HYPERKÄHLER MANIFOLDS

OLIVIER DEBARRE

Abstract. The aim of these notes is to acquaint the reader with important objects in complex algebraic geometry: K3 surfaces and their higher-dimensional analogs, hyperkähler manifolds. These manifolds are interesting from several points of view: dynamical (some have interesting automorphism groups), arithmetical (although we will not say anything on this aspect of the theory), and geometric. It is also one of those rare cases where the Torelli theorem allows for a powerful link between the geometry of these manifolds and lattice theory.

We do not prove all the results that we state. Our aim is more to provide, for specific families of hyperkähler manifolds (which are projective deformations of punctual Hilbert schemes of K3 surfaces), a panorama of results about projective embeddings, automorphisms, moduli spaces, period maps and domains, rather than a complete reference guide. These results are mostly not new, except perhaps those of Appendix B (written with E. Macrì), where we give in Theorem B.1 an explicit description of the image of the period map for these polarized manifolds.

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1. Introduction

As explained in the abstract, the aim of these notes is to gather some results on projective K3 surfaces and hyperkähler manifolds, in particular their projective embeddings, their biregular and birational automorphism groups, and their moduli spaces. For K3 surfaces, these results have been known for more than twenty years, whereas they are much more recent for hyperkähler manifolds; for example, the Torelli theorem for K3 surfaces was proved more than 35 years ago, whereas its version for all hyperkähler manifolds (Theorem 3.18) was published by M. Verbitsky in 2013. The results on the image of the period map (Theorems 3.27 and B.1) are new and were obtained on collaboration with E. Macrì.

Section 2 is devoted to K3 surfaces: complex compact surfaces with vanishing irregularity and whose space of holomorphic 2-forms is generated by a nowhere vanishing form.

After describing their topological invariants, we state some characterizations of ample and very ample line bundles on projective K3 surfaces. We then describe general K3 surfaces with a polarization of low degree (most are complete intersections in a homogeneous space). The Torelli theorem says that K3 surfaces are characterized by their Hodge structure; more precisely, any automorphism between the Hodge structures on their second cohomology groups is induced by an isomorphism. This means that their period map, a regular map between their (quasi-projective) moduli space (whose construction we explain in Section 2.5 for polarized K3 surfaces) and their (quasi-projective) period domain (the quotient of a Hermitian symmetric domain by an arithmetic group of automorphisms) is an open embedding (Section 2.8). Its image is also described in that Section, using the description of the ample cone given in Section 2.4: it is the complement of the union of one or two Heegner divisors. The Torelli theorem is also extremely useful to study automorphisms groups of K3 surfaces. We give examples in Section 2.10.

The rest of the notes deals with hyperkähler manifolds, which are generalizations of K3 surfaces in higher (even) dimensions and for which many results are still unknown. They are defined in Section 3 as simply connected compact Kähler manifolds whose space of holomorphic 2-forms is generated by an everywhere non-degenerate form. Their second integral cohomology group carries a canonical integral valued non-degenerate quadratic form defined by Fujiki and Beauville. Examples are provided by punctual Hilbert schemes of K3 surfaces and generalized Kummer varieties. Most of the results that we state will concern only deformations of the former type (called *hyperkähler manifolds of* $\mathrm{K3}^{[m]}$*-type*).

Polarized hyperkähler manifolds of that type admit quasi-projective moduli spaces whose irreducibility we discuss in Section 3.5. Even in low degrees and dimension 4, their projective embeddings are only known in a few cases, through beautiful but quite involved geometric constructions (Section 3.6). In the next (long) Section 3.7, we define and study two important cones attached to a hyperkähler manifold: the nef and the movable cones. These cones are closed convex cones in a real vector space of dimension the rank of the Picard group of the manifold. Their determination is a very difficult question, only recently settled by works of Bayer, Macrì, Hassett, and Tschinkel. Many examples are given in Section 3.7, essentially when the Picard rank is 2, where their description involves playing with some equations of Pell-type.

The next section (Section 3.8) contains two versions of the Torelli theorem. We miss all the subtleties (and difficulties) of this general result by restricting ourselves to polarized hyperkähler manifolds. Even with the "classical" version, one has to be careful (see Theorem 3.18). When one states the Torelli theorem in terms of the injectivity of a period map (Theorem 3.19), the situation is tricky and more complicated than for K3 surfaces: the moduli spaces, although still quasi-projective, may be reducible, and the period domain is obtained by quotienting (still a Hermitian symmetric domain) by a restricted automorphism group. At the same time, this makes the situation richer: the period domains may have nontrivial involutions and may be isomorphic to each other. This implies that the some moduli spaces of polarized hyperkähler manifolds are birationally isomorphic, a phenomenon which we call *strange duality* (Remarks 3.21 and 3.22).

In the rest of Section 3, we determine explicitly the image of the period map for polarized hyperkähler fourfolds. This involves going through a rather lengthy description of the Heegner divisors in the period domain (Section 3.10). As for K3 surfaces, the image is the complement of a finite number of Heegner divisors which we describe precisely. This result is a simple consequence of the description of the ample cone (which is the interior of the nef cone) given earlier.

In the final Section 4, we use our knowledge of the movable and nef cones of hyperkähler manifolds given in Section 3.7 to describe explicitely the birational and biregular automorphism groups of hyperkähler manifolds of Picard number 1 or 2 in some cases. For Picard number 1 (Section 4.2), the result is a rather simple consequence of the Torelli theorem and some easy lattice-theoretic considerations. For Picard number 2, a general result was proved by Oguiso (Theorem 4.6). We end the section with many explicit calculations.

In the final short Section 5, we use the Torelli theorem (and a deep result of Clozel and Ullmo on Shimura varieties) to prove that in each moduli space of polarized hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type, the points corresponding to Hilbert squares of K3 surfaces form a dense subset (our Proposition 5.1 provides a more explicit statement).

In Appendix A, we go through a few elementary facts about Pell-type equations. In the more difficult Appendix B, written with E. Macrì, we revisit the description of the ample cone of a projective hyperkähler manifold in terms of its *Mukai lattice* and use it to describe explicitly the image of the period map for polarized hyperkähler manifolds in all dimensions.

2. K3 surfaces

Although K3 surfaces in positive characteristics are interesting in their own (and have been used to prove results about complex K3 surfaces), we will restrict ourselves to complex K3 surfaces.[1] The reader will find in [Hu1] a very complete account of these surfaces. The reference [L] is more elementary and covers more or less the same topics as this section (with more details).

2.1. **Definition and first properties.**

Definition 2.1. A *K3 surface* is a compact surface S such that $H^0(S,\Omega_S^2)=\mathbf{C}\omega$, where ω is a nowhere vanishing holomorphic 2-form on S, and $H^1(S,\mathscr{O}_S)=0$.

K3 surfaces are interesting for many reasons:

- they have interesting dynamics: up to bimeromorphism, they are the only compact complex surfaces which can have an automorphism with positive entropy (see footnote 11) and no fixed points;[2]
- they have interesting arithmetic properties;
- they have interesting geometric properties: it is conjectured (and known except when the Picard number is 2 or 4; see [LL]) that any K3 surface contains countably many rational curves.

[1] This strange name was coined by Weil in 1958: "ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire."

[2] A theorem of Cantat ([C]) says that a smooth compact Kähler surface S with an automorphism of positive entropy is bimeromorphic to either $\mathbf{P}^2$, or to a 2-dimensional complex torus, or to an Enriques surface, or to a K3 surface. If the automorphism has no fixed points, S is birational to a projective K3 surface of Picard number greater than 1 and conversely, there is a projective K3 surface of Picard number 2 with a fixed-point-free automorphism of positive entropy (see [Og1] or Theorem 2.16).

Let S be a K3 surface. The exponential sequence

$$0 \to \mathbf{Z} \xrightarrow{\cdot 2i\pi} \mathscr{O}_S \xrightarrow{\exp} \mathscr{O}_S^* \to 1$$

implies that $H^1(S, \mathbf{Z})$ is a subgroup of $H^1(S, \mathscr{O}_S)$, hence $b_1(S) = 0$. Moreover, there is an exact sequence

$$0 \longrightarrow \mathrm{Pic}(S) \longrightarrow H^2(S, \mathbf{Z}) \longrightarrow H^2(S, \mathscr{O}_S). \tag{1}$$

Lemma 2.2. *The Picard group of a K3 surface is torsion-free.*

Proof. Let M be a torsion element in $\mathrm{Pic}(S)$. The Riemann–Roch theorem and Serre duality give

$$h^0(S, M) - h^1(S, M) + h^0(S, M^{-1}) = \chi(S, M) = \chi(S, \mathscr{O}_S) = 2.$$

In particular, either M or M^{-1} has a non-zero section s. If m is a positive integer such that $M^{\otimes m}$ is trivial, the non-zero section s^m of $M^{\otimes m}$ vanishes nowhere, hence so does s. The line bundle M is therefore trivial. □

Since $H^2(S, \mathscr{O}_S)$ is also torsion-free, the lemma implies that $H^2(S, \mathbf{Z})$ is torsion-free, hence so is $H_2(S, \mathbf{Z})$ by Poincaré duality. By the Universal Coefficient Theorem, $H_1(S, \mathbf{Z})$ is torsion-free hence 0 (because $b_1(S) = 0$) , hence so is $H^3(S, \mathbf{Z})$ by Poincaré duality again. So the whole (co)homology of S is torsion-free.

We have $c_2(S) = \chi_{\mathrm{top}}(S) = b_2(S) + 2$. Noether's formula

$$12\chi(S, \mathscr{O}_S) = c_1^2(S) + c_2(S)$$

then implies

$$c_2(S) = 24 \quad , \quad b_2(S) = 22.$$

The abelian group $H^2(S, \mathbf{Z})$ is therefore free of rank 22. The intersection form is unimodular and *even*.[3] Its signature is given by Hirzebruch's formula

$$\tau(S) = \frac{1}{3}(c_1^2(S) - 2c_2(S)) = -16.$$

Recall from Lemma 2.2 that the Picard group is therefore a free abelian group; we let $\rho(S)$ be its rank (the *Picard number* of S).

The Lefschetz $(1,1)$-theorem tells us that $\rho(S) \leq h^1(S, \Omega_S^1)$. If S is Kähler, we have

$$h^1(S, \Omega_S^1) = b_2(S) - 2h^0(S, \Omega_S^2) = 20$$

[3] Let X be a smooth compact real manifold of (real) dimension 4. Its second *Wu class* $v_2(X) \in H^2(X, \mathbf{Z}/2\mathbf{Z})$ is characterized by the property (Wu's formula)

$$\forall x \in H^2(X, \mathbf{Z}/2\mathbf{Z}) \qquad v_2(X) \cup x = x \cup x.$$

The class v_2 can be expressed as $w_2 + w_1^2$, where w_1 and w_2 are the first two Stiefel–Whitney classes. When X is a complex surface, one has $w_1(X) = 0$ and $w_2(X)$ is the reduction modulo 2 of $c_1(X)$ (which is 0 for a K3 surface).

and this holds even if S is not Kähler (which does not happen anyway!).[4] In particular, we have

$$0 \le \rho(S) \le 20.$$

Finally, one can show that all K3 surfaces are diffeomorphic and that they are all simply connected.

Exercise 2.3. Let $S \subset \mathbf{P}^3$ be the Fermat surface defined by the quartic equation $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$.

(a) Show that S is smooth and that it is a K3 surface.

(b) Show that S contains (at least) 48 lines $L_1, \dots, L_{48}$.

(c) How would you show (using a computer) that the rank of the Picard group $\operatorname{Pic}(S)$ is at least 20?

(d) Prove that the rank of the Picard group $\operatorname{Pic}(S)$ is exactly 20.[5]

2.2. **Properties of lines bundles.** Let L be an ample line bundle on S. By Kodaira vanishing, we have $H^1(S, L) = 0$ hence the Riemann–Roch theorem reads

$$h^0(S, L) = \chi(S, L) = \frac{1}{2}L^2 + 2.$$

Most of the times, L is generated by global sections, by the following result that we will not prove.

Theorem 2.4. *Let L be an ample line bundle on a K3 surface S. The line bundle L is generated by global sections if and only if there are no divisors D on S such that $LD = 1$ and $D^2 = 0$.*

The hypothesis of the theorem holds in particular when $\rho(S) = 1$. When L is generated by global sections, it defines a morphism

$$\varphi_L \colon S \longrightarrow \mathbf{P}^{e+1},$$

where $e := \frac{1}{2}L^2$. General elements $C \in |L|$ are smooth irreducible (by Bertini's theorem) curves of genus $e + 1$ and the restriction of φ_L to C is the canonical map $C \to \mathbf{P}^e$.

