Rational curves on hypersurfaces

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Contents

Abstract

In the past few decades, it has become more and more obvious that rational curves are the primary player in the study of the birational geometry of projective varieties. Studying rational curves on algebraic varieties is actually a very old subject which started in the nineteenth century with the study of lines on hypersurfaces in the projective space. In these notes, we review results on rational curves on hypersurfaces, some classical, some more recent, briefly introducing along the way tools such as Schubert calculus and Chern classes. We end with a discussion of ongoing research on the moduli space of rational curves of fixed degree on a general hypersurface.

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Introduction

The set of zeroes in the projective space of a homogeneous polynomial of degree d with coefficients in a field is called a projective hypersurface of degree d and the study of their geometry is a very classical subject.

For example, Cayley wrote in the 1869 memoir [C] that a smooth complex cubic ($d = 3$) surface contains exactly 27 projective lines. In 1904, Fano published the article [F] on the variety of lines contained in a general complex cubic hypersurface of dimension 3.[1]

After recalling in Chapter 1 basic facts on projective spaces and Grassmannians, we concentrate in Chapter 2 on the variety of lines contained in a hypersurface. We take this opportunity to review quickly Chern classes and to introduce a little bit of Schubert calculus. We define free lines (those whose normal bundle is generated by global sections) and discuss the particular cases of quadric and cubic hypersurfaces. In particular, we prove that the variety of lines contained in a cubic hypersurface of dimension 4 is a holomorphic symplectic manifold ([BD]). In the examples, we discuss several base fields: $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$, and we finish the chapter with the case of finite fields, where, surprisingly, some basic questions, like the existence of lines on cubic hypersurfaces of dimension 3 or 4, remain open.

In Chapter 3, we discuss analogous questions about conics instead of lines. We construct a parameter space for conics in the projective space and prove that a general hypersurface of dimension n and degree $> (3n + 1)/2$ contains no conics. We also discuss S. Katz's result that a general quintic threefold contains 609,205 conics. Again, we examine more closely the case of quadric and cubic hypersurfaces. We close the chapter with the construction of the space of rational curves in a given projective scheme, state a theorem about its local structure, and discuss a conjecture and several recent results on its global structure.

In Chapter 4, we review standard fact about varieties covered by rational curves: we define uniruledness and separable uniruledness and characterize the latter by the existence of a free rational curve.

Instead of proving every result that we state, we have tried instead to give a taste of the many tools that are used in modern classical algebraic geometry. The bibliography provides a few references where the reader can find more detailed expositions. Throughout this text,

[1] This is why these varieties of lines are often called "Fano varieties," although this is confusing since this terminology is nowadays used more often for varieties whose anticanonical bundle is ample.

I offer exercises and I discuss several conjectures concerning very concrete and elementary questions which are however still open.

Chapter 1

Projective spaces and Grassmannians

Let $\mathbf{k}$ be a field. A $\mathbf{k}$-*variety* X is an integral and separated scheme of finite type over $\mathbf{k}$. We denote by $X(\mathbf{k})$ the set of points of X defined over $\mathbf{k}$, also called $\mathbf{k}$-rational points.

The Picard group $\mathrm{Pic}(X)$ is the group of isomorphism classes of invertible (or locally free of rank 1) sheaves $\mathscr{L}$ on X, under the operation given by tensor product; the inverse $\mathscr{L}^{-1}$ of $\mathscr{L}$ is then its dual $\mathscr{L}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$. It is also the group of Cartier divisors on X modulo linear equivalence $\underset{\mathrm{lin}}{\equiv}$ (two Cartier divisors are linear equivalent if their difference is the divisor of a regular function).

When X is smooth of dimension n, the sheaf $\omega_X := \Omega_X^n$ of regular n-forms on X is invertible. Its class in $\mathrm{Pic}(X)$ is called the canonical class, often written as K_X. The tangent sheaf of X is denoted by T_X; it is the dual of the sheaf Ω_X of regular 1-forms on X.

We fix a $\mathbf{k}$-vector space V of dimension N.

1.1 Projective spaces

The projective space

$$\mathbf{P}(V) := \{\text{1-dimensional vector subspaces in } V\}$$

is a smooth projective $\mathbf{k}$-variety of dimension $N-1$. It is endowed with a very ample invertible sheaf $\mathscr{O}_{\mathbf{P}(V)}(1)$; seen as a line bundle, its fiber at a point $[V_1]$ is the dual vector space $V_1^\vee$. It corresponds to the (Cartier) divisors defined by hyperplanes in $\mathbf{P}(V)$.

We define $\mathscr{O}_{\mathbf{P}(V)}(-1)$ as the dual of $\mathscr{O}_{\mathbf{P}(V)}(1)$ and, for any $m \in \mathbf{N}$, we set $\mathscr{O}_{\mathbf{P}(V)}(m) := \mathscr{O}_{\mathbf{P}(V)}(1)^{\otimes m}$ and $\mathscr{O}_{\mathbf{P}(V)}(-m) := \mathscr{O}_{\mathbf{P}(V)}(-1)^{\otimes m}$. Whis this notation, the map

$$\begin{aligned} \mathbf{Z} &\longrightarrow \mathrm{Pic}(\mathbf{P}(V)) \\ m &\longmapsto [\mathscr{O}_{\mathbf{P}(V)}(m)] \end{aligned}$$

is a group isomorphism.

The space of global sections of $\mathscr{O}_{\mathbf{P}(V)}(1)$ is isomorphic to $V^\vee$ by the map

$$\begin{array}{rcl} V^\vee & \xrightarrow{\sim} & H^0\big(\mathbf{P}(V), \mathscr{O}_{\mathbf{P}(V)}(1)\big) \\ v^\vee & \longmapsto & ([V_1] \mapsto v^\vee|_{V_1}). \end{array}$$

More generally, for any $m \in \mathbf{Z}$, the space of global sections of $\mathscr{O}_{\mathbf{P}(V)}(m)$ is isomorphic to $\mathsf{S}^m V^\vee$ for $m \geq 0$, and to 0 for $m < 0$.

1.2 The Euler sequence

The variety $\mathbf{P}(V)$ is smooth and its tangent bundle fits into an exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}(V)} \to \mathscr{O}_{\mathbf{P}(V)}(1) \otimes_{\mathbf{k}} V \to T_{\mathbf{P}(V)} \to 0. \tag{1.1}$$

At a point $[V_1]$, this exact sequence induces the following exact sequence of $\mathbf{k}$-vector spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{k} & \longrightarrow & V_1^\vee \otimes_{\mathbf{k}} V & \longrightarrow & T_{\mathbf{P}(V),[V_1]} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & \mathrm{Hom}_{\mathbf{k}}(V_1, V_1) & \to & \mathrm{Hom}_{\mathbf{k}}(V_1, V) & \to & \mathrm{Hom}_{\mathbf{k}}(V_1, V/V_1) & \to & 0. \end{array}$$

By taking determinants in the exact sequence (1.1), we obtain

$$\omega_{\mathbf{P}(V)} \simeq \det(T_{\mathbf{P}(V)})^\vee = \det(\mathscr{O}_{\mathbf{P}(V)}(1) \otimes_{\mathbf{k}} V)^\vee = \mathscr{O}_{\mathbf{P}(V)}(-N).$$

1.3 Grassmannians

For any integer r such that $0 \leq r \leq N = \dim_{\mathbf{k}}(V)$, the Grassmannian

$$G := \mathsf{Gr}(r, V) := \{r\text{-dimensional vector subspaces in } V\}$$

is a smooth projective $\mathbf{k}$-variety of dimension $r(N-r)$ (when $r = 1$, this is just $\mathbf{P}(V)$; when $r = N-1$, this is the *dual projective space* $\mathbf{P}(V^\vee)$).

There is on G a *tautological rank-r subbundle* $\mathscr{S}$ whose fiber at a point $[V_r]$ of G is V_r (when $G = \mathbf{P}(V)$, this is $\mathscr{O}_{\mathbf{P}(V)}(-1)$). It fits into an exact sequence

$$0 \to \mathscr{S} \to \mathscr{O}_G \otimes_{\mathbf{k}} V \to \mathscr{Q} \to 0, \tag{1.2}$$

where $\mathscr{Q}$ is the *tautological rank-$(N-r)$ quotient bundle.*

As in the case of the projective space, for any $m \in \mathbf{N}$, the space of global sections of $\mathsf{S}^m \mathscr{S}^\vee$ is isomorphic to $\mathsf{S}^m V^\vee$.

Let $[V_r]$ be a point of G and choose a decomposition $V = V_r \oplus V_{N-r}$. The subset of G consisting of subspaces complementary to V_{N-r} is an open neighborhood $G_{V_{N-r}}$ of $[V_r]$

in G whose elements can be written as $\{x + u(x) \mid x \in V_r\}$ for some uniquely defined $u \in \mathrm{Hom}_{\mathbf{k}}(V_r, V_{N-r})$. This implies that there is an isomorphism $\varphi_{V_r,V_{N-r}} : G_{V_{N-r}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{k}}(V_r, V_{N-r})$. One checks that the composed isomorphism

$$T_{G,[V_r]} \xrightarrow{T_{\varphi_{V_r,V_{N-r}},[V_r]}} \mathrm{Hom}_{\mathbf{k}}(V_r, V_{N-r}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{k}}(V_r, V/V_r) \tag{1.3}$$

is independent of the choice of the complementary subspace V_{N-r}. Therefore,

$$T_G \simeq \mathscr{H}om_{\mathscr{O}_S}(\mathscr{S}, \mathscr{Q}) \simeq \mathscr{S}^\vee \otimes_{\mathscr{O}_S} \mathscr{Q}.$$

The generalization of the Euler sequence (1.1) for the Grassmannian is therefore

$$0 \to \mathscr{S}^\vee \otimes_{\mathscr{O}_S} \mathscr{S} \to \mathscr{S}^\vee \otimes_{\mathbf{k}} V \to T_G \to 0. \tag{1.4}$$

The invertible sheaf

$$\mathscr{O}_G(1) := \bigwedge^r \mathscr{S}^\vee = \det(\mathscr{S}^\vee) \tag{1.5}$$

is again very ample, with space of global sections isomorphic to $\bigwedge^r V^\vee$. It induces the *Plücker embedding*

$$\begin{array}{rcl} \mathsf{Gr}(r,V) & \longrightarrow & \mathbf{P}(\bigwedge^r V) \\ [V_r] & \longmapsto & [\bigwedge^r V_r]. \end{array}$$

We define the invertible sheaves $\mathscr{O}_G(m)$ for all $m \in \mathbf{Z}$ as in the case of $\mathbf{P}(V)$ and again, the map $m \mapsto [\mathscr{O}_G(m)]$ induces a group isomorphism

$$\mathbf{Z} \xrightarrow{\sim} \mathrm{Pic}(G).$$

By taking determinants in the exact sequence (1.4), we obtain

$$\omega_G \simeq \det(T_G)^\vee = \det(\mathscr{S}^\vee \otimes_{\mathscr{O}_S} \mathscr{S}) \otimes \det(\mathscr{S}^\vee \otimes_{\mathbf{k}} V)^\vee = \det(\mathscr{S} \otimes_{\mathbf{k}} V^\vee) = \mathscr{O}_G(-N). \tag{1.6}$$

Example 1.1 When $N = 4$, the image of the Plücker embedding $\mathsf{Gr}(2,V) \hookrightarrow \mathbf{P}(\bigwedge^2 V) = \mathbf{P}^5_{\mathbf{k}}$ is the smooth quadric with equation $\eta \mapsto \eta \wedge \eta$.

More generally, the image of the Plücker embedding $\mathsf{Gr}(2,V) \hookrightarrow \mathbf{P}(\bigwedge^2 V)$ is defined by the intersection of all *Plücker quadrics* $\eta \mapsto \eta \wedge \eta \wedge \omega$, as ω describes $\bigwedge^{N-4} V$ (it consists of the decomposable tensors in $\bigwedge^2 V$).

1.4 Linear spaces contained in a subscheme of $\mathbf{P}(V)$

We can also interpret the isomorphism (1.3) as follows. We write two Euler exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{O}_{\mathbf{P}(V_r)} & \to & \mathscr{O}_{\mathbf{P}(V_r)}(1) \otimes_{\mathbf{k}} V_r & \longrightarrow & T_{\mathbf{P}(V_r)} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{O}_{\mathbf{P}(V_r)} & \to & \mathscr{O}_{\mathbf{P}(V_r)}(1) \otimes_{\mathbf{k}} V & \to & T_{\mathbf{P}(V)}|_{V_r} & \to & 0, \end{array}$$

from which we obtain a formula for the normal bundle of $\mathbf{P}(V_r)$ in $\mathbf{P}(V)$ (the cokernel of the rightmost vertical map)

$$N_{\mathbf{P}(V_r)/\mathbf{P}(V)} \simeq \mathscr{O}_{\mathbf{P}(V_r)}(1) \otimes_{\mathbf{k}} (V/V_r). \tag{1.7}$$

We can therefore rewrite (1.3) as

$$T_{\mathsf{Gr}(r,V),[V_r]} \simeq H^0(\mathbf{P}(V_r), N_{\mathbf{P}(V_r)/\mathbf{P}(V)}). \tag{1.8}$$

This is a particular case of a more general result.

Let $f \in \mathsf{S}^d V^\vee$ (a homogeneous polynomial in degree d) and let $X \subset \mathbf{P}(V)$ be the hypersurface $Z(f)$ defined by f. Set

$$F_r(X) := \{[V_r] \in \mathsf{Gr}(r, V) \mid V_r \subset X\}.$$

We define a scheme structure on this closed subset as follows. By definition, $[V_r] \in F(X)$ if and only if $f|_{V_r}$ is identically 0. On the other hand, f defines a section s_f of $\mathsf{S}^d\mathscr{S}^\vee$ by $[V_r] \mapsto f|_{V_r}$. We set

$$F_r(X) := Z(s_f) \subset \mathsf{Gr}(r, V), \tag{1.9}$$

the zero-scheme of the section s_f. Since the rank of $\mathsf{S}^d\mathscr{S}^\vee$ is $\binom{d+r-1}{r-1}$, this has the important consequence that either $F_r(X)$ is empty, or it has everywhere codimension at most $\binom{d+r-1}{r-1}$ in at each point.

If $X \subset \mathbf{P}(V)$ is now a general subscheme defined by the equations $f_1 = \cdots = f_m = 0$, we set

$$F_r(X) := F_r(Z(f_1)) \cap \cdots \cap F_r(Z(f_m)) \subset \mathsf{Gr}(r, V)$$

as a (projective) scheme.

We can now generalize (1.8).

Theorem 1.2 *Let $X \subset \mathbf{P}(V)$ be a subscheme containing $\mathbf{P}(V_r)$. If X is smooth along $\mathbf{P}(V_r)$, one has*

$$T_{F_r(X),[V_r]} \simeq H^0(\mathbf{P}(V_r), N_{\mathbf{P}(V_r)/X}).$$

Proof. We will only do the case where $X \subset \mathbf{P}(V)$ is a hypersurface of degree d. The general case is an easy consequence of that particular case.

Choose a decomposition $V = V_r \oplus V_{N-r}$ and adapted coordinates $x_1, \ldots, x_r, x_{r+1}, \ldots, x_N$ on V. Since X contains V_r, we may write its equation as

$$f = x_{r+1}f_{r+1} + \cdots + x_N f_N,$$

where $f_{r+1}, \ldots, f_N$ are homogeneous polynomials of degree $d - 1$.

We may represent a first-order deformation of V_r in the direction of the tangent vector $u \in \operatorname{Hom}_{\mathbf{k}}(V_r, V_{N-r})$ as the graph of $x \mapsto x + \varepsilon u(x)$, with $\varepsilon^2 = 0$. We then have

$$\forall x \in V_r \qquad f(x + \varepsilon u(x)) = \varepsilon u_{r+1}(x) f_{r+1}(x, 0) + \cdots + \varepsilon u_N(x) f_N(x, 0)$$

so this first-order deformation is contained in X if and only if

$$\forall x \in V_r \qquad u_{r+1}(x) f_{r+1}(x,0) + \cdots + u_N(x) f_N(x,0) = 0. \tag{1.10}$$

The global section of $N_{\mathbf{P}(V_r)/\mathbf{P}(V)}$ corresponding to the tangent vector u via (1.8) is $x \mapsto (u_{r+1}(x), \ldots, u_N(x))$. Since

$$\frac{\partial f}{\partial x_i}(x,0) = \begin{cases} 0 & \text{if } 1 \leq i \leq r, \\ f_i(x,0) & \text{if } r+1 \leq i \leq N, \end{cases}$$

(1.10) is exactly the condition for this section to have values in $N_{\mathbf{P}(V_r)/X}$. This proves the theorem in the case where X is a hypersurface. □

Chapter 2

Projective lines contained in a hypersurface

If X is a subscheme of the projective space $\mathbf{P}(V)$, we defined in Section 1.4 the scheme $F_2(X) \subset \mathsf{Gr}(2, V)$ of projective lines contained in X. From now on, we will use instead the notation $F(X)$, or $M(X, 1)$ (for "moduli space of curves of degree 1 contained in X"). It is a projective scheme.

