|\nabla \alpha\|_{L^2\Lambda^r(U)} \leq 1 \right\}.$$

Note that this norm is a slightly stronger than the standard H^{-1} norm which only requires to have test functions in H^1_0. In this step, we prove that for each $w \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \cap \left(C^r_{\mathrm{d},0}\right)^\perp$, one has

$$\|w\|_{L^2\Lambda^{r-1}(U)} \leq \|\mathrm{d}w\|_{\underline{H}^{-1}\Lambda^r(U)}.$$

To do so, let v be the unique solution in $H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \cap \left(C^r_{\mathrm{d},0}\right)^\perp$ of the problem

$$\begin{cases} \delta \mathrm{d}v = w \text{ in } U \\ \quad \mathbf{t}v = 0 \text{ on } \partial U. \end{cases} \tag{4.5.9}$$

The existence and uniqueness of such a solution are obtained by minimizing the quantity

$$\mathcal{J}(v) := \langle \mathrm{d}v, \mathrm{d}v \rangle_U - \langle w, v \rangle_U$$

on the space $H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \cap \left(C^r_{\mathrm{d},0}\right)^\perp$ and requires to use the Poincaré inequality, Proposition 4.3.1. The details are left to the reader.

If v is a solution (4.5.9), note that $\mathrm{d}\mathrm{d}v = 0 \in L^2\Lambda^{r+1}(U)$, $\delta\mathrm{d}v = w \in L^2\Lambda^{r-1}(U)$ and $\mathbf{t}\mathrm{d}w = 0$ (this last property is implied by the condition $\mathbf{t}v = 0$, see for instance [**123**]). As a consequence, the Gaffney-Friedrichs inequality, Proposition 4.3.3, implies that $\mathrm{d}v \in H^1\Lambda^r(U)$, together with the estimate

$$\|\mathrm{d}v\|_{H^1\Lambda^r(U)} \leq C\left(\|w\|_{L^2\Lambda^{r-1}(U)} + \|\mathrm{d}v\|_{L^2\Lambda^r(U)} \right).$$

Testing (4.5.9) with v and using the Poincaré inequality also shows

$$\|\mathrm{d}v\|_{L^2\Lambda^r(U)} \leq C \|w\|_{L^2\Lambda^{r-1}(U)}.$$

Combining the two previous displays shows

$$\|\mathrm{d}v\|_{H^1\Lambda^r(U)} \leq C \|w\|_{L^2\Lambda^{r-1}(U)}. \tag{4.5.10}$$

Testing (4.5.9) with w then shows,

$$\langle \mathrm{d}w, \mathrm{d}v\rangle_U = \langle w, w\rangle_U = \|w\|_{L^2\Lambda^{r-1}(U)}.$$

On the other hand, by the definition of the $\underline{H}^{-1}$ norm and (4.5.10), one has

$$\begin{aligned}\langle \mathrm{d}w, \mathrm{d}v\rangle_U &\le \|\mathrm{d}w\|_{\underline{H}^{-1}\Lambda^r(U)}\|\mathrm{d}v\|_{H^1\Lambda^r(U)}\\ &\le \|\mathrm{d}w\|_{\underline{H}^{-1}\Lambda^r(U)}\|w\|_{L^2\Lambda^{r-1}(U)}.\end{aligned}$$

Combining the two previous displays completes the proof of Step 4.

Step 5. We prove that

$$\|\mathrm{d}w^\varepsilon - \mathrm{d}u\|_{\underline{H}^{-1}\Lambda^r(U)} \le \mathcal{O}_1\left(\frac{C\,\|\mathrm{d}f\|_{H^1\Lambda^r(U)}\,\varepsilon^\alpha}{l^{2+d/2}}\right). \tag{4.5.11}$$

One has that

$$\mathrm{d}w^\varepsilon - \mathrm{d}u = \mathrm{d}\left(\varepsilon\zeta_l \sum_{|I|=r}(\mathrm{d}u)_I\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\right)$$

and therefore, since $w^\varepsilon - u \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$, one has

$$\|\mathrm{d}w^\varepsilon - \mathrm{d}u\|_{\underline{H}^{-1}\Lambda^r(U)} \le C\,\|\mathrm{d}u\|_{L^\infty(U_r)}\sum_{|I|=r}\varepsilon\left\|\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2\Lambda^r(U)}.$$

But with the same proof as in Step 4, with $\varepsilon\square_m$ instead of U, and Proposition 4.4.10, one has

$$\varepsilon\left\|\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2\Lambda^r(U)} \le \varepsilon\left\|\phi_{m,I}\left(\frac{\cdot}{\varepsilon}\right)\right\|_{L^2\Lambda^r(\varepsilon\square_m)} \le \mathcal{O}_1\left(C\varepsilon^\alpha\right),$$

then, by Proposition 4.A.3, one has

$$\|\mathrm{d}w^\varepsilon - \mathrm{d}w\|_{\underline{H}^{-1}\Lambda^r(U)} \le \|\mathrm{d}f\|_{L^2\Lambda^r(U)}\mathcal{O}_1\left(\frac{C\varepsilon^\alpha}{l^{1+d/2}}\right).$$

This completes the proof of (4.5.11).

Step 6. The conclusion. By Steps 2 an 3, one can compute

$$\begin{aligned}\|\mathrm{d}u^\varepsilon - \mathrm{d}u\|_{\underline{H}^{-1}\Lambda^r(U)} &\le \|\mathrm{d}u^\varepsilon - \mathrm{d}w^\varepsilon\|_{\underline{H}^{-1}\Lambda^r(U)} + \|\mathrm{d}w^\varepsilon - \mathrm{d}u\|_{\underline{H}^{-1}\Lambda^r(U)}\\ &\le \|\mathrm{d}u^\varepsilon - \mathrm{d}w^\varepsilon\|_{L^2\Lambda^r(U)} + \|\mathrm{d}w^\varepsilon - \mathrm{d}u\|_{\underline{H}^{-1}\Lambda^r(U)}.\end{aligned}$$

This yields

$$\|\mathrm{d}u^\varepsilon - \mathrm{d}u\|_{\underline{H}^{-1}\Lambda^r(U)} \le \begin{cases} C\,\|\mathrm{d}f\|_{H^1\Lambda^r(U)}\left(l^{\frac{1}{d-2}} + \mathcal{O}_1\left(\frac{\varepsilon^\alpha}{l^{3+d/2}}\right)\right) & \text{if } d \ge 3,\\ C\,\|\mathrm{d}f\|_{H^1\Lambda^r(U)}\left(l^{\frac{1}{4}} + \mathcal{O}_1\left(\frac{\varepsilon^\alpha}{l^{3+d/2}}\right)\right) & \text{if } d = 2.\end{cases}$$

Finally, the bound for $\|u^\varepsilon - u\|_{L^2(U)}$ is obtained from the previous inequality and Step 4. Indeed, since $u - u^\varepsilon \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U) \cap \left(C^r_{\mathrm{d},0}\right)^\perp$, one has

$$\|\mathrm{d}u^\varepsilon - \mathrm{d}u\|_{L^2\Lambda^r(U)} \le C\,\|\mathrm{d}u^\varepsilon - \mathrm{d}u\|_{\underline{H}^{-1}\Lambda^r(U)}.$$

Step 7. The conclusion. The estimate obtained is valid for any $0 < l \le 1$, in particular we can choose l to be a small power of ε such that $\frac{\varepsilon^\alpha}{l^{3+d/2}}$ is still a small power of ε. This completes the proof of Theorem 4.1.2. □

4.6. Duality

The goal of this section is to study a duality property between the homogenization of r-forms and $(d-r)$-forms. We note that similar results were obtained independently by Serre [**143**] in the case of periodic coefficients. For each $\mathbf{a} \in \Omega_r$ and each $x \in \mathbb{R}^d$, the operator $\mathbf{a}(x) \in \mathcal{L}\left(\Lambda^r(\mathbb{R}^d), \Lambda^{(d-r)}(\mathbb{R}^d)\right)$ satisfies the ellipticity assumption (4.1.21), so it is invertible and one can define the inverse operator $(\mathbf{a}(x))^{-1} \in \mathcal{L}\left(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r(\mathbb{R}^d)\right)$, which satisfies the symmetry assumption (4.1.20) and the following ellipticity condition

$$\frac{1}{\lambda}|p|^2 \leq \mathbf{a}(x)^{-1} p \wedge p \leq \lambda |p|^2, \; \forall p \in \Lambda^{(d-r)}(\mathbb{R}^d). \tag{4.6.1}$$

We denote by

$$\Omega'_{d-r} := \Big\{ \mathbf{a}(\cdot) \,:\, \mathbf{a} : \mathbb{R}^d \to \mathcal{L}\left(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r\left(\mathbb{R}^d\right)\right) \text{ is Lebesgue measurable} \\ \text{and satisfies (4.1.20) and (4.6.1)} \Big\}.$$

We equip this set with a family of sigma-algebras, for each $U \subseteq \mathbb{R}^d$,

$$\mathcal{F}'_r(U) := \Big\{ \sigma\text{-algebra on } \Omega'_r \text{ generated by the family of maps} \\ \mathbf{a} \to \int_U p \wedge \mathbf{a}(x) q \phi(x), \, p, q \in \Lambda^r(\mathbb{R}^d), \, \phi \in C_c^\infty(U) \Big\}.$$

One also defines inv to be the mapping

$$\operatorname{inv} : \begin{cases} \Omega_r \to \Omega'_{d-r}, \\ \mathbf{a} \to \mathbf{a}^{-1}. \end{cases}$$

We then define $\operatorname{inv}_* \mathbb{P}$ the probability measure defined on the measured space $(\Omega'_{d-r}, \mathcal{F}'_{d-r})$ by, for each $A \in \mathcal{F}'_{d-r}$,

$$\operatorname{inv}_* \mathbb{P}_r(A) := \mathbb{P}_r\left(\operatorname{inv}^{-1} A\right),$$

the probability space $(\Omega'_{d-r}, \mathcal{F}'_{d-r}, \operatorname{inv}_* \mathbb{P}_r)$ satisfies the stationarity assumption (4.1.24) and the independence assumption (4.1.25). The idea is then to define, for each $(p,q) \in \Lambda^{(d-r)}(\mathbb{R}^d) \times \Lambda^r(\mathbb{R}^d)$ and each $m \in \mathbb{N}$,

$$J_{\mathrm{inv}}(\square_m, p, q) := \sup_{u \in \mathcal{A}^{\mathrm{inv}}(\square_m)} \fint_{\square_m} \left(-\frac{1}{2} \mathbf{a}^{-1} \mathrm{d}u \wedge \mathrm{d}u - \mathbf{a}^{-1} \mathrm{d}u \wedge p + q \wedge \mathrm{d}u \right),$$

where $\mathcal{A}^{\mathrm{inv}}(\square_m)$ is the set of solutions under the environment $\mathbf{a}^{-1}$, i.e.

$$\mathcal{A}_{\mathrm{inv}}(\square_m) := \left\{ u \in H^1_{\mathrm{d}} \Lambda^{(d-r-1)}(\square_m) \,:\, \forall v \in C_c^\infty \Lambda^{r-1}(\square_m), \int_{\square_m} du \wedge \mathbf{a}^{-1} dv = 0 \right\},$$

and this quantity satisfies the conclusions of Proposition 4.4.1 and Theorem 4.1.1. In particular, there exist a constant $C(d,\lambda) < \infty$, an exponent $\alpha(d,\lambda) > 0$ and a linear operator

$$\overline{\operatorname{inv} \mathbf{a}} \in \mathcal{L}\left(\Lambda^{(d-r)}(\mathbb{R}^d), \Lambda^r(\mathbb{R}^d)\right)$$

such that, for each $m \in \mathbb{N}$,

$$\sup_{p \in B_1 \Lambda^{(d-r)}(\mathbb{R}^d)} \mathbb{E}\left[J_{\mathrm{inv}}(\square_m, p, \overline{\operatorname{inv} \mathbf{a}}\, p) \right] \leq C 3^{-m\alpha}.$$

The following theorem determines $\overline{\operatorname{inv} \mathbf{a}}$.

Theorem 4.1.3 (Duality). *The homogenized linear maps $\bar{\mathbf{a}}$ and $\overline{\operatorname{inv} \mathbf{a}}$ satisfy*

$$\overline{\operatorname{inv} \mathbf{a}} = (\bar{\mathbf{a}})^{-1}.$$

PROOF. First we need to prove the following result, for each $0 \le r \le d$ and each bounded $m \in \mathbb{N}$,

$$\mathcal{A}_{\mathrm{inv}}(\square_m) = \left\{ v \in H^1_{\mathrm{d}}\Lambda^{(d-r-1)}(\square_m) \ : \ \mathrm{d}v = \mathbf{a}\mathrm{d}u \text{ with } u \in \mathcal{A}(\square_m) \right\}.$$

We split the proof into 2 steps

- We prove that each $v \in H^1_{\mathrm{d}}\Lambda^{(d-r-1)}(\square_m)$ satisfying $\mathrm{d}v = \mathbf{a}\mathrm{d}u$ for some $u \in \mathcal{A}(\square_m)$ belongs to $\mathcal{A}_{\mathrm{inv}}(\square_m)$. Indeed, for each $w \in C^\infty_c\Lambda^{r-1}(\square_m)$, one has, by the symmetry assumption (4.1.20), and (4.1.18),

$$\int_{\square_m} \mathrm{d}v \wedge \mathbf{a}^{-1}\mathrm{d}w = \int_{\square_m} \mathrm{d}w \wedge \mathbf{a}^{-1}\mathrm{d}v = \int_{\square_m} \mathrm{d}w \wedge \mathrm{d}u = 0. \tag{4.6.2}$$

- We prove that for each $v \in \mathcal{A}_{\mathrm{inv}}(\square_m)$, there exists $u \in \mathcal{A}(\square_m)$ such that $\mathrm{d}v = \mathbf{a}\mathrm{d}u$. Indeed, if $v \in \mathcal{A}_{\mathrm{inv}}(\square_m)$, then $\mathbf{a}^{-1}\mathrm{d}v$ belongs to $L^2\Lambda^r(\square_m)$ and satisfies

$$d\left(\mathbf{a}^{-1}\mathrm{d}v\right) = 0 \text{ in } \square_m.$$

Consequently $\mathbf{a}^{-1}\mathrm{d}v \in H^1_{\mathrm{d}}\Lambda^r(\square_m)$. We can apply Proposition 4.2.4, to prove that there exists $u \in H^1_{\mathrm{d}}\Lambda^r(\square_m)$ such that

$$\mathbf{a}^{-1}\mathrm{d}v = \mathrm{d}u \text{ in } \square_m.$$

There only remains to prove that $u \in \mathcal{A}(\square_m)$, it is a consequence of the following computation: for each $w \in C^\infty_c\Lambda^{(d-r-1)}(\square_m)$, one has, by the symmetry assumption (4.1.20) and (4.1.18),

$$\int_{\square_m} \mathrm{d}u \wedge \mathbf{a}\mathrm{d}w = \int_{\square_m} \mathrm{d}w \wedge \mathbf{a}\mathrm{d}u = \int_{\square_m} \mathrm{d}w \wedge \mathrm{d}v = 0.$$

Using (4.6.2), one has

$$\begin{aligned}
J_{\mathrm{inv}}(\square_m, p, q) &= \sup_{u \in \mathcal{A}^{\mathrm{inv}}(\square_m)} \fint_{\square_m} \left(-\frac{1}{2}\mathbf{a}^{-1}\mathrm{d}u \wedge \mathrm{d}u - \mathbf{a}^{-1}\mathrm{d}u \wedge p + q \wedge \mathrm{d}u \right) \\
&= \sup_{v \in \mathcal{A}(\square_m)} \fint_{\square_m} \left(-\frac{1}{2}(\mathbf{a}\mathrm{d}v) \wedge (\mathbf{a}\mathrm{d}v) - \mathbf{a}^{-1}(\mathbf{a}\mathrm{d}v) \wedge p + q \wedge (\mathbf{a}\mathrm{d}v) \right) \\
&= \sup_{v \in \mathcal{A}(\square_m)} \fint_{\square_m} \left(-\frac{1}{2}\mathrm{d}v \wedge \mathbf{a}\mathrm{d}v - \mathrm{d}v \wedge p + q \wedge \mathbf{a}\mathrm{d}v \right) \\
&= J(\square_m, -q, -p) \\
&= J(\square_m, q, p).
\end{aligned}$$

Thus, by Theorem 4.1.1, for each $m \in \mathbb{N}$,

$$\sup_{p \in B_1\Lambda^r(\mathbb{R}^d)} \mathbb{E}\left[J_{\mathrm{inv}}(\square_m, \bar{\mathbf{a}}p, p) \right] = \sup_{p \in B_1\Lambda^r(\mathbb{R}^d)} \mathbb{E}\left[J(\square_m, p, \bar{\mathbf{a}}p) \right] \le C3^{-\alpha m}.$$

The previous inequality can be rewritten

$$\sup_{q \in B_1\Lambda^{(d-r)}(\mathbb{R}^d)} \mathbb{E}\left[J_{\mathrm{inv}}(\square_m, p, \bar{\mathbf{a}}^{-1}p) \right] \le C3^{-\alpha m}.$$

Since the homogenized matrix is unique, we have $\bar{\mathbf{a}}^{-1} = \overline{\mathrm{inv}\,\mathbf{a}}$. This gives the expected result. □

REMARK 4.6.1. The previous result can be applied in the particular case $d = 2, r = 1$ and the standard homogenization problem

$$\nabla \cdot (\mathbf{a}\nabla u) = 0$$

can be rewritten with the formalism of forms

$$\mathrm{d}(\star\mathbf{a}\mathrm{d}u) = 0$$

(we identify the space $\mathbb{R}^2$ with the space $\Lambda^1(\mathbb{R}^2)$ canonically). Thus, one can compute the dual problem,

$$\mathrm{d}(\mathbf{a}^{-1} \star \mathrm{d}u) = 0,$$

which can be rewritten in the standard formalism,

$$\nabla^\perp \cdot (\mathbf{a}^{-1} \nabla^\perp u) = 0, \tag{4.6.3}$$

where we used the notation

$$\nabla^\perp f = \begin{pmatrix} -\partial_2 f \\ \partial_1 f \end{pmatrix}.$$

Denote by

$$P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Performing the change of variables $u(x) \to u(Px)$, the equation (4.6.3) becomes

$$\nabla \cdot \left(\left(\mathbf{a}^{-1} \circ P \right) \nabla u \right) = 0, \tag{4.6.4}$$

where $\mathbf{a}^{-1} \circ P$ is defined by, for each $x \in \mathbb{R}^d$,

$$\mathbf{a}^{-1} \circ P(x) = \mathbf{a}^{-1}(Px).$$

With this in mind, one can compute the homogenized matrix $\overline{\mathbf{a}^{-1} \circ P}$ of the problem (4.6.4). We obtain according to Theorem 4.1.3,

$$\overline{\mathbf{a}^{-1} \circ P} = \bar{\mathbf{a}}^{-1}. \tag{4.6.5}$$

In particular, if we assume that the environment satisfies, for some positive constant k and for each $x \in \mathbb{R}^d$,

$$\mathbf{a}(x)\mathbf{a}(Px) = kI_d,$$

then $\mathbf{a}(x) = k\mathbf{a}^{-1}(Px)$ and (4.6.5) gives

$$\frac{\bar{\mathbf{a}}}{k} = \bar{\mathbf{a}}^{-1},$$

which implies

$$\bar{\mathbf{a}} = \sqrt{k} I_d.$$

This formula is known as the *Dykhne formula* which was originally proved in [**62**].

4.A. Regularity estimates for differential forms

In this appendix, we record some properties about the regularity of the solutions of the constant coefficient equation $\mathrm{d}\bar{\mathbf{a}}\mathrm{d}u$. The two main results are the pointwise interior estimate, Proposition 4.A.3 and the H^2 boundary estimate, Proposition 4.A.4. Both results are used in the proof of Theorem 4.1.2. Most of these proofs are an adaptation of the classical proofs of the regularity theory of uniformly elliptic equations (c.f. [**74**]).

We first state two propositions, Proposition 4.A.1, an interior Gaffney-Friedrich inequality, and Proposition 4.A.2, an interior H^2 regularity estimate. We then use these two ingredients to prove the pointwise interior estimate, Proposition 4.A.3. We finally prove a global H^2 regularity result for the solutions of $\mathrm{d}\bar{\mathbf{a}}\mathrm{d}u = 0$, Proposition 4.A.4.

The following proposition is an interior version of the Gaffney-Friedrich inequality, Propositions 4.3.3 and 4.3.4. The result is weaker then the ones aforementioned because it is only an interior estimate, but it does not require any regularity for the domain U nor any assumption on the value of the form on the boundary of the domain.

PROPOSITION 4.A.1 (Interior Gaffney-Friedrich inequality). *There exists a constant $C := C(d) < \infty$ such that, for every $0 \le r \le d$, every open bounded subsets $V, U \subseteq \mathbb{R}^d$ satisfying $\overline{V} \subseteq U$, and every $u \in L^2\Lambda^r(U)$ such that $\mathrm{d}u \in L^2\Lambda^{r+1}(U)$ and $\delta u \in L^2\Lambda^{r-1}(U)$, one has $u \in H^1\Lambda^r(V)$ with the estimate,*

$$\|\nabla u\|_{L^2\Lambda^r(V)} \le C \left(\|\mathrm{d}u\|_{L^2\Lambda^r(U)} + \|\delta u\|_{L^2\Lambda^r(U)} + \frac{1}{\mathrm{dist}(V, \partial U)} \|u\|_{L^2\Lambda^r(U)} \right). \tag{4.1.1}$$

PROOF. The proof relies on the following observation, given a form $u = \sum_{|I|=r} u_I \mathrm{d}x_I \in C^\infty \Lambda^r(U)$, one has

$$(\mathrm{d}\delta + \delta \mathrm{d})u = \sum_{|I|=r} \Delta u_I \mathrm{d}x_I.$$

Select a function $\eta \in C_c^\infty(U)$ such that

$$\mathbb{1}_V \le \eta \le 1, \ |\nabla \eta| \le \frac{C}{\mathrm{dist}(V, \partial U)}.$$

We then compute

$$\begin{aligned}
\|\nabla u\|^2_{L^2\Lambda^r(V)} &= \sum_I \int_V |\nabla u_I|^2(x)\,dx \\
&\le \sum_I \int_U |\nabla u_I \eta|^2(x)\,dx \\
&= \sum_I \int_U (u_I\eta)(x)\Delta(u_I\eta)(x)\,dx \\
&= \langle u\eta, (\delta d + d\delta)u\eta\rangle_U \\
&= \langle \mathrm{d}(u\eta), \mathrm{d}(u\eta)\rangle_U + \langle \delta(u\eta), \delta(u\eta)\rangle_U
\end{aligned}$$

By (4.1.6), one has

$$\begin{aligned}
\langle \mathrm{d}(u\eta), \mathrm{d}(u\eta)\rangle_U &= \langle \eta \mathrm{d}u + \mathrm{d}\eta \wedge u, \eta \mathrm{d}u + \mathrm{d}\eta \wedge u\rangle_U \\
&\le 2\langle \eta \mathrm{d}u, \eta \mathrm{d}u\rangle_U + 2\langle \mathrm{d}\eta \wedge u, \mathrm{d}\eta \wedge u\rangle_U \\
&\le C\left(\|\mathrm{d}u\|^2_{L^2\Lambda^{r+1}(U)} + \frac{1}{\mathrm{dist}(V,\partial U)^2}\|u\|^2_{L^2\Lambda^r(U)}\right).
\end{aligned}$$

A similar computation yields

$$\langle \delta(u\eta), \delta(u\eta)\rangle_U \le C\left(\|\delta u\|^2_{L^2\Lambda^{r-1}(U)} + \frac{1}{\mathrm{dist}(V,\partial U)^2}\|u\|^2_{L^2\Lambda^r(U)}\right).$$

Combining the three previous displays completes the proof of (4.1.1). □

We then use the previous interior Gaffney-Friedrich inequality to prove the following interior H^2 estimate. The proof of the following proposition is an adaptation of the standard interior H^2 estimate for the solutions of uniformly elliptic equations, cf [**74**, Theorem 8.8].

PROPOSITION 4.A.2 (Interior H^2 regularity estimate). *For every open bounded subsets $U, V \subseteq \mathbb{R}^d$ such that $\overline{V} \subseteq U$, every $1 \le r \le d$ and every $u \in H^1_{\mathrm{d}}\Lambda^{(r-1)}(U)$ solution of the equation*

$$\mathrm{d}(\bar{\mathbf{a}}\mathrm{d}u) = 0 \ \text{in } U, \tag{4.1.2}$$

one has $\mathrm{d}u \in H^1\Lambda^r(V)$ and it satisfies the interior estimate

$$\|\nabla \mathrm{d}u\|_{L^2\Lambda^r(V)} \le C\left(\frac{1}{\mathrm{dist}(V,\partial U)}\|\mathrm{d}u\|_{L^2\Lambda^r(U)} + \frac{1}{\mathrm{dist}(V,\partial U)^2}\|u\|_{L^2\Lambda^{(r-1)}(U)}\right)$$

for some constant $C := C(d,\lambda) < \infty$.

PROOF. The main idea of this proof is to follow the proof of [**74**, Theorem 8.8] and combine it with the interior Gaffney-Friedrich inequality.

First note that without loss of generality, one can assume that $u \in C^{r-1}_{\mathrm{d}}(U)^\perp$. Select an open subset $V \subseteq U$ such that $\overline{V} \subseteq U$ and select two other open spaces W, W_1 such that $V \subseteq W \subseteq \overline{W} \subseteq W_1 \subseteq \overline{W_1} \subseteq U$ and such that

$$\mathrm{dist}(V,\partial W) = \mathrm{dist}(W,\partial W_1) = \mathrm{dist}(W_1,\partial U) = \frac{\mathrm{dist}(V,\partial W)}{3}. \tag{4.1.3}$$

Additionally, we select a cutoff function $\eta \in C_c^\infty(U)$ such that

$$\mathbb{1}_V \leq \eta \leq \mathbb{1}_W, \ \|\nabla \eta\| \leq \frac{C}{\operatorname{dist}(V, \partial U)}. \tag{4.1.4}$$

Let $h > 0$ be small, choose $k \in \{1, \ldots, d\}$ and denote by

$$v := D_k^{-h}\left(\eta^2 D_k^h u\right),$$

where D_k^h is the difference quotient, defined by

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}.$$

If h is small enough then $v \in H^1_{\mathrm{d},0}\Lambda^{r-1}(U)$ can be used as a test function in (4.1.2). We obtain

$$\langle \mathrm{d}u, \bar{\mathbf{a}}\mathrm{d}v\rangle_U = 0.$$

Thanks to (4.1.6) and the equality $\mathrm{d}D_k^h u = D_k^h \mathrm{d}u$, one computes

$$\mathrm{d}v = D_k^{-h}\left(\mathrm{d}\left(\eta^2 D_k^h u\right)\right) = D_k^{-h}\left(2\eta \mathrm{d}\eta \wedge D_k^h u\right) + D_k^{-h}\left(\eta^2 D_k^h \mathrm{d}u\right).$$

Combining the two previous displays yields

$$\left\langle \mathrm{d}u, \bar{\mathbf{a}} D_k^{-h}\left(\eta^2 D_k^h \mathrm{d}u\right)\right\rangle_U = -\left\langle \mathrm{d}u, \bar{\mathbf{a}} D_k^{-h}\left(2\eta \mathrm{d}\eta \wedge D_k^h u\right)\right\rangle_U.$$

Performing a discrete integration by parts shows

$$\left\langle D_k^h \mathrm{d}u, \bar{\mathbf{a}}\left(\eta^2 D_k^h \mathrm{d}u\right)\right\rangle_U = -\left\langle D_k^h \mathrm{d}u, \bar{\mathbf{a}}\left(2\eta \mathrm{d}\eta \wedge D_k^h u\right)\right\rangle_U.$$

By the Cauchy-Schwarz inequality, one obtains

$$\left\langle D_k^h \mathrm{d}u, \bar{\mathbf{a}}\left(\eta^2 D_k^h \mathrm{d}u\right)\right\rangle_U \leq 2\left(\left\langle D_k^h \mathrm{d}u, \bar{\mathbf{a}}\left(\eta^2 D_k^h \mathrm{d}u\right)\right\rangle_U\right)^{\frac{1}{2}}\left(\left\langle \mathrm{d}\eta \wedge D_k^h u, \bar{\mathbf{a}} \mathrm{d}\eta \wedge D_k^h u\right\rangle_U\right)^{\frac{1}{2}}.$$

and consequently, by (4.A.1) and the ellipticity assumption (4.1.21),

$$\left\langle D_k^h \mathrm{d}u, D_k^h \mathrm{d}u\right\rangle_V \leq \frac{C}{\operatorname{dist}(V, \partial U)^2}\left\langle D_k^h u, D_k^h u\right\rangle_W.$$

Since we assumed $u \in C_d^{r-1}(U)^\perp$, one has $\delta u = 0$ in U and in particular $\delta u \in L^2\Lambda^{r-2}(V)$. From this we deduce that u satisfies the assumptions of Proposition 4.A.1 and consequently u is in $H^1\Lambda^{r-1}(W)$. Moreover it satisfies the estimate

$$\begin{aligned}
\|\nabla u\|_{L^2\Lambda^r(W_1)} &\leq C\left(\|\mathrm{d}u\|_{L^2\Lambda^r(U)} + \frac{1}{\operatorname{dist}(W_1, \partial U)}\|u\|_{L^2\Lambda^{(r-1)}(U)}\right) \\
&\leq C\left(\|\mathrm{d}u\|_{L^2\Lambda^r(U)} + \frac{1}{\operatorname{dist}(V, \partial U)}\|u\|_{L^2\Lambda^{(r-1)}(U)}\right),
\end{aligned}$$

where we used (4.1.4) in the second inequality. Additionally, according to [**74**, Lemma 7.23], one has the inequality

$$\left\|D_k^h u\right\|_{L^2(W)} \leq C\|\nabla u\|_{L^2\Lambda^r(W_1)} \tag{4.1.5}$$

for $h > 0$ small enough. Combining the three previous displays shows

$$\left\langle D_k^h \mathrm{d}u, D_k^h \mathrm{d}u\right\rangle_V \leq \frac{C}{\operatorname{dist}(V, \partial U)^2}\left(\|\mathrm{d}u\|^2_{L^2\Lambda^r(U)} + \frac{1}{\operatorname{dist}(W_1, \partial U)^2}\|u\|^2_{L^2\Lambda^{(r-1)}(U)}\right).$$

Since this inequality is true for every $|h| > 0$ small enough, one has, according to [**74**, Lemma 7.24], $\mathrm{d}u \in H^1\Lambda^{r-1}(V)$ and

$$\|\nabla \mathrm{d}u\|^2_{L^2\Lambda^r(V)} \leq \frac{C}{\operatorname{dist}(V, \partial U)^2}\left(\|\mathrm{d}u\|^2_{L^2\Lambda^r(U)} + \frac{1}{\operatorname{dist}(V, \partial U)^2}\|u\|^2_{L^2\Lambda^{(r-1)}(U)}\right)$$

and the proof is complete. □

PROPOSITION 4.A.3 (Elliptic regularity). *There exists a constant* $C := C(d,k,\lambda) < \infty$ *such that for every open bounded subset* $U \subseteq \mathbb{R}^d$, *every* $0 \le r \le d$, *every* $k \in \mathbb{N}$, *every* $R > 0$, *and every solution of the equation*

$$\mathrm{d}\left(\bar{\mathbf{a}}\mathrm{d}u\right) = 0 \ \text{in } U,$$

the following pointwise estimate holds

$$\left\|\nabla^k \mathrm{d}u\right\|_{L^\infty\Lambda^r(U_R)} \le \frac{C}{R^{k+d/2}} \left\|\mathrm{d}u\right\|_{L^2\Lambda^r(U)}, \tag{4.1.6}$$

where we denoted by $U_R := \{x \in U : \operatorname{dist}(x,\partial U) > R\}$.

PROOF. Select an integer $k \in \mathbb{N}$, a non-negative real number $R > 0$, and a point $x \in U_R$. It is sufficient to prove (4.1.6), to show the estimate

$$\left|\nabla^k \mathrm{d}u(x)\right| \le \frac{C}{R^{k+d/2}} \left\|\mathrm{d}u\right\|_{L^2\Lambda^r(B_R(x))}, \tag{4.1.7}$$

for some constant $C = C(d,k,\Lambda) < \infty$. We split the proof into two steps

Step 1. We prove that there exists a constant $C = C(d,\lambda) < \infty$, such that for every $l \in \mathbb{N}$, $\mathrm{d}u \in H^l\Lambda^r\left(B_{R/2^l}(x)\right)$ and

$$\left\|\nabla^l \mathrm{d}u\right\|_{L^2\Lambda^r\left(B_{R/2^l}(x)\right)} \le \frac{C^l 2^{l^2/2}}{R^l} \left\|\mathrm{d}u\right\|_{L^2\Lambda^r(B_R(x))}. \tag{4.1.8}$$

This inequality can be proved by induction on l. It is true for $l = 0$. We can use Proposition 4.A.2 to go from l to $l+1$. Assume that (4.1.8) holds with l. In that case, one has $\nabla^l \mathrm{d}u \in L^2\Lambda^r\left(B_{R/2^l}(x)\right)$. It is easy to check

$$\mathrm{d}\left(\nabla^l \mathrm{d}u\right) = 0.$$

Thus by Proposition 4.2.4, there exists a form $v_l \in H^1_{\mathrm{d}}\Lambda^{r-1}\left(B_{R/2^l}(x)\right)$ such that

$$v_l \in C^{r-1}_{\mathrm{d}}\left(B_{R/2^l}(x)\right)^\perp \text{ and } \mathrm{d}v_l = \nabla^l \mathrm{d}u.$$

It is moreover a straightforward computation to check

$$\mathrm{d}\left(\bar{\mathbf{a}}\mathrm{d}v_l\right) = 0.$$

Consequently, one can apply Proposition 4.A.2 to v_l with $U = B_{R/2^l}(x)$ and $V = B_{R/2^{l+1}}(x)$. This gives $\nabla^{l+1}\mathrm{d}u \in H^1\Lambda^r\left(B_{R/2^{l+1}}(x)\right)$, and thus $\mathrm{d}u \in H^{l+1}\Lambda^r\left(B_{R/2^{l+1}}(x)\right)$ with the estimate

$$\left\|\nabla^{l+1}\mathrm{d}u\right\|_{L^2\Lambda^r\left(B_{R/2^{l+1}}(x)\right)} \le \frac{C2^{l+1}}{R}\left(\left\|\nabla^l \mathrm{d}u\right\|_{L^2\Lambda^r\left(B_{R/2^l}(x)\right)} + \frac{2^{l+1}}{R}\left\|v_l\right\|_{L^2\Lambda^{(r-1)}\left(B_{R/2^l}(x)\right)}\right).$$

By Proposition 4.3.1, v_l satisfies the Poincaré inequality

$$\left\|v_l\right\|_{L^2\Lambda^{(r-1)}(B_{R/2^l}(x))} \le C\frac{2^l}{R}\left\|\nabla^l u\right\|_{L^2\Lambda^{(r-1)}\left(B_{R/2^l}(x)\right)}.$$

Combining the two previous displays yields

$$\left\|\nabla^{l+1}\mathrm{d}u\right\|_{L^2\Lambda^r\left(B_{R/2^{l+1}}(x)\right)} \le \frac{C2^{l+1}}{R}\left\|\nabla^l \mathrm{d}u\right\|_{L^2\Lambda^r\left(B_{R/2^l}(x)\right)}$$

for some $C := C(d,\Lambda) < \infty$. Applying the induction hypothesis completes the proof.

Step 2. From the first step, we get that for every $l \in \mathbb{N}$, $\nabla^{k+l}\mathrm{d}u \in L^2\Lambda^r\left(B_{R/2^{k+l}}(x)\right)$. In particular, by the Sobolev injection, see for instance [**1**, Chapter 4], one has

$$\nabla^k \mathrm{d}u \in L^\infty\Lambda^r\left(B_{R/2^{k+d/2+1}}(x)\right),$$

together with the estimate

$$\left\|\nabla^k \mathrm{d}u\right\|_{L^\infty\Lambda^r\left(B_{R/2^{k+d/2+1}}(x)\right)} \le \frac{C}{R^{k+d/2}}\left\|\mathrm{d}u\right\|_{L^2\Lambda^r(B_R(x))}.$$

This completes the proof of (4.1.7). □

We then establish the following global H^2 estimate for the solutions of $\mathrm{d}\bar{\mathbf{a}}\mathrm{d}u = 0$.

Proposition 4.A.4 (Global H^2 regularity). *Let $U \subseteq \mathbb{R}^d$ be a smooth bounded domain of $\mathbb{R}^d$. For $0 \leq r \leq d$, let $f \in H^1_{\mathrm{d}}\Lambda^{r-1}(U)$ be such that $\mathrm{d}f \in H^1\Lambda^{r-1}(U)$. Let $u \in H^1_{d,0}\Lambda^{r-1}(U)$ be a solution of the equation*

$$\begin{cases} \mathrm{d}\left(\bar{\mathbf{a}}\mathrm{d}u\right) = 0 \ \text{in } U, \\ \quad \mathbf{t}u = f \ \text{on } \partial U, \end{cases} \tag{4.1.9}$$

then $\mathrm{d}u \in H^1\Lambda^r(U)$ and one has the estimate

$$\|\mathrm{d}u\|_{H^1\Lambda^r(U)} \leq \|\mathrm{d}f\|_{H^1\Lambda^r(U)}\,. \tag{4.1.10}$$

Proof. First note that two solutions of (4.1.9) differ by a form in $C^{r-1}_{\mathrm{d},0}$, this implies that two solutions of (4.1.9) have the same exterior derivative. Thus to prove (4.1.10), it is enough to prove it for a particular solution of (4.A.4).

The strategy of the proof is the following: one want to apply the result from the regularity theory of strongly elliptic operators to the differential form u, see (4.1.18) for a definition and [**116**] for a reference on the topic of strongly differential operators. Unfortunately the operator $\mathrm{d}\bar{\mathbf{a}}\mathrm{d}$ is not strongly elliptic, thus the result cannot directly apply. The strategy is then to solve the problem $\mathrm{d}\bar{\mathbf{a}}\mathrm{d} + (-1)^r \star \mathrm{d}\delta u = 0$ with appropriate boundary conditions so that $\star d\delta u = 0$ and u is in fact a solution of (4.1.9). Contrary to $\mathrm{d}\bar{\mathbf{a}}\mathrm{d}$, the operator $\mathrm{d}\bar{\mathbf{a}}\mathrm{d} + (-1)^r \star \mathrm{d}\delta$ is strongly elliptic and a regularity theory exists for these operators. Thanks to this, one is able to derive H^2 boundary regularity for the function u. This implies (4.1.10) by the previous remark.

The main ideas of the proof are standard and can be found in [**142**, Chapter 2] and [**116**, Chapter 4]. We recall the notation for the set of harmonic forms with Dirichlet boundary condition introduced in Proposition 4.3.4,

$$\mathcal{H}^{r-1}_D(U) \coloneqq \mathcal{H}^{r-1}(U) \cap H^1_{\mathrm{d},0}(U) \coloneqq \left\{ u \in L^2\Lambda^{r-1}(U) \,:\, \mathrm{d}u = 0,\ \delta u = 0 \text{ in } U \text{ and } \mathbf{t}u = 0 \text{ on } \partial U \right\}.$$

We split the proof into 8 steps

- In Step 1, we show that there exists a unique solution in $u \in \mathcal{H}^{r-1}_D(U)^\perp$ to the elliptic system

$$\begin{cases} \mathrm{d}\bar{\mathbf{a}}\mathrm{d}u + (-1)^r \star \mathrm{d}\delta u = \mathrm{d}\bar{\mathbf{a}}\mathrm{d}f, \\ \mathbf{t}u = 0 \text{ on } \partial U, \\ \mathbf{t}\delta u = 0 \text{ on } \partial U. \end{cases} \tag{4.1.11}$$

- In Step 2, we show that the form u defined in Step 1 satisfies $\mathrm{d}\delta u = 0$ and is actually a solution of (4.1.9).
- Steps 3 to 6 are the technical steps, we show the H^2 boundary regularity for the solution of the more general problem,

$$\begin{cases} \mathrm{d}\bar{\mathbf{a}}\mathrm{d}u + (-1)^r \star \mathrm{d}\delta u = \mathrm{d}\bar{\mathbf{a}}\mathrm{d}f, \\ \mathbf{t}u = 0 \text{ on } \partial U, \\ \mathbf{t}\delta u = 0 \text{ on } \partial U, \end{cases}$$

  using the theory of strongly elliptic operators developed in [**116**].
- In Steps 7 and 8, we combine the results of the previous steps to prove (4.1.10).

Step 1. First, we prove that there exists a unique solution $u \in \mathcal{H}^{r-1}_D(U)^\perp$ of the system

$$\begin{cases} \mathrm{d}\bar{\mathbf{a}}\mathrm{d}u + (-1)^r \star \mathrm{d}\delta u = \mathrm{d}\bar{\mathbf{a}}\mathrm{d}f, \\ \mathbf{t}u = 0 \text{ on } \partial U, \\ \mathbf{t}\delta u = 0 \text{ on } \partial U. \end{cases}$$

This equation can be rewritten variationally the following way, there exists $u \in H^1\Lambda^{r-1}(U) \cap \mathcal{H}_D^{r-1}(U)^\perp$ such that $\mathbf{t}u = 0$ and for each $v \in H^1\Lambda^{r-1}(U)$ satisfying $\mathbf{t}v = 0$,

$$\int_U \mathrm{d}u \wedge \bar{\mathbf{a}}\mathrm{d}v + \int_U \delta u \wedge \star\delta v = \int_U \mathrm{d}f \wedge \bar{\mathbf{a}}\mathrm{d}v. \tag{4.1.12}$$

To solve this, we look at the associated energy: for $v \in H^1\Lambda^{r-1}(U) \cap \mathcal{H}_D^{r-1}(U)^\perp$ satisfying the boundary condition $\mathbf{t}v = 0$, we define

$$\mathcal{J}(v) := \int_U \mathrm{d}v \wedge \bar{\mathbf{a}}\mathrm{d}v + \int_U \delta v \wedge \star\delta v - \int_U \mathrm{d}f \wedge \bar{\mathbf{a}}\mathrm{d}v.$$

Since $\bar{\mathbf{a}}$ satisfies the ellipticity assumption (4.1.21), one has

$$\int_U \mathrm{d}v \wedge \bar{\mathbf{a}}\mathrm{d}v + \int_U \delta v \wedge \star\delta v \geq \lambda \left\|\mathrm{d}v\right\|_{L^2\Lambda^r(U)} + \left\|\delta v\right\|_{L^2\Lambda^r(U)}.$$

Moreover, by the Gaffney-Friedrich inequality, Proposition 4.3.4, we have

$$\begin{aligned}\int_U \mathrm{d}v \wedge \bar{\mathbf{a}}\mathrm{d}v + \int_U \delta v \wedge \star\delta v &\geq \lambda \left\|\mathrm{d}v\right\|_{L^2\Lambda^r(U)} + \left\|\delta v\right\|_{L^2\Lambda^r(U)} \\ &\geq c\left\|\nabla v\right\|_{L^2\Lambda^{r-1}(U)},\end{aligned}$$

for some $c := c(d, \lambda, U) > 0$. Arguing by contradiction, it is straightforward to prove the following Poincaré inequality: there exists $C := C(d, U) < \infty$ such that for each $u \in H^1\Lambda^{r-1}(U) \cap \mathcal{H}_D^{r-1}(U)^\perp$ satisfying $\mathbf{t}u = 0$ on ∂U,

$$\left\|v\right\|_{L^2\Lambda^{r-1}(U)} \leq \left\|\nabla v\right\|_{L^2\Lambda^{r-1}(U)}. \tag{4.1.13}$$

This implies that the functional $\mathcal{J}$ is coercive on the space

$$\left\{u \in H^1\Lambda^{r-1}(U) \cap \mathcal{H}_D^{r-1}(U)^\perp \ : \ \mathbf{t}u = 0 \text{ on } \partial U\right\},$$

equipped with the H^1 norm. Moreover, this functional is also uniformly convex. The standard techniques of the calculus of variations then show that there exists a unique minimizer of $\mathcal{J}$ denoted by u. By the first variation formula, one has for each $v \in H^1\Lambda^{r-1}(U) \cap \mathcal{H}_D^{r-1}(U)^\perp$ satisfying the boundary condition $\mathbf{t}v = 0$,

$$\int_U \mathrm{d}u \wedge \bar{\mathbf{a}}\mathrm{d}v + \int_U \delta u \wedge \star\delta v = \int_U \mathrm{d}f \wedge \bar{\mathbf{a}}\mathrm{d}v.$$

Also, for each $v \in \mathcal{H}_D^{r-1}(U)$, one has

$$\int_U \mathrm{d}u \wedge \bar{\mathbf{a}}\mathrm{d}v + \int_U \delta u \wedge \star\delta v = \int_U \mathrm{d}f \wedge \bar{\mathbf{a}}\mathrm{d}v = 0.$$

Thus for each $v \in H^1\Lambda^{r-1}(U)$ satisfying $\mathbf{t}v = 0$, we have

$$\int_U \mathrm{d}u \wedge \bar{\mathbf{a}}\mathrm{d}v + \int_U \delta u \wedge \star\delta v = \int_U \mathrm{d}f \wedge \bar{\mathbf{a}}\mathrm{d}v$$

and the proof of Step 1 is complete. As a remark, note that since $\mathrm{d}f \in H^1\Lambda^r(U)$, $\mathrm{d}\bar{\mathbf{a}}\mathrm{d}f \in L^2\Lambda^{d-r+1}(U)$. Thus, if we denote by $g := \mathrm{d}\bar{\mathbf{a}}\mathrm{d}f \in L^2\Lambda^{d-r+1}(U)$, one has

$$\int_U \mathrm{d}u \wedge \bar{\mathbf{a}}\mathrm{d}v + \int_U \delta u \wedge \star\delta v = \int_U g \wedge v, \tag{4.1.14}$$

for each $v \in H^1\Lambda^{r-1}(U)$ satisfying $\mathbf{t}v = 0$.

Step 2. We show that the solution u constructed in the previous step satisfies

$$\begin{cases}\mathrm{d}\bar{\mathbf{a}}\mathrm{d}u = \mathrm{d}\bar{\mathbf{a}}\mathrm{d}f \text{ in } U, \\ \qquad \mathbf{t}u = 0 \text{ on } \partial U.\end{cases}$$

To prove this, it is enough, by Proposition 4.2.3, to show that for each $v \in H^1\Lambda^{r-1}(U)$ satisfying the boundary condition $\mathbf{t}v = 0$,

$$\int_U \mathrm{d}u \wedge \bar{\mathbf{a}}\mathrm{d}v = \int_U \mathrm{d}f \wedge \bar{\mathbf{a}}\mathrm{d}v.$$

To this end, select some $v \in H^1\Lambda^{r-1}(U)$ satisfying $\mathbf{t}v = 0$. Denote by α_v the form of $C^{r-1}_{\mathrm{d},0}(U)$ such that

$$\alpha_v = \operatorname*{argmin}_{\alpha \in C^{r-1}_{\mathrm{d},0}(U)} \|v - \alpha\|_{L^2\Lambda^{r-1}(U)},$$

and set $w = v - \alpha_v$. In particular, this form satisfies, for each $\gamma \in C^\infty_c\Lambda^{r-2}(U)$,

$$\langle w, \mathrm{d}\gamma\rangle_{L^2\Lambda^{r-1}(U)} = 0,$$

which implies $\delta w = 0$. Moreover it is clear that $\mathrm{d}w = \mathrm{d}v$ and that $\mathbf{t}w = 0$. Thus, by the Gaffney-Friedrich inequality, $w \in H^1\Lambda^{r-1}(U)$ and w can be tested in (4.1.12). This gives

$$\int_U \mathrm{d}u \wedge \bar{\mathbf{a}}\mathrm{d}w = \int_U \mathrm{d}f \wedge \bar{\mathbf{a}}\mathrm{d}w,$$

and since $\mathrm{d}w = \mathrm{d}v$, the previous equality can be rewritten

$$\int_U \mathrm{d}u \wedge \bar{\mathbf{a}}\mathrm{d}v = \int_U \mathrm{d}f \wedge \bar{\mathbf{a}}\mathrm{d}v,$$

which is the desired result. The proof of Step 2 is complete.

Step 3. In this step, we follow the arguments of the proofs of [**142**, Section 2.3]. Let X be a smooth vector field supported in U, tangent to the boundary of U. This vector field generates a global flow ψ^X_t such that for every $t \in \mathbb{R}$, ψ^X_t is a smooth diffeomorphism of U. The pullback $\left(\psi^X_t\right)^*$ gives rise to the following linear mapping, for $t \in \mathbb{R} \setminus \{0\}$,

$$\Sigma^X_t := \begin{cases} L^2\Lambda^{r-1}(U) \to L^2\Lambda^{r-1}(U) \\ \qquad\qquad \omega \mapsto \frac{1}{t}\left(\left(\psi^X_t\right)^* \omega - \omega\right). \end{cases}$$

This operator satisfies a number of convenient properties which are listed below. Most of these properties can be found in [**142**, Section 2.3 and Lemma 2.3.1].

Lemma 4.A.5 (Properties of Σ^X_t). *The operator Σ^X_t satisfies the following properties*

- *Since $\left(\psi^X_t\right)^*$ commutes with the exterior derivative* d, *so does Σ^X_t,*

$$\mathrm{d}\Sigma^X_t\omega = \Sigma^X_t\mathrm{d}\omega.$$

- *Since $\left(\psi^X_t\right)^*$ commutes with the projection to the tangential component, so does Σ^X_t,*

$$\mathbf{t}\omega = 0 \implies \mathbf{t}\Sigma^X_t\omega = 0.$$

- *There exists a constant $C := C(d, U, X) < \infty$ such that for each $t \in [-1, 1]$ and each $\omega \in H^1\Lambda^{r-1}(U)$,*

$$\left\|\Sigma^X_t\omega\right\|_{L^2\Lambda^{r-1}(U)} \le C\|\omega\|_{H^1\Lambda^{r-1}(U)}.$$

- *Let $\Theta^X_t : L^2\Lambda^{r-1}(U) \to L^2\Lambda^{r-1}(U)$ be the operator defined according to*

$$\Sigma^X_t \star \omega = \star\Sigma^X_t + \star\Theta^X_t\omega,$$

then there exists a constant $C := C(d, U, X) < \infty$ such that for each $t \in [-1, 1]$,

$$\left\|\Theta^X_t\omega\right\|_{L^2\Lambda^{r-1}} \le C\|\omega\|_{L^2\Lambda^{r-1}}.$$

- *Let $\Theta^{\bar{\mathbf{a}},X}_t : L^2\Lambda^r(U) \to L^2\Lambda^{d-r}(U)$ be the operator defined according to the formula*

$$\Sigma^X_t\bar{\mathbf{a}} = \bar{\mathbf{a}}\Sigma^X_t - \left(\psi^t_X\right)^*\Theta^{\bar{\mathbf{a}},X}_t\omega,$$

then there exists a constant $C := C(d, U, X) < \infty$ such that for each $t \in [-1, 1]$,

$$\left\|\Theta^{\bar{\mathbf{a}},X}_t\omega\right\|_{L^2\Lambda^{d-r}(U)} \le C\|\omega\|_{L^2\Lambda^r(U)}.$$

PROOF. All these properties are proved in [**142**, Section 2.3 and Lemma 2.3.1] except for the last one, which we now prove.