One can show the morphism φ_L is then an embedding (in which case all smooth curves in $|L|$ are non-hyperelliptic), that is, L is very ample, except in the following cases, where φ_L is a double cover of its image, a smooth surface of minimal degree e in $\mathbf{P}^{e+1}$:

- $L^2 = 2$;
- $L^2 = 8$ and $L = 2D$;
- there is a divisor D on S such that $LD = 2$ and $D^2 = 0$ (all smooth curves in $|L|$ are then hyperelliptic).

The last case does not occur when $\rho(S) = 1$. This implies the following result.

[4] As explained in the proof of [BHPvV, Lemma IV.2.6], there is an inclusion $H^0(S, \Omega_S^1) \subset H^1(S, \mathbf{C})$ for any compact complex surface S, so that $H^0(S, \Omega_S^1) = 0$ for a K3 surface S. We then have $H^2(S, \Omega_S^1) = 0$ by Serre duality and the remaining number $h^1(S, \Omega_S^1)$ can be computed using the Riemann–Roch theorem $\chi(S, \Omega_S^1) = \int_S \operatorname{ch}(\Omega_S^1)\operatorname{td}(S) = \frac{1}{2}c_1^2(\Omega_S^1) - c_2(\Omega_S^1) + 2\operatorname{rank}(\Omega_S^1) = -20$ (because $c_1(\Omega_S^1) = 0$ and $c_2(\Omega_S^1) = 24$).

[5] It was proved by Mizukami in 1975 that the group $\operatorname{Pic}(S)$ is generated by $L_1, \dots, L_{48}$ and that its discriminant is -64.

Theorem 2.5. *Let L be an ample line bundle on a K3 surface. The line bundle $L^{\otimes 2}$ is generated by global sections and the line bundle $L^{\otimes k}$ is very ample for all $k \geq 3$.*

2.3. **Polarized K3 surfaces of low degrees.** A polarization L on a K3 surface S is an isomorphism class of ample line bundles on S or equivalently, an ample class in $\mathrm{Pic}(S)$, which is not divisible in $\mathrm{Pic}(S)$. Its degree is $2e := L^2$. We give below descriptions (mostly due to Mukai; see [Mu1, Mu2, Mu3, Mu4, Mu5, Mu6]) of the morphism $\varphi_L \colon S \to \mathbf{P}^{e+1}$ associated with a general polarized K3 surface (S, L) of degree $2e \leq 14$ (it is a morphism by Theorem 2.4) and of S for $e \in \{8, \ldots, 12, 15, 17, 19\}$.[6]

Several of these descriptions involve the Grassmannian $\mathsf{Gr}(r, n)$, the smooth projective variety of dimension $r(n-r)$ that parametrizes r-dimensional subspaces in $\mathbf{C}^n$, and its vector bundles $\mathscr{S}$, the tautological rank-r subbundle, and $\mathscr{Q}$, the tautological rank-$(n-r)$ quotient bundle. It embeds via the Plücker embedding into $\mathbf{P}(\bigwedge^r \mathbf{C}^n)$ and the restriction of $\mathscr{O}_{\mathbf{P}(\bigwedge^r \mathbf{C}^n)}(1)$ is a generator of $\mathrm{Pic}(\mathsf{Gr}(r, n))$.

$\mathbf{L^2 = 2}$. The morphism $\varphi_L \colon S \to \mathbf{P}^2$ is a double cover branched over a smooth plane sextic curve. Conversely, any such double cover is a polarized K3 surface of degree 2.

$\mathbf{L^2 = 4}$. The morphism $\varphi_L \colon S \to \mathbf{P}^3$ induces an isomorphism between S and a smooth quartic surface. Conversely, any smooth quartic surface in $\mathbf{P}^3$ is a polarized K3 surface of degree 4.

$\mathbf{L^2 = 6}$. The morphism $\varphi_L \colon S \to \mathbf{P}^4$ induces an isomorphism between S and the intersection of a quadric and a cubic. Conversely, any smooth complete intersection of a quadric and a cubic in $\mathbf{P}^4$ is a polarized K3 surface of degree 6.

$\mathbf{L^2 = 8}$. The morphism $\varphi_L \colon S \to \mathbf{P}^5$ induces an isomorphism between S and the intersection of 3 quadrics. Conversely, any smooth complete intersection of 3 quadrics in $\mathbf{P}^5$ is a polarized K3 surface of degree 8.

$\mathbf{L^2 = 10}$. The morphism $\varphi_L \colon S \to \mathbf{P}^6$ is a closed embedding. Its image is obtained as the transverse intersection of the Grassmannian $\mathsf{Gr}(2, 5) \subset \mathbf{P}^9$, a quadric $Q \subset \mathbf{P}^9$, and a $\mathbf{P}^6 \subset \mathbf{P}^9$. Conversely, any such smooth complete intersection is a polarized K3 surface of degree 10.

$\mathbf{L^2 = 12}$. The morphism $\varphi_L \colon S \to \mathbf{P}^7$ is a closed embedding. Its image is obtained as the transverse intersection of the orthogonal Grassmannian[7] $\mathsf{OGr}(5, 10) \subset \mathbf{P}^{15}$ and a $\mathbf{P}^8 \subset \mathbf{P}^{15}$. Conversely, any such smooth complete intersection is a polarized K3 surface of degree 12.

One can also describe $\varphi_L(S)$ as the zero-locus of a general section of the rank-4 vector bundle $\mathscr{O}(1)^{\oplus 2} \oplus \mathscr{S}(2)$ on $\mathsf{Gr}(2, 5)$ ([Mu5, Theorem 9]).

$\mathbf{L^2 = 14}$. The morphism $\varphi_L \colon S \to \mathbf{P}^8$ is a closed embedding. Its image is obtained as the transverse intersection of the Grassmannian $\mathsf{Gr}(2, 6) \subset \mathbf{P}^{14}$ and a $\mathbf{P}^8 \subset \mathbf{P}^{14}$. Conversely, any such smooth complete intersection is a polarized K3 surface of degree 14.

[6] It might be a bit premature to talk about "general polarized K3 surfaces" here but the statements below hold whenever $\mathrm{Pic}(S) \simeq \mathbf{Z}L$, a condition that can be achieved by slightly perturbing (S, L); we will come back to that in Section 2.9.

[7] This is one of the two components of the family of all 5-dimensional isotropic subspaces for a non-degenerate quadratic form on $\mathbf{C}^{10}$.

$\mathbf{L^2 = 16}$. General K3 surfaces of degree 16 are exactly the zero loci of general sections of the of the rank-6 vector bundle $\mathscr{O}(1)^{\oplus 4} \oplus \mathscr{S}(1)$ on $\mathsf{Gr}(3,6)$ ([Mu2, Example 1]).

$\mathbf{L^2 = 18}$. General K3 surfaces of degree 18 are exactly the zero loci of general sections of the rank-7 vector bundle $\mathscr{O}(1)^{\oplus 3} \oplus \mathscr{Q}^\vee(1)$ on $\mathsf{Gr}(2,7)$ ([Mu2, Example 1]).

$\mathbf{L^2 = 20}$. General K3 surfaces of degree 20 are described in [Mu3] as Brill–Noether loci on curves of genus 11.

$\mathbf{L^2 = 22}$. General K3 surfaces of degree 22 are exactly the zero loci of general sections of the rank-10 vector bundle $\mathscr{O}(1) \oplus \mathscr{S}(1)^{\oplus 3}$ on $\mathsf{Gr}(3,7)$ ([Mu5, Theorem 10]).

$\mathbf{L^2 = 24}$. General K3 surfaces of degree 24 are exactly the zero loci of general sections of the rank-10 vector bundle $\mathscr{S}(1)^{\oplus 2} \oplus \mathscr{Q}^\vee(1)$ on $\mathsf{Gr}(3,7)$ ([Mu5, Theorem 1]).

$\mathbf{L^2 = 30}$. Let T be the 12-dimensional GIT quotient of $\mathbf{C}^2 \otimes \mathbf{C}^3 \otimes \mathbf{C}^4$ by the action of $\mathrm{GL}(2) \times GL(3)$ on the first and second factors. There are tautological vector bundles $\mathscr{E}$ and $\mathscr{F}$ on T, of respective ranks 3 and 2. General K3 surfaces of degree 30 are exactly the zero loci of general sections of the rank-10 vector bundle $\mathscr{E}^{\oplus 2} \oplus \mathscr{F}^{\oplus 2}$ on T ([Mu6]).

$\mathbf{L^2 = 34}$. General K3 surfaces of degree 34 are exactly the zero loci of general sections of the rank-10 vector bundle $\bigwedge^2 \mathscr{S}^\vee \oplus \bigwedge^2 \mathscr{Q} \oplus \bigwedge^2 \mathscr{Q}$ on $\mathsf{Gr}(4,7)$ ([Mu5, Theorem 1]).

$\mathbf{L^2 = 38}$. General K3 surfaces of degree 38 are exactly the zero loci of general sections of the rank-18 vector bundle $(\bigwedge^2 \mathscr{S}^\vee)^{\oplus 3}$ on $\mathsf{Gr}(4,9)$.

2.4. **The ample cone of a projective K3 surface.** Let X be a projective manifold. We define the *nef cone*

$$\mathrm{Nef}(X) \subset \mathrm{NS}(X) \otimes \mathbf{R}$$

as the closed convex cone generated by classes of nef line bundles.[8] Its interior $\mathrm{Amp}(X)$ is not empty and consists of ample classes.

Let now S be a projective K3 surface and let M be a line bundle on S with $M^2 \geq -2$. The Riemann–Roch theorem and Serre duality imply

$$h^0(S,M) - h^1(S,M) + h^0(S,M^{-1}) = 2 + \frac{1}{2}M^2 \geq 1,$$

so that either M or M^{-1} has a non-zero section. If we fix an ample class L on S, the line bundle M has a non-zero section if and only if $L \cdot M \geq 0$. Let $\mathrm{Pos}(S)$ be that of the two components of the cone $\{x \in \mathrm{Pic}(S) \otimes \mathbf{R} \mid x^2 > 0\}$ that contains an (hence all) ample class (recall that the signature of the lattice $\mathrm{Pic}(S)$ is $(1, \rho(S) - 1)$).

We set

$$\Delta^+ := \{M \in \mathrm{Pic}(S) \mid M^2 = -2,\ H^0(S,M) \neq 0\}.$$

If L is a ample class, we have $L \cdot M > 0$ for all $M \in \Delta^+$. If $M \in \Delta^+$, one shows that the fixed part of the linear system $|M|$ contains a smooth rational curve C (with $C^2 = -2$).

Theorem 2.6. *Let S be a projective K3 surface. We have*

$$\mathrm{Amp}(S) \quad = \quad \{x \in \mathrm{Pos}(S) \mid \forall M \in \Delta^+ \quad x \cdot M > 0\}$$

[8] The group $\mathrm{NS}(X)$ is the group of line bundles on X modulo numerical equivalence; it is a finitely generated abelian group. When $H^1(X, \mathscr{O}_X) = 0$ (e.g., for K3 surfaces), it is the same as $\mathrm{Pic}(X)$.

$$= \{x \in \mathrm{Pos}(S) \mid \text{ for all smooth rational curves } C \subset S,\ x \cdot C > 0\}.$$

For the proof, which is elementary, we refer to [Hu1, Corollary 8.1.6].

2.5. **Moduli spaces for polarized K3 surfaces.** We will indicate here the main steps for the construction of quasi-projective coarse moduli space for polarized complex K3 surfaces of fixed degree. Historically, the first existence proof was given in 1971 by Pjatecki-Shapiro–Shafarevich and relied on the Torelli theorem (see next section), but it makes more sense to reverse the course of history and explain a construction based on later work of Viehweg.

Let (S, L) be a polarized K3 surface of degree $2e$. The fact that a fixed multiple (here $L^{\otimes 3}$) is very ample implies that S embeds into some projective space of fixed dimension (here $\mathbf{P}^{9e+1}$), with fixed Hilbert polynomial (here $9eT^2+2$). The Hilbert scheme that parametrizes closed subschemes of $\mathbf{P}^{9e+1}$ with that Hilbert polynomial is projective (Grothendieck) and its subscheme $\mathscr{H}$ parametrizing K3 surfaces is open and smooth. The question is now to take the quotient of $\mathscr{H}$ by the canonical action of $\mathrm{PGL}(9e+2)$. The usual technique for taking this quotient, Geometric Invariant Theory, is difficult to apply directly in that case but Viehweg managed to go around this difficulty to avoid a direct check of GIT stability and still construct a quasi-projective coarse moduli space (over $\mathbf{C}$ only! See [Vi]).

Theorem 2.7. *Let e be a positive integer. There exists an irreducible 19-dimensional quasi-projective coarse moduli space $\mathscr{K}_{2e}$ for polarized complex K3 surfaces of degree $2e$.*

The constructions explained in Section 2.3 imply that $\mathscr{K}_{2e}$ is unirational for $e \le 7$. It is in fact known to be unirational for $e \le 19$ and $e \notin \{14, 18\}$ by work of Mukai (and more recently Nuer). At the opposite end, $\mathscr{K}_{2e}$ is of general type for $e \ge 31$ (and for a few lower values of e; see [GHS1, GHS3]).