2.1 Local study of $F(X)$

Let L be a projective line contained in X. We proved in Theorem 1.2 that if X is smooth along L, the tangent space to $F(X)$ at the point $[L]$ is isomorphic to $H^0(L, N_{L/X})$.

We have a more precise result, which we will not prove.

Theorem 2.1 *Let $X \subset \mathbf{P}(V)$ be a subscheme containing a projective line L. If X is smooth along L, the scheme $F(X)$ can be defined, in a neighborhood of its point $[L]$, by $h^1(L, N_{L/X})$ equations in a smooth scheme of dimension $h^0(L, N_{L/X})$. In particular, any (geometric) component of $F(X)$ through the point $[L]$ has dimension at least*

$$\chi(L, N_{L/X}) := h^0(L, N_{L/X}) - h^1(L, N_{L/X}) = \deg(N_{L/X}) + \dim(X) - 1.$$

The last equality follows from the Riemann-Roch theorem applied to the vector bundle $N_{L/X}$ on the genus-0 curve L.

The number $\deg(N_{L/X}) + \dim(X) - 1$ is called the *expected dimension* of $F(X)$. When $H^1(L, N_{L/X}) = 0$, the scheme $F(X)$ is smooth of the expected dimension at $[L]$.

Assume that $X \subset \mathbf{P}(V)$ is a hypersurface of degree d containing L. We explained in Section 1.4 that $F(X)$ has dimension at least

$$\dim(\mathsf{Gr}(2, V)) - \operatorname{rank}(\mathsf{S}^d \mathscr{S}^\vee) = 2(N-2) - (d+1) = 2N - 5 - d. \tag{2.1}$$

When X is moreover smooth along L, we have the normal exact sequence

$$0 \to N_{L/X} \to N_{L/\mathbf{P}(V)} \to N_{X/\mathbf{P}(V)}|_L \to 0. \tag{2.2}$$

By (1.7), the normal bundle $N_{L/\mathbf{P}(V)}$ is isomorphic to $\mathscr{O}_L(1)^{\oplus(N-2)}$, hence has degree $N-2$, whereas $N_{X/\mathbf{P}(V)}$ is isomorphic to $\mathscr{O}_X(d)$. It follows that $N_{L/X}$ has rank $N-3$ and degree $N-2-d$. The number in (2.1) is therefore the expected dimension of $F(X)$ as defined above.

2.2 Schubert calculus

To prove results about the *existence* of a projective line in a hypersurface, we will use cohomological calculations with Chern classes, using *Schubert calculus.*

Let X be a smooth projective variety. The Chow ring $CH(X) = \bigoplus_{i \in \mathbf{N}} CH^i(X)$ is the ring of cycles on X modulo rational equivalence, graded by codimension, where the product is given by intersection. I do not want to explain here what this means because we will only use it as a formal tool. The group $CH^1(X)$ is the group $\mathrm{Pic}(X)$ of divisors modulo linear equivalence defined at the beginning of Chapter 1; for higher i, the theory is more subtle. For our purposes, the Chow ring can be replaced by any good cohomology theory that you like, such as the singular cohomology ring $H(X, \mathbf{Z})$ when $\mathbf{k} = \mathbf{C}$.

To any coherent sheaf $\mathscr{F}$ on X, one can associate a (total) Chern class $c(\mathscr{F}) \in CH(X)$ which behave nicely by pullbacks and such that, for any exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0,$$

one has

$$c(\mathscr{F}) = c(\mathscr{F}')c(\mathscr{F}''). \tag{2.3}$$

We write $c(\mathscr{F}) = \sum_{i \geq 0} c_i(\mathscr{F})$, with $c_i(\mathscr{F}) \in CH^i(X)$. When $\mathscr{F}$ is locally free, we have $c_i(\mathscr{F}) = 0$ for $i > \mathrm{rank}(\mathscr{F})$ and $c_1(\mathscr{F})$ is the class in $CH^1(X) = \mathrm{Pic}(X)$ of $\det(\mathscr{F})$.

We will need the following result.

Theorem 2.2 *Let X be a smooth irreducible projective scheme and let $\mathscr{E}$ be a locally free sheaf on X of rank r. Assume that the zero-scheme $Z(s)$ of some global section s of $\mathscr{E}$ is empty or has codimension exactly r in X. Then its class $[Z(s)] \in CH^r(X)$ is equal to $c_r(\mathscr{E})$. In particular, if $c_r(\mathscr{E})$ is nonzero, $Z(s)$ is nonempty.*

If X is a hypersurface of degree d of $\mathbf{P}(V)$, the subscheme $F(X) \subset G := \mathsf{Gr}(2, V)$ of lines contained in X is defined as the zero locus of a section of $\mathsf{S}^d\mathscr{S}^\vee$, a locally free sheaf on G of rank $d+1$.

To compute $c_{d+1}(\mathsf{S}^d\mathscr{S}^\vee)$, we need to know the ring $CH(G)$. To describe it, we define the Schubert cycles.

Let a and b be integers such that $N-2 \geq a \geq b \geq 0$. Choose vector subspaces

$$V_{N-1-a} \subset V_{N-b} \subset V$$

such that $\dim(V_{N-1-a}) = N-1-a$ and $\dim(V_{N-b}) = N-b$. We define a subvariety of G, called a *Schubert variety,* by

$$\Sigma_{a,b} := \{[V_2] \in G \mid V_2 \cap V_{N-1-a} \neq 0,\ V_2 \subset V_{N-b}\}. \tag{2.4}$$

It is irreducible of codimension $a+b$ in G and its class $\sigma_{a,b} := [\Sigma_{a,b}] \in CH^{a+b}(G)$ only depends on a and b. It is usual to write σ_a (resp. Σ_a) for $\sigma_{a,0}$ (resp. $\Sigma_{a,0}$) and to set $\sigma_{a,b} = 0$ whenever (a,b) does not satisfy $N-2 \geq a \geq b \geq 0$.

Exercise 2.3 Set as above $G := \mathsf{Gr}(2,V)$.

a) Prove that via the identification $CH^1(G) \simeq \mathrm{Pic}(G)$, the class σ_1 corresponds to the (isomorphism class of the) invertible sheaf $\mathscr{O}_G(1)$ defined in (1.5) (*Hint:* use the Plücker embedding).

b) Prove that in the linear sytem $\mathbf{P}(H^0(G, \mathscr{O}_G(1))) = \mathbf{P}(\bigwedge^2 V^\vee)$, the Schubert divisor Σ_1 associated via (2.4) with $V_{N-2} \subset V$ corresponds to the point $[V_{N-2}]$ of $\mathsf{Gr}(N-2, V) \simeq \mathsf{Gr}(2, V^\vee) \overset{\text{Plücker}}{\subset} \mathbf{P}(\bigwedge^2 V^\vee)$.

Theorem 2.4 *The group $CH(\mathsf{Gr}(2,V))$ is a free abelian group with basis $(\sigma_{a,b})_{N-2\geq a\geq b\geq 0}$.*

For example, the group $CH^1(G) \simeq \mathrm{Pic}(G)$ has rank 1, generated by σ_1. This class is the first Chern class of the invertible sheaf $\mathscr{O}_G(1)$ (Exercise 2.3). We also have

$$c(\mathscr{Q}) = 1 + \sigma_1 + \cdots + \sigma_{N-2},$$

hence also, using (1.2) and (2.3),

$$c(\mathscr{S}) = (1 + \sigma_1 + \cdots + \sigma_{N-2})^{-1} = 1 - \sigma_1 + \sigma_1^2 - \sigma_2.$$

(The rank of $\mathscr{S}$ is 2 so there are no higher Chern classes.) The compute this class, we need to know the multiplicative structure of $CH(G)$: whenever $N-2 \geq a \geq b \geq 0$ and $N-2 \geq c \geq d \geq 0$, there must exist formulas

$$\sigma_{a,b} \cdot \sigma_{c,d} = \sum_{\substack{x+y=a+b+c+d \\ N-2\geq x\geq y\geq 0}} n_{a,b,c,d,x,y}\sigma_{x,y},$$

where the $n_{a,b,c,d,x,y}$ are integers. This is the content of Schubert calculus, which we will only illustrate in some particular cases (the combinatorics are quite involved in general).

Poincaré duality. If $a+b+c+d=2N-4$, one has

$$\sigma_{a,b}\cdot\sigma_{c,d}=\begin{cases}1 & \text{if } a+d=b+c=N-2\\ 0 & \text{otherwise.}\end{cases}$$

(The class $\sigma_{N-2,N-2}$ is the class of a point and generates $CH^{2N-4}(G)$; we usually drop it.) In other words, the Poincaré dual of $\sigma_{a,b}$ is $\sigma_{N-2-b,N-2-a}$.

Pieri's formula. This is the relation

$$\sigma_m\cdot\sigma_{a,b}=\sum_{\substack{x+y=a+b+m\\ x\geq a,\ y\geq b}}\sigma_{x,y}.$$

For example, we have

$$\sigma_1\cdot\sigma_{a,b}=\sigma_{a+1,b}+\sigma_{a,b+1} \tag{2.5}$$

(where the last term is 0 when $a=b$), which implies

$$c(\mathscr{S}^\vee)=1+\sigma_1+\sigma_{1,1}.$$

The following formula can be deduced from Pieri's formula (using $\sigma_{1,1}=\sigma_1^2-\sigma_2$):

$$\sigma_{1,1}\cdot\sigma_{a,b}=\sigma_{a+1,b+1}. \tag{2.6}$$

Example 2.5 How many lines meet 4 general lines L_1, L_2, L_3, and L_4 in $\mathbf{P}^3_{\mathbf{C}}$? One can answer this question geometrically as follows: through any point of L_3, there is a unique line meeting L_1 and L_2 and one checks by explicit calculations that the union of these lines is a smooth quadric surface, which meets L_4 in 2 points. Through each of these 2 points, there is a unique line meeting all 4 lines.

But we can also use Schubert calculus: the set of lines meeting L_i has class σ_1, hence the answer is (use (2.5))

$$\sigma_1^4=\sigma_1^2(\sigma_2+\sigma_{1,1})=\sigma_1(\sigma_{2,1}+\sigma_{2,1})=2\sigma_{2,2}.$$

(To be honest, this calculation only shows that either there are either 2 such lines "counted with multiplicities" or infinitely many of them.)

2.3 Projective lines contained in a hypersurface

2.3.1 Existence of lines in a hypersurface

We use Schubert calculus to show the existence of lines in hypersurfaces of small enough degrees.

Theorem 2.6 *When* $\mathbf{k}$ *is algebraically closed and* $d \le 2N-5$*, any hypersurface of degree* d *in* $\mathbf{P}(V)$ *contains a projective line.*

Proof. By Theorem 2.2, it is enough to prove that the top Chern class $c_{d+1}(\mathsf{S}^d\mathscr{S}^\vee)$ does not vanish.

The method for computing the Chern classes of the symmetric powers of $\mathscr{S}^\vee$ is the following: pretend that $\mathscr{S}^\vee$ is the direct sum of two invertible sheaves $\mathscr{L}_1$ and $\mathscr{L}_2$, with first Chern classes ℓ_1 and ℓ_2 (the *Chern roots* of $\mathscr{S}^\vee$), so that

$$c(\mathscr{S}^\vee) = (1+\ell_1)(1+\ell_2).$$

Then

$$\mathsf{S}^d\mathscr{S}^\vee \simeq \bigoplus_{i=0}^{d}(\mathscr{L}_1^{\otimes i}\otimes\mathscr{L}_2^{\otimes(d-i)})$$

and, using (2.3), we obtain

$$c(\mathsf{S}^d\mathscr{S}^\vee) = \prod_{i=0}^{d}\big(1+i\ell_1+(d-i)\ell_2\big).$$

This symmetric polynomial in ℓ_1 and ℓ_2 can be expressed as a polynomial in

$$\begin{aligned}\ell_1+\ell_2 &= c_1(\mathscr{S}^\vee)=\sigma_1\\ \ell_1\ell_2 &= c_2(\mathscr{S}^\vee)=\sigma_{1,1}.\end{aligned}$$

One obtains in particular

$$\begin{aligned}c_{d+1}(\mathsf{S}^d\mathscr{S}^\vee) &= \prod_{i=0}^{d}\big(i\ell_1+(d-i)\ell_2\big)\\ &= \prod_{0\le i\le d/2}\big(i(d-i)\big(\ell_1+\ell_2\big)^2+(d-2i)^2\ell_1\ell_2\big)\\ &= \prod_{0\le i\le d/2}\big(i(d-i)\sigma_1^2+(d-2i)^2\sigma_{1,1}\big)\\ &= \prod_{0\le i\le d/2}\big(i(d-i)\sigma_2+\big((d-2i)^2+i(d-i)\big)\sigma_{1,1}\big).\end{aligned}$$

Formulas (2.5) and (2.6) imply that this is a sum of Schubert classes with nonnegative coefficients which, since $(d-2i)^2+i(d-i)\ge 1$ for all i, is "greater than or equal to"

$$\prod_{0\le i\le d/2}\sigma_{1,1}=\sigma_{\lfloor d/2\rfloor+1,\lfloor d/2\rfloor+1},$$

which is nonzero for $\lfloor d/2\rfloor+1\le N-2$. This inequality is equivalent to $d\le 2N-5$. □

Exercise 2.7 Prove the relations

a) $c_4(\mathsf{S}^3\mathscr{S}^\vee) = 9(2\sigma_{3,1} + 3\sigma_{2,2})$, $c_5(\mathsf{S}^4\mathscr{S}^\vee) = 32(3\sigma_{4,1} + 10\sigma_{3,2})$, $c_6(\mathsf{S}^5\mathscr{S}^\vee) = 25(24\sigma_{5,1} + 130\sigma_{4,2} + 115\sigma_{3,3})$.

b) $\det(\mathsf{S}^d\mathscr{S}^\vee) \simeq \mathscr{O}_G\big(\frac{d(d+1)}{2}\big)$ and $c_1(\mathsf{S}^d\mathscr{S}^\vee) = \frac{d(d+1)}{2}\sigma_1$.

Assume $2N - 5 - d \geq 0$. Under the hypotheses of the theorem, it follows from (2.1) that $F(X)$ has everywhere dimension $\geq 2N - 5 - d$.

One can show that when $X \subset \mathbf{P}(V)$ is a *general* hypersurface of degree d, the scheme $F(X)$ is smooth projective of (the expected) dimension $2N - 5 - d$ (empty whenever this number is < 0).

Exercise 2.8 Deduce from Exercise 2.7 that a general quintic hypersurface in $\mathbf{P}^4_{\mathbf{k}}$ contains 2,875 lines.

Whenever $F(X)$ is smooth of the expected dimension, its canonical class is given by an adjunction formula: if a section of a vector bundle $\mathscr{E}$ on a smooth projective variety G has smooth zero locus Z, we have $K_Z = \big(K_G + c_1(\mathscr{E})\big)|_Z$. In our case, we obtain, using (1.6) and Exercise 2.7.b),

$$K_{F(X)} = \Big(-N\sigma_1 + \frac{d(d+1)}{2}\sigma_1\Big)|_{F(X)}. \tag{2.7}$$

In particular, when $N > d(d+1)/2$, the smooth variety $F(X)$ is a *Fano variety:* its anticanonical class $-K_{F(X)}$ is ample.

When $d \geq 4$, even if X is smooth, the scheme $F(X)$ may be singular, and even reducible or nonreduced (this does not happen when $d = 2$ or 3; see Sections 2.4 and 2.5). However, we have the following conjecture.

