GUSHEL–MUKAI VARIETIES

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Abstract. *Gushel–Mukai* (or GM for short) *varieties* are smooth (complex) dimensionally transverse intersections of a cone over the Grassmannian $\mathsf{Gr}(2,5)$ with a linear space and a quadratic hypersurface. They occur in each dimension 1 through 6 and they are Fano varieties (their anticanonical bundle is ample) in dimensions 3, 4, 5, and 6. The aim of this series of lectures is to discuss the geometry, moduli, Hodge structures, and categorical aspects of these varieties. It is based on joint work with Alexander Kuznetsov and earlier work of Logachev, Iliev, Manivel, and O'Grady.

These varieties first appeared in the classification of complex Fano threefolds: Gushel showed that any smooth prime Fano threefold of degree 10 is a GM variety. Mukai later extended Gushel's results and proved that all Fano varieties of coindex 3, degree 10, and Picard number 1, all Brill–Noether-general polarized K3 surfaces of degree 10, and all Clifford-general curves of genus 6 are GM varieties.

Why are people interested in these varieties? One of the reasons why their geometry is so rich is that, to any GM variety is canonically attached a sextic hypersurface in $\mathbf{P}^5$, called an *Eisenbud–Popescu–Walter* (or EPW for short) *sextic* and a canonical double cover thereof, a hyperkähler fourfold called a *double EPW sextic.* In some sense, the pair consisting of a GM variety and its double EPW sextic behaves very much (but with more complications, and also some differences) like a cubic hypersurface in $\mathbf{P}^5$ and its (hyperkähler) variety of lines. Not many examples of these pairs—a Fano variety and a hyperkähler variety—are known (another example are the hyperplane sections of the Grassmannian $\mathsf{Gr}(3,10)$ and their associated Debarre–Voisin hyperkähler fourfold), so it is worth looking in depth into one of these.

The first main difference is that there is a positive-dimensional family of GM varieties attached to the same EPW sextic; this family can be precisely described. Another distinguishing feature is that each EPW sextic has a dual EPW sextic (its projective dual). GM varieties associated with isomorphic or dual EPW sextics are all birationally isomorphic.

A common feature with cubic fourfolds is that the middle Hodge structure of a GM variety of even dimension is isomorphic (up to a Tate twist) to the second cohomology of its associated double EPW sextic. Together with the Verbitsky–Torelli theorem for hyperkähler fourfolds, this leads to a complete description of the period map for even-dimensional GM varieties.

In odd dimensions, say 3 or 5, a GM variety has a 10-dimensional intermediate Jacobian and we show that it is isomorphic to the Albanese variety of a surface of general type canonically attached to the EPW sextic.

Another aspect of GM varieties that makes them very close to cubic fourfolds is the rationality problem. Whereas (most) GM varieties of dimensions at most 3 are not rational, and GM varieties of dimensions at least 5 are all rational, the situation for GM fourfolds is still mysterious: some rational examples are known but, although one expects, as for cubic fourfolds, that a very general GM fourfold should be irrational, not a single irrational example is known.

Finally, the derived category of GM varieties was studied by Kuznetsov and Perry with this rationality problem in mind. They show that this category contains a special semiorthogonal component, which is a K3 or Enriques category according to whether the dimension of the variety is even or odd.

All my contributions are joint work with Alexander Kuznetsov.

These notes were written for a series of lectures given at the Shanghai Center for Mathematical Sciences at Fudan University in the fall of 2019. I would like to thank this institution and Zhi Jiang for the invitation and support, Qixiao Ma for his many questions and remarks, Alexander Perry for his comments about competing conjectures for rationality of Gushel–Mukai fourfolds, Jean-Louis Colliot-Thélène for his explanations about conic bundles, and, last but certainly not least, Alexander Kuznetsov for correcting some of my mistakes and for the pleasure of working with him all these years.

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(Preliminay version – December 14, 2019)

1. Definition of Gushel–Mukai varieties

In these notes, we always work over $\mathbf{C}$ and U_k, V_k, W_k all denote complex vector spaces of dimension k.

Let X be a (smooth complex) Fano variety of dimension n and Picard number 1, so that $\operatorname{Pic}(X) = \mathbf{Z}H$, with H ample. The positive integer such that $-K_X \underset{\text{lin}}{\equiv} rH$ is called the *index* of X and the integer $d := H^n$ its *degree*. For example, a smooth hypersurface $X \subseteq \mathbf{P}^{n+1}$ of degree $d \le n+1$ is a Fano variety of index $r = n+2-d$ and degree d.

The index satisfies the following properties:

- if $r \ge n+1$, then $X \simeq \mathbf{P}^n$ (Kobayashi–Ochiai, 1973);
- if $r = n$, then X is a quadric in $\mathbf{P}^{n+1}$ (Kobayashi–Ochiai, 1973);
- if $r = n-1$ (del Pezzo varieties), classification by Fujita and Iskovskikh (1977–1988);

- if $r = n-2$, classification by Mukai (1989–1992), modulo a result proved later by Shokurov in dimension 3, and Mella in all dimensions (1997).

Mukai used the *vector bundle method* (initiated by Gushel in 1982) to prove that in the last case, $d \in \{2,4,6,8,10,12,14,16,18,22\}$.

We will restrict ourselves in these lectures to the case $d = 10$. We call the corresponding varieties *Gushel–Mukai* (or GM for short) varieties.

Theorem 1.1 (Mukai)**.** *Any smooth Fano n-fold X_n with Picard number 1, index $n-2$, and degree* 10 *has dimension $n \in \{3,4,5,6\}$ and can be obtained as follows:*

- *either $n \in \{3,4,5\}$ and $X_n = \mathsf{Gr}(2,V_5) \cap \mathbf{P}(W_{n+5}) \cap Q \subseteq \mathbf{P}(\bigwedge^2 V_5)$, where Q is a quadric and $W_{n+5} \subseteq \bigwedge^2 V_5$ is a vector subspace of dimension $n+5$ (*ordinary *case);*
- *or $X_n \to \mathsf{Gr}(2,V_5) \cap \mathbf{P}(W_{n+4})$ is a double cover branched along an X_{n-1} (*special *case).*

In fact, Mukai had to make the extra assumption that a general element of $|H|$ is smooth, a fact that was proved only later on by Mella (who actually showed that H is very ample).

In both cases, the morphism $\gamma\colon X_n \to \mathsf{Gr}(2,V_5) \subseteq \mathbf{P}(\bigwedge^2 V_5)$ (called the *Gushel morphism*) is defined by a linear subsystem of $|H|$.

In dimension $n = 1$, these constructions provide genus-6 curves which are general in the sense that they are neither hyperelliptic, nor trigonal, nor plane quintics (this is also termed *Clifford general* because the Clifford index has the maximal value 2); in dimension $n = 2$, these constructions provide K3 surfaces of degree 10 which are general in the sense that their general hyperplane sections are Clifford general genus-6 curves (they are also called *Brill–Noether general* K3 surfaces; see [M, Definition 3.8]).

Sketch of proof. Let $X := X_n$. In order to construct the Gushel morphism $\gamma\colon X \to \mathsf{Gr}(2,V_5)$ with $\gamma^*\mathscr{O}_{\mathsf{Gr}(2,V_5)}(1) = H$, one needs to construct a rank-2 globally generated vector bundle $\mathscr{E}$ on S with 5 independent sections and $c_1(\mathscr{E}) = H$; if $\mathscr{U}$ is the tautological rank-2 vector bundle on $\mathsf{Gr}(2,V_5)$, one then has $\mathscr{E} = \gamma^*\mathscr{U}^\vee$. By Mella's result, the line bundle H is very ample hence defines an embedding $\varphi_H\colon X \hookrightarrow \mathbf{P}(W_{n+5})$, where $W_{n+5} := H^0(X,H)^\vee$.

We give in [DK1] a construction of $\mathscr{E}$ based on the excess conormal sheaf: we prove that $X \subseteq \mathbf{P}(W_{n+5})$ is the base-locus of the space of quadrics $V_6 := H^0(\mathbf{P}(W_{n+5}), \mathscr{I}_X(2))$ and we define $\mathscr{E}$ as the dual of the (rank-2) kernel of the canonical surjection

$$V_6 \otimes \mathscr{O}_X \twoheadrightarrow (\mathscr{I}_X/\mathscr{I}_X^2)(2) = N^\vee_{X/\mathbf{P}(W_{n+5})}(2).$$

Mukai had a different construction and his original proof ([M]) went roughly as follows. The intersection of $n-2$ general elements of $|H|$ is a smooth K3 surface S and the restriction $\mathscr{E}_S$ of $\mathscr{E}$ to S must satisfy $c_1(\mathscr{E}_S) = H_S$ and $c_2(\mathscr{E}_S) = 4$.

So we start out by constructing such a vector bundle on the genus-6 K3 surface $S \subseteq \mathbf{P}^6$ using Serre's construction. A general hyperplane section of S is a canonical curve C of genus 6, hence has a g^1_4. Let $Z \subseteq C$ be an element of this pencil, so that

$$\begin{aligned} h^0(S,\mathscr{I}_Z(H_S)) &= 1 + h^0(C, H_S - Z) \\ &= 1 + h^1(C, g^1_4) \\ &= 1 + h^0(C, g^1_4) - (4+1-6) = 4. \end{aligned}$$

By Serre's construction, there is a vector bundle $\mathscr{E}_S$ on S that fits into an exact sequence

$$0 \to \mathscr{O}_S \to \mathscr{E}_S \to \mathscr{I}_Z(H_S) \to 0.$$

Assume $n = 3$. One proves that $\mathscr{E}_S$ extends to a locally free sheaf $\mathscr{E}$ on X. Then one checks that $V_5 := H^0(X, \mathscr{E})^\vee$ has dimension 5 and that $\mathscr{E}$ is generated by global sections hence defines a morphism $\gamma\colon X \to \mathsf{Gr}(2, V_5)$.

We have a linear map

$$\eta\colon \bigwedge^2 H^0(X, \mathscr{E}) \to H^0(X, \bigwedge^2 \mathscr{E}) \simeq H^0(X, H)$$

and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, hook, "\varphi_H"'] & \mathsf{Gr}(2, V_5) \arrow[d, hook] \\ \mathbf{P}(H^0(X, H)^\vee) \arrow[r, dashed, "\eta^\vee"] & \mathbf{P}(\bigwedge^2 V_5). \end{tikzcd}$$

If η is surjective, the map $\eta^\vee$ in the diagram is a morphism, which is moreover injective. The image of $\eta^\vee$ is a $\mathbf{P}(W_8) \subseteq \mathbf{P}(\bigwedge^2 V_5)$. The intersection $M_X := \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_8)$ has dimension 4 (because all hyperplane sections of $\mathsf{Gr}(2, V_5)$ are irreducible) and degree 5, and $\varphi_H(X) \subseteq M_X$ is a smooth hypersurface. It follows that M_X must be smooth (otherwise, one easily sees that its singular locus would have positive dimension hence would meet $\varphi_H(X)$, which is impossible). Thus, $\varphi_H(X)$ is a Cartier divisor, of degree 10, hence is the intersection of M_X with a quadric.

If the corank of η is 1, consider the cone $\mathsf{CGr} \subseteq \mathbf{P}(\mathbf{C} \oplus \bigwedge^2 V_5)$, with vertex $v = \mathbf{P}(\mathbf{C})$, over $\mathsf{Gr}(2, V_5)$. In the above diagram, we may view $H^0(X, H)^\vee$ as embedded in $\mathbf{C} \oplus \bigwedge^2 V_5$ by mapping $(\mathrm{Im}(\eta))^\perp$ to $\mathbf{C}$, and $\eta^\vee$ as the projection from v, with image a $\mathbf{P}(W_7) \subseteq \mathbf{P}(\bigwedge^2 V_5)$. Again, the intersection $M_X := \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_7) = \mathrm{Im}(\psi)$ is a smooth threefold and $\varphi_H(X)$ is the intersection of the (smooth locus of the) degree-5 cone over M_X with a quadric.

If the corank of η is ≥ 2, one shows that $\varphi_H(X)$ must be singular, which is impossible. So this proves the theorem when $n = 3$.

For $n \geq 4$, one extends $\mathscr{E}$ to successive hyperplane sections of X all the way up to X and proceed similarly (see [IP, Proposition 5.2.7]). $\square$

Why are people interested in GM varieties? As the rest of these notes will, I hope, show, for several reasons:

- they have interesting geometries (Section 2);
- they have intriguing rationality properties in dimension 4 similar to that of cubic fourfolds (most X_3 are irrational, all X_5 and X_6 are rational) (Section 3);
- they have interesting period maps (Section 4);
- they have interesting derived categories (Section 5).

2. Gushel–Mukai varieties, EPW sextics, and moduli

In this section, we explain an intricate (but still elementary) connection (discovered by O'Grady and Iliev–Manivel) between Gushel–Mukai varieties and Lagrangian subspaces in a 20-dimensional symplectic complex vector space.