An explicit computation gives the following formula for $\Theta_t^{\mathbf{\bar{a}},X}$,

$$\Theta_t^{\mathbf{\bar{a}},X} = \frac{\left(\psi_X^{-t}\right)^* \mathbf{\bar{a}} \left(\psi_X^{t}\right)^* - \mathbf{\bar{a}}}{t}.$$

Then with this formula note that there exists smooth functions $\phi_{I,J} : [-1,1] \times U \to \mathbb{R}$, with $I, J \subseteq \{1, \ldots, d\}$ satisfying $|I| = r$ and $|J| = d - r$, such that, for each $\omega \in L^2\Lambda^r(U)$,

$$\left(\psi_X^{-t}\right)^* \mathbf{\bar{a}} \left(\psi_X^{t}\right)^* \omega = \sum_{I,J} \omega_I(x)\phi_{I,J}(t,x)\mathrm{d}x_J.$$

Using that all the functions $\phi_{I,J}$ are smooth and that $\left(\psi_X^{0}\right)^* \mathbf{\bar{a}} \left(\psi_X^{0}\right)^* = \mathbf{\bar{a}}$, we obtain the result. □

With these properties, one can prove the following estimate, which allows to perform integration by parts: there exists a constant $C := C(d, \lambda, X) < \infty$ such that for each $t \in [-1,1]$, and for each $v, w \in H^1\Lambda^{r-1}(U)$,

$$\left| \int_U \mathrm{d}v \wedge \mathbf{\bar{a}}\mathrm{d}\Sigma_{-t}^X w - \int_U \mathrm{d}\Sigma_t^X v \wedge \mathbf{\bar{a}}\mathrm{d}w \right| + \left| \int_U \delta\Sigma_t^X v \wedge \star\delta w - \int_U \delta v \wedge \star\delta\Sigma_{-t}^X w \right| \leq C \left\| v \right\|_{H^1\Lambda^{r-1}(U)} \left\| w \right\|_{H^1\Lambda^{r-1}(U)}. \tag{4.1.15}$$

To prove this, it is straightforward to show that, for each $\omega, \xi \in L^2\Lambda^r(U)$

$$\Sigma_t^X(\omega \wedge \mathbf{\bar{a}}\xi) = \Sigma_t^X \omega \wedge \mathbf{\bar{a}}\xi - \left(\psi_t^X\right)^* \left(\omega \wedge \mathbf{\bar{a}}\Sigma_{-t}^X \xi\right) + \left(\psi_t^X\right)^* \left(\omega \wedge \left(\psi_{-t}^X\right)^* \Theta_{-t}^{\mathbf{\bar{a}},X} \xi\right).$$

Integrating this equation over U and using the formula (4.1.12) gives

$$\int_U \Sigma_t^X w \wedge \mathbf{\bar{a}}\xi - \int_U \left(w \wedge \mathbf{\bar{a}}\Sigma_{-t}^X \xi\right) + \int_U \left(w \wedge \left(\psi_{-t}^X\right)^* \Theta_{-t}^{\mathbf{\bar{a}},X} \xi\right) = 0.$$

Applying the previous formula with $\omega = \mathrm{d}v$ and $\xi = \mathrm{d}w$ and using Lemma 4.A.5 gives

$$\left| \int_U \mathrm{d}v \wedge \mathbf{\bar{a}}\mathrm{d}\Sigma_{-t}^X w - \int_U \mathrm{d}\Sigma_t^X v \wedge \mathbf{\bar{a}}\mathrm{d}w \right| \leq C \left\| v \right\|_{H^1\Lambda^{r-1}(U)} \left\| w \right\|_{H^1\Lambda^{r-1}(U)}.$$

The proof of the second inequality

$$\left| \int_U \delta\Sigma_t^X v \wedge \star\delta w - \int_U \delta v \wedge \star\delta\Sigma_{-t}^X w \right| \leq C \left\| v \right\|_{H^1\Lambda^{r-1}(U)} \left\| w \right\|_{H^1\Lambda^{r-1}(U)}$$

is similar and we omit the details. They can be found in [**142**, Lemma 2.3.1].

We then apply (4.1.15) with $v = u$ and $w = \Sigma_t^X u$, this gives

$$\left| \int_U \mathrm{d}u \wedge \mathbf{\bar{a}}\mathrm{d}\Sigma_{-t}^X\Sigma_t^X u - \int_U \mathrm{d}\Sigma_t^X u \wedge \mathbf{\bar{a}}\mathrm{d}\Sigma_t^X u \right| + \left| \int_U \delta\Sigma_t^X u \wedge \star\delta\Sigma_t^X u - \int_U \delta u \wedge \star\delta\Sigma_{-t}^X\Sigma_t^X u \right| \leq C \left\| u \right\|_{H^1\Lambda^{r-1}(U)} \left\| \Sigma_t^X u \right\|_{H^1\Lambda^{r-1}(U)}.$$

But, by the definition of u given in Step 1, one has

$$\int_U \mathrm{d}u \wedge \mathbf{\bar{a}}\mathrm{d}\Sigma_{-t}^X\Sigma_t^X u + \int_U \delta u \wedge \star\delta\Sigma_{-t}^X\Sigma_t^X u = \int_U \mathrm{d}f \wedge \mathbf{\bar{a}}\mathrm{d}\Sigma_{-t}^X\Sigma_t^X u.$$

The term on the right-hand side can be estimated by (4.1.15),

$$\begin{aligned} \left| \int_U \mathrm{d}f \wedge \mathbf{\bar{a}}\mathrm{d}\Sigma_{-t}^X\Sigma_t^X u \right| &\leq \left| \int_U \mathrm{d}\Sigma_t^X f \wedge \mathbf{\bar{a}}\mathrm{d}\Sigma_t^X u \right| + C \left\| \mathrm{d}f \right\|_{H^1\Lambda^{r-1}(U)} \left\| \Sigma_t^X u \right\|_{H^1\Lambda^{r-1}(U)} \\ &\leq C \left\| \mathrm{d}f \right\|_{H^1\Lambda^{r-1}(U)} \left\| \Sigma_t^X u \right\|_{H^1\Lambda^{r-1}(U)}. \end{aligned}$$

Combining the few previous displays implies

$$\int_U \mathrm{d}\Sigma_t^X u \wedge \mathbf{\bar{a}}\mathrm{d}\Sigma_t^X u + \int_U \delta\Sigma_t^X u \wedge \star\delta\Sigma_t^X u \leq C \left(\left\| u \right\|_{H^1\Lambda^{r-1}(U)} + \left\| \mathrm{d}f \right\|_{H^1\Lambda^{r-1}(U)} \right) \left\| \Sigma_t^X u \right\|_{H^1\Lambda^{r-1}(U)}.$$

By the version of the Gaffney-Friedrich inequality stated in (4.3.4), one obtains

$$\left\|\Sigma_t^X u\right\|_{H^1\Lambda^{r-1}(U)}^2 \leq C\left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)}\right)\left\|\Sigma_t^X u\right\|_{H^1\Lambda^{r-1}(U)} + C\left\|\Sigma_t^X u\right\|_{L^2\Lambda^{r-1}(U)}^2.$$

The previous inequality can be further refined

$$\left\|\Sigma_t^X u\right\|_{H^1\Lambda^{r-1}(U)}^2 \leq C\left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)}\right)\left\|\Sigma_t^X u\right\|_{H^1\Lambda^{r-1}(U)} + C\|u\|_{H^1\Lambda^{r-1}(U)}^2$$

and thus

$$\left\|\Sigma_t^X u\right\|_{H^1\Lambda^{r-1}(U)} \leq C\left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)}\right). \tag{4.1.16}$$

Step 4. Interior regularity. Using the result of Step 3, we prove that u is locally H^2 in U. To this end, fix $x \in U$ and consider an open subset V_x such that

$$x \in V_x \subseteq \overline{V}_x \subseteq U.$$

Consider d vector fields $X_1, \ldots, X_d$ compactly supported in U such that for each $k \in \{1, \ldots, d\}$

$$X_k(y) = \mathfrak{e}_k, \qquad \forall y \in V.$$

We then recall the notation for the finite difference operator used in Proposition 4.A.2: for $h > 0$ small, and $k \in \{1, \ldots, d\}$, we denote by

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}.$$

By (4.1.16), we deduce that for $t > 0$ small enough and each $k \in \{1, \ldots, d\}$,

$$\left\|D_k^h u\right\|_{H^1\Lambda^{r-1}(V)} \leq C\left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)}\right).$$

Thus according to [**74**, Lemma 7.24], $u \in H^2\Lambda^r(V)$ with the estimate

$$\|u\|_{H^2\Lambda^{r-1}(V)} \leq C\left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)}\right).$$

Step 5. Boundary regularity I. The first part of this step is to reduce the problem to the half-ball denoted by $B^+ := \{x \in B(0,1) : x_n \geq 0\}$. For further notice, we introduce the notation $B_{\frac{1}{2}}^+ := \left\{x \in B\left(0, \frac{1}{2}\right) : x_n \geq 0\right\}$.

Select $x \in \partial U$. Since ∂U is assumed to be smooth there exists an open set $V \subseteq \mathbb{R}^d$ such that $x \in V$ and a smooth positively oriented diffeomorphism $\Phi : B(0,1) \to V$ such that

$$\Phi(B^+) = V \cap \overline{U} \text{ and } \Phi(0) = x.$$

Using the change of variables formula (4.1.12) and the definition of the tangential trace (4.2.2), one obtains that,

$$v \in H_{\mathrm{d},0}^1\Lambda^{r-1}(U) \implies \mathbf{t}\Phi^* v = 0 \text{ on } \{x \in B(0,1) : x_n = 0\}.$$

To ease the notation, we denote by $u_\Phi := (\Phi)^* u$. It is a form defined on B^+. The purpose of this step is to prove that for $k \in \{1, \ldots, d-1\}$, $\partial_k \nabla u_\Phi \in L^2\Lambda^r(B^+)$. As in the previous step, consider $(d-1)$ vector field $X_1, \ldots, X_{d-1}$ compactly supported in B^+, tangent to the boundary and satisfying

$$X_k(y) = \mathfrak{e}_k, \quad \forall y \in B_{\frac{1}{2}}^+.$$

Note then that one has the identity, for each $k \in \{1, \ldots, d-1\}$,

$$\left(\psi_t^{X_k}\right)^* u_\Phi = (\Phi)^*\left(\psi_t^{\widetilde{X}_k}\right)^* u,$$

where $\widetilde{X}_k$ is the vector field defined on V according to

$$\widetilde{X}_k(\Phi(x)) := \mathrm{d}\Phi(x)(X_k(x)), \quad \forall x \in B^+.$$

Thanks to (4.1.16), one has

$$\left\| \left(\psi_t^{\widetilde{X}_k}\right)^* u \right\|_{H^1\Lambda^{r-1}(V)} \leq C \left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)} \right),$$

for some $C := C(d, \lambda, \Phi, X_k) < \infty$. By (4.1.8), the previous display can be further refined

$$\left\| (\Phi)^* \left(\psi_t^{\widetilde{X}_k}\right)^* u \right\|_{H^1\Lambda^{r-1}(V)} \leq C \left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)} \right),$$

for some $C := C(d, \lambda, \Phi, X_k) < \infty$. This can be rewritten

$$\left\| \left(\psi_t^{X_k}\right)^* u_\Phi \right\|_{H^1\Lambda^{r-1}(V)} \leq C \left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)} \right),$$

for some $C := C(d, \lambda, \Phi, X_k) < \infty$. As in Step 4, we now apply [**74**, Lemma 7.24] to get that for each $k \in \{1, \ldots, d-1\}$, $\partial_k \nabla u_\Phi \in L^2\Lambda^r(B^+)$, with the estimate

$$\|\partial_k \nabla u_\Phi\|_{L^2\Lambda^{r-1}\left(B^+_{\frac{1}{2}}\right)} \leq C \left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)} \right),$$

for some $C := C(d, \lambda, \Phi) < \infty$.

Step 6. Boundary regularity II. The purpose of this step is to prove that u_Φ belongs to $H^2\Lambda^{r-1}\left(B^+_{1/2}\right)$. To this end, we see that, thanks to the previous step, there only remains to prove that $\partial_d\partial_d u_\Phi$ belongs to $L^2\Lambda^{r-1}\left(B^+_{1/2}\right)$. This is what is proved in this step, along with the estimate

$$\|\partial_d\partial_d u_\Phi\|_{L^2\Lambda^{r-1}\left(B^+_{1/2}\right)} \leq C \left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)} \right), \tag{4.1.17}$$

for some constant $C := C(d, \lambda, \Phi) < \infty$. The main ingredient is the uniform ellipticity of the operator

$$\mathrm{d}\bar{\mathbf{a}}\mathrm{d} + (-1)^r \star \mathrm{d}\delta.$$

Since $u_\Phi = (\Phi)^* u$, one sees that this differential form is a solution of the following equation

$$\Phi^* \left(\mathrm{d}\bar{\mathbf{a}}\mathrm{d} + (-1)^r \star \mathrm{d}\delta\right) \left(\Phi^{-1}\right)^* u_\Phi = (\Phi)^* \mathrm{d}\bar{\mathbf{a}}\mathrm{d}f \quad \text{in } B^+.$$

This second order differential operator can be written in the form

$$\Phi^* \left(\mathrm{d}\bar{\mathbf{a}}\mathrm{d} + (-1)^r \star \mathrm{d}\delta\right) \left(\Phi^{-1}\right)^* u = \sum_{j,k=1}^d A_{j,k}\partial_j\partial_k u + \sum_{j=1}^d A_j \partial_j u + Au,$$

where the coefficients $A_{j,k}, A_j$ and A are smooth functions from B^+ to the space of matrices of size $\binom{d}{r-1} \times \binom{d}{r-1}$ (or equivalently the space of endomorphisms of $\Lambda^{r-1}\left(\mathbb{R}^d\right)$). Since this operator is self-adjoint, one knows that the matrices $A_{i,j}$ are symmetric. The idea to prove (4.1.17) is to show that this operator is strongly elliptic, i.e.

$$\sum_{j,k=1}^d \left(\eta^\top A_{j,k}(x)\eta\right) \xi_j\xi_k \geq c|\eta|^2|\xi|^2 \qquad \forall x \in U, \forall \eta \in \mathbb{R}^{\binom{d}{r-1}}, \forall \xi \in \mathbb{R}^d. \tag{4.1.18}$$

To prove the strong ellipticity, it is enough, by [**116**, Theorem 4.6], to prove that, for each $w \in C_c^\infty\Lambda^{r-1}(B^+)$,

$$\int_{B^+} \Phi^* \left(\mathrm{d}\bar{\mathbf{a}}\mathrm{d} + (-1)^r \star \mathrm{d}\delta\right) \left(\Phi^{-1}\right)^* w \wedge w \geq c \|w\|^2_{H^1\Lambda^{r-1}(B^+)} - C \|w\|^2_{L^2\Lambda^{r-1}(B^+)}. \tag{4.1.19}$$

This is a consequence of the following computation

$$\begin{aligned}
&\int_{B^+} \Phi^* \left(\mathrm{d}\bar{\mathbf{a}}\mathrm{d} + (-1)^r \star \mathrm{d}\delta\right) \left(\Phi^{-1}\right)^* w \wedge w \\
&\quad = \int_V \left(\mathrm{d}\bar{\mathbf{a}}\mathrm{d} + (-1)^r \star \mathrm{d}\delta\right) \left(\Phi^{-1}\right)^* w \wedge \left(\Phi^{-1}\right)^* w \\
&\quad = \int_V \bar{\mathbf{a}}\mathrm{d} \left(\Phi^{-1}\right)^* w \wedge \mathrm{d} \left(\Phi^{-1}\right)^* w + \int_V \delta \left(\Phi^{-1}\right)^* w \wedge \star\delta \left(\Phi^{-1}\right)^* w \\
&\quad \geq \lambda \left\| \mathrm{d} \left(\Phi^{-1}\right)^* w \right\|^2_{L^2\Lambda^r(V)} + \left\| \delta \left(\Phi^{-1}\right)^* w \right\|^2_{L^2\Lambda^{r-2}(V)}
\end{aligned}$$

Since $w \in C^\infty_c \Lambda^{r-1}(B^+)$, we also have $\left(\Phi^{-1}\right)^* w \in C^\infty_c \Lambda^{r-1}(V)$ and thus by the Gaffney-Friedrich inequality,

$$\left\| \mathrm{d} \left(\Phi^{-1}\right)^* w \right\|^2_{L^2\Lambda^r(V)} + \left\| \delta \left(\Phi^{-1}\right)^* w \right\|^2_{L^2\Lambda^{r-2}(V)} \geq c \left\| \left(\Phi^{-1}\right)^* w \right\|^2_{H^1\Lambda^r(V)} - C \left\| \left(\Phi^{-1}\right)^* w \right\|^2_{L^2\Lambda^r(V)}.$$

We then note that

$$\|w\|_{H^1\Lambda^r(V)} \leq C \left\| \left(\Phi^{-1}\right)^* w \right\|_{H^1\Lambda^r(B^+)}$$

and

$$\left\| \left(\Phi^{-1}\right)^* w \right\|_{L^2\Lambda^r(B^+)} \leq C \|w\|_{L^2\Lambda^r(V)}$$

for some constant $C := C(d, \Phi) < \infty$. This implies (4.1.19). Now that one knows that the operator is strongly elliptic, one knows, by [**116**, Lemma 4.17], that the coefficient $A_{n,n}$ has a uniformly bounded inverse. As a consequence, one has

$$\begin{aligned}
\|\partial_d \partial_d u_\Phi\|_{L^2\Lambda^r(B^+)} &\leq \|A_{d,d} \partial_d \partial_d u_\Phi\|_{L^2\Lambda^r(B^+)} \\
&\leq \sum_{j=1}^d \sum_{k=1}^{d-1} \|A_{j,k} \partial_j \partial_k u\|_{L^2\Lambda^r(B^+)} + \sum_{j=1}^d \|A_j \partial_j u\|_{L^2\Lambda^r(B^+)} \\
&\qquad + \|Au\|_{L^2\Lambda^r(B^+)} + \|\Phi^* \mathrm{d}\bar{\mathbf{a}}\mathrm{d} f\|_{L^2\Lambda^{d-r+1}(B^+)}.
\end{aligned}$$

Using the main result of Step 3, this gives, for some constant $C := C(d, \lambda, \Phi) < \infty$,

$$\|\partial_d \partial_d u_\Phi\|_{L^2\Lambda^r(B^+)} \leq C \left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^r(U)} \right),$$

and the proof of Step 6 is complete.

Step 7. The main results of Steps 5 and 6 show that the function u_Φ belongs to $H^2\Lambda^{r-1}(B^+)$ together with the estimate

$$\|u_\Phi\|_{H^2\Lambda^r(B^+)} \leq C \left(\|u\|_{H^1\Lambda^{r-1}(U)} + \|\mathrm{d}f\|_{H^1\Lambda^r(U)} \right),$$

with $C := C(d, \lambda, \Phi) < \infty$. This implies

$$\|u\|_{H^2\Lambda^r(V\cap U)} \leq C \|u\|_{H^1\Lambda^{r-1}(U)} + C \|\mathrm{d}f\|_{H^1\Lambda^r(V\cap U)}. \tag{4.1.20}$$

Since ∂U is compact, we can cover ∂U with finitely many open sets $V_1, \ldots, V_N$ as above. We sum the resulting estimates, along with the interior estimate proved in Step 3, and obtain $u \in H^2\Lambda^r(U)$ with the estimate

$$\|u\|_{H^2\Lambda^r(U)} \leq C \|u\|_{H^1\Lambda^{r-1}(U)} + C \|\mathrm{d}f\|_{H^1\Lambda^{r-1}(U)},$$

for some $C := C(d, \lambda, U) < \infty$. We then simplify the right-hand side. Since we assumed $u \in \mathcal{H}^{r-1}_D(U)^\perp$, one has, by the Gaffney-Friedrich inequality, Proposition 4.3.3,

$$\|\nabla u\|_{L^2\Lambda^{r-1}(U)} \leq C \|\mathrm{d}u\|_{L^2\Lambda^r(U)} + C \|\delta u\|_{L^2\Lambda^{r-2}(U)}.$$

This inequality can be further refined, thanks to the version of the Poincaré inequality stated in (4.1.13), into

$$\|u\|_{H^1\Lambda^{r-1}(U)} \leq C \|\mathrm{d}u\|_{L^2\Lambda^r(U)} + C \|\delta u\|_{L^2\Lambda^{r-2}(U)}.$$

By (4.1.14) and the ellipticity assumption (4.1.21), one has

$$\|\mathrm{d}u\|^2_{L^2\Lambda^r(U)} + \|\delta u\|^2_{L^2\Lambda^{r-2}(U)} \leq C \|\mathrm{d}f\|_{H^1\Lambda^r(U)} \|u\|_{L^2\Lambda^{d-r+1}(U)} .$$

Combining the two previous displays with (4.1.20) shows

$$\|u\|_{H^2\Lambda^{r-1}(U)} \leq C \|\mathrm{d}f\|_{H^1\Lambda^r(U)} .$$

and the proof of Step 7 is complete.

Step 8. The conclusion. Note that, if u is a solution to (4.1.11), then $f + u$ is a solution of (4.1.9). Note also that, since two solutions of (4.1.9) differ by a form of $C^{r-1}_{\mathrm{d},0}$, so they have the same exterior derivative. From these remarks and the previous estimate, one obtains (4.1.10). The proof is complete. □

CHAPTER 5

Quantitative homogenization of the disordered $\nabla\phi$ model

We study the $\nabla\phi$ model with uniformly convex Hamiltonian $\mathcal{H}(\phi) := \sum V(\nabla\phi)$ and prove a quantitative rate of convergence for the properly rescaled partition function as well as a quantitative rate of convergence for the field ϕ subject to affine boundary condition in the L^2 norms. One of our motivations is to develop a new toolbox for studying this problem that does not rely on the Helffer-Sjöstrand representation. Instead, we make use of the variational formulation of the partition function, the notion of displacement convexity from the theory of optimal transport, and the recently developed theory of quantitative stochastic homogenization.

This chapter corresponds to the article [**53**].

Contents

5.1. Introduction, notation and main results

Let $(\mathbb{Z}^d, \mathbf{B}_d)$ be the graph $\mathbb{Z}^d$ in dimension $d \geq 2$ with its nearest-neighbor unoriented edges. Let $\lambda \in (0,1)$ be a fixed parameter, and let $(V_i)_{i=1,\ldots,d}$ be a family of functions in $C^2(\mathbb{R})$ satisfying

(1) $V_i(x) = V_i(-x)$
(2) $0 < \lambda \leq V_i''(x) \leq \frac{1}{\lambda} < \infty$.

In this article, we wish to study the Ginzburg-Landau or $\nabla\phi$ model associated to this potential. Specifically, we fix a bounded discrete set $U \subseteq \mathbb{Z}^d$ and define the boundary ∂U of U to be the set of vertices of U which are connected to $\mathbb{Z}^d \setminus U$. With this notation, we define for each function ϕ from U to $\mathbb{R}$ the following convex Hamiltonian

$$\mathcal{H}(\phi) := \sum_{e \subseteq U} V_e(\nabla\phi(e)),$$

where $\nabla\phi(e) = \phi(y) - \phi(x)$ if the edge is given by $e = (x,y)$ and the function V_e is equal to V_i is the edge e is of the form $(x, x + \mathfrak{e}_i)$, for $x \in \mathbb{Z}^d$. The goal of this article is to derive some quantitative information about the large scale behaviour of the Gibbs measure associated to this Hamiltonian and defined by, for some $p \in \mathbb{R}^d$,

$$d\mu_U = Z_U^{-1} \exp\left(-\sum_{e \subseteq U} V_e\left(\nabla\phi(e)\right)\right) \prod_{x \in U \setminus \partial U} d\phi(x) \prod_{x \in \partial U} \delta_{p \cdot x}(\phi(x)),$$

as well as some quantitative information about the normalization factor Z_U, also referred to as the partition function, which makes μ_U a probability measure.

This model and its large scale behavior have already been studied in several works. A common tool to study the $\nabla\phi$ model is the Helffer-Sjöstrand PDE representation which originates in [**93**]. Naddaf and Spencer in [**132**] were able to obtain a central limit theorem for this model by homogenizing the infinite dimensional elliptic PDE obtained from the Helffer-Sjöstrand representation. Funaki and Spohn in [**70**] studied the dynamics of this model. Deuschel Giacomin and Ioffe in [**56**] established the large scale L^2 convergence of the surface shape to some deterministic function, which can be characterized as the solution of an elliptic equation, as well as a large deviation principle. These result were later extended by Funaki and Sakagawa in [**69**]. In 2001, Giacomin, Olla and Spohn established in [**72**] a central limit theorem for the Langevin dynamic associated to this model. More recently Miller in [**119**] proved a central limit theorem for the fluctuation field around a macroscopic tilt.

The Helffer-Sjöstrand representation is a very powerful tool, but may also face some limitations. The PDE operator arising in this representation contains a divergence-form part whose coefficients are given by V''. In case when V'' is singular, or of a varying sign, then it is rather unclear how to proceed (see however [**48, 34, 47**]). Besides the specific results to be proved in this paper, we are interested in developing new tools to study the $\nabla\phi$ model that completely forego any reference to the Helffer-Sjöstrand representation. We will rely instead on the variational formulation of the free energy, and of the displacement convexity of the associated functional. To the best of our knowledge, it is the first time that tools from optimal transport are being used to study this model.

The mechanism by which we will obtain a rate of convergence, as opposed to a qualitative homogenization result, is inspired by recent developments in the homogenization of divergence-form operators with random coefficients. The first results in this context date bach to the early 1980s, with the results of Kozlov [**107**], Papanicolaou-Varadhan [**135**] and Yurinskiĭ [**150**] who were able to prove qualitative homogenization for linear elliptic equations under very general assumptions on the coefficient field. These results were later extended by Dal Maso and Modica in [**49, 50**] to the nonlinear setting. Obtaining quantitative rates of convergence has been the subject of much recent study over the past few years. Some notable progress were achieved by Gloria, Neukamm and Otto [**82, 83, 81**] and by Armstrong, Kuusi, Mourrat and Smart [**16, 17, 18, 21, 20**].

While most of the theory developed to understand stochastic homogenization focuses on linear elliptic equations, the closest analogy with the $\nabla\phi$ interface model is the stochastic homogenization of nonlinear equations. In this setting, the results are more sparse: one can mention the work of Armstrong, Mourrat and Smart [**20, 21**] who quantified the work of Dal Maso and Modica [**49, 50**]. More recently, Arsmtrong, Ferguson and Kuusi [**14**] were able to adapt part of the theory developed in the linear setting to the nonlinear setting.

The main results of this article are a sub-optimal algebraic rate of convergence for the logarithm of the partition function, cf Theorem 5.1.1, and to deduce from the previous result an algebraic rate of convergence of the field ϕ with affine boundary conditions in the L^2 norm to an affine function, cf Theorem 5.1.2. The analysis relies on the study of two subadditive quantities, denoted by ν and ν^*, which are approximately convex dual to one another. These quantities are reminiscent of those used in stochastic homogenization (cf [**18**, Chapters 1 and 2]) and which were key ingredients to develop this theory.

5.1.1. Notations and assumptions.

5.1.1.1. *Notations for the lattice and cubes.* In dimension $d \geq 2$, let $\mathbb{Z}^d$ be the standard d-dimensional hypercubic lattice, $\mathbf{B}_d := \left\{(x,y) \, : \, x,y \in \mathbb{Z}^d, |x-y| = 1\right\}$ the set of unoriented nearest neighbors and E_d be the set of oriented nearest neighbors. We denote the canonical basis of $\mathbb{R}^d$ by $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_d\}$. For $x,y \in \mathbb{Z}^d$, we write $x \sim y$ if x and y are nearest neighbors. For $x \in \mathbb{Z}^d$ and $r > 0$, we denote by $B(x,r) \subseteq \mathbb{Z}^d$ the discrete ball of center x and of radius r. We usually denote a generic edge by e. For a given subset U of $\mathbb{Z}^d$, we denote by $\mathbf{B}_d(U)$ the unoriented edges of U, i.e,

$$\mathbf{B}_d(U) := \{(x,y) \in \mathbf{B}_d \ : \ x \in U, y \in U \text{ and } x \sim y\}.$$

If we wish to talk about an oriented edge we will use an arrow to distinguish them from the unoriented edge. We denote by $\vec{e}$ a generic oriented edge. Similarly, we denote by E_d the oriented edges of U,

$$E_d(U) := \left\{\overrightarrow{xy} \in E_d \ : \ x \in U, y \in U \text{ and } x \sim y\right\}$$

We also denote by ∂U the discrete boundary of U, defined by

$$\partial U := \left\{x \in U : \exists y \in \mathbb{Z}^d, \ y \sim x \text{ and } y \notin U\right\}$$

and by U^{o} the discrete interior of U,

$$U^{\mathrm{o}} := U \setminus \partial U.$$

We also denote by $|U|$ the cardinality of U, we may refer to this quantity as the (discrete) volume of U. For $N \in \mathbb{N}$, we write $N\mathbb{Z}^d$ to refer to the set $\{Nx : x \in \mathbb{Z}^d\} \subseteq \mathbb{Z}^d$. A cube of $\mathbb{Z}^d$ is a set of the form

$$\mathbb{Z}^d \cap \left(z + [0,N]^d\right), \ z \in \mathbb{Z}^d, \ N \in \mathbb{N}.$$

We define the size of a cube given in the previous display above to be $N+1$. For $n \in \mathbb{N}$, we denote by $\square_n$ the discrete triadic cube of size 3^n,

$$\square_n := \left(-\frac{3^n}{2}, \frac{3^n}{2}\right)^d \cap \mathbb{Z}^d.$$

We say that a cube $\square$ is a triadic cube if it can be written

$$\square = z + \square_n, \text{ for some } n \in \mathbb{N}, \text{ and } z \in 3^n\mathbb{Z}^d.$$

Note that two triadic cubes are either disjoint or included in one another. Moreover for each $n \in \mathbb{N}$, the family of triadic cubes of size 3^n forms a partition of $\mathbb{Z}^d$. A caveat must be mentioned here, the family of triadic cubes $(z + \square_n)_{z \in 3^n\mathbb{Z}^d}$ forms a partition of $\mathbb{Z}^d$ but the family of edges $(\mathbf{B}(z + \square_n))_{z \in 3^n\mathbb{Z}^d}$ does not form a partition of $\mathbf{B}_d$, indeed we are missing the edges connecting two triadic cubes, i.e., the edges of the set

$$\left\{(x,y) \in \mathbf{B}_d \ : \ \exists z \in 3^n\mathbb{Z}^d, x \in z + \square_n \text{ and } y \notin z + \square_n\right\}.$$

We mention that the volume of a discrete triadic cube of the form $z + \square_n$ is 3^{dn} since this will be extensively used in this article.

Given two integers $m, n \in \mathbb{N}$ with $m < n$, we denote by

$$\mathcal{Z}_{m,n} := 3^m\mathbb{Z}^d \cap \square_n, \tag{5.1.1}$$

we also frequently use the shortcut notation

$$\mathcal{Z}_n := \mathcal{Z}_{n,n+1} = 3^n\mathbb{Z}^d \cap \square_{n+1}.$$

These sets have the property that $(z + \square_m)_{z \in \mathcal{Z}_{m,n}}$ is a partition of $\square_m$. In particular $(z + \square_n)_{z \in \mathcal{Z}_n}$ is a partition of $\square_{n+1}$.

5.1.1.2. *Notations for functions.* For a bounded subset $U \subseteq \mathbb{Z}^d$, and a function $\phi : U \to \mathbb{R}$, we denote by $(\phi)_U$ its mean value, that is to say

$$(\phi)_U := \frac{1}{|U|} \sum_{x \in U} \phi(x).$$

We let $h^1_0(U)$ and $\mathring{h}^1(U)$ be the set of functions from U to $\mathbb{R}$ with value zero on the boundary of U and mean value zero respectively, i.e

$$h^1_0(U) := \{\phi : U \to \mathbb{R} \ : \ u = 0 \text{ on } \partial U\}$$

and

$$\mathring{h}^1(U) := \{\psi : U \to \mathbb{R} \ : \ (\psi)_U = 0\}.$$

These spaces are finite dimensional and their dimension is given by the formulas

$$\dim h_0^1(U) = |U \setminus \partial U| \text{ and } \dim \mathring{h}^1(U) = |U| - 1.$$

We sometimes need to restrict functions, to this end we introduce the following notation, for any subsets $U, V \subseteq \mathbb{Z}^d$ satisfying $V \subseteq U$, and any function $\phi : U \to \mathbb{R}$, we denote by $\phi_{|V}$ the restriction of ϕ to V.

Let $U \subseteq \mathbb{Z}^d$, a vector field G on U is a function

$$G : E_d(U) \to \mathbb{R}$$

which is antisymmetric, that is, $G\left(\overrightarrow{xy}\right) = -G\left(\overrightarrow{yx}\right)$ for each $x, y \in E_d(U)$. Given a function $\phi : U \to \mathbb{R}$, we define its gradient by, for each $\overrightarrow{e} = \overrightarrow{xy} \in E_d(U)$,

$$\nabla\phi\left(\overrightarrow{e}\right) = \phi(y) - \phi(x).$$

The divergence of a vector field G is the function from U to $\mathbb{R}$ defined by, for each $x \in U$

$$\operatorname{div} G(x) = \sum_{y \in U, y \sim x} G\left(\overrightarrow{xy}\right).$$

We also define the Laplacian Δ of a function $\phi : U \to \mathbb{R}$ to be, for each $x \in U$

$$\Delta\phi(x) = \sum_{y \in U, y \sim x} (\phi(y) - \phi(x))$$

For $p \in \mathbb{R}^d$, we also denote by p the constant vector field given by

$$p(x, y) \coloneqq p \cdot (x - y). \tag{5.1.2}$$

Given two vector fields F and G, we define their product to be the function defined on the set of unoriented edges by

$$F \cdot G(x, y) = F(\overrightarrow{xy})G(\overrightarrow{xy}).$$

This notation will be frequently applied when F is a constant vector q and when G is the gradient of a function $\nabla\psi$, so we frequently write, for each $(x, y) \in \mathbf{B}_d$

$$q \cdot \nabla\psi(x, y) = q(x, y)\nabla\psi(x, y).$$

We also often use the shortcut notation

$$\sum_{e \subseteq U} \text{ to mean } \sum_{e \in \mathbf{B}_d(U)}.$$

If one assumes additionally that U is bounded, then for any vector field $F : E_d(U) \to \mathbb{R}$, we denote by $\langle F \rangle_U$ the unique vector in $\mathbb{R}^d$ such that, for each $p \in \mathbb{R}^d$

$$p \cdot \langle F \rangle_U = \frac{1}{|U|} \sum_{e \subseteq U} p \cdot F(e).$$

5.1.1.3. *Notations for vector spaces and scalar products.* Let V be a finite dimensional real vector space equipped with a scalar product $(\cdot,\cdot)_V$, this space can be endowed with a canonical Lebesgue measure denoted by Leb_V. This measure will be simply denoted by dx when we are integrating on V, i.e we write, for any measurable, integrable or non-negative, function $f : V \to \mathbb{R}$,

$$\int_V f(x)\,dx \text{ to mean } \int_V f(x)\,\mathrm{Leb}_V(dx). \tag{5.1.3}$$

For any linear subspace $H \subseteq V$, we denote by $H^\perp$ the orthogonal complement of H. Given $H, K \subseteq V$, we use the notation

$$V = H \overset{\perp}{\oplus} K \text{ if } V = H \oplus K \text{ and } \forall (h, k) \in H \times K,\ (h, k)_V = 0.$$

Note that from the scalar product on V, one can define scalar products on H and $H^\perp$ naturally by restricting the scalar product on V to these spaces. Consequently the spaces H and $H^\perp$ are equipped with Lebesgue measures denoted by Leb_H and $\mathrm{Leb}_{H^\perp}$. These measures are related to the Lebesgue measure on V by the relation

$$\mathrm{Leb}_V = \mathrm{Leb}_H \otimes \mathrm{Leb}_{H^\perp}, \tag{5.1.4}$$

where the notation $\otimes$ is used to denote the standard product of measures. Note that we used in the previous notation the equality $V = H \overset{\perp}{\oplus} H^\perp$ to obtain a canonical isomorphism between the spaces V (on which Leb_V is defined), and the space $H \times H^\perp$ (on which $\mathrm{Leb}_H \otimes \mathrm{Leb}_{H^\perp}$ is defined).

Given a bounded subset $U \subseteq \mathbb{Z}^d$, we equip any linear space V of functions from U to $\mathbb{R}$ with the standard L^2 scalar product, i.e., for any $\phi, \psi \in V$, we define

$$(\phi, \psi)_V := \sum_{x \in U} \phi(x)\psi(x).$$

This in particular applies to the spaces $h_0^1(U)$ and $\mathring{h}^1(U)$. From now on, we consider that these spaces are equipped with a scalar product and consequently with Lebesgue measure denoted by $\mathrm{Leb}_{h_0^1(U)}$ and $\mathrm{Leb}_{\mathring{h}^1(U)}$, or simply by dx when we use the notation convention (5.1.3).

5.1.1.4. *Notations for measures and random variables.* For any finite dimensional real vector space V, we denote by $\mathcal{P}(V)$ the set of probability measures on V equipped with its Borel σ-algebra denoted by $\mathcal{B}(V)$. For a pair of finite dimensional real vector spaces V and W, and a measure $\pi \in \mathcal{P}(V \times W)$, the first marginal of π is the probability measure $\mu \in \mathbb{P}(V)$ defined by, for each $A \in \mathcal{B}(V)$,

$$\mu(A) := \pi\left(A \times W\right),$$

we similarly define the second marginal as a measure in $\mathcal{P}(W)$. Given two probability measures $\mu \in \mathcal{P}(V)$ and $\nu \in \mathcal{P}(W)$, we denote by $\Pi(\mu, \nu)$ the set of probability measures of $\mathbb{P}\left(V \times W\right)$ whose first marginal is μ and second marginal is ν, i.e.,

$$\Pi(\mu,\nu) := \left\{\pi \in \mathcal{P}(V \times W) \,:\, \forall (A,B) \in (\mathcal{B}(V), \mathcal{B}(W)),\ \pi\left(A \times W\right) = \mu(A) \text{ and } \pi\left(V \times B\right) = \nu(B)\right\}.$$

We define a coupling between two probability measures $\mu \in \mathcal{P}(V)$ and $\nu \in \mathcal{P}(W)$ to be a measure in $\Pi(\mu,\nu)$. For a generic random variable X, we denote by $\mathbb{P}_X$ its law. Given two random variables X and Y, a coupling between X and Y is a random variable whose law belongs to $\Pi(\mathbb{P}_X, \mathbb{P}_Y)$.

An important caveat must be mentioned here, in this article we do not assume that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all the random variables are defined. In short, we use the random variables as proxy for their laws to simplify the notations. A consequence of this is that given two random variables X and Y, we have to be careful to first fix a coupling between X and Y if one wants to define the random variables $X + Y$, XY etc.

Given a measurable space $(X, \mathcal{F})$ and two σ-finite measures μ and ν on X, we write $\mu \ll \nu$ to mean that μ is absolutely continuous with respect to ν, and denote by $\frac{d\mu}{d\nu}$ the Radon-Nikodym derivative of μ with respect to ν. If we are given a second measurable space $(Y, \mathcal{F}')$ and a measurable map $T : X \to Y$, we denote by $T_* \mu$ the pushforward of the measure μ by the map T.

5.1.1.5. *Notations for the $\nabla\phi$ model.* We consider a family of function $(V_i)_{i=1,\ldots,d} \in C^2(\mathbb{R})$ satisfying the following assumptions, for each $i \in \{1, \ldots, d\}$

(1) *Symmetry:* for each $x \in \mathbb{R}$, $V_i(x) = V_i(-x)$,
(2) *Uniform convexity:* there exists $\lambda \in (0,1)$ such that $\lambda < V_i'' < \frac{1}{\lambda}$.
(3) *Normalization:* The value of V_i at 0 are fixed: $V_i(0) = 0$

Assumption (2) implies, for each $p_1, p_2 \in \mathbb{R}$,

$$\lambda \left|p_1 - p_2\right|^2 \leq V_i\left(p_1\right) + V_i\left(p_1\right) - 2V_i\left(\frac{p_1 + p_2}{2}\right) \leq \frac{1}{\lambda}\left|p_1 - p_2\right|^2. \tag{5.1.5}$$

From the first and second assumptions, we see that the functions V_i have a unique minimum achieved in 0. The third assumption is not necessary and can be easily removed, but thanks to this assumption we have the convenient inequality, for each $x \in \mathbb{R}$,

$$\lambda |x|^2 \le V_i(x) \le \frac{1}{\lambda}|x|^2.$$

For each edge $e \in \mathbf{B}_d$, we define $V_e := V_i$ where i is the unique integer in $\{1, \ldots, d\}$ such that e can be written $(x, x + \mathfrak{e}_i)$ for some $x \in \mathbb{Z}^d$. We then define the partition function, for each bounded subset $U \subseteq \mathbb{Z}^d$ and for each $p \in \mathbb{R}^d$,

$$Z_p(U) := \int_{h_0^1(U)} \exp\left(- \sum_{e \subseteq U} V_e(p(e) + \nabla\phi(e)) \right) d\phi, \tag{5.1.6}$$

where we recall that the notation $p(e)$ denotes the constant vector field introduced in (5.1.2) and $d\phi$ stands for the Lebesgue measure on $h_0^1(U)$. Note that thanks to the symmetry of the functions V_i, even if the vector field $p + \nabla\phi$ is defined for oriented edges, the quantity $V_e(p(e) + \nabla\phi(e)))$ can be defined for unoriented edges. From this one can define

$$\nu(U, p) := -\frac{1}{|U|} \ln Z_p(U) \tag{5.1.7}$$

and the probability measure on $h_0^1(U)$

$$\mathbb{P}_{U,p}(d\phi) := \frac{\exp\left(-\sum_{e\subseteq U} V_e(p(e) + \nabla\phi(e))\right) d\phi}{Z_p(U)}.$$

We also denote by $\phi_{U,p}$ a random variable of law $\mathbb{P}_{U,p}$. These objects will frequently be used with triadic cubes, we thus define the shortcut notations, for $n \in \mathbb{N}$,

$$\mathbb{P}_{n,p} := \mathbb{P}_{\square_n, p}$$

and by $\phi_{n,p}$ a random variable of law $\mathbb{P}_{n,p}$. We also define the quantity, for each $q \in \mathbb{R}^d$,

$$Z_q^*(U) := \int_{\mathring{h}^1(U)} \exp\left(- \sum_{e \subseteq U} \left(V_e(\nabla\psi(e)) - q \cdot \nabla\psi(e)\right) \right) d\psi, \tag{5.1.8}$$

as well as the quantity

$$\nu^*(U, q) := \frac{1}{|U|} \ln Z_q^*(U) \tag{5.1.9}$$

and the probability measure

$$\mathbb{P}_{U,q}^*(d\psi) := \frac{\exp\left(-\sum_{e\subseteq U} \left(V_e(\nabla\psi(e)) - q\cdot\nabla\psi(e)\right)\right) d\psi}{Z_q^*(U)}.$$

We denote by $\psi_{U,p}$ a random variable of law $\mathbb{P}_{U,p}$ and we frequently write $\mathbb{P}_{n,q}^*$ and $\psi_{n,p}$ instead of $\mathbb{P}_{\square_n,q}^*$ and $\psi_{\square_n,p}$.

5.1.1.6. *Convention for constants and exponents.* Throughout this article, the symbols c and C denote positive constants which may vary from line to line. These constants may depend solely on the parameters d, the dimension of the space, and λ, the ellipticity bound on the second derivative of the function V_e. Similarly we use the symbols α and β to denote positive exponents which may vary from line to line and depend only on d and λ. Usually, we use C for large constants (whose value is expected to belong to $[1, \infty)$) and c for small constants (whose value is expected to be in $(0, 1]$). The values of the exponents α and β are always expected to be small.

We also frequently write $C := C(d, \lambda) < \infty$ to mean that the constant C depends only on the parameters d, λ and that its value is expected to be large. We may also write $C := C(d) < \infty$ or $C := C(\lambda) < \infty$ if the constant C depends only on d (resp. λ). For small constants or exponents we use the notations $c := c(d, \lambda) > 0$, $\alpha := \alpha(d, \lambda) > 0$, $\beta := \beta(d, \lambda) > 0$.

5.1.2. Main result. The goal of this article is to show the quantitative convergence of the quantities $\nu(\square_n, p)$ and $\nu^*(\square_n, q)$. This is stated in the following theorem.

THEOREM 5.1.1 (Quantitative convergence to the Gibbs state). *There exists a constant $C := C(d,\lambda) < \infty$ and an exponent $\alpha := \alpha(d,\lambda) > 0$ such that for each $p, q \in \mathbb{R}^d$, there exist two real numbers $\overline{\nu}(p)$ and $\overline{\nu}^*(q)$ such that*

$$|\nu(\square_n, p) - \overline{\nu}(p)| \leq C 3^{-\alpha n}(1 + |p|^2) \tag{5.1.10}$$

and

$$|\nu^*(\square_n, q) - \overline{\nu}^*(q)| \leq C 3^{-\alpha n}(1 + |q|^2). \tag{5.1.11}$$

Moreover the functions $p \to \overline{\nu}(p)$ and $q \to \overline{\nu}^*(q)$ are uniformly convex, there exists a constant $C := C(d,\lambda) < \infty$ such that for each $p_1, p_2 \in \mathbb{R}^d$

$$\frac{1}{C}|p_1 - p_2|^2 \leq \overline{\nu}(p_1) + \overline{\nu}(p_2) - 2\overline{\nu}\left(\frac{p_1 + p_2}{2}\right) \leq C|p_1 - p_2|^2, \tag{5.1.12}$$

for each $q_1, q_2 \in \mathbb{R}^d$,

$$\frac{1}{C}|q_1 - q_2|^2 \leq \overline{\nu}^*(q_1) + \overline{\nu}^*(q_2) - 2\overline{\nu}^*\left(\frac{q_1 + q_2}{2}\right) \leq C|q_1 - q_2|^2 \tag{5.1.13}$$

and are dual convex, i.e. for each $q \in \mathbb{R}^d$

$$\overline{\nu}^*(q) = \sup_{p \in \mathbb{R}^d} \left(-\overline{\nu}(p) + p \cdot q\right). \tag{5.1.14}$$

From this, we deduce that the random variables $\phi_{n,p}$ and $\psi_{n,q}$ are close to affine functions in the expectation of the L^2 norm. Note that (5.1.12) and (5.1.13) imply that the functions $p \to \overline{\nu}(p)$ and $q \to \overline{\nu}^*(q)$ are $C^{1,1}$ and we denote their gradients by $\nabla_p \overline{\nu}$ and $\nabla_q \overline{\nu}^*$.

THEOREM 5.1.2 (L^2 contraction of the Gibbs measure). *There exist a constant $C := C(d,\lambda) < \infty$ and an exponent $\alpha := \alpha(d,\lambda) > 0$ such that for each $n \in \mathbb{N}$, $p, q \in \mathbb{R}^d$,*

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{x \in \square_n}\left(|\phi_{n,p}(x)|^2 + |\psi_{n,q}(x) - \nabla_q \overline{\nu}^*(q) \cdot x|^2\right)\right] \leq C 3^{n(2-\alpha)}\left(1 + |p|^2 + |q|^2\right).$$

5.1.3. Strategy of the proof. The strategy of the proof is to use the ideas from the theory of quantitative stochastic homogenization, and in particular the ideas developed by Armstrong, Kuusi, Mourrat and Smart in [**18**] and [**21**] to the setting of the $\nabla\phi$ model. To this end, we introduce the two quantities ν and ν^*, which are in some sort equivalent to the subadditive quantities with the same notation used in [**18**, Chapters 1 and 2].

The idea is then to find a variational formulation for these quantities to rewrite them as a minimization problem of a convex functional, this in done in the Subsection 5.2.2. Nevertheless this functionnal involves a term of entropy and it is not a priori clear that the functional is convex. To solve this issue, we appeal to optimal transport and more specifically to the notion of displacement convexity to obtain some sort of convexity for the entropy.

Once this is done, we are able to collect some properties about ν and ν^* which match the basic properties of the equivalent quantities in stochastic homogenization, to see this, one can for instance compare Propostion 5.3.1 with Lemma 1.1 of [**18**]. One can then exploit the convex dualiy between ν and ν^* to obtain a quantitative rate of convergence for these quantities as it is done in Section 5.4.

5.1.4. Outline of the paper. The rest of the article is organized as follows. In Section 5.2, we collect some preliminary results which will be useful to prove the main theorems. Specifically, we introduce the differential entropy of a measure and state some of its properties, we also record some definitions from the theory of optimal transport and state the main result we need to borrow from this theory, namely the displacement convexity of the entropy, Proposition 5.2.10. We also introduce the variational formulation for ν and $\overline{\nu}^*$ in Subsection 5.2.2. In Subsection 5.2.4 we state and prove a technical lemma which allows to construct suitable coupling between random variables. We then complete Section 5.2 by stating some functional inequalities on the lattice $\mathbb{Z}^d$, in particular the multiscale Poincaré inequality which is an important ingredient in the theory of stochastic homogenization.

In Section 5.3, we use the tools from Section 5.2 to prove a series of properties on the quantities ν and ν^*, summed up in Proposition 5.3.1. These estimates, though not particularly difficult, are technical and the details are many.

In Section 5.4, we combine the tools collected in Section 5.2 with the result proved in Section 5.3 to first prove that the variance of the random variable $(\psi_{n,q})_{\square_n}$ contracts, this is done in Lemma 5.4.2. We then deduce from this result and the multiscale Poincaré inequality that the random variable $\psi_{n,q}$ is close to an affine function in the L^2 norm, this is Proposition 5.4.4. We then use these results combined with a patching construction, reminiscent to the one performed in [**21**], to prove Theorems 5.1.1 and 5.1.13.

Appendix 5.A is devoted to the proof of some technical estimates useful in Sections 5.2 and 5.3.

Appendix 5.B is devoted to the proof of some inequalities from the theory of elliptic equations adapted to the setting of the $\nabla\phi$ model. Namely we prove a version of the Caccioppoli inequality, the reverse Hölder inequality and the Meyers estimate for the $\nabla\phi$ model.

5.1.5. Acknowledgements. I would like to thank Jean-Christophe Mourrat for helpful discussions and comments.

5.2. Preliminaries

5.2.1. The entropy and some of its properties. In this section, we define one of the main tools used in this article, the differential entropy, we then collect a few properties of this quantity which will be useful in the rest of the article. We first give the definition of the entropy.

Definition 5.2.1 (Differential entropy). Let V be a finite dimensional vector space equipped with a scalar product. Denote by $\mathcal{B}_V$ the associated Borel set. Consider the Lebesgue measure on V and denote it by Leb. For each probability measure $\mathbb{P}$ on V, we define its entropy according to

$$H(\mathbb{P}) := \begin{cases} \int_V \frac{d\mathbb{P}}{d\mathrm{Leb}}(x) \ln\left(\frac{d\mathbb{P}}{d\mathrm{Leb}}(x)\right) dx \text{ if } \mathbb{P} \ll \mathrm{Leb} \text{ and } \frac{d\mathbb{P}}{d\mathrm{Leb}} \ln\left(\frac{d\mathbb{P}}{d\mathrm{Leb}}\right) \in L^1(V, \mathcal{B}(V), \mathrm{Leb}), \\ +\infty \text{ otherwise.} \end{cases}$$

Remark 5.2.2.
- In this article we implicitely extend the function $x \to x \ln x$ by 0 at 0.
- We emphasize that the usual definition of the differential entropy is stated with the function $x \to -x \ln x$ instead of the function $x \to x \ln x$. Adopting the other sign convention is more meaningful in this article because we want the entropy to be convex in the sense of displacement convexity as it will be explained in the following subsections.