Conjecture 2.9 (de Jong–Debarre) *Assume $N > d \geq 3$ and that* char$(\mathbf{k})$ *is either 0 or $\geq d$. For* any *smooth hypersurface $X \subset \mathbf{P}(V)$ of degree d, the scheme $F(X)$ has the expected dimension $2N - 5 - d$.*

We will see in Section 2.5 that the conjecture holds for $d = 3$. When char$(\mathbf{k}) = 0$, the conjecture is known for $d \leq 6$ or for $d \ll N$ (Collino ($d = 4$), Debarre ($d \leq 5$), Beheshti ($d \leq 6$), Harris et al. ($d \ll N$)).

Example 2.11 shows that the hypothesis char$(\mathbf{k}) \geq d$ is necessary. The assumption $N > d$ is also necessary: one can show that when char$(\mathbf{k})$ is either 0 or $\geq d$, the dimension of the scheme of projective lines contained in a Fermat hypersurface of degree $d \geq N - 1$ (see equation (2.8) below) is $N - 4$ (which is the expected dimension only if $d = N - 1$). In particular, Fermat quintic threefolds contain infinitely many lines (compare with Exercise 2.8).

Example 2.10 (Real lines) When d is even, the Fermat hypersurface

$$x_1^d + \cdots + x_N^d = 0 \tag{2.8}$$

contains no real points, hence no real lines, whereas the diagonal hypersurface

$$x_1^d + \cdots + x_{N-1}^d - x_N^d = 0$$

contains infinitely many real points, but no real lines.

Example 2.11 (Positive characteristic) Over $\overline{\mathbf{F}_p}$, we consider the smooth Fermat hypersurface $X \subset \mathbf{P}(V)$ with equation

$$x_1^{p^r+1} + \cdots + x_N^{p^r+1} = 0,$$

with $r \geq 1$. The line L joining two points x and y of X is contained in X if and only if

$$\begin{aligned} 0 &= \sum_{j=1}^N (x_j + ty_j)^{p^r+1} \\ &= \sum_{j=1}^N (x_j^{p^r} + t^{p^r} y_j^{p^r})(x_j + ty_j) \\ &= \sum_{j=1}^N (x_j^{p^r+1} + tx_j^{p^r} y_j + t^{p^r} x_j y_j^{p^r} + t^{p^r+1} y_j^{p^r+1}) \\ &= t\sum_{j=1}^N x_j^{p^r} y_j + t^{p^r} \sum_{j=1}^N x_j y_j^{p^r} \end{aligned} \tag{2.9}$$

for all t. This is equivalent to the two equations

$$\sum_{j=1}^N x_j^{p^r} y_j = \sum_{j=1}^N x_j y_j^{p^r} = 0, \tag{2.10}$$

hence $F(X)$ has dimension at least $2\dim(X) - 2 - 2 = 2N - 8$ at every point $[L]$.

It is known that any locally free sheaf on $\mathbf{P}^1$ split as a direct sum of invertible sheaves, so we can write

$$N_{L/X} \simeq \bigoplus_{i=1}^{N-3} \mathscr{O}_L(a_i), \tag{2.11}$$

where $a_1 \geq \cdots \geq a_{N-3}$ and $a_1 + \cdots + a_{N-3} = N - p^r - 3$. By (2.2), $N_{L/X}$ is a subsheaf of $N_{L/\mathbf{P}(V)} \simeq \mathscr{O}_L(1)^{\oplus(N-2)}$, hence $a_1 \leq 1$. We have

$$\begin{aligned} 2N - 8 &\leq \dim(F(X)) \\ &\leq h^0(L, N_{L/X}) \qquad \text{by Theorem 1.2} \\ &= 2\,\mathrm{Card}\{i \mid a_i = 1\} + \mathrm{Card}\{i \mid a_i = 0\} \\ &\leq \mathrm{Card}\{i \mid a_i = 1\} + N - 3. \end{aligned}$$

The only possibility is

$$N_{L/X} \simeq \mathscr{O}_L(1)^{\oplus(N-4)} \oplus \mathscr{O}_L(1 - p^r), \tag{2.12}$$

which implies $h^0(L, N_{L/X}) = 2N - 8$. Since this is $\leq \dim(F(X))$, Theorem 1.2 implies that $F(X)$ is smooth of (nonexpected if $p^r \neq 2$) dimension $2N - 8$.

Exercise 2.12 (Pfaffian hypersurfaces) Let $\mathbf{k}$ be an algebraically closed field of characteristic $\neq 2$ and let $W := \mathbf{k}^{2d}$. In $\mathbf{P}(\bigwedge^2 W^\vee)$, the *Pfaffian hypersurface* X_d of degenerate skew-symmetric bilinear forms (defined by the vanishing of the Pfaffian polynomial) has degree d.

a) Let m be a positive integer. Given a 2-dimensional vector space of skew-symmetric forms on $\mathbf{k}^m$, prove that there exists a subspace of dimension $\lfloor (m+1)/2 \rfloor$ which is isotropic for all forms in that space (*Hint:* proceed by induction on m).

b) Given a 2-dimensional vector space of *degenerate* skew-symmetric forms on $\mathbf{k}^{2d}$, prove that there exists a subspace of dimension $d+1$ which is isotropic for all forms in that space (*Hint:* proceed by induction on d and use a)).

c) Show that the scheme $F(X_d)$ of projective lines contained in X_d is irreducible of the expected dimension (see (2.1))[1] (*Hint:* prove that the locus $\{([L],[V_{d+1}]) \in \mathsf{Gr}(2, \bigwedge^2 W^\vee) \times \mathsf{Gr}(d+1, W) \mid V_{d+1}$ is isotropic for all forms in $L\}$ is irreducible of dimension $4d^2 - 3d - 5$ and apply b)).

The hypersurface X_d is singular (when $d \geq 3$): its singular locus is the locus of nonzero skew-symmetric bilinear forms of rank $\leq 2d-4$, which has codimension 6 in $\mathbf{P}(\bigwedge^2 W^\vee)$. If $\mathbf{k}$ is infinite, a linear section $X := X_d \cap \mathbf{P}(V_6)$, where $V_6 \subset \bigwedge^2 W^\vee$ is a general 6-dimensional vector subspace, is a smooth hypersurface by Bertini's theorem.

d) Prove that $F(X)$ is of the expected dimension $7-d$.

2.3.2 Lines through a point

Given a subscheme $X \subset \mathbf{P}(V)$ and a point $x \in X(\mathbf{k})$, it is sometimes useful to look at the $\mathbf{k}$-scheme $F(X,x)$ of projective lines passing through x and contained in X. This is actually a simpler problem, because, if x corresponds to a one-dimensional $\mathbf{k}$-vector subspace $V_1 \subset V$, projective lines in $\mathbf{P}(V)$ passing through x are parametrized by $\mathbf{P}(V/V_1)$, a projective space of dimension one less. This projective space can also be viewed, inside the Grassmannian $\mathsf{Gr}(2,V)$, as a Schubert cycle Σ_{N-2} defined in (2.4). We may then define $F(X,x)$ as the scheme-theoretic intersection

$$F(X,x) := F(X) \cap \mathbf{P}(V/V_1).$$

In practice, it is easy to write down explicit equations for $F(X,x)$. Assume first that X is a degree-d hypersurface with equation f. Let $x \in X(\mathbf{k})$; choose coordinates so that $x = (0,\dots,0,1)$. One can write f as

$$f(x_1,\dots,x_N) = x_N^{d-1} f^{(1)}(x_1,\dots,x_{N-1}) + \dots + x_N f^{(d-1)}(x_1,\dots,x_{N-1}) + f^{(d)}(x_1,\dots,x_{N-1}),$$

where $f^{(i)}$ is homogeneous of degree i. The line joining x to $y := (x_1,\dots,x_{N-1},0)$ is contained in X if and only

$$f^{(1)}(x_1,\dots,x_{N-1}) = \dots = f^{(d-1)}(x_1,\dots,x_{N-1}) = f^{(d)}(x_1,\dots,x_{N-1}) = 0. \tag{2.13}$$

[1]Note that X_d is singular for $d \geq 3$: its singular locus is the set of skew-symmetric forms of rank $\leq 2d-4$.

These are the equations that define the scheme $F(X,x)$ in $\mathbf{P}(V/V_1)$. If $X \subset \mathbf{P}(V)$ is a general subscheme defined by the equations $f_1 = \cdots = f_m = 0$, we set

$$F(X,x) := F(Z(f_1),x) \cap \cdots \cap F(Z(f_m),x) \subset \mathbf{P}(V/V_1).$$

Proposition 2.13 *Assume that* $\mathbf{k}$ *is algebraically closed and let* $X \subset \mathbf{P}(V)$ *be a hypersurface of degree* d*. If* $d \le N-2$*, any point of* X *is on a line contained in* X*.*

We say that X is *covered by lines.*

PROOF. This is because for any $x \in X$, the scheme $F(X,x)$ is defined by the d equations (2.13) in an $(N-2)$-dimensional projective space. □

Remark 2.14 Assume $d = N-2$. When X is general, $F(X)$ has dimension $2N-d-5 = N-4$ hence the union of these lines has dimension $\le N-3$: they cannot cover X. If the characteristic is either 0 or $\ge N-2$, this should remain true whenever X is smooth if we believe Conjecture 2.9.

Exercise 2.15 Let $\mathbf{k}$ be an algebraically closed field and let $X \subset \mathbf{P}(V)$ be a subscheme defined by the equations $f_1 = \cdots = f_m = 0$. If $\deg(f_1) + \cdots + \deg(f_m) \le N-2$, prove that any point of X is on a line contained in X.

Exercise 2.16 Let $X \subset \mathbf{P}(V)$ be the smooth Fermat hypersurface discussed in Example 2.11. When $N \ge 5$, prove that for any $x \in X$, the scheme $F(X,x)$ is not reduced (*Hint:* use (2.10)).

2.3.3 Free lines

Assume that an n-dimensional variety $X \subset \mathbf{P}(V)$ contains a projective line L and is smooth along L. Consider the decomposition (2.11)

$$N_{L/X} \simeq \bigoplus_{i=1}^{n-1} \mathscr{O}_L(a_i),$$

where $a_1 \ge \cdots \ge a_{n-1}$. We say that the line L is *free* (on X) if $a_{n-1} \ge 0$. This is equivalent to saying that $N_{L/X}$ is generated by its global sections. It is a condition which is strictly stronger than the vanishing of $H^1(L, N_{L/X})$ (which, by Theorem 2.1, ensures the smoothness of $F(X)$ at $[L]$).

Theorem 2.17 *Let* $X \subset \mathbf{P}(V)$ *be a subvariety.*

a) *If* L *is a free line contained in the smooth locus of* X*, the deformations of* L *in* X *cover* X*. In particular,*

$$\bigcup_{[L]\in F(X)} L = X.$$

b) *Conversely, if* $\mathbf{k}$ *is algebraically closed of characteristic 0 and* X *is covered by lines, a general line contained in* X *is free.*

A couple of comments are in order about b). First, the characteristic-0 hypothesis is necessary: when $N \geq 5$, the smooth Fermat hypersurface of Example 2.11 is covered by lines but none of them are free (see (2.12)). Second, b) does *not* say that all lines in X are free: when $N \geq 5$, a smooth cubic hypersurface is covered by lines, but some of them ("lines of the second type"; see footnote 2) are not free.

A corollary of Proposition 2.13 and the theorem is that in characteristic 0, *a general line contained in the smooth locus of a hypersurface* $X \subset \mathbf{P}(V)$ *of degree* $\leq N-2$ *is free.*

We will only sketch the proof of the theorem, because it is a particular case of a more general result (Theorem 4.7).

Sketch of proof. Let us introduce the *incidence* variety

$$I := \{(x, L) \in X \times F(X) \mid x \in L\}$$

with its projections $\mathsf{pr}_1 : I \to X$ and $\mathsf{pr}_2 : I \to F(X)$ (a $\mathbf{P}^1$-bundle).

If L is free, it corresponds, by Theorem 2.1, to a smooth point of $F(X)$. Moreover, for any $x \in L$, the point (x, L) of I is smooth on I (because $\mathsf{pr}_2 : I \to F(X)$ is smooth). One then computes the tangent map $T_{\mathsf{pr}_1,(x,L)} : T_{I,(x,L)} \to T_{X,x}$ and prove that it is surjective if and only if L is free. The map pr_1 is therefore dominant, hence surjective since I is projective.

Conversely, if X is covered by lines, pr_1 is surjective. If the characteristic is 0, the projection $I_{\mathrm{red}} \to X$ is generically smooth; this means that for L general in $F(X)$ and x general in L, the tangent map $T_{I_{\mathrm{red}},(x,L)} \to T_{X,x}$ is surjective. Since it factors through $T_{\mathsf{pr}_1,(x,L)} : T_{I,(x,L)} \to T_{X,x}$, this last map is also surjective, hence L is free. □

Exercise 2.18 Assume that $\mathbf{k}$ is algebraically closed of characteristic 0, let X be a smooth hypersurface of degree $d \leq N-2$, and let L be a general line contained in X. Prove

$$N_{L/X} \simeq \mathscr{O}_L(1)^{\oplus(N-2-d)} \oplus \mathscr{O}_L^{\oplus(d-1)} \tag{2.14}$$

(*Hint:* use (2.2), (2.11), Proposition 2.13, and Theorem 2.17).

Note that the conclusion does not hold in characteristic $p > 0$ for the Fermat hypersurfaces of degree $p^r + 1$ by (2.12).

2.4 Projective lines contained in a quadric hypersurface

Assume that $\mathbf{k}$ is algebraically closed of characteristic different from 2. All smooth quadric hypersurfaces $X \subset \mathbf{P}(V)$ are isomorphic to the hypersurface defined by the quadratic form

$$f(x_1, \ldots, x_N) = x_1^2 + \cdots + x_N^2.$$

When $N = 4$, the quadric X is isomorphic to $\mathbf{P}^1_{\mathbf{k}} \times \mathbf{P}^1_{\mathbf{k}}$ and the scheme $F(X)$ which parametrizes lines contained in X is the disjoint union of two smooth conics in $\mathsf{Gr}(2, V)$ (itself a smooth quadric in $\mathbf{P}(\bigwedge^2 V) \simeq \mathbf{P}^5_{\mathbf{k}}$; see Example 1.1).

For $N \geq 4$, there are several ways to study $F(X)$. The first one is elementary: let $L \subset X$ be a line. If $x \in L$, the line L lies in $T_{X,x} \cap X$, which is a cone over an $(N-4)$-dimensional smooth quadric. It follows that the family of pairs (x, L), with $x \in L \subset X$, is smooth, has dimension $N - 2 + N - 4$, and is irreducible when $N > 4$. This implies that $F(X)$ is smooth, has dimension $2N - 7$, and is irreducible when $N > 4$.

But we can also use the machinery of Section 2.1: if $L \subset X$, we have from (2.2) an exact sequence

$$0 \to N_{L/X} \to \mathscr{O}_L(1)^{\oplus(N-2)} \to \mathscr{O}_L(2) \to 0.$$

We write as in (2.11)

$$N_{L/X} \simeq \bigoplus_{i=1}^{N-3} \mathscr{O}_L(a_i),$$

where $1 \geq a_1 \geq \cdots \geq a_{N-3}$ and $a_1 + \cdots + a_{N-3} = N - 4$. We have

$$a_{N-3} = (N-4) - a_1 - \cdots - a_{N-4} \geq 0.$$

This implies $H^1(L, N_{L/X}) = 0$, hence, by Theorem 2.1, $F(X)$ is smooth of the expected dimension $2N - 7$ (we have actually shown $N_{L/X} \simeq \mathscr{O}_L(1)^{\oplus(N-4)} \oplus \mathscr{O}_L$: the line L is free).

In the next exercise, we determine $F(X)$ when $N = 5$ or 6.