2.6. **The Torelli theorem.** The Torelli theorem (originally stated and proved for curves) answers the question as to whether a smooth Kähler complex manifold is determined (up to isomorphism) by (part of) its Hodge structure. In the case of polarized K3 surfaces, this property holds.

Theorem 2.8 (Torelli theorem, first version)**.** *Let (S, L) and (S', L') be polarized complex K3 surfaces. If there exists an isometry of lattices*

$$\varphi\colon H^2(S', \mathbf{Z}) \xrightarrow{\sim} H^2(S, \mathbf{Z})$$

such that $\varphi(L') = L$ and $\varphi_{\mathbf{C}}(H^{2,0}(S')) = H^{2,0}(S)$, there exists an isomorphism $\sigma\colon S \xrightarrow{\sim} S'$ such that $\varphi = \sigma^*$.

We will see later that the isomorphism u is uniquely determined by φ (Proposition 2.14).

2.7. **A bit of lattice theory.** A very good introduction to this theory can be found in the superb book [Se]. More advanced results are proved in the difficult article [Ni].

A lattice is a free abelian group Λ of finite rank endowed with an integral valued quadratic form q. Its discriminant group is the finite abelian group

$$D(\Lambda) := \Lambda^\vee / \Lambda,$$

where

$$\Lambda \subset \Lambda^\vee := \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{Z}) = \{x \in \Lambda \otimes \mathbf{Q} \mid \forall y \in \Lambda \quad x \cdot y \in \mathbf{Z}\} \subset \Lambda \otimes \mathbf{Q}.$$

The lattice Λ is *unimodular* if the group $D(\Lambda)$ is trivial; it is *even* if the quadratic form q only takes even values.

If t is an integer, we let $\Lambda(t)$ be the group Λ with the quadratic form tq. Finally, for any integers $r, s \geq 0$, we let I_1 be the lattice $\mathbf{Z}$ with the quadratic form $q(x) = x^2$ and we let $I_{r,s}$ be the lattice $I_1^{\oplus r} \oplus I_1(-1)^{\oplus s}$.

The only unimodular lattice of rank 1 is the lattice I_1. The only unimodular lattices of rank 2 are the lattices $I_{2,0}, I_{1,1}, I_{0,2}$ and the hyperbolic plane U. There is a unique positive definite even unimodular lattice of rank 8, which we denote by E_8. The signature $\tau(\Lambda)$ of an even unimodular lattice Λ is divisible by 8; if Λ is not definite (positive or negative), it is a direct sum of copies of U and E_8 if $\tau(\Lambda) \geq 0$ (resp. of U and $E_8(-1)$ if $\tau(\Lambda) < 0$).

If x is a non-zero element of Λ, we define its divisibility $\gamma(x)$ as the positive generator of the subgroup $x \cdot \Lambda$ of $\mathbf{Z}$. We also consider $x/\gamma(x)$, a primitive (i.e., non-zero and non-divisible) element of $\Lambda^\vee$, and its class $x_* = [x/\gamma(x)] \in D(\Lambda)$, an element of order $\gamma(x)$.

Assume now that the lattice Λ is even. We extend the quadratic form to a $\mathbf{Q}$-valued quadratic form on $\Lambda \otimes \mathbf{Q}$, hence also on $\Lambda^\vee$. We then define a quadratic form $\bar{q} \colon D(\Lambda) \to \mathbf{Q}/2\mathbf{Z}$ as follows: let $x \in \Lambda^\vee, y \in \Lambda$; then $q(x+y) = q(x) + 2x \cdot y + q(y)$, modulo $2\mathbf{Z}$, does not depend on y. We may therefore set

$$\bar{q}([x]) := q(x) \in \mathbf{Q}/2\mathbf{Z}.$$

The *stable orthogonal group* $\widetilde{O}(\Lambda)$ is the kernel of the canonical map

$$O(\Lambda) \longrightarrow O(D(\Lambda), \bar{q}).$$

This map is surjective when Λ is indefinite and $\mathrm{rank}(\Lambda)$ is at least the minimal number of generators of the finite abelian group $D(\Lambda)$ plus 2.

We will use the following result very often (see [GHS2, Lemma 3.5]). When Λ is even unimodular (and contains at least two orthogonal copies of the hyperbolic plane U), it says that the $O(\Lambda)$-orbit of a primitive vector is exactly characterized by the square of this vector.

Theorem 2.9 (Eichler's criterion)**.** *Let Λ be an even lattice that contains at least two orthogonal copies of U. The $\widetilde{O}(\Lambda)$-orbit of a primitive vector $x \in \Lambda$ is determined by the integer $q(x)$ and the element x_* of $D(\Lambda)$.*

2.8. **The period map.** The Torelli theorem (Theorem 2.8) says that a polarized K3 surface is determined by its Hodge structure. We want to express this as the injectivity of a certain morphism, the period map, which we now construct.

Let S be a complex K3 surface. The lattice $(H^2(S, \mathbf{Z}), \cdot)$ was shown in Section 2.1 to be even unimodular with signature -16; by the discussion in Section 2.7, it is therefore isomorphic to the rank-22 lattice

$$\Lambda_{\mathrm{K3}} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

Since we will restrict ourselves to polarized K3 surfaces, we fix, for each positive integer e, a primitive vector $h_{2e} \in \Lambda_{\mathrm{K3}}$ with $h_{2e}^2 = 2e$ (by Eichler's criterion (Theorem 2.9), they are all in the same $O(\Lambda_{\mathrm{K3}})$-orbit). For example, if (u, v) is a standard basis for one copy of U, we may take $h_{2e} = u + ev$. We then have

$$\Lambda_{\mathrm{K3},2e} := h_{2e}^{\perp} = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-2e). \tag{2}$$

Let now (S, L) be a polarized K3 surface of degree $2e$ and let $\varphi\colon H^2(S, \mathbf{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$ be an isometry of lattices such that $\varphi(L) = h_{2e}$ (such an isometry exists by Eichler's criterion). The "period" $p(S, L) := \varphi_{\mathbf{C}}(H^{2,0}(S)) \in \Lambda_{\mathrm{K3}} \otimes \mathbf{C}$ is then in $h_{2e}^{\perp}$; it also satisfies the Hodge–Riemann bilinear relations

$$p(S, L) \cdot p(S, L) = 0 \quad , \quad p(S, L) \cdot \overline{p(S, L)} > 0.$$

This leads us to define the 19-dimensional (non-connected) complex manifold

$$\Omega_{2e} := \{[x] \in \mathbf{P}(\Lambda_{\mathrm{K3},2e} \otimes \mathbf{C}) \mid x \cdot x = 0,\ x \cdot \overline{x} > 0\},$$

so that $p(S, L)$ is in Ω_{2e}. However, the point $p(S, L)$ depends on the choice of the isometry φ, so we would like to consider the quotient of Ω_{2e} by the image of the (injective) restriction morphism

$$\begin{array}{rcl} \{\Phi \in O(\Lambda_{\mathrm{K3}}) \mid \Phi(h_{2e}) = h_{2e}\} & \longrightarrow & O(\Lambda_{\mathrm{K3},2e}) \\ \Phi & \longmapsto & \Phi|_{h_{2e}^{\perp}}. \end{array}$$

It turns out that this image is equal to the special orthogonal group $\widetilde{O}(\Lambda_{\mathrm{K3},2e})$, so we set[9]

$$\mathscr{P}_{2e} := \widetilde{O}(\Lambda_{\mathrm{K3},2e}) \backslash \Omega_{2e}.$$

Everything goes well: by the Baily–Borel theory, $\mathscr{P}_{2e}$ is an irreducible quasi-projective normal non-compact variety of dimension 19 and one can define a *period map*

$$\begin{array}{rcl} \wp_{2e}\colon \mathscr{K}_{2e} & \longrightarrow & \mathscr{P}_{2e} \\ [(S, L)] & \longmapsto & [p(S, L)] \end{array}$$

which is an algebraic morphism. The Torelli theorem now takes the following form.

Theorem 2.10 (Torelli theorem, second version)**.** *Let e be a positive integer. The period map*

$$\wp_{2e}\colon \mathscr{K}_{2e} \longrightarrow \mathscr{P}_{2e}$$

is an open embedding.

It is this description of $\mathscr{K}_{2e}$ (as an open subset of the quotient of the Hermitian symmetric domain Ω_{2e} by an arithmetic subgroup acting properly and discontinuously) that Gritsenko–Hulek–Sankaran used to compute the Kodaira dimension of $\mathscr{K}_{2e}$ ([GHS1, GHS3]).

Let us discuss what the image of $\wp_{2e}$ is. Let $y \in \Lambda_{\mathrm{K3},2e}$ be such that $y^2 = -2$ (one says that y is a *root* of $\Lambda_{\mathrm{K3},2e}$). If a period $p(S, L)$ is orthogonal to y, the latter is, by the Lefschetz $(1,1)$-theorem, the class of a line bundle M with $M^2 = -2$ and $L \cdot M = 0$. Since either $|M|$

[9] This presentation is correct but a bit misleading: Ω_{2e} has in fact two irreducible components (interchanged by complex conjugation) and one usually chooses one component Ω_{2e}^{+} and considers the subgroups of the various orthogonal groups that preserve this component (denoted by O^{+} in [M2, GHS2, GHS3]), so that $\mathscr{P}_{2e} = \widetilde{O}^{+}(\Lambda_{\mathrm{K3},2e}) \backslash \Omega_{2e}^{+}$.

or $|M^{-1}|$ is non-empty (Section 2.4), this contradicts the ampleness of L and implies that the image of the period map is contained in the complement $\mathscr{P}^0_{2e}$ of the image of

$$\bigcup_{y\in\Lambda_{\mathrm{K3},2e},\ y^2=-2} y^\perp \subset \Omega_{2e}$$

in $\mathscr{P}_{2e}$. It turns out that the image of the period map is exactly $\mathscr{P}^0_{2e}$ ([Hu1, Remark 6.3.7]). Let us describe $\mathscr{P}^0_{2e}$ using Eichler's criterion.

Proposition 2.11. *Let e be a positive integer. The image of the period map $\wp_{2e}\colon \mathscr{K}_{2e} \to \mathscr{P}_{2e}$ is the complement of one irreducible hypersurface if $e \not\equiv 1 \pmod 4$ and of two irreducible hypersurfaces if $e \equiv 1 \pmod 4$.*

Proof. By Eichler's criterion, the $\widetilde{O}(\Lambda_{\mathrm{K3},2e})$-orbit of a root y of $\Lambda_{\mathrm{K3},2e}$ is characterized by $y_* \in D(\Lambda_{\mathrm{K3},2e})$. Let w be a generator of $I_1(-2e)$ in a decomposition (2) of $\Lambda_{\mathrm{K3},2e}$ (if (u,v) is a standard basis for one copy of U in Λ_{K3}, we may take $w = u - ev$). We then have $\mathrm{div}(w) = 2e$ and w_* generates $D(\Lambda_{\mathrm{K3},2e}) \simeq \mathbf{Z}/2e\mathbf{Z}$, with $\bar q(w_*) = q\bigl(\frac{1}{2e}w\bigr) = -\frac{1}{2e} \pmod{2\mathbf{Z}}$. A root y has divisibility 1 or 2; if $\mathrm{div}(y) = 1$, we have $y_* = 0$; if $\mathrm{div}(y) = 2$, then $y_* = \frac12 y$ has order 2 in $D(\Lambda_{\mathrm{K3},2e})$, hence $y_* = ew_*$. In the latter case, we have $\bar q(y_*) \equiv q(\frac12 y) = -\frac12$, whereas $\bar q(ew_*) \equiv e^2\bigl(-\frac{1}{2e}\bigr) = -\frac{e}{2}$; this implies $e \equiv 1 \pmod 4$. Conversely, if this condition is realized, we write $e = 4e' + 1$ and we let (u', v') be a standard basis for one copy of U in $\Lambda_{\mathrm{K3},2e}$. The vector

$$y := w + 2(u' + e'v')$$

is then a root with divisibility 2. This proves the proposition. ∎

These irreducible hypersurfaces, or more generally any hypersurface in $\mathscr{P}_{2e}$ which is the (irreducible) image $\mathscr{D}_x$ of a hyperplane section of Ω_{2e} of the form $x^\perp$, for some non-zero $x \in \Lambda_{\mathrm{K3},2e}$, is usually called a *Heegner divisor*. By Eichler's criterion, the Heegner divisors can be indexed by the integer x^2 and the element x_* of $D(\Lambda_{\mathrm{K3},2e})$, for x primitive in $\Lambda_{\mathrm{K3},2e}$.

Remark 2.12 (Period map over the integers)**.** The construction of the period domain and period map can be done over $\mathbf{Q}$, and even over $\mathbf{Z}[\frac12]$. We refer the reader to [Li, Section 8.6] for a summary of results and for references.