Let

$$X = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_{n+5}) \cap Q \subseteq \mathbf{P}(\bigwedge^2 V_5)$$

be a (smooth) ordinary[1] GM n-fold, with $n \in \{3,4,5\}$. We set

$$M := \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_{n+5}),$$

a variety of degree 5 and dimension $n+1$, so that $X = M \cap Q$.

Lemma 2.1. *If X is smooth, so is M, and each nonzero element of the orthogonal complement $W_{n+5}^{\perp} \subseteq \bigwedge^2 V_5^{\vee}$, viewed as a skew-symmetric form on V_5, has maximal rank 4.*

Proof. Let $W := W_{n+5}$. One has (see [PV, Corollary 1.6])

$$\mathrm{Sing}(M) = \mathbf{P}(W) \ \cap \bigcup_{\omega \in W^{\perp} \smallsetminus \{0\}} \mathsf{Gr}(2, \mathrm{Ker}(\omega)). \tag{1}$$

Since X is smooth, $\mathrm{Sing}(M)$ must be finite.

If $\dim(W) = 10$, then $M = \mathsf{Gr}(2, V_5)$ is smooth.

If $\dim(W) = 9$, either a generator ω of $W^{\perp}$ has rank 2 and $\mathrm{Sing}(M)$ is a 2-plane, which is impossible, or else M is smooth.

If $\dim(W) = 8$, either some $\omega \in W^{\perp}$ has rank 2, in which case $\mathsf{Gr}(2, \mathrm{Ker}(\omega))$ is a 2-plane contained in the hyperplane $\omega^{\perp}$, whose intersection with $\mathbf{P}(W)$ therefore contains a line along which M is singular, which is impossible, or else M is smooth. ☐

2.1. **GM data sets.** Note that M is cut out in $\mathbf{P}(W)$ by the traces of the Plücker quadrics

$$\mathbf{q}(v)(w, w) := v \wedge w \wedge w,$$

for $v \in V_5$ (we choose an isomorphism $\bigwedge^5 V_5 \simeq \mathbf{C}$). More precisely, there is a (not quite canonical) identification

$$V_5 \simeq H^0(\mathbf{P}(W), \mathscr{I}_M(2)).$$

One then has

$$V_6 :\simeq H^0(\mathbf{P}(W), \mathscr{I}_X(2)) \simeq V_5 \oplus \mathbf{C}Q.$$

We formalize this by defining a *GM data set* $(W_{n+5}, V_6, V_5, \mathbf{q})$ (of dimension n) by

- V_6 is a 6-dimensional vector space;
- $V_5 \subseteq V_6$ is hyperplane;
- $W_{n+5} \subseteq \bigwedge^2 V_5$ is a vector subspace of dimension $n+5$;
- $\mathbf{q}\colon V_6 \to \mathrm{Sym}^2 W_{n+5}^{\vee}$ is a linear map such that

$$\forall v \in V_5 \ \ \forall w \in W_{n+5} \qquad \mathbf{q}(v)(w, w) = v \wedge w \wedge w.$$

Given such a data set, we set

$$X := \bigcap_{v \in V_6} Q(v) \subseteq \mathbf{P}(W_{n+5}),$$

where $Q(v) \subseteq \mathbf{P}(W_{n+5})$ is the quadric hypersurface defined by $\mathbf{q}(v)$. When X is smooth of the expected dimension n, it is a GM variety.

[1]We make this assumption for the sake of simplicity, but our constructions also work for special GM varieties, with some modifications. Of course, by making this assumption, we unfortunately completely miss the case $n = 6$. Also, the theory also works in dimensions $n \in \{1, 2\}$, also with some modifications. We refer to [DK1] for details.

2.2. **Lagrangian data sets.** We now set up a more complicated correspondance. We will endow $\bigwedge^3 V_6$ with the (conformal) symplectic form given by wedge product. A *Lagrangian data set* is a triple (V_6, V_5, A), where $V_5 \subseteq V_6$ is a hyperplane and $A \subseteq \bigwedge^3 V_6$ is a (10-dimensional) Lagrangian subspace.

Theorem 2.2. *There is a bijection (explained in the proof) between GM data sets $(W, V_6, V_5, \mathbf{q})$ and Lagrangian data sets (V_6, V_5, A).*

One has the following relations

$$\begin{aligned} &\dim(W) = 10 - \dim(A \cap \textstyle\bigwedge^3 V_5), \\ \forall v \in V_6 \smallsetminus V_5 \qquad &\mathrm{Ker}(\mathbf{q}(v)) \simeq A \cap (v \wedge \textstyle\bigwedge^2 V_6) = A \cap (v \wedge \textstyle\bigwedge^2 V_5) \subseteq W. \end{aligned}$$

Proof. This construction is explained in great details and more generality in the proof of [DK1, Theorem 3.6]. We present a simplified version.

For one direction, let us start with a Lagrangian data set (V_6, V_5, A). Let $\lambda \in V_6^\vee$ be a linear form with kernel V_5. By the Leibniz rule, it induces $\lambda_p \colon \bigwedge^p V_6 \to \bigwedge^{p-1} V_5$, with kernel $\bigwedge^p V_5$. Define $\mathbf{q} \colon V_6 \otimes \mathrm{Sym}^2 A \to \mathbf{C}$ by

$$\forall v \in V_6 \;\; \forall \xi, \xi' \in A \qquad \mathbf{q}(v \otimes (\xi, \xi')) := -\lambda_4(v \wedge \xi) \wedge \lambda_3(\xi') \in \textstyle\bigwedge^5 V_5.$$

The fact that A is Lagrangian implies $\xi \wedge \xi' = 0$, hence, by the Leibniz rule,

$$0 = \lambda_6(\xi \wedge \xi') = \lambda_3(\xi) \wedge \xi' - \xi \wedge \lambda_3(\xi').$$

Therefore, we have, by the Leibniz rule again,

$$\mathbf{q}(v \otimes (\xi, \xi')) := -\lambda_1(v) \wedge \xi \wedge \lambda_3(\xi') + v \wedge \lambda_3(\xi) \wedge \lambda_3(\xi'),$$

which is symmetric in ξ and ξ'.

Moreover, if $\xi' \in A \cap \bigwedge^3 V_5$, one has $\lambda_3(\xi') = 0$, hence $\mathbf{q}(v \otimes (\xi, \xi')) = 0$, so that $\mathbf{q}$ descends to

$$\mathbf{q} \colon V_6 \otimes \mathrm{Sym}^2 W \to \mathbf{C},$$

where $W := A/(A \cap \bigwedge^3 V_5)$.

Finally, if $v \in V_5$, one has $\lambda(v) = 0$, hence $\mathbf{q}(v \otimes (\xi, \xi')) = v \wedge \lambda_3(\xi) \wedge \lambda_3(\xi')$ by the calculation above. The canonical factorization of the composition

$$A \subseteq \textstyle\bigwedge^3 V_6 \xrightarrow{\lambda_3} \textstyle\bigwedge^2 V_5$$

induces an injection $W \hookrightarrow \bigwedge^2 V_5$ and all the conditions are met for $(W, V_6, V_5, \mathbf{q})$ to be a GM data set.

For the other direction, we start from a GM data set $(W, V_6, V_5, \mathbf{q})$. Choose $v_0 \in V_6 \smallsetminus V_5$. We define A by the short exact sequence

$$\begin{array}{ccccccc} 0 & \to & A & \to & \bigwedge^3 V_5 \oplus W & \longrightarrow & W^\vee \\ & & & & (\xi, w) & \longmapsto & \big(w' \mapsto \xi \wedge w' + \mathbf{q}(v_0)(w, w')\big), \end{array} \tag{2}$$

where the middle term is seen inside $\bigwedge^3 V_5 \oplus (v_0 \wedge \bigwedge^2 V_5) = \bigwedge^3 V_6$.

Let us check that A is Lagrangian. If $(\xi, w), (\xi', w') \in A$, we have

$$\begin{aligned} (\xi, w) \wedge (\xi', w') &= (\xi + v_0 \wedge w) \wedge (\xi' + v_0 \wedge w') \\ &= \xi \wedge v_0 \wedge w' + v_0 \wedge w \wedge \xi' \\ &= v_0 \wedge (-\xi \wedge w' + \xi' \wedge w). \end{aligned}$$

But this vanishes because (ξ, w) goes to 0 in $W^\vee$, hence $\xi \wedge w' + \mathbf{q}(v_0)(w, w') = 0$, and similarly $\xi \wedge w + \mathbf{q}(v_0)(w', w) = 0$. So the fact that A is isotropic follows from the symmetry of $\mathbf{q}(v_0)$.

Since A has dimension ≥ 10 by construction, it is Lagrangian (and the rightmost map in (2) is surjective).

One checks that A is independent of the choice of v_0 and that the two constructions are inverse of each other.

The formula for $\dim(W)$ was given in the course of the proof. By construction of A, the kernel of $\mathbf{q}(v_0)$ is the set of $(0,w)$ that belong to A, which is isomorphic to $A \cap (v_0 \wedge \bigwedge^2 V_5)$. Since A is independent of the choice of v_0, this remains valid for all $v_0 \in V_6 \smallsetminus V_5$. □

Putting everything together, we obtain a bijection between the set of isomorphism classes of ordinary GM varieties and the set of isomorphism classes of certain Lagrangian data sets (V_6, V_5, A). However, we need to worry about the smoothness of the associated GM variety. It turns out, and this is what makes this construction so efficient, that there is a very simple criterion that only involves the Lagrangian A and not the hyperplane V_5.

Theorem 2.3. *Let $n \in \{3,4,5\}$. The bijections constructed above combine to give a bijection between the set of isomorphism classes of (smooth) ordinary GM varieties of dimension n and the set of isomorphism classes of Lagrangian data sets (V_6, V_5, A) such that $\dim(A \cap \bigwedge^3 V_5) = 5-n$ and A* contains no decomposable vectors.

The condition on A means $\mathbf{P}(A) \cap \mathsf{Gr}(3, V_6) = \varnothing$ inside $\mathbf{P}(\bigwedge^3 V_6)$.

Proof. Assume first that X is smooth, so that, by Lemma 2.1, $M = \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W)$ is smooth and $W^\perp$ contains no rank-2 forms, that is, $\mathbf{P}(W^\perp) \cap \mathsf{Gr}(2, V_5^\vee) = \varnothing$.

If $\xi \in \mathbf{P}(A) \cap \mathsf{Gr}(3, V_5)$, the linear form $w' \mapsto \xi \wedge w'$ is in $W^\vee$ but it has rank 2, and this is absurd.

If $\xi \in A$ is decomposable, it is therefore not in $\mathsf{Gr}(3, V_5)$, so we can write $\xi = v_0 \wedge v_1 \wedge v_2$ for some $v_0 \in V_6 \smallsetminus V_5$ and $v_1, v_2 \in V_5$. From the exact sequence (2) defining A, we obtain that $v_1 \wedge v_2$ is in W and that $(0, v_1 \wedge v_2)$ goes to 0 in $W^\vee$. The bivector $v_1 \wedge v_2$ is therefore in the kernel of the quadratic form $\mathbf{q}(v_0)$. In other words, the quadric $Q(v_0)$ is singular at the point $[v_1 \wedge v_2]$ of $\mathbf{P}(W)$. But this point is on M, hence it is singular on $X = M \cap Q(v_0)$, which is absurd.

All this shows that A contains no decomposable vectors.

Assume for the converse that A contains no decomposable vectors. What we just showed implies that M is smooth of dimension $n+1$ (because $A \cap \mathsf{Gr}(3, V_5) = \varnothing$). If X is singular at a point $[w]$, this therefore means that $\mathbf{T}_{Q(v),[w]} \supseteq \mathbf{T}_{M,[w]}$ for all $v \in V_6 \smallsetminus V_5$. Since M is the intersection of Plücker quadrics, there exists $v' \in V_5$ such that $\mathbf{T}_{Q(v),[w]} = \mathbf{T}_{Q(v'),[w]}$. Replacing v by some $v + tv'$, we see that there exists $v \in V_6 \smallsetminus V_5$ for which the quadric $Q(v)$ is singular at $[w]$. The second relation in Theorem 2.2 implies that $v \wedge w$ is in A, which is absurd since it is decomposable. □

2.3. **EPW sextics.** Eisenbud–Popescu–Walter (EPW) sextics are special hypersurfaces of degree 6 in $\mathbf{P}(V_6)$ constructed from Lagrangian subspaces $A \subseteq \bigwedge^3 V_6$. More generally, given such a Lagrangian A, we set, for any integer ℓ,

$$Y_A^{\geq \ell} := \bigl\{[v] \in \mathbf{P}(V_6) \mid \dim\bigl(A \cap (v \wedge \textstyle\bigwedge^2 V_6)\bigr) \geq \ell\bigr\} \tag{3}$$

and endow it with a scheme structure as in [O1, Section 2]. We have inclusions

$$\mathbf{P}(V_6) = Y_A^{\geq 0} \supseteq Y_A^{\geq 1} \supseteq Y_A^{\geq 2} \supseteq \cdots$$

and when the scheme $Y_A := Y_A^{\geq 1}$ is not the whole space $\mathbf{P}(V_6)$, it is a sextic hypersurface ([O1, (1.8)]) called an *EPW sextic*. When A contains no decomposable vectors, its geometry was described by O'Grady.