We now record a few properties about the entropy. We first study how the entropy behaves under translation and affine change of variables. These properties are standard and fairly simple to prove, the details are thus omitted.

Proposition 5.2.3 (Translation and linear change of variables of the entropy). *Let X be a random variable taking values in a finite dimensional real vector space V equipped with a scalar* *products. Denote by* $\mathbb{P}_X$ *the law of* X*. For each* $a \in V$*, if we denote by* $\mathbb{P}_{X+a}$ *the law of the random variable* $X + a$ *then we have*

$$H(\mathbb{P}_{X+a}) = H(\mathbb{P}_X) \tag{5.2.1}$$

Now consider L *a linear map from* V *to* V*. Then if we denote by* $\mathbb{P}_{L(X)}$ *the law of the random variable* $L(X)$*, then we have*

$$H\left(\mathbb{P}_{L(X)}\right) = H(\mathbb{P}_X) - \ln|\det L|. \tag{5.2.2}$$

In particular if L *is non invertible then* $\det L = 0$ *and* $H\left(\mathbb{P}_{L(X)}\right) = \infty$*.*

Let V, W be two finite dimensional real vector spaces equipped with scalar product denoted by $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$. Denote by Leb_V and Leb_W the Lebesgue measure on V and W. Consider the space $V \times W$. Define a scalar product on this space by, for each $v, v' \in V$ and each $w, w' \in W$,

$$\left((v,w),(v',w')\right)_{V\times W} = \left(v,v'\right)_V + \left(w,w'\right)_W.$$

Then the Lebesgue measure on $V \oplus W$ satisfies

$$\mathrm{Leb}_{V\times W} = \mathrm{Leb}_V \otimes \mathrm{Leb}_W.$$

The following proposition gives a property about the entropy of a pair of random variables.

Proposition 5.2.4. *Let* V, W *be two finite dimensional real vector spaces equipped with scalar product. Consider the space* $V \times W$ *equipped with the scalar product defined above. Let* X *and* Y *be two random variables valued in respectively* V *and* W*. Assume that* $H(\mathbb{P}_X) < \infty$*,* $H(\mathbb{P}_Y) < \infty$ *and that we are given a coupling* (X,Y) *between* X *and* Y*, then we have*

$$H\left(\mathbb{P}_{(X,Y)}\right) \geq H(\mathbb{P}_X) + H(\mathbb{P}_Y),$$

with equality if and only if X *and* Y *are independent.*

Remark 5.2.5. This inequality states that the entropy of two random variable is minimal when X and Y are independent while the reader may be used for the entropy to be maximal when the random variables are independent. This is due to the sign convention adopted in Definition 5.2.1.

Proof. These estimates can be obtained using the convexity of the function $x \to x\ln x$ and the Jensen inequality. The proof is standard and the details are omitted. □

Frequently in this article, the previous proposition will be used with the following formulation

Proposition 5.2.6. *Let* U *be a finite dimensional vector space equipped with a scalar product and assume that we are given two linear spaces of* U*, denoted by* V *and* W *such that*

$$U = V \overset{\perp}{\oplus} W.$$

Assume moreover that we are given two random variables X *and* Y *taking values respectively in* V *and* W*. Assume that* $H(\mathbb{P}_X) < \infty$*,* $H(\mathbb{P}_Y) < \infty$ *and that we are given a coupling* (X,Y) *between* X *and* Y*, then we have*

$$H(\mathbb{P}_{X+Y}) \geq H(\mathbb{P}_X) + H(\mathbb{P}_Y),$$

where the entropy of $X + Y$ *(resp.* X *and* Y*) is computed with respect to the Lebesgue measure on* U *(resp.* V *and* W*). Moreover there is equality in the previous display if and only if* X *and* Y *are independent.*

Proof. This proposition is a consequence of Proposition 5.2.4 and the fact that there exists a canonical isometry between U and $V \times W$ given by

$$\Phi : \begin{cases} V \times W \to U \\ (v,w) \mapsto v + w. \end{cases} \tag{5.2.3}$$

□

5.2.2. Variational formula for ν and ν^*. One of the key ideas of this article is to introduce a convex functional $\mathcal{F}_{n,p}$ (resp. $\mathcal{F}^*_{n,q}$) defined on set of the probability measures on the space $h^1_0(\square_n)$ (resp. $\mathring{h}^1(\square_n)$) such that $\mathbb{P}_{n,p}$ (resp. $\mathbb{P}^*_{n,q}$) is the minimizer of $\mathcal{F}_{n,p}$ (resp. $\mathcal{F}^*_{n,q}$). The convexity of $\mathcal{F}_{n,p}$ (resp. $\mathcal{F}^*_{n,q}$) allows to perform a perturbative analysis around its minimizer, i.e. the measure $\mathbb{P}_{n,p}$ (resp. $\mathbb{P}^*_{n,q}$), and to obtain quantitative estimates which will turn out to be crucial in the proof of Theorem 5.1.1.

Definition 5.2.7. For each $n \in \mathbb{N}$ and each $p, q \in \mathbb{R}^d$, we define

$$\mathcal{F}_{n,p} : \begin{cases} \mathcal{P}\left(h^1_0(\square_n)\right) \to \mathbb{R} \\ \qquad \mathbb{P} \mapsto \mathbb{E}\left[\sum_{e \subseteq \square_n} V_e\left(p\cdot e + \nabla\phi_e\right)\right] + H\left(\mathbb{P}\right), \end{cases}$$

where ϕ is a random variable of law $\mathbb{P}$. Similarly, we define

$$\mathcal{F}^*_{n,q} : \begin{cases} \mathcal{P}\left(\mathring{h}^1(\square_n)\right) \to \mathbb{R} \\ \qquad \mathbb{P}^* \mapsto \mathbb{E}\left[\sum_{e \subseteq \square_n} \left(V_e\left(\nabla\psi(e)\right) - q\nabla\psi(e)\right)\right] + H\left(\mathbb{P}^*\right), \end{cases}$$

where ψ is a random variable of law $\mathbb{P}^*$.

The main property about this functional is stated in the following proposition.

Proposition 5.2.8. *Let V be a finite dimensional real vector space equipped with a scalar product. We denote by $\mathcal{B}_V$ be the Borel set associated to V. For any measurable function $f : V \to \mathbb{R}$ bounded from below, one has the formula*

$$-\log\int_V \exp\left(-f(x)\right)\,dx = \inf_{\mathbb{P}\in\mathcal{P}(V)}\left(\int_{\mathbb{P}} f(x)\mathbb{P}(dx) + H\left(\mathbb{P}\right)\right) \tag{5.2.4}$$

where the integral in the left-hand side is computed with respect to the Lebesgue measure on V.

As a consequence, one has the following formula, for each $n \in \mathbb{N}$ and each $p \in \mathbb{R}^d$,

$$\nu\left(\square_n, p\right) = \inf_{\mathbb{P}\in\mathcal{P}\left(h^1_0(\square_n)\right)}\left(\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(p\cdot e + \nabla\phi(e)\right)\right] + \frac{1}{|\square_n|}H\left(\mathbb{P}\right)\right), \tag{5.2.5}$$

where in the previous formula, ϕ is a random variable of law $\mathbb{P}$. Moreover the minimum is attained for the measure $\mathbb{P}_{n,p}$. Similarly, one has, for each $q \in \mathbb{R}^d$,

$$\nu^*\left(\square_n, q\right) = \sup_{\mathbb{P}^*\in\mathcal{P}\left(\mathring{h}^1(\square_n)\right)}\left(\frac{1}{|\square_n|}\mathbb{E}\left[-\sum_{e\subseteq\square_n}\left(V_e\left(\nabla\psi(e)\right) - q\cdot\nabla\psi(e)\right)\right] - \frac{1}{|\square_n|}H\left(\mathbb{P}^*\right)\right), \tag{5.2.6}$$

where in the previous formula, ψ is a random variable of law $\mathbb{P}^*$. Moreover the minimum is attained for the measure $\mathbb{P}^*_{n,q}$.

Proof. We first prove (5.2.4) and decompose the proof into two steps.

Step 1. Let $\mathbb{P}$ be a probability measure on V, we want to show that

$$\int_V f(x)\mathbb{P}(dx) + H\left(\mathbb{P}\right) \geq -\log\int_V \exp\left(-f(x)\right)\,dx. \tag{5.2.7}$$

First note that if $H(\mathbb{P}) = \infty$, then the term on the left-hand side is equal to ∞ and the inequality is satisfied. Thus one can assume that $H\left(\mathbb{P}\right) < \infty$, this implies that $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure on V and we denote by h its density. In particular, we have

$$H\left(\mathbb{P}\right) = \int_V h(x)\ln h(x)\,dx.$$

Similarly, one can assume that $\int_V f(x)h(x)\,dx < \infty$ otherwise the estimate (5.2.7) is automatically verified. Using that h is a probability density and the Jensen inequality, one obtains

$$\exp\left(-\int_V f(x)h(x)\,dx - H(\mathbb{P})\right) \le \int_V \exp\left(-f(x) - \ln h(x)\right) h(x)\,dx.$$

We then denote

$$A := \{x \in V \,:\, h(x) > 0\} \in \mathcal{B}_V,$$

so that

$$\begin{aligned}\exp\left(-\int_V f(x)h(x)\,dx - H(\mathbb{P})\right) &\le \int_V \mathbb{1}_A(x)\exp\left(-f(x)\right) h(x)^{-1}h(x)\,dx \\ &\le \int_V \mathbb{1}_A(x)\exp\left(-f(x)\right)\,dx \\ &\le \int_V \exp\left(-f(x)\right)\,dx.\end{aligned}$$

This is precisely (5.2.7).

Step 2. We assume that

$$\int_V \exp\left(-f(x)\right)\,dx < \infty \text{ and } \int_V |f(x)|\exp\left(-f(x)\right)\,dx < \infty.$$

and construct a measure $\mathbb{P} \in \mathcal{P}(V)$ satisfying

$$\int_{\mathbb{P}} f(x)\mathbb{P}(dx) + H(\mathbb{P}) = -\log\int_V \exp\left(-f(x)\right)\,dx. \tag{5.2.8}$$

In this case, we define

$$\mathbb{P} := \frac{\exp\left(-f(x)\right)}{\int_V \exp\left(-f(x)\right)\,dx}\,dx.$$

It is clear that $\mathbb{P} \ll \mathrm{Leb}_V$ and, from the assumption required on the function f for this step, that $H(\mathbb{P}) < \infty$. An explicit computation gives

$$\begin{aligned}\int_V f(x)\mathbb{P}(dx) + H(\mathbb{P}) &= \int_V f(x) - f(x) - \ln\left(\int_V \exp\left(-f(x)\right)\,dx\right)\mathbb{P}(dx) \\ &= \ln\left(\int_V \exp\left(-f(x)\right)\,dx\right)\mathbb{P}(dx).\end{aligned}$$

This completes the proof of (5.2.4).

Step 3. In this step, we assume that

$$\int_V \exp\left(-f(x)\right)\,dx = \infty \text{ or } \int_V |f(x)|\exp\left(-f(x)\right)\,dx = \infty$$

and we construct a sequence of probability measures $\mathbb{P}_n$ such that

$$\int_V f(x)\mathbb{P}_M(dx) + H(\mathbb{P}_M) \underset{n\to\infty}{\longrightarrow} -\log\int_V \exp\left(-f(x)\right)\,dx,$$

where we used the convention

$$-\log\int_V \exp\left(-f(x)\right)\,dx = -\infty \text{ if } \int_V \exp\left(-f(x)\right)\,dx = \infty.$$

To this end, we define, for each $n \in \mathbb{N}$,

$$\mathbb{P}_n := \frac{\exp\left(-f(x)\right)\mathbb{1}_{\{|x|\le n \text{ and } f(x)\le n\}}}{\int_V \exp\left(-f(x)\right)\mathbb{1}_{\{|x|\le n \text{ and } f(x)\le n\}}\,dx}\,dx.$$

With this definition, we compute

$$\int_V f(x)\mathbb{P}_n(dx) + H(\mathbb{P}_n) = -\ln\left(\int_V \exp\left(-f(x)\right)\mathbb{1}_{\{|x|\le n \text{ and } f(x)\le n\}}\,dx\right).$$

Sending $n \to \infty$ gives the result.

Step 4. In this step, we prove (5.2.5). First note that by the bound $V_e(x) \leq \frac{1}{\lambda}|x|^2$, there exists a constant $C := C(d,\lambda) < \infty$ such that for each $n \in \mathbb{N}$ and each $\phi \in h_0^1(\square_n)$,

$$\sum_{e\subseteq\square_n} V_e\left(p\cdot e + \nabla\phi(e)\right) \leq C|p|^2 + C\sum_{x\in\square_n}|\phi(x)|^2. \tag{5.2.9}$$

Using this inequality, we deduce that one can apply (5.2.4) with $V = h_0^1(\square_n)$ equipped with the standard scalar product and $f(\phi) = \sum_{e\subseteq\square_n} V_e\left(p\cdot e + \nabla\phi(e)\right)$. Moreover, using the estimate (5.2.9), one sees

$$\int_{h_0^1(\square_n)} \exp\left(-f(\phi)\right)\,dx < \infty \text{ and } \int_{h_0^1(\square_n)} |f(\phi)|\exp\left(-f(\phi)\right)\,d\phi < \infty.$$

Thus applying the result proved in Steps 1 and 2, one has

$$\nu\left(\square_n, p\right) = \inf_{\mathbb{P}\in\mathcal{P}\left(h_0^1(\square_n)\right)} \frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(p\cdot e + \nabla\phi(e)\right)\right] + \frac{1}{|\square_n|}H\left(\mathbb{P}\right)$$

and the minimum is attained for the measure

$$\mathbb{P}_{n,p}(d\phi) = \frac{\exp\left(-\sum_{e\subseteq U} V_e(p\cdot e + \nabla\phi(e))\right)\,d\phi}{\int_{h_0^1(\square_n)}\exp\left(-\sum_{e\subseteq U} V_e(p\cdot e + \nabla\phi(e))\right)\,d\phi}.$$

The proof of (5.2.6) is similar and the details are left to the reader. ☐

5.2.3. Optimal transport and displacement convexity. In this section, we introduce a few definitions about optimal transport and state one of the main tools of this article, namely the displacement convexity. We first give a definition of the optimal coupling. The existence of this coupling is rather standard and the uniqueness is more involved and is a byproduct of Brenier's theorem. We refer to [**146**, Proposition 2.1 and Theorem 2.12] for this definition.

DEFINITION 5.2.9. Let U be a finite dimensional real vector space equipped with a scalar product. We denote by $|\cdot|$ the norm associated to this scalar product. Let X and Y be two random variables taking values in U and denote their laws by $\mathbb{P}_X$ and $\mathbb{P}_Y$ respectively. Assume additionally that $\mathbb{P}_X$ and $\mathbb{P}_Y$ have a finite second moment, i.e,

$$\mathbb{E}\left[|X|^2\right] < \infty \text{ and } \mathbb{E}\left[|Y|^2\right] < \infty, \tag{5.2.10}$$

and that they are absolutely continuous with respect to the Lebesgue measure on V, then the minimization problem

$$\inf_{\mu\in\Pi(\mathbb{P}_X,\mathbb{P}_Y)} \int_U |x-y|^2\,\mu(dx,dy)$$

admits a unique minimizer denoted by $\mu_{(X,Y)}$ and called the optimal coupling between X and Y.

For $t\in[0,1]$, we denote by T_t the mapping

$$T_t := \begin{cases} U\times U \to U \\ (x,y)\mapsto (1-t)x + ty, \end{cases}$$

and for two random variables X and Y taking values in U with finite second moment, we denote by

$$\mu_t := \left(T_t\right)_* \mu_{(X,Y)},$$

this is the law of $(1-t)X + tY$ when the coupling between X and Y is the optimal coupling. The main property we need to use is called the displacement convexity and stated in the following proposition. We refer to [**146**, Chapter 5] for this theorem but it is mostly due to McCann [**115**].

PROPOSITION 5.2.10 (Displacement convexity, Theorem 5.15 of [**146**]). *Let U be a finite dimensional real vector space equipped with a scalar product, let X and Y be two random variables taking values in U with finite second moment, i.e satisfying* (5.2.10)*, then the function $t\to H(\mu_t)$ is convex, i.e., for each $t\in[0,1]$,*

$$H\left(\mu_t\right) \leq (1-t)H\left(\mathbb{P}_X\right) + tH\left(\mathbb{P}_Y\right).$$

5.2.4. Coupling lemmas. Thanks to optimal transport theory and particularly thanks to Definition 5.2.9, we are able to couple two random variables. The next question which arises, and which needs to be answered to prove Theorem 5.1.1, is to find a way to couple three random variables. Broadly speaking, the question we need to answer is the following: assume that we are given three random variables X, Y and Z, a coupling between X and Y and another coupling between Y and Z, can we find a coupling between X, Y and Z? This question can be positively answered thanks to the following proposition.

PROPOSITION 5.2.11. *Let $(E_1, \mathcal{B}_1)$, $(E_2, \mathcal{B}_2)$, $(E_3, \mathcal{B}_3)$ be three Polish spaces equipped with their Borel σ-algebras. Assume that we are given three probability measures $\mathbb{P}_X$ on E_1, $\mathbb{P}_Y$ on E_2 and $\mathbb{P}_Z$ on E_3 as well as a coupling $\mathbb{P}_{(X,Y)}$ between $\mathbb{P}_X$, $\mathbb{P}_Y$ and a coupling $\mathbb{P}_{(Y,Z)}$ between $\mathbb{P}_Y$, $\mathbb{P}_Z$, that is to say two measures on $(E_1 \times E_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$ and $(E_2 \times E_3, \mathcal{B}_2 \otimes \mathcal{B}_3)$ satisfying, for each $(B_1, B_2, B_3) \in (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$,*

$$\mathbb{P}_{(X,Y)}(B_1 \times E_2) = \mathbb{P}_X(B_1), \ \mathbb{P}_{(X,Y)}(E \times B_2) = \mathbb{P}_Y(B_2) \text{ and}$$
$$\mathbb{P}_{(Y,Z)}(B_2 \times E) = \mathbb{P}_Y(B_2), \ \mathbb{P}_{(Y,Z)}(E \times B_3) = \mathbb{P}_Z(B_3),$$

then there exists a probability measure $\mathbb{P}_{(X,Y,Z)}$ on $(E_1 \times E_2 \times E_3, \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3)$ such that for each $B_{12} \in \mathcal{B}_1 \otimes \mathcal{B}_2$ and each $B_{23} \in \mathcal{B}_2 \otimes \mathcal{B}_3$,

$$\mathbb{P}_{(X,Y,Z)}(B_{12} \times E_3) = \mathbb{P}_{(X,Y)}(B_{12}) \text{ and } \mathbb{P}_{(X,Y,Z)}(E_1 \times B_{23}) = \mathbb{P}_{(Y,Z)}(B_{23}). \tag{5.2.11}$$

REMARK 5.2.12. As was mentioned earlier, in this article we think of random variables as laws and we do not assume that there is an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all the random variables are already defined. For instance, we will say that we are given two random variables (X, Y) and (Y, Z) to mean that we are given two measures $\mathbb{P}_{(X,Y)}$ and $\mathbb{P}_{(Y,Z)}$ such that the marginals of $\mathbb{P}_{(X,Y)}$ are $\mathbb{P}_X$ and $\mathbb{P}_Y$ and the marginals of $\mathbb{P}_{(Y,Z)}$ are $\mathbb{P}_Y$ and $\mathbb{P}_Z$ without assuming that there exists an implicit probability space on which X, Y and Z are defined, indeed in that case the statement of the proposition would be trivial.

This convention allows to simplify the notation in the proofs and has the following consequence: when we are given two random variables X and Y, we need to be careful to always construct a coupling between X and Y before introducing the random variables $X + Y$, XY or any other display involving both X and Y.

The proof of Proposition 5.2.11 relies on the existence of the conditional law which is recalled below.

PROPOSITION 5.2.13 (Theorem 33.3 and Theorem 34.5 of [**31**]). *Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be two Polish spaces equipped with their Borel σ-algebras. Assume that we are given two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on E and F respectively. Let $\mathbb{P}_{12}$ be a probability measure on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ whose first and second marginals are $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively, then there exists a mapping $\nu : E_1 \times \mathcal{F} \to \mathbb{R}_+$ such that*

1. *for each $x \in E$, $\nu(x, \cdot)$ is a probability measure on $(F, \mathcal{F})$*
2. *for each $A \in \mathcal{F}$, the mapping*

$$\nu(\cdot, A) \begin{cases} (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \\ \quad x \mapsto \nu(x, A) \end{cases}$$

   is measurable.

3. *For each $A_1 \in \mathcal{E}$ and each $A_2 \in \mathcal{F}$,*

$$\mathbb{P}_{12}(A_1 \times A_2) = \int_{A_1} \nu(x, A_2)\, \mathbb{P}_1(dx).$$

We can now prove Proposition 5.2.11.

Proof of Proposition 5.2.11. The idea is to apply Proposition 5.2.13 to the two laws $\mathbb{P}_{(X,Y)}$ and $\mathbb{P}_{(Y,Z)}$. This gives the existence of two conditional laws denoted by ν_X and ν_Z such that for each $B_1, B_2, B_3 \in (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$,

$$\mathbb{P}_{(X,Y)}(B_1 \times B_2) = \int_{B_2} \nu_X(y, B_1)\, \mathbb{P}_Y(dy) \text{ and } \mathbb{P}_{(Y,Z)}(B_2 \times B_3) = \int_{B_2} \nu_Z(y, B_3)\, \mathbb{P}_Y(dy).$$

We can the define for each $B_1, B_2, B_3 \in (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$,

$$\mathbb{P}_{(X,Y,Z)}(B_1 \times B_2 \times B_3) = \int_{B_2} \nu_X(y, B_1)\, \nu_Z(y, B_3)\, \mathbb{P}_Y(dy).$$

Using standard tools from measure theory, one can then extend $\mathbb{P}_{(X,Y,Z)}$ into a measure on $\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3$ and verify that this measure satisfies (5.2.11). □

5.2.5. Functional inequalities on the lattice. In this section, we want to prove a few functional inequalities for functions on the lattice $\mathbb{Z}^d$, namely the Poincaré inequality, and the multiscale Poincaré inequality. These inequalities are known on $\mathbb{R}^d$ so the strategy of the proof is to extend functions defined on $\mathbb{Z}^d$ to $\mathbb{R}^d$, to apply the inequalities to the extended functions and then show that the inequality obtained for the extended function is enough to prove the inequality for the discrete function.

The second inequality presented, called the multiscale Poincaré inequality, is a convenient tool to control the L^2 norm of a function by the spatial average of its gradient. It is proved in [**18**, Proposition 1.7 and Lemma 1.8]. The philosophy behind it comes from the theory of stochastic homogenization and roughly states that the usual Poincaré inequality can be refined by estimating the L^2 norm of a function by the spatial average of the gradient. This inequality is useful when one is dealing with rapidly oscillating functions, which frequently appear in homogenization. Indeed for these functions, the oscillations cancel out in the spatial average of the gradient, as a result these spatial averages are much smaller than the L^2 norm of the gradient. The resulting estimate is thus much more precise than the standard Poincaré inequality. We recall the definition of $\mathcal{Z}_{m,n}$ given in (5.1.1).

Proposition 5.2.14 (Poincaré and Multiscale Poincaré inequalities). *Let $\square$ be a cube of $\mathbb{Z}^d$ of size R and $u : \square \to \mathbb{R}$, then one has the inequality, for some $C := C(d) < \infty$*

$$\sum_{x \in \square} \left| u(x) - (u)_{\square} \right|^2 \le CR^2 \sum_{e \subseteq \square} |\nabla u(e)|^2, \tag{5.2.12}$$

if one assumes that $u = 0$ on $\partial\square$, then one has

$$\sum_{x \in \square} |u(x)|^2 \le CR^2 \sum_{e \subseteq \square} |\nabla u(e)|^2. \tag{5.2.13}$$

For each $n \in \mathbb{N}$, there exists a constant $C := C(d) < \infty$ such that for each $u : \square_n \to \mathbb{R}$,

$$\frac{1}{|\square_n|} \sum_{x \in \square_n} \left| u(x) - (u)_{\square_n} \right|^2 \le C \sum_{e \subseteq \square_n} |\nabla u(e)|^2 + C 3^n \sum_{k=1}^{n} 3^k \left(\frac{1}{|\mathcal{Z}_{k,n}|} \sum_{y \in \mathcal{Z}_{k,n}} \left| \langle \nabla u \rangle_{z + \square_k} \right|^2 \right). \tag{5.2.14}$$

If one assume that $u \in h_0^1(\square_n)$, then one has

$$\frac{1}{|\square_n|} \sum_{x \in \square_n} |u(x)|^2 \le C \sum_{e \subseteq \square_n} |\nabla u(e)|^2 + C 3^n \sum_{k=1}^{n} 3^k \left(\frac{1}{|\mathcal{Z}_{k,n}|} \sum_{y \in \mathcal{Z}_{k,n}} \left| \langle \nabla u \rangle_{z + \square_k} \right|^2 \right). \tag{5.2.15}$$

Proof. The idea is to construct a smooth function $\widetilde{u}$ which is close to u by first extending it to be piecewise constant on the cubes $z + \left(-\frac{1}{2}, \frac{1}{2}\right)^d$, where $z \in \square_m$. We then make this function smooth by taking the convolution with a smooth approximation of the identity supported in the ball $B_{1/2}$. It follows that $\widetilde{u}(z) = u(z)$ for each $z \in \square_m$ and that the following estimate on holds: for each $z \in \square_m$,

$$\sup_{z + \left(-\frac{1}{2}, \frac{1}{2}\right)^d} |\nabla \widetilde{u}(x)| \le C \sum_{y \sim x} |u(y) - u(x)|.$$

One can then apply the Poincaré inequalities to the function $\widetilde{u}$ and then check that this is enough to obtain (5.2.12) and (5.2.13). The proof of (5.2.14) follows the same lines, a proof of this inequality is stated in Proposition 2.A.2 of Chapter 2. Note that the version stated here is a slight modification of the one which can be found there but can be deduced from it by applying the Cauchy-Schwarz inequality.

The version of the multiscale Poincaré inequality with 0 boundary condition given in (5.2.15) cannot be found in Chapter 2. Nevertheless the continuous version of this inequality is a consequence of [**18**, Proposition 1.7 and Lemma 1.8]. The transposition to the discrete setting is identical to the proof given in Proposition 2.A.2. □

5.3. Subadditive quantities and their basic properties

The goal of this section is to study the quantities ν and ν^* introduced in the previous sections. We prove a series of results about these quantities, which are reminiscent of the basic properties of ν and ν^* in stochastic homogenization, see [**18**, Lemma 1.1 and Lemma 2.2]. We first state these properties in the same proposition. Most of them are already known in the literature and in Remark 5.3.2, we provide references for these results. We then prove the remaining results.

PROPOSITION 5.3.1 (Properties of ν and $\overline{\nu}^*$). *There exists a constant $C := C(d,\lambda) < \infty$ such that the following properties hold*

- Subadditivity. *For each $n \in \mathbb{N}$ and each $p \in \mathbb{R}^d$,*

$$\nu\left(\square_{n+1}, p\right) \leq \nu\left(\square_n, p\right) + C\left(1+|p|^2\right)3^{-n}. \tag{5.3.1}$$

  Similarly, for each $q \in \mathbb{R}^d$,

$$\nu^*\left(\square_{n+1}, q\right) \leq \nu^*\left(\square_n, q\right) + C\left(1+|q|^2\right)3^{-n}. \tag{5.3.2}$$

- One-sided convex duality. *For each $p, q \in \mathbb{R}^d$ and each $n \in \mathbb{N}$,*

$$\nu(\square_n, p) + \nu^*(\square_n, q) \geq p \cdot q - C3^{-n}. \tag{5.3.3}$$

- Quadratic bounds. *there exists a small constant $c := c(d,\lambda) > 0$ such that for each $n \in \mathbb{N}$ and each $p \in \mathbb{R}^d$*

$$-C + c|p|^2 \leq \nu\left(\square_n, p\right) \leq C\left(1+|p|^2\right), \tag{5.3.4}$$

  and for each $q \in \mathbb{R}^d$,

$$-C + c|q|^2 \leq \nu^*\left(\square_n, q\right) \leq C\left(1+|q|^2\right), \tag{5.3.5}$$

- Uniform convexity of ν. *For each $p_0, p_1 \in \mathbb{R}^d$,*

$$\frac{1}{C}|p_0-p_1|^2 \leq \frac{1}{2}\nu(\square_n, p_0) + \frac{1}{2}\nu(\square_n, p_1) - \nu\left(\square_n, \frac{p_0+p_1}{2}\right) \leq C|p_0-p_1|^2. \tag{5.3.6}$$

- Convexity of ν^*. *The mapping $q \to \nu^*(\square_n, q)$ is convex.*
- L^2 bounds for the minimizers. *For each $p \in \mathbb{R}^d$*

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}|\nabla\phi_{n,p}(e)|^2\right] \leq C(1+|p|^2). \tag{5.3.7}$$

  Similarly, for each $q \in \mathbb{R}^d$, one has

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}|\nabla\psi_{n,q}(e)|^2\right] \leq C(1+|q|^2). \tag{5.3.8}$$

REMARK 5.3.2. The reader can find in the literature the proofs of the following results.

(1) The subadditivity of ν stated in (5.3.1) is essentially proved by Funaki and Spohn in [**70**, Lemma II.1].
(2) In Proposition 5.3.9, we prove a quantitative version of the subadditivity inequality (5.3.2) which is strictly stronger than the estimate (5.3.2).
(3) The quadratic bounds for ν stated in (5.3.4) are elementary and we refer to the monograph of Funaki [**68**, Section 5.2].
(4) The uniform convexity of the finite volume surface tension ν stated in (5.3.6) was established by Deuschel Giacomin and Ioffe in [**56**, Lemma 3.6].
(5) The convexity of the mapping $q \to \nu^*(\square_n, q)$ is a straightforward application of the Cauchy-Schwarz inequality.
(6) The L^2 bounds for the minimizers (5.3.7) and (5.3.7) can be obtained the following way. Using the explicit formula for ν and ν^* together with a computation similar to [**56**, Lemma 2.11], one derives the bounds

$$\mathbb{E}\left[\exp\left(\varepsilon \sum_{e\subseteq\square_n} |\nabla\phi_{n,q}|^2\right)\right] \leq \exp\left(C\left|\square_n\right|(1+|p|^2)\right)$$
$$\text{and } \mathbb{E}\left[\exp\left(\varepsilon \sum_{e\subseteq\square_n} |\nabla\psi_{n,q}|^2\right)\right] \leq \exp\left(C\left|\square_n\right|(1+|p|^2)\right).$$

for some $\varepsilon := \varepsilon(d,\lambda) > 0$ and $C := C(d,\lambda) < \infty$. By the Jensen inequality, this implies, and is in fact much stronger than, the desired estimates.

The two statements of Proposition 5.3.1 which remain to be proved are the one-sided convex duality (5.3.3) and the quadratic bound (5.3.8) for ν^*. They are established in Propositions 5.3.6 and 5.3.7 respectively.

Remark 5.3.3. From the subadditivity properties and the quadratic bounds, we obtain that for each $p, q \in \mathbb{R}^d$, the quantities $\nu(\square_n, p)$ and $\nu^*(\square_n, q)$ converge as $n \to \infty$. Moreover the limit satisfies the convexity and one-sided duality properties. This is summarized in the following proposition.

Proposition 5.3.4. *For each $p \in \mathbb{R}^d$ and $q \in \mathbb{R}^d$, the quantities $\nu(\square_n, p)$ and $\nu^*(\square_n, q)$ converge as $n \to \infty$. We denote by $\overline{\nu}(p)$ and $\overline{\nu}^*(q)$ their respective limits. Moreover, there exists a constant $C := C(d,\lambda) > \infty$ such that the following properties hold*

- One-sided convex duality. *For each $p, q \in \mathbb{R}^d$,*
$$\overline{\nu}(p) + \overline{\nu}^*(q) \geq p \cdot q.$$
- Quadratic bounds. *There exists a small constant $c := c(d,\lambda) > 0$ such that for each $p \in \mathbb{R}^d$,*
$$-C + c|p|^2 \leq \overline{\nu}(p) \leq C\left(1+|p|^2\right),$$
and for each $q \in \mathbb{R}^d$,
$$-C + c|q|^2 \leq \overline{\nu}^*(q) \leq C\left(1+|q|^2\right).$$
- Convexity and uniform convexity. *The mapping $q \to \overline{\nu}^*(q)$ is convex and for each $p_1, p_2 \in \mathbb{R}^d$,*
$$\frac{1}{C}|p_0 - p_1|^2 \leq \frac{1}{2}\overline{\nu}(p_0) + \frac{1}{2}\overline{\nu}(p_1) - \overline{\nu}\left(\frac{p_0+p_1}{2}\right) \leq C|p_0-p_1|^2.$$

Proof. The properties mentioned in the proposition are valid for the quantities $\nu(\square_n, p)$ and $\nu^*(\square_n, q)$. Sending n to infinity implies the results. For the case of the surface tension ν, this result was originally obtained by Funaki and Spohn [**70**, Proposition 1.1 (i)]. $\square$

REMARK 5.3.5. The previous proposition proves the estimate (5.1.12) of Theorem 5.1.1. Also by the previous proposition, to prove (5.1.14), there remains to show the upper bound

$$\overline{\nu}^* (q) \leq \sup_{p \in \mathbb{R}^d} -\overline{\nu} (p) + p \cdot q,$$

since the lower bound follows from the one-sided convex duality. This upper bound will be proved later in the article. The uniform convexity of $\overline{\nu}^*$ stated in (5.1.13) will then be deduced from (5.1.14) and (5.1.12).

5.3.1. Convex duality: lower bound. We now turn to the proof of the convex duality for ν and ν^*.

PROPOSITION 5.3.6 (Convex duality). *There exists a constant $C := C(d, \lambda) < \infty$ such that for each $p, q \in \mathbb{R}^d$ and each $n \in \mathbb{N}$,*

$$\nu(\square_n, p) + \nu^* (\square_n, q) \geq p \cdot q - C3^{-n}.$$

PROOF. We recall the notation $\partial \square_n$ and $\square_n^\circ$ to denote respectively the boundary and the interior of the cube $\square_n$. We decompose the space $\mathring{h}^1(\square_n)$ into three orthogonal subspaces

$$\mathring{h}^1(\square_n) = \mathring{h}^1 (\partial \square_n) \overset{\perp}{\oplus} \mathring{h}^1 (\square_n^\circ) \overset{\perp}{\oplus} \mathbb{R}v, \tag{5.3.9}$$

where we use a slight abuse of notation and denote by

$$\mathring{h}^1 (\partial \square_n) := \left\{ \psi \in \mathring{h}^1(\square_n) \ : \ \psi_{|\square_n^\circ} = 0 \right\} \text{ and } \mathring{h}^1 (\square_n^\circ) := \left\{ \psi \in \mathring{h}^1(\square_n) \ : \ \psi_{|\partial \square_n} = 0 \right\},$$

and where v is the function defined by

$$v = \frac{1}{| \square_n |} \left(\sqrt{\frac{|\partial \square_n|}{|\square_n^\circ|}} \mathbb{1}_{\square_n^\circ} - \sqrt{\frac{|\square_n^\circ|}{|\partial \square_n|}} \mathbb{1}_{\partial \square_n} \right),$$

so that

$$\sum_{x \in \square_n} v(x) = 0 \text{ and } \sum_{x \in \square_n} v(x)^2 = 1.$$

Since it will be important later in the proof, we note that, for each $n \in \mathbb{N}$,

$$\dim \mathring{h}^1 (\partial \square_n) = |\partial \square_n| - 1 \leq C3^{(d-1)n}. \tag{5.3.10}$$

We split the proof into 4 steps.

- In Step 1, we show that, for some $C := C(d, \lambda) < \infty$

$$\nu^* (\square_n, q) \geq \frac{1}{|\square_n|} \log \left(\int_{\mathring{h}^1 (\square_n^\circ) \oplus \mathbb{R}v} \exp \left(- \sum_{e \subseteq \square_n} V_e(\nabla \psi(e)) \right) d\psi \right) - C3^{-n}. \tag{5.3.11}$$

- In Step 2, we show that, for some $C := C(d) < \infty$

$$\nu (\square_n, 0) \geq -\frac{1}{|\square_n|} \log \left(\int_{\mathring{h}^1 (\square_n^\circ) \oplus \mathbb{R}v} \exp \left(- \sum_{e \subseteq \square_n} V_e(\nabla \psi(e)) \right) d\psi \right) - Cn3^{-dn}. \tag{5.3.12}$$

- In Step 3, we combine the results of Steps 1 and 2 to obtain that there exists $C := C(d, \lambda) < \infty$ such that,

$$\nu^* (\square_n, q) + \nu (\square_n, 0) \geq -C3^{-n}.$$

- In Step 4, we remove the assumption $p = 0$ and prove for each $p, q \in \mathbb{R}^d$,

$$\nu^* (\square_n, q) + \nu (\square_n, p) \geq p \cdot q - C3^{-n}.$$

Step 1. First, by (5.3.9), any function $\psi \in \mathring{h}^1(\square_n)$ can be uniquely decomposed according to

$$\psi = \psi_1 + \psi_2 + tv$$

with $\psi_1 \in \mathring{h}^1(\partial\square_n)$, $\psi_2 \in \mathring{h}^1(\square_n^{\circ})$ and $t \in \mathbb{R}$. Note that each function ψ_2 in $\mathring{h}^1(\square_n^{\circ})$ is equal to 0 on the boundary of $\square_n$, thus we have, for each $q \in \mathbb{R}^d$,

$$\sum_{e \subseteq \square_n} q \cdot \nabla\psi_2(e) = 0.$$

Since the function v is constant on $\partial\square_n$, we also have

$$\sum_{e \subseteq \square_n} q \cdot \nabla v(e) = 0.$$

To prove (5.3.11), it is sufficient to prove, for each $\psi_2 \in \mathring{h}^1(\square_n^{\circ})$ and each $t \in \mathbb{R}$,

$$\int_{\mathring{h}^1(\partial\square_n)} \exp\left(-\sum_{e \subseteq \square_n} \left(V_e\left(\nabla\left(\psi_1 + \psi_2 + tv\right)(e)\right) - q \cdot \nabla\psi_1(e)\right)\right) d\psi_1 \geq c^{3^{(d-1)n}} \exp\left(-\sum_{e \subseteq \square_n} V_e\left(\nabla\left(\psi_2 + tv\right)(e)\right)\right), \tag{5.3.13}$$

for some $c := c(d,\lambda) > 0$. Indeed, the estimate (5.3.11) is then obtained by integrating the previous inequality over $\mathring{h}^1(\square_n^{\circ}) \oplus \mathbb{R}v$. To prove (5.3.13), we use the following Taylor expansion

$$V_e(\nabla\left(\psi_1 + \psi_2 + tv\right)(e)) \leq V_e\left(\nabla\left(\psi_2 + tv\right)(e)\right) + V_e'\left(\nabla\left(\psi_2 + tv\right)(e)\right)\nabla\psi_1(e) + \frac{1}{2\lambda}\left|\nabla\psi_1(e)\right|^2.$$

This implies

$$\exp\left(-\sum_{e \subseteq \square_n} \left(V_e\left(\nabla\left(\psi_1 + \psi_2 + tv\right)(e)\right) - q \cdot \nabla\psi_1(e)\right)\right) \geq \exp\left(-\sum_{e \subseteq \square_n} \left(V_e\left(\nabla\left(\psi_2 + tv\right)(e)\right) + \left(V_e'\left(\nabla\left(\psi_2 + tv\right)(e)\right) - q(e)\right)\nabla\psi_1(e) + \frac{1}{2\lambda}\left|\nabla\psi_1(e)\right|^2\right)\right).$$

Using the crude inequality for an edge $e = (x,y) \subseteq \square_n$

$$\left|\nabla\psi_1(e)\right|^2 = \left|\psi_1(x) - \psi_1(y)\right|^2 \leq 2\left|\psi_1(x)\right|^2 + 2\left|\psi_1(y)\right|^2,$$

and summing over all the edges of $\square_n$ yields

$$\sum_{e \subseteq \square_n} \left|\nabla\psi_1(e)\right|^2 \leq 2d \sum_{x \in \partial\square_n} \psi_1(x)^2.$$

But note that for each $a \in \mathbb{R}$,

$$\int_{\mathbb{R}} \exp\left(ax - \frac{d}{\lambda}x^2\right) dx \geq \sqrt{\frac{\lambda\pi}{d}}.$$

With the previous estimate and (5.3.10), one proves

$$\int_{\mathring{h}^1(\partial\square_n)} \exp\left(-\sum_{e \subseteq \square_n} \left(V_e'\left(\nabla\left(\psi_2 + tv\right)(e)\right)\nabla\psi_1(e) - q \cdot \nabla\psi_1(e)\right) + \frac{d}{\lambda}\sum_{x \in \partial\square_n} \left|\psi_1(x)\right|^2\right) d\psi_1 \geq c^{3^{(d-1)n}},$$

for some $c := c(d,\lambda) > 0$. Combining the few previous displays gives

$$\int_{\mathring{h}^1(\partial\square_n)} \exp\left(-\sum_{e \subseteq \square_n} V_e\left(\nabla\left(\psi_1 + \psi_2 + tv\right)(e)\right) - q \cdot \nabla\psi_1(e)\right) d\psi_1 \geq c^{3^{(d-1)n}} \exp\left(-\sum_{e \subseteq \square_n} V_e\left(\nabla\left(\psi_2 + tv\right)(e)\right)\right).$$

This is precisely (5.3.13).

Step 2. We denote by

$$\widetilde{v} := \frac{1}{\sqrt{|\square_n^\circ|}} \mathbb{1}_{\square_n^\circ}$$

so that $\sum_{x \in \square_n} \widetilde{v}(x)^2 = 1$. Note that the two functions v and $\widetilde{v}$ are related by

$$\begin{aligned} v + \frac{1}{|\square_n|}\sqrt{\frac{|\square_n^\circ|}{|\partial \square_n|}} \mathbb{1}_{\square_n} &= \frac{1}{|\square_n|}\left(\sqrt{\frac{|\partial\square_n|}{|\square_n^\circ|}} + \sqrt{\frac{|\square_n^\circ|}{|\partial\square_n|}}\right)\mathbb{1}_{\square_n^\circ} \\ &= \frac{\sqrt{|\square_n^\circ|}}{|\square_n|}\left(\sqrt{\frac{|\partial\square_n|}{|\square_n^\circ|}} + \sqrt{\frac{|\square_n^\circ|}{|\partial\square_n|}}\right)\widetilde{v}. \end{aligned}$$

To shorten the notation, we denote by

$$\alpha_n := \frac{\sqrt{|\square_n^\circ|}}{|\square_n|}\left(\sqrt{\frac{|\partial\square_n|}{|\square_n^\circ|}} + \sqrt{\frac{|\square_n^\circ|}{|\partial\square_n|}}\right).$$

Combining the two previous displays we obtain, for each $e \subseteq \square_n$,

$$\nabla v(e) = \alpha_n \nabla \widetilde{v}(e). \tag{5.3.14}$$

Note also that, there exist $c := c(d) > 0$ and $C := C(d) < \infty$ such that

$$c 3^{-\frac{(d-1)n}{2}} \le \alpha_n \le C 3^{-\frac{(d-1)n}{2}}. \tag{5.3.15}$$

We then use the orthogonal decomposition $h_0^1(\square_n) = \mathring{h}^1(\square_n^\circ) \overset{\perp}{\oplus} \mathbb{R}\widetilde{v}$ and the decomposition of the Lebesgue measure explained in (5.1.4) to obtain

$$\int_{h_0^1(\square_n)} \exp\left(-\sum_{e\subseteq\square_n} V_e(\nabla\phi(e))\right) d\phi = \int_{\mathbb{R}}\int_{\mathring{h}^1(\square_n^\circ)} \exp\left(-\sum_{e\subseteq\square_n} V_e(\nabla\phi(e) + t\nabla\widetilde{v}(e))\right) d\phi dt.$$

Using (5.3.14), we obtain

$$\begin{aligned} &\int_{h_0^1(\square_n)} \exp\left(-\sum_{e\subseteq\square_n} V_e(\nabla\phi(e))\right) d\phi \\ &\quad = \int_{\mathbb{R}}\int_{\mathring{h}^1(\square_n^\circ)} \exp\left(-\sum_{e\subseteq\square_n} V_e\left(\nabla\phi(e) + \frac{t}{\alpha_n}\nabla v(e)\right)\right) d\phi dt \\ &\quad = \alpha_n \int_{\mathbb{R}}\int_{\mathring{h}^1(\square_n^\circ)} \exp\left(-\sum_{e\subseteq\square_n} V_e\left(\nabla\phi(e) + t\nabla v(e)\right)\right) d\phi dt \\ &\quad = \alpha_n \int_{\mathring{h}^1(\square_n^\circ)\oplus\mathbb{R}v} \exp\left(-\sum_{e\subseteq\square_n} V_e(\nabla\phi(e))\right) d\phi. \end{aligned}$$

Taking the logarithm, dividing by $|\square_n|$ and using (5.3.15), we obtain (5.3.12).

Step 3. Combining the main results of Steps 1 and 2 gives

$$\nu^*(\square_n, q) + \nu(\square_n, 0) \ge -C3^{-n} - Cn3^{-dn} \ge -C3^{-n}.$$

Step 4. Let $p \in \mathbb{R}^d$, define $\widetilde{V}_e := V_e(p(e) + \cdot)$ and denote by

$$\widetilde{\nu}(\square_n, 0) := -\frac{1}{|\square_n|}\log\int_{h_0^1(\square_n)} \exp\left(-\sum_{e\subseteq U} \widetilde{V}_e(\nabla\phi(e))\right) d\phi,$$

and, for every $q \in \mathbb{R}^d$,

$$\widetilde{\nu}^*(\square_n, q) := \frac{1}{|\square_n|}\log\int_{\mathring{h}^1(\square_n)} \exp\left(-\sum_{e\subseteq\square_n} \left(\widetilde{V}_e(\nabla\phi(e)) - q\cdot\nabla\phi(e)\right)\right) d\phi.$$

The functions $\widetilde{V}_e$ satisfy the same property of uniform convexity property (5.1.5) as V_e, thus one can apply the result of Steps 1, 2 and 3 with these functions. This gives, for every $q \in \mathbb{R}^d$,

$$\widetilde{\nu}(\square_n, 0) + \widetilde{\nu}^*(\square_n, q) \geq -C3^{-n}.$$

But note that

$$\begin{aligned}\widetilde{\nu}(\square_n, 0) &= -\frac{1}{|\square_n|}\log\int_{h_0^1(\square_n)}\exp\left(-\sum_{e\subseteq U}\widetilde{V}_e(\nabla\phi(e))\right)d\phi\\ &= -\frac{1}{|\square_n|}\log\int_{h_0^1(\square_n)}\exp\left(-\sum_{e\subseteq U}V_e(p\cdot e+\nabla\phi(e))\right)d\phi\\ &= \nu(\square_n, p).\end{aligned}$$

Note also that, by translation invariance of the Lebesgue measure on $\mathring{h}^1(\square_n)$, one can perform the change of variables $\phi := \phi - l_p$, where $l_p \in \mathring{h}^1(\square_n)$ is affine the function defined by $l_p(x) = p\cdot x$. This gives, for each $q \in \mathbb{R}^d$,

$$\begin{aligned}\widetilde{\nu}^*(\square_n, q) &= \frac{1}{|\square_n|}\log\int_{\mathring{h}^1(\square_n)}\exp\left(-\sum_{e\subseteq\square_n}\left(\widetilde{V}_e(\nabla\phi(e)) - q\cdot\nabla\phi(e)\right)\right)d\phi\\ &= \frac{1}{|\square_n|}\log\int_{\mathring{h}^1(\square_n)}\exp\left(-\sum_{e\subseteq\square_n}\left(V_e(p\cdot e+\nabla\phi(e)) - q\cdot\nabla\phi(e)\right)\right)d\phi\\ &= \frac{1}{|\square_n|}\log\int_{\mathring{h}^1(\square_n)}\exp\left(-\sum_{e\subseteq\square_n}\left(V_e(\nabla\psi(e)) + (q\cdot e)(p\cdot e) - q\cdot\nabla\psi(e)\right)\right)d\psi\\ &= \nu^*(\square_n, q) - p\cdot q.\end{aligned}$$

Combining the few previous displays yields, for each $p, q \in \mathbb{R}^d$,

$$\nu(\square_n, p) + \nu^*(\square_n, q) \geq p\cdot q - C3^{-n}.$$

The proof of Proposition 5.3.6 is complete. □

5.3.2. Quadratic bounds for ν^*. We now prove the quadratic bound property for ν^*.

PROPOSITION 5.3.7 (Quadratic bounds for ν^*). *There exist two constants $c := c(d,\lambda) > 0$ and $C := C(d,\lambda) < \infty$ such that, for each $q \in \mathbb{R}^d$ and each $n \in \mathbb{N}$,*

$$-C + c|q|^2 \leq \nu^*(\square_n, q) \leq C(1+|q|^2). \tag{5.3.16}$$

PROOF. We now prove (5.3.16). We start with the upper bound. By (5.3.2), we have for each integer $n \in \mathbb{N}$ and each $q \in \mathbb{R}^d$,

$$\nu^*(\square_n, q) \leq \nu^*(\square_1, q) + C(1+|q|^2).$$

A straightforward computation gives the bound

$$\nu^*(\square_1, q) \leq C(1+|q|^2).$$

Combining the two previous displays gives the upper bound of (5.3.16).

We then prove the lower bound, the idea is to use the convex duality proved in Proposition 5.3.6 combined with the upper bound estimate (5.3.4). By Proposition 5.3.6, for each $p, q \in \mathbb{R}^d$,

$$\nu(\square_n, p) + \nu^*(\square_n, q) \geq p\cdot q - C3^{-n}.$$

Using (5.3.4) and the crude bound $3^{-n} \leq 1$, the previous estimate becomes

$$\nu^*(\square_n, q) \geq p\cdot q - C(1+|p|^2).$$

Picking $p = q/2C$ gives

$$\nu^*(\square_n, q) \geq \frac{|q|^2}{4C} - C.$$

This is the desired lower bound. □

5.3.3. Two scales comparison. The goal of this section is obtain a quantitative version of the subadditivity for the ν and ν^* quantities stated in Proposition 5.3.1. More precisely one wishes to derive a second variation type of statement, following the techniques of the calculus of variations, for the surface tensions ν and ν^*: the objective is to construct, for $m < n$, a coupling between the random variables $\phi_{m,p}$ and $\phi_{n,p}$ (resp. $\psi_{m,p}$ and $\psi_{n,p}$) such that the L^2 norm of the gradient of their difference is controlled by $\nu(\square_m, p) - \nu(\square_n, p)$.

An interesting consequence of this estimate is that, since the sequence $\nu(\square_n, p)$ converges, the difference $\nu(\square_m, p) - \nu(\square_n, p)$ will be small when m and n are large: this implies that the gradient of the fields on two different scales are close in the L^2 norm.

For any pair of integers $m, n \in \mathbb{N}$ with $m < n$, the triadic cube $\square_n$ can be split into $3^{(n-m)d}$ cubes of the form $z + \square_m$, with $z \in \mathcal{Z}_{m,n}$. Denote by $(\phi_z)_{z \in \mathcal{Z}_{m,n}}$ a family of random variables such that

- For each $z \in \mathcal{Z}_{m,n}$, ϕ_z takes values in $h_0^1(z + \square_m)$ and the law of $\phi_z(\cdot - z)$ is $\mathbb{P}_{m,p}$
- The random variables ϕ_z are independent.