Remark 2.13. As noted by Brendan Hassett, the results of Section 2.2 imply, by the same reasoning as in the proof of Proposition 2.11, that, inside $\mathscr{K}_{2e}$, the locus where the polarization is globally generated (resp. very ample) is the complement of the union of finitely many Heegner divisors.

2.9. **The Noether–Lefschetz locus.** Given a polarized K3 surface (S, L) with period $p(S, L) \in \Omega_{2e}$, the Picard group of S can be identified, by the Lefschetz $(1,1)$-theorem, with the saturation of the subgroup of Λ_{K3} generated by h_{2e} and

$$p(S, L)^\perp \cap \Lambda_{\mathrm{K3},2e}.$$

This means that if the period of (S, L) is outside the countable union

$$\bigcup_{x \text{ primitive in } \Lambda_{\mathrm{K3},2e}} \mathscr{D}_x$$

of Heegner divisors, the Picard number $\rho(S)$ is 1 (and $\mathrm{Pic}(S)$ is generated by L). The inverse image in $\mathscr{K}_{2e}$ of this countable union is called the *Noether–Lefschetz locus.*

2.10. **Automorphisms.** Many works deal with automorphism groups of K3 surfaces and we will only mention a couple of related results. Let S be a K3 surface. The first remark is that since $T_S \simeq \Omega^1_S \otimes (\Omega^2_S)^\vee \simeq \Omega^1_S$, we have

$$H^0(S, T_S) \simeq H^0(S, \Omega^1_S) = 0.$$

In particular, the group $\mathrm{Aut}(S)$ of biregular automorphisms of S is discrete (note that since S is a minimal surface which has a unique minimal model, any birational automorphism of S is biregular).

Proposition 2.14. *Let S be a K3 surface. Any automorphism of S that acts trivially on $H^2(S, \mathbf{Z})$ is trivial.*

Sketch of proof. We follow [Hu1, Section 15.1.1]. Let σ be a non-trivial automorphism of S of finite order n that acts trivially on $H^2(S, \mathbf{Z})$ (hence on $H^\bullet(S, \mathbf{Z})$). By the Hodge decomposition, σ also acts trivially on $H^0(S, \Omega^2_S)$, hence $\sigma^*\omega = \omega$. Around any fixed point of σ, there are analytic local coordinates (x_1, x_2) such that $\sigma(x_1, x_2) = (\lambda x_1, \lambda^{-1} x_2)$, where λ is a primitive nth root of 1 [Hu1, Lemma 15.1.4]. In particular, the number $N(\sigma)$ of fixed points of σ is finite.

The holomorphic Lefschetz fixed point formula, which relates $N(\sigma)$ to the trace of the action of σ on the $H^i(S, \mathscr{O}_S)$, implies $N(\sigma) \le 8$ ([Hu1, Corollary 15.1.5]). The topological Lefschetz fixed point formula, which relates $N(\sigma)$ to the trace of the action of σ on the $H^i(S, \mathbf{Z})$, implies that since σ acts trivially on $H^\bullet(S, \mathbf{Z})$, we have $N(\sigma) = \chi_{\mathrm{top}}(S) = 24$ ([Hu1, Corollary 15.1.6]). This gives a contradiction, hence no non-trivial automorphism of S of finite order acts trivially on $H^2(S, \mathbf{Z})$.

To prove that any automorphism σ that acts trivially on $H^2(S, \mathbf{Z})$ has finite order, one can invoke [F, Theorem 4.8] which says that the group of automorphisms of a compact Kähler manifold that fix a Kähler class has only finitely many connected components: this implies in our case that σ has finite order, hence is trivial. □

The conclusion of the proposition can be restated as the injectivity of the map

$$\Psi_S \colon \mathrm{Aut}(S) \longrightarrow O(H^2(S, \mathbf{Z}), \cdot). \tag{3}$$

One may also consider the morphism

$$\overline{\Psi}_S \colon \mathrm{Aut}(S) \longrightarrow O(\mathrm{Pic}(S), \cdot).$$

Even if S is projective, this morphism is not necessarily injective. An easy example is provided by polarized K3 surfaces (S, L) of degree 2: we saw in Section 2.3 that $\varphi_L \colon S \to \mathbf{P}^2$ is a double cover; the associated involution ι of S satisfies $\iota^* L \simeq L$ and when (S, L) is very general, $\mathrm{Pic}(S) = \mathbf{Z}L$ is acted on trivially by ι.

However, the kernel of $\overline{\Psi}_S$ is always finite: if L is an ample line bundle on S, any automorphism σ in the kernel satisfies $\sigma^* L \simeq L$. Since $L^{\otimes 3}$ is very ample (Theorem 2.5), it defines an embedding $\varphi_{L^{\otimes 3}} \colon S \hookrightarrow \mathbf{P}(H^0(S, L^{\otimes 3}))$ and σ acts linearly on $\mathbf{P}(H^0(S, L^{\otimes 3}))$ while globally preserving $\varphi_{L^{\otimes 3}}(S)$. The group $\ker(\overline{\Psi}_S)$ is therefore a quasi-projective linear algebraic group hence has finitely many components. Since it is discrete, it is finite.

Here a simple application of this result: the automorphism group of a K3 surface with Picard number 1 is not very interesting!

Proposition 2.15. *Let S be a K3 surface whose Picard group is generated by an ample class L. The automorphism group of S is trivial when $L^2 \geq 4$ and has order 2 when $L^2 = 2$.*

Proof. Let σ be an automorphism of S. One has $\sigma^* L = L$ and σ^* induces a Hodge isometry of the *transcendental lattice* $L^\perp$. One can show that $\sigma^*|_{L^\perp} = \varepsilon \operatorname{Id}$, where $\varepsilon \in \{-1, 1\}$.[10] Choose as above an identification of the lattice $H^2(S, \mathbf{Z})$ with Λ_{K3} such that $L = u + ev$, where $L^2 = 2e$ and (u, v) is a standard basis of a hyperbolic plane $U \subset \Lambda_{\mathrm{K3}}$. One then has

$$\sigma^*(u + ev) = u + ev \qquad \text{and} \qquad \sigma^*(u - ev) = \varepsilon(u - ev).$$

This implies $2e\sigma^*(v) = (1 - \varepsilon)u + e(1 + \varepsilon)v$, so that $2e \mid (1 - \varepsilon)$. If $e > 1$, this implies $\varepsilon = 1$, hence $\sigma^* = \operatorname{Id}$, and $\sigma = \operatorname{Id}$ by Proposition 2.14. If $e = 1$, there is also the possibility $\varepsilon = -1$, and σ, if non-trivial, is a uniquely defined involution of S. But in that case, such an involution always exists (Section 2.3). □

To actually construct automorphisms of a K3 surface S, one needs to know the image of the map Ψ_S. This is provided by the Torelli theorem, or rather an extended version of Theorem 2.8 to the non-polarized setting: any Hodge isometry of $H^2(S, \mathbf{Z})$ that maps one Kähler class to a Kähler class is induced by an automorphism of S ([Hu1, Theorem 7.5.3]).

Automorphisms get more interesting when the Picard number increases. There is a huge literature on the subject and we will only sketch one construction.

Theorem 2.16 (Cantat, Oguiso)**.** *There exists a projective K3 surface of Picard number 2 with a fixed-point-free automorphism of positive entropy.*[11]

Sketch of proof. One first shows that there exists a projective K3 surface S with Picard lattice isomorphic to the rank-2 lattice $K = \mathbf{Z}^2$ with intersection matrix[12]

$$\begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}.$$

By Theorem 2.6 and the fact that the lattice K does not represent -2, the ample cone of S is one component of its positive cone.

[10] If (S, L) is very general, this follows from standard deformation theory; the general argument is clever and relies on Kronecker's theorem and the fact that 21, the rank of $L^\perp$, is odd ([Hu1, Corollary 3.3.5]).

[11] If σ is an automorphism of a metric space (X, d), we set, for all positive integers m and all $x, y \in X$,

$$d_m(x, y) := \max_{0 \leq i < m} d(\sigma^i(x), \sigma^i(y)).$$

For $\varepsilon > 0$, let $s_m(\varepsilon)$ be the maximal number of disjoint balls in X of radius $\varepsilon/2$ for the distance d_m. The *topological entropy* of σ is defined by

$$h(\sigma) := \lim_{\varepsilon \to 0} \limsup_{m \to \infty} \frac{\log s_m(\varepsilon)}{m} \geq 0.$$

Automorphisms with positive entropy are the most interesting from a dynamical point of view.

[12] This can be deduced as in [H, Lemma 4.3.3] from the Torelli theorem and the surjectivity of the period map, because this lattice does not represent -2; the square-4 class given by the first basis vector is even very ample on S since the lattice does not represent 0 either. Oguiso gives in [Og1] a geometric construction of the surface S as a quartic in $\mathbf{P}^3$. It was later realized in [FGvGvL] that S is a determinantal Cayley surface (meaning that its equation can be written as the determinant of a 4×4-matrix of linear forms) and that the automorphism σ had already been constructed by Cayley in 1870 (moreover, $\operatorname{Aut}(S) \simeq \mathbf{Z}$, generated by σ); that article gives very clear explanations about the various facets of these beautiful constructions.

The next step is to check that the isometry φ of $K \oplus K^\perp$ that acts as the matrix

$$\begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$$

on K and as $-\mathrm{Id}$ on $K^\perp$ extends to an isometry of Λ_{K3} (one can do it by hand as in the proof above). This isometry obviously preserves the ample cone of S, and a variation of the Torelli Theorem 2.8 implies that it is induced by an antisymplectic automorphism σ of S.

Since 1 is not an eigenvalue of φ, the fixed locus of σ contains no curves: it is therefore a finite set of points, whose cardinality c can be computed by the Lefschetz fixed point formula

$$\begin{aligned} c &= \sum_{i=0}^{4} \mathrm{Tr}(\sigma^*|_{H^i(S,\mathbf{Q})}) \\ &= 2 + \mathrm{Tr}(\sigma^*|_{H^2(S,\mathbf{Q})}) \\ &= 2 + \mathrm{Tr}(\sigma^*|_{K\otimes\mathbf{Q}}) + \mathrm{Tr}(\sigma^*|_{K^\perp\otimes\mathbf{Q}}) = 2 + 18 - 20 = 0. \end{aligned}$$

The fact that the entropy $h(\sigma)$ is positive is a consequence of results of Gromov and Yomdim that say that $h(\sigma)$ is the logarithm of the largest eigenvalue of σ^* (here $9 + 4\sqrt{5}$) acting on $H^{1,1}(S, \mathbf{R})$. □

Remark 2.17. It is known that for any K3 surface S, the group $\mathrm{Aut}(S)$ is finitely generated. The proof uses the injective map Ψ_S defined in (3) ([Hu1, Corollary 15.2.4]).

3. Hyperkähler manifolds

We study in the section generalizations of complex K3 surfaces to higher (even) dimensions.

3.1. Definition and first properties.

Definition 3.1. A *hyperkähler manifold* is a simply connected compact Kähler manifold X such that $H^0(X, \Omega_X^2) = \mathbf{C}\omega$, where ω is a holomorphic 2-form on X which is nowhere degenerate (as a skew symmetric form on the tangent space).

These properties imply that the canonical bundle is trivial, the dimension of X is even, say $2m$, and the abelian group $H^2(X, \mathbf{Z})$ is torsion-free.[13] Hyperkähler manifolds of dimension 2 are K3 surfaces. The following result follows from the classification of compact Kähler manifolds with vanishing real first Chern class (see [B2]).

Proposition 3.2. *Let X be a hyperkähler manifold of dimension $2m$ and let ω be a generator of $H^0(X, \Omega_X^2)$. For each $r \in \{0, \dots, 2m\}$, we have*

$$H^0(X, \Omega_X^r) = \begin{cases} \mathbf{C}\omega^{\wedge(r/2)} & \textit{if } r \textit{ is even;} \\ 0 & \textit{if } r \textit{ is odd.} \end{cases}$$

In particular, $\chi(X, \mathscr{O}_X) = m + 1$.

[13] The fact that X is simply connected implies $H_1(X, \mathbf{Z}) = 0$, hence the torsion of $H^2(X, \mathbf{Z})$ vanishes by the Universal Coefficient Theorem.

Hyperkähler manifolds of fixed dimension $2m \geq 4$ do not all have the same topological type. The various possibilities (even the possible Betti numbers) are still unknown and it is also not known whether there are finitely many deformation types.

A fundamental tool in the study of hyperkähler manifolds is the *Beauville–Fujiki* form, a canonical integral non-divisible quadratic form q_X on the free abelian group $H^2(X, \mathbf{Z})$. Its signature is $(3, b_2(X) - 3)$, it is proportional to the quadratic form

$$x \longmapsto \int_X \sqrt{\operatorname{td}(X)}\, x^2,$$

and it satisfies

$$\forall x \in H^2(X, \mathbf{Z}) \qquad x^{2m} = c_X q_X(x)^m,$$

where c_X (the *Fujiki constant*) is a positive rational number and $m := \frac{1}{2}\dim(X)$ (in dimension 2, q_X is of course the cup-product). Moreover, one has $q_X(x) > 0$ for all Kähler (e.g., ample) classes $x \in H^2(X, \mathbf{Z})$.