Theorem 2.4 (O'Grady). *Let $A \subseteq \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors.*

(a) *Y_A is an integral normal sextic hypersurface in $\mathbf{P}(V_6)$;*
(b) *$Y_A^{\geq 2} = \mathrm{Sing}(Y_A)$ is an integral normal Cohen–Macaulay surface of degree 40;*
(c) *$Y_A^{\geq 3} = \mathrm{Sing}(Y_A^{\geq 2})$ is finite and smooth, and is empty for A general;*
(d) *$Y_A^{\geq 4}$ is empty.*

Given a (smooth) GM variety X, Iliev and Manivel described in [IM] a direct way to construct its associated EPW sextic Y_A. It goes as follows. The space V_6 of quadrics in $\mathbf{P}(W)$ containing X is 6-dimensional. We define the discriminant locus $\widetilde{\mathrm{Disc}}(X)$ as the subscheme of $\mathbf{P}(V_6)$ of *singular* quadrics containing X. It is a hypersurface of degree $\dim(W) = n+5$ in which the multiplicity of the hyperplane $\mathbf{P}(V_5)$ of Plücker quadrics is at least the corank of a general such quadric, which is at least $\dim(W) - 6 = n - 1$. We set

$$\mathrm{Disc}(X) := \widetilde{\mathrm{Disc}}(X) - (n-1)\mathbf{P}(V_5). \tag{4}$$

This is a sextic hypersurface in $\mathbf{P}(V_6)$.

Proposition 2.5. *Let X be a (smooth) ordinary GM variety of dimension $n \in \{3,4,5\}$ with associated Lagrangian A. The subschemes Y_A and $\mathrm{Disc}(X)$ of $\mathbf{P}(V_6)$ are equal.*

Proof. For all $v \in V_6 \smallsetminus V_5$, we have $\mathrm{Ker}(\mathbf{q}(v)) = A \cap (v \wedge \bigwedge^2 V_5)$ by Theorem 2.2, hence, Y_A and $\mathrm{Disc}(X)$ coincide as sets on the complement of $\mathbf{P}(V_5)$. Since they are both sextics and Y_A is integral and not contained (as a set) in $\mathbf{P}(V_5)$, they are equal. $\square$

2.4. **Dual EPW sextics.** If $A \subseteq \bigwedge^3 V_6$ is a Lagrangian subspace, so is its orthogonal $A^\perp \subseteq \bigwedge^3 V_6^\vee$ (which we also call its *dual*) and A contains decomposable vectors if and only if $A^\perp$ does. The EPW sequence (3) for $A^\perp$ can be described in terms of A as

$$Y_{A^\perp}^{\geq \ell} = \big\{[V_5] \in \mathsf{Gr}(5, V_6) = \mathbf{P}(V_6^\vee) \mid \dim(A \cap \textstyle\bigwedge^3 V_5) \geq \ell\big\}. \tag{5}$$

and the sextic hypersurface $Y_{A^\perp} \subseteq \mathbf{P}(V_6^\vee)$ is the projective dual of the hypersurface $Y_A \subseteq \mathbf{P}(V_6)$.

In particular, for each $n \in \{3,4,5\}$, we can rewrite the bijection of Theorem 2.3 as a bijection between the set of isomorphism classes of (smooth) ordinary GM varieties of dimension n and the set of isomorphism classes of Lagrangian data sets (V_6, V_5, A) such that A contains no decomposable vectors and $[V_5] \in Y_{A^\perp}^{5-n}$.

2.5. **Moduli.** A GIT moduli space for EPW sextics was constructed by O'Grady in [O4]: consider the natural action of the group $\mathrm{PGL}(V_6)$ on the Lagrangian Grassmannian $\mathsf{LGr}(\bigwedge^3 V_6)$ and its natural linearization in the line bundle $\mathscr{O}(2)$ (the line bundle $\mathscr{O}(1)$ does not admit a linearization). The GIT quotient

$$\overline{\mathbf{M}}^{\mathrm{EPW}} := \mathsf{LGr}(\textstyle\bigwedge^3 V_6) /\!\!/ \mathrm{PGL}(V_6)$$

is a projective, irreducible, 20-dimensional, coarse moduli space for EPW sextics. The hypersurface

$$\Sigma := \{[A] \in \mathsf{LGr}(\textstyle\bigwedge^3 V_6) \mid A \text{ has decomposable vectors}\}$$

is $\mathrm{PGL}(V_6)$-invariant. Consider its complement: it is affine, consists of stable points ([O4, Corollary 2.5.1]), and its image $\mathbf{M}^{\mathrm{EPW}}$ in $\overline{\mathbf{M}}^{\mathrm{EPW}}$ is affine and is a coarse moduli space for EPW sextics Y_A such that A has no decomposable vectors.

The map $A \mapsto A^\perp$ described in Section 2.4 induces a nontrivial involution τ on $\overline{\mathbf{M}}^{\mathrm{EPW}}$ that satisfies $\tau(\Sigma) = \Sigma$, hence also an involution on $\mathbf{M}^{\mathrm{EPW}}$.

We do not know how to construct directly a moduli space for GM varieties. Instead, we extend the constructions of Section 2.2 to families of GM varieties (this is not an easy task, not least because we need to include special GM varieties). Here is the main result of [DK3].

Theorem 2.6. *Let $n \in \{3,4,5,6\}$. There is a quasiprojective, irreducible, coarse moduli space $\mathbf{M}_n^{\mathrm{GM}}$ for (smooth) GM varieties of dimension n. Its dimension is $25-(5-n)(6-n)/2$, there is a surjective morphism*

$$\pi_n \colon \mathbf{M}_n^{\mathrm{GM}} \longrightarrow \mathbf{M}^{\mathrm{EPW}},$$

and the fiber $\pi_n^{-1}([A])$ of a Lagrangian A with no decomposable vectors is isomorphic to $Y^{5-n}_{A^\perp} \cup Y^{6-n}_{A^\perp}$.[2]

More exactly, the part $Y^{5-n}_{A^\perp}$ of the fiber corresponds to ordinary GM varieties and the part $Y^{6-n}_{A^\perp}$ corresponds to special GM varieties. Also, the schemes $\mathbf{M}_6^{\mathrm{GM}}$ and $\mathbf{M}_5^{\mathrm{GM}}$ are affine[3] and there is an open embedding $\mathbf{M}_6^{\mathrm{GM}} \hookrightarrow \mathbf{M}_5^{\mathrm{GM}}$ whose image is the moduli space of ordinary GM fivefolds.

3. Birationalities and rationality

In this section, we will show that all GM varieties of the same dimension $n \in \{3,4,5,6\}$ associated with the same Lagrangian (that is, in the same fiber of the map π_n defined in Theorem 2.6), or with dual associated Lagrangians, are birationally isomorphic. Actually, since it is easily seen that all GM varieties of dimensions 5 or 6 are rational (see Proposition 3.3), we will concentrate on the cases $n \in \{3,4\}$.

Let X be a GM variety of dimension $n \in \{3,4,5\}$, which we assume to be ordinary for simplicity, and let $\mathscr{U}_X$ be the restriction to X of the tautological rank-2 vector bundle on $\mathsf{Gr}(2,V_5)$. There are maps

$$\begin{tikzcd} & \mathbf{P}(\mathscr{U}_X) \arrow[dl] \arrow[dr, "\rho"] & \\ X & & \mathbf{P}(V_5). \end{tikzcd} \tag{6}$$

Let

$$\Sigma_1(X) \subseteq \mathbf{P}(V_5)$$

be the union of the kernels of all nonzero elements of $W^\perp$, seen as rank-4 skew forms on V_5. It is

- empty for $n=5$ ($W^\perp = 0$);
- a point for $n=4$;
- a smooth conic for $n=3$.

Proposition 3.1. *Let X be a smooth ordinary GM variety of dimension $n \in \{3,4,5\}$. The map ρ is surjective and*

- *over $\mathbf{P}(V_5) \smallsetminus \Sigma_1(X)$, it is a relative quadratic hypersurface in a $\mathbf{P}^{n-2}$-fibration, whose fiber over a point $[v]$ has corank k if and only if $[v] \in Y^k_A \cap \mathbf{P}(V_5)$;*

[2]Modulo the finite group of linear automorphisms of V_6 that preserve $A^\perp$, which is trivial for A very general ([DK1, Proposition B.9]).

[3]It should be remarked that $\mathbf{M}_6^{\mathrm{GM}}$ can be directly constructed as a standard GIT quotient of an affine space ([DK3, Proposition 5.16]).

- *over $\Sigma_1(X)$, it is a relative quadratic hypersurface in a $\mathbf{P}^{n-1}$-fibration, whose fiber over a point $[v]$ of $\Sigma_1(X)$ has corank k if and only if $[v] \in Y_A^{k+1} \cap \mathbf{P}(V_5)$. In particular, $\Sigma_1(X) \subseteq Y_A$.*

Proof. Choose $v_0 \in V_6 \smallsetminus V_5$ and let $v \in V_5 \smallsetminus \{0\}$. The fiber $\rho^{-1}([v])$ is the set of $V_2 \subseteq V_5$ containing v which correspond to points of X. This is the intersection $Q'(v)$ of the nonPlücker quadric $Q(v_0) \subseteq \mathbf{P}(W)$ defining X with the subspace $\mathbf{P}\big((v \wedge V_5) \cap W\big) \subseteq \mathbf{P}(W)$.

If $[v] \notin \Sigma_1(X)$, it is not in the kernel of any 2-form defining W, hence $\mathbf{P}\big((v \wedge V_5) \cap W\big)$ has codimension $5-n$ in $\mathbf{P}(v \wedge V_5) \simeq \mathbf{P}^3$: it is a linear space of dimension $n-2$.

If $[v] \in \Sigma_1(X)$, it is in the kernel of a unique 2-form defining W, hence $\mathbf{P}\big((v \wedge V_5) \cap W\big)$ has codimension $4-n$ in $\mathbf{P}(v \wedge V_5) \simeq \mathbf{P}^3$: it is a linear space of dimension $n-1$.

The statements about the rank of $Q'(v)$ are not so obvious and we skip their proofs (see [DK1, Proposition 4.5]). □

3.1. **Birationalities.** Our main result is the following ([DK1, Corollary 4.16 and Theorem 4.20]).

Theorem 3.2. *GM varieties of the same dimension $n \geq 3$ whose associated Lagrangians are either the same or dual one to another are birationally isomorphic.*

Proof. Because of Proposition 3.3, the only interesting cases are when $n \in \{3, 4\}$. We will only sketch the proof in the case when X and X' are smooth ordinary GM threefolds associated with the same Lagrangian subspace A and different hyperplanes $[V_5], [V_5'] \in Y^2_{A^\perp}$ (the other cases are more difficult). Set $V_4 := V_5 \cap V_5'$ and restrict the diagrams

$$\begin{tikzcd} & \mathbf{P}(\mathscr{U}_X) \arrow[dl] \arrow[dr, "\rho"] & \\ X & & \mathbf{P}(V_5) \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{P}(\mathscr{U}_{X'}) \arrow[dl] \arrow[dr, "\rho'"] & \\ X' & & \mathbf{P}(V_5') \end{tikzcd}$$

to

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\varepsilon"'] \arrow[dr, "\tilde\rho"] & & \widetilde{X}' \arrow[dl, "\tilde\rho'"'] \arrow[dr, "\varepsilon'"] & \\ X & & \mathbf{P}(V_4) & & X', \end{tikzcd}$$

where $\widetilde{X} := \rho^{-1}(\mathbf{P}(V_4))$ and similarly for $\widetilde{X}'$. One can show that ε and ε' are birational maps. Moreover, by Proposition 3.1, $\tilde\rho$ is, outside of $(\Sigma_1(X) \cup Y_A^{\geq 2}) \cap \mathbf{P}(V_4)$,[4] a double cover branched along the sextic surface $Y_A \cap \mathbf{P}(V_4)$. It follows that above the complement of $(\Sigma_1(X) \cup \Sigma_1(X') \cup \cup Y_A^{\geq 2}) \cap \mathbf{P}(V_4)$, the morphisms $\tilde\rho$ and $\tilde\rho'$ are double covers branched along *the same* sextic surface. The schemes $\widetilde{X}$ and $\widetilde{X}'$ are therefore birationally isomorphic, and so are the varieties X and X'.

A similar proof works in case X and X' are smooth ordinary GM fourfolds associated with the same Lagrangian subspace A and different hyperplanes $[V_5], [V_5'] \in Y^1_{A^\perp}$: with the same notation, $\tilde\rho$ and $\tilde\rho'$ are, outside a subvariety of $\mathbf{P}(V_4)$ of dimension ≤ 1, conic bundles with the same discriminant $Y_A \cap \mathbf{P}(V_4)$. Over a general point of every component of that surface, the fiber is the union of two distinct lines, and the double cover of $Y_A \cap \mathbf{P}(V_4)$ that this defines is the same for X and for X': one can show it is induced by restriction of the canonical double cover $\widetilde{Y}_A \to Y_A$. A general argument based on Brauer groups (see [C-T]) then shows that the fields of functions of $\widetilde{X}$ and $\widetilde{X}'$ are the same extension of the field of functions of $\mathbf{P}(V_4)$. Therefore, $\widetilde{X}$ and $\widetilde{X}'$ are birationally isomorphic, and so are X and X'. □

[4] By [DK1, Lemma B.6], this has dimension ≤ 1.