We can then, for each $z \in \mathcal{Z}_{m,n}$, see ϕ_z as a random variable taking value in $h_0^1(\square_n)$, by extending it to be 0 on $\square_n \setminus (z + \square_m)$. We also denote by $\phi := \sum_{z \in \mathcal{Z}_{m,n}} \phi_z$.

Proposition 5.3.8 (Two scale comparison for ν). *Given $m, n \in \mathbb{N}$ with $m < n$ and $p \in \mathbb{R}^d$, consider the random variable ϕ defined in the previous paragraph and taking values in $h_0^1(\square_n)$. There exists a coupling between ϕ and $\phi_{n,p}$ and a constant $C := C(d, \lambda) < \infty$ such that,*

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e \subseteq \square_n} |\nabla\phi(e) - \nabla\phi_{n,p}(e)|^2\right] \le C\left(\nu(\square_m, q) - \nu(\square_n, q)\right) + C3^{-\frac{m}{2}}(1 + |p|^2). \tag{5.3.17}$$

Proof. We first introduce the set of vertices $\partial_{int}\square_{m,n} \subseteq \square_m$ which are on the boundary of one of the $z + \square_m$ but not on the boundary of $\square_n$,

$$\partial_{int}\square_{m,n} := \left(\bigcup_{z \in \mathcal{Z}_{m,n}} \partial\,(z + \square_m)\right) \setminus \partial\,\square_n\,. \tag{5.3.18}$$

Note that the cardinality of this set satisfies the upper bound estimate

$$|\partial_{int}\square_{m,n}| \le C3^{dn-m}$$

An idea to obtain (5.3.17) relies on the second variation formula from calculus of variations applied to the functional $\mathcal{F}_{n,p}$: by considering the optimal coupling between the laws of ϕ and $\phi_{n,p}$, one can use the displacement convexity of the entropy and the uniform convexity of V_e to obtain uniform convexity for the functional $\mathcal{F}_{n,p}$ and then apply the standard proof of the second variation formula for uniformly convex functionals.

Unfortunately a technical problem has to be treated along the way: with the current definition of the function ϕ, one has

$$\forall x \in \partial_{int}\square_{m,n},\ \phi(x) = 0.$$

A consequence of the previous identity is that the law of ϕ is not absolutely continuous with respect to the Lebesgue measure on $h_0^1(\square_n)$ and thus its entropy is infinite. Nevertheless, this is the only obstruction and to remedy this the idea is to add a few extra random variables which are small and whose only purpose is to make the entropy of ϕ finite. We consequently introduce a random variable X taking values in $h_0^1(\square_n)$ and satisfying

- for each $x \in \partial_{int}\square_{m,n}$, the law of $X(x)$ is uniform on $[0,1]$ and for each $x \in \square_{n+1} \setminus \partial_{int}\square_{m,n}$, $X(x) = 0$,
- the $\mathbb{R}$-valued random variables $(X(x))_{x \in \partial_{int}\square_{m,n}}$ are independent,
- the random variables X and ϕ are independent.

We then consider the random variable $\phi' := \phi + X$. It is a random variable taking value in $h_0^1(\square_n)$. Moreover, by construction, we see that this random variable is absolutely continuous with respect to the Lebesgue measure on $h_0^1(\square_n)$. We denote by $\mathbb{P}_{\phi'}$ its law. The idea to keep in mind is that the random variable ϕ' is a small perturbation of the random variable ϕ and thanks to that it will be possible to obtain estimates on ϕ from estimates on ϕ'. This is carried out in Steps 3 and 4 of the proof.

We then split the proof into 6 steps.

- In Step 1, we compute the entropy of $\mathbb{P}_{\phi'}$ and prove that

$$H\left(\mathbb{P}_{\phi'}\right) = 3^{(n-m)d} H\left(\mathbb{P}_{m,p}\right), \tag{5.3.19}$$

  where the entropy on the left-hand side is computed with respect to the Lebesgue measure on $h_0^1(\square_n)$ and the entropy on the right-hand side is computed with respect to the Lebesgue measure on $h_0^1(\square_m)$.
- In Step 2, we consider the optimal coupling between ϕ' and $\phi_{n,p}$ and prove that, under this coupling,

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}\left|\nabla\phi'(e)-\nabla\phi_{n,p}(e)\right|^2\right] \\ \leq C\left(\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e\left(p(e)+\nabla\phi'(e)\right)\right]+\frac{1}{|\square_n|}H\left(\mathbb{P}_{\phi'}\right)-\nu(\square_n,p)\right).\end{aligned} \tag{5.3.20}$$

- In Step 3, we estimate the term on the right-hand side of (5.3.20) and prove

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e\left(p(e)+\nabla\phi'(e)\right)\right] \leq \frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e\left(p(e)+\nabla\phi(e)\right)\right]+C3^{-\frac{m}{2}}(1+|p|). \tag{5.3.21}$$

- In Step 4, we combine the main results of Steps 1 and 3 to obtain

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e\left(p(e)+\nabla\phi'(e)\right)\right]+\frac{1}{|\square_n|}H\left(\mathbb{P}_{\phi'}\right) \leq \nu\left(\square_n,p\right)+C3^{-\frac{m}{2}}(1+|p|^2).$$

- In Step 5, we estimate the term on the left-hand side of (5.3.20). We replace the ϕ' by ϕ and show that this operation can be performed up to a small error term:

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}\left|\nabla\phi(e)-\nabla\phi_{n,p}(e)\right|^2\right] \leq C\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}\left|\nabla\phi'(e)-\nabla\phi_{n,p}(e)\right|^2\right]+C3^{-\frac{m}{2}}(1+|p|). \tag{5.3.22}$$

  Note that to compute the expectation on the left-hand side, we used the coupling between ϕ and $\phi_{n,p}$ which is induced by the coupling between ϕ' and $\phi_{n,p}$.
- In Step 6, the conclusion, we combine the main results of Steps 2, 3, 4 and 5 to obtain (5.3.17).

Step 1. The idea to obtain (5.3.19) is to use Proposition 5.2.4 pertaining to the entropy of a pair of random variables. To this end, note that one has the orthogonal decomposition of $h_0^1(\square_n)$

$$h_0^1(\square_n) = \bigoplus_{z\in\mathcal{Z}_{m,n}} h_0^1(z+\square_m) \overset{\perp}{\oplus} \mathbb{R}^{\partial_{int}\square_{m,n}},$$

where $\mathbb{R}^{\partial_{int}\square_{m,n}}$ stands for the set of functions from $\partial_{int}\square_{m,n}$ to $\mathbb{R}$.

Using the previous remark, one can apply Proposition 5.2.6 with $Y := \phi$, $Z := X$ and consequently $Y + Z = \phi'$. This leads to

$$H\left(\mathbb{P}_{\phi'}\right) = H\left(\mathbb{P}_{\phi}\right) + H\left(\mathbb{P}_X\right),$$

where the entropy of ϕ' (resp. ϕ and X) is computed with respect to the Lebesgue measure on $h_0^1(\square_n)$ (resp. $\bigoplus_{z\partial_{int}\square_{m,n}} h_0^1(z+\square_n)$ and $\mathbb{R}^{\partial_{int}\square_{m,n}}$).

On the one hand, since the random variables $(X(x))_{x\in\partial_{int}\square_{m,n}}$ are independent and of law uniform on $[0,1]$, one has

$$H\left(\mathbb{P}_X\right) = 0.$$

On the other hand, one also has the equality $\phi' = \sum_{z\in\partial_{int}\square_{m,n}} \phi_z$. Using the independence of the family $(\phi_z)_{z\in\mathcal{Z}_{m,n}}$, that for each $z \in \mathcal{Z}_{m,n}$, $\phi_z(\cdot - z)$ has law $\mathbb{P}_{m,p}$ and Proposition 5.2.6, with $3^{(n-m)d}$ random variables instead of two, one deduces

$$H\left(\mathbb{P}_{\phi}\right) = \sum_{z\in\mathcal{Z}_{m,n}} H\left(\mathbb{P}_{\phi_z}\right) = 3^{(n-m)d} H\left(\mathbb{P}_{m,p}\right).$$

Combining the few previous displays yields (5.3.19) and completes the proof of Step 1.

Step 2. Consider the optimal coupling with respect to the L^2 scalar product on $h_0^1(\square_n)$ between ϕ' and $\phi_{n,p}$ and denote by $\mathbb{P}_{\frac{\phi'+\phi_{n,p}}{2}}$ the law of the random variable $\frac{\phi'+\phi_{n,p}}{2}$ under this coupling. Using Proposition 5.2.10 about the displacement convexity of the entropy, one has

$$H\left(\mathbb{P}_{\frac{\phi'+\phi_{n,p}}{2}}\right) \leq \frac{1}{2} H\left(\mathbb{P}_{\phi'}\right) + \frac{1}{2} H\left(\mathbb{P}_{n,p}\right).$$

Also by the uniform convexity of V_e, one has

$$\begin{aligned} \lambda \mathbb{E}\left[\sum_{e\subseteq\square_n} \left|\nabla\phi'(e) - \nabla\phi_{n,p}(e)\right|^2\right] &\leq \mathbb{E}\left[\sum_{e\subseteq\square_n} V\left(p(e) + \nabla\phi'(e)\right)\right] + \mathbb{E}\left[\sum_{e\subseteq\square_n} V\left(p(e) + \nabla\phi_{n,p}(e)\right)\right] \\ &\quad - 2\mathbb{E}\left[\sum_{e\subseteq\square_n} V\left(p(e) + \frac{\nabla\phi_{n,p}(e) + \nabla\phi'(e)}{2}\right)\right]. \end{aligned}$$

By definition of ν, the following equality holds

$$\mathbb{E}\left[\sum_{e\subseteq\square_n} V\left(p\cdot e + \nabla\phi_{n,p}(e)\right)\right] + H\left(\mathbb{P}_{n,p}\right) = \nu(\square_n, p).$$

Using the variational formulation for ν given in Proposition 5.2.8, one derives the inequality

$$\nu(\square_n, p) \leq \mathbb{E}\left[\sum_{e\subseteq\square_n} V\left(p(e) + \frac{\nabla\phi_{n,p}(e) + \nabla\phi'(e)}{2}\right)\right] + H\left(\mathbb{P}_{\frac{\phi'+\phi_{n,p}}{2}}\right).$$

Combining the few previous displays provides the inequality

$$\begin{aligned} &\mathbb{E}\left[\sum_{e\subseteq\square_n} \left|\nabla\phi'(e) - \nabla\phi_{n,p}(e)\right|^2\right] \\ &\qquad \leq C\left(\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(p(e) + \nabla\phi'(e)\right)\right] + \frac{1}{|\square_n|} H\left(\mathbb{P}_{\phi'}\right) - \nu(\square_n, p)\right). \end{aligned}$$

This is precisely (5.3.20) and the proof of Step 2 is complete.

Step 3. The main goal of this step is to prove the following estimate

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(p(e) + \nabla\phi'(e)\right)\right] \leq \frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(p(e) + \nabla\phi(e)\right)\right] + C3^{-\frac{n}{2}}(1 + |p|).$$

To achieve this, we recall that ϕ' is defined from ϕ according to the formula

$$\phi' := \phi + X,$$

and that the random variable X is supported in the vertices of $\partial_{int}\square_{m,n}$. We denote by $B^X_{m,n}$ the set of edges of $\square_n$ where ∇X is supported, i.e.

$$B^X_{m,n} := \left\{(x,y) \in \mathbf{B}_d(\square_n) \,:\, x \in \partial_{int}\,\square_{m,n} \text{ or } y \in \partial_{int}\square_{m,n}\right\}.$$

One can estimate the cardinality of $B^X_{m,n}$ according to

$$\left|B^X_{m,n}\right| \leq C3^{dn-m}. \tag{5.3.23}$$

We then split the sum and use that ∇X is supported on $B^X_{m,n}$ to get

$$\sum_{e\subseteq\square_n} V_e\left(p(e)+\nabla\phi'(e)\right) = \sum_{e\subseteq\square_n\setminus B^X_{m,n}} V_e\left(p(e)+\nabla\phi'(e)\right) + \sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi'(e)\right)$$
$$= \sum_{e\subseteq\square_n\setminus B^X_{m,n}} V_e\left(p(e)+\nabla\phi(e)\right) + \sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi'(e)\right). \tag{5.3.24}$$

The second term on the right-hand side can be estimated by using the uniform convexity of V_e and a Taylor expansion,

$$\sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi'(e)\right) = \sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi(e)+\nabla X(e)\right)$$
$$\leq \sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi(e)\right) + V_e'\left(p(e)+\nabla\phi(e)\right)\nabla X(e) + \frac{1}{\lambda}|\nabla X(e)|^2.$$

By definition of X, its gradient is bounded by 1 and by the assumption made on the elastic potential V_e, one has

$$\forall x\in\mathbb{R}, \quad |V_e'(x)| \leq \frac{1}{\lambda}|x|.$$

Using these ideas, the previous estimate can be rewritten

$$\sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi'(e)\right) \leq \sum_{e\in B^X_{m,n}} \left(V_e\left(p(e)+\nabla\phi(e)\right) + |\nabla\phi(e)|\right) + C\left|B^X_{m,n}\right|(1+|p|).$$

Using the estimate on the cardinality of $B^X_{m,n}$ given in (5.3.23) and the Cauchy-Schwarz inequality, one has

$$\sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi'(e)\right) \leq \sum_{e\in B^X_{n}} V_e\left(p(e)+\nabla\phi(e)\right) + 3^{\frac{dn-m}{2}}\left(\sum_{e\in B^X_{m,n}} |\nabla\phi(e)|^2\right)^{\frac12} + C3^{dn-m}(1+|p|).$$

Taking the expectation and dividing by $|\square_n|$ gives

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi'(e)\right)\right] \leq \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi(e)\right)\right]$$
$$+ 3^{-\frac{dn+m}{2}}\mathbb{E}\left[\left(\sum_{e\in B^X_{m,n}} |\nabla\phi(e)|^2\right)^{\frac12}\right] + C3^{-m}(1+|p|).$$

By the Cauchy-Schwarz inequality, we further obtain

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi'(e)\right)\right] \leq \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\in B^X_{m,n}} V_e\left(p(e)+\nabla\phi(e)\right)\right]$$
$$+ 3^{-\frac{dn+m}{2}}\mathbb{E}\left[\sum_{e\in B^X_{m,n}} |\nabla\phi(e)|^2\right]^{\frac12} + C3^{-m}(1+|p|).$$

Combining the previous display with (5.3.24) gives

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n} V_e\left(p(e)+\nabla\phi'(e)\right)\right] \leq \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n} V_e\left(p(e)+\nabla\phi(e)\right)\right]$$
$$+ 3^{-\frac{dn+m}{2}}\mathbb{E}\left[\sum_{e\in B^X_{m,n}} |\nabla\phi(e)|^2\right]^{\frac12} + C3^{-m}(1+|p|).$$

The proof of (5.3.21) is almost complete: note that by the bound (5.3.7) stated in Propostion 5.3.1 and the construction of the random variable ϕ, one has the energy estimate

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}|\nabla\phi(e)|^2\right]\leq C(1+|p|^2). \tag{5.3.25}$$

This immediately implies the desired result since

$$\mathbb{E}\left[\sum_{e\in B^X_{m,n}}|\nabla\phi(e)|^2\right]^{\frac{1}{2}}\leq\mathbb{E}\left[\sum_{e\subseteq\square_n}|\nabla\phi(e)|^2\right]^{\frac{1}{2}}.$$

Step 4. With the same proof as in Step 2, we can decompose

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e\left(p(e)+\nabla\phi(e)\right)\right]$$
$$=\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}V_e\left(p(e)+\nabla\phi_z(e)\right)+\sum_{e\in B_{m,n}}V_e\left(p(e)\right)\right].$$

Using the estimate $V(x)\leq\frac{1}{\lambda}|x|^2$ and the bound on the cardinality of $B_{m,n}$

$$|B_{m,n}|\leq C3^{dn-m},$$

one obtains

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e\left(p(e)+\nabla\phi(e)\right)\right]$$
$$\leq\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq(z+\square_m)}V_e\left(p(e)+\nabla\phi_z(e)\right)\right]+C3^{-m}(1+|p|^2).$$

Combining this inequality with the main result (5.3.21) of Step 3, we obtain

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e\left(p(e)+\nabla\phi'(e)\right)\right]\leq\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\sum_{e\subseteq(z+\square_m)}V_e\left(p(e)+\nabla\phi_z(e)\right)\right]$$
$$+C3^{-m}(1+|p|^2)+C3^{-\frac{m}{2}}(1+|p|).$$

The error term can be simplified according to

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e\left(p(e)+\nabla\phi'(e)\right)\right]$$
$$\leq\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\sum_{e\subseteq(z+\square_m)}V_e\left(p\cdot e+\nabla\phi_z(e)\right)\right]+C3^{-\frac{m}{2}}(1+|p|^2).$$

Adding the entropy of $\mathbb{P}_{\phi'}$ and using the equality proved in Step 1, we obtain

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e\left(p(e)+\nabla\phi'(e)\right)\right]+\frac{1}{|\square_n|}H\left(\mathbb{P}_{\phi'}\right)$$
$$\leq 3^{-d(n-m)}\sum_{z\in\mathcal{Z}_{m,n}}\left(\mathbb{E}\left[\frac{1}{|\square_m|}\sum_{e\subseteq(z+\square_m)}V_e\left(p(e)+\nabla\phi_z(e)\right)\right]+\frac{1}{|\square_m|}H\left(\mathbb{P}_{m,p}\right)\right)$$
$$+C3^{-\frac{m}{2}}(1+|p|^2).$$

Since the law of ϕ_z in $z+\square_m$ is $\mathbb{P}_{m,p}$, we have, for each $z\in\mathcal{Z}_{m,n}$,

$$\mathbb{E}\left[\frac{1}{|\square_m|}\sum_{e\subseteq z+\square_m}V_e\left(p(e)+\nabla\phi_z(e)\right)\right]+\frac{1}{|\square_m|}H\left(\mathbb{P}_{m,p}\right)=\nu(\square_m,p).$$

Combining the two previous displays shows

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(p(e)+\nabla\phi'(e)\right)\right]+\frac{1}{|\square_n|}H\left(\mathbb{P}_{\phi'}\right)\le\nu(\square_m,p)+C3^{-\frac{m}{2}}(1+|p|^2)$$

and the proof of Step 4 is complete.

Step 5. We proceed as in Step 3 and use that the gradient of X is supported in $B^X_{m,n}$ to decompose the sum

$$\sum_{e\subseteq\square_n}\left|\nabla\phi'(e)-\nabla\phi_{n,p}(e)\right|^2=\sum_{e\in B^X_{m,n}}|\nabla\phi(e)+\nabla X(e)-\nabla\phi_{n,p}(e)|^2 + \sum_{e\subseteq\square_n\setminus B^X_{m,n}}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2. \tag{5.3.26}$$

We then expand the first term on the right-hand side

$$\begin{aligned}\sum_{e\in B^X_{m,n}}&|\nabla\phi(e)+\nabla X(e)-\nabla\phi_{n,p}(e)|^2\\ &=\sum_{e\in B^X_{m,n}}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2+2\sum_{e\in B^X_{m,n}}\left(\nabla\phi(e)-\nabla\phi_{n,p}(e)\right)\nabla X(e)+\sum_{e\in B^X_{m,n}}|\nabla X(e)|^2.\end{aligned}$$

Using that the gradient of X is bounded by 1, the Cauchy-Schwarz inequality and the upper bound on the cardinality of $B^X_{m,n}$, one further obtains

$$\begin{aligned}\sum_{e\in B^X_{m,n}}&|\nabla\phi(e)+\nabla X(e)-\nabla\phi_{n,p}(e)|^2\\ &\ge\sum_{e\in B^X_{m,n}}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2-C3^{\frac{dn-m}{2}}\left(\sum_{e\in B^X_{m,n}}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2\right)^{\frac12}-C3^{dn-m}.\end{aligned}$$

Dividing by $|\square_n|$ on both sides of the previous inequality and taking the expectation gives

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\in B^X_{m,n}}|\nabla\phi(e)+\nabla X(e)-\nabla\phi_{n,p}(e)|^2\right]&\ge\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\in B^X_{m,n}}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2\right]\\ &\quad-C3^{-\frac{dn+m}{2}}\mathbb{E}\left[\left(\sum_{e\in B^X_{m,n}}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2\right)^{\frac12}\right]-C3^{-m}.\end{aligned}$$

By (5.3.26) and the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}\mathbb{E}&\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}\left|\nabla\phi'(e)-\nabla\phi_{n,p}(e)\right|^2\right]\\ &\ge\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2\right]-C3^{-\frac{dn+m}{2}}\mathbb{E}\left[\sum_{e\in B^X_{m,n}}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2\right]^{\frac12}-C3^{-m}.\end{aligned}$$

There remains to estimate the second term on the right-hand side of the previous equation. To do so, we recall the energy estimate already introduced in Step 4,

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}|\nabla\phi(e)|^2\right]\le C(1+|p|^2),$$

and by (5.3.7) of Proposition 5.3.1, one has

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}|\nabla\phi_{n,p}(e)|^2\right]\leq C(1+|p|^2).$$

Combining the three previous displays shows

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}\left|\nabla\phi'(e)-\nabla\phi_{n,p}(e)\right|^2\right]\geq\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2\right]-C3^{-\frac{m}{2}}(1+|p|),$$

and the proof of Step 5 is complete.

Step 6. The conclusion. First, combining the main results of Steps 2 and 4 gives,

$$\mathbb{E}\left[\sum_{e\subseteq\square_n}\left|\nabla\phi'(e)-\nabla\phi_{n,p}(e)\right|^2\right]\leq C\left(\nu(\square_m,p)-\nu(\square_n,p)\right)+C3^{-\frac{m}{2}}(1+|p|^2).$$

Then by the main result of Step 5, we obtain

$$\mathbb{E}\left[\sum_{e\subseteq\square_n}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2\right]\leq C\left(\nu(\square_m,p)-\nu(\square_n,p)\right)+C3^{-\frac{m}{2}}(1+|p|^2).$$

This is exactly (5.3.17) and the proof of Proposition 5.3.8 is complete. □

We now want to prove a version of the two scale comparison for ν^*. Similarly to what was performed in Proposition 5.3.8, we fix two integers $m,n\in\mathbb{N}$ such that $m<n$ and define a family of random variables ψ_z for $z\in\mathcal{Z}_{m,n}$ according to

- For each $z\in\mathcal{Z}_{m,n}$, ψ_z takes values in $\mathring{h}^1(z+\square_n)$, is equal to 0 in $\square_n\setminus(z+\square_m)$ and the law of $\psi_z(\cdot-z)$ is $\mathbb{P}^*_{m,q}$.
- The random variables ψ_z, for $z\in\mathcal{Z}_{m,n}$, are independent.

We also denote by $\psi':=\sum_{z\in\mathcal{Z}_{m,n}}\psi_z$ and by $\mathbb{P}_{\psi'}$ its law, it is a probability measure on $\mathring{h}^1(\square_n)$.

Proposition 5.3.9 (Two scales comparison for ν^*). *Given $n,m\in\mathbb{N}$ satisfying $m<n$ and $q\in\mathbb{R}^d$, consider the random variables ψ_z and ψ' defined in the previous paragraph. There exist a coupling between ψ' and $\psi_{n,q}$ and a constant $C:=C(d,\lambda)<\infty$ such that,*

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}|\nabla\psi_z(e)-\nabla\psi_{n,q}(e)|^2\right] \leq C\left(\nu^*(\square_m,q)-\nu^*(\square_n,q)\right)+C(1+|q|^2)3^{-m}. \tag{5.3.27}$$

Proof. The first idea of the proof is to consider the following decomposition of the space $\mathring{h}^1(\square_n)$:

$$\mathring{h}^1(\square_n)=\bigoplus_{z\in\mathcal{Z}_{m,n}}\mathring{h}^1(z+\square_n)\overset{\perp}{\oplus}H, \tag{5.3.28}$$

where we denote by

$$\mathring{h}^1(z+\square_m):=\left\{\psi\in\mathring{h}^1(\square_n)\ :\ \psi_{|\square_n\setminus(z+\square_m)}=0\right\},$$

with a slight abuse of notation: we extend the functions of $\mathring{h}^1(z+\square_m)$ by 0 outside the cube $(z+\square_m)$.

The remaining space H is the space of functions of $\mathring{h}^1(\square_n)$ which are constant on the subcubes $(z+\square_n)_{z\in\mathcal{Z}_{m,n}}$. It is a space of dimension $3^{d(n-m)}-1$ and each function $h\in H$ can be written in the following form

$$h=\sum_{z\in\mathcal{Z}_{m,n}}\lambda_z\mathbb{1}_{z+\square_m},$$

for some real constants $(\lambda_z)_{z\in\mathcal{Z}_{m,n}}$ satisfying $\sum_{z\in\mathcal{Z}_{m,n}}\lambda_z=0$.

For $z \in \mathcal{Z}_{m,n}$, we denote by $\psi_{n,q}^z$ orthogonal projections of the random variable $\psi_{n,q}$ on the space $\mathring{h}^1(z + \square_m)$. This defines a random variable taking values in $\mathring{h}^1(z + \square_m)$.

We also let h the orthogonal projection of the random variable $\psi_{n,q}$ on the space H. Finally, we introduce the notation

$$\psi'_{n,q} := \sum_{z \in \mathcal{Z}_{m,n}} \psi_{n,q}^z.$$

This is a random variable taking values in $\bigoplus_{z \in \mathcal{Z}_{m,n}} \mathring{h}^1(z + \square_m) \subseteq \mathring{h}^1(\square_n)$. Its law is a measure defined on the vector space $\bigoplus_{z \in \mathcal{Z}_{m,n}} \mathring{h}^1(z + \square_m)$ and is denoted by $\mathbb{P}_{\psi'_{n+1,q}}$.

As in the proof of the previous proposition, we introduce $B_{m,n}$ the set of edges connecting two subcubes of the form $z + \square_m$, i.e.

$$B_{m,n} := \left\{(x,y) \, : \, \exists z, z' \in \mathcal{Z}_{m,n}, \, z \neq z' \text{ such that } x \in z + \square_m \text{ and } y \in z' + \square_m \right\}, \tag{5.3.29}$$

so as to have the decomposition of the sum

$$\sum_{e \subseteq \square_n} = \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq z + \square_m} + \sum_{e \in B_{m,n}}. \tag{5.3.30}$$

Note also that for every $h \in H$, the gradient ∇h is supported on the edges of $B_{m,n}$ and

$$\text{for each } z, z' \in \mathcal{Z}_{m,n} \text{ with } z \neq z' \text{ and for each } e \subseteq \left(z' + \square_m\right), \nabla \psi_{n,q}^z(e) = 0. \tag{5.3.31}$$

This implies, for each $z \in \mathcal{Z}_{m,n}$ and each $e \subseteq z + \square_m$,

$$\nabla \psi'_{n,q}(e) = \nabla \psi_{n,q}^z(e).$$

The same result is valid for ψ': for each $z \in \mathcal{Z}_{m,n}$ and each $e \subseteq z + \square_m$,

$$\nabla \psi'(e) = \nabla \psi_z(e).$$

We now split the proof into 5 Steps. In Steps 1 to 4, we assume $q = 0$. We then remove this additional assumption in Step 5.

- In Step 1, we show that the law of ψ' is the minimizer of the variational problem

$$\inf_{\mathbb{P}} \mathbb{E}\left[\sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e(\nabla \psi(e))\right] + H(\mathbb{P}),$$

  where the infimum is chosen over all the probability measures on $\bigoplus_{z \in \mathcal{Z}_{m,n}} \mathring{h}^1(z + \square_m)$ and the entropy is computed with respect to the Lebesgue measure on this space.
- In Step 2, we consider the optimal coupling between ψ' and $\psi'_{n,0}$ and derive the following inequality,

$$\mathbb{E}\left[\frac{1}{|\square_n|} \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} \left|\nabla \psi_z(e) - \nabla \psi_{n,0}^z(e)\right|^2\right] \\ \leq C\left(\nu^*(\square_n, 0) + \mathbb{E}\left[\frac{1}{|\square_n|} \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq (z + \square_n)} V_e\left(\nabla \psi_{n,0}^z(e)\right)\right] + \frac{1}{|\square_n|} H\left(\mathbb{P}_{\psi'_{n,0}}\right)\right).$$

- In Step 3, we prove the following estimate pertaining to the random variables $\psi_{n,0}^z$,

$$\sum_{z \in \mathcal{Z}_{m,n}} \mathbb{E}\left[\frac{1}{|\square_n|} \sum_{e \subseteq (z + \square_m)} V\left(\nabla \psi_{n,0}^z(e)\right)\right] + \frac{1}{|\square_n|} H\left(\mathbb{P}_{\psi_{n,0}^z}\right) \leq -\nu^*(\square_m, 0) + Cm3^{-dm}. \tag{5.3.32}$$

- In Step 4, we complete the proof and show that there exists a coupling between $\psi_{n,0}$ and ψ' such that

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}\left|\nabla\psi'(e)-\nabla\psi_{n,0}(e)\right|^2\right]\leq C(\nu^*(\square_m,0)-\nu^*(\square_n,0))+Cm3^{-dm},$$

- In Step 5, we remove the assumption $q=0$ and prove the more general result: for each $q\in\mathbb{R}^d$, there exists a coupling between the random variables ψ' and $\psi_{n,q}$ such that

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}|\nabla\psi_z(e)-\nabla\psi_{n,q}(e)|^2\right]\leq C(\nu^*(\square_m,q)-\nu^*(\square_n,q))+C(1+|q|^2)3^{-m}.$$

Step 1. By Proposition 5.2.8, one has

$$\inf_{\mathbb{P}}\left(\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}V_e(\nabla\psi(e))\right]+H(\mathbb{P})\right)$$
$$=-\log\int_{\oplus_{z\in\mathcal{Z}_{m,n}}\mathring{h}^1(z+\square_m)}\exp\left(-\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_n)}V_e(\nabla\psi(e))\right)d\psi, \tag{5.3.33}$$

where the infimum is considered over all probability measure on $\bigoplus_{z\in\mathcal{Z}_{m,n}}\mathring{h}^1(z+\square_m)$.

But on the one hand, one has the equality

$$\int_{\oplus_{z\in\mathcal{Z}_{m,n}}\mathring{h}^1(z+\square_m)}\exp\left(-\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}V_e(\nabla\psi(e))\right)d\psi$$
$$=\left(\int_{\mathring{h}^1(\square_m)}\exp\left(-\sum_{e\subseteq\square_m}V_e(\nabla\psi(e))\right)d\psi\right)^{3^{d(n-m)}}. \tag{5.3.34}$$

On the other hand, since by assumption the random variables ψ_z, for $z\in\mathcal{Z}_{m,n}$ are independent, one has,

$$H\left(\mathbb{P}_{\psi'}\right)=\sum_{z\in\mathcal{Z}_{m,n}}H\left(\mathbb{P}_{\psi_z}\right)=3^{d(n-m)}H\left(\mathbb{P}^*_{m,0}\right).$$

As a remark, note that the entropy of $\mathbb{P}_{\psi'}$ is computed with respect to the Lebesgue measure on $\oplus_{z\in\mathcal{Z}_{m,n}}\mathring{h}^1(z+\square_m)$, while the entropies of $\mathbb{P}_{\psi_z}$ and of $\mathbb{P}^*_{m,0}$ are computed with respect to the Lebesgue measures on $\mathring{h}^1(z+\square_m)$ and on $\mathring{h}^1(\square_m)$ respectively. The energy part of the random variable ψ' can be computed explicitly and one derives

$$\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}V_e(\nabla\psi'(e))\right]=\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\sum_{e\subseteq(z+\square_m)}V_e(\nabla\psi_z(e))\right]$$
$$=3^{d(n-m)}\mathbb{E}\left[\sum_{e\subseteq\square_n}V_e(\nabla\psi_{n,0}(e))\right].$$

Consequently

$$\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}V_e(\nabla\psi'(e))\right]+H\left(\mathbb{P}_{\psi'}\right)=3^{d(n-m)}\left(\mathbb{E}\left[\sum_{e\subseteq\square_m}V_e(\nabla\psi_{m,0}(e))\right]+H\left(\mathbb{P}^*_{m,0}\right)\right).$$

By definition of law $\mathbb{P}^*_{m,0}$, the term on the right-hand side can be explicitly computed, and one obtains

$$\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}V_e(\nabla\psi'(e))\right]+H\left(\mathbb{P}_{\psi'}\right)=-3^{d(n-m)}\log\left(\int_{\mathring{h}^1(\square_m)}\exp\left(-\sum_{e\subseteq\square_m}V_e(\nabla\psi(e))\right)d\psi\right).$$

Combining the previous display with (5.3.33) and (5.3.34), we obtain

$$\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e(\nabla\psi'(e))\right] + H\left(\mathbb{P}_{\psi'}\right) = \inf_{\mathbb{P}}\left(\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e(\nabla\psi(e))\right] + H\left(\mathbb{P}\right)\right).$$

The proof of Step 1 is complete.

Step 2. The idea of this step is similar to the strategy adopted in Step 2 of Proposition 5.3.8: one uses the uniform convexity of the elastic potential V_e and the displacement convexity of the entropy to prove a uniform convexity result for the functional $\mathcal{F}^*_{n,0}$ defined in Definition 5.2.7. One then applies the techniques used to obtain a second variation formula.

First, we consider the optimal coupling between the random variables ψ' and $\psi'_{n,0}$. In particular, by the displacement convexity of the entropy, the law of $\frac{\psi'+\psi'_{n,0}}{2}$ satisfies the inequality

$$H\left(\mathbb{P}_{\frac{\psi'+\psi'_{n,0}}{2}}\right) \leq \frac{H\left(\mathbb{P}_{\psi'}\right) + H\left(\mathbb{P}_{\psi'_{n,0}}\right)}{2}. \tag{5.3.35}$$

Moreover, by the uniform convexity of V_e, one has

$$\begin{aligned} 2\mathbb{E}&\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\frac{\nabla\psi'(e)+\nabla\psi'_{n,0}(e)}{2}\right)\right] \\ &\leq \mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\nabla\psi'_{n,0}(e)\right)\right] + \mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\nabla\psi'(e)\right)\right] \\ &\quad - \lambda\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} \left|\nabla\psi'(e)-\nabla\psi'_{n,0}(e)\right|^2\right]. \end{aligned} \tag{5.3.36}$$

We can then use Step 1 to compute

$$\begin{aligned} \mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e(\nabla\psi'(e))\right] + H\left(\mathbb{P}_{\psi'}\right) &= \inf_{\mathbb{P}}\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e(\nabla\psi(e))\right] + H\left(\mathbb{P}\right) \\ &\leq \mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\frac{\nabla\psi'(e)+\nabla\psi'_{n,0}(e)}{2}\right)\right] \\ &\quad + H\left(\mathbb{P}_{\frac{\nabla\psi'(e)+\nabla\psi'_{n,0}(e)}{2}}\right). \end{aligned}$$

One can then apply (5.3.35) and (5.3.36) to deduce

$$\begin{aligned} \mathbb{E}&\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e(\nabla\psi'(e))\right] + H\left(\mathbb{P}_{\psi'}\right) \\ &\leq \frac{1}{2}\left(\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e(\nabla\psi'(e))\right] + H\left(\mathbb{P}_{\psi'}\right)\right) \\ &\quad + \frac{1}{2}\left(\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\nabla\psi'_{n,0}(e)\right)\right] + H\left(\mathbb{P}_{\psi'_{n,0}}\right)\right) \\ &\quad - \frac{\lambda}{2}\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} \left|\nabla\psi'(e)-\nabla\psi'_{n,0}(e)\right|^2\right]. \end{aligned}$$

Note that from Step 1, one also has

$$\frac{1}{|\square_n|}\left(\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e(\nabla\psi'(e))\right] + H\left(\mathbb{P}_{\psi'}\right)\right) = -\frac{3^{d(n-m)}|\square_m|}{|\square_n|}\nu^*\left(\square_m,0\right) = -\nu^*\left(\square_m,0\right).$$

Combining the two previous displays and dividing by $|\square_n|$ gives

$$\frac{\lambda}{2}\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}\left|\nabla\psi'(e)-\nabla\psi'_{n,0}(e)\right|^2\right]$$
$$\leq \frac{1}{2}\nu^*\left(\square_m,0\right) + \frac{1}{2}\left(\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\nabla\psi'_{n,0}(e)\right)\right] + \frac{1}{|\square_n|}H\left(\mathbb{P}_{\psi'_{n,0}}\right)\right).$$

This completes the proof of Step 2.

Step 3. Before starting this step, we recall the notation for the edges between two subcubes $B_{m,n}$ introduced in (5.3.29) and the decomposition of the sum (5.3.30). This step is the most technical one, and the main difficulty is to compare the entropies of the random variables $\psi_{n,0}$ and $\psi'_{n,0}$. The only property which is known about these random variables comes from their definitions: one has

$$\psi_{n,0} - \psi'_{n,0} \in H,$$

where H is the space of functions of mean zero which are constant on the cubes $z+\square_m$, it was first introduced in (5.3.28). In this situation, it is difficult to compare the two entropies. Fortunately, one can use the elastic energy of the random variable $\nabla\psi_{n,0}$ on the edges of $B_{m,n}$, where the gradient of $\psi_{n,0}-\psi'_{n,0}$ is supported to obtain some information: more specifically, we prove the inequality

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\in B_{m,n}} V_e(\nabla\psi_{n,0}(e))\right] + \frac{1}{|\square_n|}H\left(\mathbb{P}^*_{n,0}\right) \geq \frac{1}{|\square_n|}H\left(\mathbb{P}_{\psi'_{n,0}}\right) - Cm3^{-dm}. \tag{5.3.37}$$

The previous inequality is the crucial argument of this step and once it is established, it is relatively to conclude. In the next paragraph, we explain how to obtain the main result (5.3.32) of this step, assuming that (5.3.37) holds: first one deduces from (5.3.37)

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\nabla\psi_{n,0}(e)\right)\right] + \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\in B_{m,n}} V_e\left(\nabla\psi_{n,0}\right)\right] + \frac{1}{|\square_n|}H\left(\mathbb{P}_{n,0}\right)$$
$$\geq \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\nabla\psi_{n,0}(e)\right)\right] + \frac{1}{|\square_n|}H\left(\mathbb{P}_{\psi'_{n,0}}\right) - Cm3^{-dm}.$$

By the splitting of the sum identity stated in (5.3.30), the term on the left-hand side can be rewritten

$$\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\nabla\psi_{n,0}(e)\right)\right] + \mathbb{E}\left[\sum_{e\in B_{m,n}} V_e(\nabla\psi_{n,0}(e))\right] = \mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(\nabla\psi_{n,0}(e)\right)\right].$$

We then use the equality

$$\mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(\nabla\psi_{n,0}(e)\right)\right] + H\left(\mathbb{P}^*_{n,0}\right) = -|\square_n|\nu^*\left(\square_n,0\right).$$

Combining the few previous results and dividing by $|\square_n|$ then yields

$$-\nu^*\left(\square_n,0\right) \geq \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\nabla\psi_{n,0}(e)\right)\right] + \frac{1}{|\square_n|}H\left(\mathbb{P}_{\psi'_{n,0}}\right) - Cm3^{-dm}. \tag{5.3.38}$$

This is (5.3.32) and completes Step 3 up to two details:

(1) First there should be a $\psi^z_{n,0}$ instead of a $\psi_{n,0}$ on the right-hand side. By definition $\psi^z_{n,0}$ is the orthogonal projection of $\psi_{n,0}$ on the space $\mathring{h}^1(z+\square_m)$, and using the property (5.3.31), one has

$$\text{for each } z \in \mathcal{Z}_{m,n} \text{ and each } e \subseteq z+\square_m,\ \nabla\psi_{n,0}(e) = \nabla\psi^z_{n,0}(e).$$

With this identity, the inequality (5.3.38) can be rewritten

$$-\nu^*(\square_n, 0) \geq \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V_e\left(\nabla\psi^z_{n,0}\right)\right] + \frac{1}{|\square_n|} H\left(\mathbb{P}_{\psi'_{n,0}}\right) - Cm3^{-dm}.$$

This settles the first problem.

(2) The second detail which needs to be fixed is the entropy which is not exactly the same as in (5.3.32). By Proposition 5.2.6, one has

$$H\left(\mathbb{P}_{\psi'_{n,0}}\right) \geq \sum_{z\in\mathcal{Z}_{m,n}} H\left(\mathbb{P}_{\psi^z_{n,0}}\right),$$

which provides a solution.

We now turn to the proof of (5.3.37). We recall the notation $Z^*_0(\square_n)$ introduced in (5.1.8). We also let ρ be the density associated to the law $\mathbb{P}^*_{n,0}$; it is defined on $\mathring{h}^1(\square_n)$ by

$$\rho : \begin{cases} \mathring{h}^1(\square_n) \to & \mathbb{R} \\ \psi \quad \mapsto & \dfrac{1}{Z^*_0(\square_n)} \exp\left(-\sum_{e\subseteq\square_n} V_e(\nabla\psi(e))\right). \end{cases}$$

Using the orthogonal decomposition (5.3.28), and the definition of $\psi'_{n,0}$, we can compute the density ρ' of the random variable $\psi'_{n,0}$. It is defined on the space $\bigoplus_{z\in\mathcal{Z}_{m,n}} \mathring{h}^1(z+\square_m)$ according to

$$\rho' : \begin{cases} \bigoplus_{z\in\mathcal{Z}_{m,n}} \mathring{h}^1(z+\square_m) \to & \mathbb{R} \\ \psi \quad \mapsto & \dfrac{1}{Z^*_0(\square_n)} \displaystyle\int_H \exp\left(-\sum_{e\subseteq\square_n} V_e(\nabla\psi(e)+\nabla h(e))\right) dh, \end{cases}$$

where the integral is considered with respect to the Lebesgue measure on the space H defined in (5.3.28). Using that for every $h \in H$, ∇h is supported in $B_{m,n}$, we can use the splitting of the sum stated in (5.3.30) to obtain, for each $\psi \in \bigoplus_{z\in\mathcal{Z}_{m,n}} \mathring{h}^1(z+\square_m)$,

$$\rho'(\psi) = \frac{\exp\left(-\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq z+\square_m} V(\nabla\psi(e))\right)}{Z^*_0(\square_n)} \int_H \exp\left(-\sum_{e\in B_{m,n}} V_e(\nabla\psi(e)+\nabla h(e))\right) dh. \tag{5.3.39}$$

The next idea of the proof is to provide an upper bound for the term inside the integral and prove that its logarithm is relatively small: one can show that there exists $C := C(d,\lambda) < \infty$ such that, for each $\psi \in \bigoplus_{z\in\mathcal{Z}_{m,n}} \mathring{h}^1(z+\square_m)$,

$$\log \int_H \exp\left(-\sum_{e\in B_{m,n}} V_e(\nabla\psi(e)+\nabla h(e))\right) dh \leq Cm3^{d(n-m)}. \tag{5.3.40}$$

The proof of this estimate is essentially technical and relies on the fact that the dimension of H is $3^{d(n-m)} - 1$, which explains the form of the right-hand side. The proof is postponed to

Appendix A, Proposition 5.A.1. We now show how to deduce (5.3.37) from (5.3.40). We first compute the entropy of the law $\mathbb{P}^*_{n,0}$, this gives

$$H\left(\mathbb{P}^*_{n,0}\right) = \int_{\mathring{h}^1(\square_n)} \rho(\psi) \log \rho(\psi)\, d\psi$$
$$= \int_{\mathring{h}^1(\square_n)} \rho(\psi) \left(- \sum_{e \subseteq \square_n} V_e\left(\nabla \psi\right) \right) d\psi - \log Z_0^*\left(\square_n\right).$$

Adding the term $\mathbb{E}\left[\sum_{e \in B_{m,n}} V_e(\nabla\psi(e))\right]$ and using the splitting of the sum stated in (5.3.30) gives

$$\mathbb{E}\left[\frac{1}{|\square_n|} \sum_{e \in B_{m,n}} V_e(\nabla \psi_{n,0}(e))\right] + H\left(\mathbb{P}^*_{n,0}\right)$$
$$= \int_{\mathring{h}^1(\square_n)} \rho(\psi) \left(- \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq z + \square_m} V\left(\nabla \psi(e)\right) \right) d\psi - \log Z_0^*\left(\square_n\right).$$

We focus on the integral on the right-hand side: using the decomposition (5.3.28), one can apply Fubini's Theorem and first integrate over H then over $\bigoplus_{z \in \mathcal{Z}_{m,n}} \mathring{h}^1\left(z + \square_m\right)$. This gives

$$\int_{\mathring{h}^1(\square_n)} \rho(\psi) \left(- \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e\left(\nabla \psi\right) \right) d\psi$$
$$= \int_{\bigoplus_{z \in \mathcal{Z}_{m,n}} \mathring{h}^1(z+\square_m)} \int_H \rho(\psi + h) \left(- \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e\left(\nabla \psi\right) \right) dh\, d\psi.$$

Note that here we have used that the gradient of an element of H is supported on $B_{m,n}$. Using the definition of ρ' stated in 5.3.39, the previous equality can be rewritten

$$\int_{\mathring{h}^1(\square_n)} \rho(\psi) \left(- \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e\left(\nabla \psi\right) \right) d\psi$$
$$= \int_{\bigoplus_{z \in \mathcal{Z}_{m,n}} \mathring{h}^1(z+\square_m)} \rho'(\psi) \left(- \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e\left(\nabla \psi\right) \right) d\psi.$$

Combining the few previous displays then gives

$$\mathbb{E}\left[\sum_{e \in B_{m,n}} V_e(\nabla \psi_{n,0}(e))\right] + H\left(\mathbb{P}^*_{n,0}\right) \tag{5.3.41}$$
$$= \int_{\bigoplus_{z \in \mathcal{Z}_{m,n}} \mathring{h}^1(z+\square_m)} \rho'(\psi) \left(- \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V_e\left(\nabla \psi\right) \right) d\psi - \log Z^*_{\square_n}.$$

But note that by the definition of ρ' in (5.3.39) and the technical estimate (5.3.40), one has, for each $\psi \in \bigoplus_{z \in \mathcal{Z}_{m,n}} \mathring{h}^1\left(z + \square_m\right)$,

$$\log \rho'(\psi) \leq - \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq (z + \square_m)} V\left(\nabla \psi(e)\right) d\psi - \log Z_0^*\left(\square_n\right) + C m 3^{d(n-m)}.$$

This allows to perform the following computation

$$H\left(\mathbb{P}_{\psi'_{n,0}}\right) = \int_{\bigoplus_{z\in\mathcal{Z}_{m,n}} \mathring{h}^1(z+\square_m)} \rho'(\psi)\log\rho'(\psi)\,d\psi \tag{5.3.42}$$
$$\leq \int_{\bigoplus_{z\in\mathcal{Z}_{m,n}} \mathring{h}^1(z+\square_m)} \rho'(\psi)\left(-\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)} V\left(\nabla\psi(e)\right)\right)d\psi$$
$$-\log Z_0^*\left(\square_n\right) + Cm3^{d(n-m)}.$$

Combining (5.3.41) and (5.3.42) gives

$$H\left(\mathbb{P}_{\psi'_{n,0}}\right) \leq \mathbb{E}\left[\sum_{e\in B_{m,n}} V_e(\nabla\psi_{n,0}(e))\right] + H\left(\mathbb{P}^*_{n,0}\right) + Cm3^{d(n-m)}.$$

This is precisely (5.3.37) and the proof of Step 3 is complete.

Step 4. Combining the main results of Steps 2 and 3, one obtains the existence of a coupling between the random variables ψ' and $\psi'_{n,0}$, such that

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}\left|\nabla\psi_z(e)-\nabla\psi'_{n,0}(e)\right|^2\right] \leq C\left(\nu^*(\square_m,0)-\nu^*(\square_n,0)\right) + Cm3^{-dm}, \tag{5.3.43}$$

The main objective of this step is to find a coupling between the random variables $\psi_{n,0}$ and ψ', instead of $\psi'_{n,0}$, and ψ'. Recall that we denoted by h the orthogonal projection of $\psi_{n,0}$ on the space H. Denote by $\mathbb{P}_h$ its law, it is a probability measure on H.

By Lemma 5.2.11, there exists a coupling between the three probability measures $\mathbb{P}_{\psi'}, \mathbb{P}_{\psi'_{n,0}}$ and $\mathbb{P}_h$ such that, under this coupling, the law of $(\psi', \psi'_{n,0})$ is the optimal coupling between $\mathbb{P}_{\psi'}$ and $\mathbb{P}_{\psi'_{n,0}}$ and the law of $(\psi'_{n,0}, h)$ is $\mathbb{P}^*_{n,0}$. This provides the desired coupling between the random variables ψ' and $\psi_{n,0}$: under this coupling the inequality (5.3.43) is satisfied. Step 4 is complete.

Step 5. We remove the assumption $q = 0$. This can be achieved by applying the result obtained for $q = 0$ with the tilted elastic potential

$$V_{e,q}(x) := V_e(x) - q(e)x.$$

This new elastic potential satisfies the same uniform ellipticity assumption as V_e and one can perform the same proof with $V_{e,q}$ instead of V_e, the difference is mostly notational: the only place where it as an impact is in the estimate (5.3.40) and it provides an additional error term which can be proved to be bounded by $C(1+|q|^2)3^{dn-m}$. Dividing by the volume of the triadic cube $|\square_n|$ eventually shows

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}\left|\nabla\psi'(e)-\nabla\psi_{n,q}(e)\right|^2\right]$$
$$\leq C\left(\nu^*(\square_m,q)-\nu^*(\square_n,q)\right) + Cm3^{-dm} + C(1+|q^2|)3^{-m}.$$

This can be simplified into

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq(z+\square_m)}\left|\nabla\psi'(e)-\nabla\psi_{n,q}(e)\right|^2\right] \leq C\left(\nu^*(\square_m,q)-\nu^*(\square_n,q)\right) + C(1+|q^2|)3^{-m}.$$

The proof of Proposition 5.3.9 is complete. □

5.4. Convergence of the subadditive quantities

The main goal of this section is to use the tools developed in Section 5.3 to prove the main result of this article, namely Theorem 5.1.1. To this end, we first introduce a notation for the subadditivity defect of the surface tensions ν and ν^*.

DEFINITION 5.4.1. For each $p \in \mathbb{R}^d$ and each $n \in \mathbb{N}$, we define

$$\tau_n(p) := \nu(\square_n, p) - \nu(\square_{n+1}, p)$$

and for each $q \in \mathbb{R}^d$,

$$\tau_n^*(q) := \nu^*(\square_n, q) - \nu^*(\square_{n+1}, q).$$

These terms correspond to the subadditivity defect in the subadditivity of the surface tensions and are an important object because they appear in the right-hand side of the main estimates of Propositions 5.3.8 and 5.3.9. These two propositions are central in the analysis of this sections and the subadditivity defect quantities $\tau_n(p)$ and $\tau_n^*(q)$ will be used frequently in the proofs.

We also recall the following notation from the introduction: for a bounded subset $U \subseteq \mathbb{Z}^d$ and a vector field $F : E_d(U) \to \mathbb{R}$, we let $\langle F \rangle_U$ be the unique vector in $\mathbb{R}^d$ such that, for each $p \in \mathbb{R}^d$

$$p \cdot \langle F \rangle_U = \frac{1}{|U|} \sum_{e \subseteq U} p \cdot F(e).$$

In the rest of this section, this will be applied when U is a triadic cube and when F is the gradient of a function. We may also refer to the quantity $\langle \nabla \phi \rangle_U$ as the slope of the function ϕ over the set U.