3.2. **Examples.** A few families of hyperkähler manifolds are known: in each even dimension ≥ 4, two deformations types, which we describe below, were found by Beauville ([B1, Sections 6 and 7]), and two other types (in dimensions 6 and 10) were later found by O'Grady ([O1, O2]).

3.2.1. *Hilbert powers of K3 surfaces.* Let S be a K3 surface and let m be a positive integer. The Hilbert–Douady space $S^{[m]}$ parametrizes analytic subspaces of S of length m. It is a smooth (Fogarty) compact Kähler (Varouchas) manifold of dimension $2m$ and Beauville proved in [B1, Théorème 3] that it is a hyperkähler manifold. When $m \geq 2$, he also computed the Fujiki constant $c_{S^{[m]}} = \frac{(2m)!}{m!2^m}$ and the second cohomology group

$$H^2(S^{[m]}, \mathbf{Z}) \simeq H^2(S, \mathbf{Z}) \oplus \mathbf{Z}\delta,$$

where 2δ is the class of the divisor in $S^{[m]}$ that parametrizes non-reduced subspaces. This decomposition is orthogonal for the Beauville form $q_{S^{[m]}}$, which restricts to the intersection form on $H^2(S, \mathbf{Z})$ and satisfies $q_{S^{[m]}}(\delta) = -2(m-1)$. In particular, we have

$$\begin{aligned}(H^2(S^{[m]}, \mathbf{Z}), q_{S^{[m]}}) &\simeq \Lambda_{\mathrm{K3}} \oplus I_1(-2(m-1)) \\ &\simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-2(m-1)) =: \Lambda_{\mathrm{K3}^{[m]}}.\end{aligned} \tag{4}$$

The second Betti number of $S^{[m]}$ is therefore 23. This is the maximal possible value for all hyperkähler fourfolds ([Gu]) and sixfolds ([S, Theorem 3]). Odd Betti numbers of $S^{[m]}$ all vanish.

The geometric structure of $S^{[m]}$ is explained in [B1, Section 6]. It is particularly simple when $m = 2$: the fourfold $S^{[2]}$ is the quotient by the involution that interchanges the two factors of the blow-up of the diagonal in S^2.

Finally, any line bundle M on S induces a line bundle on each $S^{[m]}$; we denote it by M_m.

3.2.2. *Generalized Kummer varieties.* Let A be a complex torus of dimension 2. The Hilbert–Douady space $A^{[m+1]}$ again carries a nowhere-degenerate holomorphic 2-form, but it is not simply connected. We consider instead the sum morphism

$$\begin{array}{ccc} A^{[m+1]} & \longrightarrow & A \\ (a_1, \dots, a_m) & \longmapsto & a_1 + \dots + a_m \end{array}$$

and the inverse image $K_m(A)$ of $0 \in A$. Beauville proved in [B1, Théorème 4] that it is a hyperkähler manifold (of dimension $2m$). When $m = 1$, the surface $K_1(A)$ is isomorphic to the blow-up of the surface $A/\pm 1$ at its 16 singular points; this is the Kummer surface of A. For this reason, the $K_m(A)$ are called generalized Kummer varieties. When $m \geq 2$, we have $c_{K_m(A)} = \frac{(2m)!(m+1)}{m!2^m}$ and there is again a decomposition

$$H^2(K_m(A), \mathbf{Z}) \simeq H^2(A, \mathbf{Z}) \oplus \mathbf{Z}\delta,$$

which is orthogonal for the Beauville form $q_{K_m(A)}$, and $q_{K_m(A)}(\delta) = -2(m+1)$. The second Betti number of $K_m(A)$ is therefore 7 and

$$(H^2(K_m(A), \mathbf{Z}), q_{K_m(A)}) \simeq U^{\oplus 3} \oplus I_1(-2(m+1)) =: \Lambda_{K_m}.$$

As for all hyperkähler manifolds, the first Betti number vanishes, but not all odd Betti numbers vanish (for example, one has $b_3(K_2(A)) = 8$).

3.3. **The Hirzebruch–Riemann–Roch theorem.** The Hirzebruch–Riemann–Roch theorem takes the following form on hyperkähler manifolds.

Theorem 3.3 (Huybrechts)**.** *Let X be a hyperkähler manifold of dimension $2m$. There exist rational constants $a_0, a_2, \dots, a_{2m}$ such that, for every line bundle M on X, one has*

$$\chi(X, M) = \sum_{i=0}^{m} a_{2i} q_X(M)^i.$$

The relevant constants have been computed for the two main series of examples:

- when X is the mth Hilbert power of a K3 surface (or a deformation), we have (Ellingsrud–Göttsche–M. Lehn)

$$\chi(X, M) = \binom{\frac{1}{2}q_X(M) + m + 1}{m}; \tag{5}$$

- when X is a generalized Kummer variety of dimension $2m$ (or a deformation), we have (Britze)

$$\chi(X, M) = (m+1)\binom{\frac{1}{2}q_X(M) + m}{m}.$$

3.4. **Moduli spaces for polarized hyperkähler manifolds.** Quasi-projective coarse moduli spaces for polarized hyperkähler manifolds (X, H) of fixed dimension $2m$ and fixed degree H^{2m} can be constructed using the techniques explained in Section 2.5: Matsusaka's Big Theorem implies that there is a positive integer $k(m)$ such that, for any hyperkähler manifold

X of dimension $2m$ and any ample line bundle H on X, the line bundle $H^{\otimes k}$ is very ample for all $k \geq k(m)$,[14] and Viehweg's theorem works in any dimension.

Theorem 3.4. *Let m and d be positive integers. There exists a quasi-projective coarse moduli space for polarized complex hyperkähler manifolds of dimension $2m$ and degree d.*

The dimension of the moduli space at a point (X, H) is $h^1(X, T_X) - 1 = h^{1,1}(X) - 1 = b_2(X) - 3$. The matter of its irreducibility will be discussed in the next section.

3.5. **Hyperkähler manifolds of** $\mathrm{K3}^{[m]}$**-type.** A hyperkähler manifold X is said to be *of* $\mathrm{K3}^{[m]}$*-type* if it is a smooth deformation of the mth Hilbert power of a K3 surface. This fixes the topology of X and its Beauville form. In particular, we have

$$(H^2(X, \mathbf{Z}), q_X) \simeq \Lambda_{\mathrm{K3}^{[m]}} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus I_1(-(2m-2)).$$

Let ℓ be a generator for $I_1(-(2m-2))$. The lattice $\Lambda_{\mathrm{K3}^{[m]}}$ has discriminant group $\mathbf{Z}/(2m-2)\mathbf{Z}$, generated by $\ell_* = \ell/(2m-2)$, with $\bar{q}(\ell_*) = -1/(2m-2) \pmod{2\mathbf{Z}}$.

A *polarization type* is the $O(\Lambda_{\mathrm{K3}^{[m]}})$-orbit of a primitive element h with positive square. It certainly determines the positive integers $2n := h^2$ and $\gamma := \mathrm{div}(h)$; the converse is not true in general, but it is when $\gamma = 2$ or when $\gcd(\frac{2n}{\gamma}, \frac{2m-2}{\gamma}, \gamma) = 1$ ([GHS2, Corollary 3.7 and Example 3.10]), e.g., when $\gcd(n, m-1)$ is square-free and odd.

If we write $h = ax + b\ell$, where $a, b \in \mathbf{Z}$ are relatively prime and x is primitive in Λ_{K3}, we have $\mathrm{div}(h) = \gcd(a, 2m-2)$.

Even fixing the polarization type does not always give rise to *irreducible* moduli spaces of polarized hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type, but the number of irreducible components is known ([Ap1, Corollary 2.4 and Proposition 3.1]).[15] We have in particular the following result.

[14] The integer $k(m)$ was made explicit (but very large) by work of Demailly and Siu, but is still far from the value $k(m) = 2m + 2$ conjectured by Fujita in general, or for the optimistic value $k(m) = 3$ conjectured (for hyperkähler manifolds) by Huybrechts ([Hu1, p. 34]) and O'Grady.

[15] Apostolov computed the number of irreducible components of the moduli space ${}^m\mathscr{M}_{2n}^{(\gamma)}$. From [Ap1, Proposition 3.1] (note that the first two cases of [Ap1, Proposition 3.2] have an unnecessary supplementary hypothesis and the last case is wrong), we get some cases where ${}^m\mathscr{M}_{2n}^{(\gamma)}$ is irreducible non-empty (if p is a prime and d a non-zero integer, we denote by $v_p(d)$ the exponent of p in the prime decomposition of d):

- $\gamma = 1$ (see [GHS2, Example 3.8]);
- $\gamma = 2$ and $n + m \equiv 1 \pmod 4$ (see [GHS2, Example 3.10]);
- $\gamma = 3$ and $9 \mid \gcd(m-1, n)$;
- $\gamma = 4$, and
  - either $v_2(m-1) = v_2(n) = 2$ and $n + m \equiv 1 \pmod{16}$;
  - either $v_2(m-1) = v_2(n) = 3$;
  - or $16 \mid \gcd(m-1, n)$;
- $\gamma = p^a$ (p odd prime, $a > 0$), $v_p(m-1) = v_p(n) = a$, and $-(m-1)/n$ is a square modulo p^a;
- $\gamma = 2^a$ ($a \geq 2$), $v_2(m-1) = v_2(n) = a-1$, and $-(m-1)/n$ is a square modulo 2^{a+1}.

At the other end, when $\gamma = p_1 \cdots p_r$, where $p_1, \dots, p_r$ are distinct odd primes, $m - 1 = a\gamma$, and $n = b\gamma$, where $\gcd(a, b, \gamma) = 1$ and $-ab$ is a square modulo n, the polarization type is determined by n and γ (because $\gcd(\frac{2n}{\gamma}, \frac{2m-2}{\gamma}, \gamma) = 1$) but the number of irreducible components of ${}^m\mathscr{M}_{2n}^{(\gamma)}$ is 2^{r-1} ([Ap1, Remark 3.3(1)]).

Theorem 3.5 (Gritsenko–Hulek–Sankaran, Apostolov). *Let n and m be integers with $m \geq 2$ and $n > 0$. The quasi-projective moduli space ${}^m\mathscr{M}_{2n}^{(\gamma)}$ which parametrizes hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type with a polarization of square $2n$ and divisibility γ is irreducible of dimension 20 whenever $\gamma = 1$, or $\gamma = 2$ and $n + m \equiv 1 \pmod 4$.*

We can work out in those two cases what the lattice $h^\perp$ is. Let (u, v) be a standard basis for a copy of U in $\Lambda_{\mathrm{K3}^{[m]}}$ and set $M := U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$.

- When $\gamma = 1$, we have $h_* = 0$ and we may take $h = u + nv$, so that

$$h^\perp \simeq M \oplus I_1(-(2m-2)) \oplus I_1(-2n) =: \Lambda^{(1)}_{\mathrm{K3}^{[m]},2n}, \tag{6}$$

  with discriminant group $\mathbf{Z}/(2m-2)\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$.[16]
- When $\gamma = 2$ (and $n + m \equiv 1 \pmod 4$), we have $h_* = (m-1)\ell_*$ and we may take $h = 2(u + \frac{n+m-1}{4}v) + \ell$, where (u, v) is a standard basis for a copy of U inside $\Lambda_{\mathrm{K3}^{[m]}}$, so that

$$h^\perp \simeq M \oplus \begin{pmatrix} -(2m-2) & -(m-1) \\ -(m-1) & -\frac{n+m-1}{2} \end{pmatrix} =: \Lambda^{(2)}_{\mathrm{K3}^{[m]},2n}, \tag{7}$$

  with discriminant group of order $n(m-1)$ and isomorphic to $\mathbf{Z}/(m-1)\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ when n (hence also $m-1$) is odd, or when $n = m-1$ (and n even).[17]

Remark 3.6. Note in both cases the symmetry $\Lambda^{(\gamma)}_{\mathrm{K3}^{[m]},2n} \simeq \Lambda^{(\gamma)}_{\mathrm{K3}^{[n+1]},2(m-1)}$: it is obvious when $\gamma = 1$; when $\gamma = 2$, the change of coordinates $(x, y) \leftrightarrow (-x, 2x + y)$ interchanges the matrices $\begin{pmatrix} 2m-2 & m-1 \\ m-1 & \frac{n+m-1}{2} \end{pmatrix}$ and $\begin{pmatrix} 2(n+1)-2 & (n+1)-1 \\ (n+1)-1 & \frac{(m-1)+(n+1)-1}{2} \end{pmatrix}$.