In [DIM1, Section 7], we described explicit birational maps between ordinary GM threefolds associated with the same Lagrangian ("conic transformations") or with dual Lagrangians ("lines transformations").

3.2. **Rationality.** We begin by proving that all GM varieties of dimension $n \in \{5, 6\}$ are rational.

Proposition 3.3. *All GM varieties of dimension* $n \in \{5, 6\}$ *are rational.*

Proof. Let us do the proof for an ordinary GM fivefold X, corresponding to a Lagrangian $A \subseteq \bigwedge^3 V_6$ with no decomposable vectors and a hyperplane $[V_5] \in \mathbf{P}(V_6) \smallsetminus Y_{A^\perp}$. First, one can show that the surface $Y_A^{\geq 2}$ is not contained in a hyperplane, so we can choose $[v] \in Y_A^2 \smallsetminus \mathbf{P}(V_5)$. By Theorem 2.2, the corresponding 8-dimensional nonPlücker quadric $Q(v)$ has corank 2 and $X = \mathsf{Gr}(2, V_5) \cap Q(v)$. One shows that for a general 6-dimensional $\mathbf{q}(v)$-isotropic subspace I, the intersection $S := \mathbf{P}(I) \cap X = \mathbf{P}(I) \cap \mathsf{Gr}(2, V_5)$ is a smooth quintic del Pezzo surface. Consider the linear projection

$$\pi_S \colon X \dashrightarrow \mathbf{P}^3$$

from $\mathbf{P}(I)$. A general fiber is also a quintic del Pezzo surface: indeed, the intersection of a general $\mathbf{P}^6$ containing $\mathbf{P}(I) = \mathbf{P}^5$ with $Q(v)$ is a union $\mathbf{P}(I) \cup \mathbf{P}(I')$, where I' is another Lagrangian subspace, and its intersection with X is the union of S and another smooth quintic del Pezzo surface $\mathbf{P}(I') \cap \mathsf{Gr}(2, V_5)$.

Therefore, the field of rational functions of X is the field of rational functions of a smooth quintic del Pezzo surface defined over the field of rational functions on $\mathbf{P}^3$. But a smooth quintic del Pezzo surface is rational over any field by a theorem of Enriques (see [S-B]). $\square$

The situation for GM threefolds is the following. First, any (smooth) GM threefold X is unirational (there is a degree-2 rational map $\mathbf{P}^3 \dashrightarrow X$). However, using the Clemens–Griffiths criterion and a degeneration argument, one obtains that a general GM threefold is *not rational.* People believe that this should be true for *all* smooth GM threefolds, but a description of the singular locus of the theta divisor of the intermediate Jacobian is lacking.

Again, any (smooth) GM fourfold X is unirational (there is a degree-2 rational map $\mathbf{P}^4 \dashrightarrow X$; see [DIM2, Proposition 3.1]). The question of their rationality is not settled and is very much like the case of cubic fourfolds in $\mathbf{P}^5$: some (smooth) rational examples are known (see below) and it is expected that a very general GM fourfold is not rational, but not a single example is known. Note that by Theorem 3.2, the rationality of a GM fourfold only depends on its associated Lagrangian.

Example 3.4 (GM fourfolds containing a quintic del Pezzo surface)**.** Let X be a smooth GM fourfold. If $Y_A^3 \not\subseteq \mathbf{P}(V_5)$, the same argument as in the proof of Proposition 3.3 shows that X contains a quintic del Pezzo surface and is rational. More generally, using Theorem 3.2, one can show that X is rational as soon as $Y_A^3 \neq \varnothing$ (see [KP, Lemma 4.7] for more details). This is a codimension-1 condition in the moduli space $\mathbf{M}^{\mathrm{EPW}}$ of Lagrangians with no decomposable vectors: the set

$$\Delta := \{[A] \in \mathsf{LGr}(\textstyle\bigwedge^3 V_6) \mid Y_A^3 \neq \varnothing\} \tag{7}$$

is a $\mathrm{PGL}(V_6)$-invariant irreducible hypersurface. Likewise, this is a codimension-1 condition in the moduli space $\mathbf{M}_4^{\mathrm{GM}}$ of GM fourfolds.

Example 3.5 (GM fourfolds containing a σ-plane)**.** A σ-plane is a plane of the type $P = \mathbf{P}(V_1 \wedge V_4) \subseteq \mathbf{P}(\bigwedge^2 V_5)$; it is contained in $\mathsf{Gr}(2, V_5)$. Let X be a smooth GM fourfold, corresponding to a Lagrangian $A \subseteq \bigwedge^3 V_6$ with no decomposable vectors and a hyperplane $[V_5] \in Y_{A^\perp} \smallsetminus Y_{A^\perp}^{\geq 2}$. We will see later that X contains a σ-plane if and only if $Y^3_A \cap \mathbf{P}(V_5) \neq \varnothing$. In particular, $[A]$ is in the hypersurface Δ defined in (7); if $[A]$ is general in Δ, the scheme $Y^{\geq 3}_A$ consists of a single point $[v]$ and one needs $[v] \in \mathbf{P}(V_5)$, or equivalently, $[V_5] \in (Y_{A^\perp} \smallsetminus Y^{\geq 2}_{A^\perp}) \cap \mathbf{P}(v^\perp)$. So this is a codimension-1 condition on A in $\mathbf{M}^{\mathrm{EPW}}$ and a codimension-2 condition on X in $\mathbf{M}^{\mathrm{GM}}_4$.

It is easy to see directly that X is rational: the image of the linear projection $\pi_P \colon X \dashrightarrow \mathbf{P}^5$ from P is a smooth quadric $Q \subseteq \mathbf{P}^5$ and the induced morphism $\mathrm{Bl}_P X \to Q$ is the blow up of a smooth degree-9 surface $\widetilde{S} \subseteq Q$ (itself the blow up of a degree-10 K3 surface $S \subseteq \mathbf{P}^6$ at a point). This was known to Roth ([R, Section 4]) and Prokhorov ([Pr, Section 3]).

Example 3.6 (New examples of rational GM fourfolds)**.** The image of $\mathbf{P}^2$ by the linear system of quartic curves through three simple points and one double point in general position is a smooth surface of degree 9 in $\mathbf{P}^8$. Very recently, it was proved in [HS] that any smooth GM fourfold containing such a surface is rational. This is a codimension-1 condition in $\mathbf{M}^{\mathrm{GM}}_4$.

3.3. **Linear spaces contained in GM varieties.** For any partial flag $V_1 \subseteq V_{r+2} \subseteq V_5$, we consider the r-dimensional linear space $\mathbf{P}(V_1 \wedge V_{r+2}) \subseteq \mathsf{Gr}(2, V_5)$; it is said to be *of σ-type.* For $r \in \{1, 3\}$, all $\mathbf{P}^r$ contained in $\mathsf{Gr}(2, V_5)$ are of σ-type; for $r = 2$, all planes contained in $\mathsf{Gr}(2, V_5)$ are of σ-type or of the form $\mathsf{Gr}(2, V_3)$ (called τ-planes).

Let X be a (smooth) ordinary GM variety and let $F_r(X)$ be the Hilbert scheme of r-linear spaces of σ-type contained in X. There are morphisms

$$\begin{array}{rcl} \sigma\colon F_r(X) & \longrightarrow & \mathbf{P}(V_5) \\ {[\mathbf{P}(V_1 \wedge V_{r+2})]} & \longmapsto & [V_1]. \end{array}$$

We will relate these schemes $F_r(X)$ to the quadric fibration $\rho\colon \mathbf{P}(\mathscr{U}_X) \to \mathbf{P}(V_5)$ defined in (6).

Proposition 3.7. *Let X be a smooth ordinary GM variety of dimension $n \geq 3$, with associated Lagrangian data set (V_6, V_5, A). The maps $\sigma\colon F_r(X) \to \mathbf{P}(V_5)$ lift to isomorphisms*

$$F_r(X) \simeq \mathrm{Hilb}^{\mathbf{P}^r}(\mathbf{P}(\mathscr{U}_X)/\mathbf{P}(V_5))$$

with the relative Hilbert schemes for the quadric fibration ρ.

Proof. Assume that $P = \mathbf{P}(V_1 \wedge V_{r+2})$ is contained in X. The inclusion $P \hookrightarrow X$ lifts to an inclusion $P \hookrightarrow \mathbf{P}(\mathscr{U}_X)$ by mapping $x \in P$ to $(x, [V_1])$, and the image is contained in the fiber $\rho^{-1}([V_1])$. Conversely, any $\mathbf{P}^r$ contained in a $\rho^{-1}([V_1])$ projects isomorphically onto a $\mathbf{P}^r = \mathbf{P}(V_1 \wedge V_{r+2})$ of σ-type contained in X. □

Corollary 3.8. *Let X be a smooth ordinary GM n-fold. One has $F_3(X) = \varnothing$ and, if $n \leq 3$, one has $F_2(X) = \varnothing$.*

Proof. By Proposition 3.7, any $\mathbf{P}^3$ contained in X is contained in a fiber $\rho^{-1}([v])$, which, by Proposition 3.1, is a quadric of corank ≤ 3 in $\mathbf{P}^{n-2}$ if $v \notin \Sigma_1(X)$, or a quadric of corank ≤ 2 in $\mathbf{P}^{n-1}$ if $v \in \Sigma_1(X)$. But no such quadric contains a $\mathbf{P}^3$, hence $F_3(X) = \varnothing$. The reasoning for $F_2(X)$ is analogous. □

One could also deduce the corollary from Hodge theory and the Lefschetz theorem (see Section 4.1). The next results are more interesting; we include the case $n = 6$ although it has been excluded from our discussions up to now (all GM sixfolds are special) and we assume $Y^3_A = \varnothing$ to make the statement cleaner (but then we miss the case of GM fourfolds containing a σ-plane, as explained in Example 3.5).

Theorem 3.9 (σ-planes on a GM variety)**.** *Let X be a smooth GM variety of dimension $n \in \{4, 5, 6\}$, with associated Lagrangian data set (V_6, V_5, A). We assume $Y^3_A = \varnothing$.*

- *If $n = 6$, there is a factorization*
$$\sigma\colon F_2(X) \xrightarrow{\tilde\sigma} \widetilde{Y}_{A,V_5} \xrightarrow{f_{A,V_5}} Y_A \cap \mathbf{P}(V_5) \hookrightarrow \mathbf{P}(V_5),$$
where $\tilde\sigma$ is a $\mathbf{P}^1$-bundle and f_{A,V_5} is an irreducible ramified double cover (between threefolds). In particular, $F_2(X)$ is irreducible of dimension 4.
- *If $n = 5$ and X is ordinary, there is a factorization*
$$\sigma\colon F_2(X) \xrightarrow{g_{A,V_5}} Y_A^2 \cap \mathbf{P}(V_5) \hookrightarrow \mathbf{P}(V_5),$$
where g_{A,V_5} is a connected étale double cover. In particular, $F_2(X)$ is a connected curve.
- *If $n = 4$, one has $F_2(X) = \varnothing$.*

Proof. This is again a direct consequence of Propositions 3.7 and 3.1. Let us do the case $n = 5$ and X ordinary, where $\Sigma_1(X) = \varnothing$. We have to look at planes contained in quadrics of corank $k \le 2$ in $\mathbf{P}^3$. They only exist if $k = 2$ and there are exactly 2 of them; the conclusion follows (except for the connectedness of $F_2(X)$, which has to be shown by other means). ☐

The next theorem is proved similarly.

Theorem 3.10 (Lines on a GM variety)**.** *Let X be a smooth GM variety of dimension $n \in \{3, 4\}$, with associated Lagrangian data set (V_6, V_5, A). We assume $Y_A^3 = \varnothing$.*

- *If $n = 4$, the scheme $F_1(X)$ is integral of dimension 3 and there is a factorization*
$$\sigma\colon F_1(X) \xrightarrow{\tilde\sigma} \widetilde{Y}_{A,V_5} \xrightarrow{f_{A,V_5}} Y_A \cap \mathbf{P}(V_5) \hookrightarrow \mathbf{P}(V_5),$$
where $\tilde\sigma$ is an isomorphism over the complement of the inverse image by f_{A,V_5} of the point $\Sigma_1(X)$ of $Y_A \cap \mathbf{P}(V_5)$ and has $\mathbf{P}^1$ fibers over that set, and f_{A,V_5} is an irreducible ramified double cover.[5]
- *If $n = 3$, the scheme $F_1(X)$ is reduced of pure dimension 1 and there is a factorization*
$$\sigma\colon F_1(X) \xrightarrow{\tilde\sigma} Y_A^2 \cap \mathbf{P}(V_5) \hookrightarrow \mathbf{P}(V_5),$$
where $\tilde\sigma$ is an isomorphism over the complement of $Y_A^2 \cap \Sigma_1(X)$ and a double étale cover over that set.