This section is organized as follows, we first prove, using Proposition 5.3.9, that the variance of the slope of the random variable $\psi_{n,q}$ over the cube $\square_n$ contracts as n tends to infinity. More precisely, one shows a quantitative control of the variance of the slope by the subadditivity defect $\tau_n^*(q)$, which is expected to be small as n tends to infinity. This is performed in Proposition 5.4.2 and essentially relies of the two scale comparison for ν^* stated in Proposition 5.3.9.

Once the slope of the random variable $\psi_{n,q}$ is controlled, we apply the multiscale Poincaré inequality, stated in Proposition 5.2.14, to prove that $\psi_{n,q}$ is in fact close, in the expectation of the L^2-norm over the cube $\square_n$, to an affine function. With all these tools at hand, we prove the technical estimate of this article, Proposition 5.4.5, thanks to a patching construction. This technical lemma, combined with the convex duality property proved in Proposition 5.3.6, shows that on a large scale, the functions $p \to \nu(\square_n, p)$ and $q \to \nu^*(\square_n, q)$ are approximately convex dual to one another, i.e. they satisfy up to a small error

$$\nu^*(\square_n, q) \simeq \sup_{p \in \mathbb{R}^d} -\nu(\square_n, p) + p \cdot q.$$

Once such a result is established, we turn to the proof of the quantitative convergence of the surface tension, Theorem 5.1.1.

5.4.1. Contraction of the variance of the slope of $\psi_{n,q}$. We first prove the contraction of the variance of the slope of the random interface $\psi_{n,q}$. This is stated in the following proposition.

PROPOSITION 5.4.2 (Contraction of the slope of $\psi_{n,q}$). *There exists a constant $C := C(d, \lambda) < \infty$ such that for each $n \in \mathbb{N}$, and each $q \in \mathbb{R}^d$,*

$$\operatorname{var}\left[\langle \nabla \psi_{n+1,q} \rangle_{\square_{n+1}}\right] \leq C(1 + |q|^2) 3^{-n} + C \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q). \tag{5.4.1}$$

PROOF. The proof of this inequality relies crucially on Proposition 5.3.9 and it is necessary to reintroduce the objects used in the statement of this proposition. This is performed in the following paragraph.

Consider the family of random variables ψ_z, for $z \in \mathcal{Z}_{n,n+1}$ and the random variable $\psi' := \sum_{z \in \mathcal{Z}_{n,n+1}} \psi_z$ which were introduced before the statement of Proposition 5.3.9. We recall that it satisfies the following properties:

- for each $z \in \mathcal{Z}_{n,n+1}$, ψ_z is a random variable valued in $\mathring{h}^1(\square_{n+1})$. It is equal to 0 outside $z + \square_n$ and has law $\mathbb{P}^*_{n,q}$ in $z + \square_n$,
- the ψ_z are independent.

We also consider the coupling between ψ' and $\psi_{n+1,q}$ which was introduced in Proposition 5.3.9. In particular estimate (5.3.27) holds.

The main idea of the proof is the following: by Proposition 5.3.9, one knows that, up to an error of size $\tau_n^*(q)$, the gradient of the random interface $\psi_{n+1,q}$ is close to the gradients of 3^d independent random variables: the random variables ψ_z. As a consequence, the slope of the random interface $\psi_{n+1,q}$ can be written, up to an error of size $\tau_n^*(q)$, as a sum of independent random variables. One can then apply a standard concentration inequality for the variance of independent random variables to show the contraction of the slope.

We split the proof into 2 steps

- In Step 1, we use Proposition 5.3.9 to prove

$$\operatorname{var}^{\frac12}\left[\left\langle \nabla\psi_{n+1,q}\right\rangle_{\square_{n+1}}\right] \leq 3^{-\frac{d}{2}} \operatorname{var}^{\frac12}\left[\left\langle \nabla\psi_{n,q}(e)\right\rangle_{\square_n}\right] + C\tau_n^*(q)^{\frac12} + C3^{-\frac{n}{2}}(1+|q|). \tag{5.4.2}$$

- In Step 2, we iterate the inequality obtained in Step 1 to derive (5.4.1).

Step 1. First we recall the definition of the set of edges connecting two subcubes of the form $z + \square_n$,

$$B_n := \left\{(x,y) \,:\, \exists z, z' \in 3^n\mathbb{Z}^d \cap \square_{n+1} \,\text{such that}\, z \neq z',\, x \in z + \square_n \,\text{and}\, y \in z' + \square_n\right\}$$

and the decomposition of the sum (5.3.30)

$$\sum_{e\subseteq\square_{n+1}} = \sum_{z\in\mathcal{Z}_n}\sum_{e\subseteq z+\square_n} + \sum_{e\in B_n}.$$

This set is equal to the set $B_{n,n+1}$ from the previous sections, but since it depends only on one parameter, we use the shortcut notation B_n. From the previous decomposition of the sum, one has the estimate

$$\left|\left\langle\nabla\psi_{n+1,q}\right\rangle_{\square_{n+1}} - 3^{-d}\sum_{z\in\mathcal{Z}_n}\left\langle\nabla\psi_{n+1,q} - \nabla\psi_z\right\rangle_{z+\square_n} - 3^{-d}\sum_{z\in\mathcal{Z}_n}\left\langle\nabla\psi_z\right\rangle_{z+\square_n}\right| \leq \frac{1}{|\square_{n+1}|}\sum_{e\in B_n}|\nabla\psi_{n+1,q}(e)|.$$

Taking the square-root of the variance and using the triangle inequality, one obtains

$$\operatorname{var}^{\frac12}\left[\left\langle\nabla\psi_{n+1,q}\right\rangle_{\square_{n+1}}\right] \leq \operatorname{var}^{\frac12}\left[3^{-d}\sum_{z\in\mathcal{Z}_n}\left\langle\nabla\psi_{n+1} - \nabla\psi_z\right\rangle_{z+\square_n}\right] + \operatorname{var}^{\frac12}\left[3^{-d}\sum_{z\in\mathcal{Z}_n}\left\langle\nabla\psi_z\right\rangle_{z+\square_n}\right] + \mathbb{E}\left[\left(\frac{1}{|\square_{n+1}|}\sum_{e\in B_n}|\nabla\psi_{n+1,q}(e)|\right)^2\right]^{\frac12}.$$

We then estimate the three terms on the right-hand side separately. The first term is an error term which can be estimated by Proposition 5.3.9,

$$\begin{aligned}\operatorname{var}\left[3^{-d}\sum_{z\in\mathcal{Z}_n}\left\langle\nabla\psi_{n+1} - \nabla\psi_z\right\rangle_{z+\square_n}\right] &\leq \mathbb{E}\left[\frac{1}{|\square_{n+1}|}\sum_{z\in\mathcal{Z}_n}\sum_{e\subseteq z+\square_n}|\nabla\psi_{n+1}(e) - \nabla\psi_z(e)|^2\right]\\ &\leq C\tau_n^*(q) + C(1+|q|^2)3^{-n}.\end{aligned}$$

To estimate the second term we use the independence of the random variables ψ_z and the concentration inequality for a sum of independent random variable,

$$\operatorname{var}\left[3^{-d}\sum_{z\in\mathcal{Z}_n}\left\langle\nabla\psi_z\right\rangle_{z+\square_n}\right] = 3^{-2d}\sum_{z\in\mathcal{Z}_n}\operatorname{var}\left[\left\langle\nabla\psi_z\right\rangle_{z+\square_n}\right].$$

Using that the law of ψ_z on the subcube $z + \square_n$ is $\mathbb{P}^*_{n,q}$, one obtains

$$\operatorname{var}\left[3^{-d} \sum_{z \in \mathcal{Z}_n} \langle \nabla \psi_z \rangle_{z+\square_n}\right] = 3^{-d} \operatorname{var}\left[\langle \nabla \psi_{n,q} \rangle_{\square_n}\right].$$

The third term is also an error term and can be estimated thanks to the Cauchy-Schwarz inequality together with the bound $|B_n| \leq C 3^{-n} |\square_{n+1}|$,

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{|\square_{n+1}|} \sum_{e \in B_n} \nabla \psi_{n+1}(e)\right|^2\right] &\leq \mathbb{E}\left[\frac{|B_n|}{|\square_{n+1}|^2} \sum_{e \in B_n} |\nabla \psi_{n+1}(e)|^2\right] \\ &\leq C 3^{-n} \mathbb{E}\left[\frac{1}{|\square_{n+1}|} \sum_{e \in B_n} |\nabla \psi_{n+1}(e)|^2\right].\end{aligned}$$

By the bound on the L^2 norm of $\nabla \psi_{n+1}$ obtained in Proposition 5.3.1, this yields

$$\mathbb{E}\left[\left|\frac{1}{|\square_{n+1}|} \sum_{e \in B_n} \nabla \psi_{n+1}(e)\right|^2\right] \leq C 3^{-n} (1 + |q|^2),$$

for some $C := C(d, \lambda) < \infty$. Combining the few previous displays gives the estimate

$$\operatorname{var}^{\frac{1}{2}}\left[\langle \nabla \psi_{n+1,q} \rangle_{\square_{n+1}}\right] \leq 3^{-\frac{d}{2}} \operatorname{var}^{\frac{1}{2}}\left[\langle \nabla \psi_n(e) \rangle_{\square_n}\right] + C \tau_n^*(q)^{\frac{1}{2}} + C 3^{-\frac{n}{2}} (1 + |q|).$$

Step 2. Iteration and conclusion. We denote by

$$\sigma_n := \operatorname{var}^{\frac{1}{2}}\left[\langle \nabla \psi_n \rangle_{\square_n}\right].$$

The main estimate of Step 1 can be rewritten with this new notation

$$\sigma_{n+1} \leq 3^{-\frac{d}{2}} \sigma_n + C \tau_n^*(q)^{\frac{1}{2}} + C 3^{-\frac{n}{2}} (1 + |q|).$$

An iteration of the previous display gives

$$\begin{aligned}\sigma_n &\leq 3^{-\frac{dn}{2}} \sigma_0 + C \sum_{m=0}^{n} 3^{-\frac{d(n-m)}{2}} \tau_m^*(q)^{\frac{1}{2}} + C(1 + |q|) \sum_{m=0}^{n} 3^{-\frac{d(n-m)}{2}} 3^{-\frac{m}{2}} \\ &\leq 3^{-\frac{dn}{2}} \sigma_0 + C \sum_{m=0}^{n} 3^{-\frac{d(n-m)}{2}} \tau_m^*(q)^{\frac{1}{2}} + C(1 + |q|) 3^{-\frac{n}{2}}.\end{aligned}$$

To simplify the previous estimate, we note that by (5.3.5) of Proposition 5.3.1, one has the bound $\sigma_0 \leq C(1 + |q|)$,

$$\sigma_n \leq C(1 + |q|) 3^{-\frac{n}{2}} + C \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q)^{\frac{1}{2}},$$

Squaring the previous inequality gives

$$\sigma_n^2 \leq C(1 + |q|^2) 3^{-n} + C\left(\sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q)^{\frac{1}{2}}\right)^2 \leq C(1 + |q|^2) 3^{-n} + C \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q).$$

The proof of Step 2 is complete. □

To finish this section, we record and prove another useful property of the slope of the random interface $\psi_{n,q}$: thanks to an explicit computation, one can relate the expectation of the slope of the random interface $\psi_{n,q}$ to the gradient in the q variable of the dual surface tension ν^* according to the formula

$$\nabla_q \nu^*(\square_n, q) = \mathbb{E}\left[\langle \nabla \psi_{n,q} \rangle_{\square_n}\right].$$

This has some interesting consequences: first one derives the bound, for each $q \in \mathbb{R}^d$,

$$
\begin{aligned}
(5.4.3) \qquad |\nabla_q \nu^*(\square_n, q)| &\le \left(\mathbb{E}\left[\frac{1}{|\square_n|} \sum_{e \subseteq \square_n} |\nabla \psi_{n,q}(e)|^2 \right] \right)^{\frac{1}{2}} \\
&\le C(1 + |q|).
\end{aligned}
$$

Second and for future reference, we record that the difference of the gradient of ν^* between different scales can be controlled by the subadditivity defect. This is stated in the following lemma.

Lemma 5.4.3. *For each $m \le n$ and each $q \in \mathbb{R}^d$*

$$
(5.4.4) \qquad |\nabla_q \nu^*(\square_m, q) - \nabla_q \nu^*(\square_n, q)|^2 \le C \sum_{k=m}^{n} \tau_k^*(q) + C(1 + |q|^2) 3^{-m}.
$$

Proof. The arguments for the proof of this lemma are essentially contained in the proof of Proposition 5.4.2. The core idea is to apply Proposition 5.3.9 and thus we consider the coupling between the random variables $\psi_{n,q}$ and ψ_z introduced in this proposition. With this notation, one has

$$
(5.4.5) \qquad |\nabla_q \nu^*(\square_m, q) - \nabla_q \nu^*(\square_n, q)|^2 = \left| E\left[\langle \nabla \psi_{n,q}(e) \rangle_{\square_n} - \frac{1}{|\mathcal{Z}_{m,n}|} \sum_{z \in \mathcal{Z}_{m,n}} \langle \nabla \psi_z \rangle_{z + \square_m} \right] \right|^2 .
$$

We reintroduce the set of edges connecting two cubes of the form $z + \square_m$,

$$
B_{m,n} := \left\{ e = (x,y) \subseteq \square_n \, : \, \exists z, z' \in 3^m \mathbb{Z}^d \cap \square_n, \, z \neq z', \, x \in z + \square_m \text{ and } y \in z' + \square_m \right\}.
$$

We also recall that its cardinality can be estimated according to the formula

$$
|B_{m,n}| \le C 3^{-m} |\square_n| .
$$

and that one can partition the set of edges of $\square_n$ according to the identity

$$
e \subseteq \square_n \implies \exists z \in 3^m \mathbb{Z}^d \cap \square_n, \, e \subseteq z + \square_n \, \text{or} \, e \in B_{n,m},
$$

which can be restated with sum: one has the following splitting of the sum

$$
\sum_{e \subseteq \square_n} = \sum_{e \in B_{m,n}} + \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq z + \square_m} .
$$

Combining this with (5.4.5), one obtains

$$
\begin{aligned}
|\nabla_q \nu^*(\square_m, q) - \nabla_q \nu^*(\square_n, q)|^2 \le{}& 2\mathbb{E}\left[\frac{1}{|\square_n|} \sum_{z \in \mathcal{Z}_{m,n}} \sum_{e \subseteq z + \square_m} |\nabla \psi_{n,q}(e) - \nabla \psi_z(e)|^2 \right] \\
&+ 2\mathbb{E}\left[\left| \frac{1}{|\square_n|} \sum_{e \in B_{m,n}} \nabla \psi_{n,q}(e) \right|^2 \right].
\end{aligned}
$$

The first term on the right-hand side can be estimated by Proposition 5.3.9, and the second one by the Cauchy-Schwarz inequality, similarly to what is written in Proposition 5.4.2. This implies the desired estimate. □

5.4.2. L^2 contraction of the field $\psi_{n,q}$ to an affine function. The main objective of this section is to combine the multiscale Poincaré inequality with the contraction of the variance of the slope of $\psi_{n,q}$ proved in the previous section, to obtain that the field $\psi_{n,q}$ is close in the L^2 norm to an affine function. The right-hand side of the estimate still depends on the subadditivity defects $\tau_n^*(q)$ which are expected to be small as n tends to infinity. This is stated in the following proposition.

PROPOSITION 5.4.4. *There exists a constant $C := C(d,\lambda) < \infty$ such that, for every $n \in \mathbb{N}$ and every $q \in \mathbb{R}^d$,*

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{x\in\square_n}\left|\psi_{n,q}(x) - \nabla_q \nu^*(\square_n,q)\cdot x\right|^2\right] \le C3^{2n}\left((1+|q|^2)3^{-\frac{n}{2}} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}}\tau_m^*(q)\right). \tag{5.4.6}$$

PROOF. By the discrete version of the multiscale Poincaré inequality stated in Proposition 5.2.14, one has

$$\begin{aligned}\frac{1}{|\square_n|}\sum_{x\in\square_n}\left|\psi_{n,q}(x) - \nabla_q \nu^*(\square_n,q)\cdot x\right|^2 &\le C\frac{1}{|\square_n|}\sum_{e\subseteq\square_{n+1}}\left|\nabla\psi_{n,q}(e) - \nabla_q\nu^*(\square_n,q)\cdot e\right|^2 \\ &\quad + C3^n\sum_{m=0}^{n}3^m\left(\frac{1}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\left|\left\langle\nabla\psi_{n,q}\right\rangle_{z+\square_m} - \nabla_q\nu^*(\square_n,q)\right|^2\right).\end{aligned} \tag{5.4.7}$$

By Proposition 5.3.1 and (5.4.3), we can bound the expectation of the first term on the right-hand side

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}\left|\nabla\psi_{n,q}(e) - \nabla_q\nu^*(\square_n,q)\cdot e\right|^2\right] &\le \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}\left|\nabla\psi_{n,q}(e)\right|^2 + \left|\nabla_q\nu^*(\square_n,q)\right|^2\right] \\ &\le C(1+|q|^2).\end{aligned}$$

We then split the proof into 2 steps

- In Step 1, we estimate the expectation of the second term on the right-hand side and prove the estimate, for any integer $m \le n$,

$$\begin{aligned}&\frac{1}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\left|\left\langle\nabla\psi_{n,q}\right\rangle_{z+\square_m} - \nabla_q\nu^*(\square_n,q)\right|^2\right] \\ &\qquad \le C(1+|q|^2)3^{-m} + C\sum_{k=0}^{m}3^{\frac{(k-m)}{2}}\tau_k^*(q) + C\sum_{k=m}^{n}\tau_k^*(q).\end{aligned} \tag{5.4.8}$$

- In Step 2, we deduce (5.4.6) from the previous display.

Step 1. To prove (5.4.8), the two main ingredients are Proposition 5.4.2, proved in the previous section, and Proposition 5.3.9. We first recall the notations which were used in this proposition. To apply Proposition 5.3.9, we first recall the family random variables ψ_z, for $z \in \mathbb{Z}_{m,n}$ introduced in this proposition as well as the coupling between ψ and ψ_z which satisfies

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{z\in\mathcal{Z}_{m,n}}\sum_{e\subseteq z+\square_m}\left|\nabla\psi_z(e) - \nabla\psi_{n,q}(e)\right|^2\right] \le C\sum_{k=m}^{n}\tau_k^*(q) + C(1+|q|^2)3^{-m}. \tag{5.4.9}$$

With this in mind, we split (5.4.8)

$$\begin{aligned}&\frac{1}{|\mathcal{Z}_{m,n}|}\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\left|\left\langle\nabla\psi_{n,q}\right\rangle_{z+\square_m} - \nabla_q\nu^*(\square_n,q)\right|^2\right] \\ &\qquad\le \frac{3}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\frac{1}{|\square_m|}\mathbb{E}\left[\sum_{e\subseteq z+\square_m}\left|\nabla\psi_{n,q}(e) - \nabla\psi_z(e)\right|^2\right] \\ &\qquad\quad + \frac{3}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\left|\left\langle\nabla\psi_z\right\rangle_{z+\square_m} - \nabla_q\nu^*(\square_m,q)\right|^2\right] \\ &\qquad\quad + 3\left|\nabla_q\nu^*(\square_n,q) - \nabla_q\nu^*(\square_m,q)\right|^2,\end{aligned}$$

and estimate the three terms separately. The first and third terms term on the right-hand side are an error terms which can be estimated thanks to (5.4.9) and (5.4.4) respectively. This gives

$$\begin{aligned}\frac{1}{|\mathcal{Z}_{m,n}|}\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{m,n}}\left|\left\langle\nabla\psi_{n,q}\right\rangle_{z+\square_m}-\nabla_q\nu^*(\square_n,q)\right|^2\right] \\ \leq \frac{3}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\frac{1}{|\square_m|}\mathbb{E}\left[\sum_{e\subseteq z+\square_m}\left|\nabla\psi_z(e)-\nabla_q\nu^*(\square_m,q)\cdot e\right|^2\right] \\ + C\sum_{k=m}^{n}\tau_k^*(q)+C(1+|q|^2)3^{-m}.\end{aligned}$$

Thanks to the identity $\nabla_q\nu^*(\square_m,q) := \mathbb{E}\left[\left\langle\nabla\psi_{m,q}\right\rangle_{\square_m}\right]$, one can estimate the remaining term thanks to Proposition 5.4.2,

$$\begin{aligned}\frac{1}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\left|\left\langle\nabla\psi_z\right\rangle_{z+\square_m}-\nabla_q\nu^*(\square_m,q)\right|^2\right] = \mathrm{var}\left[\left\langle\nabla\psi_{n,q}\right\rangle_{\square_m}\right] \\ \leq C(1+|q|^2)3^{-m}+C\sum_{k=0}^{m}3^{\frac{(k-m)}{2}}\tau_k^*(q).\end{aligned}$$

Combining the previous displays yields

$$\frac{1}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\left|\left\langle\nabla\psi_{n,q}\right\rangle_{z+\square_m}-\nabla_q\nu^*(\square_n,q)\right|^2\right] \leq C(1+|q|^2)3^{-m}+C\sum_{k=0}^{m}3^{\frac{(k-m)}{2}}\tau_k^*(q)+C\sum_{k=m}^{n}\tau_k^*(q).$$

This is (5.4.8). The proof of Step 1 is complete.

Step 2. To ease the notation, we denote by, for each $m\in\{1,\ldots,n\}$,

$$X_m := \frac{1}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\left|\left\langle\nabla\psi_{n,q}\right\rangle_{z+\square_m}-\nabla_q\nu^*(\square_n,q)\right|^2.$$

The main result of Step 1 can be reformulated with this new notation

$$\mathbb{E}\left[X_m\right] \leq C(1+|q|^2)3^{-m}+C\sum_{k=0}^{m}3^{\frac{(k-m)}{2}}\tau_k^*(q)+C\sum_{k=m}^{n}\tau_k^*(q).$$

and by the multiscale Poincaré inequality stated in (5.4.7), one has

$$\frac{1}{|\square_n|}\sum_{x\in\square_n}\left|\psi_{n,q}(x)-\nabla_q\nu^*(\square_n,q)\cdot x\right|^2 \leq C(1+|q|^2)+C3^n\sum_{m=0}^{n}3^m X_m.$$

Taking the expectation on the right-hand side gives

$$\begin{aligned}\mathbb{E}\left[3^n\sum_{m=0}^{n}3^m X_m\right] &\leq C3^n\sum_{m=0}^{n}3^m\left(C(1+|q|^2)3^{-m}+C\sum_{k=0}^{m}3^{\frac{(k-m)}{2}}\tau_k^*(q)+C\sum_{k=m}^{n}\tau_k^*(q)\right) \\ &\leq C3^{2n}\left((1+|q|^2)n3^{-n}+\sum_{k=0}^{n}3^{\frac{(k-n)}{2}}\tau_k^*(q)+\sum_{k=0}^{n}3^{(k-n)}\tau_k^*(q)\right).\end{aligned}$$

The previous display can be further simplified by appealing to the crude estimates $n\leq C3^{\frac{n}{2}}$ and $3^{(k-n)}\leq 3^{\frac{(k-n)}{2}}$. Then a combination of the few previous displays shows

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{x\in\square_n}\left|\psi_{n,q}(x)-\nabla_q\nu^*(\square_n,q)\cdot x\right|^2\right] \leq C3^{2n}\left((1+|q|^2)3^{-\frac{n}{2}}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).$$

The proof of Proposition 5.4.4 is complete. □

5.4.3. Convex duality: upper bound. The objective of this section is to use the results of the previous sections, and in particular Proposition 5.4.4 to show that the two surface tensions ν and ν^* are approximately convex dual to one another. First, we introduce the following notation: for each $n \in \mathbb{N}$, we denote by $\square_n^+$ the triadic cube $\square_n$ to which one has added a boundary layer of size 1, i.e.

$$\square_n^+ := \left(-\frac{3^n + 1}{2}, \frac{3^n + 1}{2} \right)^d .$$

It is a cube of size $3^n + 2$ and satisfies the following convenient property

$$\left(\square_n^+ \right)^{\mathrm{o}} = \square_n.$$

The statement of the next proposition can be formulated as follows: if the $\tau_n^*(q)$ are small, then for each $q \in \mathbb{R}^d$, there exists $p \in \mathbb{R}^d$ such that

$$\nu\left(\square_{2n}^+, p\right) + \nu^*\left(\square_n, q\right) - p \cdot q \quad \text{is small.}$$

Moreover we have an explicit value of p which is $\nabla_q \nu^*(\square_n, q)$. In a later statement, we will remove the condition $\square_{2n}^+$ and prove that for each $q \in \mathbb{R}^d$, there exists $p \in \mathbb{R}^d$ such that

$$\nu\left(\square_n, p\right) + \nu^*\left(\square_n, q\right) - p \cdot q \quad \text{is small.}$$

Combining this result with the lower bound on the convex duality proved in Proposition 5.3.6, one obtains

$$\nu^*\left(\square_n, q\right) = \inf_{p \in \mathbb{R}^d} \left(-\nu\left(\square_n, p\right) + p \cdot q \right) \quad \text{up to a small error.}$$

The main argument in the proof of Proposition 5.4.5 is a patching construction: we need to patch functions of laws $\mathbb{P}_{n,q}^*$ in the (much larger) cube $\square_{2n}^+$, to build a law on the space $h_0^1\left(\square_{2n}^+\right)$ and test it in the variational formulation for ν.

Proposition 5.4.5. *There exist a constant $C := C(d, \lambda) < \infty$ and an exponent $\beta := \beta(d, \lambda) > 0$ such that for each $q \in \mathbb{R}^d$ and each $n \in \mathbb{N}$,*

$$\nu\left(\square_{2n}^+, \nabla_q \nu^*(\square_n, q)\right) + \nu^*\left(\square_n, q\right) - \nabla_q \nu^*(\square_n, q) \cdot q \leq C \left((1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q) \right).$$

Proof. Fix $q \in \mathbb{R}^d$ and for each $p \in \mathbb{R}^d$, we denote by l_p the linear function of slope p, defined by, for each $x \in \mathbb{R}^d$,

$$l_p(x) = p \cdot x.$$

To simplify the notation, we also write, for each $q \in \mathbb{R}^d$ and each $n \in \mathbb{N}$,

$$\nabla \nu_n^*(q) := \nabla_q \nu^*(\square_n, q) \in \mathbb{R}^d$$

We also recall the notation (5.1.2) introduced in Section 5.1 which will be used with $p = \nabla \nu_n^*(q)$ frequently in the proof.

The strategy of the proof is the following: we construct a random variable taking values in $h_0^1\left(\square_{2n}^+\right)$, denoted by κ_{2n}^+ in the proof. This random variable is essentially constructed by patching together independent random variables, which are defined on the triadic cubes $z + \square_n$, for $z \in \mathcal{Z}_{n,2n}$ and whose laws are the law of $\psi_{n,q} - l_{\nabla \nu_n^*(q)}$. The technical details are carried out in Step 1 below. We then prove that κ_{2n}^+ satisfies

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|} \sum_{e \subseteq \square_{2n}^+} V_e\left(\nabla \nu_n^*(q)(e) + \nabla \kappa_{2n}^+(e)\right) \right] + \frac{1}{\left|\square_{2n}^+\right|} H\left(\mathbb{P}_{\kappa_{2n}^+}\right) \leq -\nu^*\left(\square_n, q\right) + \nabla \nu_n^*(q) \cdot q$$
$$+ C \left((1 + |q|^2) 3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q) \right).$$

We finally use κ_{2n}^+ as a test function in the variational formulation of $\nu\left(\square_{2n}^+, \nabla\nu_n^*(q)\right)$, to obtain the inequality

$$\nu\left(\square_{2n}^+, \nabla\nu_n^*(q)\right) \leq \mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e\subseteq\square_{2n}^+} V_e\left(\nabla\nu_n^*(q) + \nabla\kappa_{2n}^+(e)\right)\right] + \frac{1}{\left|\square_{2n}^+\right|} H\left(\mathbb{P}_{\kappa_{2n}^+}\right).$$

Combining the two previous displays will complete the proof.

We split the proof into 4 steps.

- In Step 1, we construct the random variable κ_{2n}^+ taking values in $h_0^1\left(\square_{2n}^+\right)$.
- In Step 2, we show that the entropy of κ_{2n}^+ is controlled by the entropy of $\mathbb{P}_{n,q}^*$. Precisely, we prove

$$\frac{1}{\left|\square_{2n}^+\right|} H\left(\mathbb{P}_{\kappa_{2n}^+}\right) \leq \frac{1}{\left|\square_n\right|} H\left(\mathbb{P}_{n,q}^*\right) + Cn3^{-n}, \tag{5.4.10}$$

  where the entropy on the left-hand side is computed with respect to the Lebesgue measure on $h_0^1\left(\square_{2n}^+\right)$ and the entropy on the right-hand side is computed with respect to the Lebesgue measure on $\mathring{h}^1(\square_n)$.
- In Steps 3 and 4, we show that the energy of the random variable κ_{2n}^+ is controlled by the energy of $\psi_{n,q}$. Precisely, we prove

$$\begin{aligned}\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e\subseteq\square_{2n}^+} V_e\left(\nabla\nu_n^*(q) + \nabla\kappa_{2n}^+(e)\right)\right] &\leq \mathbb{E}\left[\frac{1}{\left|\square_n\right|}\sum_{e\subseteq\square_n} V_e\left(\nabla\psi_{n,q}(e)\right)\right] \\ &\quad + C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).\end{aligned} \tag{5.4.11}$$

- In Step 5, we combine the results of Steps 3 and 4 to prove

$$\nu\left(\square_{2n}^+, \nabla\nu_n^*(q)\right) - \nu^*(\square_n, q) + q\cdot\nabla\nu_n^*(q) \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).$$

Step 1. Denote by $h^1(\square_{2n})$ the set of functions from $\square_{2n}$ to $\mathbb{R}$. There exists a canonical bijection between $h^1(\square_{2n})$ and $h_0^1(\square_{2n}^+)$ obtained by extending the functions of $h^1(\square_{2n})$ to be 0 on the boundary of $\square_{2n}^+$. We first explain the strategy to construct the random interface κ_{2n}^+. Consider a family $(\psi_z)_{z\in\mathcal{Z}_{n,2n}}$ of random variables satisfying:

- for each $z\in\mathcal{Z}_{n,2n}$, ψ_z is valued in $\mathring{h}^1\left(z+\square_n\right)$ and its law is $\mathbb{P}_{n,q}^*$,
- the random variables ψ_z are independent.

As it is customary, we recall the definition for the set of edges connecting two triadic cubes of the form $z+\square_n$ in $\square_{2n}$,

$$B_{n,2n} := \left\{e=(x,y)\subseteq\square_{2n} : \exists z,z'\in\mathcal{Z}_{n,2n},\, z\neq z',\, x\in z+\square_n \,\text{and}\, y\in z'+\square_n\right\},$$

as well as the corresponding partition of edges of $\square_{2n}$,

$$e\subseteq\square_{2n} \implies \exists z\in 3^n\mathbb{Z}^d\cap\square_{2n},\, e\subseteq z+\square_n \,\text{or}\, e\in B_{n,2n}.$$

With this in mind, we construct a random vector field $\mathbf{f}$ defined on the edges $\square_{2n}$ by patching together the vector fields $\nabla\psi_z - \nabla\nu_n^*(q)$ defined on the edges of $z+\square_n$. Precisely, the vector field $\mathbf{f}$ is defined as follows, for each edge $e\subseteq\square_{2n}$,

$$\mathbf{f}(e) = \begin{cases}\nabla\psi_z(e) - \nabla\nu_n^*(q)(e) & \text{if } e\subseteq z+\square_n, \text{ for some } z\in\mathcal{Z}_{n,2n}, \\ 0 & \text{if } e\in B_{n,2n}.\end{cases} \tag{5.4.12}$$

The objective of this demonstration is to construct a function κ_{2n}^+ with Dirichlet boundary condition whose gradient is close to the vector field $\mathbf{f}$: one wishes to have, for each $e\subseteq\square_{2n}$,

$$\nabla\kappa_{2n}^+(e) \approx \mathbf{f}(e).$$

The meaning of the symbol "$\approx$" will be made precise in the following paragraphs.

A first obstruction is that the vector field $\mathbf{f}(e)$ is not in general the gradient of a function in $h^1_0(\square^+_{2n})$. To remedy this, a natural idea is to consider the orthogonal projection the vector field $\mathbf{f}$ on the space of gradients of functions in $h^1_0(\square^+_{2n})$. This is equivalent to solving the Dirichlet boundary value problem

$$\begin{cases} \Delta\kappa = \operatorname{div}\mathbf{f} \text{ in } \square_{2n}, \\ \kappa \in h^1_0(\square^+_{2n}). \end{cases} \tag{5.4.13}$$

This solves the first obstruction.

The second obstruction is that the random variable κ defined in (5.4.13) almost surely belongs to a strict linear subspace of $h^1_0(\square^+_{2n})$, consequently its law is not absolutely continuous with respect to the Lebesgue measure on $h^1_0(\square^+_{2n})$ and its entropy is infinite. To remedy this we add some independent random variables whose law uniform on $[0,1]$, as was done in Proposition 5.3.8.

We now turn to the details of the construction. Consider the orthogonal decomposition with respect to the standard L^2 scalar product

$$h^1(\square_{2n}) = \bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n) \overset{\perp}{\oplus} H, \tag{5.4.14}$$

where $H := \left(\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)\right)^\perp$ is the vector space of functions which are constant on the subcubes $z+\square_n$. Its dimension is 3^{dn}.

Then consider L the linear operator defined on $\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$ valued in $h^1_0(\square^+_{2n})$ defined according to the following procedure.

For each $\psi \in \bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$, let $L(\psi)$ be the unique solution to

$$\begin{cases} \Delta L(\psi) = \operatorname{div} f \text{ in } \square_{2n}, \\ L(\psi) \in h^1_0(\square^+_{2n}). \end{cases} \tag{5.4.15}$$

where f is the vector field defined by, for each $e \subseteq \square_{2n}$,

$$f(e) = \begin{cases} \nabla\psi(e) & \text{if } e \subseteq z+\square_n, \text{ for some } z\in\mathcal{Z}_{n,2n}, \\ 0 & \text{if } e \in B_{2n,n}. \end{cases}$$

We first verify that the operator L is injective. To this end, we check that the kernel of L is reduced to $\{0\}$. Let $\psi \in \bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$ such that $L(\psi) = 0$, we want to prove that $\psi = 0$.

First by definition of L, the condition $L(\psi) = 0$ implies $\operatorname{div} f = 0$. But the function $\operatorname{div} f$ can be computed explicitly and we have, for each $z \in \mathcal{Z}_{n,2n}$

$$\Delta_{z+\square_n}\psi = 0 \text{ in } z+\square_n,$$

where $\Delta_{z+\square_n}$ is the Laplacian on the graph $(z+\square_n)$ and is defined by, for each $x \in \square_n$,

$$\Delta_{z+\square_n}\psi(x) = \sum_{y\sim x,\, y\in z+\square_n} (\psi(y)-\psi(x)).$$

Note that this Laplacian is different from the standard Laplacian on $\square^+_{2n}$ which is used in (5.4.15) and defined by, for each $x \in \square_{2n}$,

$$\Delta\psi(x) = \sum_{y\sim x} (\psi(y)-\psi(x)).$$

This difference is fundamental and comes from the fact that f was set to be 0 on the edges of $B_{2n,n}$: for the Laplacian on the cube $z+\square_n$, one is allowed to apply the maximum principle to derive, for each $z \in \mathcal{Z}_{n,2n}$,

$$\Delta_{z+\square_n}\psi = 0 \text{ in } z+\square_n \implies \psi \text{ is constant in } z+\square_n.$$

Combining the previous remark with the assumption $\psi \in \bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$ gives $\psi = 0$ and thus $\ker L = \{0\}$. In particular, if we denote by $\operatorname{im} L$ the image of L, one has

$$\dim(\operatorname{im} L) = \dim\left(\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)\right) = 3^{2dn} - 3^{dn}.$$

We then extend L into an isomorphism of $h^1(\square_{2n})$ into $h^1_0(\square^+_{2n})$. Recall that one has the orthogonal decomposition

$$h^1(\square_{2n}) = \bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n) \overset{\perp}{\oplus} H,$$

and consider an orthonormal basis $h_1, \ldots, h_{3^{nd}}$ of H. Consider now the L^2 orthogonal decomposition

$$h^1(\square_{2n}) = \operatorname{im} L \oplus (\operatorname{im} L)^\perp. \tag{5.4.16}$$

By the injectivity of L, we have $\dim(\operatorname{im} L)^\perp = 3^{dn}$. Let $\widetilde{h}_1, \ldots, \widetilde{h}_{3^{nd}}$ be an orthonormal basis of $(\operatorname{im} L)^\perp$. the linear operator L is then extended to the full space $h^1(\square_{2n})$ by setting

$$L(h_i) = \widetilde{h}_i, \ \forall i \in \{1, \ldots, 3^{dn}\}. \tag{5.4.17}$$

By construction, the linear mapping L is an isomorphism between $h^1(\square_{2n})$ and $h^1_0(\square^+_{2n})$.

We now construct the random variable κ^+_{2n} using the operator L. To this end, consider two families $(\psi_z)_{z\in\mathcal{Z}_{n,2n}}$ and $(X_i)_{i=1,\ldots,3^{nd}}$ of random variables satisfying

- for each $z \in \mathcal{Z}_{n,2n}$, ψ_z is valued in $\mathring{h}^1(z+\square_n)$ and its law is $\mathbb{P}^*_{n,q}$. We extend it by 0 outside $z+\square_n$ so that it can be seen as a random variable taking values in $h^1(\square_{2n})$,
- for each $i \in \{1, \ldots, 3^{nd}\}$, X_i is valued in $[0,1]$ and its law is $\mathrm{Unif}[0,1]$,
- the random variables ψ_z and X_i are independent.

We also define for each $z \in \mathcal{Z}_{n,2n}$ the random variable σ_z taking values in $\mathring{h}^1(\square_{2n})$ defined by subtracting the affine function of slope $\nabla\nu^*(q)$ to ψ_z, i.e. for each $x \in \square_{2n}$,

$$\sigma_z(x) := \begin{cases} \psi_z(x) - \nabla\nu^*_n(q)\cdot(x-z) & \text{if } x \in z+\square_n, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.18}$$

Let κ and κ^+_{2n} be the random variables valued in $h^1(\square_{2n})$ defined by

$$\kappa := L\left(\sum_{z\in\mathcal{Z}_{n,2n}} \sigma_z\right) \quad \text{and} \quad \kappa^+_{2n} := L\left(\sum_{z\in\mathcal{Z}_{n,2n}} \sigma_z + \sum_{i=1}^{3^{nd}} X_i h_i\right). \tag{5.4.19}$$

With this definition, it is clear that κ^+_{2n} is a random variable valued in $h^1_0(\square^+_{2n})$ and that its law is absolutely continuous with respect to the Lebesgue measure on this space. Moreover by construction of the operator L, the random variable κ^+_{2n} is one of the functions of $h^1_0(\square^+_{2n})$ which is the closest in the L^2 norm to the vector field $\mathbf{f}$ defined in (5.4.12). The definition of κ^+_{2n} presented in the previous paragraph required two additional technical steps:

(1) First one had to extend the operator L by considering orthonormal basis of the spaces H and $(\operatorname{im} L)^\perp$ in (5.4.16),
(2) Second one had to add a sum $\sum_{i=1}^{3^{nd}} X_i h_i$.

These two additional steps are required only to be sure that the entropy of κ^+_{2n} is finite and they are constructed so as not to impact the analysis.

Step 2. The main objective of this step is to compute the entropy of the random variable κ^+_{2n} constructed in the previous step.

Using the canonical bijection between $h^1(\square_{2n})$ and $h^1_0(\square^+_{2n})$, one can see L as an automorphism of $h^1_0(\square^+_{2n})$. Combining this remark with the change of variables formula for the differential entropy, one computes the entropy of κ^+_{2n},

$$H\left(\mathbb{P}_{\kappa^+_{2n}}\right) = H\left(\mathbb{P}_{\sum_{z\in\mathcal{Z}_{n,2n}}\sigma_z+\sum_{i=1}^{3^{nd}}X_ih_i}\right) - \ln|\det L|. \tag{5.4.20}$$

We first focus on the first term on the right-hand side. By construction of σ_z, X_i, h_i and using the formula to compute the entropy of two independent random variables given in Proposition 5.2.4, one has

$$H\left(\mathbb{P}_{\sum_{z\in\mathcal{Z}_{n,2n}}\sigma_z+\sum_{i=1}^{3^{nd}}X_ih_i}\right) = \sum_{z\in\mathcal{Z}_{n,2n}} H\left(\mathbb{P}_{\sigma_z}\right) + \sum_{i=1}^{3^{nd}} H\left(\mathbb{P}_{X_i}\right).$$

Using that the law of X_i is uniform in $[0,1]$ and since the entropy of a random variable is translation invariant the previous display can be further simplified

$$H\left(\mathbb{P}_{\sum_{z\in\mathcal{Z}_{n,2n}}\sigma_z+\sum_{i=1}^{3^{nd}}X_ih_i}\right) = \sum_{z\in\mathcal{Z}_{n,2n}} H\left(\mathbb{P}_{\psi_z}\right).$$

Since the law of ψ_z is $\mathbb{P}^*_{n,q}$, one obtains

$$H\left(\mathbb{P}_{\kappa^+_{2n}}\right) = 3^{dn} H\left(\mathbb{P}^*_{n,q}\right) - \ln|\det L|. \tag{5.4.21}$$

We now focus on the second term on the right-hand side of (5.4.20). More precisely, we prove that the logarithm of the determinant of L is small compared to $|\square_n|$: more precisely, we show the bound,

$$|\ln|\det L|| \leq C3^{(2d-1)n}n. \tag{5.4.22}$$

Combining (5.4.21) with (5.4.22) gives the main result (5.4.10) of Step 2.

To prove (5.4.22), note that the dimension of the vector space $h^1_0(\square^+_{2n})$ is 3^{2dn}. Denote by $(l_1,\ldots,l_{3^{2dn}})$ the (potentially complex) eigenvalues of L. Note that since L is bijective, none of these eigenvalues is equal to 0. With this notation the determinant of L can be computed,

$$\ln|\det L| = \sum_{i=1}^{3^{2dn}} \ln|l_i|. \tag{5.4.23}$$

To prove that the logarithm of the determinant of L is small, the strategy relies on the two following ingredients:

(1) One shows that most of the eigenvalues of L are equal to 1,
(2) One then shows that the remaining eigenvalues are bounded from above and below by $C3^{4n}$ and $c3^{-4n}$ respectively.

We first show the first item: most of the eigenvalues l_i are equal to 1. To this end, we consider the interior of the cube $\square_n$:

$$\square^\circ_n := \left(-\frac{3^n-2}{2}, \frac{3^n-2}{2}\right)^d \cap \mathbb{Z}^d = \square_n \setminus \partial\square_n.$$

In particular, one has $\square^\circ_n \subseteq \square_n$ and thus $\mathring{h}^1(\square^\circ_n)$ is a linear subspace of $\mathring{h}^1(\square_n)$. This implies that $\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square^\circ_n)$ is a linear subspace of $\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$. The important observation is that for each $\psi \in \bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square^\circ_n)$ and each edge $e\in B_{2n,n}$,

$$\nabla\psi(e) = 0.$$

This is due to the fact that, by definition of the space $\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square^\circ_n)$, any function belonging to this space is equal to 0 on the boundaries of the cubes $(z+\square_n)$, for any $z\in\mathcal{Z}_{n,2n}$. Consequently the vector field f defined from ψ according to the formula

$$f(e) = \begin{cases} \nabla\psi(e) & \text{if } e \subseteq z + \square_n, \text{ for some } z \in \mathcal{Z}_{n,2n}, \\ 0 & \text{if } e \in B_{2n,n}. \end{cases} \tag{5.4.24}$$

satisfy

$$f = \nabla\psi.$$

Thus $L(\psi)$ is the solution of

$$\begin{cases} \Delta L(\psi) = \Delta\psi \text{ in } \square_{2n}, \\ L(\psi) \in h_0^1\left(\square_{2n}^+\right). \end{cases}$$

This implies $L(\psi) = \psi$ and we just proved

$$\psi \in \bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1\left(z + \square_n^{\mathrm{o}}\right), \ L(\psi) = \psi.$$

Consequently, the vector space $\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1\left(z + \square_n^{\mathrm{o}}\right)$ is an eigenspace for L associated to the eigenvalue 1, its dimension can be estimated by the following computation

$$\begin{aligned} \dim\left(\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1\left(z + \square_n^{\mathrm{o}}\right)\right) &= \sum_{z\in\mathcal{Z}_{n,2n}} \dim\left(\mathring{h}^1\left(z + \square_n^{\mathrm{o}}\right)\right) \\ &= 3^{dn} \dim\left(\mathring{h}^1\left(\square_n^{\mathrm{o}}\right)\right) \\ &= 3^{dn}\left(\left|\square_n^{\mathrm{o}}\right| - 1\right). \end{aligned}$$

The volume of $\square_n^{\mathrm{o}}$ can then be estimated according to

$$\left|\square_n^{\mathrm{o}}\right| \geq 3^{dn} - C3^{(d-1)n}.$$

Combining the two previous displays gives

$$\dim\left(\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1\left(z + \square_n^{\mathrm{o}}\right)\right) \geq 3^{2dn} - C3^{(2d-1)n}. \tag{5.4.25}$$

Thus we can, without loss of generality, assume that for each $i \geq C3^{(2d-1)n}$, $l_i = 1$. Using this, the equality (5.4.23) can be rewritten

$$\ln|\det L| = \sum_{i=1}^{C3^{(2d-1)n}} \ln|l_i|.$$

We then use the inequalities

$$\left|\left|\left|L^{-1}\right|\right|\right|^{-1} \leq \inf_{i\in\{1,\ldots,3^{2dn}\}} |l_i| \leq \sup_{i\in\{1,\ldots,3^{2dn}\}} |l_i| \leq |||L|||,$$

where $|||L|||$ (resp. $|||L^{-1}|||$) denotes the operator norm of L (resp. L^{-1}) with respect to the L^2 norm on $h_0^1\left(\square_{2n}^+\right)$. A combination of the two previous displays gives

$$|\ln|\det L|| \leq C3^{(2d-1)n} \max\left(\ln|||L|||, \ln\left|\left|\left|L^{-1}\right|\right|\right|\right). \tag{5.4.26}$$

To complete the proof, there remains to prove an estimate on the operator norms of L and L^{-1}. Specifically, we are going to prove,

$$|||L||| \leq C3^{2n} \qquad \text{and} \qquad \left|\left|\left|L^{-1}\right|\right|\right| \leq C3^{2n}. \tag{5.4.27}$$

We first focus on the estimate of the operator norm of L. Let $\phi \in h_0^1\left(\square_{2n}^+\right)$ such that $\sum_{x\in\square_{2n}^+} \phi(x)^2 \leq 1$, one aims to prove

$$\sum_{x\in\square_{2n}^+} |L(\phi)(x)|^2 \leq C3^{4n}. \tag{5.4.28}$$

To this end, we decompose ϕ according to the orthogonal decomposition (5.4.14). This gives

$$\phi = \psi + h, \text{ with } \psi \in \bigoplus_{z \in \mathcal{Z}_{n,2n}} \mathring{h}^1(z + \square_n) \text{ and } h \in H.$$

In particular,

$$\sum_{x \in \square_{2n}^+} |\psi(x)|^2 + \sum_{x \in \square_{2n}^+} |h(x)|^2 = \sum_{x \in \square_{2n}^+} |\phi(x)|^2 \le 1$$

By definition of L, $L(\psi)$ and $L(h)$ are orthogonal in $h_0^1(\square_{2n}^+)$ and

$$\sum_{x \in \square_{2n}^+} |L(h)(x)|^2 = \sum_{x \in \square_{2n}^+} |h(x)|^2$$

From this we deduce

$$\begin{aligned}\sum_{x \in \square_{2n}^+} |L(\phi)(x)|^2 = \sum_{x \in \square_{2n}^+} |L(\psi)(x)|^2 + \sum_{x \in \square_{2n}^+} |L(h)(x)|^2 &= \sum_{x \in \square_{2n}^+} |L(\psi)(x)|^2 + \sum_{x \in \square_{2n}^+} |h(x)|^2 \\ &\le \sum_{x \in \square_{2n}^+} |L(\psi)(x)|^2 + 1.\end{aligned}$$

Thus to prove (5.4.28), it is sufficient to prove:

$$\forall \psi \in \bigoplus_{z \in \mathcal{Z}_{n,2n}} \mathring{h}^1(z + \square_n) \text{ such that } \sum_{x \in \square_{2n}^+} |\psi(x)|^2 \le 1, \ \sum_{x \in \square_{2n}^+} |L(\psi)(x)|^2 \le C 3^{4n}.$$

For each $\psi \in \bigoplus_{z \in \mathcal{Z}_{n,2n}} \mathring{h}^1(z + \square_n)$, we know that $L(\psi)$ is a solution to

$$\begin{cases} \Delta L(\psi) = \operatorname{div} f \text{ in } \square_{2n}, \\ L(\psi) \in h_0^1(\square_{2n}^+). \end{cases} \tag{5.4.29}$$

where f is defined by

$$f(e) = \begin{cases} \nabla \psi(e) & \text{if } e \subseteq z + \square_n, \text{ for some } z \in \mathcal{Z}_{n,2n}, \\ 0 & \text{if } e \in B_{2n,n}. \end{cases}$$

Consequently testing $L(\psi)$ against itself in (5.4.29) shows

$$\sum_{e \subseteq \square_{2n}^+} |\nabla L(\psi)(e)|^2 = \sum_{e \subseteq \square_{2n}^+} \nabla L(\psi)(e) f(e).$$

By the Cauchy-Schwarz inequality, this implies

$$\sum_{e \subseteq \square_{2n}^+} |\nabla L(\psi)(e)|^2 \le \sum_{e \subseteq \square_{2n}^+} |f(e)|^2$$

and by definition of f, one obtains

$$\sum_{e \subseteq \square_{2n}^+} |\nabla L(\psi)(e)|^2 \le \sum_{z \in \mathcal{Z}_{n,2n}} \sum_{e \subseteq z + \square_n} |\nabla \psi(e)|^2. \tag{5.4.30}$$

Using the crude inequality, for each $e = (x, y) \subseteq \square_{2n}$

$$|\nabla \psi(e)|^2 = |\psi(x) - \psi(y)|^2 \le 2|\psi(x)|^2 + 2|\psi(y)|^2,$$

we derive the estimate,

$$\sum_{z \in 3^n \mathbb{Z}^d \cap \square_{2n}} \sum_{e \subseteq z + \square_n} |\nabla \psi(e)|^2 \le C \sum_{x \in \square_{2n}} |\psi(x)|^2.$$

Combining the previous displays and using the assumption $\sum_{x \in \square_{2n}} |\psi(x)|^2 \le 1$ shows

$$\sum_{e \subseteq \square_{2n}^+} |\nabla L(\psi)(e)|^2 \le C \sum_{x \in \square_{2n}} |\psi(x)|^2 \le C.$$

Also, since $L(\psi) \in h_0^1(\square_{2n}^+)$, one has by the Poincaré inequality

$$\sum_{x\in\square_{2n}^+} |L(\psi)(x)|^2 \leq 3^{4n} \sum_{e\subseteq\square_{2n}^+} |\nabla L(\psi)(e)|^2 .$$

Combining the two previous displays gives

$$\sum_{x\in\square_{2n}^+} |L(\psi)(x)|^2 \leq C3^{4n}.$$

This is the desired result. We now turn to the bound on the operator norm of L^{-1}, we aim to prove

$$\left|\left|\left|L^{-1}\right|\right|\right| \leq C3^{2n}.$$

To this end, let $\psi \in h_0^1(\square_{2n}^+)$, we will prove

$$\sum_{x\in\square_{2n}^+} |\psi(x)|^2 \leq C3^{4n} \sum_{x\in\square_{2n}^+} |L(\psi)(x)|^2 . \tag{5.4.31}$$

First, using the same idea as in the proof of the bound for the operator norm of L, we see that it is enough to prove (5.4.31) under the additional assumption $\psi \in \bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$. In this case, one has

$$\begin{cases} \Delta L(\psi) = \operatorname{div} f \text{ in } \square_{2n}, \\ \quad L(\psi) \in h_0^1(\square_{2n}^+) . \end{cases}$$

where f is the vector field defined by

$$f(e) = \begin{cases} \nabla\psi(e) & \text{if } e \subseteq z+\square_n, \text{ for some } z \in 3^n\mathbb{Z}^d \cap \square_{2n}, \\ \quad 0 & \text{if } e \in B_{2n,n}. \end{cases}$$

Testing this equation against ψ gives

$$\sum_{x\in\square_{2n}} \Delta L(\psi)(x)\psi(x) = \sum_{x\in\square_{2n}} \operatorname{div} f(x)\psi(x) = \sum_{e\subseteq\square_{2n}} f(e)\nabla\psi(e).$$

We then use the definition of ψ to get

$$\sum_{e\subseteq\square_{2n}} f(e)\nabla\psi(e) = \sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla\psi(e)|^2 .$$

Since ψ belongs to $\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$, it has mean 0 on each of the subcubes $z+\square_n$. We can thus apply the Poincaré inequality on each of the subcubes $z+\square_n$ to get, for some $C := C(d) < \infty$,

$$\sum_{x\in z+\square_n} |\psi(x)|^2 \leq C3^{2n} \sum_{e\subseteq z+\square_n} |\nabla\psi(e)|^2 , \qquad \forall z \in \mathcal{Z}_{n,2n}.$$

Summing the previous inequality over each $z \in \mathcal{Z}_{n,2n}$ gives

$$\sum_{x\in\square_{2n}} |\psi(x)|^2 \leq C3^{2n} \sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla\psi(e)|^2 .$$

Combining the few previous displays gives

$$\sum_{x\in\square_{2n}} |\psi(x)|^2 \leq C3^{2n} \sum_{x\in\square_{2n}} \Delta L(\psi)(x)\psi(x).$$

By the Cauchy-Schwarz inequality, one further obtains

$$\sum_{x\in\square_{2n}} |\psi(x)|^2 \leq C3^{4n} \sum_{x\in\square_{2n}} |\Delta L(\psi)(x)|^2 .$$

But by definition of the Laplacian, one has, for each $x \in \square_{2n}$

$$|\Delta L(\psi)(x)|^2 = \left|\sum_{y\sim x} (\psi(y) - \psi(x))\right|^2 \leq C \sum_{y\sim x} |\psi(y)|^2 + C|\psi(x)|^2.$$

From this obtain,

$$\sum_{x\in\square_{2n}} |\Delta L(\psi)(x)|^2 \leq C \sum_{x\in\square_{2n}} |L(\psi)(x)|^2$$

and consequently

$$\sum_{x\in\square_{2n}} |\psi(x)|^2 \leq C3^{4n} \sum_{x\in\square_{2n}} |L(\psi)(x)|^2 .$$

This is (5.4.31).