Exercise 2.19 Let W be a 4-dimensional vector space. The image of the Plücker embedding $G := \mathsf{Gr}(2, W) \subset \mathbf{P}(\bigwedge^2 W)$ is a smooth 5-dimensional quadric. Let $\omega \in \bigwedge^2 W^\vee$ be a nondegenerate symplectic form; it defines a hyperplane $\omega^\perp \subset \mathbf{P}(\bigwedge^2 W)$.

a) Prove that $X := G \cap \omega^\perp$ is a smooth 5-dimensional quadric.

b) Prove that any line contained in G is of the form $\{[W_2] \in G \mid W_1 \subset W_2 \subset W_3\}$ for some partial flag $W_1 \subset W_3 \subset W$ and that this line is contained in X if and only if W_3 is the ω-orthogonal of W_1.

c) Deduce that $F(X)$ is isomorphic to $\mathbf{P}(W)$ and that $F(G)$ is isomorphic to the *incidence variety*

$$\{(x, x^\vee) \in \mathbf{P}(W) \times \mathbf{P}(W^\vee) \mid x^\vee(x) = 0\}.$$

2.5 Projective lines contained in a cubic hypersurface

Assume that $\mathbf{k}$ is algebraically closed and let $X \subset \mathbf{P}(V)$ be a smooth cubic hypersurface. When $N = \dim(V) \geq 4$, it follows from Theorem 2.6 that X contains a line L. We write as in (2.11)

$$N_{L/X} \simeq \bigoplus_{i=1}^{N-3} \mathscr{O}_L(a_i),$$

where $1 \geq a_1 \geq \cdots \geq a_{N-3}$ and $a_1 + \cdots + a_{N-3} = N - 5$, and we deduce as in Section 2.4 $a_{N-3} = (N-5) - a_1 - \cdots - a_{N-4} \geq -1$. This implies $H^1(L, N_{L/X}) = 0$, hence $F(X)$ is smooth of the expected dimension $2N - 8$ (Theorem 2.1).[2]

We have proved the following.

Theorem 2.20 *Let $X \subset \mathbf{P}(V)$ be a smooth cubic hypersurface. If $N \geq 4$, the scheme $F(X)$ of lines contained in X is smooth of dimension $2N - 8$.*

Remark 2.21 When $N \geq 5$, the scheme $F(X)$ is (geometrically) connected. Indeed, $F(X)$ is the zero locus of a section s_f of the locally free sheaf $\mathscr{E}^\vee := \mathsf{S}^3\mathscr{S}^\vee$ on $G := \mathsf{Gr}(2, N)$ and it has the expected codimension $\operatorname{rank}(\mathscr{E}) = 4$ (see (1.9)). In this situation, we have a *Koszul resolution* of its structure sheaf:

$$0 \longrightarrow \bigwedge^4 \mathscr{E} \longrightarrow \bigwedge^3 \mathscr{E} \longrightarrow \bigwedge^2 \mathscr{E} \longrightarrow \mathscr{E} \xrightarrow{s_f^\vee} \mathscr{O}_G \longrightarrow \mathscr{O}_{F(X)} \longrightarrow 0. \qquad (2.15)$$

(This complex is exact because locally, $\mathscr{E}$ is free and the components of s_f in a basis of $\mathscr{E}$ form a regular sequence.) Using this sequence and, *in characteristic zero,* the Borel–Weil theorem, which computes the cohomology of homogeneous vector bundles such as $\bigwedge^r \mathscr{E}$ on G, one can compute some of the cohomology of $\mathscr{O}_{F(X)}$ and obtain for example $h^0(F(X), \mathscr{O}_{F(X)}) = 1$ for $N \geq 5$ ([DM, th. 3.4]), hence the connectedness of $F(X)$. This is obtained in all characteristics in [AK, Theorem (5.1)] by direct computations.

Under the hypotheses of the theorem, by Exercice 2.7.a), the subscheme $F(X) \subset \mathsf{Gr}(2, V)$ has class $9(2\sigma_{3,1} + 3\sigma_{2,2})$. Moreover, its canonical class, given by formula (2.7), is

$$K_{F(X)} = (6 - N)\sigma_1|_{F(X)}.$$

When $N = 5$, the smooth variety $F(X)$ is a surface of general type; when $N = 6$, its canonical class is trivial (see Section 2.5.2 for more details); when $N \geq 7$, it is a Fano variety.

2.5.1 Lines on a smooth cubic surface

When $N = 4$, the class $\sigma_{3,1}$ vanishes in $CH(\mathsf{Gr}(2, V))$, hence $[F(X)] = 27\sigma_{2,2}$, where $\sigma_{2,2}$ is the class of a point. Since $F(X)$ is smooth (Theorem 2.20), this proves the famous classical result:

Every smooth cubic surface over an algebraically closed field contains 27 lines.

[2] A little more work shows there are two possible types for normal bundles:

$$\begin{aligned} N_{L/X} &\simeq \mathscr{O}_L(1)^{\oplus(N-5)} \oplus \mathscr{O}_L^{\oplus 2} && \text{(lines of the first type, free);} \\ N_{L/X} &\simeq \mathscr{O}_L(1)^{\oplus(N-4)} \oplus \mathscr{O}_L(-1) && \text{(lines of the second type, not free).} \end{aligned}$$

When $N \geq 4$, there are always lines of the second type. When $N \geq 5$, a general line in X is of the first type if $\operatorname{char}(\mathbf{k}) \neq 2$ (in characteristic 0, this is because X is covered by lines (Proposition 2.13 and Theorem 2.17; see also Exercise 2.18); this is not true for the Fermat cubic in characteristic 2 by (2.12)).

Remark 2.22 Let X be an irreducible normal cubic surface in $\mathbf{P}^3$ which is not a cone. It can be shown that X contains only finitely many lines. The scheme $F(X)$ then has class $27\sigma_{2,2}$, but might not be reduced, so that X contains at most 27 lines. In fact, over an algebraically closed field, X is smooth if and only if it contains exactly 27 lines; it follows that when X is singular, $F(X)$ is not reduced.

Example 2.23 (Real lines) The 27 complex lines contained in a smooth real cubic surface X are either real or complex conjugate. Since 27 is odd, X always contains a real line. One can prove that the set of real lines contained in X has either 3, 7, 15, or 27 elements.[3] In many mathematics departments around the world, there are plaster models of (real!) cubic surfaces with 27 (real) lines on them; it is usually the Clebsch cubic (1871), with equations in $\mathbf{P}^4$:

$$x_0 + \cdots + x_4 = x_0^3 + \cdots + x_4^3 = 0.$$

Among its 27 lines, 15 are defined over $\mathbf{Q}$, and the other 12 over the field $\mathbf{Q}(\sqrt{5})$.[4]

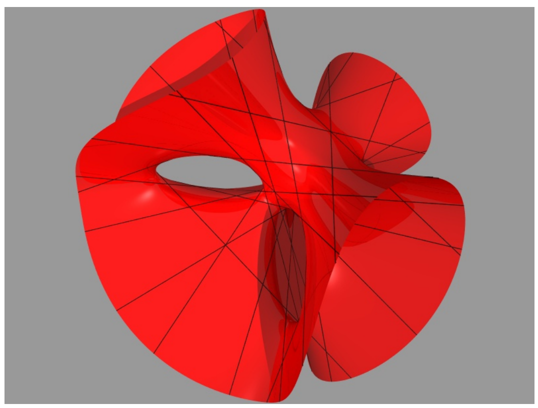

Figure 2.1: The Clebsch cubic with its 27 real lines

Example 2.24 (Q-lines) An explicit[5] surface defined over $\mathbf{Q}$ with all its 27 lines defined over $\mathbf{Q}$ was found by Tetsuji Shioda in 1995. Its equation is

$$x_2^2x_4 + 2x_2x_3^2 = x_1^3 - x_1(59475x_4^2 + 78x_3^2) + 2848750x_4^3 + 18226x_3^2x_4.$$

All 27 lines have explicit rational equations.

[3]Actually, lines on real cubic surfaces should be counted with signs, in which case one gets that the total number is always 3.

[4]The permutation group $\mathfrak{S}_5$ acts on X. The lines defined over $\mathbf{Q}$ are $\langle(1,-1,0,0,0),(0,0,1,-1,0)\rangle$ and its images by $\mathfrak{S}_5$, and the 12 other lines are the real line $\langle(1,\zeta,\zeta^2,\zeta^3,\zeta^4),(1,\bar\zeta,\bar\zeta^2,\bar\zeta^3,\bar\zeta^4)\rangle$, where $\zeta := \exp(2i\pi/5)$, and its images by $\mathfrak{S}_5$.

[5]For any field $\mathbf{k}$, there exists a smooth cubic surface defined over $\mathbf{k}$ with 27 $\mathbf{k}$-lines as soon as one can find 6 points in $\mathbf{P}^2_{\mathbf{k}}$ in general position (no 3 collinear, no 6 on a conic); this is possible whenever $\mathrm{Card}(\mathbf{k}) \geq 4$ (there are not enough points in $\mathbf{P}^2_{\mathbf{F}_2}$ or $\mathbf{P}^2_{\mathbf{F}_3}$). So the issue here is to give an explicit equation for this cubic.

2.5.2 Lines on a smooth cubic fourfold

When $N = 5$, the smooth projective fourfold $F(X)$ has trivial canonical class. When $\mathbf{k} = \mathbf{C}$, these varieties were classified by Beauville ([B]) and Bogomolov ([Bo]).

Theorem 2.25 (Beauville–Bogomolov Decomposition Theorem) *Let F be a smooth projective complex variety such that $c_1(T_F)$ vanishes in $H^2(F, \mathbf{R})$. There exists a finite étale cover of F which is isomorphic to the product of*

- *nonzero abelian varieties;*
- *simply connected* Calabi-Yau varieties;
- *simply connected* holomorphic symplectic varieties.

Here, a Calabi-Yau variety Y is a variety of dimension $n \geq 3$ such that $H^i(Y, \mathscr{O}_Y) = 0$ for all $0 < i < n$. In particular, $\chi(Y, \mathscr{O}_Y) = 1 + (-1)^n$.

A holomorphic symplectic variety Y is a variety carrying a holomorphic 2-form η which is everywhere nondegenerate. The dimension of Y is even and Y is simply connected if and only if $H^0(Y, \Omega_Y^2) = \mathbf{C}[\eta]$, in which case

$$H^0(Y, \Omega_Y^r) = \begin{cases} \mathbf{C}[\eta^{\wedge(r/2)}] & \text{if } r \text{ is even and } 0 \leq r \leq \dim(Y), \\ 0 & \text{otherwise,} \end{cases}$$

so that $\chi(Y, \mathscr{O}_Y) = 1 + \frac{1}{2}\dim(Y)$. When Y is a surface, it is called a K3 surface.

In our case, one finds[6]

$$\chi(F(X), \mathscr{O}_{F(X)}) = 3.$$

Since the holomorphic Euler characteristic of a nonzero abelian variety vanishes, there can be no such factor in the decomposition of the theorem applied to $F = F(X)$; moreover, there are only 3 possibilities for its universal cover $\pi : \widetilde{F(X)} \to F(X)$:

- either $\widetilde{F(X)}$ is a Calabi-Yau fourfold, in which case

  $$2 = \chi(\widetilde{F(X)}, \mathscr{O}_{\widetilde{F(X)}}) = \deg(\pi)\chi(F(X), \mathscr{O}_{F(X)}) = 3\deg(\pi),$$

  which is impossible;

- or $\widetilde{F(X)}$ is a product of two K3 surfaces, in which case $\chi(\widetilde{F(X)}, \mathscr{O}_{\widetilde{F(X)}}) = 4 = 3\deg(\pi)$, which is also impossible;

[6]The Koszul resolution (2.15) gives $\chi(F(X), \mathscr{O}_{F(X)}) = \sum_{i=0}^4 (-1)^i \chi(\mathsf{Gr}(2, \mathbf{C}^6), \bigwedge^i(\mathsf{S}^3\mathscr{S}))$ and this sum can be computed unsing a computer software such as Macaulay2.

- or $\widetilde{F(X)}$ is a simply connected holomorphic symplectic fourfold, in which case

$$3 = \chi(\widetilde{F(X)}, \mathscr{O}_{\widetilde{F(X)}}) = \deg(\pi)\chi(F(X), \mathscr{O}_{F(X)}) = 3\deg(\pi).$$

It follows that we are in the third case and that

when $\mathbf{k} = \mathbf{C}$*, the variety* $F(X)$ *is a simply connected holomorphic symplectic fourfold.*

We will determine its structure in the particular case when X is a *Pfaffian cubic.* We go back to a general field $\mathbf{k}$, which we assume is *infinite and of characteristic* $\neq 2$.

The construction is the following. Let W_6 be a 6-dimensional $\mathbf{k}$-vector space. We defined and studied in Exercise 2.12 the cubic hypersurface $X_3 \subset \mathbf{P}(\bigwedge^2 W_6^\vee)$ of degenerate skew-symmetric bilinear forms on W_6. Its singular locus corresponds to skew forms of corank 4, which, since W_6 has dimension 6, is just the Grassmannian $\mathsf{Gr}(2, W_6^\vee)$. It has codimension 6 in $\mathbf{P}(\bigwedge^2 W_6^\vee)$. For a general 6-dimensional $\mathbf{k}$-vector subspace $V_6 \subset \bigwedge^2 W_6^\vee$, the intersection

$$X := \mathbf{P}(V_6) \cap X_3 \subset \mathbf{P}(\textstyle\bigwedge^2 W_6^\vee)$$

is a cubic fourfold, which is smooth by the Bertini theorem, called a *Pfaffian cubic fourfold.*

Consider now in the dual space the intersection

$$S := \mathsf{Gr}(2, W_6) \cap \mathbf{P}(V_6^\perp) \subset \mathbf{P}\big(\textstyle\bigwedge^2 W_6\big).$$

Since V_6 is general and $\operatorname{codim}(V_6^\perp) = \dim(V_6) = 6$, we obtain a surface and since $K_{\mathsf{Gr}(2,W_6)} = -6\sigma_1$, its canonical sheaf is trivial (by adjunction). It is in fact a K3 surface.

Consider the blow up $\widetilde{S \times S} \to S \times S$ of the diagonal $\{(y_1, y_2) \in S \times S \mid y_1 = y_2\}$. The involution of $S \times S$ which exchanges the two factors lifts to an involution ι of $\widetilde{S \times S}$. We define the *symmetric square* $S^{[2]}$ as the smooth quotient $\widetilde{S \times S}/\iota$.

Fujiki proved that $S^{[2]}$ is a holomorphic symplectic fourfold defned over $\mathbf{k}$, in the sense that the $\mathbf{k}$-vector space $H^0(S^{[2]}, \Omega^2_{S^{[2]}})$ is generated by a form η which is nondegenerate at every point and $H^0(S^{[2]}, \Omega^4_{S^{[2]}})$ is generated by the nowhere vanishing form $\eta \wedge \eta$ (this generalizes the definition given earlier when $\mathbf{k} = \mathbf{C}$). It parametrizes subschemes of S of length 2.

Proposition 2.26 (Beauville–Donagi, [BD]) *Assume that the field* $\mathbf{k}$ *is infinite of characteristic* $\neq 2$*. Let* $X \subset \mathbf{P}(V_6)$ *be a general*[7] *Pfaffian cubic fourfold. Then* $F(X)$ *is isomorphic to* $S^{[2]}$*.*

PROOF. We construct a morphism $F(X) \to S^{[2]}$.

[7] More precisely, we assume that X is smooth and contains no projective planes and S contains no projective lines.

A projective line contained in X corresponds to a 2-dimensional vector subspace $V_2 \subset V_6$ of skew-symmetric forms on W_6, all degenerate of rank 4 (except for the zero form). There exists a 4-dimensional subspace $W_4 \subset W_6$ which is isotropic for all these forms (Exercise 2.12.b)) and one can show that it is unique.[8] The vector space V_2 is then contained in $(\bigwedge^2 W_4)^\perp \cap V_6$. Conversely, this intersection defines a projective space of dimension ≥ 1 contained in X, since any form in $(\bigwedge^2 W_4)^\perp$ must be degenerate.

A "count of parameters" shows that *for a general choice of* V_6*, the fourfold* X *contains no projective planes.* If we make this assumption, we obtain

$$\dim\big((\textstyle\bigwedge^2 W_4)^\perp \cap V_6\big) = 2.$$

By duality, this means $\dim(\bigwedge^2 W_4 + V_6^\perp) = 13$, hence

$$\dim\big((\textstyle\bigwedge^2 W_4) \cap V_6^\perp\big) = 2.$$

In other words, $\mathbf{P}(\bigwedge^2 W_4) \cap \mathbf{P}(V_6^\perp)$ is a projective line. Its intersection with the quadric $\mathsf{Gr}(2, W_4) \subset \mathsf{Gr}(2, W_6)$ (see Exercice 1.1) is then contained in S. Again, a "count of parameters" shows that *for a general choice of* V_6*, the surface* S *contains no projective lines.* If we make this further assumption, the intersection is a subscheme of S of length 2, hence a point of $S^{[2]}$.