In the next section, we will identify the double cover $g_{A,V_5}\colon F_2(X) \to Y_A^2 \cap \mathbf{P}(V_5)$ and the double covers $f_{A,V_5}\colon \widetilde{Y}_{A,V_5} \to Y_A \cap \mathbf{P}(V_5)$ occurring in both theorems: they are induced by restriction to $\mathbf{P}(V_5)$ of canonical double covers $g_A\colon \widetilde{Y}_A^2 \to Y_A^2$ and $f_A\colon \widetilde{Y}_A \to Y_A$.

Remark 3.11. All GM varieties of dimension ≥ 3 contain lines ([DK2, Theorem 4.7]). All GM varieties of dimension ≥ 5 contain σ-planes and τ-planes ([DK2, Theorems 4.3 and 4.5]).

3.4. **Double EPW sextics.** We start from a Lagrangian $A \subseteq \bigwedge^3 V_6$ with no decomposable vectors and its associated EPW sextic $Y_A \subseteq \mathbf{P}(V_6)$ as defined by (3). Following O'Grady ([O1]), we define a double cover

$$f_A\colon \widetilde{Y}_A := \mathbf{Spec}(\mathscr{O}_{Y_A} \oplus \mathscr{R}) \longrightarrow Y_A, \tag{8}$$

where $\mathscr{R}$ is a certain reflexive self-dual rank-1 sheaf on Y_A, as follows.

In the trivial symplectic sheaf $\mathscr{V} := \bigwedge^3 V_6 \otimes \mathscr{O}_{\mathbf{P}(V_6)}$, one considers the Lagrangian subsheaves $\mathscr{A}_1$, with constant fiber $A \subseteq \bigwedge^3 V_6$, and $\mathscr{A}_2$, with fiber $v \wedge \bigwedge^2 V_6$ at $[v] \in \mathbf{P}(V_6)$. With the notation of (5), the schemes $Y_A^{\ge \ell}$ are the *Lagrangian degeneration schemes*
$$D^\ell(\mathscr{A}_1, \mathscr{A}_2) := \{[v] \in \mathbf{P}(V_6) \mid \dim(\mathscr{A}_{1,[v]} \cap \mathscr{A}_{2,[v]}) \ge \ell\}.$$

[5]The same as in Theorem 3.9.

More precisely, $D^\ell(\mathscr{A}_1,\mathscr{A}_2)$ is defined as the corank-ℓ degeneracy locus of the morphism

$$\omega_{\mathscr{A}_1,\mathscr{A}_2}\colon \mathscr{A}_1 \hookrightarrow \mathscr{V} \xrightarrow{\sim} \mathscr{V}^\vee \twoheadrightarrow \mathscr{A}_2^\vee,$$

where the middle isomorphism is induced by the symplectic form on $\mathscr{V}$. For example, $D^1(\mathscr{A}_1,\mathscr{A}_2) = Y_A^{\geq 1}$ is the scheme-theoretic support of the sheaf $\mathscr{C} := \operatorname{Coker}(\omega_{\mathscr{A}_1,\mathscr{A}_2})$. At a point $[v]\in Y_A^1$, the morphism $\omega_{\mathscr{A}_1,\mathscr{A}_2}$ induces an isomorphism

$$\mathscr{A}_{1,[v]}/(\mathscr{A}_{1,[v]}\cap\mathscr{A}_{2,[v]}) \xrightarrow{\sim} (\mathscr{A}_{1,[v]}\cap\mathscr{A}_{2,[v]})^\perp \subseteq \mathscr{A}_{2,[v]}^\vee \tag{9}$$

hence the sheaf $\mathscr{C}$ has rank 1 on Y_A^1, with fiber $(\mathscr{A}_{1,[v]}\cap\mathscr{A}_{2,[v]})^\vee$ at $[v]$.

The fiber of the dual $\mathscr{C}^\vee$ at a point $[v]\in Y_A^1$ is therefore $\mathscr{A}_{1,[v]}\cap\mathscr{A}_{2,[v]}$, and the map (9) induces an isomorphism between $\mathscr{A}_1/\mathscr{C}^\vee$ and $(\mathscr{A}_2/\mathscr{C}^\vee)^\vee$ on Y_A^1, hence between their determinants, which are $\mathscr{C}^{\vee\vee}$ and $\mathscr{C}^\vee(-6)$ (because $\mathscr{A}_1$ is trivial and $\det(\mathscr{A}_2)\simeq\mathscr{O}(6)$). If we set

$$\mathscr{R} := \mathscr{C}^{\vee\vee}(-3),$$

this gives an isomorphism between the invertible sheaf $\mathscr{R}|_{Y_A^1}$ and its dual. Since $\mathscr{R}$ and $\mathscr{R}^\vee$ are reflexive and $Y_A^{\geq 2}$ has codimension 2 in Y_A (Theorem 2.4), this self-duality extends uniquely to a self-duality on Y_A, which we see as a symmetric multiplication map

$$\mathscr{R}\otimes\mathscr{R} \longrightarrow \mathscr{O}_{Y_A}$$

that defines the $\mathscr{O}_{Y_A}$-algebra structure on $\mathscr{O}_{Y_A}\oplus\mathscr{R}$ used to define the double cover f_A in (8).

The main point is to be able to study the singularities of $\widetilde{Y}_A$.

Theorem 3.12 (O'Grady)**.** *Let $A\subseteq\bigwedge^3 V_6$ be a Lagrangian with no decomposable vectors. The morphism $f_A\colon \widetilde{Y}_A\to Y_A$ is branched along the surface $Y_A^{\geq 2}$. The fourfold $\widetilde{Y}_A$, called a* double EPW sextic, *is irreducible and smooth outside the finite set $f_A^{-1}(Y_A^3)$.*

Using similar ideas (see [DK4]), one can also construct a canonical irreducible double cover

$$g_A\colon \widetilde{Y}_A^{\geq 2} \longrightarrow Y_A^{\geq 2} \tag{10}$$

branched along the finite set Y_A^3, such that

$$(g_A)_*\mathscr{O}_{\widetilde{Y}_A^{\geq 2}} \simeq \mathscr{O}_{Y_A^{\geq 2}}\oplus\omega_{Y_A^{\geq 2}}(-3).$$

The surface $Y_A^{\geq 2}$ is singular along the finite set $f_A^{-1}(Y_A^3)$. When $Y_A^3=\varnothing$, it is a smooth surface of general type of irregularity $h^1(\widetilde{Y}_A^2,\mathscr{O}_{\widetilde{Y}_A^2})=10$.

Finally, one can prove (see [DK4]) that the double covers $g_{A,V_5}\colon F_2(X)\to Y_A^2\cap\mathbf{P}(V_5)$ and $f_{A,V_5}\colon \widetilde{Y}_{A,V_5}\to Y_A\cap\mathbf{P}(V_5)$ that appear in Theorems 3.9 and 3.10 are the restrictions to $\mathbf{P}(V_5)$ of the canonical double covers $g_A\colon\widetilde{Y}_A^2\to Y_A^2$ and $f_A\colon\widetilde{Y}_A\to Y_A$.

Corollary 3.13. *Let $A\subseteq\bigwedge^3V_6$ be a Lagrangian subspace with no decomposable vectors and such that $Y_A^3=\varnothing$. Let X be the ordinary GM fourfold associated with a general $[V_5]$ in $Y_{A^\perp}$.*

- *The threefold $\widetilde{Y}_{A,V_5}$ has two singular points.*
- *The threefold $F_1(X)$ is smooth irreducible and $\tilde\sigma\colon F_1(X)\to\widetilde{Y}_{A,V_5}$ is a small resolution.*

Proof. We already mentioned that $Y_{A^\perp}$ is the projective dual of Y_A. Projective duality implies that the inverse image by $\widetilde{Y}_A\to Y_A\subseteq\mathbf{P}(V_6)$ of a general $[V_5]$ in $Y_{A^\perp}$ has at least two singular points. More precisely, the single point $[v]$ of $\Sigma_1(X)$ is in $Y_A^1\cap\mathbf{P}(V_5)$ (see Proposition 3.1) and the singular locus of $\widetilde{Y}_{A,V_5}$ consists of its two preimages in $\widetilde{Y}_A$. The conclusion then follows from Theorem 3.10 and the verification via deformation theory that $F_1(X)$ is smooth. ☐

Corollary 3.14. *Let $A \subseteq \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and such that $Y_A^3 = \varnothing$. Let X be the ordinary GM fivefold associated with a general hyperplane $V_5 \subseteq V_6$.*

The curve $F_2(X)$ is smooth connected of genus 161 and $\tilde{\sigma}\colon F_2(X) \to Y_A^2 \cap \mathbf{P}(V_5)$ is an étale double cover.

Proof. Since V_5 is general, $Y_A^2 \cap \mathbf{P}(V_5)$ is smooth by Bertini's theorem. The conclusion then follows from Theorem 3.9 and a direct computation of the genus using the numerical invariants of the smooth surface Y_A^2. □

Corollary 3.15. *Let $A \subseteq \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors and such that $Y_A^3 = \varnothing$. Let X be the GM sixfold associated with a general hyperplane $[V_5]$.*

The threefold $F_2(X)$ is smooth irreducible and $\tilde{\sigma}\colon F_2(X) \to f_A^{-1}(Y_A \cap \mathbf{P}(V_5))$ is a $\mathbf{P}^1$-fibration.

Proof. Since V_5 is general, $\widetilde{Y}_{A,V_5} = f_A^{-1}(Y_A \cap \mathbf{P}(V_5))$ is smooth by Bertini's theorem. The conclusion then follows from Theorem 3.9. □

Remark 3.16. When X is a general GM threefold, $F_1(X)$ is a smooth connected curve of genus 71 ([IP, Theorem 4.2.7]), $Y_A^2 \cap \mathbf{P}(V_5)$ is an irreducible curve with 10 nodes as singularities, and $\tilde{\sigma}\colon F_1(X) \to Y_A^2 \cap \mathbf{P}(V_5)$ is its normalization.

4. Periods of Gushel–Mukai varieties

4.1. **Cohomology and period maps of GM varieties.**
With some work, the Hodge numbers of GM varieties can be computed (see [DK2, Propositions 3.1 and 3.4]).

Proposition 4.1. *The integral cohomology of a smooth complex GM variety of dimension n is torsion-free and its Hodge diamond is*

$(n=1)$
$$\begin{array}{c} 1 \\ 6 \quad 6 \\ 1 \end{array}$$

$(n=2)$
$$\begin{array}{c} 1 \\ 0 \quad 0 \\ 1 \quad 20 \quad 1 \\ 0 \quad 0 \\ 1 \end{array}$$

$(n=3)$
$$\begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 10 \quad 10 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \\ 1 \end{array}$$

$(n=4)$
$$\begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 22 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \\ 1 \end{array}$$

$(n=5)$
$$\begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 2 \quad 0 \quad 0 \\ 0 \quad 0 \quad 10 \quad 10 \quad 0 \quad 0 \\ 0 \quad 0 \quad 2 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \\ 1 \end{array}$$

$(n=6)$
$$\begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 2 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 1 \quad 22 \quad 1 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 2 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \\ 1 \end{array}$$

In particular, only the middle cohomology $H^n(X;\mathbf{Z})$ is interesting (in other degrees, it is induced from the cohomology of $\mathsf{Gr}(2,V_5)$). We define the *vanishing cohomology* as

$$H^n(X;\mathbf{Z})_{\mathrm{van}} := \big(\gamma^* H^n(\mathsf{Gr}(2,V_5);\mathbf{Z})\big)^{\perp} \subseteq H^n(X;\mathbf{Z}),$$

where $\gamma\colon X \to \mathsf{Gr}(2,V_5)$ is the Gushel map. The Hodge numbers for the vanishing cohomology are therefore

$(n=1)$	$(n=2)$	$(n=3)$	$(n=4)$	$(n=5)$	$(n=6)$
6 6	1 19 1	0 10 10 0	0 1 20 1 0	0 0 10 10 0 0	0 0 1 20 1 0 0

In other words,

- when $n \in \{3,5\}$, the Hodge structure has weight 1 and there is a 10-dimensional principally polarized intermediate Jacobian

$$J(X) := H^n(X,\mathbf{C})/\big(H^{(n+1)/2,(n-1)/2}(X) + H^n(X,\mathbf{Z})\big)$$

and a period map

$$\begin{array}{rcl} \wp_n\colon \mathbf{M}_n^{\mathrm{GM}} & \longrightarrow & \mathscr{A}_{10} \\ [X] & \longmapsto & [J(X)]; \end{array}$$

- when $n \in \{4,6\}$, the Hodge structure is of K3 type and there is a period map

$$\begin{array}{rcl} \wp_n\colon \mathbf{M}_n^{\mathrm{GM}} & \longrightarrow & \mathscr{D} \\ [X] & \longmapsto & [H^{n/2+1,n/2-1}(X)], \end{array}$$

  where $\mathscr{D}$ is a 20-dimensional quasiprojective variety (the same for both $n=4$ and $n=6$) which will be defined in Section 4.5.