We now complete the proof of the bound $|\ln|\det L||$ stated in (5.4.22). Indeed combining (5.4.26) and (5.4.27) gives

$$\begin{aligned}|\ln|\det L|| &\leq C3^{(2d-1)n} \max\left(\ln |||L|||, \ln \left|\left|\left|L^{-1}\right|\right|\right|\right) \\ &\leq C3^{(2d-1)n} \ln\left(C3^{2n}\right) \\ &\leq C3^{(2d-1)n} n.\end{aligned}$$

The proof of Step 2 is complete.

Step 3. The goal of this step is to show the following estimate

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|} \sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e) - \nabla\kappa_{2n}^+(e)|^2\right] \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q)\right), \tag{5.4.32}$$

where we recall that $\mathbf{f}$ is the random vector field defined in (5.4.12). To achieve this, we proceed as follows.

- We first remove the additional random variable $L\left(\sum_{i=1}^{3^{nd}} X_i h_i\right)$. Precisely we prove

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|} \sum_{x\subseteq\square_{2n}^+} |\kappa_{2n}^+(x) - \kappa(x)|^2\right] \leq 3^{-dn},$$

  where the random variable κ is defined in (5.4.19).
- Then we construct a random function Ψ taking values in $h_0^1\left(\square_{2n}^+\right)$ such that

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|} \sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e) - \nabla\Psi(e)|^2\right] \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q)\right), \tag{5.4.33}$$

  for some small exponent $\beta := \beta(d) > 0$.
- We deduce from (5.4.33) that

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|} \sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e) - \nabla\kappa(e)|^2\right] \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q)\right).$$

Step 1. We first prove that we can remove the additional random variable $L\left(\sum_{i=1}^{3^{nd}} X_i h_i\right)$ which was added to κ to obtain κ_{2n}^+. These random variables were added so that the law of κ_{2n}^+ is absolutely continuous with respect to the Lebesgue measure on $h_0^1\left(\square_{2n}^+\right)$, in order not to obtain an infinite entropy. They were also chosen in a way that their role in the energy is negligible. More precisely we prove the following statement

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|} \sum_{x\subseteq\square_{2n}^+} |\kappa_{2n}^+(x) - \kappa(x)|^2\right] \leq 3^{-dn}.$$

We first recall the definition of L on the subvector space H given in (5.4.17). The previous estimate is then a consequence of the following computation

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{x\in\square_{2n}^{+}}\left|\kappa_{2n}^{+}(x)-\kappa(x)\right|^{2}\right]=\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{x\in\square_{2n}^{+}}\left|\sum_{i=1}^{3^{nd}}X_i\widetilde{h}_i(x)\right|^{2}\right]$$
$$=\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{i=1}^{3^{nd}}\left|X_i\right|^{2}\right],$$

Since the family $\widetilde{h}_i$, for $i\in\{1,\ldots,3^{nd}\}$ is orthonormal with respect to the standard L^2 scalar product in $h_0^1\left(\square_{2n}^{+}\right)$ and since the random variables $(X_i)_{i\in\{1,\ldots,3^{nd}\}}$ are i.i.d of law uniform in $[0,1]$, one can complete the computation

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{x\in\square_{2n}^{+}}\left|\kappa_{2n}^{+}(x)-\kappa(x)\right|^{2}\right]=\frac{3^{nd}}{3\left|\square_{2n}^{+}\right|}\le C3^{-dn}. \tag{5.4.34}$$

Using this and the inequality, for each $e=(x,y)\subseteq\square_{2n}$

$$\left|\nabla\left(\kappa_{2n}^{+}-\kappa\right)(e)\right|^{2}=\left|\left(\kappa_{2n}^{+}-\kappa\right)(x)-\left(\kappa_{2n}^{+}-\kappa\right)(y)\right|^{2}\le 2\left|\left(\kappa_{2n}^{+}-\kappa\right)(x)\right|^{2}+2\left|\left(\kappa_{2n}^{+}-\kappa\right)(y)\right|^{2},$$

one derives

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{e\subseteq\square_{2n}^{+}}\left|\nabla\kappa_{2n}^{+}(e)-\nabla\kappa(e)\right|^{2}\right]\le C3^{-dn}. \tag{5.4.35}$$

The proof of Step 1 is complete

Step 2. We now prove (5.4.33) and construct the random variable Ψ. We recall the definition of the family (ψ_z), for $z\in\mathcal{Z}_{n,2n}$ and extend it to each $z\in 3^n\mathbb{Z}^d$ according to

(i) for each $z\in 3^n\mathbb{Z}^d$, ψ_z is a random function from $\mathbb{Z}^d$ to $\mathbb{R}$ equal to 0 outside $z+\square_n$ and the law of $\psi_z(\cdot-z)$ restricted to $\square_n$ is $\mathbb{P}_{n,q}^{*}$.
(ii) the random variables ψ_z are independent.

It is the same family as in Step 1, except that it was extended to each $z\in 3^n\mathbb{Z}^d$ and not only for $z\in\mathcal{Z}_{n,2n}$. The reason behind this extension will become clear later in the proof.

We also define

$$\psi:=\sum_{z\in 3^n\mathbb{Z}^d}\psi_z.$$

Moreover for each $z\in\mathcal{Z}_{n,2n}$, we let $\psi_{z,n+1}$be a random variable such that

$$\psi_{z,n+1}\text{ is valued in }\mathring{h}^1\left(z+\square_{n+1}\right)\text{ and the law of }\psi_{z,n+1}\left(\cdot-z\right)\text{ is }\mathbb{P}_{n+1,q}^{*}.$$

As usual we extend this function by 0 outside $z+\square_{n+1}$ so that one can see $\psi_{z,n+1}$ as a random function from $\mathbb{Z}^d$ to $\mathbb{R}$.

The goal of the following argument is to construct a suitable coupling between the random variables $\psi_{z,n+1}$, for $z\in\mathcal{Z}_{n,2n}$.

For some fixed $z\in\mathcal{Z}_{n,2n}$, we apply Proposition 5.3.9 and Proposition 5.2.11, with the random variables $X=\psi$, $Y=\sum_{z'\in 3^n\mathbb{Z}^d\setminus(z+\square_{n+1})}\psi_{z'}$ and $Z=\psi_{z,n+1}$; we obtain that there exists a coupling between the random variables ψ and $\psi_{z,n+1}$ such that

$$\mathbb{E}\left[\frac{1}{\left|\square_n\right|}\sum_{z'\in 3^n\mathbb{Z}^d\cap(z+\square_{n+1})}\sum_{e\subseteq z'+\square_n}\left|\nabla\psi_{z,n+1}(e)-\nabla\psi_{z'}(e)\right|^{2}\right]\le C\tau_n^{*}(q)+C(1+|q|^2)3^{-n}. \tag{5.4.36}$$

This is where we used that $\psi_{z'}$ is defined for some z' outside $\square_{2n}$. Indeed for some $z\in 3^n\mathbb{Z}^d\cap\square_{2n}$, close to the boundary of $\square_{2n}$, the set $3^n\mathbb{Z}^d\cap(z+\square_{n+1})$ is not included in $\square_{2n}$.

Thanks to the previous argument, we have constructed, for each $z \in \mathcal{Z}_{n,2n}$, a coupling between ψ and $\psi_{z,n+1}$. Let $(z_1, \ldots, z_{3^{nd}})$ be an enumeration of the elements of $\mathcal{Z}_{n,2n}$.

Applying Proposition 5.2.11 with $X = \psi$, $Y = \psi_{z_1,n+1}$ and $Z = \psi_{z_2,n+1}$ constructs a coupling between ψ, $\psi_{z_1,n+1}$ and $\psi_{z_2,n+1}$ such that (5.4.36) is satisfied.

We then apply Proposition 5.2.11 a second time, with this time the random variables $X = (\psi_{z_1,n+1}, \psi_{z_2,n+1})$, $Y = \psi$ and $Z = \psi_{z_3,n+1}$ to construct a coupling between $\psi, \psi_{z_1,n+1}, \psi_{z_2,n+1}$ and $\psi_{z_3,n+1}$ such that (5.4.36) is satisfied.

Iterating this construction 3^{nd} times constructs a coupling between the random variables ψ and $\psi_{z,n+1}$, for $z \in \mathcal{Z}_{n,2n}$, such that (5.4.36) is satisfied.

From the previous construction, we derive a coupling between the random variables $\psi_{z,n+1}$, for $z \in \mathcal{Z}_{n,2n}$, which is what we wanted to design. Moreover this coupling satisfies (5.4.36), which will be a key ingredient later in the proof.

We now build the function Ψ by patching together the random variables $\psi_{z,n+1}$. The argument relies on a partition of unity: we let $\chi_0 \in h_0^1(\square_n)$ be a cutoff function satisfying

$$0 \leq \chi_0 \leq C3^{-dn}, \qquad \sum_{x \in \mathbb{Z}^d} \chi_0(x) = 1, \qquad |\nabla \chi_0| \leq C3^{-(d+1)n}, \qquad \operatorname{supp} \chi_0 \subseteq \frac{1}{2} \square_n .$$

We then define, for each $z \in \mathbb{Z}^d$

$$\chi(y) := \sum_{x \in \square_n} \chi_0(y - x).$$

Note that χ is supported in $\frac{3}{4}\square_{n+1}$, satisfies $0 \leq \chi \leq 1$ and the translations of χ form a partition of unity:

$$\sum_{z \in 3^n \mathbb{Z}^d} \chi(\cdot - z) = 1.$$

Moreover, one has the bound on the gradient of χ

$$|\nabla \chi| \leq C3^{-n}. \tag{5.4.37}$$

We next consider the cutoff function $\zeta \in h_0^1(\square_{2n}^+)$ satisfying

$$0 \leq \zeta \leq 1, \qquad \zeta = 1 \text{ on } \left\{x \in \square_{2n} \ : \ \operatorname{dist}(x, \partial \square_{2n}) \geq 3^n\right\}, \qquad |\nabla \zeta| \leq C3^{-n}, \tag{5.4.38}$$

which will be used to remove a boundary layer in the patching construction. We also define the following discrete set

$$\mathcal{Z}_{n,2n} := \left\{z \in 3^n \mathbb{Z}^d \ : \ z \in \mathcal{Z}_{n,2n} \text{ or } \operatorname{dist}(z, \partial \square_{2n}) \leq 3^n\right\}.$$

It represents the set $\mathcal{Z}_{n,2n}$ with an additional boundary layer of size 1 of points in $3^n\mathbb{Z}^d$ around it. We are going to use this set because it satisfies the following property

$$\forall y \in \square_{2n}^+, \quad \sum_{z \in \mathcal{Z}_{n,2n}^+} \chi(y - z) = 1$$

We then define the function Ψ by

$$\Psi(x) = \zeta(x) \sum_{z \in \mathcal{Z}_{n,2n}^+} \chi(x - z) \left(\psi_{z,n+1}(x) - \nabla \nu_{n+1}^*(q) \cdot (x - z)\right).$$

Now that Ψ has been constructed, we prove (5.4.33). The main ingredients to prove this estimate are (5.4.36), Proposition 5.4.4 and the interior Meyers estimate, Proposition 5.B.5 stated in Appendix 5.B.

We first compute the derivative of Ψ. An explicit computation gives, for each $e = (x,y) \subseteq \square_{2n}$,

$$\nabla\Psi(e) = \zeta(y) \sum_{z\in\mathcal{Z}^+_{n,2n}} \chi(y-z)\left(\nabla\psi_{z,n+1}(e) - \nabla\nu^*_{n+1}(q)(e)\right) + \zeta(y)\sum_{z\in\mathcal{Z}^+_{n,2n}} \nabla\chi(\cdot - z)(e)\left(\psi_{z,n+1}(x) - \nabla\nu^*_{n+1}(q)\cdot x\right) + \nabla\zeta(e)\sum_{z\in\mathcal{Z}^+_{n,2n}}\chi(x-z)\left(\psi_{z,n+1}(x) - \nabla\nu^*_{n+1}(q)\cdot x\right) \tag{5.4.39}$$

The second and third terms in the previous display are error terms, which will be proved to be small: the interesting term is the first one. The L^2 norm of second term can be estimated thanks to the bound (5.4.37) on the gradient of χ,

$$\begin{aligned}\mathbb{E}&\left[\frac{1}{\left|\square^+_{2n}\right|}\sum_{e=(x,y)\subseteq\square^+_{2n}}\left|\zeta(y)\sum_{z\in\mathcal{Z}^+_{n,2n}}\nabla\chi(\cdot-z)(e)\left(\psi_{z,n+1}(x)-\nabla\nu^*_{n+1}(q)\cdot x\right)\right|^2\right]\\ &\leq C3^{-2n}\mathbb{E}\left[\frac{1}{\left|\square^+_{2n}\right|}\sum_{z\in\mathcal{Z}^+_{n,2n}}\sum_{x\in z+\square_{n+1}}\left|\psi_{z,n+1}(x)-\nabla\nu^*_{n+1}(q)\cdot x\right|^2\right]\\ &\leq C3^{-2n}\mathbb{E}\left[\frac{1}{\left|\square_{n+1}\right|}\sum_{x\in\square_{n+1}}\left|\psi_{n+1,q}(x)-\nabla\nu^*_{n+1}(q)\cdot x\right|^2\right].\end{aligned}$$

We then apply Proposition 5.4.4 to obtain

$$\mathbb{E}\left[\frac{1}{\left|\square^+_{2n}\right|}\sum_{e=(x,y)\subseteq\square^+_{2n}}\left|\zeta(y)\sum_{z\in\mathcal{Z}^+_{n,2n}}\nabla\chi(\cdot-z)(e)\left(\psi_{z,n+1}(x)-\nabla\nu^*_{n+1}(q)\cdot x\right)\right|^2\right] \leq C\left((1+|q|^2)3^{-\frac{n}{2}} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}}\tau^*_m(q)\right).$$

The third term of (5.4.39) can be estimated in a similar manner, using (5.4.38) this time,

$$\begin{aligned}\mathbb{E}&\left[\frac{1}{\left|\square^+_{2n}\right|}\sum_{e=(x,y)\subseteq\square^+_{2n}}\left|\nabla\zeta(e)\sum_{z\in\mathcal{Z}^+_{n,2n}}\chi(x-z)\left(\psi_{z,n+1}(x)-\nabla\nu^*_{n+1}(q)\cdot x\right)\right|^2\right]\\ &\leq C3^{-2n}\mathbb{E}\left[\frac{1}{\left|\square^+_{2n}\right|}\sum_{z\in\mathcal{Z}^+_{n,2n}}\sum_{x\in z+\square_{n+1}}\left|\psi_{z,n+1}(x)-\nabla\nu^*_{n+1}(q)\cdot x\right|^2\right]\\ &\leq C3^{-2n}\mathbb{E}\left[\frac{1}{\left|\square_{n+1}\right|}\sum_{x\in\square_{n+1}}\left|\psi_{n+1,q}(x)-\nabla\nu^*_{n+1}(q)\cdot x\right|^2\right].\end{aligned}$$

We then apply Proposition 5.4.4 again to obtain

$$\mathbb{E}\left[\frac{1}{\left|\square^+_{2n}\right|}\sum_{e=(x,y)\subseteq\square^+_{2n}}\left|\nabla\zeta(e)\sum_{z\in\mathcal{Z}^+_{n,2n}}\chi(x-z)\left(\psi_{z,n+1}(x)-\nabla\nu^*_{n+1}(q)\cdot x\right)\right|^2\right] \leq C\left((1+|q|^2)3^{-\frac{n}{2}} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}}\tau^*_m(q)\right).$$

Combining the few previous displays then yields

$$\mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|\nabla\Psi(e)-\zeta(y)\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right] \leq C\left((1+|q|^2)3^{-\frac{n}{2}}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).$$

Thus to prove (5.4.33), it is sufficient to prove

$$\mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|\mathbf{f}(e)-\zeta(y)\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right] \leq C\tau_n^*(q)+C(1+|q|^2)3^{-\beta n}, \tag{5.4.40}$$

for some small exponent $\beta := \beta(d,\lambda) > 0$. We simplify the previous display by removing the function ζ. Note that if we denote

$$\partial\mathcal{Z}_{n,2n} := \left\{z\in\mathcal{Z}_{n,2n}^+ \ : \ \operatorname{dist}(z,\partial\square_{2n})\leq 2\cdot 3^n\right\},$$

then we have the following computation, using the properties of the functions ζ and χ,

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|(1-\zeta(y))\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right] \\
&\quad\leq \mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|\sum_{z\in\partial\mathcal{Z}_{n,2n}}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right] \\
&\quad\leq \mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\sum_{z\in\partial\mathcal{Z}_{n,2n}}\chi(y-z)\left|\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right|^2\right]
\end{aligned}$$

Using that, by definition, the function ζ is equal to 1 on the set $\{x\in\square_{2n} \ : \ \operatorname{dist}(x,\partial\square_{2n})\geq 3^n\}$, we can simplify the previous display and obtain

$$\mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|(1-\zeta(y))\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right] \leq \mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{z\in\partial\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_{n+1}}\left|\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right|^2\right].$$

Using that all the $\psi_{z,n+1}$ have the same law, which is $\mathbb{P}_{n+1,q}^*$, one has

$$\mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|(1-\zeta(y))\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right] \leq \frac{|\partial\mathcal{Z}_{n,2n}|}{|\square_{2n}^+|}\mathbb{E}\left[\sum_{e\subseteq\square_{n+1}}\left|\nabla\psi_{n+1,q}(e)-\nabla\nu_{n+1}^*(q)(e)\right|^2\right].$$

But by (5.3.8) of Proposition 5.3.1, the term on the right-hand side is bounded and one has

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|(1-\zeta(y))\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right] \leq \frac{\left|\partial\mathcal{Z}_{n,2n}\right|\cdot\left|\square_{n+1}\right|}{\left|\square_{2n}^+\right|}C(1+|q|^2).$$

One then appeals to the estimates

$$\left|\partial\mathcal{Z}_{n,2n}\right| \leq C3^{(d-1)n} \quad \text{and} \quad \left|\square_{n+1}\right| = 3^{d(n+1)}.$$

Consequently, one has

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|(1-\zeta(y))\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right] \leq C3^{-n}(1+|q|^2).$$

By the previous display and (5.4.40), it is enough to prove (5.4.33) to show

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|\mathbf{f}(e)-\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right] \leq C\tau_n^*(q)+C(1+|q|^2)3^{-\beta n}, \tag{5.4.41}$$

for some small exponent $\beta := \beta(d,\lambda) > 0$. We now prove this estimate. Using that χ is a partition of unity, we rewrite

$$\begin{aligned}\mathbb{E}&\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|\mathbf{f}(e)-\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right]\\ &= \mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\mathbf{f}(e)-\nabla\psi_{z,n+1}(e)+\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right].\end{aligned}$$

Using that the function χ is supported in $\frac{3}{4}\square_{n+1}$, one obtains

$$\begin{aligned}\mathbb{E}&\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|\sum_{z\in\mathcal{Z}_{n,2n}^+}\chi(y-z)\left(\mathbf{f}(e)-\nabla\psi_{z,n+1}(e)+\nabla\nu_{n+1}^*(q)(e)\right)\right|^2\right]\\ &\leq \mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{z\in\mathcal{Z}_{n,2n}^+}\sum_{e\subseteq\left(z+\frac{3}{4}\square_{n+1}\right)\cap\square_{2n}}\left|\mathbf{f}(e)-\nabla\psi_{z,n+1}(e)+\nabla\nu_{n+1}^*(q)(e)\right|^2\right].\end{aligned}$$

Using the definition of $\mathbf{f}$ given in (5.4.24), and splitting the sum according to the partition of edges,

$$e\subseteq\square_{2n} \implies e\in B_{2n,n} \quad \text{or} \quad \exists z\in\mathcal{Z}_{n,2n},\ e\subseteq z+\square_n,$$

one derives

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{e=(x,y)\subseteq\square_{2n}^{+}}\left|\mathbf{f}(e)-\sum_{z\in\mathcal{Z}_{n,2n}}\chi(y-z)\left(\nabla\psi_{z,n+1}(e)-\nabla\nu_{n+1}^{*}(q)(e)\right)\right|^{2}\right] \tag{5.4.42}$$
$$\leq\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{z,z'\in\mathcal{Z}_{n,2n},\,z\in z'+\square_{n+1}}\sum_{e\subseteq z'+\square_{n}}\left|\nabla\psi_{z'}(e)-\nabla\psi_{z,n+1}(e)\right|^{2}\right]$$
$$+\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\in B_{2n,n}\cap(z+\frac{3}{4}\square_{n+1})}\left|\nabla\psi_{z,n+1}(e)\right|^{2}\right]$$
$$+C\left|\nabla\nu_{n+1}^{*}(q)-\nabla\nu_{n}^{*}(q)\right|^{2}.$$

The first term on the right-hand side is estimated thanks to (5.4.36). This gives

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{z,z'\in\mathcal{Z}_{n,2n},\,z\in z'+\square_{n+1}}\sum_{e\subseteq z'+\square_{n}}\left|\nabla\psi_{z'}(e)-\nabla\psi_{z,n+1}(e)\right|^{2}\right] \tag{5.4.43}$$
$$=\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{z\in\mathcal{Z}_{n,2n}}\mathbb{E}\left[\sum_{z'\in 3^{n}\mathbb{Z}^{d}\cap(z+\square_{n+1})}\sum_{e\subseteq z'+\square_{n}}\left|\nabla\psi_{z,n+1}(e)-\nabla\psi_{z'}(e)\right|^{2}\right]$$
$$\leq C\tau_{n}^{*}(q)+C(1+|q|^{2})3^{-n}.$$

The third term can be estimated thanks to (5.4.4),

$$\left|\nabla\nu_{n+1}^{*}(q)-\nabla\nu_{n}^{*}(q)\right|^{2}\leq C\tau_{n}^{*}(q)+C(1+|q|^{2})3^{-n}.$$

To estimate the second term on the right-hand side of (5.4.42), we first use that all the $\psi_{z,n+1}$ have the same law, which is $\mathbb{P}_{n+1,q}^{*}$. This gives

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\in B_{2n,n}\cap(z+\frac{3}{4}\square_{n+1})}\left|\nabla\psi_{z,n+1}(e)\right|^{2}\right]$$
$$=\frac{1}{\left|\square_{2n}^{+}\right|}\sum_{z\in\mathcal{Z}_{n,2n}}\mathbb{E}\left[\sum_{e\in B_{2n,n}\cap\frac{3}{4}\square_{n+1}}\left|\nabla\psi_{n+1,q}(e)\right|^{2}\right]$$
$$\leq C\mathbb{E}\left[\frac{1}{\left|\square_{n+1}\right|}\sum_{e\in B_{2n,n}\cap\frac{3}{4}\square_{n+1}}\left|\nabla\psi_{n+1,q}(e)\right|^{2}\right].$$

We then estimate this term by the Meyers estimate, Proposition 5.B.5 with $\gamma=\frac{3}{4}$. We denote by δ the exponent of Proposition 5.B.5 and compute

$$\mathbb{E}\left[\frac{1}{\left|\square_{n+1}\right|}\sum_{e\in B_{2n,n}\cap\frac{3}{4}\square_{n+1}}\left|\nabla\psi_{n+1,q}(e)\right|^{2}\right]$$
$$\leq\frac{C}{\left|\frac{3}{4}\square_{n+1}\right|}\sum_{e\in B_{2n,n}\cap\frac{3}{4}\square_{n+1}}\mathbb{E}\left[\left|\nabla\psi_{n+1,q}(e)\right|^{2}\right]$$
$$\leq\frac{C\left|B_{2n,n}\cap\frac{3}{4}\square_{n+1}\right|^{\frac{\delta}{1+\delta}}}{\left|\frac{3}{4}\square_{n+1}\right|}\left(\sum_{e\subseteq\frac{3}{4}\square_{n+1}}\mathbb{E}\left[\left|\nabla\psi_{n+1,q}(e)\right|^{2}\right]^{1+\delta}\right)^{\frac{1}{1+\delta}}$$
$$\leq C\left(\frac{\left|B_{2n,n}\cap\frac{3}{4}\square_{n+1}\right|}{\left|\frac{3}{4}\square_{n+1}\right|}\right)^{\frac{\delta}{1+\delta}}(1+|q|^{2}).$$

We then use that

$$\frac{\left|B_{2n,n} \cap \frac{3}{4}\square_{n+1}\right|}{\left|\frac{3}{4}\square_{n+1}\right|} \leq C3^{-n},$$

to derive

$$\mathbb{E}\left[\frac{1}{|\square_{n+1}|}\sum_{e\in B_{2n,n}\cap\frac{3}{4}\square_{n+1}} |\nabla\psi_{n+1,q}(e)|^2\right] \leq C3^{-\frac{\delta}{1+\delta}n}(1+|q|^2).$$

Combining the few previous displays gives the following estimate for the second term on the right-hand side of (5.4.42)

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\in B_{2n,n}\cap(z+\frac{3}{4}\square_{n+1})} |\nabla\psi_{z,n+1}(e)|^2\right] \leq C3^{-\frac{\delta}{1+\delta}n}(1+|q|^2).$$

Combining (5.4.42) with (5.4.43), the previous displays and setting $\beta := \frac{\delta}{1+\delta}$ yields

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e=(x,y)\subseteq\square_{2n}^+}\left|\mathbf{f}(e) - \sum_{z\in 3^n\mathbb{Z}^d}\chi(y-z)\left(\nabla\psi_{z,n+1}(e) - \nabla\nu_{n+1}^*(q)\cdot e\right)\right|^2\right] \leq C\tau_n^*(q) + C(1+|q|^2)3^{-\beta n}.$$

This is precisely (5.4.41). The proof of (5.4.33) is complete.

Step 3. We now deduce from (5.4.33) that

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e\subseteq\square_{2n}^+}|\mathbf{f}(e) - \nabla\kappa(e)|^2\right] \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).$$

We recall that κ was defined as the solution of the problem

$$\begin{cases}\Delta\kappa = \operatorname{div}\mathbf{f} \text{ in } \square_{2n}, \\ \kappa \in h_0^1\left(\square_{2n}^+\right).\end{cases}$$

This implies the almost sure inequality

$$\sum_{e\subseteq\square_{2n}^+}|\mathbf{f}(e) - \nabla\kappa(e)|^2 = \inf_{\kappa'\in h_0^1(\square_{2n}^+)}\sum_{e\subseteq\square_{2n}^+}\left|\mathbf{f}(e) - \nabla\kappa'(e)\right|^2 \leq \sum_{e\subseteq\square_{2n}^+}|\mathbf{f}(e) - \nabla\Psi(e)|^2. \tag{5.4.44}$$

Taking the expectation and using (5.4.33) gives

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e\subseteq\square_{2n}^+}|\mathbf{f}(e) - \nabla\kappa(e)|^2\right] \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).$$

Combining the previous display with (5.4.34) proves the estimate

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e\subseteq\square_{2n}^+}|\mathbf{f}(e) - \nabla\kappa_{2n}^+(e)|^2\right] \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}}\tau_m^*(q)\right). \tag{5.4.45}$$

Step 4. The goal of this step is to use the main result (5.4.45) of Step 3 to prove

$$\mathbb{E}\left[\frac{1}{\left|\square_{2n}^+\right|}\sum_{e\subseteq\square_{2n}^+}V_e\left(\nabla\nu_n^*(q)(e) + \nabla\kappa_{2n}^+(e)\right)\right] \leq \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}V_e\left(\nabla\psi_{n,q}(e)\right)\right] + C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}}\tau_m^*(q)\right). \tag{5.4.46}$$

We recall the definition of the random variable $\sigma_z(x) := \psi_z(x) - \nabla \nu_n^*(q) \cdot (x - z)$ introduced in (5.4.18). The proof relies on the following technical estimate, the proof of which is postponed to Appendix 5.A, Proposition 5.A.2.

$$\begin{aligned}
(5.4.47)\quad \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu_n^*(q)(e) + \nabla\kappa(e)\right)\right] &\le \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\psi_z(e)\right)\right] \\
&\quad + C(1+|q|^2)3^{-\frac{n}{2}} + C\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} |\nabla\kappa(e) - \nabla\sigma_z(e)|^2\right].
\end{aligned}$$

We now show how to deduce (5.4.46) from the previous inequality. First, since all the ψ_z have the same law which is $\mathbb{P}^*_{n,q}$, one can simplify the first term on the right-hand side,

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\psi_z(e)\right)\right] &= \frac{|\mathcal{Z}_{n,2n}|}{|\square_{2n}|}\mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(\nabla\psi_{n,q}(e)\right)\right] \\
&= \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n} V_e\left(\nabla\psi_{n,q}(e)\right)\right].
\end{aligned}$$

We now estimate the last term on the right-hand side of (5.4.47). One has

$$\begin{aligned}
(5.4.48)\quad \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} |\nabla\sigma_z(e) - \nabla\kappa(e)|^2\right] &\le \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e) - \nabla\kappa(e)|^2\right] \\
&\le \mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e) - \nabla\kappa(e)|^2\right] \\
&\quad + \left(\frac{|\square_{2n}| - |\square_{2n}^+|}{|\square_{2n}^+|\cdot|\square_{2n}|}\right)\mathbb{E}\left[\sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e) - \nabla\kappa(e)|^2\right].
\end{aligned}$$

We first estimate the second term on the right-hand side of the previous display. Note that

$$\frac{|\square_{2n}| - |\square_{2n}^+|}{|\square_{2n}^+|\cdot|\square_{2n}|} \le 3^{-2n}\frac{1}{|\square_{2n}|}$$

and using (5.4.44) and the fact that all the ψ_z have the same law, one has

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e) - \nabla\kappa(e)|^2\right] &\le \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e)|^2\right] \\
&\le \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} |\nabla\sigma_z(e)|^2\right] \\
&\le C\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n} |\nabla\psi_{n,q}(e)|^2\right].
\end{aligned}$$

We can then bound the last term on the right-hand side thanks to Proposition 5.3.1. This gives

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e) - \nabla\kappa(e)|^2\right] \le C\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n} |\nabla\psi_{n,q}(e)|^2\right] \le C(1+|q|^2).$$

Combining the few previous displays shows

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} |\nabla\sigma_z(e) - \nabla\kappa(e)|^2\right] \le \mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e) - \nabla\kappa(e)|^2\right] + C3^{-2n}(1+|q|^2).$$

We then use (5.4.32) to complete the estimate

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}|\nabla\kappa(e)-\nabla\sigma_z(e)|^2\right]\le C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).$$

Combining this estimate with (5.4.47) shows

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa(e)\right)\right]\le\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}V_e\left(\nabla\psi_z(e)\right)\right]$$
$$+C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).$$

To complete the proof of (5.4.46), it is thus sufficient to prove

(5.4.49)
$$\mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e\subseteq\square_{2n}^+}V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa_{2n}^+(e)\right)\right]\le\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa(e)\right)\right]$$
$$+C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau_m^*(q)\right),$$

for some constant $C:=C(d,\lambda)<\infty$ and some exponent $\beta:=\beta(d,\lambda)>0$. To this end, we prove the two following inequalities:

(1) We first prove that

(5.4.50)
$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+}V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa(e)\right)\right]\le\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa(e)\right)\right]$$
$$+C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).$$

(2) We then prove

(5.4.51)
$$\mathbb{E}\left[\frac{1}{|\square_{2n}^+|}\sum_{e\subseteq\square_{2n}^+}V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa_{2n}^+(e)\right)\right]\le\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+}V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa(e)\right)\right]$$
$$+C3^{-\frac{d}{2}n}(1+|q|).$$

Proof of (1). We define $B_{n,2n}^+$ to be the set of edges of $\square_{2n}^+$ which do not belong to a cube of the form $z+\square_n$, for $z\in\mathcal{Z}_{n,2n}$, i.e.

$$B_{n,2n}^+:=\left\{e\subseteq\square_{2n}^+\ :\ \forall z\in\mathcal{Z}_{n,2n},\,e\nsubseteq z+\square_n\right\}.$$

This set has been defined to have the following splitting of the sum

$$\sum_{e\subseteq\square_{2n}^+}=\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}+\sum_{e\in B_{n,2n}^+}.$$

Note also that the set $B_{n,2n}^+$ is almost equal to the set $B_{n,2n}$, the only difference is that we added the edges which belong to $\square_{2n}^+$ but not $\square_{2n}$, which is a small boundary layer of edges. Additionally, one has the estimate on the cardinality of $B_{n,2n}^+$,

$$\left|B_{n,2n}^+\right|\le C3^{-n}\left|\square_{2n}\right|.$$

We use the splitting of the sum mentioned above to prove the estimate

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B^+_{2n,n}} V_e\left(\nabla\nu^*_n(q)(e)+\nabla\kappa(e)\right)\right]\leq C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau^*_m(q)\right).$$

We first use that for each $x\in\mathbb{R}$, $V_e(x)\leq\frac{1}{\lambda}x^2$ to prove

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B^+_{2n,n}} V_e\left(\nabla\nu^*_n(q)(e)+\nabla\kappa(e)\right)\right]&\leq C\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B^+_{2n,n}}|\nabla\kappa(e)|^2\right]+C\frac{\left|B^+_{2n,n}\right|}{|\square_{2n}|}|\nabla\nu^*_n(q)|^2\\&\leq C\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B^+_{2n,n}}|\nabla\kappa(e)|^2\right]+C(1+|q|^2)3^{-2n}.\end{aligned}$$

Since for each $e\in B^+_{2n,n}$, one has $\mathbf{f}(e)=0$, one derives the estimate

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B^+_{2n,n}}|\nabla\kappa(e)|^2\right]\leq\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square^+_{2n}}|\mathbf{f}(e)-\nabla\kappa(e)|^2\right].$$

Using the exact same computation as in (5.4.48), one obtains

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square^+_{2n}}|\mathbf{f}(e)-\nabla\kappa(e)|^2\right]\leq C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau^*_m(q)\right).$$

Combining the few previous displays shows

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B^+_{2n,n}} V_e\left(\nabla\nu^*_n(q)(e)+\nabla\kappa(e)\right)\right]\leq C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau^*_m(q)\right)$$

and consequently

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square^+_{2n}} V_e\left(\nabla\nu^*_n(q)(e)+\nabla\kappa(e)\right)\right]&=\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu^*_n(q)(e)+\nabla\kappa(e)\right)\right]\\&\quad+\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B^+_{2n,n}} V_e\left(\nabla\nu^*_n(q)(e)+\nabla\kappa(e)\right)\right]\\&\leq\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu^*_n(q)(e)+\nabla\kappa(e)\right)\right]\\&\quad+C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau^*_m(q)\right).\end{aligned}$$

This is (5.4.50).

Proof of (2). The main tool is the estimate (5.4.35), which we recall

$$\mathbb{E}\left[\frac{1}{|\square^+_{2n}|}\sum_{e\subseteq\square^+_{2n}}|\nabla\kappa^+_{2n}(e)-\nabla\kappa(e)|^2\right]\leq C3^{-dn}.$$

Using this inequality and a Taylor expansion, together with the assumption $V_e'' \leq \frac{1}{\lambda}$, one obtains

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} V_e\left(\nabla\nu_n^*(q)(e) + \nabla\kappa_{2n}^+(e)\right)\right] \\
&\quad \leq \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} V_e\left(\nabla\nu_n^*(q)(e) + \nabla\kappa_{2n}\right)\right] \\
&\qquad + \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} V_e'\left(\nabla\nu_n^*(q)(e) + \nabla\kappa(e)\right)\left(\nabla\kappa_{2n}^+(e) - \nabla\kappa(e)\right)\right] \\
&\qquad + \frac{1}{2\lambda}\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} \left|\nabla\kappa_{2n}^+(e) - \nabla\kappa(e)\right|^2\right].
\end{aligned}$$

First combining the two previous displays, one has

(5.4.52)

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} V_e\left(\nabla\nu_n^*(q)(e) + \nabla\kappa_{2n}^+(e)\right)\right] &\leq \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} V_e\left(\nabla\nu_n^*(q)(e) + \nabla\kappa_{2n}\right)\right] + C3^{-dn} \\
&\quad + \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} V_e'\left(\nabla\nu_n^*(q)(e) + \nabla\kappa(e)\right)\left(\nabla\kappa_{2n}^+(e) - \nabla\kappa(e)\right)\right],
\end{aligned}$$

so that there only remains to study the last term on the right-hand side of the previous display. This is achieved thanks to the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left|\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} V_e'\left(\nabla\nu_n^*(q)(e) + \nabla\kappa(e)\right)\left(\nabla\kappa_{2n}^+(e) - \nabla\kappa(e)\right)\right]\right| \\
&\quad \leq \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} \left|V_e'\left(\nabla\nu_n^*(q)(e) + \nabla\kappa(e)\right)\right|^2\right]^{\frac{1}{2}} \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} \left|\nabla\kappa_{2n}^+(e) - \nabla\kappa(e)\right|^2\right]^{\frac{1}{2}} \\
&\quad \leq C3^{-\frac{d}{2}n}\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} \left|V_e'\left(\nabla\nu_n^*(q)(e) + \nabla\kappa(e)\right)\right|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

We then use that, for each $x \in \mathbb{R}$, $|V_e'(x)| \leq \frac{1}{\lambda}|x|$ to get

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} \left|V_e'\left(\nabla\nu_n^*(q)(e) + \nabla\kappa(e)\right)\right|^2\right] \leq C\left|\nabla\nu_n^*(q)\right|^2 + C\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+} |\nabla\kappa(e)|^2\right]$$

and by the definition of κ given in (5.4.19), we have

$$\begin{aligned}
\sum_{e\subseteq\square_{2n}^+} |\nabla\kappa(e)|^2 &\leq \sum_{e\subseteq\square_{2n}^+} |\mathbf{f}(e)|^2 \\
&\leq \sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla\psi_z(e)|^2.
\end{aligned}$$

Taking the expectation and using (5.3.8) of Proposition 5.3.1, one derives

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\,\square_{2n}\,|}\sum_{e\subseteq\square_{2n}^{+}}|\nabla\kappa(e)|^2\right] &\leq \mathbb{E}\left[\frac{1}{|\,\square_{2n}\,|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}|\nabla\psi_z(e)|^2\right]\\ &\leq \mathbb{E}\left[\frac{1}{|\,\square_n\,|}\sum_{e\subseteq\square_n}|\nabla\psi_{n,q}(e)|^2\right]\\ &\leq C(1+|q|^2).\end{aligned}$$

Combining the few previous displays and the bound $|\nabla\nu_n^*(q)|^2 \leq C(1+|q|^2)$ proved in (5.4.3) gives

$$\left|\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^{+}}V_e'\left(\nabla\nu_n^*(q)+\nabla\kappa(e)\right)\left(\nabla\kappa_{2n}^{+}(e)-\nabla\kappa(e)\right)\right]\right| \leq C3^{-\frac{d}{2}n}(1+|q|).$$

Combining this with (5.4.52) gives

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^{+}}V_e\left(\nabla\nu_n^*(q)+\nabla\kappa_{2n}^{+}(e)\right)\right] \leq \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^{+}}V_e\left(\nabla\nu_n^*(q)+\nabla\kappa_{2n}\right)\right]+C(1+|q|)3^{-\frac{d}{2}n}.$$

We complete the proof of (5.4.51) by noting that V_e is positive and that $|\,\square_{2n}\,| \leq |\,\square_{2n}^{+}\,|$ so that

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_{2n}^{+}|}\sum_{e\subseteq\square_{2n}^{+}}V_e\left(\nabla\nu_n^*(q)+\nabla\kappa_{2n}^{+}(e)\right)\right] &\leq \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^{+}}V_e\left(\nabla\nu_n^*(q)+\nabla\kappa_{2n}^{+}(e)\right)\right]\\ &\leq \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^{+}}V_e\left(\nabla\nu_n^*(q)+\nabla\kappa(e)\right)\right]+C3^{-\frac{d}{2}n}(1+|q|).\end{aligned}$$

This completes the proof of (5.4.51).

We can now conclude this step. Combining (5.4.50) and (5.4.51) implies (5.4.49) and thus completes the proof of (5.4.46). Step 4 is complete.

Step 5. The conclusion. Combining the main results (5.4.10) of Step 2 and (5.4.11) of Steps 3 and 4, one obtains

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_{2n}^{+}|}\sum_{e\subseteq\square_{2n}^{+}}V_e\left(\nabla\nu_n^*(q)+\nabla\kappa_{2n}^{+}(e)\right)\right]+\frac{1}{|\square_{2n}^{+}|}H\left(\mathbb{P}_{\kappa_{2n}^{+}}\right) &\leq \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}V_e\left(\nabla\psi_{n,q}(e)\right)\right]\\ &\quad+\frac{1}{|\square_n|}H\left(\mathbb{P}_{n,q}^*\right)+C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).\end{aligned}$$

But one knows that

$$\begin{aligned}\nu\left(\square_{2n}^{+},\nabla\nu_n^*(q)\right) &= \inf_{\mathbb{P}\in\mathcal{P}(h_0^1(\square_{2n}^{+}))}\mathbb{E}\left[\frac{1}{|\,\square_{2n}^{+}\,|}\sum_{e\subseteq\square_{2n}^{+}}V_e\left(\nabla\nu_n^*(q)(e)+\nabla\phi(e)\right)\right]+\frac{1}{|\,\square_{2n}^{+}\,|}H\left(\mathbb{P}\right)\\ &\leq \mathbb{E}\left[\frac{1}{|\square_{2n}^{+}|}\sum_{e\subseteq\square_{2n}^{+}}V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa_{2n}^{+}(e)\right)\right]+\frac{1}{|\square_{2n}^{+}|}H\left(\mathbb{P}_{\kappa_{2n}^{+}}\right).\end{aligned}$$

Moreover, by the definition of $\mathbb{P}_{n,q}^*$ and the equality $\nabla\nu_n^*(q)=\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}\nabla\psi_{n,q}(e)\right]$, one knows that

$$\nu^*(\square_n,q) = -\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e\subseteq\square_n}V_e\left(\nabla\psi_{n,q}(e)\right)\right]+q\cdot\nabla\nu_n^*(q)-\frac{1}{|\square_n|}H\left(\mathbb{P}_{n,q}^*\right).$$

Combining the three previous displays shows

$$\nu\left(\square_{2n}^{+},\nabla\nu_n^*(q)\right)+\nu^*(\square_n,q)-q\cdot\nabla\nu_n^*(q) \leq C\left((1+|q|^2)3^{-\beta n}+\sum_{m=0}^{n}3^{\frac{(m-n)}{2}}\tau_m^*(q)\right).$$

The proof of Proposition 5.4.5 is complete. □

5.4.4. Quantitative convergence of the partitions functions. Now that Proposition 5.4.5 is proved, we can deduce the main result of the article. The theorem is recalled below.

THEOREM 5.1.1 (Quantitative convergence to the Gibbs state). *There exist a constant $C := C(d,\lambda) < \infty$ and an exponent $\alpha := \alpha(d,\lambda) > 0$ such that for each $p,q \in \mathbb{R}^d$*

$$|\nu(\square_n, p) - \overline{\nu}(p)| \leq C3^{-\alpha n}(1+|p|^2)$$

and

$$|\nu^*(\square_n, q) - \overline{\nu}^*(q)| \leq C3^{-\alpha n}(1+|q|^2).$$

Before starting the proof, we mention that the proof of this proposition only relies on the basic properties of ν and ν^* together with upper bound for the convex duality given in Proposition 5.4.5. In particular, we do not use any specific properties of the gradient field model in the rest of this section.

PROOF. From Proposition 5.3.4, we know that, for each $p,q \in \mathbb{R}^d$, the sequences $(\nu(\square_n,p))_{n\in\mathbb{N}}$ and $(\nu^*(\square_n,p))_{n\in\mathbb{N}}$ converge and that, for each $p,q \in \mathbb{R}^d$,

$$\overline{\nu}(p) + \overline{\nu}^*(q) \geq p \cdot q. \tag{5.4.53}$$

We split the proof into 5 steps.

- In Step 1, the objective is to remove the $\square_{2n}^+$ condition which appears in Proposition 5.4.5: the idea is to appeal to the subadditivity of the surface tension ν to prove the estimate, for each $p \in \mathbb{R}^d$,

$$\nu(\square_{3n}, p) \leq \nu(\square_{2n}^+, p) + C3^{-n}(1+|p|^2). \tag{5.4.54}$$

- In Step 2, we show that for each $n \in \mathbb{N}$ and each $q \in \mathbb{R}^d$

$$|\nu^*(\square_n, q) - \overline{\nu}^*(q)| \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q)\right).$$

- In Step 3, we deduce that there exists an exponent $\alpha := \alpha(d,\lambda) > 0$ such that,

$$|\nu^*(\square_n, q) - \overline{\nu}^*(q)| \leq C3^{-\alpha n}. \tag{5.4.55}$$

- In Step 4, we show that the limiting surface tensions $\overline{\nu}$ and $\overline{\nu}^*$ are dual convex to one another: one has the equality, for each $q \in \mathbb{R}^d$,

$$\overline{\nu}^*(q) = \sup_{p\in\mathbb{R}^d} -\overline{\nu}(p) + p \cdot q. \tag{5.4.56}$$

- In Step 5, we show that there exists an exponent $\alpha := \alpha(d,\lambda) > 0$ such that,

$$|\nu(\square_n, q) - \overline{\nu}(q)| \leq C3^{-\alpha n}. \tag{5.4.57}$$

Step 1. The main idea of this step is to consider a cube $\square \subseteq \mathbb{Z}^d$ satisfying the two following properties

(1) The cube $\square$ is included in $\square_{3n}$ and is almost as large as $\square_{3n}$ in the sense that it satisfies the volume estimate

$$|\,\square_{3n}\,| \leq |\,\square\,| + C3^{3dn} \times 3^{-n}. \tag{5.4.58}$$

(2) The cube $\square$ can be decomposed as a disjoint union of cubes of the same size than $\square_{2n}^+$. i.e. the union of disjoint translated of $\square_{2n}^+$.

More precisely, the cube $\square$ can be constructed as follows: denote by $\mathcal{Z}$ the set

$$\mathcal{Z} := \left\{ z \in (3^{2n} + 2)\mathbb{Z}^d \ : \ z + \square_{2n}^+ \subseteq \square_{3n} \right\}$$

and then define

$$\square = \bigcup_{z \in \mathcal{Z}} (z + \square_{2n}^+).$$

With this definition, it is clear that the cube $\square$ satisfies the two properties (1) and (2). Following the proof of the subadditivity of the finite volume surface tension ν given by Funaki and Spohn in [**70**], on obtains the estimate

$$\nu(\square_{3n}, p) \leq \frac{|\square_{3n} \setminus \square|}{|\square_{3n}|} \nu(\square_{3n} \setminus \square, p) + \sum_{z \in \mathcal{Z}} \frac{|z + \square_{2n}^+|}{|\square_{3n}|} \nu(z + \square_{2n}^+, p) + C3^{-2n}(1 + |p|^2). \tag{5.4.59}$$

Using Proposition 5.A.3, proved in Appendix 5.A, one knows that $\nu(\square_{3n} \setminus \square, p)$ is bounded by $C(1 + |p|^2)$, thus by (5.4.58), one can estimate

$$\frac{|\square_{3n} \setminus \square|}{|\square_{3n}|} \nu(\square_{3n} \setminus \square, p) \leq C3^{-n}(1 + |p|^2).$$

From (5.4.59) and the previous display, one obtains

$$\nu(\square_{3n}, p) \leq \sum_{z \in \mathcal{Z}} \frac{|z + \square_{2n}^+|}{|\square_{3n}|} \nu(z + \square_{2n}^+, p) + C3^{-2n}(1 + |p|^2).$$

But, one has for each $z \in \mathcal{Z}$, $\nu(z + \square_{2n}^+, p) = \nu(\square_{2n}^+, p)$, thus

$$\sum_{z \in \mathcal{Z}} \frac{|z + \square_{2n}^+|}{|\square_{3n}|} \nu(z + \square_{2n}^+, p) = \frac{|\square_{3n} \setminus \square|}{|\square_{3n}|} \nu(\square_{2n}^+, p).$$

Using Proposition 5.3.7, we have $\nu(\square_{2n}^+, p) \geq -C + \lambda |p|^2$. Combining this bound with (5.4.58), the previous display can be refined

$$\frac{|\square_{3n} \setminus \square|}{|\square_{3n}|} \nu(\square_{2n}^+, p) \leq \nu(\square_{2n}^+, p) + C(1 + |p|^2)3^{-n}.$$

Combining the few previous displays eventually shows

$$\nu(\square_{3n}, p) \leq \nu(\square_{2n}^+, p) + C3^{-n}(1 + |p|^2),$$

which is the desired result. The proof of Step 1 is complete.