Let us show that φ is birational by constructing an inverse. Consider two distinct points in S. We can see them as distinct 2-dimensional vector subspaces P_1 and P_2 of W_6. They also define a projective line in $\mathbf{P}(V_6^\perp)$. Since S contains no lines, this line cannot be contained in $\mathsf{Gr}(2, P_1 + P_2)$. This implies in particular that $P_1 + P_2$ has dimension 4. Any skew-symmetric form in V_6 vanishes on P_1 and P_2, hence those forms which vanish on $P_1 + P_2$ form a vector space of dimension ≥ 2 which corresponds, because of the assumption on X, to a projective line contained in X, hence to a point of $F(X)$. This defines an inverse to φ on the complement of the exceptional divisor in $S^{[2]}$.

To finish the proof, one can either see that this construction of the inverse extends to the whole of $S^{[2]}$, so that φ is an isomorphism, or argue that the pull-back by φ of a nowhere zero 4-form on $S^{[2]}$ (which exists because $S^{[2]}$ is a symplectic variety) is a non identically zero 4-form on $F(X)$. But then, this form vanishes nowhere, since $F(X)$ is also a symplectic variety. This implies that the tangent map to φ is everywhere an isomorphism, hence the birational morphism φ is an isomorphism. □

[8] One way to check that is to use a result of Jordan–Kronecker, which gives normal forms for any pair of skew-symmetric forms on a finite-dimensional vector space over an algebraically closed field of characteristic $\neq 2$. In our case, one sees that there is a basis of W_6 in which a given pair of generators of the pencil is given by the matrices $\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0 \end{pmatrix}$ and either $\begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ -1&0&0&0&0&0\\ 0&-1&0&0&0&0 \end{pmatrix}$, or $\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0 \end{pmatrix}$. In the first case, the kernels K_1 and K_2 of the forms form a direct sum and $W_4 = K_1 + K_2$ is defined over $\mathbf{k}$. In the second case, W_4 is spanned by all the kernels of the forms in the pencil and it is again defined over $\mathbf{k}$.

2.6 Cubic hypersurfaces over finite fields

As usual, V is a $\mathbf{k}$-vector space of dimension N. If $X \subset \mathbf{P}(V)$, we let $X(\mathbf{k})$ be the set of points of X defined over $\mathbf{k}$, also called $\mathbf{k}$-rational points. Our basic tool here will be the following famous result.

Theorem 2.27 (Chevalley–Warning) *If $f_1, \ldots, f_r$ are polynomials in N variables with coefficients in a finite field $\mathbf{k}$, of respective degrees $d_1, \ldots, d_r$, and if $d_1 + \cdots + d_r < N$, the number of solutions in $\mathbf{k}^N$ of the system of equations $f_1(x) = \cdots = f_r(x) = 0$ is divisible by the characteristic of $\mathbf{k}$.*

The proof is clever but elementary. A refinement by Ax says that the number of solutions is in fact divisible by $\mathrm{Card}(\mathbf{k})$.

When $f_1, \ldots, f_r$ are homogeneous, $(0, \ldots, 0)$ is always a solution, hence the theorem implies that the scheme X defined by the equations $f_1, \ldots, f_r$ in $\mathbf{P}(V)$ always has a $\mathbf{k}$-point. The Ax refinement even implies

$$\mathrm{Card}(X(\mathbf{k})) \equiv 1 \pmod{\mathrm{Card}(\mathbf{k})}.$$

The bound in the theorem is sharp: the plane cubic X defined in characteristic 2 by the cubic equation

$$A(x_1, x_2, x_3) := x_1^3 + x_2^3 + x_3^3 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1 + x_1 x_2 x_3 \tag{2.16}$$

has no $\mathbf{F}_2$-point: it is the union of 3 lines defined over $\mathbf{F}_8$ and permuted by the action of the cyclic Galois group $\mathrm{Gal}(\mathbf{F}_8/\mathbf{F}_2)$.

However, for any *smooth* plane cubic curve C defined over a finite field $\mathbf{F}_q$, the Hasse bound

$$\left|\mathrm{Card}\big(C(\mathbf{F}_q)\big) - q - 1\right| \le 2\sqrt{q} \tag{2.17}$$

implies that C always has an $\mathbf{F}_q$-point.

Exercise 2.28 Let $X \subset \mathbf{P}(V)$ be a hypersurface of degree $d < N$ defined over $\mathbf{F}_q$. Show $\mathrm{Card}(X(\mathbf{F}_q)) \ge q^{N-1-d} + \cdots + q + 1$.

Exercise 2.29 (The Swinnerton–Dyer cubic surface over $\mathbf{F}_2$) The bound in Exercise 2.28 is sharp: show that the cubic surface $X \subset \mathbf{P}^3_{\mathbf{F}_2}$ defined by the equation

$$A(x_1, x_2, x_3) + x_1 x_4^2 + x_1^2 x_4$$

(see (2.16)) is smooth and that $X(\mathbf{F}_2) = \{(0, 0, 0, 1)\}$.

Corollary 2.30 *Let $X \subset \mathbf{P}(V)$ be a hypersurface of degree d defined over a finite field $\mathbf{k}$. If $N - 1 > d(d+1)/2$, every $\mathbf{k}$-rational point of X is contained in a line defined over $\mathbf{k}$ and contained in X.*

Proof. Let $x \in X(\mathbf{k})$. By (2.13), the scheme $F(X,x)$ of lines contained in X passing through x is defined, in an $(N-2)$-dimensional projective space, by equations of degrees $1,\dots,d-1,d$. Under the hypothesis $N-1 > d(d+1)/2$, the set $F(X,x)(\mathbf{k})$ is nonempty by the Chevalley–Warning theorem. □

Exercise 2.31 Let $X \subset \mathbf{P}(V)$ be a subscheme defined over a finite field $\mathbf{k}$ by equations $f_1,\dots,f_m$. If $\sum_{i=1}^m \deg(f_i)(\deg(f_i)+1) < 2(N-1)$, prove that every $\mathbf{k}$-rational point of X is contained in a line defined over $\mathbf{k}$ and contained in X.

In particular, any cubic hypersurface $X \subset \mathbf{P}(V)$ contains a $\mathbf{k}$-line when $N-1>6$. One can actually do better: when $N>6$ and X is smooth, the smooth projective variety $F(X)$ is a Fano variety by (2.7), and a difficult theorem of Esnault ([E]) then implies $F(X)(\mathbf{k}) \neq \varnothing$, hence X contains a $\mathbf{k}$-line. This still holds for *any* cubic when $N>6$ by [FR, Corollary 1.4].

Using a vast generalization of the Hasse bound (2.17) (the Weil conjectures, proved by Deligne), I can show that any *smooth* cubic threefold defined over a finite field $\mathbf{k}$ with ≥ 11 elements contains a $\mathbf{k}$-line.

In the examples below, we show that when $N \in \{4,5,6\}$, some cubic hypersurfaces defined over small fields contain no (or few) lines.

Example 2.32 (Diagonal cubic surfaces) Consider the cubic surface $X \subset \mathbf{P}^3_{\mathbf{k}}$ defined by

$$a_1x_1^3 + a_2x_2^3 + a_3x_3^3 + a_4x_4^3,$$

where $a_1,\dots,a_4 \in \mathbf{k}$ are all nonzero. It is smooth whenever the characteristic of $\mathbf{k}$ is not 3, which we assume. Let b_{ij} be such that $b_{ij}^3 = -a_i/a_j$. If $\{1,2,3,4\} = \{i,j,k,l\}$, the projective line joining $e_i + b_{ij}e_j$ and $e_k + b_{kl}e_l$ is contained in X. Since we have 3 choices for $\{i,j\}$ and 3 choices for each b_{ij}, the 27 lines of the cubic $X_{\overline{\mathbf{k}}} := X \times_{\mathbf{k}} \overline{\mathbf{k}}$ are all obtained in this fashion hence are defined over $\mathbf{k}[\sqrt[3]{a_i/a_j}, 1 \leq i < j \leq 4]$.

In particular, the 27 lines of the smooth Fermat cubic defined by

$$x_1^3 + x_2^3 + x_3^3 + x_4^3$$

in characteristic 2 are defined over $\mathbf{F}_4$ (but only 3 of them over $\mathbf{F}_2$), whereas, if $a \in \mathbf{F}_4 \smallsetminus \{0,1\}$, the smooth cubic surface defined by

$$x_1^3 + x_2^3 + x_3^3 + ax_4^3 \tag{2.18}$$

contains no lines defined over $\mathbf{F}_4$. (Its 27 lines are all defined over $\mathbf{F}_{64}$ and the orbits of the cyclic Galois group $\mathrm{Gal}(\mathbf{F}_{64}/\mathbf{F}_4)$ all consist of 3 points.)

Example 2.33 (A smooth cubic threefold over $\mathbf{F}_2$ with no lines) With a computer, one checks that the cubic threefold $X \subset \mathbf{P}^4_{\mathbf{F}_2}$ defined by

$$A(x_1,x_2,x_3) + x_1x_4^2 + x_1^2x_4 + x_2x_5^2 + x_2^2x_5 + x_4^2x_5$$

is smooth and contains no $\mathbf{F}_2$-lines (but 9 $\mathbf{F}_2$-points and 8 $\mathbf{F}_4$-lines).

Example 2.34 (A smooth cubic threefold over $\mathbf{F}_3$ with no lines (Laface)) With a computer, one checks that the cubic threefold $X \subset \mathbf{P}^4_{\mathbf{F}_3}$ defined by

$$x_1^3 + x_2^3 - x_1x_3^2 - x_2^2x_4 + x_3^2x_4 - x_1^2x_5 - x_2x_3x_5 + x_1x_4x_5 + x_2x_4x_5 + x_4^2x_5 + x_4x_5^2 - x_5^3$$

is smooth and contains no $\mathbf{F}_3$-lines (and 25 $\mathbf{F}_3$-points[9]).

Example 2.35 (A smooth cubic fourfold over $\mathbf{F}_2$ with one line) With a computer, one checks that the cubic fourfold $X \subset \mathbf{P}^5_{\mathbf{F}_2}$ defined by

$$A(x_1,x_2,x_3)+x_1x_4^2+x_1^2x_4+x_2x_5^2+x_2^2x_5+x_4x_5^2+x_4^2x_5+x_3x_6^2+x_3^2x_6+x_4x_6^2+x_4^2x_6+x_5x_6^2+x_5^2x_6+x_4x_5x_6$$

is smooth and contains a single $\mathbf{F}_2$-line (and 13 $\mathbf{F}_2$-points).

Questions 2.36 Does there exist smooth fourfolds defined over $\mathbf{F}_2$ which contain no $\mathbf{F}_2$-lines? Does there exist smooth threefolds defined over $\mathbf{F}_4$ which contain no $\mathbf{F}_4$-lines?

[9] These points are $(0,0,0,1,0)$, $(0,0,1,0,0)$, $(0,1,0,0,1)$, $(0,1,0,1,0)$, $(0,1,0,-1,1)$, $(0,1,-1,-1,1)$, $(0,-1,0,-1,1)$, $(0,-1,1,1,1)$, $(0,-1,1,-1,1)$, $(0,-1,-1,0,1)$, $(1,0,1,0,0)$, $(1,1,0,0,1)$, $(1,1,0,1,1)$, $(1,1,-1,0,1)$, $(1,-1,1,1,1)$, $(-1,0,0,0,1)$, $(-1,0,1,0,0)$, $(-1,0,1,1,1)$, $(-1,0,-1,1,1)$, $(-1,1,0,0,0)$, $(-1,1,1,1,1)$, $(-1,1,-1,0,1)$, $(-1,1,-1,-1,1)$, $(-1,-1,0,1,0)$, $(-1,-1,1,-1,1)$.

Chapter 3

Conics and curves of higher degrees

After lines, the next simplest curves in a projective space are (plane) conics. We analyze conics on subschemes of the projective space $\mathbf{P}(V)$. The main difference with lines is that smooth conics can degenerate to singular conics, which are pair of lines meeting at a point, or double lines.

3.1 Conics in the projective space

A conic C in $\mathbf{P}(V)$ is defined by a nonzero homogeneous polynomial of degree 2 in a projective plane $\mathbf{P}(V_3)$ contained in $\mathbf{P}(V)$. The plane $\mathbf{P}(V_3)$ is the linear span of C: it is the smallest linear space containing C (as a scheme). It follows that the conics in $\mathbf{P}(V)$ can be parametrized by the total space $M(\mathbf{P}(V), 2)$ of the projective bundle

$$\pi : M(\mathbf{P}(V), 2) := \mathbf{P}(\mathsf{S}^2\mathscr{S}^\vee) \longrightarrow \mathsf{Gr}(3, V). \tag{3.1}$$

This space is therefore a smooth projective scheme of dimension $5 + 3(N - 3) = 3N - 4$. It carries a "tautological" line bundle $\mathscr{O}_{M(\mathbf{P}(V),2)}(1)$ and when $N \geq 4$, its Picard group has rank 2, generated by the classes of $\mathscr{O}_{M(\mathbf{P}(V),2)}(1)$ and $\pi^* \det(\mathscr{S}^\vee)$.

The subset of singular conics is an irreducible divisor $M_s(\mathbf{P}(V), 2)$ in $M(\mathbf{P}(V), 2)$; we denote its complement by $M^0(\mathbf{P}(V), 2)$.

Exercise 3.1 Let C be a smooth conic in $\mathbf{P}(V)$, with span P. Prove that the normal exact sequence

$$0 \to N_{C/P} \to N_{C/\mathbf{P}(V)} \to N_{P/\mathbf{P}(V)}|_C \to 0$$

splits and that $N_{C/\mathbf{P}(V)} \simeq \mathscr{O}_C(2)^{\oplus(N-3)} \oplus \mathscr{O}_C(4)$ (*Hint:* note that the restriction of $\mathscr{O}_{\mathbf{P}(V)}(1)$ to C is $\mathscr{O}_C(2)$).

Exercise 3.2 a) Find the class of the divisor $M_s(\mathbf{P}(V), 2)$ in $M(\mathbf{P}(V), 2)$.

b) Is the divisor $M_s(\mathbf{P}(V), 2)$ ample?

c) Find complete positive-dimensional families of smooth conics in $\mathbf{P}(V)$ when $N \geq 4$ (see Example 3.3 below).

Example 3.3 (O. Benoist) Assume $N \geq 3$. We construct a complete family of dimension $N-3$ of pairwise distinct smooth conics in $\mathbf{P}(V)$. The space of linear embeddings of $\mathbf{P}^2$ into $\mathbf{P}(V)$ can be identified, inside the projective space associated with the vector space $\mathbf{P}^{3N-1}_{\mathbf{k}}$ of $3 \times N$ matrices, with the open set U consisting of matrices of maximal rank 3. The complement of this open set has codimension $N-2$. By the Bertini theorem, the intersection M of U with a general linear space of dimension $N-3$ is projective. In this way, we obtain of projective family of dimension $N-3$ of morphisms $\mathbf{P}^2_{\mathbf{k}} \to \mathbf{P}(V)$, whose images are pairwise distinct. It is then enough to consider the images of a fixed smooth conic in $\mathbf{P}^2_{\mathbf{k}}$.

Question 3.4 Is $N-3$ the maximal dimension of a complete family of generically pairwise distinct smooth conics in $\mathbf{P}(V)$?

3.2 Conics contained in a hypersurface

If $X \subset \mathbf{P}(V)$ is a hypersurface defined by an equation f, we define the subscheme $M(X,2) \subset M(\mathbf{P}(V),2)$ as the space of conics C contained in X. More precisely, we require that the equation f of X, when restricted to the plane $\langle C \rangle$, vanishes on C.

The formal definition is as follows. There is a canonical inclusion $\mathscr{O}_{M(\mathbf{P}(V),2)}(-1) \hookrightarrow \pi^* \mathsf{S}^2 \mathscr{S}^\vee$ which induces an exact sequence

$$0 \to \mathscr{O}_{M(\mathbf{P}(V),2)}(-1) \otimes \pi^* \mathsf{S}^{d-2} \mathscr{S}^\vee \to \pi^* \mathsf{S}^d \mathscr{S}^\vee \to \mathscr{E} \to 0$$

of vector bundles on $M(\mathbf{P}(V),2)$. The vector bundle $\mathscr{E}$ has rank $2d+1$ and its fiber at a point corresponding to a conic C is $H^0(C, \mathscr{O}_C(2d))$ (note that the restriction of $\mathscr{O}_{\mathbf{P}(V)}(1)$ to C is $\mathscr{O}_C(2)$). On the other hand, f defines by restriction a section of $\pi^* \mathsf{S}^d \mathscr{S}^\vee$, hence a section t_f of $\mathscr{E}$. We define the scheme $M(X,2)$ to be the zero-scheme $Z(t_f)$. It is either empty or of codimension $\leq \operatorname{rank}(\mathscr{E}) = 2d+1$ in $M(\mathbf{P}(V),2)$. We say as usual that

$$\dim(M(\mathbf{P}(V),2)) - 2d - 1 = 3N - 2d - 5$$

is the *expected dimension* of $M(X,2)$. We set $M^0(X,2) := M(X,2) \cap M^0(\mathbf{P}(V),2)$, the scheme of smooth conics contained in X.