None of these period maps

$$\begin{tikzcd}
\text{dim.} & & & \text{dim.} \\
22 & \mathbf{M}_3^{\mathrm{GM}} \arrow[dr, "\wp_3"] & & \\
24 & \mathbf{M}_4^{\mathrm{GM}} \arrow[dr, "\wp_4"] & \mathscr{A}_{10} & 55 \\
25 & \mathbf{M}_5^{\mathrm{GM}} \arrow[ur, "\wp_5"'] & \mathscr{D} & 20 \\
25 & \mathbf{M}_6^{\mathrm{GM}} \arrow[ur, "\wp_6"'] & &
\end{tikzcd}$$

are injective. They are dominant when n is even. To go further, we need to go back to the double EPW sextics introduced in Section 3.4.

Remark 4.2 (The Hodge conjectures for GM varieties)**.** The (rational) Hodge conjecture holds for GM varieties of odd dimensions (because the Hodge classes all come from the Grassmannian) and for GM fourfolds (since it does for any smooth projective fourfold covered by rational curves; see [CM]); it can also be proved for GM sixfolds by using the constructions of [DK2] ([KP, Remark 2.26]).

The *integral* Hodge conjecture holds for GM varieties of dimension $n \geq 3$, except perhaps for $(2,2)$ classes on GM fourfolds and for $(3,3)$-classes on GM sixfolds (use Remark 3.11). It holds for rational GM fourfolds or for GM fourfolds birationally isomorphic to a cubic fourfold and more generally, if it holds for one GM fourfold X, it holds for any GM fourfold birationally isomorphic to X ([V, Lemma 15 and Theorem 18]): it therefore holds for all GM fourfolds whose period is in the hypersurface $\mathscr{D}_{10} \cup \mathscr{D}_{12} \cup \mathscr{D}_{20}$ of $\mathscr{D}$ (Examples 4.9 and 4.10, [DIM2, Proposition 7.2]; see Section 4.5 for the definition of the hypersurfaces $\mathscr{D}_d$).

4.2. **Double EPW sextics and hyperkähler fourfolds.** The reason why O'Grady made the construction explained in Section 3.4 of double EPW sextics is that they provide examples of hyperkähler fourfolds.

Definition 4.3. A hyperkähler variety is a smooth, compact, simply connected, Kähler variety whose space of holomorphic 2-forms is generated by a symplectic form.

Hyperkähler varieties of dimension 2 are K3 surfaces. If S is a K3 surface, it was shown by Beauville that the punctual Douady space $S^{[n]} := \mathrm{Hilb}^n(S)$ parametrizing length-n subschemes of S is a hyperkähler variety of dimension $2n$. The main result of [O1] is the following (the double EPW sextic $\widetilde{Y}_A$ was defined in Theorem 3.12).

Theorem 4.4. *Let $A \subseteq \bigwedge^3 V_6$ be a Lagrangian with no decomposable vectors and such that $Y_A^3 = \varnothing$. The double EPW sextic $\widetilde{Y}_A$ is a (smooth) hyperkähler fourfold.*

Moreover, O'Grady proved that $\widetilde{Y}_A$ is a deformation of the Douady square $S^{[2]}$ of a K3 surface S, so that its Hodge numbers are known by a computation of Beauville: for $H^2(\widetilde{Y}_A)$, they are

$$1 \quad 21 \quad 1$$

Moreover, Beauville defined an integral, nondegenerate quadratic form q_{BB} on $H^2(\widetilde{Y}_A;\mathbf{Z})$. The class $h_A := f_A^*\mathscr{O}_{\mathbf{P}(V_6)}(1)$ is ample on $\widetilde{Y}_A$ with $q_{BB}(h_A)=2$, and we define the *primitive cohomology* as

$$H^2(\widetilde{Y}_A;\mathbf{Z})_0 := h_A^\perp \subseteq H^2(\widetilde{Y}_A;\mathbf{Z}).$$

The primitive cohomology gives rise to a period map

$$\begin{array}{rccc}\wp\colon & \mathbf{M}^{\mathrm{EPW}} & \longrightarrow & \mathscr{D}\\ & [A] & \longmapsto & [H^{2,0}(\widetilde{Y}_A)],\end{array}$$

where $\mathscr{D}$ is the period domain already mentioned in Section 4.1 and which will be defined in Section 4.5 (strictly speaking, this map is only defined on the complement of the hypersurface Δ defined in (7) but O'Grady showed that it extends over Δ). By Verbitsky's global Torelli Theorem and Markman's monodromy results, *the map $\wp$ is an open embedding.*

4.3. **Factorization of the period maps of GM varieties.** We explain in this section that the period maps $\wp_n\colon \mathbf{M}_n^{\mathrm{GM}} \to (\mathscr{A}_{10}$ ou $\mathscr{D})$ for GM n-folds, defined in Section 4.1, factor through the surjective morphisms

$$\pi_n\colon \mathbf{M}_n^{\mathrm{GM}} \longrightarrow \mathbf{M}^{\mathrm{EPW}}$$

defined in Theorem 2.6 that send a GM variety to its Lagrangian.

For that, we relate the Hodge structure of a GM variety X of dimension n with associated Lagrangian A with the Hodge structure $H^2(\widetilde{Y}_A)_0$ when $n\in\{4,6\}$, and with the Hodge structure $H^1(\widetilde{Y}_A^2)$ when $n\in\{3,5\}$.

Theorem 4.5 ([DK2, DK5])**.** *Let X be a smooth GM variety of dimension $n\in\{3,4,5,6\}$, with associated Lagrangian A. We assume $Y_A^3=\varnothing$.*[6]

(a) *When $n\in\{4,6\}$, there is an isomorphism of polarized Hodge structures*

$$(H^n(X;\mathbf{Z})_{\mathrm{van}},\smile) \xrightarrow{\ \sim\ } (H^2(\widetilde{Y}_A;\mathbf{Z})_0,(-1)^{n/2-1}q_{BB}).$$

(b) *There is a canonical principal polarization on the abelian variety* $\mathrm{Alb}(\widetilde{Y}_A^2)$ *and, when $n\in\{3,5\}$, there is an isomorphism*

$$J(X) \xrightarrow{\ \sim\ } \mathrm{Alb}(\widetilde{Y}_A^2)$$

of principally polarized abelian varieties.

When $n\in\{4,6\}$, the period map $\wp_n\colon \mathbf{M}_n^{\mathrm{GM}}\to\mathscr{D}$ therefore factors as

$$\begin{tikzcd}[row sep=small] \text{dim.} & & & & & \text{dim.}\\ 24 & \mathbf{M}_4^{\mathrm{GM}} \arrow[dr, two heads, "\pi_4"] & & & & \\ & & \mathbf{M}^{\mathrm{EPW}} \arrow[r, hook, "\wp"] & \mathscr{D} & & 20 \\ 25 & \mathbf{M}_6^{\mathrm{GM}} \arrow[ur, two heads, "\pi_6"'] & & & & \end{tikzcd} \tag{11}$$

[6]This hypothesis ensures that both $\widetilde{Y}_A$ and $\widetilde{Y}_A^2$ are smooth.

where the period map $\wp$ for double EPW sextics is an open embedding. When $n \in \{3,5\}$, the period map $\wp_n \colon \mathbf{M}_n^{\mathrm{GM}} \to \mathscr{A}_{10}$ factors as

$$\begin{tikzcd}
\text{dim.} & & & & & \text{dim.} \\
22 & \mathbf{M}_3^{\mathrm{GM}} \arrow[dr, two heads, "\pi_3"] & & & & \\
& & \mathbf{M}^{\mathrm{EPW}} \arrow[r] & \mathbf{M}^{\mathrm{EPW}}/r \arrow[r, "\mathsf{q}"] & \mathscr{A}_{10} & 55 \\
25 & \mathbf{M}_5^{\mathrm{GM}} \arrow[ur, two heads, "\pi_5"'] & & & &
\end{tikzcd} \tag{12}$$

where r is the (nontrivial) duality involution of $\mathbf{M}^{\mathrm{EPW}}$. The morphism q is known to be unramified ([DIM1]) and it is expected to be (generically) injective.

4.4. **Sketches of proofs.** The standard argument for this kind of results goes back to Clemens–Griffiths ([CG]) and Beauville–Donagi ([BD]), who treated the case of a smooth cubic fourfold $W \subseteq \mathbf{P}(V_6)$ and its (smooth) hyperkähler fourfold of lines $F_1(W) \subseteq \mathsf{Gr}(2,V_6)$. There is an incidence diagram

$$\begin{tikzcd}
& I \arrow[dl, "q"'] \arrow[dr, "p"] & \\
W & & F_1(W),
\end{tikzcd}$$

where p is the universal line, a $\mathbf{P}^1$-bundle, and q is dominant and generically finite. Beauville and Donagi prove that the Abel–Jacobi map $p_*q^* \colon H^4(W;\mathbf{Z}) \to H^2(F_1(W);\mathbf{Z})$ is an isomorphism of Hodge structures which induces an isometry between the primitive cohomologies $(H^4(W;\mathbf{Z})_0, \smile)$ and $(H^2(F_1(W);\mathbf{Z})_0, -q_{BB})$. Two ingredients used here are:

- $F_1(W)$ parametrizes curves on W, so there is a correspondence between W and $F_1(W)$;
- $F_1(W)$ and I are smooth, so one can define p_* in singular cohomology.

We use a similar approach, using Hilbert schemes of lines or σ-planes on GM varieties, depending on the dimension. In what follows, X is a GM n-fold with associated Lagrangian A. We assume $Y_A^3 = \varnothing$.

Dimension 4. By Corollary 3.13, for $[V_5]$ general in $Y_{A^\perp}$, the scheme of lines $F_1(X)$ is a smooth threefold and $\tilde\sigma \colon F_1(X) \to f_A^{-1}(Y_A \cap \mathbf{P}(V_5))$ is a small resolution. One shows that the induced composition

$$a \colon H^2(\widetilde{Y}_A;\mathbf{Z}) \xrightarrow{\sim} H^2(f_A^{-1}(Y_A \cap \mathbf{P}(V_5));\mathbf{Z}) \xrightarrow{\tilde\sigma^*} H^2(F_1(X);\mathbf{Z})$$

(where the first map is an isomorphism by the Lefschetz theorem) is injective.

The Abel–Jacobi map $p_*q^* \colon H^4(X;\mathbf{Z})_{\mathrm{van}} \to H^2(F_1(X);\mathbf{Z})$ is also injective (as in [BD]) and induces an antiisometry between $H^4(X;\mathbf{Z})_{\mathrm{van}}$ and $a(H^2(\widetilde{Y}_A;\mathbf{Z})_0) \subseteq H^2(F_1(X);\mathbf{Z})$.

Dimension 6. For a general GM 6-fold X, the proof is similar: one uses instead the smooth fourfold $F_2(X)$ parametrizing σ-planes contained in X and Corollary 3.15.

Dimension 5. By Corollary 3.14, for $[V_5]$ general in $Y_{A^\perp}$, the curve $F_2(X)$ is a connected double étale cover of a general genus-81 hyperplane section of the smooth surface $Y_A^2 \subseteq \mathbf{P}(V_6)$. A generalization of an old argument of Clemens (written by Tjurin in [T]) shows that the corresponding Abel–Jacobi map $q_*p^* \colon H_1(F_2(X);\mathbf{Z}) \to H_5(X;\mathbf{Z})$ in homology is surjective. It induces a surjective morphism

$$a \colon J(F_2(X)) \longrightarrow J(X)$$

with connected kernel. By the Lefschetz theorem, there is another surjective morphism

$$b\colon J(F_2(X)) \longrightarrow \mathrm{Alb}(\widetilde{Y}^2_A)$$

with connected kernel. We want to show that the morphisms a and b are the same.

We use a trick: it follows from Deligne–Picard–Lefschetz theory that since the surface Y^2_A is regular, for a very general choice of hyperplane V_5, the Jacobian $J(Y^2_A \cap \mathbf{P}(V_5))$ is simple (of dimension 81). It is therefore contracted by both a and b, and this induces surjective morphisms

$$a'\colon \text{Prym} \longrightarrow J(X) \qquad \text{and} \qquad b'\colon \text{Prym} \longrightarrow \mathrm{Alb}(\widetilde{Y}^2_A)$$

with connected kernels. Using again monodromy arguments, one shows that the kernel of b' is simple (of dimension 70). It is therefore contracted by a', and this induces an isomorphism

$$\mathrm{Alb}(\widetilde{Y}^2_A) \xrightarrow{\sim} J(X).$$

Dimension 3. By Remark 3.16, lines on a general GM threefold X are parametrized by a smooth connected curve $F_1(X)$ of genus 71 which is the normalization of the singular curve $Y^2_A \cap \mathbf{P}(V_5)$ (the hyperplane V_5 is not general anymore), but it is hard to relate this curve with the surface $\widetilde{Y}^2_A$.