Step 2. First, by the formula, for each $q \in \mathbb{R}^d$,

$$\nabla_q \nu^*(\square_n, q) = \mathbb{E}\left[\frac{1}{|\square_n|} \sum_{e \subseteq \square_n} \nabla \psi_{n,q}(e) \right]$$

and by the estimate (5.3.8) of Proposition 5.3.1, one has

$$|\nabla_q \nu^*(\square_n, q)| \leq C(1 + |q|).$$

As a consequence, by the main result (5.4.54) of Step 1, one has

$$\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) \leq \nu(\square_{2n}^+, \nabla_q \nu^*(\square_n, q)) + C3^{-n}(1 + |q|^2).$$

Combining the previous display with Proposition 5.4.5, one obtains

$$\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) + \nu^*(\square_n, q) + q \cdot \nabla_q \nu^*(\square_n, q) \leq C\left((1 + |q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q) \right). \tag{5.4.60}$$

Moreover using the inequality (5.4.53) applied with $p = \nabla_q \nu^*(\square_n, q)$ and q gives

$$0 \leq \overline{\nu}(\nabla_q \nu^*(\square_n, q)) + \overline{\nu}^*(q) - \nabla_q \nu^*(\square_n, q) \cdot q.$$

A combination of the two previous displays gives

$$(\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) - \overline{\nu}(\nabla_q \nu^*(\square_n, q))) + (\nu^*(\square_n, q) - \overline{\nu}^*(q)) \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q)\right). \tag{5.4.61}$$

But by the subadditivity for ν stated in Proposition 5.3.1, one sees that there exists a constant $C := C(d,\lambda) < \infty$ (in particular larger than the one appearing in the proposition) such the sequence $n \to \nu(\square_n, p) + C(1+|p|^2)3^{-n}$ is decreasing. As a consequence, for each $n \in \mathbb{N}$, and each $p \in \mathbb{R}^d$,

$$\nu(\square_n, p) \geq \overline{\nu}(p) - C(1+|p|^2)3^{-n}.$$

This implies, for each $q \in \mathbb{R}^d$,

$$\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) - \overline{\nu}(\nabla_q \nu^*(\square_n, q)) \geq -C(1+|q|^2)3^{-3n}.$$

Combining the previous inequality with (5.4.61) shows

$$\nu^*(\square_n, q) - \overline{\nu}^*(q) \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q)\right). \tag{5.4.62}$$

The proof of Step 2 is complete.

Step 3. Let $C := C(d,\lambda) < \infty$ be a constant large enough so that the sequence $\nu^*(\square_n, q) + C(1+|q|^2)3^{-n}$ is decreasing. To shorten the notation, we denote by, for $q \in \mathbb{R}^d$,

$$F_n(q) := \nu^*(\square_n, q) + C(1+|q|^2)3^{-n} - \overline{\nu}^*(q), \tag{5.4.63}$$

so that $F_n(q)$ is decreasing and tends to 0 as n tends to infinity. Moreover, one has the following inequality

$$\tau_n^*(q) \leq F_n(q) - F_{n+1}(q).$$

We can then rewrite the main result (5.4.62) of Step 3 with this notation

$$F_n(q) \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} (F_m(q) - F_{m+1}(q))\right). \tag{5.4.64}$$

We then define

$$\widetilde{F}_n(q) := 3^{-\frac{n}{4}} \sum_{m=0}^{n} 3^{\frac{m}{4}} F_m(q).$$

We next show that there exist $\theta := \theta(d,\lambda) \in (0,1)$, $C := C(d,\lambda) < \infty$ and $\beta := \beta(d,\lambda) > 0$ such that for every $n \in \mathbb{N}$,

$$\widetilde{F}_{n+1}(q) \leq \theta \widetilde{F}_n(q) + C3^{-\beta n}. \tag{5.4.65}$$

Using the inequality $F_0(q) \leq C(1+|q|^2)$, one has

$$\widetilde{F}_n(q) - \widetilde{F}_{n+1}(q) \geq 3^{-\frac{n}{4}} \sum_{m=0}^{n} 3^{\frac{m}{4}} (F_m(q) - F_{m+1}(q)) - C(1+|q|^2)3^{-\frac{n}{4}}. \tag{5.4.66}$$

Since $F_n(q)$ is a decreasing sequence, we deduce from the previous display that, for each $n \in \mathbb{N}$,

$$\widetilde{F}_{n+1}(q) \leq \widetilde{F}_n(q) + C(1+|q|^2)3^{-\frac{n}{4}}.$$

By (5.4.64) and reducing the exponent β if necessary, one deduces

$$\begin{aligned}
\widetilde{F}_{n+1}(q) &\le \widetilde{F}_n(q) + C(1+|q|^2)3^{-\frac{n}{4}} \\
&\le 3^{-\frac{n}{4}} \sum_{m=0}^{n} 3^{\frac{m}{4}} F_m(q) + C(1+|q|^2)3^{-\frac{n}{4}} \\
&\le C3^{-\frac{n}{4}} \sum_{m=0}^{n} 3^{\frac{m}{4}} \left(\left((1+|q|^2)3^{-\beta m} + \sum_{k=0}^{m} 3^{\frac{(k-m)}{2}} \left(F_k(q) - F_{k+1}(q)\right) \right) \right) + C(1+|q|^2)3^{-\frac{n}{4}} \\
&\le C3^{-\frac{n}{4}} \sum_{m=0}^{n} \sum_{k=0}^{m} 3^{-\frac{m}{4}} 3^{\frac{k}{2}} \left(F_k(q) - F_{k+1}(q)\right) + C(1+|q|^2)3^{-\beta n} \\
&\le C3^{-\frac{n}{4}} \sum_{k=0}^{n} \sum_{m=k}^{n} 3^{-\frac{m}{4}} 3^{\frac{k}{2}} \left(F_k(q) - F_{k+1}(q)\right) + C(1+|q|^2)3^{-\beta n} \\
&\le C3^{-\frac{n}{4}} \sum_{k=0}^{m} 3^{\frac{k}{4}} \left(F_k(q) - F_{k+1}(q)\right) + C(1+|q|^2)3^{-\beta n}.
\end{aligned}$$

Comparing the previous display with (5.4.66) gives

$$\widetilde{F}_{n+1}(q) \le C\left(\widetilde{F}_n(q) - \widetilde{F}_{n+1}(q)\right) + C(1+|q|^2)3^{-\beta n}.$$

A rearrangement of this inequality gives (5.4.65). An iteration of (5.4.65) yields

$$\widetilde{F}_n(q) \le \theta^n \widetilde{F}_0 + C(1+|q|^2) \sum_{k=0}^{n} \theta^k 3^{-\beta(n-k)}$$

By making θ closer to 1 if necessary, one has

$$\sum_{k=0}^{n} \theta^k 3^{-\beta(n-k)} \le C\theta^n.$$

Combining the few previous displays shows

$$\widetilde{F}_n(q) \le C(1+|q|^2)\theta^n.$$

Setting $\alpha := -\frac{\ln\theta}{\ln 3}$ so that $\theta = 3^{-\alpha}$ gives the bound $\widetilde{F}_n(q) \le C3^{-\alpha n}$. By the definition of $\widetilde{F}_n(q)$, one has the clear inequality

$$F_n(q) \le \widetilde{F}_n(q),$$

and thus

$$F_n(q) \le C(1+|q|^2)3^{-\alpha n}.$$

We conclude the proof by noting that $F_n(q)$ was defined so that it is decreasing and tends to 0. In particular, it is positive. By the explicit formula (5.4.63) for F_n and the previous display, one obtains

$$-C(1+|q|^2)3^{-n} \le \nu^*(\square_n, q) - \overline{\nu}^*(q) \le C(1+|q|^2)3^{-\alpha n},$$

for some $C := C(d,\lambda) < \infty$ and $\alpha := \alpha(d,\lambda) > 0$. By making α closer to 0 if necessary, one eventually obtains

$$|\nu^*(\square_n, q) - \overline{\nu}^*(q)| \le C(1+|q|^2)3^{-\alpha n}.$$

The proof of Step 3 is complete.

Step 4. First note that, by (5.4.53), for each $p, q \in \mathbb{R}^d$

$$0 \le \overline{\nu}(p) + \overline{\nu}^*(q) - p \cdot q. \tag{5.4.67}$$

This implies

$$\overline{\nu}^*(q) \ge \sup_{p\in\mathbb{R}^d} -\overline{\nu}(p) + p \cdot q$$

The main idea of this step is to use Proposition 5.4.5 to show the two following results:

(1) For each $q \in \mathbb{R}^d$, the sequence $\nabla_q \nu^*(\square_n, q)$ converges as n tends to infinity. We denote its limit by $\overline{P}(q)$. Moreover one has that the following quantitative estimate

$$\left|\nabla_q \nu^*(\square_n, q) - \overline{P}(q)\right| \leq C(1+|q|)3^{-\alpha n}. \tag{5.4.68}$$

REMARK 5.4.6. We would like to say that the limit is in fact $\nabla_q \overline{\nu}^*(q)$ but at this point of the argument, one only knows that the function $q \to \overline{\nu}^*(q)$ is convex and in particular we do not know that it is differentiable everywhere. We will prove later that $\overline{\nu}^*(q)$ is in fact $C^1(\mathbb{R})$ and this will imply $\overline{P}(q) = \nabla_q \overline{\nu}^*(q)$.

(2) We deduce from (1) that for each $q \in \mathbb{R}^d$, one has the following quantitative convergence estimate

$$\left|\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) - \overline{\nu}\left(\overline{P}(q)\right)\right| \leq C(1+|q|^2)3^{-\alpha n} \tag{5.4.69}$$

We first prove (1). From the main result of the previous step (5.4.56), we deduce that, for each $q \in \mathbb{R}^d$,

$$\tau_n^*(q) = \nu^*\left(\square_n, q\right) - \nu^*\left(\square_{n+1}, q\right) \leq C(1+|q|^2)3^{-\alpha n}.$$

Combining this result with (5.4.4) gives, for each $q \in \mathbb{R}^d$,

$$|\nabla_q \nu^*(\square_{n+1}, q) - \nabla_q \nu^*(\square_n, q)| \leq C(1+|q|^2)3^{-\alpha n}.$$

The previous display implies that the sequence $\nabla_q \nu^*(\square_{n+1}, q)$ converges for each $q \in \mathbb{R}^d$ together with the quantitative rate of convergence (5.4.68). We denote by $\overline{P}(q)$ its limit. The proof of (1) is complete.

We now prove (2). We first split (5.4.69) thanks to the triangle inequality

$$\begin{aligned}\left|\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) - \overline{\nu}\left(\overline{P}(q)\right)\right| \leq & \left|\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) - \nu\left(\square_{3n}, \overline{P}(q)\right)\right| \\ & + \left|\nu\left(\square_{3n}, \overline{P}(q)\right) - \overline{\nu}\left(\overline{P}(q)\right)\right|.\end{aligned}$$

Since one has the bound, for each $n \in \mathbb{N}$, $|\nabla_q \nu^*(\square_n, q)| \leq C(1+|q|)$, one obtains by taking the limit n tends to infinity,

$$\left|\overline{P}(q)\right| \leq C(1+|q|). \tag{5.4.70}$$

Combining the previous bound with (5.4.55) gives

$$\left|\nu(\square_{3n}, \overline{P}(q)) - \overline{\nu}\left(\overline{P}(q)\right)\right| \leq C(1+|q|^2)3^{-\alpha n}.$$

To prove (5.4.69), it is sufficient to prove

$$\left|\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) - \nu\left(\square_{3n}, \overline{P}(q)\right)\right| \leq C(1+|q|^2)3^{-\alpha n}.$$

To this end, we first prove the following property: for each $p, p' \in \mathbb{R}^d$, and each $n \in \mathbb{N}$,

$$|\nu(\square_n, p) - \nu(\square_n, p')| \leq C(|p| + |p'|)|p - p'|.$$

The idea to prove the previous inequality is to compute the gradient of $p \mapsto \nu(\square_n, p)$. A straightforward computation gives

$$|\nabla_p \nu(\square_n, p)| \leq \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e \in \square_n}\left|V_e'(p \cdot e + \nabla\phi_{n,p}(e))\right|\right].$$

Using the bound, for each $x \in \mathbb{R}$ $V_e'(x) \leq \frac{1}{\lambda}|x|$ and the Jensen inequality, we obtain

$$\begin{aligned}|\nabla_p \nu(\square_n, p)| &\leq |p| + \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e \in \square_n}|\nabla\phi_{n,p}(e)|\right] \\ &\leq |p| + \mathbb{E}\left[\frac{1}{|\square_n|}\sum_{e \in \square_n}|\nabla\phi_{n,p}(e)|^2\right]^{\frac{1}{2}}.\end{aligned}$$

We then apply the estimate (5.3.7) Proposition 5.3.1 to derive the bound

$$|\nabla_p \nu(\square_n, p)| \leq C(1+|p|). \tag{5.4.71}$$

This implies that, for each $n \in \mathbb{N}$ and each $p, p' \in \mathbb{R}^d$, $\nu(\square_n, \cdot)$ is $C(1+|p|+|p'|)$-Lipschitz in the ball $B(0,|p|+|p'|)$. Since both p and p' belongs to $B(0,|p|+|p'|)$, one has

$$|\nu(\square_n, p) - \nu(\square_n, p')| \leq C(1+|p|+|p'|)|p-p'|. \tag{5.4.72}$$

This is the desired result. Applying the previous estimate with $p = \nabla_p \nu^*(\square_n, q)$ and $p' = \overline{P}(q)$ gives

$$\left|\nu(\square_{3n}, \nabla_p \nu^*(\square_n, q)) - \nu(\square_{3n}, \overline{P}(q))\right| \leq C(1+|\nabla_q \nu^*(\square_n, q)| + |\overline{P}(q)|)|\nabla_q \nu^*(\square_n, q) - \overline{P}(q)|.$$

By (5.4.70), for each $q \in \mathbb{R}^d$,

$$\left|\overline{P}(q)\right| \leq C(1+|q|).$$

Combining the three previous displays with (5.4.68) gives

$$\left|\nu(\square_{3n}, \nabla_p \nu^*(\square_n, q)) - \nu(\square_{3n}, \overline{P}(q))\right| \leq C(1+|q|^2)3^{-\alpha n} \tag{5.4.73}$$

and completes the proof of (2).

We now prove the main result (5.4.56) of Step 4. By (5.4.60) and the main result (5.4.55) of Step 3, one has, for each $q \in \mathbb{R}^d$,

$$\nu\left(\square_{3n}, \nabla_q \nu^*\left(\square_n, q\right)\right) + \nu^*\left(\square_n, q\right) - \nabla_q \nu^*\left(\square_n, q\right) \cdot q \leq C(1+|q|^2)3^{-\alpha n}.$$

By (5.4.68) and (5.4.69), one also has the convergence

$$\nu\left(\square_{3n}, \nabla_q \nu^*\left(\square_n, q\right)\right) + \nu^*\left(\square_n, q\right) - \nabla_q \nu^*\left(\square_n, q\right) \cdot q \underset{n\to\infty}{\longrightarrow} \overline{\nu}\left(\overline{P}(q)\right) + \overline{\nu}^*\left(q\right) - \overline{P}(q) \cdot q.$$

A combination of the two previous displays gives

$$\overline{\nu}\left(\overline{P}(q)\right) + \overline{\nu}^*\left(q\right) - \overline{P}(q) \cdot q \leq 0.$$

Together with (5.4.67), the previous estimate gives in particular

$$\overline{\nu}\left(\overline{P}(q)\right) + \overline{\nu}^*\left(q\right) - \overline{P}(q) \cdot q = 0,$$

and thus

$$\overline{\nu}^*(q) = \sup_{p\in\mathbb{R}^d} -\overline{\nu}(p) + p \cdot q.$$

This is precisely (5.4.56) and the proof of Step 4 is complete.

Step 5. The main result (5.4.56) of Step 4 asserts that $\overline{\nu}^*$ is the Legendre-Fenchel transform of $\overline{\nu}$. But by Proposition 5.3.4, one knows that for each $p_1, p_2 \in \mathbb{R}^d$,

$$\frac{1}{C}|p_0 - p_1|^2 \leq \frac{1}{2}\overline{\nu}(\square_n, p_0) + \frac{1}{2}\overline{\nu}(\square_n, p_1) - \overline{\nu}\left(\square_n, \frac{p_0+p_1}{2}\right) \leq C|p_0-p_1|^2.$$

With the two previous ideas, one deduces that $\overline{\nu}^*$ is also uniformly convex. As a consequence, it is in $C^{1,1}(\mathbb{R}^d)$ and one has the following equalities, for each $p, q \in \mathbb{R}^d$,

$$\nabla_p \overline{\nu}\left(\nabla_q \overline{\nu}^*(q)\right) = q, \ \nabla_q \overline{\nu}^*\left(\nabla_p \overline{\nu}(q)\right) = q, \text{ and } \overline{P}(q) = \nabla_q \overline{\nu}^*(q). \tag{5.4.74}$$

We are now ready to prove (5.4.57). We start from (5.4.61), which reads for each $q \in \mathbb{R}^d$,

$$\begin{aligned}(\nu\left(\square_{3n}, \nabla_q \nu^*\left(\square_n, q\right)\right) - \overline{\nu}(\nabla_q \nu^*\left(\square_n, q\right))) + (\nu^*(\square_n, q) - \overline{\nu}^*(q)) \\ \leq C\left((1+|q|^2)3^{-\beta n} + \sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q)\right).\end{aligned} \tag{5.4.75}$$

We then apply (5.4.55) which allows to estimate most of the terms in the previous display. Precisely, one has the inequalities

$$\nu^*(\square_n, q) - \overline{\nu}^*(q) \leq C(1+|q|^2)3^{-\alpha n}$$

and

$$\sum_{m=0}^{n} 3^{\frac{(m-n)}{2}} \tau_m^*(q) \leq C(1+|q|^2)3^{-\alpha n}.$$

With these estimates, the inequality (5.4.75) becomes

$$\nu\left(\square_{3n}, \nabla_q \nu^*\left(\square_n, q\right)\right) - \overline{\nu}(\nabla_q \nu^*\left(\square_n, q\right)) \leq C(1+|q|^2)3^{-\alpha n}.$$

Then by (5.4.73), one has

$$\left|\nu(\square_{3n}, \nabla_q \nu^*(\square_n, q)) - \nu(\square_{3n}, \overline{P}(q))\right| \leq C(1+|q|^2)3^{-\alpha n}.$$

Then by sending n to infinity in (5.4.72), one obtains for each $p, p' \in \mathbb{R}^d$

$$|\overline{\nu}(p) - \overline{\nu}(p')| \leq C(|p| + |p'|)|p - p'|. \tag{5.4.76}$$

With the same proof as the one which gives (5.4.73), one obtains

$$\left|\overline{\nu}(\nabla_q \nu^*(\square_n, q)) - \overline{\nu}(\overline{P}(q))\right| \leq C(1+|q|^2)3^{-\alpha n}.$$

Combining the few previous displays shows, for each $q \in \mathbb{R}^d$,

$$\nu\left(\square_{3n}, \overline{P}(q)\right) - \overline{\nu}\left(\overline{P}(q)\right) \leq C(1+|q|^2)3^{-\alpha n}.$$

Applying the previous inequality with $q = \nabla_p \overline{\nu}(p)$ gives, thanks to (5.4.74),

$$\nu\left(\square_{3n}, p\right) - \overline{\nu}\left(p\right) \leq C(1+|\nabla_p \overline{\nu}(p)|^2)3^{-\alpha n}.$$

We then simplify the term on the right-hand side. Thanks to (5.4.76), one obtains the bound on the gradient of $\overline{\nu}$, for each $p \in \mathbb{R}^d$,

$$|\nabla_p \overline{\nu}\left(p\right)| \leq C(1+|p|).$$

A combination of the two previous displays gives, for each $n \in \mathbb{N}$, and each $p \in \mathbb{R}^d$,

$$\nu\left(\square_{3n}, p\right) - \overline{\nu}\left(p\right) \leq C(1+|p|^2)3^{-\alpha n}.$$

We now want to remove the $3n$ term. To this end, we use the subadditivity of ν stated in Proposition 5.3.1, to obtain, for each $p \in \mathbb{R}^d$ and each $n \in \mathbb{N}$,

$$\begin{aligned}\nu\left(\square_{3n+2}, p\right) - \overline{\nu}\left(p\right) \leq \nu\left(\square_{3n+1}, p\right) - \overline{\nu}\left(p\right) + C(1+|p|^2)3^{-n} &\leq \nu\left(\square_{3n}, p\right) - \overline{\nu}\left(p\right) + C(1+|p|^2)3^{-n} \\ &\leq C(1+|p|^2)3^{-\alpha n}.\end{aligned}$$

From the previous display and by making α smaller, one obtains for each $n \in \mathbb{N}$ and each $p \in \mathbb{R}^d$,

$$\nu\left(\square_n, p\right) - \overline{\nu}\left(p\right) \leq C(1+|p|^2)3^{-\alpha n}.$$

The proof of (5.4.57) is almost complete, there only remains to prove a lower bound for $\nu\left(\square_n, p\right) - \overline{\nu}\left(p\right)$. But one knows that there exists a constant $C := C(d, \lambda) < \infty$ such that the sequence $\nu\left(\square_n, p\right) + C(1+|p|^2)3^{-n}$ is decreasing and converges to $\overline{\nu}\left(p\right)$. This implies in particular that, for each $n \in \mathbb{N}$ and each $p \in \mathbb{R}^d$,

$$\nu\left(\square_n, p\right) - \overline{\nu}\left(p\right) \geq -C(1+|p|^2)3^{-n}$$

and provides the lower bound. Indeed a combination of the two previous displays shows

$$|\nu\left(\square_n, p\right) - \overline{\nu}\left(p\right)| \leq C(1+|p|^2)3^{-\alpha n}$$

and completes the proof of Step 5 and of Theorem 5.1.1. □

5.4.5. Quantitative contraction of the fields $\phi_{n,p}$ and $\psi_{n,q}$ to affine functions. Now that Theorem 5.1.1 is proved, we deduce the L^2 estimate on the random variables $\phi_{n,p}$ and $\psi_{n,q}$ stated in Theorem 5.1.2. The theorem is recalled below.

THEOREM 5.1.2 (L^2 contraction of the Gibbs measure). *There exist a constant $C := C(d,\lambda) < \infty$ and an exponent $\alpha := \alpha(d,\lambda) > 0$ such that for each $n \in \mathbb{N}$, $p, q \in \mathbb{R}^d$,*

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{x\in\square_n}\left(|\phi_{n,p}(x)|^2 + |\psi_{n,q}(x) - \nabla_q \overline{\nu}^*(q)\cdot x|^2\right)\right] \le C3^{n(2-\alpha)}\left(1+|p|^2+|q|^2\right).$$

PROOF. We first prove the estimate for the random variable $\psi_{n,q}$, i.e.

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{x\in\square_n}|\psi_{n,q}(x) - \nabla_q \overline{\nu}^*(q)\cdot x|^2\right] \le C3^{n(2-\alpha)}\left(1+|q|^2\right). \tag{5.4.77}$$

Indeed in that case all the tools have already been developed and this allows for a short proof. First by Theorem 5.1.1, one knows that, for each $q \in \mathbb{R}^d$,

$$\tau_n^*(q) \le 3^{-\alpha n}(1+|q|^2).$$

Using the previous display together with Proposition 5.4.4, we obtain

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{x\in\square_n}|\psi_{n,q}(x) - \nabla_q \nu^*(\square_n, q)\cdot x|^2\right] \le C3^{(2-\alpha)n}(1+|q|^2).$$

But by (5.4.68) and (5.4.74), one also has

$$|\nabla_q \nu^*(\square_n, q) - \nabla_q \overline{\nu}^*(q)| \le C3^{-\alpha n}(1+|q|^2).$$

A combination of the two previous displays gives (5.4.77) and completes the proof.

We now want to prove the estimate with the random variable $\phi_{n,p}$, i.e.,

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{x\in\square_n}|\phi_{n,p}(x) - p\cdot x|^2\right] \le C3^{(2-\alpha)n}\left(1+|p|^2\right).$$

The proof follows the same lines as the proof of (5.4.77) except that we have not proved an equivalent version of Proposition 5.4.4. The proof of this statement is split into 2 steps.

- In Step 1, we show that, for each $m \in \mathbb{N}$ with $m \le n$,

$$\frac{1}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\left|\langle\nabla\phi_{n,p}\rangle_{z+\square_m}\right|^2\right] \le C(1+|p|^2)3^{-\alpha m}.$$

- In Step 2, we deduce from the previous step and the multiscale Poincaré inequality

$$\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{x\in\square_n}|\phi_{n,p}|^2\right] \le C(1+|p|^2)3^{(2-\alpha)n}.$$

Step 1. Consider the random variable $\phi = \sum_{z\in\mathcal{Z}_{m,n}}\phi_z$ introduced in Proposition 5.3.8 as well as the coupling between ϕ and $\phi_{n,p}$ introduced in the same proposition. The following estimate holds

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}|\nabla\phi(e) - \nabla\phi_{n,p}(e)|^2\right] \le C\left(\nu(\square_m, q) - \nu(\square_n, q)\right) + C3^{-\frac{m}{2}}(1+|p|^2).$$

Using Theorem 5.1.1, the previous estimate can be refined and one obtains

$$\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}|\nabla\phi(e) - \nabla\phi_{n,p}(e)|^2\right] \le C(1+|p|^2)3^{-\alpha m}.$$

Using this estimate, one has

$$\begin{aligned}\frac{1}{|\mathcal{Z}_{m,n}|}&\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\left|\langle\nabla\phi_{n,p}\rangle_{z+\square_m}\right|^2\right]\\&\leq\frac{1}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\left|\langle\nabla\phi\rangle_{z+\square_m}\right|^2\right]+\frac{1}{|\square_n|}\mathbb{E}\left[\sum_{e\subseteq\square_n}|\nabla\phi(e)-\nabla\phi_{n,p}(e)|^2\right]\\&\leq\frac{1}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\left|\langle\nabla\phi\rangle_{z+\square_m}\right|^2\right]+C(1+|p|^2)3^{-\alpha m}.\end{aligned}$$

We then note that for each $z\in\mathcal{Z}_{m,n}$, $\langle\nabla\phi\rangle_{z+\square_m}=\langle\nabla\phi_z\rangle_{z+\square_m}$ and that, since $\phi_z\in h_0^1(z+\square_m)$, one has $\langle\nabla\phi_z\rangle_{z+\square_m}=0$. Consequently, the previous display can be simplified and one obtains

$$\frac{1}{|\mathcal{Z}_{m,n}|}\sum_{z\in\mathcal{Z}_{m,n}}\mathbb{E}\left[\left|\langle\nabla\phi_{n,p}\rangle_{z+\square_m}\right|^2\right]\leq C(1+|p|^2)3^{-\alpha m}.$$

Step 2. We now apply the multiscale Poincaré inequality stated in Proposition 5.2.14 for functions in $h_0^1(\square_n)$, this gives

$$\frac{1}{|\square_n|}\sum_{x\in\square_n}|\phi_{n,p}(x)|^2\leq C\sum_{e\subseteq\square_n}|\nabla\phi_{n,p}(e)|^2+C3^n\sum_{m=1}^n3^m\left(\frac{1}{|\mathcal{Z}_{m,n}|}\sum_{y\in\mathcal{Z}_{m,n}}\left|\langle\nabla\phi_{n,p}\rangle_{z+\square_m}\right|^2\right).$$

Taking the expectation and using Proposition 5.3.7 to estimate the first term on the right-hand side and the main result of Step 1 to estimate the second term gives

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_n|}\sum_{x\in\square_n}|\phi_{n,p}(x)|^2\right]&\leq C(1+|p|^2)+C(1+|p|^2)3^n\sum_{m=1}^n3^m3^{-\alpha m}\\&\leq C(1+|p|^2)3^{(2-\alpha)n}.\end{aligned}$$

This is the desired result. The proof of Theorem 5.1.2 is complete. □

5.A. Technical estimates

Before stating the first proposition of this appendix, we recall that the space H mentioned in the following propsoition is the space of functions of $\mathring{h}^1(\square_n)$ which are constant on the subcubes $z+\square_m$, for $z\in\mathcal{Z}_{m,n}$. It is a space of dimension $3^{d(n-m)}-1$ and each function $h\in H$ can be written in the following form

$$h=\sum_{z\in\mathcal{Z}_{m,n}}\lambda_z\mathbb{1}_{z+\square_m},$$

for some constants $(\lambda_z)_{z\in\mathcal{Z}_{m,n}}$ satisfying $\sum_{z\in\mathcal{Z}_{m,n}}\lambda_z=0$.

PROPOSITION 5.A.1. *There exists a constant $C:=C(d,\lambda)<\infty$ such that the following estimate holds, for each $\psi\in\mathring{h}^1(\square_n)$,*

$$\log\int_H\exp\left(-\sum_{e\in B_{m,n}}V_e\left(\nabla\psi(e)+\nabla h(e)\right)\right)dh\leq Cm3^{d(n-m)}.$$

PROOF. By the assumptions made on the elastic potential V_e, one has, for each $x\in\mathbb{R}$,

$$V_e(x)\geq\lambda x^2.$$

This gives the following estimate, for each $\psi\in\bigoplus_{z\in\mathcal{Z}_{m,n}}\mathring{h}^1(z+\square_m)$,

$$-\sum_{e\in B_{m,n}}V_e\left(\nabla\psi(e)+\nabla h(e)\right)\leq-\sum_{e\in B_{m,n}}\lambda\left(\nabla\psi(e)+\nabla h(e)\right)^2. \tag{5.1.1}$$

For the rest of the proof, we introduce the following notation: for $z,z'\in\mathcal{Z}_{m,n}$, we write

$$z\sim z'\text{ if and only if }|z-z'|_1=3^m.$$

That is to say, we write $z \sim z'$ if and only if z and z' are neighbors in the rescaled lattice $3^m\mathbb{Z}^d$. With this notation, the set $B_{m,n}$ can be partitioned according to

$$B_{m,n} = \bigcup_{z,z' \in \mathcal{Z}_{m,n},\, z \sim z'} F_{z,z'},$$

where we introduced the notation

$$F_{z,z'} := \left\{ e = (x,y) \subseteq \square_{n+1} \, : \, x \in z + \square_n \text{ and } y \in z' + \square_n \right\}.$$

With this notation, the right-hand side of (5.1.1) can be rewritten

$$\sum_{e \in B_{m,n}} (\nabla \psi(e) + \nabla h(e))^2 = \sum_{z,z' \in \mathcal{Z}_{m,n},\, z \sim z'} \sum_{e \in F_{z,z'}} (\nabla \psi(e) + \lambda_{z'} - \lambda_z)^2 .$$

Expanding the square gives, for each $z, z' \in \mathcal{Z}_{m,n}$ satisfying $z \sim z'$,

$$\begin{aligned}
\sum_{e \in F_{z,z'}} \left(\nabla \psi(e) + \lambda_z' - \lambda_z \right)^2 &= \sum_{e \in F_{z,z'}} |\nabla \psi(e)|^2 + 2 \nabla \psi(e) \left(\lambda_{z'} - \lambda_z \right) + |\lambda_{z'} - \lambda_z|^2 \\
&= |F_{z,z'}| \left(\lambda_{z'} - \lambda_z + \frac{1}{|F_{z,z'}|} \sum_{e \in F_{z,z'}} \nabla \psi(e) \right)^2 \\
&\quad - \frac{1}{|F_{z,z'}|} \left| \sum_{e \in F_{z,z'}} \nabla \psi(e) \right|^2 + \sum_{e \in F_{z,z'}} |\nabla \psi(e)|^2.
\end{aligned}$$

But one has

$$- \frac{1}{|F_{z,z'}|} \left| \sum_{e \in F_{z,z'}} \nabla \psi(e) \right|^2 + \sum_{e \in F_{z,z'}} |\nabla \psi(e)|^2 \geq 0,$$

thus one derives

$$\sum_{e \in F_{z,z'}} \left(\nabla \psi(e) + \lambda_z' - \lambda_z \right)^2 \geq |F_{z,z'}| \left(\lambda_{z'} - \lambda_z + \frac{1}{|F_{z,z'}|} \sum_{e \in F_{z,z'}} \nabla \psi(e) \right)^2 .$$

Note that the cardinality $|F_{z,z'}|$ is the same for each $z, z' \in \mathcal{Z}_{m,n}$ such that $z \sim z'$. It is indeed the cardinality of a face of the cube $\square_m$ and is equal to $3^{(d-1)m}$. This cardinality will be denoted by $|F_m|$ in the rest of the proof.

The next step of the proof is to construct an isometry between the spaces H and $\mathring{h}^1(\square_{n-m})$. To do so, note that one has the equality

$$\mathcal{Z}_{m,n} = 3^m \square_{n-m},$$

in particular if $z \in \mathcal{Z}_{m,n}$ then $z/3^m \in \square_{n-m}$. From this one obtains that the existence of an isometry between the spaces H and $\mathring{h}^1(\square_{n-m})$ given by

$$\Phi : \left\{ \begin{array}{ccc} H & \to & \mathring{h}^1 \left(\square_{n-m} \right) \\ h := \sum_{z \in \mathcal{Z}_{m,n}} \lambda_z \mathbb{1}_{\{z + \square_m\}} & \mapsto & \Phi(h) = 3^{\frac{dm}{2}} \sum_{z \in \mathcal{Z}_{m,n}} \lambda_z \delta_{z/3^m}, \end{array} \right. \tag{5.1.2}$$

where δ_z is the function defined by $\delta_z(z') = 1$ if $z = z'$ and $\delta_z(z') = 0$ otherwise. The scalar $3^{\frac{dm}{2}}$ is here to ensure that

$$\sum_{x \in \square_n} h(x)^2 = \sum_{z \in \mathcal{Z}_{m,n}} 3^{dm} |\lambda_z|^2$$

is equal to

$$\sum_{x \in \square_{x \in \square_{n-m}}} \Phi(h)(x)^2 = \sum_{z \in \mathcal{Z}_{m,n}} \left| 3^{\frac{dm}{2}} \lambda_z \right|^2 .$$

We also denote by X_ψ the vector field defined on the edges of $\square_{n-m}$ by

$$X_\psi\left(\frac{z}{3^m}, \frac{z'}{3^m}\right) = \frac{1}{|F_m|} \sum_{e \in F_{z,z'}} \nabla\psi(e).$$

Performing the change of variables by the isometry Φ shows

$$\int_H \exp\left(-\sum_{e \in B_{m,n}} \lambda\left(\nabla\psi(e) + \nabla h(e)\right)^2\right) dh \leq \int_{\mathring{h}^1(\square_{n-m})} \exp\left(-\sum_{e \subseteq \square_{n-m}} \lambda |F_m| \left(3^{-\frac{dm}{2}} \nabla h(e) + X_\psi(e)\right)^2\right) dh$$

Using the equality $|F_m| = 3^{(d-1)m}$, one obtains

$$\int_H \exp\left(-\sum_{e \in B_{m,n}} \lambda\left(\nabla\psi(e) + \nabla h(e)\right)^2\right) dh \leq \int_{\mathring{h}^1(\square_{n-m})} \exp\left(-\sum_{e \subseteq \square_{m,n}} \lambda 3^{(d-1)m} \left(3^{-\frac{dm}{2}} \nabla h(e) - X_\psi(e)\right)^2\right) dh. \tag{5.1.3}$$

We denote by $V(\square_{n-m})$ the vector space of vector fields of $\square_{n-m}$ and equip it with the standard L^2 scalar product. The idea is then to consider the following orthogonal decomposition

$$V(\square_{n-m}) = \nabla\mathring{h}^1(\square_{n-m}) \overset{\perp}{\oplus} \left(\nabla\mathring{h}^1(\square_{n-m})\right)^\perp.$$

so that the vector field X_ψ can be decomposed according to the formula

$$X_\psi = \nabla h_\psi + X_\psi^\perp,$$

where $h_\psi \in \mathring{h}^1(\square_{n-m})$ and $X_\psi^\perp \in \left(\nabla\mathring{h}^1(\square_{n-m})\right)^\perp$. Using this decomposition, one has

$$\begin{aligned}\sum_{e \subseteq \square_{n-m}} \left(3^{-\frac{dm}{2}} \nabla h(e) - X_\psi(e)\right)^2 &= \sum_{e \subseteq \square_{n-m}} \left(3^{-\frac{dm}{2}} \nabla h(e) - \nabla h_\psi\right)^2 + \sum_{e \subseteq \square_{n-m}} \left(X_\psi^\perp\right)^2 \\ &\geq \sum_{e \subseteq \square_{n-m}} \left(3^{-\frac{dm}{2}} \nabla h(e) - \nabla h_\psi\right)^2.\end{aligned}$$

Using the previous inequality, the estimate (5.1.3) becomes

$$\int_H \exp\left(-\sum_{e \in B_{m,n}} \lambda\left(\nabla\psi(e) + \nabla h(e)\right)^2\right) dh \leq \int_{\mathring{h}^1(\square_{n-m})} \exp\left(-\sum_{e \subseteq \square_{n-m}} \lambda 3^{(d-1)m} \left(3^{-\frac{dm}{2}} \nabla h(e) - \nabla h_\psi\right)^2\right) dh.$$

We then use the translation invariance of the Lebesgue measure to prove

$$\int_{\mathring{h}^1(\square_{n-m})} \exp\left(-\sum_{e \subseteq \square_{n-m}} \lambda 3^{(d-1)m} \left(3^{-\frac{dm}{2}} \nabla h(e) - \nabla h_\psi\right)^2\right) dh = \int_{\mathring{h}^1(\square_{n-m})} \exp\left(-\sum_{e \subseteq \square_{n-m}} \lambda 3^{-m} \nabla h(e)^2\right) dh.$$

Combining the two previous displays yields

$$\int_H \exp\left(-\sum_{e \in B_{m,n}} \lambda\left(\nabla\psi(e) + \nabla h(e)\right)^2\right) dh \leq \int_{\mathring{h}^1(\square_{n-m})} \exp\left(-\sum_{e \subseteq \square_{n-m}} \lambda 3^{-m} \nabla h(e)^2\right) dh.$$

We then perform a change of variables to get,

$$\int_H \exp\left(-\sum_{e\in B_{m,n}} \lambda\left(\nabla\psi(e)+\nabla h(e)\right)^2\right) dh \le \left(\frac{3^m}{\lambda}\right)^{\frac{\left(3^{d(n-m)}-1\right)}{2}} \int_{\mathring{h}^1(\square_{n-m})} \exp\left(-\sum_{e\subseteq \square_{n-m}} \nabla h(e)^2\right) dh,$$

since $\dim \mathring{h}^1(\square_{n-m}) = 3^{d(n-m)}-1$. Taking the logarithm and applying Proposition 5.3.7, one sees that

$$\log \int_H \exp\left(-\sum_{e\in B_{m,n}} \lambda\left(\nabla\psi(e)+\nabla h(e)\right)^2\right) dh \le Cm\left(3^{d(n-m)}-1\right) + C|\square_{n-m}| \le Cm 3^{d(n-m)}.$$

This completes the proof of Proposition 5.A.1. □

We now turn to the proof of the second technical lemma of the appendix which allows to use the uniform convexity of V to obtain an L^2estimate when perturbing around ψ_z. This lemma and the notation are used in Step 4 of the proof of Proposition 5.4.5.

Proposition 5.A.2. *There exists a constant $C := C(d,\lambda) < \infty$ such that for each $n\in\mathbb{N}$,*

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa(e)\right)\right] \le \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\psi_z(e)\right)\right] + C(1+|q|^2)3^{-\frac{n}{2}} + C\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} |\nabla\kappa(e)-\nabla\sigma_z(e)|^2\right]. \tag{5.1.4}$$

Proof. Denote by σ the random variable taking values in $\oplus_{z\in\mathcal{Z}_{n,2n}}\mathring{h}^1(z+\square_n)$ given by

$$\sigma = \sum_{z\in\mathcal{Z}_{n,2n}} \sigma_z,$$

we also recall that the random variable σ_z is defined by the identity

$$\forall z\in\mathcal{Z}_{n,2n}, \forall x\in z+\square_n,\ \sigma_z(x) = \psi_z(x) - \nabla\nu_n^*(q)\cdot x.$$

Let P be the orthogonal projection from $h^1(\square_{2n})$ to $\oplus_{z\in\mathcal{Z}_{n,2n}}\mathring{h}^1(z+\square_n)$. Note the operator P satisfies the following property

$$\forall g\in h^1(\square_{2n}),\ \forall z\in\mathcal{Z}_{n,2n},\ \forall e\subseteq z+\square_n,\ \nabla Pg(e) = \nabla g(e). \tag{5.1.5}$$

Denote by ξ the random variable taking values in $\oplus_{z\in\mathcal{Z}_{n,2n}}\mathring{h}^1(z+\square_n)$, defined according to

$$\xi := 2\sigma - P\kappa,$$

so that $\sigma = \frac{\xi+P\kappa}{2}$. Using the uniform convexity of the elastic potential V_e, one has

$$2\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\psi_z(e)\right)\right] \ge \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu_n^*(q)(e)+\nabla\xi(e)\right)\right] + \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu_n^*(q)(e)+\nabla P\kappa(e)\right)\right] - C\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} |\nabla\sigma_z(e)-\nabla P\kappa(e)|^2\right].$$

We then use (5.1.5) to get

$$
(5.1.6)\quad 2\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\psi_z(e)\right)\right] \geq \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu_n^*(q)(e)+\nabla\xi(e)\right)\right]
$$
$$
+\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu_n^*(q)(e)+\nabla\kappa(e)\right)\right]
$$
$$
-C\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}|\nabla\psi(e)-\nabla\kappa(e)|^2\right].
$$

Note that the random variable $\sum_{z\in\mathcal{Z}_{n,2n}}\psi_z$ is the minimizer of the problem

$$
\inf_{\mathbb{P}}\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}\left(V_e(\nabla\psi')-q\cdot\nabla\psi'(e)\right)\right]+H\left(\mathbb{P}\right),
$$

where the infimum is taken over all the probability measures on $\oplus_{z\in\mathcal{Z}_{n,2n}}\mathring{h}^1(z+\square_n)$ and ψ' is a random variable of law $\mathbb{P}$. In particular, using the translation invariance of the entropy gives

$$
(5.1.7)\quad \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu_n^*(q)(e)+\nabla\xi(e)\right)-q\cdot\left(\nabla\nu_n^*(q)+\nabla\xi\right)(e)\right]+\frac{1}{|\square_{2n}|}H\left(\mathbb{P}_\xi\right)
$$
$$
\geq\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\psi_z(e)\right)-q\cdot\nabla\psi_z(e)\right]+\frac{1}{|\square_n|}H\left(\mathbb{P}_{n,q}^*\right).
$$

We first simplify a slightly the previous display by removing the linear terms in the left and right-hand side. Using that $\nabla\nu_n^*(q)=\mathbb{E}\left[\langle\nabla\psi_{n,q}\rangle_{\square_n}\right]$, we obtain

$$
\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} q\cdot\nabla\psi_z(e)\right]=q\cdot\nabla\nu_n^*(q),
$$

and, using the definition of σ,

$$
\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} q\cdot\left(\nabla\nu_n^*(q)+\nabla\xi\right)(e)\right]=q\cdot\nabla\nu_n^*(q)+\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} q\cdot\nabla P\kappa(e)\right].
$$

We denote by $B^+_{n,2n}$ the set of edges of $\square^+_{2n}$ which do not belong to a cube of the form $z+\square_n$, for $z\in\mathcal{Z}_{n,2n}$, i.e.

$$
B^+_{n,2n}:=\left\{e\subseteq\square^+_{2n}\;:\;\forall z\in\mathcal{Z}_{n,2n},\,e\nsubseteq z+\square_n\right\}.
$$

Using this set, one can split the sum

$$
\sum_{e\subseteq\square^+_{2n}}=\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n}+\sum_{e\in B^+_{n,2n}}.
$$

Note also that the set $B^+_{n,2n}$ is almost equal to the set $B_{n,2n}$, the only difference is that we added the edges which belong to $\square^+_{2n}$ but not $\square_{2n}$, which is a small boundary layer of edges. Additionally, one has the estimate on the cardinality of $B^+_{n,2n}$,

$$
\left|B^+_{n,2n}\right|\leq C3^{-n}\left|\square_{2n}\right|.
$$

Using the partition of the sum, the fact that $\kappa \in h_0^1(\square_{2n}^+)$ and the property (5.1.5), one obtains,

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} q\cdot(\nabla P\kappa)(e)\right] = \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} q\cdot\nabla\kappa(e)\right]$$
$$= \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B_{n,2n}^+} q\cdot\nabla\kappa(e)\right].$$

We then apply the Cauchy-Schwarz inequality as well as the definition of κ given in (5.4.19) to obtain

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B_{n,2n}^+} q\cdot\nabla\kappa(e)\right] &\le |q|\left(\frac{|B_{n,2n}^+|}{|\square_{2n}|}\right)^{\frac12}\left(\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\in B_{n,2n}^+}|\nabla\kappa(e)|^2\right]\right)^{\frac12}\\
&\le C|q|3^{-\frac{n}{2}}\left(\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+}|\nabla\kappa(e)|^2\right]\right)^{\frac12}\\
&\le C|q|3^{-\frac{n}{2}}\left(\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+}|\mathbf{f}(e)|^2\right]\right)^{\frac12}\\
&\le C|q|3^{-\frac{n}{2}}(1+|q|).\end{aligned}$$

But using the definition of $\mathbf{f}$ given in (5.4.12) together with the estimate (5.3.8) of Proposition 5.3.1, one has

$$\left(\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{e\subseteq\square_{2n}^+}|\mathbf{f}(e)|^2\right]\right)^{\frac12} \le C(1+|q|).$$

A combination of the previous displays gives

$$\left|\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} q\cdot\nabla P\kappa(e)\right]\right| \le C3^{-\frac{n}{2}}(1+|q|^2).$$

Using the previous estimate in (5.1.7), one obtains the simplified display

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\nu_n^*(q)(e)+\nabla\xi(e)\right)\right] + \frac{1}{|\square_{2n}|}H\left(\mathbb{P}_\xi\right)\\
\ge \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\psi_z(e)\right)\right] + \frac{1}{|\square_{2n}|}H\left(\mathbb{P}_\psi\right) - C3^{-\frac{n}{2}}(1+|q|^2).\end{aligned} \tag{5.1.8}$$

We now show the following estimate comparing the entropies of $\mathbb{P}_\xi$ and $\mathbb{P}_\psi$,

$$\frac{1}{|\square_{2n}|}H\left(\mathbb{P}_\xi\right) \le \frac{1}{|\square_n|}H\left(\mathbb{P}_{n,q}^*\right) + C3^{-n}. \tag{5.1.9}$$

First we recall the definition of the linear operator L given in (5.4.15) and that the random variable κ is defined by

$$\kappa := L\left(\sum_{z\in\mathcal{Z}_{n,2n}}\sigma_z\right).$$

We consequently have

$$\xi = (2\mathrm{Id} - P\circ L)\psi,$$

where $2\mathrm{Id} - P \circ L$ is seen as a linear operator from $\oplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$ into itself. Using the change of variables formula for the entropy, one obtains

$$\frac{1}{|\square_{2n}|} H\left(\mathbb{P}_\xi\right) = \frac{1}{|\square_n|} H\left(\mathbb{P}^*_{n,q}\right) - \ln|\det(2\mathrm{Id} - P\circ L)|. \tag{5.1.10}$$

Since the dimension of $\oplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$ is $3^{2dn}-3^{dn}$, we denote by $l_1, \ldots, l_{3^{2dn}-3^{dn}}$ the eigenvalues (potentially complex and with repetition) of $P\circ L$. We thus have

$$\ln|\det(2\mathrm{Id} - P\circ L)| = \sum_{i=0}^{3^{2dn}-3^{dn}} \ln|2 - l_i|.$$

We now prove the two following statements on l_i

(1) for each $i \in \{1, \ldots, 3^{2dn} - 3^{dn}\}$, $|l_i| \le 1$.
(2) There exists a constant $C := C(d) < \infty$ such that at least $3^{2dn} - C3^{dn}$ eigenvalues l_i are equal to 1.

To prove the first fact, note that, by (5.1.5), for each $\psi \in \oplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$,

$$\sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla P\circ L(\psi)(e)|^2 = \sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla L(\psi)(e)|^2.$$

Moreover by (5.4.30), one has

$$\sum_{e\subseteq\square_{2n}^+} |\nabla L(\psi)(e)|^2 \le \sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla\psi(e)|^2.$$

Since one clearly has

$$\sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla L(\psi)(e)|^2 \le \sum_{e\subseteq\square_{2n}^+} |\nabla L(\psi)(e)|^2,$$

one obtains

$$\sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla P\circ L(\psi)(e)|^2 \le \sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla\psi(e)|^2.$$

Thus if l_i is an eigenvalue of $P\circ L$, consider ψ_i an eigenvector (which may be complex) associated to this eigenvalue. Then one has

$$|l_i|^2 \sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla\psi_i(e)|^2 \le \sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla\psi_i(e)|^2,$$

which implies $|l_i| \le 1$ as soon as $\sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla\psi_i(e)|^2$ is not 0, but since $\psi_i \in \oplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1(z+\square_n)$, one also has

$$\sum_{z\in\mathcal{Z}_{n,2n}} \sum_{e\subseteq z+\square_n} |\nabla\psi_i(e)|^2 = 0 \iff \psi_i = 0.$$

This completes the proof of the first item.

We now prove the second item. To this end we proceed exactly as in Step 1 of Proposition 5.4.5, where it is proved proved that if one considers the interior $\square_n^{\circ}$,

$$\square_n^{\circ} := \left(-\frac{3^n-2}{2}, \frac{3^n-2}{2}\right)^d \cap \mathbb{Z}^d = \square_n \setminus \partial\square_n,$$

then for each $\psi \in \oplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1\left(z+\square_n^{\circ}\right)$ one has

$$L(\psi) = \psi.$$

Moreover, from (5.4.25) one has the estimate on the dimension of $\oplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1\left(z+\square_n^-\right)$,

$$\dim\left(\bigoplus_{z\in\mathcal{Z}_{n,2n}} \mathring{h}^1\left(z+\square_n^-\right)\right) \ge 3^{2dn} - C3^{(2d-1)n}.$$

This implies that among the l_i, at least $3^{2dn} - C3^{(2d-1)n}$ of them are equal to 1. Without loss of generality, we can thus assume that for each $i \in \{1, \ldots, 3^{2dn} - C3^{(2d-1)n}\}$, $l_i = 1$.

Combining (1) and (2), we obtain

$$\begin{aligned}\frac{1}{|\square_{2n}|}\left|\ln|\det(2\mathrm{Id}-P\circ L)|\right| &\leq \sum_{i=0}^{3^{2dn}-3^{dn}} \left|\ln|2-l_i|\right| \\ &\leq \sum_{i=3^{2dn}-C3^{(2d-1)n}}^{3^{2dn}-3^{dn}} \left|\ln|2-l_i|\right| \\ &\leq C3^{(2d-1)n}.\end{aligned}$$

Thus

$$\frac{1}{|\square_{2n}|}\left|\ln|\det(2\mathrm{Id}-P\circ L)|\right| \leq C3^{-n}.$$

Combining this estimate with (5.1.10) gives

$$\frac{1}{|\square_{2n}|}H\left(\mathbb{P}_\xi\right) \leq \frac{1}{|\square_n|}H\left(\mathbb{P}^*_{n,q}\right) + C3^{-n}.$$

This is precisely (5.1.9).