Another way to see this is to note that the lattice $\Lambda_{\mathrm{K3}^{[m]}}$ is the orthogonal in the larger even unimodular lattice $\widetilde{\Lambda}_{\mathrm{K3}}$ defined in (11) of any primitive vector of square $2m-2$. The lattice $h^\perp$ considered above is then the orthogonal in $\widetilde{\Lambda}_{\mathrm{K3}}$ of a (possibly non-primitive) rank-2 lattice $\Lambda_{m-1,n}$ with intersection matrix $\begin{pmatrix} 2m-2 & 0 \\ 0 & 2n \end{pmatrix}$. The embedding of $\Lambda_{m-1,n}$ is primitive if and only if $\gamma = 1$ and any two such embedding differ by an isometry of $\widetilde{\Lambda}_{\mathrm{K3}}$; the symmetry is then explained by the isomorphism $\Lambda_{m-1,n} \simeq \Lambda_{n,m-1}$. The embedding of $\Lambda_{m-1,n}$ has index 2 in its saturation if and only if $\gamma = 2$; this explains the symmetry in the same way ([Ap1, Proposition 2.2]).

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[16]One has $h^\perp \simeq M \oplus \mathbf{Z}\ell \oplus \mathbf{Z}(u - nv)$, hence $D(h^\perp) \simeq \mathbf{Z}/(2m-2)\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$, with generators $\frac{1}{2m-2}\ell$ and $\frac{1}{2n}(u - nv)$, and intersection matrix $\begin{pmatrix} -\frac{1}{2m-2} & 0 \\ 0 & -\frac{1}{2n} \end{pmatrix}$.

[17]We have $\Lambda^{(2)}_{\mathrm{K3}^{[m]},2n} \simeq M \oplus \langle e_1, e_2 \rangle$, with $e_1 = (m-1)v + \ell$ and $e_2 := -u + \frac{n+m-1}{4}v$, with intersection matrix as desired. It contains $M \oplus \mathbf{Z}e_1 \oplus \mathbf{Z}(e_1 - 2e_2) \simeq \Lambda^{(1)}_{\mathrm{K3}^{[m]},2n}$ as a sublattice of index 2 and discriminant group $\mathbf{Z}/(2m-2)\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$. This implies that $D(h^\perp)$ is the quotient by an element of order 2 of a subgroup of index 2 of $\mathbf{Z}/(2m-2)\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$. When n is odd, it is therefore isomorphic to $\mathbf{Z}/(m-1)\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$. One can choose as generators of each factor $\frac{1}{m-1}e_1$ and $\frac{1}{n}(e_1 - 2e_2)$, with intersection matrix $\begin{pmatrix} -\frac{2}{m-1} & 0 \\ 0 & -\frac{2}{n} \end{pmatrix}$. When $n = m-1$ (which implies n even), the lattices $(\mathbf{Z}^2, \begin{pmatrix} -2n & -n \\ -n & -n \end{pmatrix})$ and $(\mathbf{Z}^2, \begin{pmatrix} -n & 0 \\ 0 & -n \end{pmatrix})$ are isomorphic, hence $D(h^\perp) \simeq \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$, with generators $\frac{1}{n}(e_1 - e_2)$ and $\frac{1}{n}e_2$, and intersection matrix $\begin{pmatrix} -\frac{1}{n} & 0 \\ 0 & -\frac{1}{n} \end{pmatrix}$.

More generally, one checks that $D(h^\perp) \simeq \mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/(m-1)\mathbf{Z}$ if and only if $v_2(n) = v_2(m-1)$. If $v_2(n) > v_2(m-1)$, one has $D(h^\perp) \simeq \mathbf{Z}/2n\mathbf{Z} \times \mathbf{Z}/((m-1)/2)\mathbf{Z}$, and analogously if $v_2(m-1) > v_2(n)$.

3.6. **Projective models of hyperkähler manifolds.** In this section, we consider exclusively hyperkähler manifolds (mostly fourfolds) X of $\mathrm{K3}^{[m]}$-type with a polarization H of divisibility $\gamma \in \{1,2\}$ and $q_X(H) = 2n$ (that is, the pair (X,H) represents a point of the irreducible moduli space ${}^m\mathscr{M}_{2n}^{(\gamma)}$; the case $\gamma = 2$ only occurs when $n + m \equiv 1 \pmod 4$). We want to know whether H is very ample (at least for general (X,H)) and describe the corresponding embedding of X into a projective space. We start with a general construction.

Let $S \subset \mathbf{P}^{e+1}$ be a K3 surface. There is a morphism

$$\varphi_2 \colon S^{[2]} \longrightarrow \mathsf{Gr}(2, e+2)$$

that takes a pair of points to the line that it spans in $\mathbf{P}^{e+1}$. More generally, if the line bundle L defining the embedding $S \subset \mathbf{P}^{e+1}$ is $(m-1)$-*very ample,*[18] one can define a morphism

$$\varphi_m \colon S^{[m]} \longrightarrow \mathsf{Gr}(m, e+2)$$

sending a 0-dimensional subscheme of S of length m to its linear span in $\mathbf{P}^{e+1}$, and φ_m is an embedding if and only if L is m-very ample. The pull-back by φ_m of the Plücker line bundle on the Grassmannian has class $L_m - \delta$ on $S^{[m]}$ (with the notation of Section 3.2.1).

Proposition 3.7. *Let (S,L) be a polarized K3 surface with* $\mathrm{Pic}(S) \simeq \mathbf{Z}L$ *and $L^2 = 2e$. The line bundle $L^{\otimes a}$ is k-very ample if and only if either $a = 1$ and $k \le e/2$, or $a \ge 2$ and $k \le 2(a-1)e - 2$.*

Proof. We follow the proof of [BCNS, Proposition 3.1] and use the numerical characterization of k-ample line bundles on S given in [Kn, Theorem 1.1]. Set $H := L^{\otimes a}$; that theorem implies the following.

If $a = 1$ and $k \le e/2$, the line bundle H is k-very ample unless there exist a positive integer n and a non-zero divisor $D \in |L^{\otimes n}|$ satisfying various properties, including $2D^2 \le HD$, which is absurd. If $a \ge 2$ and $k \le 2e(a-1) - 2$, we have $H^2 = 2ea^2 \ge 2e(4a-4) > 4k$, hence H is k-very ample unless there exist a non-zero divisor $D \in |L^{\otimes n}|$ satisfying various properties, including $2D^2 \le HD$, i.e., $2n \le a/2$, and $HD \le D^2 + k + 1$, i.e., $2ane \le 2en^2 + k + 1$. These two inequalities imply $2e(a-1) \le 2en(a-n) \le k+1$, which contradicts our hypothesis. The divisor D therefore does not exist and this proves the proposition.

The proof that these conditions are optimal is left to the reader. □

Corollary 3.8. *Let (S,L) be a polarized K3 surface with* $\mathrm{Pic}(S) \simeq \mathbf{Z}L$ *and $L^2 = 2e$. The line bundle $aL_m - \delta$ on $S^{[m]}$ is base-point-free if and only if either $a = 1$ and $m - 1 \le e/2$, or $a \ge 2$ and $m \le 2(a-1)e - 1$; it is very ample if and only if either $a = 1$ and $m \le e/2$, or $a \ge 2$ and $m \le 2(a-1)e - 2$.*

Let us restrict ourselves to the case $a = 1$ (resp. $a = 2$). The class $aL_m - \delta$ then has divisibility a and square $2(e - m + 1)$ (resp. $2(4e - m + 1)$); it is very ample when $e \ge 2m$ (resp. $e \ge (m+2)/2$).

Corollary 3.9. *Let m, n, and γ be integers with $m \ge 2$, $n \ge 1$, and $\gamma \in \{1,2\}$. Let (X,H) be a polarized hyperkähler $2m$-fold corresponding to a general point of the (irreducible) moduli space ${}^m\mathscr{M}_{2n}^{(\gamma)}$.*

[18] A line bundle L on S is m-very ample if, for every 0-dimensional scheme $Z \subset S$ of length $\le m+1$, the restriction map $H^0(S,L) \to H^0(Z, L|_Z)$ is surjective (in particular, 1-very ample is just very ample).

- *When $\gamma = 1$, the line bundle H is base-point-free when $n \geq m-1$ and very ample when $n \geq m+1$.*
- *When $\gamma = 2$, the line bundle H is very ample.*

When H is very ample it defines an embedding

$$X \hookrightarrow \mathbf{P}^{\binom{n+m+1}{m}-1}.$$

Proof. Assume $\gamma = 1$ (the proof in the case $\gamma = 2$ is completely similar and left to the reader). If (S, L) is a very general K3 surface of degree $2e$, the class $L_m - \delta$ on $S^{[m]}$ has divisibility 1 and square $2e - (2m-2) =: 2n$. By Corollary 3.8, it is base-point-free as soon as $2(m-1) \leq e = n + m - 1$. It then defines a morphism

$$S^{[m]} \xrightarrow{\varphi_m} \mathsf{Gr}(m, n+m+1) \xhookrightarrow{\text{Plücker}} \mathbf{P}^{\binom{n+m+1}{m}-1}$$

which is the morphism associated with the complete linear system $|L_m - \delta|$ (compare with (5)). By Corollary 3.8 again, this morphism is a closed embedding when $2m \leq e = n + m - 1$.

Since base-point-freeness and very ampleness are open properties, they still hold for a general deformation of $(S^{[m]}, L_m - \delta)$, that is, for a general element of ${}^m\mathscr{M}_{2n}^{(1)}$. ☐

Example 3.10. A general polarized K3 surface (S, L) of degree 4 is a smooth quartic surface in $\mathbf{P}^3$. Points Z_1 and Z_2 of $S^{[2]}$ have same image by $\varphi_2 \colon S^{[2]} \to \mathsf{Gr}(2,4)$ if and only if they span the same line. If (S, L) is very general, S contains no lines and φ_2 is finite of degree $\binom{4}{2}$ (so that the class $L_2 - \delta$ is ample on $S^{[2]}$, of square 2).

Example 3.11. A general polarized K3 surface (S, L) of degree 6 is the intersection of a smooth quadric Q and a cubic C in $\mathbf{P}^4$. Two points Z_1 and Z_2 of $S^{[2]}$ have same image by $\varphi_2 \colon S^{[2]} \to \mathsf{Gr}(2,5)$ if and only if they span the same line. If $Z_1 \neq Z_2$, this line lies in Q. Conversely, if (S, L) is very general, S contains no lines, any line contained in Q (and there is a $\mathbf{P}^3$ of such lines) meets C in 3 points and gives rise to 3 points of $S^{[2]}$ identified par φ_2. The morphism φ_2 is therefore finite, birational, but not an embedding (so that the class $L_2 - \delta$ is ample, but not very ample, on $S^{[2]}$).

The rational map $\varphi_3 \colon S^{[3]} \dashrightarrow \mathsf{Gr}(3,5)$ is not a morphism; it is dominant of degree $\binom{6}{3}$.

3.6.1. *Hyperkähler fourfolds of low degrees.* We review the known descriptions of the morphism $\varphi_H \colon X \to \mathbf{P}$ for (X, H) general in ${}^2\mathscr{M}_{2n}^{(\gamma)}$ and small n (recall that the case $\gamma = 2$ only occurs when $n \equiv -1 \pmod 4$). By (5) and Kodaira vanishing, we have

$$h^0(X, H) = \chi(X, H) = \binom{n+3}{2}.$$

$\mathbf{q(H) = 2}$. O'Grady showed that for (X, H) general in ${}^2\mathscr{M}_2^{(1)}$, the map $\varphi_H \colon X \to \mathbf{P}^5$ is a morphism (as predicted by Corollary 3.9) which is a double ramified cover of a singular sextic hypersurface called an *EPW sextic* (for Eisenbud–Popescu–Walter). Any such smooth *double EPW sextic* is a polarized hyperkähler fourfold of degree 2. We saw in Example 3.10 (non-general) examples where the morphism φ_H is 6:1 onto the quadric $\mathsf{Gr}(2,4) \subset \mathbf{P}^5$.

$\mathbf{q(H) = 4}$. There is no geometric description of general elements (X, H) of ${}^2\mathscr{M}_4^{(1)}$. In particular, it is not known whether $\varphi_H \colon X \to \mathbf{P}^9$ is an embedding (e.g., whether H is very ample).

Example 3.11 shows that φ_H is birational onto its image, which therefore has degree 48. The pairs $(X,H)=(S^{[2]},L_2-\delta)$, where (S,L) is a polarized K3 surface of degree 6, form a hypersurface in ${}^2\mathscr{M}_4^{(1)}$; for these pairs, $\varphi_H(X)$ is a non-normal fourfold in $\mathsf{Gr}(2,5)\subset \mathbf{P}^9$. The general elements (X,H) of another hypersurface in ${}^2\mathscr{M}_4^{(1)}$ where described in [IKKR2]; for those pairs, the morphism φ_H is a double cover of a singular quartic section of the cone over the Segre embedding $\mathbf{P}^2\times\mathbf{P}^2\hookrightarrow \mathbf{P}^8$ (a fourfold of degree 24).