However, it was proved by Logachev in [L] that the Hilbert scheme of conics contained in X is the blow up of the smooth surface $\widetilde{Y}^2_{A^\perp}$ at a point. This gives an Abel–Jacobi map $\mathrm{Alb}(\widetilde{Y}^2_{A^\perp}) \to J(X)$ which should be an isomophism.

We actually proceed differently in [DK5] and prove instead that the Abel–Jacobi map associated with a family of rational quartic curves parametrized by the surface $\widetilde{Y}^2_A$ gives the desired isomorphism $\mathrm{Alb}(\widetilde{Y}^2_A) \xrightarrow{\sim} J(X)$ (a similar method also works for fivefolds).

4.5. **Hodge special GM fourfolds.** Let us first explain in more details what the period domain $\mathscr{D}$ is. For any GM fourfold X, the unimodular cohomology lattice $(H^4(X;\mathbf{Z}), \smile)$ is isomorphic to $(1)^{\oplus 22} \oplus (-1)^{\oplus 2}$ and the vanishing cohomology lattice $(H^4(X;\mathbf{Z})_{\mathrm{van}}, \smile)$ to the even lattice

$$\Lambda := E_8^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 2} \oplus \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

where E_8 is the rank-8 positive definite even lattice. It has signature $(20,2)$.

Similarly, for any Lagrangian A, the cohomology lattice $(H^2(\widetilde{Y}_A;\mathbf{Z}), q_{BB})$ is isomorphic to the lattice

$$E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3} \oplus (-2),$$

and the primitive cohomology lattice $(H^2(\widetilde{Y}_A;\mathbf{Z})_0, q_{BB})$ is isomorphic to $\Lambda(-1)$ (in accordance with Theorem 4.5(a)).

The manifold

$$\Omega := \{\omega \in \mathbf{P}(\Lambda \otimes \mathbf{C}) \mid (\omega \cdot \omega) = 0 \ , (\omega \cdot \bar{\omega}) < 0\}$$

is a homogeneous space with two components, Ω^+ and Ω^-, both isomorphic to the 20-dimensional open complex manifold $\mathrm{SO}_0(20,2)/\,\mathrm{SO}(20) \times \mathrm{SO}(2)$, a bounded symmetric domain of type IV. Then,

$$\mathscr{D} := \widetilde{O}(\Lambda) \backslash \Omega^+,$$

where $\widetilde{O}(\Lambda)$ is a subgroup of index 2, called the *stable orthogonal group,* of the isometry group $O(\Lambda)$ of the lattice Λ, has the structure of an irreducible quasi-projective variety of dimension 20 (see [DIM2, Section 5] for more details). It is a period domain for both GM fourfolds and double EPW sextics.

The domain $\mathscr{D}$ carries a nontrivial canonical involution $r_{\mathscr{D}}$ corresponding to the double cover $\mathscr{D} \to O(\Lambda)\backslash\Omega^+$. If $\wp\colon \mathbf{M}^{\mathrm{EPW}} \to \mathscr{D}$ is the (extended) period map for double EPW sextics, it is related to the duality involution r of $\mathbf{M}^{\mathrm{EPW}}$ defined in Section 2.5 by the relation

$$\wp \circ r = r_{\mathscr{D}} \circ \wp$$

proved in [O2].

The fact that the period map $\wp$ is dominant easily implies that the Picard group of a very general EPW sextic $\widetilde{Y}_A$ is generated by the class of the polarization $h_A = f_A^*\mathscr{O}_{\mathbf{P}(V_6)}(1)$. Indeed, for $H^{1,1}(\widetilde{Y}_A) \cap H^2(\widetilde{Y}_A, \mathbf{Z})_0$ to be nonzero, the corresponding period must be in one of the (countably many) loci $\alpha^\perp \cap \mathscr{D}$, for some nonzero $\alpha \in \Lambda$. We label these loci as $\mathscr{D}_d$, where d is the discriminant of the lattice $\alpha^\perp \subseteq \Lambda$, called the *nonspecial lattice.* The Picard group of a double EPW sextic whose period point is a very general point of $\mathscr{D}_d$ (more precisely, it is not in any other $\mathscr{D}_{d'}$, for $d' \neq d$) has rank 2: it is the saturation of the subgroup $\mathbf{Z}h_A \oplus \mathbf{Z}\alpha$ of $H^2(\widetilde{Y}_A, \mathbf{Z})$.

The following result is [DIM2, Lemma 6.1 and Corollary 6.3].

Proposition 4.6. *The locus $\mathscr{D}_d$ is nonempty if and only if $d > 0$ and*

- *either $d \equiv 0 \pmod 4$, in which case $\mathscr{D}_d \subseteq \mathscr{D}$ is an irreducible hypersurface;*
- *or $d \equiv 2 \pmod 8$, in which case $\mathscr{D}_d \subseteq \mathscr{D}$ is the union of two irreducible hypersurfaces $\mathscr{D}'_d$ and $\mathscr{D}''_d$, which are interchanged by the involution $r_{\mathscr{D}}$.*

It is also known that

- the image of the period map $\wp\colon \mathbf{M}^{\mathrm{EPW}} \to \mathscr{D}$ does not meet the hypersurface $\mathscr{D}_2 \cup \mathscr{D}_4 \cup \mathscr{D}_8$ ([O5, Theorem 1.3], [DM, Example 6.3]) and is conjectured to be equal to its complement $\mathscr{D} \smallsetminus (\mathscr{D}_2 \cup \mathscr{D}_4 \cup \mathscr{D}_8)$;[7]
- the image of the hypersurface Δ defined in (7) is the hypersurface $\mathscr{D}''_{10}$ (see Example 4.7 below).

All this discussion translates in terms of GM fourfolds.[8] We say that a GM fourfold X is *Hodge-special* if its period is in $\bigcup_d \mathscr{D}_d$. For X nonHodge-special, we have $H^{2,2}(X) \cap H^4(X, \mathbf{Z}) = H^4(\mathsf{Gr}(2, V_5), \mathbf{Z})$ and, if X is very general in $\wp_4^{-1}(\mathscr{D}_d)$, the lattice $H^{2,2}(X) \cap H^4(X, \mathbf{Z})$ has rank 3 and discriminant d.

Because of Theorem 4.5(a), the images of the period maps $\wp$ and

$$\wp_4\colon \mathbf{M}_4^{\mathrm{GM}} \longrightarrow \mathscr{D}$$

are the same.

Example 4.7 (GM fourfolds containing a σ-plane)**.** This is a continuation of Example 3.5. Let $P = \mathbf{P}(V_1 \wedge V_4)$ be a σ-plane contained in a GM fourfold X. Its class in $\mathsf{Gr}(2, V_5)$ is the Schubert

[7] If this holds, the diagrams (11) and (12) say that there should be an r-invariant rational map

$$\bar{\mathsf{q}}\colon \mathscr{D} \dashrightarrow \mathscr{A}_{10},$$

regular over the complement of $\mathscr{D}_2 \cup \mathscr{D}_4 \cup \mathscr{D}_8$. General points of $\mathscr{D}_4$ are periods of Hilbert squares $S^{[2]}$, where $S \subseteq \mathbf{P}(V_4)$ is a quartic: these are smooth hyperkähler degenerations of double EPW sextics $\widetilde{Y}_A$, and the surfaces $\widetilde{Y}_A^{\geq 2}$ degenerate to the surfaces of lines in $\mathbf{P}(V_4)$ that are bitangent to S. These smooth surfaces have irregularity 10, hence $\bar{\mathsf{q}}$ is regular at general points of $\mathscr{D}_4$. General points of $\mathscr{D}_8$ are periods of nodal GM fourfolds ([DIM2, Section 7.6]), whose intermediate Jacobian is a $\mathbf{C}^\star$-extension of a principally polarized abelian variety of dimension 9 ([DIM3, Section 7.1]), so $\bar{\mathsf{q}}$ does not extend to $\mathscr{D}_8$.

[8] It also applies to GM sixfolds, but we will not discuss that here.

class $\sigma_{3,1}$ and one computes (see [DIM2, Section 7.1]) the intersection numbers

$$(\gamma^*\sigma_{1,1}\cdot P)_X = (\sigma_{1,1}\cdot P)_{\mathsf{Gr}(2,V_5)} = 0\ ,\quad (\gamma^*\sigma_2\cdot P)_X = (\sigma_2\cdot P)_{\mathsf{Gr}(2,V_5)} = 1\ ,\quad (P\cdot P)_X = 3$$

of classes in $H^4(X;\mathbf{Z})$. It follows that the intersection form on the rank-3 sublattice

$$K := \langle \gamma^* H^4(\mathsf{Gr}(2,V_5);\mathbf{Z}), [P]\rangle = \langle \gamma^*\sigma_{1,1}, \gamma^*\sigma_2, [P]\rangle$$

is given by the matrix

$$\begin{pmatrix} 2 & 2 & 0\\ 2 & 4 & 1\\ 0 & 1 & 3 \end{pmatrix},$$

whose discriminant is 10. The orthogonal of $[P]$ in $H^4(X;\mathbf{Z})_{\mathrm{van}}$ is the same as the orthogonal of the primitive sublattice K in the unimodular lattice $H^4(X;\mathbf{Z})$, and they have the same discriminant as K. The period of X therefore belongs to the hypersurface $\mathscr{D}_{10}$, and more precisely to the component $\mathscr{D}''_{10}$ (see the comment on how we label the two components after [DIM2, Corollary 6.3]). Since the periods of these fourfolds are dense in the image of Δ in $\mathscr{D}$, we see that this image is $\mathscr{D}''_{10}$.

Example 4.8 (GM fourfolds containing a quintic del Pezzo surface)**.** This is a continuation of Example 3.4. We consider quintic del Pezzo surfaces obtained as the intersection of $\mathsf{Gr}(2,V_5)$ with a $\mathbf{P}^5$; their class is $\sigma_1^4 = 3\sigma_{3,1}+2\sigma_{2,2}$ in $\mathsf{Gr}(2,V_5)$. Let X be a GM fourfold containing such a surface S. One computes (see [DIM2, Section 7.5]) the intersection numbers

$$(\gamma^*\sigma_{1,1}\cdot S)_X = (\sigma_{1,1}\cdot \sigma_1^4)_{\mathsf{Gr}(2,V_5)} = 2\ ,\quad (\gamma^*\sigma_2\cdot S)_X = (\sigma_2\cdot \sigma_1^4)_{\mathsf{Gr}(2,V_5)} = 3\ ,\quad (S\cdot S)_X = 5$$

of classes in $H^4(X;\mathbf{Z})$. The intersection form on the rank-3 sublattice $\langle \gamma^*\sigma_{1,1}, \gamma^*\sigma_2, [S]\rangle$ is therefore given by the matrix

$$\begin{pmatrix} 2 & 2 & 2\\ 2 & 4 & 3\\ 2 & 3 & 5 \end{pmatrix},$$

whose discriminant is 10. Again, the period of X belongs to the hypersurface $\mathscr{D}''_{10}$ (this was expected since the associated Lagrangians as the same as the Lagrangians associated with GM fourfolds containing a σ-plane). More precisely, GM fourfolds containing a quintic del Pezzo surface form a dense subset of $\wp_4^{-1}(\mathscr{D}''_{10})$.

Example 4.9 (GM fourfolds containing a τ-quadric surface)**.** A τ-quadric surface Q in $\mathsf{Gr}(2,V_5)$ is a linear section of a $\mathsf{Gr}(2,V_4)$ for some $V_4\subseteq V_5$; its class in $\mathsf{Gr}(2,V_5)$ is $\sigma_1^2\cdot\sigma_{1,1} = \sigma_{3,1}+\sigma_{2,2}$. If X is a GM fourfold containing Q, one computes as above (see [DIM2, Section 7.3]) that the intersection form on the rank-3 sublattice $\langle \gamma^*\sigma_{1,1}, \gamma^*\sigma_2, [Q]\rangle$ is given by the matrix

$$\begin{pmatrix} 2 & 2 & 1\\ 2 & 4 & 1\\ 1 & 1 & 3 \end{pmatrix},$$

whose discriminant is 10. This time however, the period of X belongs to the hypersurface $\mathscr{D}'_{10}$ and GM fourfolds containing a τ-quadric surface form a dense subset of $\wp_4^{-1}(\mathscr{D}'_{10})$. According to Theorem 3.2, these fourfolds, like those containing a σ-plane, are all rational, but this can be seen directly by projecting X from the $\mathbf{P}^3$ spanned by Q.

Example 4.10. The GM fourfolds constructed in Example 3.6 were shown in [HS] to fill out a dense subset of $\wp_4^{-1}(\mathscr{D}_{20})$.

4.6. **K3 surfaces associated with GM fourfolds.** For some integers d, the nonspecial lattice is isomorphic to the primitive lattice of a polarized K3 surface (with the sign of the intersection form reversed), necessarily of degree d. The precise lattice-theoretic result is the following ([DIM2, Proposition 6.6]).