Proposition 3.5 *A general hypersurface $X \subset \mathbf{P}(V)$ of degree $d > \frac{3}{2}N - \frac{5}{2}$ contains no conics.*

Proof. This is a standard dimension count. We let $\mathbf{P} := \mathbf{P}(\mathsf{S}^d V^\vee)$ be the parameter space for degree-d hypersurfaces in $\mathbf{P}(V)$ and we introduce the incidence variety

$$I := \{(C,X) \in M(\mathbf{P}(V),2) \times \mathbf{P} \mid C \subset X\} \tag{3.2}$$

with its two projections $\mathsf{pr}_1 : I \to M(\mathbf{P}(V),2)$ and $\mathsf{pr}_2 : I \to \mathbf{P}$. Given $[C] \in M(\mathbf{P}(V),2)$, the fiber $\mathsf{pr}_1^{-1}([C])$ is the projectivization of the kernel of the map

$$\mathsf{S}^d V^\vee = H^0(\mathbf{P}(V), \mathscr{O}_{\mathbf{P}(V)}(d)) \to H^0(C, \mathscr{O}_C(2d)).$$

Since this map is always surjective, I is a projective bundle over $M(\mathbf{P}(V),2)$ hence is smooth projective irreducible of codimension $2d+1$ in $M(\mathbf{P}(V),2)\times\mathbf{P}$.

Since $2d+1>3N-4=\dim(M(\mathbf{P}(V),2))$, the map $\mathsf{pr}_2: I\to\mathbf{P}$ is not surjective. □

It is likely that in the range $d\le\frac{3}{2}N-\frac{5}{2}$, any hypersurface $X\subset\mathbf{P}(V)$ of degree d contains a conic, but I was unable to find a satisfactory reference. One method would be to prove $c_{2d+1}(\mathscr{E})\neq 0$, but it is probably difficult. Another method would be to prove that, with the notation of the proof above, the projection $\mathsf{pr}_2: I\to\mathbf{P}$ is surjective. For that, it is enough to find one hypersurface X of degree d such that $M(X,2)$ has the expected dimension at one point. At least in characteristic 0, since I is smooth, this would imply additionally, by generic smoothness, that when X is general, $M(X,2)$ is smooth of the expected dimension.[1]

I will present now a sketch of proof of an older result of S. Katz ([K]), which is a particular case of these statements.

Theorem 3.6 (S. Katz) *A general quintic hypersurface $X\subset\mathbf{P}^4_{\mathbf{k}}$ contains 609,205 conics and they are all smooth.*

Sketch of proof. We introduce as in (3.2) the incidence variety $I\subset M(\mathbf{P}(V),2)\times\mathbf{P}$. We know that it is smooth projective irreducible of dimension $\dim(\mathbf{P})$. We analyze the projection $\mathsf{pr}_2: I\to\mathbf{P}$; any fiber $\mathsf{pr}_2^{-1}([X])$ is isomorphic as a scheme to $M(X,2)$. Katz constructs one quintic X smooth along a smooth conic $C\subset X$ and such that

$$N_{C/X}\simeq\mathscr{O}_C(-1)\oplus\mathscr{O}_C(-1).$$

In particular, $H^1(C,N_{C/X})=0$ and Theorem 2.1, which is still valid for conics (see Theorem 3.15), implies that $M(X,2)$ is smooth of dimension $H^0(C,N_{C/X})=0$ at $[C]$. This says that the fiber $\mathsf{pr}_2^{-1}([X])$ is smooth at the point (C,X) of I. In other words, pr_2 is étale at (C,X).

In particular, pr_2 is surjective and generically smooth: for X general, $M(X,2)$ is smooth and consists of smooth conics on X. Their number is given by the top Chern class $c_{11}(\mathscr{E})$, which a software such as Macaulay2 computes as being 609,205! □

Exercise 3.7 Given a line L contained in a smooth subvariety $X\subset\mathbf{P}(V)$, prove that the space of nonreduced conics in X with reduced structure L is isomorphic to $\mathbf{P}\big(H^0(L,N_{L/X}(-1))\big)$. Deduce that in characteristic $p>0$, the (smooth) Fermat hypersurface of degree p^r+1 studied in Exercise 2.11 contains a family of nonreduced conics of dimension $3N-13$.

Given a point $x\in X(\mathbf{k})$, we may consider the subscheme $M(X,2,x)\subset M(X,2)$ of conics contained in X and passing through x. It is not too difficult to construct $M(X,2,x)$ as the zero-scheme of a vector bundle $\mathscr{F}$ of rank $2d$ over a projective bundle of relative

[1] Assume $\mathrm{char}(\mathbf{k})=0$ and X general. Theorem 1.1 of [S] claims that for X general of degree $d\le\frac{3}{2}N-\frac{5}{2}$, the scheme $M^0(X,2)$ is smooth of the expected dimension, but the proof does not actually show that it is nonempty. If $d<N-2$, [De, Proposition 2.3.4] says that $M(X,2)$ is smooth, irreducible, of the expected dimension.

dimension 4 over $\mathsf{Gr}(2, V/V_1)$ (where $V_1 \subset V$ is the one-dimensional subspace corresponding to x).

It follows that the expected dimension of $M(X, 2, x)$ is $2(N-1-2)+4-2d = 2(N-1-d)$. In particular, we expect that for $d \leq N-1$, it is always nonempty, i.e., that X is covered by conics (recall from Proposition 2.13 that X is covered by lines when $d \leq N-2$). However, I could not find a reference in the literature for this fact. Of course, it would follow from the nonvanishing of the top Chern class of $\mathscr{F}$, but again, this is not easy to prove! We present an alternate proof of this fact in the exercise below.

Exercise 3.8 (Starr) The aim of this exercise is to prove that a hypersurface $X \subset \mathbf{P}(V)$ of degree $N-1$ is covered by conics and that for X general, $M(X,2)$ is smooth of the expected dimension. It was communicated to us by J. Starr.

Consider the homogeneous polynomial

$$f = (y_1 y_3 - y_2^2) y_1^{N-3} + y_4 y_1^{N-4} y_3^2 + y_5 y_1^{N-5} y_3^3 + \cdots + y_N y_1^0 y_3^{N-2}$$

of degree $N-1$ and the hypersurface $X \subset \mathbf{P}(V)$ that it defines. It vanishes on the smooth conic $C = Z(y_1 y_3 - y_2^2, y_4, \dots, y_N)$, whose span is the plane $P = Z(y_4, \dots, y_N)$.

We want to determine the normal bundle of C in X. Recall from Exercise 3.1 that there are isomorphisms $N_{C/\mathbf{P}(V)} \simeq N_{P/\mathbf{P}(V)}|_C \oplus N_{C/P} \simeq \mathscr{O}_C(2)^{\oplus(N-3)} \oplus \mathscr{O}_C(4)$ and an exact sequence $0 \to N_{C/X} \to N_{C/\mathbf{P}(V)} \to N_{X/\mathbf{P}(V)}|_C \to 0$, with $N_{X/\mathbf{P}(V)}|_C \simeq \mathscr{O}_C(2N-2)$. We consider its twist

$$0 \to N_{C/X}(-1) \to \mathscr{O}_C(1)^{\oplus(N-3)} \oplus \mathscr{O}_C(3) \xrightarrow{df} \mathscr{O}_C(2N-3) \to 0. \tag{3.3}$$

We parametrize C by the map $\mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}(V)$, $(t,u) \mapsto (t^2, tu, u^2, 0, \dots, 0)$, so that $H^0(C, \mathscr{O}_C(2N-3)) \simeq \langle t^{2N-3}, t^{2N-4}u, \dots, u^{2N-3} \rangle$.

a) Prove that the image of $H^0(C, \mathscr{O}_C(1)^{\oplus(N-3)})$ by the map df in (3.3) is the subspace $\langle t^{2N-7}u^4, t^{2N-8}u^5, \dots, u^{2N-3} \rangle$.
b) Prove that the image of $H^0(C, \mathscr{O}_C(3))$ by the map df in (3.3) is the remaining subspace $\langle t^{2N-3}, t^{2N-4}u, t^{2N-5}u^2, t^{2N-6}u^3 \rangle$.
c) Deduce that $H^1(C, N_{C/X}(-1)) = 0$ and that C is a free conic on X, i.e., that $N_{C/X}$ is generated by its global sections (see Section 2.3.3).

If we accept the generalization of Theorem 2.17 from lines to conics (this is implied by Theorem 4.7), it follows that X is covered by conics. Moreover, if we also accept the generalization of Theorem 2.1 to conics (this is implied by Theorem 3.15), it also follows that $M(X,2)$ is smooth of the expected dimension at $[C]$.

d) Deduce that *any* hypersurface of degree $N-1$ in $\mathbf{P}(V)$ is covered by conics and that for X general, $M(X,2)$ is smooth of the expected dimension (*Hint:* use the incidence variety (3.2) as in the proof of Theorem 3.6).

3.2.1 Conics contained in a quadric hypersurface

Assume that $\mathbf{k}$ is algebraically closed of characteristic different from 2.

We now consider a smooth quadric hypersurface $X \subset \mathbf{P}(V)$ and study the closed subscheme $M(X,2) \subset M(\mathbf{P}(V),2)$ of conics contained in X.

Proposition 3.9 *Let $M(X,2)$ be the scheme of conics contained in a smooth quadric hypersurface $X \subset \mathbf{P}(V)$. The map π of (3.1) induces a birational map $\pi_X : M(X,2) \to \mathsf{Gr}(3,V)$. In particular, $M(X,2)$ is irreducible of the expected dimension $3(N-3)$; it is moreover smooth and*

- *when $N \leq 5$, the map π_X is an isomorphism;*
- *when $N = 6$, the map π_X is the blow up of two disjoint smooth subvarieties of dimension 3;*
- *when $N > 6$, the map π_X is the blow up of a smooth subvariety of dimension 3.*

Sketch of proof. If C is a conic contained in X, either the projective plane $\langle C\rangle$ spanned by C is contained in X, or $C = \langle C\rangle \cap X$.

When $N \leq 5$, the quadric X contains no projective planes, hence π_X is an isomorphism.

When $N = 6$, the quadric X contains two disjoint families of projective planes, parametrized by two disjoint smooth subvarieties F_1, $F_2 \subset \mathsf{Gr}(3,V)$, each of dimension 3. Over a point of F_i corresponding to a projective plane $P \subset X$, the fiber is the space, isomorphic to $\mathbf{P}^5$, of all conics contained in P. Hence $\pi_X^{-1}(F_i)$ has dimension 8. Since $M(X,2)$ has everywhere dimension $\geq 3N - 2d - 5 = 9$, it follows that $M(X,2)$ is irreducible and that π_X is birational. One can check that π_X is in fact the blow up of $F_1 \sqcup F_2$ in $\mathsf{Gr}(3,V)$; it follows that $M(X,2)$ is smooth irreducible and that its Picard group is isomorphic to $\mathbf{Z}^3$.

For $N > 6$, the space $F_3(X)$ of projective planes contained in X is smooth irreducible of dimension $3(N-3) - 6$. Again, $M(X,2)$ is the blow-up of $\mathsf{Gr}(3,V)$ along $F_3(X)$; it is therefore smooth irreducible of (the expected) dimension $3N - 9$ and its Picard group is isomorphic to $\mathbf{Z}^2$. □

Exercise 3.10 Prove that the varieties F_1 and F_2 in the proof above are both isomorphic to $\mathbf{P}^3$ (*Hint:* consider a smooth hyperplane section of X and use Exercise 2.19).

3.2.2 Conics contained in a cubic hypersurface

Assume that $\mathbf{k}$ is algebraically closed.

The main remark is that if a conic C is contained in a cubic hypersurface $X \subset \mathbf{P}(V)$, either the plane $\langle C\rangle$ spanned by C is contained in X, or the "residual curve" of the intersection of $\langle C\rangle \cap X$ is a line. We denote by $M^1(X,2) \subset M(X,2)$ the open subscheme of conics $C \subset X$ such that $\langle C\rangle \not\subset X$. There is therefore a morphism

$$\rho_X : M^1(X,2) \to M(X,1)$$

which expresses $M^1(X,2)$ as an open subset of a $\mathbf{P}^{N-3}$-bundle over the smooth scheme $M(X,1)$ (Theorem 2.20). It is therefore smooth of the expected dimension $N-3+2N-8 =$

$3N-11$, irreducible when $N \geq 5$.[2]

Proposition 3.11 *Let $M(X,2)$ be the scheme of conics contained in a smooth cubic hypersurface $X \subset \mathbf{P}(V)$.*

- *When $N=4$, the scheme $M(X,2)$ is the disjoint union of 27 copies of $\mathbf{P}^1$.*
- *When $N=5$, the scheme $M(X,2)$ is smooth irreducible of the expected dimension 4.*
- *When $N \geq 6$ and X is general, $M(X,2)$ is irreducible of the expected dimension $3N-11$.*

When $N \geq 6$, the scheme $M(X,2)$ is actually irreducible for all smooth cubics X ([CS, Theorem 1.1]).

Proof. Assume first $N \leq 5$. Since X is smooth, it contains no planes,[3] hence $M^1(X,2) = M(X,2)$ and the proposition follows from the discussion above.

If $N \geq 6$ and X is general, the scheme $F_3(X)$ of projective planes contained in X has the expected dimension $3(N-3) - \binom{5}{2} = 3N-19$ (see Section 1.4; note that this does not hold for all smooth cubics X: when $N=6$, some smooth cubic fourfolds do contain planes—but they may only contain finitely many of them). This implies that $M(X,2) \smallsetminus M^1(X,2)$ has dimension at most $3N-19+5$, which is less than the dimension $3N-11$ of $M(X,2)$ at every point. It follows that $M^1(X,2)$ is dense in $M^1(X,2)$, hence the latter is irreducible by the discussion above. □

Exercise 3.12 Let C be a smooth conic contained in a smooth cubic hypersurface $X \subset \mathbf{P}(V)$ and let $P = \langle C \rangle$.

a) Prove $N_{C/X} \simeq \bigoplus_{i=1}^{N-3} \mathscr{O}_C(a_i)$, with $4 \geq a_1 \geq \cdots \geq a_{N-3}$ and $a_1 + \cdots + a_{N-3} = 2N-8$ and that $H^1(C, N_{C/X}) = 0$ except if $(a_1, \ldots, a_{N-3}) = (4, 2, \ldots, 2, -2)$ (*Hint:* use Exercise 3.1).

b) Prove that $a_1 = 4$ if and only if $P \subset X$. Deduce that $M^1(X,2) \cap M^0(X,2)$ is smooth of the expected dimension.

[2] We are actually cheating here, because this argument does not prove that the scheme structure on $M^1(X,2)$ coincides with the scheme structure induced by the $\mathbf{P}^{N-3}$-bundle map ρ_X. For the argument to be complete, we need Exercise 3.12, and even a bit more, since that exercise does not deal with singular conics...

[3] When $N=5$, this follows either from a direct computation that if $X \supset P$, the partial derivatives of an equation of X would have a common zero on P, or from the Lefschetz Hyperplane Theorem, which says that the restriction $\mathrm{Pic}(\mathbf{P}(V)) \to \mathrm{Pic}(X)$ is an isomorphism for $N \geq 5$.

3.3 Rational curves of higher degrees

After studying lines and conics on subschemes $X \subset \mathbf{P}(V)$, we can move on to rational curves of higher degrees. For that, we will shift our point of view: instead of looking at rational curves on X as subschemes of X, we will parametrize them and consider them as *morphisms*

$$\varphi : \mathbf{P}^1_{\mathbf{k}} \to X.$$

This new point of view allows us to include singular curves (we just consider their normalizations) and even "multiple" curves (φ is not required to be birational onto its image), but not reducible curves. These can be taken care of by considering more complicated morphisms, like *Kontsevich stable maps,* which we will not describe here.