Note that $\dim(\operatorname{Sym}^2 H^0(X,H))=\binom{9+2}{2}$, whereas $H^0(X,H^{\otimes 2})=\binom{8+3}{2}$ by (5). One would expect the canonical map

$$\operatorname{Sym}^2 H^0(X,H)\longrightarrow H^0(X,H^{\otimes 2})$$

to be an isomorphism for (X,H) general in ${}^2\mathscr{M}_4^{(1)}$, but this does not hold for the two families of examples we just described.

$\mathbf{q(H)=6,\gamma=1}$. There is no geometric description of general elements (X,H) of ${}^2\mathscr{M}_6^{(1)}$. The morphism $\varphi_H\colon X\to \mathbf{P}^{14}$ is a closed embedding by Corollary 3.9. Moreover, the pairs $(S^{[2]},L_2-\delta)$, where (S,L) is a polarized K3 of degree 8, form a hypersurface in ${}^2\mathscr{M}_6^{(1)}$.

$\mathbf{q(H)=6,\gamma=2}$. General elements (X,H) of ${}^2\mathscr{M}_6^{(2)}$ can be described as follows. Let $W\subset\mathbf{P}^5$ be a smooth cubic hypersurface. Beauville–Donagi showed in [BD] that the family $F(W)\subset \mathsf{Gr}(2,6)\subset\mathbf{P}^{14}$ of lines contained in W is a hyperkähler fourfold and that the Plücker polarization H has square 6 and divisibility 2. General elements of ${}^2\mathscr{M}_6^{(2)}$ are of the form $(F(W),H)$. We have $h^0(X,H)=\binom{6}{2}=15$, and φ_H is the closed embedding $F(W)\subset\mathsf{Gr}(2,6)\subset\mathbf{P}^{14}$; in particular, H is very ample.

Example 3.11 shows that the pairs $(S^{[2]},2L_2-\delta)$, where (S,L) is a polarized K3 surface of degree 2, form a hypersurface in ${}^2\mathscr{M}_6^{(2)}$. This hypersurface is actually disjoint from the family of varieties of lines in cubic fourfolds described above. It is interesting to mention that Beauville–Donagi proved that $F(W)$ is a hyperkähler fourfold by exhibiting a codimension-1 family of cubics W for which $(F(W),H)$ is isomorphic to $(S^{[2]},2L_2-5\delta)$, where (S,L) is a polarized K3 surface of degree 14 (we check $q_{S^{[2]}}(2L_2-5\delta)=2^2\cdot 14+5^2(-2)=6$).

$\mathbf{q(H)=22,\gamma=2}$. General elements (X,H) of ${}^2\mathscr{M}_{22}^{(2)}$ were described by Debarre–Voisin in [DV] as follows. Let $\mathscr{U}$ be the tautological rank-6 subbundle on the Grassmannian $\mathsf{Gr}(6,10)$. Any smooth zero locus $X\subset\mathsf{Gr}(6,10)$ (of the expected dimension 4) of a section of $\mathscr{E}:=\bigwedge^3\mathscr{U}^\vee$ is a hyperkähler fourfold, the Plücker polarization has square 22 and divisibility 2, and general elements of ${}^2\mathscr{M}_{22}^{(2)}$ are of this form.

One checks using the Koszul complex for $\mathscr{E}$ that $X\subset\mathsf{Gr}(6,10)\subset\mathbf{P}(\bigwedge^6\mathbf{C}^{10})$ is actually contained in a linear subspace $\mathbf{P}^{90}\subset\mathbf{P}(\bigwedge^6\mathbf{C}^{10})$.[19] By (5), we have $h^0(X,H)=\binom{14}{2}=91$ and φ_H is the closed embedding $X\hookrightarrow\mathbf{P}^{90}$; in particular, H is very ample, as predicted by Corollary 3.9.

[19] We need to compute $h^0(\mathsf{Gr}(6,10),\mathscr{I}_X(1))$. Only two terms from the Koszul complex contribute: $H^0(\mathsf{Gr}(6,10),\mathscr{E}^\vee(1))\simeq\bigwedge^3\mathbf{C}^{10\vee}$, of dimension $\binom{10}{3}=120$, and $H^0(\mathsf{Gr}(6,10),\bigwedge^2\mathscr{E}^\vee(1))\simeq H^0(\mathsf{Gr}(6,10),\bigwedge^6\mathscr{U}(1))\simeq H^0(\mathsf{Gr}(6,10),\mathscr{O}_{\mathsf{Gr}(6,10)})$, of dimension 1. Hence $H^0(\mathsf{Gr}(6,10),\mathscr{I}_X(1))$ has dimension $120-1=\dim(\bigwedge^6\mathbf{C}^{10})-91$ (thanks to L. Manivel for doing these computations).

$\mathbf{q(H) = 38, \gamma = 2}$. General elements (X, H) of ${}^2\mathscr{M}_{38}^{(2)}$ can be described as follows: for a general cubic polynomial P in 6 variables,

$$\mathrm{VSP}(P, 10) := \overline{\{(\ell_1, \dots, \ell_{10}) \in \mathrm{Hilb}^{10}(\mathbf{P}^5) \mid P \in \langle \ell_1^3, \dots, \ell_{10}^3 \rangle\}}$$

is a (smooth) hyperkähler fourfold with a natural embedding into $\mathsf{Gr}(4, \bigwedge^4 \mathbf{C}^6)$ ([IR]). It was checked in [Mo1, Proposition 1.4.16] that the Plücker polarization restricts to a polarization H of square 38 and divisibility 2 (by checking that this polarization is $2L_2 - 3\delta$ when $\mathrm{VSP}(P, 10)$ is isomorphic to the Hilbert square of a very general polarized K3 surface (S, L) of degree 14). A general element of ${}^2\mathscr{M}_{38}^{(2)}$ is of the form $(\mathrm{VSP}(P, 10), H)$.

The line bundle H is very ample by Corollary 3.9, and $h^0(X, H) = \binom{22}{2} = 231$. It is likely that the embedding $X \hookrightarrow \mathsf{Gr}(4, \bigwedge^4 \mathbf{C}^6) \xhookrightarrow{\text{Plücker}} \mathbf{P}(\bigwedge^4(\bigwedge^4 \mathbf{C}^6))$ factors as $X \xhookrightarrow{\varphi_H} \mathbf{P}^{230} \subset \mathbf{P}(\bigwedge^4(\bigwedge^4 \mathbf{C}^6))$.

3.6.2. *An example in dimension 6* ($\mathbf{m = 3, q(H) = 4, \gamma = 2}$). General elements (X, H) of ${}^3\mathscr{M}_4^{(2)}$ are described in [IKKR1, Theorem 1.1] as double covers of certain degeneracy loci $D \subset \mathsf{Gr}(3, 6) \subset \mathbf{P}^{19}$. By (5), we have $h^0(X, H) = 20$, and φ_H factors as

$$\varphi_H \colon X \xrightarrow{2:1} D \hookrightarrow \mathsf{Gr}(3, 6) \xhookrightarrow{\text{Plücker}} \mathbf{P}(\textstyle\bigwedge^3 \mathbf{C}^6).$$

3.6.3. *An example in dimension 8* ($\mathbf{m = 4, q(H) = 2, \gamma = 2}$). Let again $W \subset \mathbf{P}^5$ be a smooth cubic hypersurface that contains no planes. The moduli space $\mathscr{M}_3(W)$ of generalized twisted cubic curves on W is a smooth and irreducible projective variety of dimension 10, and there is a contraction $\mathscr{M}_3(W) \to X(W)$, where $X(W)$ is a projective hyperkähler manifold of type $\mathrm{K3}^{[4]}$ ([LLSvS, AL]). The maps that takes a cubic curve to its span defines a morphism from $\mathscr{M}_3(W)$ to $\mathsf{Gr}(4, 6)$ which factors through a surjective rational map $X(W) \dashrightarrow \mathsf{Gr}(4, 6)$ of degree 72. The Plücker polarization on $\mathsf{Gr}(4, 6)$ pulls back to a polarization H on $X(W)$ with $q_{X(W)}(H) = 2$. By (5), we have $h^0(X(W), H) = 15$, and φ_H is the rational map

$$\varphi_H \colon X(W) \overset{72:1}{\dashrightarrow\!\!\!\rightarrow} \mathsf{Gr}(4, 6) \xhookrightarrow{\text{Plücker}} \mathbf{P}(\textstyle\bigwedge^4 \mathbf{C}^6).$$

It follows from [AL][20] that the divisibility γ is 2 and a general element of ${}^4\mathscr{M}_2^{(2)}$ is of the form $(X(W), H)$.

Remark 3.12. It follows from the above descriptions that the moduli spaces ${}^2\mathscr{M}_2^{(1)}$, ${}^2\mathscr{M}_6^{(2)}$, ${}^2\mathscr{M}_{22}^{(2)}$, ${}^2\mathscr{M}_{38}^{(2)}$, ${}^3\mathscr{M}_4^{(2)}$, and ${}^4\mathscr{M}_2^{(2)}$ are unirational. It was proved in [GHS2] that ${}^2\mathscr{M}_{2n}^{(1)}$ is of general type for all $n \geq 12$.

3.7. **The nef and movable cones.** Let X be a hyperkähler manifold. We define its *positive cone* $\mathrm{Pos}(X)$ as that of the two components of the cone $\{x \in \mathrm{Pic}(X) \otimes \mathbf{R} \mid q_X(x) > 0\}$ that contains one (hence all) Kähler classes.

[20]The main result of [AL] is that $X(W)$ is a deformation of $S^{[4]}$, where (S, L) is a very general polarized K3 surface of degree 14. One can therefore write the polarization on $X(W)$ as $H = aL_4 - b\delta$, with $2 = H^2 = 14a^2 - 6b^2$, hence $a^2 \equiv b^2 - 1 \pmod 4$. This implies that a is even, hence so is γ. Since $\gamma \mid H^2$, we get $\gamma = 2$ (note that the condition $n + m \equiv 1 \pmod 4$ of footnote 15 holds).

The (closed) *movable cone*

$$\mathrm{Mov}(X) \subset \mathrm{Pic}(X) \otimes \mathbf{R}$$

is the closed convex cone generated by classes of line bundles on X whose base locus has codimension ≥ 2 (no fixed divisor). It is not too difficult to prove the inclusion[21]

$$\mathrm{Mov}(X) \subset \overline{\mathrm{Pos}(X)}.$$

When X is moreover projective, we defined in Section 2.4 the nef cone $\mathrm{Nef}(X) \subset \mathrm{Pic}(X) \otimes \mathbf{R}$; we have of course

$$\mathrm{Nef}(X) \subset \mathrm{Mov}(X). \tag{8}$$

The importance of the movable cone (which, for K3 surfaces is just the nef cone) stems from the following result.

Proposition 3.13. *Let X and X' be hyperkähler manifolds. Any birational isomorphism $\sigma\colon X \overset{\sim}{\dashrightarrow} X'$ induces a Hodge isometry $\sigma^*\colon (H^2(X',\mathbf{Z}), q_{X'}) \xrightarrow{\sim} (H^2(X,\mathbf{Z}), q_X)$ which satisfies*

$$\sigma^*(\mathrm{Mov}(X')) = \mathrm{Mov}(X).$$

If X and X' are moreover projective[22] *and $\sigma^*(\mathrm{Nef}(X'))$ meets $\mathrm{Amp}(X)$, the map σ is an isomorphism and $\sigma^*(\mathrm{Nef}(X')) = \mathrm{Nef}(X)$.*

Sketch of proof. Let $U \subset X$ (resp. $U' \subset X'$) be the largest open subset on which σ (resp. σ^{-1}) is defined. We have $\mathrm{codim}_X(X \smallsetminus U) \geq 2$ hence, since X is normal and Ω^2_X is locally free, restriction induces an isomorphism $H^0(X, \Omega^2_X) \xrightarrow{\sim} H^0(U, \Omega^2_U)$. These vector spaces are spanned by the symplectic form ω and, since $(\sigma|_U)^*\omega'$ is non-zero, it is a non-zero multiple of $\omega|_U$. Since this 2-form is nowhere degenerate, $\sigma|_U$ is quasi-finite and, being birational, it is an open embedding by Zariski's Main Theorem. This implies that σ induces an isomorphism between U and U'. Since $\mathrm{codim}_X(X \smallsetminus U) \geq 2$, the restriction $H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(U,\mathbf{Z})$ is an isomorphism[23] and we get an isomorphism $\sigma^*\colon H^2(X',\mathbf{Z}) \xrightarrow{\sim} H^2(X,\mathbf{Z})$ of Hodge structures.