We then combine (5.1.8) and (5.1.9) to obtain

$$\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\xi(e)\right)\right] \geq \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\psi_z(e)\right)\right] - C(1+|q|^2)3^{-\frac{n}{2}}.$$

Using this inequality together with (5.1.6) gives

$$\begin{aligned}\mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\psi(e)\right)\right] &\geq \mathbb{E}\left[\frac{1}{|\square_{2n}|}\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} V_e\left(\nabla\kappa(e)\right)\right] - C(1+|q|^2)3^{-\frac{n}{2}} \\ &\quad - C\mathbb{E}\left[\sum_{z\in\mathcal{Z}_{n,2n}}\sum_{e\subseteq z+\square_n} |\nabla\psi(e)-\nabla\kappa(e)|^2\right].\end{aligned}$$

This is (5.1.4) and thus the proof of Lemma 5.A.2 is complete. □

We then prove the last lemma of this appendix. It gives a quadratic upper bound for the value of $\nu(U,p)$ for any bounded domain $U\subseteq\mathbb{Z}^d$.

Proposition 5.A.3. *There exists a constant* $C := C(d,\lambda) < \infty$ *such that for each bounded domain* $U\subseteq\mathbb{Z}^d$ *and each* $p\in\mathbb{R}^d$,

$$\nu\left(U,p\right) \leq C(1+|p|^2).$$

Remark 5.A.4. This statement is a more general version of the upper bound for ν than the one given in Proposition 5.3.7, since it is valid for any bounded domain $U\subseteq\mathbb{Z}^d$, nevertheless the argument presented here does not give a lower bound as the one we computed in the case of cubes. It also does not give bounds on ν^*, this is why this statement is presented in the appendix.

Proof. Consider a random variable X, taking values in $h^1_0(U)$ whose law is defined by

- for each $x\in U$, the law of $X(x)$ is uniform in $[0,1]$
- the random variables $X(x)$, for $x\in U$ are independent.

Using that the entropy of the law uniform in $[0,1]$ is equal to 0 together with Proposition 5.2.4, one obtains

$$H\left(\mathbb{P}_X\right) = 0.$$

Then by Proposition 5.2.8, one has the following computation

$$\begin{aligned}\nu(U,p) &\leq \mathbb{E}\left[\frac{1}{|U|}\sum_{e\subseteq U} V_e\left(p(e)+\nabla X(e)\right)\right] + \frac{1}{|U|}H\left(\mathbb{P}_X\right)\\ &\leq \mathbb{E}\left[\frac{1}{|U|}\sum_{e\subseteq U} V_e\left(p(e)+\nabla X(e)\right)\right].\end{aligned}$$

We then use the bound $V_e(x) \leq \frac{1}{\lambda}|x|^2$ combined with the estimate $|\nabla X(e)| \leq 1$ for each $e \subseteq U$ to obtain

$$\mathbb{E}\left[\frac{1}{|U|}\sum_{e\subseteq U} V_e\left(p(e)+\nabla X(e)\right)\right] \leq C\left(1+|p|^2\right).$$

A combination of the two previous displays completes the proof of the proposition. □

5.B. Functional inequalities

The goal of this second appendix is to prove some classic inequalities from the theory of elliptic equations in the setting of the $\nabla\phi$ model. These inequalities are proved with the random variable $\psi_{n,q}$ associated to the law $\mathbb{P}^*_{n,q}$ because it is needed in the proof of Theorem 5.1.2, nevertheless similar statements, with similar proofs, should exist for the random variable $\phi_{n,p}$ associated to the law $\mathbb{P}_{n,p}$.

Proposition 5.B.1 (Interior Caccioppoli inequality). *There exists a constant $C := C(d,\lambda) < \infty$ such that for every integer $n \geq 1$, every $x \in \square_n$ and every $r \geq 1$ such that $B(x,2r) \subseteq \square_n$*

$$\mathbb{E}\left[\sum_{e\subseteq B(x,r)} |\nabla\psi_{n,q}(e)|^2\right] \leq \frac{C}{r^2}\mathbb{E}\left[\sum_{x\in B(x,2r)} \left|\psi_{n,q}(y) - (\psi_{n,q})_{B(x,2r)}\right|^2\right] + Cr^d.$$

Proof. Let η be a cutoff function defined on the discrete lattice $\square_n$ taking values in $\mathbb{R}$ satisfying

$$\mathbb{1}_{B(x,r)} \leq \eta \leq \mathbb{1}_{B(x,2r)} \text{ and } \forall e=(x,y)\subseteq \square_n,\ |\nabla\eta(e)|^2 \leq Cr^{-2}\left(\eta(x)+\eta(y)\right).$$

For $t \geq 0$, denote by L_t the following linear operator

$$L_t := \left\{\begin{array}{lcl} \mathring{h}^1(\square_n) & \to & \mathring{h}^1(\square_n) \\ \psi & \mapsto & \psi + t\eta\left(\psi - (\psi)_{B(x,2r)}\right) - \left(\psi + t\eta\left(\psi - (\psi)_{B(x,2r)}\right)\right)_{\square_n}. \end{array}\right.$$

As a remark, note that the last term on the right-hand side can be rewritten

$$\left(\psi + t\eta\left(\psi - (\psi)_{B(x,2r)}\right)\right)_{\square_n} = t\left(\eta\left(\psi - (\psi)_{B(x,2r)}\right)\right)_{\square_n},$$

since $\psi \in \mathring{h}^1(\square_n)$. Note that L_0 is the identity of $h^1(\square_n)$. We now show the following inequality which estimates the distance between L_t and the identity of $\mathring{h}^1(\square_n)$, in the L^2 operator norm:

$$\forall\psi\in\mathring{h}^1(\square_n),\ \sum_{x\in\square_n}|\psi(x) - L_t(\psi)(x)|^2 \leq |t|^2\sum_{x\in\square_n}|\psi(x)|^2.$$

This is a consequence of the computation

$$\begin{aligned}\sum_{x\in\square_n}|\psi(x) - L_t(\psi)(x)|^2 &\leq \sum_{x\in\square_n}\left|t\eta(x)\left(\psi(x) - (\psi)_{B(x,2r)}\right)\right|^2\\ &\leq |t|^2\sum_{x\in B(x,2r)}\left|\psi(x) - (\psi)_{B(x,2r)}\right|^2\\ &\leq |t|^2\sum_{x\in B(x,2r)}|\psi(x)|^2.\end{aligned}$$

This implies in particular that for each $t \in (-1,1)$, the operator L_t is bijective. Note also that by definition of L_t,

$$\forall\psi\in\mathring{h}^1(\square_n),\ \forall e\subseteq\square_n,\ \nabla L_t(\psi)(e) = \nabla\psi(e) + t\nabla\left(\eta\left(\psi - (\psi)_{B(x,2r)}\right)\right)(e) \tag{5.2.1}$$

Fix $q \in \mathbb{R}$. We use the random variable $L_t(\psi_{n,q})$ as a test random variable in Proposition 5.2.8. This yields

$$\mathbb{E}\left[-\sum_{e\subseteq\square_n}\left(V_e\left(\nabla\psi_{n,q}(e)\right) - q\cdot\nabla\psi_{n,q}(e)\right)\right] - H(\mathbb{P}^*_{n,q}) \\ \geq \mathbb{E}\left[-\sum_{e\subseteq\square_n}\left(V_e\left(\nabla L_t\left(\psi_{n,q}\right)(e)\right) - q\cdot\nabla L_t\left(\psi_{n,q}\right)(e)\right)\right] - H\left(\mathbb{P}_{L_t(\psi_{n,q})}\right).$$

First note that, since η is supported in $B(x,2r)\subseteq\square_n$,

$$\left\langle\nabla L_t\left(\psi_{n,q}\right)\right\rangle_{\square_n} = \left\langle\nabla\psi_{n,q}\right\rangle_{\square_n},$$

consequently,

$$\mathbb{E}\left[\sum_{e\subseteq\square_n} q\cdot\nabla\psi_{n,q}(e)\right] = \mathbb{E}\left[\sum_{e\subseteq\square_n} q\cdot\nabla L_t\left(\psi_{n,q}\right)(e)\right].$$

Thus one can simplify the previous display

$$\mathbb{E}\left[-\sum_{e\subseteq\square_n} V_e\left(\nabla\psi_{n,q}(e)\right)\right] - H(\mathbb{P}^*_{n,q}) \geq \mathbb{E}\left[-\sum_{e\subseteq\square_n} V_e\left(\nabla L_t\left(\psi_{n,q}\right)(e)\right)\right] - H\left(\mathbb{P}_{L_t(\psi_{n,q})}\right).$$

By Proposition 5.2.3,

$$H\left(\mathbb{P}_{L_t(\psi_{n,q})}\right) = H(\mathbb{P}^*_{n,q}) - \ln\det L_t.$$

Using the previous display and the formula for L_t, one obtains, for each $t\in(-1,1)$,

$$\mathbb{E}\left[\sum_{e\subseteq\square_n} V_e\left(\nabla\psi_{n,q}(e) + t\nabla\left(\eta\left(\psi_{n,q} - (\psi_{n,q})_{B(x,2r)}\right)\right)(e)\right) - V_e\left(\nabla\psi_{n,q}(e)\right)\right] - \ln\det L_t \geq 0.$$

It is clear that the function $t\to\ln\det L_t$ is smooth for $t\in(-1,1)$. In particular, dividing the previous display by t and sending t to 0 gives

$$\mathbb{E}\left[\sum_{e\subseteq\square_n} V'_e\left(\nabla\psi_{n,q})(e)\right)\nabla\left(\eta\left(\psi_{n,q} - (\psi_{n,q})_{B(x,2r)}\right)\right)(e)\right] - \frac{d}{dt}_{|t=0}\ln\det L_t = 0. \tag{5.2.2}$$

We first deal with the term coming from the entropy. By the chain rule, one has the formula

$$\frac{d}{dt}_{|t=0}\ln\det L_t = \operatorname{tr} L'_0,$$

where L'_0 denote the derivative of the operator L_t at $t=0$, it is given by the explicit formula

$$L'_0 := \begin{cases} \mathring{h}^1(\square_n) & \to \quad \mathring{h}^1(\square_n) \\ \psi & \mapsto \quad \eta\left(\psi - (\psi)_{B(x,2r)}\right) - \left(\eta\left(\psi - (\psi)_{B(x,2r)}\right)\right)_{\square_n}. \end{cases}$$

In particular, for each $\psi\in\mathring{h}^1(\square_n)$,

$$\begin{aligned}\sum_{x\in\square_n}\left|L'_0(\psi)(x)\right|^2 &\leq \sum_{x\in\square_n}\left|\eta(x)\left(\psi(x) - (\psi)_{B(x,2r)}\right)\right|^2 \\ &\leq \sum_{x\in B(x,2r)}\left|\psi(x) - (\psi)_{B(x,2r)}\right|^2 \\ &\leq \sum_{x\in B(x,2r)}\left|\psi(x)\right|^2.\end{aligned}$$

This implies that every function ψ supported in $\square_n \setminus B(x,2r)$ is in the kernel of L'_0 and thus one has

$$\dim\ker L'_0 \geq |\,\square_n\,| - Cr^d.$$

Combining the two previous displays shows

$$\left|\operatorname{tr} L'_0\right| \leq \dim\mathring{h}^1(\square_n) - \dim\ker L'_0 \leq Cr^d. \tag{5.2.3}$$

We now turn to the first term on the right-hand side of (5.2.2). To simplify the notation in the following computation, we set

$$\chi_{n,q} := \psi_{n,q} - (\psi_{n,q})_{B(x,2r)}$$

and compute

$$\begin{aligned}
&\sum_{e\subseteq B(x,2r)} V_e'\left(\nabla\chi_{n,q})(e)\right)\nabla(\eta\chi_{n,q})(e) \\
&\quad = \sum_{x,y\in B(x,2r),x\sim y} \left(\eta(x)\chi_{n,q}(x) - \eta(y)\chi_{n,q}(y)\right) V_{(x,y)}'\left(\chi_{n,q}(x) - \chi_{n,q}(x)\right) \\
&\quad = \sum_{x,y\in B(x,2r),x\sim y} \eta(x)\left(\chi_{n,q}(x) - \chi_{n,q}(y)\right) V_{(x,y)}'\left(\chi_{n,q}(x) - \chi_{n,q}(x)\right) \\
&\qquad + \sum_{x,y\in B(x,2r),x\sim y} \chi_{n,q}(y)\left(\eta(x) - \eta(y)\right) V_{(x,y)}'\left(\chi_{n,q}(x) - \chi_{n,q}(x)\right).
\end{aligned}$$

Using the uniform convexity of V_e and (5.2.2), one obtains, by taking the expectation,

$$\begin{aligned}
&\lambda\mathbb{E}\left[\sum_{x,y\in B(x,2r),x\sim y} \eta(x)\left(\chi_{n,q}(x) - \chi_{n,q}(y)\right)^2\right] \\
&\quad \le \mathbb{E}\left[\sum_{x,y\in B(x,2r),x\sim y} \eta(x)\left(\chi_{n,q}(x) - \chi_{n,q}(y)\right) V_{(x,y)}'\left(\chi_{n,q}(x) - \chi_{n,q}(x)\right)\right] \\
&\quad \le \mathbb{E}\left[\sum_{x,y\in B(x,2r),x\sim y} |\chi_{n,q}(y)|\,|\eta(x) - \eta(y)|\left|V_{(x,y)}'\left(\chi_{n,q}(x) - \chi_{n,q}(x)\right)\right|\right] + |\operatorname{tr} L_0'|.
\end{aligned}$$

We then use the bound $V_e'(x) \le \lambda|x|$. This yields

$$\begin{aligned}
&\lambda\mathbb{E}\left[\sum_{x,y\in B(x,2r),x\sim y} \eta(x)\left(\chi_{n,q}(x) - \chi_{n,q}(y)\right)^2\right] \\
&\quad \le C\mathbb{E}\left[\sum_{x,y\in B(x,2r),x\sim y} \frac{|\eta(x) - \eta(y)|^2}{\eta(x) + \eta(y)} |\chi_{n,q}(y)|^2\right] \\
&\qquad + \mathbb{E}\left[\frac{\lambda}{4}\sum_{x,y\in B(x,2r),x\sim y} \left(\eta(x) + \eta(y)\right)\left(\chi_{n,q}(x) - \chi_{n,q}(y)\right)^2\right] + |\operatorname{tr} L_0'| \\
&\quad \le 3^{-2n}\mathbb{E}\left[\sum_{x,y\in B(x,2r),x\sim y} |\chi_{n,q}(y)|^2\right] + \frac{\lambda}{2}\mathbb{E}\left[\sum_{x,y\in U,x\sim y} \eta(x)\,|\chi_{n,q}(x) - \chi_{n,q}(y)|^2\right] + |\operatorname{tr} L_0'|.
\end{aligned}$$

Absorbing the second term on the right back into the left-hand side and using the estimate (5.2.3) gives,

$$\mathbb{E}\left[\sum_{x,y\in B(x,2r),x\sim y} \eta(x)\left(\chi_{n,q}(x) - \chi_{n,q}(y)\right)^2\right] \le Cr^{-2}\mathbb{E}\left[\sum_x |\chi_{n,q}(x)|^2\right] + Cr^d.$$

Now we replace $\chi_{n,q}$ by $\psi_{n,q} - (\psi_{n,q})_{B(x,2r)}$ to eventually obtain

$$\mathbb{E}\left[\sum_{e\subseteq B(x,r)} |\nabla\psi_{n,q}(e)|^2\right] \le Cr^{-2}\mathbb{E}\left[\sum_{e\subseteq B(x,2r)} \left|\psi_{n,q}(e) - (\psi_{n,q})_{B(x,2r)}\right|^2\right] + Cr^d.$$

This is the desired result. □

The next statement one wishes to obtain is a reverse Hölder inequality for the random variable $\psi_{n,q}$. It is obtained by combining the Caccioppoli inequality proved in the previous proposition with the Sobolev inequality recalled below.

PROPOSITION 5.B.2 (Sobolev inequality on $\mathbb{Z}^d$). *There exists a constant $C := C(d) < \infty$ such that for each $x \in \mathbb{Z}^d$, each $r \geq 1$, each exponent $s \in \left(\frac{d}{d-1}, \infty\right)$ and each function $f : B(x,r) \to \mathbb{R}$ satisfying*

$$\sum_{x \in B(x,r)} f(x) = 0,$$

one has the estimate

$$\left(\sum_{x \in B(x,r)} |f(x)|^s \right)^{\frac{1}{s}} \leq C \left(\sum_{e \subseteq B(x,r)} |\nabla f(e)|^{s_\star} \right)^{\frac{1}{s_\star}},$$

where $s_\star$ is the Sobolev conjugate defined from s by the formula

$$s_\star := \frac{sd}{s+d}.$$

This inequality can be deduced from the continuous Sobolev inequality (on $\mathbb{R}^d$) by an interpolation argument. From the Sobolev inequality and the Cacioppoli inequality, we deduce the following reverse Hölder inequality

PROPOSITION 5.B.3 (Reverse Hölder inequality for $\mathbb{P}^*_{n,q}$). *There exists a constant $C := C(d,\lambda) < \infty$ such that for every integer $n \geq 1$, every $x \in \square_n$, every $r \geq 1$ such that $B(x,2r) \subseteq \square_n$, and every $q \in \mathbb{R}^d$,*

$$\mathbb{E}\left[\frac{1}{|B(x,r)|} \sum_{e \subseteq B(x,r)} |\nabla \psi_{n,q}(e)|^2 \right] \leq C \left(\frac{1}{|B(x,2r)|} \sum_{e \subseteq B(x,2r)} \mathbb{E}\left[|\nabla \psi_{n,q}(e)|^2 \right]^{\frac{d}{d+2}} \right)^{\frac{d+2}{d}} + C.$$

PROOF. Fix $q \in \mathbb{R}^d$, an integer $n \geq 1$, and a ball $B(x,r)$ with $x \in \square_n$ such that $B(x,2r) \subseteq \square_n$. By Proposition 5.B.1, one has the inequality

$$\mathbb{E}\left[\sum_{e \subseteq B(x,r)} |\nabla \psi_{n,q}(e)|^2 \right] \leq \frac{C}{r^2} \mathbb{E}\left[\sum_{x \in B(x,2r)} \left| \psi_{n,q}(y) - (\psi_{n,q})_{B(x,2r)} \right|^2 \right] + Cr^d.$$

We then apply Proposition 5.B.2 with $s = 2$ and $s_\star = \frac{2d}{d+2}$ and obtain

$$\mathbb{E}\left[\sum_{e \subseteq B(x,r)} |\nabla \psi_{n,q}(e)|^2 \right] \leq \frac{C}{r^2} \left(\sum_{e \subseteq B(x,r)} \mathbb{E}\left[|\nabla \psi_{n,q}(e)|^2 \right]^{\frac{d}{d+2}} \right)^{\frac{d+2}{d}} + Cr^d. \tag{5.2.4}$$

Then for each $N \in \mathbb{N}$, we introduce the following notation for the half space

$$\mathbb{R}^N_+ := \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \, : \, x_1 \geq 0, \ldots, x_N \geq 0 \right\}.$$

Note that the mapping

$$F := \begin{cases} \mathbb{R}^N_+ & \to \quad \mathbb{R} \\ (x_1, \cdots, x_N) & \mapsto \quad \left(\sum_{e \subseteq B(x,r)} |x_i|^{\frac{d}{d+2}} \right)^{\frac{d+2}{d}} \end{cases}$$

is concave. Then we pick N to be the number of edges of $\square_n$ and we apply Jensen's inequality to the random variable $\left(|\nabla \psi_{n,q}(e)|^2 \right)_{e \subseteq \mathbb{R}^N}$, which is valued in $\mathbb{R}^N_+$, to obtain

$$\mathbb{E}\left[\left(\sum_{e \subseteq B(x,r)} |\nabla \psi_{n,q}(e)|^{\frac{2d}{d+2}} \right)^{\frac{d+2}{d}} \right] \leq \left(\sum_{e \subseteq B(x,r)} \mathbb{E}\left[|\nabla \psi_{n,q}(e)|^2 \right]^{\frac{d}{d+2}} \right)^{\frac{d+2}{d}}.$$

Combining this estimate with (5.2.4) and dividing by r^d completes the proof of Proposition 5.B.3. □

We then combine the previous estimate with the discrete version of the Gehring's Lemma, which is stated in the following proposition. The continuous version of this result can be found in [**77**].

Proposition 5.B.4 (Discrete Gehring Lemma). *Fix $q < 1$, $K \geq 1$ and $R > 0$. Suppose that we are given two (discrete) functions $f, g : B(0,R) \to \mathbb{R}$, and that f satisfies the following reverse Hölder inequality, for each $z \in \mathbb{Z}^d$ and each $r \geq 1$ such that $B(z,2r) \subseteq B(0,R)$,*

$$\frac{1}{|B(x,r)|}\sum_{x\in B(x,r)}|f(x)| \leq K\left(\frac{1}{|B(x,2r)|}\sum_{x\in B(x,2r)}|f(x)|^q\right)^{\frac{1}{q}} + \frac{K}{|B(x,2r)|}\sum_{x\in B(x,2r)}|g(x)|,$$

then there exist an exponent $\delta := \delta(q,K,d) > 0$ and a constant $C := C(q,K,d) < \infty$ such that

$$\left(\frac{1}{\left|B\left(x,\frac{R}{2}\right)\right|}\sum_{x\in B\left(x,\frac{R}{2}\right)}|f(x)|^{1+\delta}\right)^{\frac{1}{1+\delta}} \leq C\left(\frac{1}{|B(x,R)|}\sum_{x\in B(x,R)}|f(x)|\right) + C\left(\frac{1}{|B(x,R)|}\sum_{x\in B(x,R)}|g(x)|^{1+\delta}\right)^{\frac{1}{1+\delta}}.$$

From this estimate, one obtains the following version of the interior Meyers estimate for the $\nabla\phi$ model. The idea is to combine the reverse Hölder inequality and the Gehring's Lemma to improve the integrability of the expectation of the field $\psi_{n,q}$ (seen as a function from the triadic cube $\square_n$ to $\mathbb{R}$) form L^2 to $L^{2+\delta}$.

Proposition 5.B.5 (Interior Meyers estimate for $\mathbb{P}^*_{n,q}$). *For each $\gamma \in (0,1]$ and each $n \in \mathbb{N}$, denote by $\gamma\square_n$ the cube*

$$\gamma\square_n := \left(-\frac{\gamma 3^n}{2}, \frac{\gamma 3^n}{2}\right)^d \cap \mathbb{Z}^d.$$

Fix $q \in \mathbb{R}^d$. For each $\gamma \in (0,1)$, each $n \in \mathbb{N}$, there exist an exponent $\delta := \delta(d,\lambda) > 0$ and a constant $C := C(d,\lambda,\gamma) < \infty$ such that

$$\left(\frac{1}{|\gamma\square_n|}\sum_{e\subseteq\gamma\square_n}\mathbb{E}\left[|\nabla\psi_{n,q}(e)|^2\right]^{1+\delta}\right)^{\frac{1}{1+\delta}} \leq \frac{C}{|\square_n|}\sum_{e\subseteq\square_n}\mathbb{E}\left[|\nabla\psi_{n,q}(e)|^2\right] + C.$$

Proof. The main idea of the proof is to apply the Gehring's Lemma, Proposition 5.B.4, with the following choice of functions

$$\forall x \in \square_n,\ f(x) := \mathbb{E}\left[\sum_{y\in\square_n, x\sim y}|\psi_{n,q}(y)|^2\right] \text{ and } g(x) = 1.$$

By Proposition 5.B.3, one has the following reverse Hölder inequality: there exists a constant $C := C(d,\lambda) < \infty$ such that for each $z \in \mathbb{Z}^d$ and each $r \geq 1$ satisfying $B(z,2r) \subseteq \square_n$,

$$\frac{1}{|B(x,r)|}\sum_{x\in B(x,r)}|f(x)| \leq C\left(\frac{1}{|B(x,2r)|}\sum_{x\in B(x,2r)}|f(x)|^{\frac{d+2}{d}}\right)^{\frac{d}{d+2}} + \frac{C}{|B(x,2r)|}\sum_{x\in B(x,2r)}|g(x)|.$$

Applying Proposition 5.B.4, there exist an exponent $\delta := \delta(d,\lambda) > 0$ and a constant $C := C(d,\lambda) < \infty$, such that for each $z \in \square_n$, $R \geq 1$ satisfying $B(x,2R) \subseteq \square_n$,

$$\left(\frac{1}{|B(x,R)|} \sum_{x \in B(x,R)} |f(x)|^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq \frac{C}{|B(x,2R)|} \sum_{x \in B(x,2R)} |f(x)| + C \left(\left(\frac{1}{|B(x,2R)|} \sum_{x \in B(x,2R)} |g(x)|^{1+\delta} \right)^{\frac{1}{1+\delta}} \right).$$

Which can be rewritten

$$\left(\frac{1}{|B(x,R)|} \sum_{e \subseteq B(x,R)} \mathbb{E}[|\nabla \psi_{n,q}(x)|^2]^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq \frac{C}{|B(x,2R)|} \sum_{x \in B(x,2R)} \mathbb{E}[|\nabla \psi_{n,q}(x)|^2] + C.$$

We then conclude that, for each $\gamma \in [0,1)$, the cube $\gamma\square_n$ ca be covered by finitely many balls of the form $B(x,R)$ such that $B(x,2R)$ is included in $\square_n$. The cardinality of this covering family can be bounded from above by a constant depending only on d and γ. This implies that, for each $\gamma \in (0,1)$, there exist an exponent $\delta := \delta(d,\lambda) > 0$ and a constant $C := C(d,\lambda,\gamma) < \infty$ such that

$$\left(\frac{1}{|\gamma\square_n|} \sum_{e \subseteq \gamma\square_n} \mathbb{E}[|\nabla \psi_{n,q}(x)|^2]^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq \frac{C}{|\square_n|} \sum_{x \in \square_n} \mathbb{E}[|\nabla \psi_{n,q}(x)|^2] + C.$$

□

REMARK 5.B.6. Combining the Meyers estimate with (5.3.8) of Proposition 5.3.1, one obtains

$$\left(\frac{1}{|\gamma\square_n|} \sum_{e \subseteq \gamma\square_n} \mathbb{E}\left[|\nabla \psi_{n,q}(e)|^2\right]^{1+\delta} \right)^{\frac{1}{1+\delta}} \leq C(1+|q|^2).$$

Bibliography

[1] R. A. Adams. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] M. Aizenman and D. Barsky. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, 108(3):489–526, 1987.

[3] M. Aizenman, F. Delyon, and B. Souillard. Lower bounds on the cluster size distribution. *J. Statist. Phys.*, 23(3):267–280, 1980.

[4] M. Aizenman, H. Kesten, and C. M. Newman. Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Comm. Math. Phys.*, 111(4):505–531, 1987.

[5] M. Aizenman, H. Kesten, and C. M. Newman. Uniqueness of the infinite cluster and related results in percolation. In *Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985)*, volume 8 of *IMA Vol. Math. Appl.*, pages 13–20. Springer, New York, 1987.

[6] K. Alexander, J. T. Chayes, and L. Chayes. The Wulff construction and asymptotics of the finite cluster distribution for two-dimensional Bernoulli percolation. *Comm. Math. Phys.*, 131(1):1–50, 1990.

[7] S. Andres, M. T. Barlow, J.-D. Deuschel, and B. M. Hambly. Invariance principle for the random conductance model. *Probab. Theory Related Fields*, 156(3-4):535–580, 2013.

[8] S. Andres, A. Chiarini, J.-D. Deuschel, and M. Slowik. Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. *The Annals of Probability*, 46(1):302–336, 2018.

[9] S. Andres, J.-D. Deuschel, and M. Slowik. Invariance principle for the random conductance model in a degenerate ergodic environment. *The Annals of Probability*, 43(4):1866–1891, 2015.

[10] S. Andres, J.-D. Deuschel, and M. Slowik. Heat kernel estimates for random walks with degenerate weights. *Electronic Journal of Probability*, 21, 2016.

[11] P. Antal and A. Pisztora. On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.*, 24(2):1036–1048, 1996.

[12] S. Armstrong, A. Bordas, and J.-C. Mourrat. Quantitative stochastic homogenization and regularity theory of parabolic equations. *Anal. PDE*, 11(8):1945–2014, 2018.

[13] S. Armstrong and P. Dario. Elliptic regularity and quantitative homogenization on percolation clusters. *Comm. Pure Appl. Math.*, 71(9):1717–1849, 2018.

[14] S. Armstrong, S. Ferguson, and T. Kuusi. Homogenization, linearization and large-scale regularity for nonlinear elliptic equations. *arXiv preprint arXiv:1805.00467*, 2018.

[15] S. Armstrong, A. Hannukainen, T. Kuusi, and J.-C. Mourrat. An iterative method for elliptic problems with rapidly oscillating coefficients. *arXiv preprint arXiv:1803.03551*, 2018.

[16] S. Armstrong, T. Kuusi, and J.-C. Mourrat. Mesoscopic higher regularity and subadditivity in elliptic homogenization. *Comm. Math. Phys.*, 347(2):315–361, 2016.

[17] S. Armstrong, T. Kuusi, and J.-C. Mourrat. The additive structure of elliptic homogenization. *Invent. Math.*, 208(3):999–1154, 2017.

[18] S. Armstrong, T. Kuusi, and J.-C. Mourrat. *Quantitative stochastic homogenization and large-scale regularity*, volume 352 of *Grundlehren der mathematischen Wissenschaften*. Springer International Publishing, 2019.

[19] S. Armstrong and J. Lin. Optimal quantitative estimates in stochastic homogenization for elliptic equations in nondivergence form. *Archive for Rational Mechanics and Analysis*, 225(2):937–991, 2017.

[20] S. Armstrong and J.-C. Mourrat. Lipschitz regularity for elliptic equations with random coefficients. *Arch. Ration. Mech. Anal.*, 219(1):255–348, 2016.

[21] S. Armstrong and C. K. Smart. Quantitative stochastic homogenization of convex integral functionals. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(2):423–481, 2016.

[22] M. Avellaneda and F.-H. Lin. Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.*, 40(6):803–847, 1987.

[23] M. Avellaneda and F.-H. Lin. L^p bounds on singular integrals in homogenization. *Comm. Pure Appl. Math.*, 44(8-9):897–910, 1991.

[24] M. T. Barlow. Random walks on supercritical percolation clusters. *Ann. Probab.*, 32(4):3024–3084, 2004.

[25] M. T. Barlow and J.-D. Deuschel. Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.*, 38(1):234–276, 2010.

[26] M. T. Barlow and B. M. Hambly. Parabolic Harnack inequality and local limit theorem for percolation clusters. *Electron. J. Probab.*, 14:no. 1, 1–27, 2009.

[27] P. Bella, A. Chiarini, and B. Fehrman. A Liouville theorem for stationary and ergodic ensembles of parabolic systems. *Probability Theory and Related Fields*, pages 1–54, 2018.

[28] P. Bella, B. Fehrman, and F. Otto. A Liouville theorem for elliptic systems with degenerate ergodic coefficients. *The Annals of Applied Probability*, 28(3):1379–1422, 2018.

[29] I. Benjamini, H. Duminil-Copin, G. Kozma, and A. Yadin. Disorder, entropy and harmonic functions. *Ann. Probab.*, 43(5):2332–2373, 2015.

[30] N. Berger and M. Biskup. Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields*, 137(1-2):83–120, 2007.

[31] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999.

[32] M. Biskup. Recent progress on the random conductance model. *Probab. Surv.*, 8:294–373, 2011.

[33] M. Biskup and T. M. Prescott. Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.*, 12:no. 49, 1323–1348, 2007.

[34] M. Biskup and H. Spohn. Scaling limit for a class of gradient fields with nonconvex potentials. *Ann. Probab.*, 39(1):224–251, 2011.

[35] T. Bodineau. The Wulff construction in three and more dimensions. *Comm. Math. Phys.*, 207(1):197–229, 1999.

[36] D. Boivin. Weak convergence for reversible random walks in a random environment. *Ann. Probab.*, 21(3):1427–1440, 1993.

[37] B. Bollobás and O. Riordan. *Percolation*. Cambridge University Press, New York, 2006.

[38] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[39] S. R. Broadbent and J. M. Hammersley. Percolation processes. I. Crystals and mazes. *Proc. Cambridge Philos. Soc.*, 53:629–641, 1957.

[40] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.*, 121(3):501–505, 1989.

[41] R. Cerf. Large deviations for three dimensional supercritical percolation. *Astérisque*, (267):vi+177, 2000.

[42] R. Cerf and A. Pisztora. On the Wulff crystal in the Ising model. *Ann. Probab.*, 28(3):947–1017, 2000.

[43] J. T. Chayes, L. Chayes, G. R. Grimmett, H. Kesten, and R. H. Schonmann. The correlation length for the high-density phase of Bernoulli percolation. *Ann. Probab.*, 17(4):1277–1302, 1989.

[44] A. Chiarini and J.-D. Deuschel. Invariance principle for symmetric diffusions in a degenerate and unbounded stationary and ergodic random medium. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 52, pages 1535–1563. Institut Henri Poincaré, 2016.

[45] J. G. Conlon and A. Naddaf. Green's functions for elliptic and parabolic equations with random coefficients. *New York J. Math.*, 6:153–225, 2000.

[46] J. G. Conlon and T. Spencer. Strong convergence to the homogenized limit of elliptic equations with random coefficients. *Trans. Amer. Math. Soc.*, 366(3):1257–1288, 2014.

[47] C. Cotar and J.-D. Deuschel. Decay of covariances, uniqueness of ergodic component and scaling limit for a class of $\nabla\phi$ systems with non-convex potential. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48(3):819–853, 2012.

[48] C. Cotar, J.-D. Deuschel, and S. Müller. Strict convexity of the free energy for a class of non-convex gradient models. *Comm. Math. Phys.*, 286(1):359–376, 2009.

[49] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization. *Ann. Mat. Pura Appl. (4)*, 144:347–389, 1986.

[50] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.*, 368:28–42, 1986.

[51] P. Dario. Optimal corrector estimates on percolation clusters. *arXiv preprint arXiv:1805.00902*, 2018.

[52] P. Dario. Quantitative homogenization of differential forms. *arXiv preprint arXiv:1806.07760*, 2018.

[53] P. Dario. Quantitative homogenization of the disordered $\nabla\phi$ model. *arXiv preprint arXiv:1810.06428*, 2018.

[54] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.

[55] A. De Masi, P. Ferrari, S. Goldstein, and W. D. Wick. An invariance principle for reversible markov processes. applications to random motions in random environments. *Journal of Statistical Physics*, 55(3-4):787–855, 1989.

[56] J.-D. Deuschel, G. Giacomin, and D. Ioffe. Large deviations and concentration properties for $\nabla\phi$ interface models. *Probab. Theory Related Fields*, 117(1):49–111, 2000.

[57] J.-D. Deuschel, T. Nguyen, and M. Slowik. Quenched invariance principles for the random conductance model on a random graph with degenerate ergodic weights. *Probability Theory and Related Fields*, 170(1-2):363–386, 2018.

[58] R. Dobrushin, R. Koteckỳ, and S. Shlosman. *Wulff construction: a global shape from local interaction*, volume 104. American Mathematical Society Providence, 1992.

[59] M. Duerinckx and A. Gloria. Stochastic homogenization of nonconvex unbounded integral functionals with convex growth. *Arch. Ration. Mech. Anal.*, 221(3):1511–1584, 2016.

[60] M. Duerinckx, A. Gloria, and F. Otto. The structure of fluctuations in stochastic homogenization. *arXiv preprint arXiv:1602.01717*, 2016.

[61] M. Duerinckx, A. Gloria, and F. Otto. Robustness of the pathways structure of fluctuations in stochastic homogenization. *arXiv preprint arXiv:1807.11781*, 2018.

[62] A. Dykhne. Conductivity of a two-dimensional two-phase system. *Sov. Phys. JETP*, 32(1):63–65, 1971.

[63] A.-C. Egloffe, A. Gloria, J.-C. Mourrat, and T. N. Nguyen. Random walk in random environment, corrector equation and homogenized coefficients: from theory to numerics, back and forth. *IMA J. Numer. Anal.*, 35(2):499–545, 2015.

[64] E. Fabes, O. Mendez, and M. Mitrea. Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains. *J. Funct. Anal.*, 159(2):323–368, 1998.

[65] J. Fischer and F. Otto. A higher-order large-scale regularity theory for random elliptic operators. *Comm. Partial Differential Equations*, 41(7):1108–1148, 2016.

[66] R. Fitzner and R. van der Hofstad. Mean-field behavior for nearest-neighbor percolation in $d > 10$. *Electron. J. Probab.*, 22:Paper No. 43, 65, 2017.

[67] K. O. Friedrichs. Differential forms on Riemannian manifolds. *Comm. Pure Appl. Math.*, 8:551–590, 1955.

[68] T. Funaki. Stochastic interface models. In *Lectures on probability theory and statistics*, volume 1869 of *Lecture Notes in Math.*, pages 103–274. Springer, Berlin, 2005.

[69] T. Funaki and H. Sakagawa. Large deviations for $\nabla\phi$ interface model and derivation of free boundary problems. In *Stochastic analysis on large scale interacting systems*, volume 39 of *Adv. Stud. Pure Math.*, pages 173–211. Math. Soc. Japan, Tokyo, 2004.

[70] T. Funaki and H. Spohn. Motion by mean curvature from the Ginzburg-Landau $\nabla\phi$ interface model. *Comm. Math. Phys.*, 185(1):1–36, 1997.

[71] M. P. Gaffney. The harmonic operator for exterior differential forms. *Proc. Nat. Acad. Sci. U. S. A.*, 37:48–50, 1951.

[72] G. Giacomin, S. Olla, and H. Spohn. Equilibrium fluctuations for $\nabla\phi$ interface model. *Ann. Probab.*, 29(3):1138–1172, 2001.

[73] M. Giaquinta and E. Giusti. On the regularity of the minima of variational integrals. *Acta Math.*, 148:31–46, 1982.

[74] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[75] A. Giunti, R. Höfer, and J. Velázquez. Homogenization for the Poisson equation in randomly perforated domains under minimal assumptions on the size of the holes. *Comm. Partial Differential Equations*, 43(9):1377–1412, 2018.
[76] A. Giunti and J.-C. Mourrat. Quantitative homogenization of degenerate random environments. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(1):22–50, 2018.
[77] E. Giusti. *Direct methods in the calculus of variations*. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
[78] J. Glimm and A. Jaffe. *Quantum physics*. Springer-Verlag, New York-Berlin, 1981.
[79] A. Gloria, S. Neukamm, and F. Otto. An optimal quantitative two-scale expansion in stochastic homogenization of discrete elliptic equations. *ESAIM Math. Model. Numer. Anal.*, 48(2):325–346, 2014.
[80] A. Gloria, S. Neukamm, and F. Otto. A regularity theory for random elliptic operators. *arXiv preprint arXiv:1409.2678*, 2014.
[81] A. Gloria, S. Neukamm, and F. Otto. Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics. *Invent. Math.*, 199(2):455–515, 2015.
[82] A. Gloria and F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.*, 39(3):779–856, 2011.
[83] A. Gloria and F. Otto. An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab.*, 22(1):1–28, 2012.
[84] A. Gloria and F. Otto. The corrector in stochastic homogenization: optimal rates, stochastic integrability, and fluctuations. *arXiv preprint arXiv:1510.08290*, 2015.
[85] A. Gloria and F. Otto. The corrector in stochastic homogenization: near-optimal rates with optimal stochastic integrability, preprint, arXiv:1510.08290.
[86] A. Gloria and F. Otto. Quantitative results on the corrector equation in stochastic homogenization. *J. Eur. Math. Soc.*, in press, arXiv:1409.0801.
[87] G. Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.
[88] Y. Gu and J.-C. Mourrat. Scaling limit of fluctuations in stochastic homogenization. *Multiscale Model. Simul.*, 14(1):452–481, 2016.
[89] J. M. Hammersley. Percolation processes: Lower bounds for the critical probability. *Ann. Math. Statist.*, 28:790–795, 1957.
[90] J. M. Hammersley. Bornes supérieures de la probabilité critique dans un processus de filtration. In *Le calcul des probabilités et ses applications. Paris, 15-20 juillet 1958*, Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII, pages 17–37. Centre National de la Recherche Scientifique, Paris, 1959.
[91] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.*, 128(2):333–391, 1990.
[92] T. E. Harris. A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.*, 56:13–20, 1960.
[93] B. Helffer and J. Sjöstrand. On the correlation for Kac-like models in the convex case. *J. Statist. Phys.*, 74(1-2):349–409, 1994.
[94] D. Ioffe. Large deviations for the 2D Ising model: a lower bound without cluster expansions. *J. Statist. Phys.*, 74(1-2):411–432, 1994.
[95] D. Ioffe. Exact large deviation bounds up to T_c for the Ising model in two dimensions. *Probab. Theory Related Fields*, 102(3):313–330, 1995.
[96] D. Ioffe and R. Schonmann. Dobrushin-Kotecký-Shlosman theorem up to the critical temperature. *Comm. Math. Phys.*, 199(1):117–167, 1998.
[97] D. Jerison and C. E. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130(1):161–219, 1995.
[98] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian [G. A. Iosif′yan].
[99] A. Jonsson and H. Wallin. Function spaces on subsets of $\mathbf{R}^n$. *Math. Rep.*, 2(1):xiv+221, 1984.
[100] H. Kesten. The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.*, 74(1):41–59, 1980.

[101] H. Kesten. Analyticity properties and power law estimates of functions in percolation theory. *J. Statist. Phys.*, 25(4):717–756, 1981.

[102] H. Kesten. *Percolation theory for mathematicians*, volume 2 of *Progress in Probability and Statistics*. Birkhäuser, Boston, Mass., 1982.

[103] H. Kesten and Y. Zhang. The probability of a large finite cluster in supercritical Bernoulli percolation. *Ann. Probab.*, 18(2):537–555, 1990.

[104] C. Kipnis and S. R. S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.*, 104(1):1–19, 1986.

[105] S. Kozlov. The method of averaging and walks in inhomogeneous environments. *Russian Math. Surveys*, 40(2):73, 1985.

[106] S. M. Kozlov. Averaging of differential operators with almost periodic rapidly oscillating coefficients. *Mat. Sb. (N.S.)*, 107(149)(2):199–217, 317, 1978.

[107] S. M. Kozlov. The averaging of random operators. *Mat. Sb. (N.S.)*, 109(151)(2):188–202, 327, 1979.

[108] A. Lamacz, S. Neukamm, and F. Otto. Moment bounds for the corrector in stochastic homogenization of a percolation model. *Electron. J. Probab.*, 20:no. 106, 30, 2015.

[109] T. Lévy and A. Sengupta. Four chapters on low-dimensional gauge theories. In *Stochastic geometric mechanics*, volume 202 of *Springer Proc. Math. Stat.*, pages 115–167. Springer, Cham, 2017.

[110] D. Marahrens and F. Otto. Annealed estimates on the Green function. *Probab. Theory Related Fields*, 163(3-4):527–573, 2015.

[111] J. Marschall. The trace of Sobolev-Slobodeckij spaces on Lipschitz domains. *Manuscripta Math.*, 58(1-2):47–65, 1987.

[112] P. Mathieu. Quenched invariance principles for random walks with random conductances. *J. Stat. Phys.*, 130(5):1025–1046, 2008.

[113] P. Mathieu and A. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 463(2085):2287–2307, 2007.

[114] P. Mathieu and E. Remy. Isoperimetry and heat kernel decay on percolation clusters. *Ann. Probab.*, 32(1A):100–128, 2004.

[115] R. J. McCann. Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.*, 80(2):309–323, 1995.

[116] W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[117] M. V. Menshikov. Coincidence of critical points in percolation problems. *Dokl. Akad. Nauk SSSR*, 288(6):1308–1311, 1986.

[118] N. Meyers. An L^p-estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:189–206, 1963.

[119] J. Miller. Fluctuations for the Ginzburg-Landau $\nabla\phi$ interface model on a bounded domain. *Comm. Math. Phys.*, 308(3):591–639, 2011.

[120] D. Mitrea, M. Mitrea, and S. Monniaux. The Poisson problem for the exterior derivative operator with Dirichlet boundary condition in nonsmooth domains. *Commun. Pure Appl. Anal.*, 7(6):1295–1333, 2008.

[121] D. Mitrea, M. Mitrea, and S. Monniaux. The Poisson problem for the exterior derivative operator with Dirichlet boundary condition in nonsmooth domains. *Commun. Pure Appl. Anal.*, 7(6):1295–1333, 2008.

[122] D. Mitrea, M. Mitrea, and M. Shaw. Traces of differential forms on Lipschitz domains, the boundary de Rham complex, and Hodge decompositions. *Indiana Univ. Math. J.*, 57(5):2061–2095, 2008.

[123] D. Mitrea, M. Mitrea, and M.-C. Shaw. Traces of differential forms on Lipschitz domains, the boundary de Rham complex, and Hodge decompositions. *Indiana Univ. Math. J.*, 57(5):2061–2095, 2008.

[124] M. Mitrea. Dirichlet integrals and Gaffney-Friedrichs inequalities in convex domains. *Forum Math.*, 13(4):531–567, 2001.

[125] J. Moser. On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14(3):577–591, 1961.

[126] J. Moser. A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.*, 17(1):101–134, 1964.

[127] J.-C. Mourrat. Variance decay for functionals of the environment viewed by the particle. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 47, pages 294–327. Institut Henri Poincaré, 2011.

[128] J.-C. Mourrat. A quantitative central limit theorem for the random walk among random conductances. *Electronic Journal of Probability*, 17, 2012.

[129] J.-C. Mourrat. Efficient methods for the estimation of homogenized coefficients. *arXiv preprint arXiv:1609.06674*, 2016.

[130] J.-C. Mourrat and J. Nolen. Scaling limit of the corrector in stochastic homogenization. *Ann. Appl. Probab.*, 27(2):944–959, 2017.

[131] J.-C. Mourrat and F. Otto. Correlation structure of the corrector in stochastic homogenization. *Ann. Probab.*, 44(5):3207–3233, 2016.

[132] A. Naddaf and T. Spencer. On homogenization and scaling limit of some gradient perturbations of a massless free field. *Comm. Math. Phys.*, 183(1):55–84, 1997.

[133] A. Naddaf and T. Spencer. Estimates on the variance of some homogenization problems, 1998, unpublished preprint.

[134] H. Osada. Homogenization of diffusion processes with random stationary coefficients. In *Probability theory and mathematical statistics (Tbilisi, 1982)*, volume 1021 of *Lecture Notes in Math.*, pages 507–517. Springer, Berlin, 1983.

[135] G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam, 1981.

[136] M. Penrose and A. Pisztora. Large deviations for discrete and continuous percolation. *Adv. in Appl. Probab.*, 28(1):29–52, 1996.

[137] C.-E. Pfister and Y. Velenik. Large deviations and continuum limit in the 2D Ising model. *Probab. Theory Related Fields*, 109(4):435–506, 1997.

[138] A. Pisztora. Surface order large deviations for Ising, Potts and percolation models. *Probab. Theory Related Fields*, 104(4):427–466, 1996.

[139] E. Procaccia, R. Rosenthal, and A. Sapozhnikov. Quenched invariance principle for simple random walk on clusters in correlated percolation models. *Probability theory and related fields*, 166(3-4):619–657, 2016.

[140] A. Sapozhnikov. Random walks on infinite percolation clusters in models with long-range correlations. *Ann. Probab.*, in press, arXiv:1410.0605.

[141] R. H. Schonmann and S. B. Shlosman. Complete analyticity for 2D Ising completed. *Comm. Math. Phys.*, 170(2):453–482, 1995.

[142] G. Schwarz. *Hodge decomposition—a method for solving boundary value problems*, volume 1607 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.

[143] D. Serre. Periodic homogenization in terms of differential forms. *Rev. Roumaine Math. Pures Appl.*, 63(4):527–546, 2018.

[144] V. Sidoravicius and A.-S. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Related Fields*, 129(2):219–244, 2004.

[145] J. Sjöstrand. Correlation asymptotics and Witten Laplacians. *Algebra i Analiz*, 8(1):160–191, 1996.

[146] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[147] W. Werner. *Percolation et modèle d'Ising*, volume 16 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2009.

[148] G. Wulff. Zur frage der geschwindigkeit des wachstums und der auflosung der kristallflächen. *Z. Kristallogr*, 34:449–530, 1901.

[149] V. V. Yurinskiĭ. Averaging of symmetric diffusion in a random medium. *Sibirsk. Mat. Zh.*, 27(4):167–180, 215, 1986.

[150] V. V. Yurinskiĭ. On a Dirichlet problem with random coefficients. In *Stochastic differential systems (Proc. IFIP-WG 7/1 Working Conf., Vilnius, 1978)*, volume 25 of *Lecture Notes in Control and Information Sci.*, pages 344–353. Springer, Berlin-New York, 1980.

RÉSUMÉ

Cette thèse est consacrée à l'homogénéisation stochastique, qui cherche à étudier le comportement d'équations aux dérivées partielles présentant des coefficients aléatoires oscillant rapidement. Elle est divisée en trois parties.
La première partie correspond aux Chapitres 2 et 3 et cherche à étendre la théorie de l'homogénéisation stochastique quantitative, développée sous une hypothèse d'uniforme ellipticité, au contexte dégénéré de la percolation de Bernoulli sur-critique. Nous obtenons dans le Chapitre 2, un théorème d'homogénéisation quantitative ainsi qu'une théorie de la régularité à grande échelle pour les fonctions harmoniques sur l'amas infini. Dans le Chapitre 3, nous obtenons des estimées spatiales optimales en toute dimension pour le correcteur sur l'amas infini.
Dans le Chapitre 4, nous étudions un autre type d'environnement dégénéré impliquant des formes différentielles et démontrons, dans ce contexte, un théorème d'homogénéisation quantitative.
Dans le Chapitre 5, nous appliquons les idées de l'homogénéisation stochastique à un modèle issu de la physique statistique : le modèle de Ginzburg-Landau discret. Nous revisitons le début de la théorie de l'homogénéisation et la combinons avec des arguments de la théorie du transport optimal afin de démontrer un théorème de convergence quantitative pour la tension de surface du modèle.

MOTS CLÉS

Homogénéisation, percolation, formes différentielles, modèle d'interface

ABSTRACT

This thesis is devoted to the study of stochastic homogenization, which aims at studying the behavior of partial differential equations with highly heterogeneous, but statistically homogeneous, random coefficients. It is divided into three parts.
The first part corresponds to Chapters 2 and 3 and tries to extend the theory of quantitative stochastic homogenization, developed under an assumption of uniform ellipticity, to the degenerate setting of supercritical Bernoulli bond percolation. In Chapter 2, we prove a quantitative homogenization theorem as well as a large scale regularity theory and Liouville results for harmonic functions on the infinite cluster. In Chapter 3, we obtain optimal spatial estimates in all dimension for the corrector on the infinite cluster.
In Chapter 4, we study another type of degenerate environment involving differential forms and prove, in this setting, a quantitative homogenization theorem.
In Chapter 5, we apply ideas from homogenization to a model of statistical physics: the discrete Ginzburg-Landau model. In this chapter, we revisit the beginning of the theory of stochastic homogenization and combine it with arguments from the theory of optimal transport to derive a quantitative rate of convergence for the finite-volume surface tension of the model.

KEYWORDS

Homogenization, percolation, differential forms, stochastic interface model