It is quite elementary to construct parameter spaces for all rational maps to X. We define the *degree* of a $\mathbf{k}$-morphism $\varphi : \mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}(V)$ as $e := \deg(\varphi^* \mathscr{O}_{\mathbf{P}(V)}(1))$. For example, the degree of an injective parametrization of a line is 1, and it is 2 for a conic (but higher degree parametrizations are also allowed).

Theorem 3.13 *Let X be a closed subscheme of $\mathbf{P}(V)$. Morphisms $\mathbf{P}^1_{\mathbf{k}} \to X$ of degree e can be parametrized by a quasi-projective scheme which we denote by $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X)$.*

The word "parametrized" in the statement above has a very precise meaning: there exists a "universal morphism" (also called "evaluation map")

$$\begin{array}{rcl} \mathsf{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X) & \longrightarrow & X \\ \big((t,u), \varphi\big) & \longmapsto & \varphi(t,u) \end{array}$$

which satisfies a universal property that the reader is invited to make precise.

Proof. [4] We first do the case $X = \mathbf{P}(V)$. Any $\mathbf{k}$-morphism $\varphi : \mathbf{P}^1_{\mathbf{k}} \to \mathbf{P}(V)$ of degree e can be written as

$$\varphi(t,u) = (\varphi_1(t,u), \dots, \varphi_N(t,u))$$

where $\varphi_1, \dots, \varphi_N \in \mathbf{k}[T,U]_e$, the space of homogeneous polynomials degree e, have *no non-constant common factor* in $\mathbf{k}[T,U]$. The "bad" locus is the set of $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathbf{P}\big(\mathbf{k}[T,U]_e^N\big)$ which are in the image of the regular map

$$\begin{array}{rcl} \displaystyle\bigcup_{1 \le m \le e} \Big(\mathbf{P}\big(\mathbf{k}[T,U]_m\big) \times \mathbf{P}\big(\mathbf{k}[T,U]_{e-m}^N\big)\Big) & \longrightarrow & \mathbf{P}\big(\mathbf{k}[T,U]_e^N\big) \\ \big(\psi, (\psi_1, \dots, \psi_N)\big) & \longmapsto & (\psi\psi_1, \dots, \psi\psi_N). \end{array}$$

This morphism is defined over $\mathbf{Z}$ and projective, hence its image is *defined over $\mathbf{Z}$ and closed.*

Therefore, morphisms of degree e from $\mathbf{P}^1_{\mathbf{k}}$ to $\mathbf{P}(V)$ are parametrized by a Zariski open set (defined over $\mathbf{Z}$) of the projective space $\mathbf{P}\big(\mathbf{k}[T,U]_e^N\big)$; we denote this quasi-projective

[4] We thank S. Druel for suggesting this proof.

scheme by $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}(V))$. Note that these morphisms fit together into the universal morphism

$$\begin{array}{ccc} \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}(V)) & \longrightarrow & \mathbf{P}(V) \\ \big((t,u),\varphi\big) & \longmapsto & \big(\varphi_1(t,u),\dots,\varphi_N(t,u)\big). \end{array}$$

Let now X be a (closed) subscheme of $\mathbf{P}^N_{\mathbf{k}}$ defined by homogeneous equations $f_1, \dots, f_m$. Degree-e morphisms $\mathbf{P}^1_{\mathbf{k}} \to X$ are then parametrized by the subscheme $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X)$ of $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}(V))$ defined by the equations $f_j(\varphi_1, \dots, \varphi_N) = 0$ for all $j \in \{1, \dots, m\}$. □

Remark 3.14 It follows from the fact that the scheme $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}(V))$ can be defined over $\mathbf{Z}$ that if $X \subset \mathbf{P}^{N-1}_{\mathbf{k}}$ can be defined by equations $f_1, \dots, f_m$ with coefficients in a subring R of $\mathbf{k}$, the scheme $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X)$ has the same property. If $\mathfrak{m}$ is a maximal ideal of R, one may consider the reduction $X_{\mathfrak{m}}$ of X modulo $\mathfrak{m}$: this is the subscheme of $\mathbf{P}^N_{R/\mathfrak{m}}$ defined by the reductions of $f_1, \dots, f_m$ modulo $\mathfrak{m}$. Because the equations defining the complement of $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, \mathbf{P}^{N-1}_{\mathbf{k}})$ in $\mathbf{P}(\mathsf{S}^e\mathbf{k}^2 \otimes \mathbf{k}^N)$ are defined over $\mathbf{Z}$ and the same for all fields, $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X_{\mathfrak{m}})$ is the reduction of the R-scheme $\mathrm{Mor}_e(\mathbf{P}^1, X)$ modulo $\mathfrak{m}$. In fancy terms, one may express this as follows: if $\mathscr{X}$ is a projective scheme over $\mathrm{Spec}(R)$, the R-morphisms $\mathbf{P}^1_R \to \mathscr{X}$ are parametrized by (the R-points of) a locally Noetherian scheme

$$\mathrm{Mor}(\mathbf{P}^1_R, \mathscr{X}) \to \mathrm{Spec}\, R$$

and the fiber of a closed point $\mathfrak{m}$ is the space $\mathrm{Mor}(\mathbf{P}^1_{R/\mathfrak{m}}, \mathscr{X}_{\mathfrak{m}})$.

The local structure of the scheme

$$\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) = \bigsqcup_{e \geq 0} \mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X)$$

can be explained as in Theorem 2.1, except that the normal bundle is replaced with the pull-back of the tangent bundle of X. Note that if C is a smooth curve and $\varphi : C \to X$ a closed embedding such that X is smooth along $\varphi(C)$, there is an exact sequence

$$0 \to T_C \to \varphi^* T_X \to N_{C/X} \to 0.$$

If C is rational, $T_C = \mathscr{O}_C(2)$ and there is an isomorphism $H^1(C, \varphi^* T_X) \xrightarrow{\sim} H^1(C, N_{C/X})$ and an exact sequence

$$0 \to H^0(C, T_C) \to H^0(C, \varphi^* T_X) \to H^0(C, N_{C/X}) \to 0.$$

The extra deformations of φ coming from the 3-dimensional vector space $H^0(C, T_C)$ correspond to the reparametrizations of C coming from the automorphism group $\mathrm{Aut}(\mathbf{P}^1_{\mathbf{k}}) = \mathrm{PGL}(2, \mathbf{k})$.

Theorem 3.15 *Let $C = \mathbf{P}^1_{\mathbf{k}}$, let X be a irreducible projective scheme, and let $\varphi : C \to X$ be a morphism such that X is smooth along $\varphi(C)$. Locally around $[\varphi]$, the scheme $\mathrm{Mor}(C, X)$ can be defined by $h^1(C, \varphi^* T_X)$ equations in a smooth scheme of dimension $h^0(C, \varphi^* T_X)$. In particular, any (geometric) irreducible component of $\mathrm{Mor}(C, X)$ through $[\varphi]$ has dimension at least*

$$h^0(C, \varphi^* T_X) - h^1(C, \varphi^* T_X) = \deg(\varphi^* T_X) + \dim(X)(1 - g(C)).$$

In fact, everything that we have said here remains valid for any smooth projective $\mathbf{k}$-curve C instead of $\mathbf{P}^1_{\mathbf{k}}$: quasi-projective $\mathbf{k}$-schemes $\mathrm{Mor}_e(C, X)$ can be constructed and Theorem 3.15 still holds.

Let us restrict ourselves to the case where $X \subset \mathbf{P}(V)$ is a smooth hypersurface of degree d. In this case, the expected dimension of $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X)$ is, by Theorem 3.15 (this can also be seen directly from the explicit construction of $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X)$ given in Theorem 3.13),

$$\deg(\varphi^* T_X) + \dim(X)(1 - g(\mathbf{P}^1_{\mathbf{k}})) = \deg(\varphi^* \mathscr{O}_{\mathbf{P}(V)}(N-d)) + N - 2 = e(N-d) + N - 2.$$

If we want to look instead at rational curves in X, i.e., if we want to forget the parametrizations of the curves, we need to take the quotient by the group $\mathrm{Aut}(\mathbf{P}^1_{\mathbf{k}}) \simeq \mathrm{PGL}(2, \mathbf{k})$. However, we need to exclude first morphisms like degree-e covers of a line. So we first restrict to the open subset of $\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X)$ which consists of morphisms $\mathbf{P}^1_{\mathbf{k}} \to X$ which are birational onto their image and then take the quotient. We obtain a scheme $M^{\mathrm{irr}}(X, e)$ whose expected dimension is

$$\mathrm{exp.dim}(M^{\mathrm{irr}}(X, e)) = \mathrm{exp.dim}(\mathrm{Mor}_e(\mathbf{P}^1_{\mathbf{k}}, X)) - \dim(\mathrm{PGL}(2, \mathbf{k})) = e(N-d) + N - 5$$

and which parametrizes (irreducible) rational curves of degree e in X. Note that these numbers coincide with the numbers obtained in Sections 2.3.1 and 3.2 for lines ($e = 1$) and conics ($e = 2$).

When $N \geq 4$, it is reasonable to expect that as usual, $M^{\mathrm{irr}}(X, e)$ should have this expected dimension when X is general of degree d (including the fact that it should be empty when this expected dimension is negative) and that it should be irreducible whenever this expected dimension is positive (with the exception of the case of conics on a cubic surface ($N = 4$, $d = 3$, and $e = 2$; see Proposition 3.11)).

Although there has been a lot of activity on that subject in the recent past, this far-reaching question is still very much open.[5] It includes the following problems:

- The Clemens conjecture ($N = d = 5$): the expected dimensions vanish for all e, hence a very general complex quintic 3-fold should contain a finite (nonzero) number of rational curves of each degree. This is known to hold up to degrees ≤ 11 (see Exercise 2.8 for lines and Theorem 3.6 for conics). These finite numbers are predicted by the physicists.

- When $d \geq 2N - 4$, a very general hypersurface of degree d in $\mathbf{P}(V)$ should contain no rational curves at all. This was proved over $\mathbf{C}$ by Voisin in [V].

- When $2N - 4 > d \geq \frac{3}{2}N - \frac{5}{2}$ and $d > N$, the only rational curves on a very general hypersurface of degree d in $\mathbf{P}(V)$ should be lines. Over $\mathbf{C}$, this was proved by Pacienza in [P] when $d = 2N - 5$. More generally, if $d \geq \left(1 + \frac{1}{e}\right)N - \frac{5}{e}$ and $d > N$, the only rational curves on a very general hypersurface of degree d in $\mathbf{P}(V)$ should have degree $\leq e$.

[5] Note however that for $e \leq d+2$, it follows from [GLP] that $M^{\mathrm{irr}}(X, e)$ is nonempty whenever its expected dimension is nonnegative.

- The situation changes radically when $d \leq N$. The conjecture then says that there should be rational curves of all degrees e whenever $N \geq 5$. It has been proved that in characteristic 0, $M(X,e)$ is irreducible of the expected dimension for X general of degree $d \leq N-3$ ([HRS] for $d < N/2$; [BK] for $d \leq 2(N-1)/3$; [RY] for $d \leq N-3$).

Some of these results require working on a compactification of $M^{\mathrm{irr}}(X,e)$. For lines ($e=1$), the situation was made easy by the fact that $M^{\mathrm{irr}}(X,1)$ is already projective. For conics ($e=2$), we considered the "Hilbert" compactification $M(X,2)$ of $M^{\mathrm{irr}}(X,2)$. In general, one considers other compactifications such as the (already mentioned) Kontsevich compactification by stable maps.

Chapter 4

Varieties covered by rational curves

In the first chapters, we encountered varieties covered by either lines or conics. We make a general study of varieties covered by rational curves.

4.1 Uniruled and separably uniruled varieties

We want to make a formal definition for varieties that are "covered by rational curves". The most reasonable approach is to make it a "geometric" property by defining it over an algebraic closure of the base field. Special attention has to be paid to the case of positive characteristic, hence the two variants of the definition.

Definition 4.1 Let $\mathbf{k}$ be a field and let $\overline{\mathbf{k}}$ be an algebraically closed extension of $\mathbf{k}$. A variety X of dimension n defined over $\mathbf{k}$ is

- *uniruled* if there exist a $\overline{\mathbf{k}}$-variety M of dimension $n-1$ and a dominant rational map $\mathbf{P}^1_{\overline{\mathbf{k}}} \times M \dashrightarrow X_{\overline{\mathbf{k}}}$;
- *separably uniruled* if there exist a $\overline{\mathbf{k}}$-variety M of dimension $n-1$ and a dominant and separable rational map $\mathbf{P}^1_{\overline{\mathbf{k}}} \times M \dashrightarrow X_{\overline{\mathbf{k}}}$.

These definitions do not depend on the choice of $\overline{\mathbf{k}}$, and in characteristic zero, both definitions are equivalent. As we will see in Example 4.11, in positive characteristic, some smooth projective varieties are uniruled, but not separably uniruled.

A point is not uniruled. Any variety birationally isomorphic to a (separably) uniruled variety is (separably) uniruled. The product of a (separably) uniruled variety with any variety is (separably) uniruled.

Here are various other characterizations and properties of (separably) uniruled varieties (for the proofs, see [D, Remarks 4.2]).

Remark 4.2 Here is a useful alternate definition. A $\mathbf{k}$-variety X is (separably) uniruled if and only if there exist a $\overline{\mathbf{k}}$-variety M and a dominant (separable) rational map $\mathbf{P}^1_{\overline{\mathbf{k}}} \times M \dashrightarrow X_{\overline{\mathbf{k}}}$ such that for some $m \in M(\overline{\mathbf{k}})$, the map e induces a *nonconstant* rational map $\mathbf{P}^1_{\overline{\mathbf{k}}} \times \{m\} \dashrightarrow X_{\overline{\mathbf{k}}}$.

Remark 4.3 Let X be a *proper* (separably) uniruled variety, with a rational map $e : \mathbf{P}^1_{\overline{\mathbf{k}}} \times M \dashrightarrow X_{\overline{\mathbf{k}}}$ as in the definition. We may compactify M then normalize it. The map e is then defined outside of a subvariety of $\mathbf{P}^1_{\overline{\mathbf{k}}} \times M$ of codimension at least 2, which therefore projects onto a proper closed subset of M. By shrinking M, we may therefore assume that the map e of the definition is a *morphism.*

Remark 4.4 Assume $\mathbf{k}$ is algebraically closed. It follows from Remark 4.3 that there is a rational curve through a general point of a proper uniruled variety (actually, by a degeneration argument, there is even a rational curve through *every* point). The converse holds *if* $\mathbf{k}$ *is uncountable.*[1] Therefore, in the definition, it is often useful to choose an uncountable algebraically closed extension $\overline{\mathbf{k}}$.

We end this section with a difficult theorem, first proved by Mori for Fano varieties (i.e., smooth projective varieties X such that $-K_X$ is ample).

Theorem 4.5 *Any smooth projective variety X with $-K_X$ nef but not numerically trivial is uniruled.*[2]

For a proof, see [Ko, Corollary IV.1.14] or [D, Theorem 3.10]. One can even prove that X is uniruled by curves of $(-K_X)$-degree $\leq \dim(X) + 1$ ([Ko, Corollary IV.1.15]).

Varieties with K_X trivial are usually not uniruled (think of abelian varieties); in fact, they are never separably uniruled by Corollary 4.10, so never uniruled in characteristic 0.

The theorem implies that all Fano varieties are uniruled, but it is expected that some are not separably uniruled. On the other hand, it is conjectured that all smooth hypersurfaces in $\mathbf{P}(V)$ which are Fano varieties (i.e., of degree $\leq N - 1$) are separably uniruled.

This is a problem in positive characteristic. Conduché proved in [Co] that the Fermat hypersurface of degree $d = p^r + 1$ in $\mathbf{P}^{N-1}_{\mathbf{F}_p}$ considered in Example 2.11 (and which often seems to exhibit the strangest behavior) is indeed separably uniruled when $d \leq N/2$. This also holds for the Fermat quintic hypersurface in $\mathbf{P}^5_{\mathbf{F}_2}$ ([BDENY]).

[1] Here is the proof. We may, after shrinking and compactifying X, assume that it is projective. There is still a rational curve through a general point, and this is exactly saying that the evaluation map $\mathsf{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}_{>0}(\mathbf{P}^1_{\mathbf{k}}, X) \to X$ is dominant. Since $\mathrm{Mor}_{>0}(\mathbf{P}^1_{\mathbf{k}}, X)$ has at most countably many irreducible components and X is not the union of countably many proper subvarieties, the restriction of ev to at least one of these components must be surjective, hence X is uniruled by Remark 4.2.