Proposition 4.11. *The positive integers d for which the nonspecial lattice is isomorphic to the (opposite of the) primitive cohomology lattice of a degree d polarized K3 surface are the integers $d \equiv 2$ or $4 \pmod 8$ such that the only odd primes that divide d are $\equiv 1 \pmod 4$.*

The first values of d that satisfy the conditions of the proposition and for which the hypersurface $\mathscr{D}_d$ meets the image of the period map are 10, 20, and 26.

Let X be a GM fourfold whose period point belongs to a hypersurface $\mathscr{D}_d$, for some positive integer d satisfying the conditions of Proposition 4.11. Since the period map for semipolarized K3 surfaces is bijective, the period point of X in $\mathscr{D}$ is the period point of a unique K3 surface S with a nef (ample when the period of X is general in $\mathscr{D}_d$) algebraic class h such that $h^2 = d$, so that the polarized Hodge structures $K^\perp$ and $H^2(S)_0(-1)$ are isomorphic. We say that the K3 surface S is *associated* with X. By Theorem 4.5, it only depends on the Lagrangian associated with X.

Example 4.12. In some cases, the associated K3 surface can be recovered geometrically from the GM fourfold X. This is the case in Example 3.5, and here is another example: when X is general in $\wp_4^{-1}(\mathscr{D}'_{10})$, it contains a τ-quadric surface Q (see Example 4.9) and it was shown in the proof of [DIM2, Proposition 7.3] that the projection $\mathrm{Bl}_Q X \to \mathbf{P}^4$ from the $\mathbf{P}^3$ spanned by Q is the blow up of the surface in $\mathbf{P}^4$ obtained as the projection of a degree-10 K3 surface $S \subseteq \mathbf{P}^6$ from the line joining two points of S.

This surface S is the K3 surface associated with X. It can be constructed in several different ways: the dual $A^\perp$ of the Lagrangian A associated with X is in the hypersurface Δ defined in (7), so that $Y^3_{A^\perp} \neq \varnothing$. Since X is general in $\wp_4^{-1}(\mathscr{D}'_{10})$, the set $Y^3_{A^\perp}$ consists of only one point and the associated GM variety has dimension 2 (see Theorem 2.2): this is the K3 surface S again; finally, a small resolution of the singular double EPW sextic $\widetilde{Y}_{A^\perp}$ is isomorphic to the Hilbert square $S^{[2]}$ (this is all explained in [O3]).

The reason why this strange notion of associated K3 surface is interesting lies in the following conjecture, originally envisioned for cubic fourfolds by Harris–Hassett.

Conjecture 4.13. A GM fourfold is rational if and only if it has an associated K3 surface.

This conjecture implies in particular that a very general GM fourfold is not rational. We will discuss it at the end of the next section.

5. Derived categories

For a very nice geometrically oriented introduction to derived categories, I refer to [Hu]. For a very detailed series of lecture notes on the topics that concern us here, but in the case of cubic fourfolds, I refer to [K3]. I will only give here a very brief and superficial overview of the situation for GM varieties.

5.1. **Semiorthogonal decompositions.** Let $\mathscr{T}$ be a ($\mathbf{C}$-linear) triangulated category. We say that a pair $(\mathscr{A}_1, \mathscr{A}_2)$ of full triangulated subcategories of $\mathscr{T}$ forms a (two-step) *semiorthogonal decomposition* (s.o.d. for short), and we write

$$\mathscr{T} = \langle \mathscr{A}_1, \mathscr{A}_2 \rangle,$$

if

- $\mathrm{Hom}(A_2, A_1) = 0$ for any $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$;
- $\mathscr{A}_1$ and $\mathscr{A}_2$ generate $\mathscr{T}$, in the sense that for any $T \in \mathscr{T}$, there is a (unique) distinguished triangle
$$A_2 \to T \to A_1 \to A_2[1],$$
with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$.

In that case, one shows ([K3, Lemma 2.2]) that $\mathscr{A}_2$ is a *right admissible subcategory* of $\mathscr{T}$: the inclusion functor $\alpha_2\colon \mathscr{A}_2 \to \mathscr{T}$ has a right adjoint $\alpha_2^!\colon \mathscr{T} \to \mathscr{A}_2$, given by $T \mapsto A_2$, that satisfies $\alpha_2^! \circ \alpha_2 \simeq \mathrm{Id}_{\mathscr{A}_2}$.

Conversely, given any right admissible triangulated functor $\alpha\colon \mathscr{A} \to \mathscr{T}$ is fully faithful and there is an s.o.d.
$$\mathscr{T} = \langle \mathscr{A}^\perp, \mathscr{A}\rangle,$$
where $\mathscr{A}^\perp := \mathrm{Ker}(\alpha^!)$ is the *right orthogonal* of $\mathscr{A}$.

This terminology is generalized to several factors: the equality
$$\mathscr{T} = \langle \mathscr{A}_1, \dots, \mathscr{A}_r\rangle$$
means

(a) $\mathrm{Hom}(A_j, A_i) = 0$ for any $A_i \in \mathscr{A}_i$, $A_j \in \mathscr{A}_j$, and $i < j$ (no morphisms from right to left);
(b) $\mathscr{A}_1, \dots, \mathscr{A}_r$ generate $\mathscr{T}$ (this is defined by induction on r).

A useful example of right admissible subcategory is the following. Let E be an object of $\mathscr{T}$ and let $\langle E\rangle$ be the subcategory of $\mathscr{T}$ generated by E: this is the smallest full triangulated subcategory of $\mathscr{T}$ containing E. If E is *exceptional,* that is, if
$$\mathrm{Ext}^\bullet(E, E) = \mathbf{C},$$
(this means $\mathrm{Hom}(E, E[m]) = \mathbf{C}$ if $m = 0$, and 0 otherwise), the category $\langle E\rangle$ is admissible ([Hu, Lemma 1.58]). In particular, we have an s.o.d. $\mathscr{T} = \langle \langle E\rangle^\perp, \langle E\rangle\rangle$ which we write simply as
$$\mathscr{T} = \langle E^\perp, E\rangle.$$

Finally, exceptional objects $E_1, \dots, E_r$ form an *exceptional collection* if the subcategories that they generate satisfy condition (a) above, which means in this case $\mathrm{Hom}(E_j, E_i[m]) = 0$ for all m and all $i < j$. There is then an s.o.d.
$$\mathscr{T} = \langle \langle E_1, \dots, E_r\rangle^\perp, E_1 \dots, E_r\rangle.$$

5.2. **Examples of semiorthogonal decompositions and exceptional collections.** We will apply these concepts in the following categories: if X is a smooth projective (complex) variety, the derived category $\mathrm{D^b}(X)$ of the category of bounded complexes of coherent sheaves on X is a triangulated category.

On the projective space $\mathbf{P}^n$, each line bundle $\mathscr{O}_{\mathbf{P}^n}(m)$ is exceptional and Beilinson proved that there is an s.o.d. ([Hu, Corollary 8.29])
$$\mathrm{D^b}(X) = \langle \mathscr{O}_{\mathbf{P}^n}, \mathscr{O}_{\mathbf{P}^n}(1), \dots, \mathscr{O}_{\mathbf{P}^n}(n)\rangle.$$
Beilinson's result was extended by Kapranov to all Grassmannians.

In the case of $G := \mathsf{Gr}(2, V_5)$ which interests us here, Kuznetsov produced the following s.o.d. ([K1, Section 6.1])
$$\mathrm{D^b}(G) = \langle \mathscr{O}_G, \mathscr{U}^\vee, \mathscr{O}_G(1), \mathscr{U}^\vee(1), \dots, \mathscr{O}_G(4), \mathscr{U}^\vee(4)\rangle,$$

where $\mathscr{U}$ is the rank-2 tautological subbundle on G. For smooth linear sections $M := \mathsf{Gr}(2, V_5) \cap \mathbf{P}(W_{n+5})$ of G of dimension $n+1 \geq 4$, he showed

$$\mathrm{D^b}(M) = \langle \mathscr{O}_M, \mathscr{U}_M^\vee, \mathscr{O}_M(1), \mathscr{U}_M^\vee(1), \dots, \mathscr{O}_M(n-1), \mathscr{U}_M^\vee(n-1)\rangle.$$

Finally, for any GM variety X of dimension $n \geq 3$, he obtained with Perry ([KP, Proposition 2.3]) that

$$\mathscr{O}_X, \mathscr{U}_X^\vee, \mathscr{O}_X(1), \mathscr{U}_X^\vee(1), \dots, \mathscr{O}_X(n-3), \mathscr{U}_X^\vee(n-3)$$

form an exceptional collection in $\mathrm{D^b}(X)$. If one sets

$$\mathscr{A}_X := \langle \mathscr{O}_X, \mathscr{U}_X^\vee, \mathscr{O}_X(1), \mathscr{U}_X^\vee(1), \dots, \mathscr{O}_X(n-3), \mathscr{U}_X^\vee(n-3)\rangle^\perp,$$

the right orthogonal of this exceptional collection, we obtain an s.o.d.

$$\mathrm{D^b}(X) = \langle \mathscr{A}_X, \mathscr{O}_X, \mathscr{U}_X^\vee, \mathscr{O}_X(1), \mathscr{U}_X^\vee(1), \dots, \mathscr{O}_X(n-3), \mathscr{U}_X^\vee(n-3)\rangle.$$

We say that the triangulated category $\mathscr{A}_X$ is associated with the GM variety X.

5.3. **Serre functors and K3/Enriques categories.** A *Serre functor* for a triangulated category $\mathscr{T}$ is an equivalence $\mathrm{S}_\mathscr{T} \colon \mathscr{T} \xrightarrow{\sim} \mathscr{T}$ with bifunctorial isomorphisms

$$\mathrm{Hom}(F, \mathrm{S}_\mathscr{T}(G)) \xrightarrow{\sim} \mathrm{Hom}(G, F)^\vee$$

for all $F, G \in \mathscr{T}$. If a Serre functor exists, it is unique (up to functorial isomorphisms), so it is an invariant of the category.

IfX is a smooth projective variety of dimension n, the category $\mathrm{D^b}(X)$ has a Serre functor given by

$$\mathrm{S}_{\mathrm{D^b}(X)}(F) = (F \otimes \omega_X)[n],$$

where ω_X is the dualizing sheaf of X. In particular, if ω_X is trivial (one often says that X is a *Calabi–Yau variety*), the Serre functor is just the shift by the dimension of X.

Bondal and Kapranov proved that an admissible subcategory of a triangulated category that has a Serre functor also has a Serre functor and Kuznetsov computed it in several interesting examples. The following is [KP, Proposition 2.6].

Proposition 5.1. *Let X be a GM variety of dimension $n \geq 3$.*

- *If n is even, the associated category $\mathscr{A}_X$ is a* K3 category*: it has a Serre functor given by the shift* [2].
- *If n is odd, the associated category $\mathscr{A}_X$ is an* Enriques category*: it has a Serre functor $\sigma \circ [2]$, where $\sigma \colon \mathscr{A}_X \xrightarrow{\sim} \mathscr{A}_X$ is a nontrivial involution.*

5.4. **The Kuznetsov–Perry conjecture.** Given a GM variety of dimension $n \in \{4, 6\}$, we have constructed a K3 category $\mathscr{A}_X$. The question is now the following: can $\mathscr{A}_X$ be equivalent to the derived category of an actual K3 surface? The answer is no for X very general ([KP, Proposition 2.29]).

Proposition 5.2. *If the category associated with a GM variety X of even dimension is equivalent to the derived category of a K3 surface, the variety X is Hodge special (see Section 4.5).*

When a GM variety X of dimension 4 has an associated K3 surface S in the Hodge theoretic sense of Section 4.6, its category $\mathscr{A}_X$ is equivalent to the category $\mathrm{D^b}(S)$. However, the converse does not hold.[9]

[9] To prove these two statements, one has to use [PPZ, Theorem 1.9], which says that the category $\mathscr{A}_X$ associated with a GM variety X of dimension 4 or 6 is equivalent to the derived category of a K3 surface if and only if a certain lattice $\widetilde{H}^{1,1}(\mathscr{A}_X, \mathbf{Z})$ associated with $\mathscr{A}_X$ contains a hyperbolic plane.

Therefore ([KP, Conjecture 3.12], [PPZ, Remark 1.10]), there are two competing conjectural conditions (existence of a categorical versus Hodge-theoretic K3 surface) for the rationality of a GM fourfold. This should be contrasted with the case of cubic fourfolds, where these conditions are known to be equivalent.

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If X is a GM fourfold with a Hodge theoretically associated K3 surface S, the lattice $\widetilde{H}^{1,1}(\mathscr{A}_X, \mathbf{Z})$ does contain a hyperbolic plane by [P, Theorem 1.2], and the categories $\mathscr{A}_X$ and $\mathrm{D^b}(S)$ are equivalent.

Then, Pertusi constructed in [P, Section 3.3] GM fourfolds X such that $\widetilde{H}^{1,1}(\mathscr{A}_X, \mathbf{Z})$ does contain a hyperbolic plane (so that the category $\mathscr{A}_X$ is equivalent to the derived category of a K3 surface), but with no Hodge theoretically associated K3 surface. Many thanks to Alex Perry for these explanations.

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