[2] A divisor D on a smooth projective variety X is nef if $D \cdot C \geq 0$ for all curves $C \subset X$, and it is numerically trivial if $D \cdot C = 0$ for all curves $C \subset X$.

4.2 Free rational curves and separably uniruled varieties

We want to generalize to general rational curves the definition of free lines given in Section 2.3.3, prove the analog of Theorem 2.17, and connect the existence of a free rational curve with separable uniruledness. The first item is easy: we just copy the definition.

Definition 4.6 Let X be a $\mathbf{k}$-variety. A $\mathbf{k}$-rational curve $\varphi : \mathbf{P}^1_{\mathbf{k}} \to X$ is *free* if its image is a curve contained in the smooth locus of X and $\varphi^* T_X$ is generated by its global sections.

If we write as usual the decomposition

$$\varphi^* T_X \simeq \bigoplus_{i=1}^{n} \mathscr{O}_L(a_i),$$

where $a_1 \geq \cdots \geq a_n$, the morphism φ is *free* if it is nonconstant and $a_n \geq 0$. It is a condition which is strictly stronger than the vanishing of $H^1(\mathbf{P}^1_{\mathbf{k}}, \varphi^* T_X)$ (which, by Theorem 3.15, ensures the smoothness of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ at $[\varphi]$).

One can express the fact that φ is nonconstant as follows. If φ is separable, $\varphi^* T_X$ then contains $T_{\mathbf{P}^1_{\mathbf{k}}} \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(2)$ and $a_1 \geq 2$. In general, decompose φ as $\mathbf{P}^1_{\mathbf{k}} \xrightarrow{\varphi''} \mathbf{P}^1_{\mathbf{k}} \xrightarrow{\varphi'} X$ where φ' is separable and φ'' is a composition of r Frobenius morphisms. Then $a_1(\varphi) = p^r a_1(\varphi') \geq 2$.

For any morphism $\varphi : \mathbf{P}^1_{\mathbf{k}} \to X$ whose image is contained in the smooth locus of X, we have $\deg(\varphi^* T_X) = -(K_X \cdot \varphi_* \mathbf{P}^1_{\mathbf{k}})$: there are no free rational curves on a smooth variety whose canonical divisor is nef (because one would then have $\deg(\varphi^* T_X) \geq a_1 \geq 2$), such as a smooth hypersurface in $\mathbf{P}(V)$ of degree $\geq N$.

Theorem 4.7 *Let X be a smooth quasi-projective variety defined over a field $\mathbf{k}$ and let $\varphi : \mathbf{P}^1_{\mathbf{k}} \to X$ be a $\mathbf{k}$-rational curve.*

a) *If φ is free, the evaluation map*

$$\mathsf{ev} : \mathbf{P}^1_{\mathbf{k}} \times \mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X) \to X$$

is smooth at all points of $\mathbf{P}^1_{\mathbf{k}} \times \{[\varphi]\}$.

b) *If there is a $\mathbf{k}$-scheme M with a $\mathbf{k}$-point m and a morphism $e : \mathbf{P}^1_{\mathbf{k}} \times M \to X$ such that $e|_{\mathbf{P}^1_{\mathbf{k}} \times \{m\}} = \varphi$ and the tangent map to e is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times \{m\}$, the curve φ is free.*

Geometrically speaking, item a) implies that the deformations of a free rational curve cover X. In b), the hypothesis that the tangent map to e is surjective is weaker than the smoothness of e, and does not assume smoothness, or even reducedness, of M; in characteristic 0, by generic smoothness, this hypothesis is satisfied if and only if e is dominant.

The theorem implies that the set of free rational curves on a quasi-projective $\mathbf{k}$-variety X is a smooth open subset of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$, possibly empty.

Finally, *when* $\mathrm{char}(\mathbf{k}) = 0$ and there is an irreducible $\mathbf{k}$-scheme M and a *dominant* morphism $e : \mathbf{P}^1_{\mathbf{k}} \times M \to X$ which does not contract one $\mathbf{P}^1_{\mathbf{k}} \times m$, the rational curves corresponding to points in some nonempty open subset of M are free (by generic smoothness, the tangent map to e is surjective on some nonempty open subset of $\mathbf{P}^1_{\mathbf{k}} \times M$).

Proof of the Theorem. The tangent map to ev at $(t, [\varphi])$ is the map

$$\begin{array}{rcl} T_{\mathbf{P}^1_{\mathbf{k}},t} \oplus H^0(\mathbf{P}^1_{\mathbf{k}}, \varphi^* T_X) & \longrightarrow & T_{X,\varphi(t)} \simeq (\varphi^* T_X)_t \\ (u, \sigma) & \longmapsto & T_{\varphi,t}(u) + \sigma(t). \end{array} \tag{4.1}$$

If φ is free, it is surjective because the evaluation map

$$H^0(\mathbf{P}^1_{\mathbf{k}}, \varphi^* T_X) \longrightarrow (\varphi^* T_X)_t$$

is. Moreover, since $H^1(\mathbf{P}^1_{\mathbf{k}}, \varphi^* T_X)$ vanishes, $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth at $[\varphi]$ (Theorem 3.15). This implies that ev is smooth at $(t, [\varphi])$ and proves a).

Conversely, by the universal property of ev, the morphism e factors through ev, whose tangent map at $(t, [\varphi])$ is therefore surjective. This implies that the map

$$H^0(\mathbf{P}^1_{\mathbf{k}}, \varphi^* T_X) \to (\varphi^* T_X)_t / \mathrm{Im}(T_{\varphi,t}) \tag{4.2}$$

is surjective. There is a commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{P}^1_{\mathbf{k}}, \varphi^* T_X) & \xrightarrow{a} & (\varphi^* T_X)_t \\ \uparrow & & \uparrow T_{\varphi,t} \\ H^0(\mathbf{P}^1_{\mathbf{k}}, T_{\mathbf{P}^1_{\mathbf{k}}}) & \xrightarrow{a'} & T_{\mathbf{P}^1_{\mathbf{k}},t}. \end{array}$$

Since a' is surjective, the image of a contains $\mathrm{Im}(T_{\varphi,t})$. Since the map (4.2) is surjective, a is surjective. Hence $\varphi^* T_X$ is generated by global sections at one point. It is therefore generated by global sections and φ is free. □

Corollary 4.8 *Let X be a smooth projective variety defined over an algebraically closed field. The variety X is separably uniruled if and only if X contains a free rational curve. If this is the case, there exists a free rational curve through a general point of X.*

Proof. Consider the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ constructed in Theorem 3.13. If $\varphi : \mathbf{P}^1_{\mathbf{k}} \to X$ is free, the evaluation map ev is smooth at $(0, [\varphi])$ by Theorem 4.7.a). It follows that the restriction of ev to the unique component of $\mathrm{Mor}_{>0}(\mathbf{P}^1_{\mathbf{k}}, X)$ that contains $[\varphi]$ is separable and dominant and X is separably uniruled by Remark 4.2.

Assume conversely that X is separably uniruled. By Remark 4.3, there exists a $\mathbf{k}$-variety M and a dominant and separable, hence generically smooth, morphism $\mathbf{P}^1_{\mathbf{k}} \times M \to X$. The rational curve corresponding to a general point of M passes through a general point of X and is free by Theorem 4.7.b). $\square$

Corollary 4.9 *Let X be a smooth projective variety defined over an algebraically closed field of characteristic zero. There exists a set $X^{\text{free}} \subset X$ which is the intersection of countably many dense open subsets of X, such that any rational curve in X meeting X^{free} is free.*

Proof. The space $\text{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ has at most countably many irreducible components, which we will denote by $(M_i)_{i \in \mathbf{N}}$. Let $\mathsf{ev}_i : \mathbf{P}^1_{\mathbf{k}} \times M_i \to X$ be the evaluation maps.

Denote by U_i a dense open subset of X such that the tangent map to ev_i is surjective at each point of $(\mathsf{ev}_i)^{-1}(U_i)$ (the existence of U_i follows from [H], III, Proposition 4.6 and uses the hypothesis that the characteristic is zero; if ev_i is not dominant, one may simply take for U_i the complement of the closure of the image of ev_i). We let X^{free} be the intersection $\bigcap_{i \in \mathbf{N}} U_i$.

Let $\varphi : \mathbf{P}^1_{\mathbf{k}} \to X$ be a rational curve whose image meets X^{free}, and let M_i be an irreducible component of $\text{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ that contains $[\varphi]$. By construction, the tangent map to ev_i is surjective at some point of $\mathbf{P}^1_{\mathbf{k}} \times \{[\varphi]\}$, hence so is the tangent map to ev; it follows from Theorem 4.7 that φ is free. $\square$

The corollary is interesting only when X is uniruled (otherwise, the set X^{free} is more or less the complement of the union of all rational curves on X); it is also useless when the ground field is countable, because X^{free} may be empty.

Corollary 4.10 *If X is a smooth projective separably uniruled variety, $H^0(X, \omega_X^{\otimes m}) = 0$ for all $m > 0$.*

Proof. Let s be a global section of $\omega_X^{\otimes m}$ and let x be a general point of X. By Corollary 4.8, there exists a free rational curve $\varphi : \mathbf{P}^1_{\mathbf{k}} \to X$ through x. The pull-back of $\omega_X = \det(T_X^\vee)$ to $\mathbf{P}^1_{\mathbf{k}}$ has negative degree, hence $\varphi^*\omega_X^{\otimes m}$ has no nonzero global sections. It follows that s vanishes on $\varphi(\mathbf{P}^1_{\mathbf{k}})$, hence at x. We conclude that s vanishes at a general point of X, hence $s = 0$. $\square$

The converse is conjectured to hold: for curves, it is obvious since $h^0(X, \omega_X)$ is the genus of X; for surfaces, we have the more precise Castelnuovo criterion: $H^0(X, \omega_X^{\otimes 12}) = 0$ if and only if X is birationally isomorphic to a ruled surface; in dimension three, it is known in characteristic zero.

Example 4.11 It follows from the last corollary that a smooth hypersurface of degree $\geq N$ in $\mathbf{P}(V)$ is not separably uniruled. In characteristic $p > 0$, we saw in Example 2.11 that the Fermat hypersurface of degree $p^r + 1$ is uniruled. When $p^r + 1 \geq N$, it is however not separably uniruled.

4.3 Minimal free rational curves

In this section, *we assume that the base field* $\mathbf{k}$ *is algebraically closed of characteristic 0.*

Let $X \subset \mathbf{P}(V)$ be a uniruled subvariety. By Corollary 4.8, there exists a free rational curve on X (whose deformations then cover a dense open subset of X by Theorem 4.7). We are interested in those free curves of minimal possible degree.

For example, if $X \subset \mathbf{P}(V)$ is a hypersurface of degree $d \le N-2$, we proved that X is covered by lines (Proposition 2.13) and the minimal degree is 1. The normal bundle of a general line L contained in X is given by (2.14); it easily implies

$$T_X|_L \simeq \mathscr{O}_L(2) \oplus \mathscr{O}_L(1)^{\oplus(N-2-d)} \oplus \mathscr{O}_L^{\oplus(d-1)}.$$

When $d = N-1$, the variety X is not covered by lines anymore (Remark 2.14) but by conics (Exercise 3.8): the minimal degree is 2. A general conic C contained in X is free by Theorem 4.7 (since we are in characteristic 0) and since $\deg(T_X|_C) = \deg(\mathscr{O}_{\mathbf{P}(V)}(1)|_C) = 2$, we have

$$T_X|_C \simeq \mathscr{O}_C(2) \oplus \mathscr{O}_C^{\oplus(N-3)}.$$

We want to show that a similar type of decomposition always holds for free rational curves of minimal degrees.

Theorem 4.12 *Let X be a smooth projective uniruled variety of dimension n defined over an algebraically closed field of characteristic zero and let $\varphi : \mathbf{P}^1_{\mathbf{k}} \to X$ be a general free rational curve of minimal degree with respect to a fixed ample class on X.*[3] *Then*

$$\varphi^* T_X \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(2) \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(1)^{\oplus(s-1)} \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}^{\oplus(n-s)}. \tag{4.3}$$

Moreover, s is the dimension of the subscheme of X swept out by the deformations of φ which pass through a general point of X.

Sketch of proof. Since the condition (4.3) on φ is open, it is enough to find one rational curve with this property. So we may assume that $\mathbf{k}$ is uncountable and that the point $x = \varphi(0)$ is in the subset X^{free} defined in Corollary 4.9.

Recall from Theorem 3.15 that since φ is free, the scheme $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ is smooth at $[\varphi]$. Let M be the unique component of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$ through $[\varphi]$. By Theorem 4.7.a), the evaluation map

$$\mathsf{ev} : \mathbf{P}^1_{\mathbf{k}} \times M \longrightarrow X$$

is dominant. Since M is a component of $\mathrm{Mor}(\mathbf{P}^1_{\mathbf{k}}, X)$, any reparametrization of any element of M is still in M. This implies that for each t in $\mathbf{P}^1_{\mathbf{k}}$, the morphism "evaluation at t"

$$\begin{aligned} \mathsf{ev}_t : M &\longrightarrow X \\ g &\longmapsto g(t) \end{aligned}$$

[3] We mean here that φ has minimal degree among all free rational curves on X.

is dominant.

Let M_x be an irreducible component of $\mathsf{ev}_0^{-1}(x)$ through $[\varphi]$. The image of

$$\mathsf{ev}_\infty : M_x \to X$$

is the subset of X swept out by the deformations of φ in M passing through x. This is the subset referred to in the statement of the theorem. We denote by s its dimension.

Assume $\dim(\mathsf{ev}_\infty^{-1}(y)) \geq 2$ for some $y \in X$. By Mori's bend-and-break ([D, Proposition 3.2]), the rational curves in $\mathsf{ev}_\infty^{-1}(y) \subset M_x$, which all pass through x and y, will deform to a nonintegral effective 1-cycle of rational curves on X. One of these curves passes through x, hence is free because $x \in X^{\mathrm{free}}$, but it has degree less than the degree of φ. Since φ is a free curve of minimal degree, this is a contradiction. Therefore, all fibers of ev_∞ have dimension ≤ 1, and in fact exactly 1, since they contain all the reparametrizations of any of their elements by multiplication by a nonzero element of $\mathbf{k}$.

It follows that we have

$$s = \dim(M_x) - 1. \tag{4.4}$$

We now invoke an analog of Theorem 3.15 (which follows from (4.1)) which says that at any point $[\psi]$ of M_x, one has

$$T_{M_x,[\psi]} = \{\sigma \in H^0(\mathbf{P}^1_{\mathbf{k}}, \psi^*T_X) \mid \sigma(0) = 0\} \simeq H^0(\mathbf{P}^1_{\mathbf{k}}, (\psi^*T_X)(-1)).$$

If we write as usual

$$\psi^*T_X \simeq \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_n)$$

with $a_1 \geq \cdots \geq a_n \geq 0$, this space is therefore $\bigoplus_{i=1}^n H^0(\mathbf{P}^1_{\mathbf{k}}, \mathscr{O}_{\mathbf{P}^1_{\mathbf{k}}}(a_i - 1))$, so that

$$\dim(M_x) = \sum_{i=1}^n a_i. \tag{4.5}$$

Moreover, the tangent map to ev_∞ is

$$\begin{aligned} T_{\mathsf{ev}_\infty,[\psi]} : T_{M_x,[\psi]} &\longrightarrow T_{X,\psi(\infty)} \\ \sigma &\longmapsto \sigma(\infty). \end{aligned}$$

Since we are in characteristic zero, the map ev_∞ is smooth onto its image at $[\psi]$ when $[\psi]$ is general in M_x, so that

$$s = \operatorname{rank}(T_{\mathsf{ev}_\infty,[\psi]}) = \operatorname{Card}\{i \in \{1, \ldots, n\} \mid a_i > 0\}. \tag{4.6}$$

Comparing (4.4), (4.5), and (4.6), we obtain the desired result. □

The integer s in the theorem may vary between 1 and n.

When $s = n$, the deformations of a general free curve φ of minimal degree which pass through a general point of X cover X and the pull-back φ^*T_X is ample (one says that the curve φ is *very free*). In that case, it follows from the proof of the main result of [Ke] that X is isomorphic to the projective space $\mathbf{P}^n$.

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