For the proof that σ^* is an isometry, we refer to [GHJ, Section 27.1]. Given a line bundle M' on X', we have isomorphisms

$$H^0(X', M') \xrightarrow{\sim} H^0(U', M') \xrightarrow{\sim} H^0(U, \sigma^* M') \xleftarrow{\sim} H^0(X, \sigma^* M') \tag{9}$$

and it is clear that σ^* maps $\mathrm{Mov}(X')$ to $\mathrm{Mov}(X)$.

Finally, if σ^* maps an (integral) ample class H' on X' to an ample class H on X, we obtain by (9) isomorphisms $H^0(X', H'^{\otimes k}) \xrightarrow{\sim} H^0(X, (\sigma^* H')^{\otimes k})) = H^0(X, H^{\otimes k})$ for all $k \geq 0$. Since $X = \mathrm{Proj}(\bigoplus_{k=0}^{+\infty} H^0(X, H^{\otimes k}))$ and $X' = \mathrm{Proj}(\bigoplus_{k=0}^{+\infty} H^0(X', H'^{\otimes k}))$, this means that σ is an isomorphism. □

[21] One may follow the argument in the proof of [HT1, Theorem 7] and compute explicitly the Beauville–Fujiki form on a resolution of the rational map induced by the complete linear system of the movable divisor.

[22] Since any compact Kähler manifold which is Moishezon is projective, X is projective if and only if X' is projective. The general statement here is that if σ^* maps a Kähler class of X' to a Kähler class of X, the map σ is an isomorphism ([GHJ, Proposition 27.6]).

[23] If $N := \dim_{\mathbf{C}}(X)$ and $Y := X \smallsetminus U$, this follows from example from the long exact sequence

$$H_{2N-2}(Y,\mathbf{Z}) \to H_{2N-2}(X,\mathbf{Z}) \to H_{2N-2}(X,Y,\mathbf{Z}) \to H_{2N-3}(Y,\mathbf{Z}),$$

the fact that $H_j(Y,\mathbf{Z}) = 0$ for $j > 2(N-2) \geq \dim_{\mathbf{R}}(Y)$, and the duality isomorphisms $H_{2N-i}(X,\mathbf{Z}) \simeq H^i(X,\mathbf{Z})$ and $H_{2N-i}(X,Y,\mathbf{Z}) \simeq H^i(X \smallsetminus Y,\mathbf{Z})$.

If X' is a (projective) hyperkähler manifold with a birational map $\sigma\colon X \overset{\sim}{\dashrightarrow} X'$, one identifies $H^2(X',\mathbf{Z})$ and $H^2(X,\mathbf{Z})$ using σ^* (Proposition 3.13). By [HT1, Theorem 7], we have

$$\mathrm{Mov}(X) = \overline{\bigcup_{\sigma\colon X \overset{\sim}{\dashrightarrow} X'} \sigma^*(\mathrm{Nef}(X'))}, \tag{10}$$

where the various cones $\sigma^*(\mathrm{Nef}(X'))$ are either equal or have disjoint interiors. It is known that for any hyperkähler manifold X of $\mathrm{K3}^{[m]}$-type, the set of isomorphism classes of hyperkähler manifolds birationally isomorphic to X is finite (Kawamata–Morrison conjecture; [MY, Corollary 1.5]).

3.7.1. *Nef and movable cones of hyperkähler fourfolds of* $\mathrm{K3}^{[2]}$*-type.* The nef and movable cones of hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type are known, although their concrete descriptions can be quite complicated. They are best explained in terms of the Markman–Mukai lattice, which we now define (see [M2, Section 9] and [BHT, Section 1] for more details).

We define the *extended K3 lattice*

$$\widetilde{\Lambda}_{\mathrm{K3}} := U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}. \tag{11}$$

It is even, unimodular of signature $(4,20)$ and the lattice $\Lambda_{\mathrm{K3}^{[m]}}$ defined in (4) is the orthogonal of any primitive vector of square $2m-2$.

Given a hyperkähler manifold X of $\mathrm{K3}^{[m]}$-type, there is a canonical extension

$$\theta_X\colon H^2(X,\mathbf{Z}) \hookrightarrow \widetilde{\Lambda}_X$$

of lattices and weight-2 Hodge structures, where the lattice $\widetilde{\Lambda}_X$ is isomorphic to $\widetilde{\Lambda}_{\mathrm{K3}}$. A generator $\mathbf{v}_X$ (with square $2m-2$) of $H^2(X,\mathbf{Z})^\perp$ is of type $(1,1)$. We denote by $\widetilde{\Lambda}_{\mathrm{alg},X}$ the algebraic (i.e., type $(1,1)$) part of $\widetilde{\Lambda}_X$, so that $\mathrm{Pic}(X) = \mathbf{v}_X^\perp \cap \widetilde{\Lambda}_{\mathrm{alg},X}$.

Given a class $\mathbf{s} \in \widetilde{\Lambda}_X$, we define a hyperplane

$$H_{\mathbf{s}} := \{x \in \mathrm{Pic}(X) \otimes \mathbf{R} \mid x \cdot \mathbf{s} = 0\}.$$

In order to keep things simple, we will limit ourselves to the descriptions of the nef and movable cones of hyperkähler manifolds of $\mathrm{K3}^{[2]}$-type ($m=2$). The general case will be treated in Theorem B.2 (see also Example 3.17).

Theorem 3.14. *Let X be a hyperkähler fourfold of* $\mathrm{K3}^{[2]}$*-type.*

(a) *The interior* $\mathrm{Int}(\mathrm{Mov}(X))$ *of the movable cone is the connected component of*

$$\mathrm{Pos}(X) \smallsetminus \bigcup_{\substack{\kappa \in \mathrm{Pic}(X) \\ \kappa^2 = -2}} H_\kappa$$

that contains the class of an ample divisor.

(b) *The ample cone* $\mathrm{Amp}(X)$ *is the connected component of*

$$\mathrm{Int}(\mathrm{Mov}(X)) \smallsetminus \bigcup_{\substack{\kappa \in \mathrm{Pic}(X) \\ \kappa^2 = -10 \\ \mathrm{div}_{H^2(X,\mathbf{Z})}(\kappa) = 2}} H_\kappa$$

that contains the class of an ample divisor.

Proof. Statement (a) follows from the general result [M2, Lemma 6.22] (see also [M2, Proposition 6.10, Proposition 9.12, Theorem 9.17]). More precisely, this lemma says that the interior of the movable cone is the connected component that contains the class of an ample divisor of the complement in $\mathrm{Pos}(X)$ of the union of the hyperplanes $H_{\mathbf{s}}$, where $\mathbf{s} \in \widetilde{\Lambda}_{\mathrm{alg},X}$ is such that $\mathbf{s}^2 = -2$ and $\mathbf{s} \cdot \mathbf{v}_X = 0$. The second relation means $\mathbf{s} \in \mathrm{Pic}(X)$, hence (a) is proved.

As to (b), the dual statement of [BHT, Theorem 1] says[24] that the ample cone is the connected component containing the class of an ample divisor of the complement in $\mathrm{Pos}(X)$ of the union of the hyperplanes $H_{\mathbf{s}}$, where $\mathbf{s} \in \widetilde{\Lambda}_{\mathrm{alg},X}$ is such that $\mathbf{s}^2 = -2$ and $\mathbf{s} \cdot \mathbf{v}_X \in \{0, 1\}$.

Writing $\mathbf{s} = a\kappa + b\mathbf{v}_X$, with $\kappa \in \mathrm{Pic}(X)$ primitive and $a, b \in \frac{1}{2}\mathbf{Z}$, we get $-2 = \mathbf{s}^2 = a^2\kappa^2 + 2b^2$ and $\mathbf{s} \cdot \mathbf{v}_X = 2b$. Inside $\mathrm{Int}(\mathrm{Mov}(X))$, the only new condition therefore corresponds to $b = \frac{1}{2}$ and $a^2\kappa^2 = -\frac{5}{2}$, hence $a = \frac{1}{2}$ and $\kappa^2 = -10$. Moreover, $\kappa \cdot H^2(X, \mathbf{Z}) = 2\mathbf{s} \cdot \widetilde{\Lambda}_X = 2\mathbf{Z}$, hence $\mathrm{div}_{H^2(X,\mathbf{Z})}(\kappa) = 2$. Conversely, given any such s, the element $\kappa + \mathbf{v}_X$ of $\widetilde{\Lambda}_X$ is divisible by 2 and $\mathbf{s} := \frac{1}{2}(\kappa + \mathbf{v}_X)$ satisfies $\mathbf{s}^2 = -2$, $\mathbf{s} \cdot \mathbf{v}_X = 1$, and $H_\kappa = H_{\mathbf{s}}$. This proves (b). □

Remark 3.15. We can make the descriptions in Theorem 3.14 more precise.

(a) As explained in [M2, Section 6], it follows from [M3] that there is a group of reflections W_{Exc} acting on $\mathrm{Pos}(X)$ that acts faithfully and transitively on the set of connected components of

$$\mathrm{Pos}(X) \smallsetminus \bigcup_{\substack{\mathbf{s} \in \mathrm{Pic}(X) \\ \mathbf{s}^2 = -2}} H_{\mathbf{s}}$$

In particular, $\overline{\mathrm{Mov}(X)} \cap \mathrm{Pos}(X)$ is a fundamental domain for the action of W_{Exc} on $\mathrm{Pos}(X)$.

(b) By [Ma, Proposition 2.1] (see also [HT1, Theorem 7]), each connected component of

$$\mathrm{Int}(\mathrm{Mov}(X)) \smallsetminus \bigcup_{\substack{\mathbf{s} \in \mathrm{Pic}(X) \\ \mathbf{s}^2 = -10 \\ \mathrm{div}_{H^2(X,\mathbf{Z})}(\mathbf{s}) = 2}} H_{\mathbf{s}}$$

corresponds to the ample cone of a hyperkähler fourfold X' of $\mathrm{K3}^{[2]}$-type via a birational isomorphism $X \overset{\sim}{\dashrightarrow} X'$ (compare with (10)).

3.7.2. *Nef and movable cones of punctual Hilbert schemes of K3 surfaces.* In this section, (S, L) is a polarized K3 surface of degree $2e$ with $\mathrm{Pic}(S) = \mathbf{Z}L$ and m is an integer such that $m \geq 2$. We describe the nef and movable cones of the mth Hilbert power $S^{[m]}$. The line bundle L_m on $S^{[m]}$ induced by L is nef and big and spans a ray which is extremal for both cones $\mathrm{Mov}(S^{[m]})$ and $\mathrm{Nef}(S^{[m]})$. Since the Picard number of $S^{[m]}$ is 2, there is just one "other" extremal ray for each cone.

Example 3.16 (Nef and movable cones of $S^{[2]}$)**.** By Theorem 3.14, cones of divisors on $S^{[2]}$ can be described as follows.

(a) The other extremal ray of the (closed) movable cone $\mathrm{Mov}(S^{[2]})$ is spanned by $L_2 - \mu_e\delta$, where

[24] This still requires some work, and reading [BM2, Sections 12 and 13] might help.

- if e is a perfect square, $\mu_e = \sqrt{e}$;
- if e is not a perfect square and (a_1, b_1) is the minimal solution of the equation $\mathscr{P}_e(1)$, $\mu_e = e\frac{b_1}{a_1}$.

(b) The other extremal ray of the nef cone $\mathrm{Nef}(S^{[2]})$ is spanned by $L_2 - \nu_e\delta$, where
- if the equation $\mathscr{P}_{4e}(5)$ is not solvable, $\nu_e = \mu_e$;
- if the equation $\mathscr{P}_{4e}(5)$ is solvable and (a_5, b_5) is its minimal solution, $\nu_e = 2e\frac{b_5}{a_5}$.[25]

The walls of the decomposition (10) of the movable cone $\mathrm{Mov}(S^{[2]})$ correspond exactly to the solutions of the equation $\mathscr{P}_{4e}(5)$ that give rays inside that cone and there are only three possibilities (see Appendix A and in particular Lemma A.1, which says that the equation $\mathscr{P}_{4e}(5)$ has at most two classes of solutions):

- either $e = 1$ or the equation $\mathscr{P}_{4e}(5)$ is not solvable, and the nef and movable cones are equal;
- or $e > 1$, the equation $\mathscr{P}_{4e}(5)$ has a minimal solution (a_5, b_5), and, if (a_1, b_1) is the minimal solution to the equation $\mathscr{P}_e(1)$,
  - either b_1 is odd, or b_1 is even and $5 \mid e$, in which cases there are two chambers and the middle wall is spanned by $a_5L_2 - 2eb_5\delta$;
  - or b_1 is even and $5 \nmid e$, in which case there are three chambers and the middle walls are spanned by $a_5L_2 - 2eb_5\delta$ and $(a_1a_5 - 2eb_1b_5)L_2 - e(a_5b_1 - 2a_1b_5)\delta$ respectively.

In the last case, for example, there are three "different" birational isomorphisms $S^{[2]} \dashrightarrow X$, where X is a hyperkähler manifold of dimension 4.

The nef and movable cones of $S^{[2]}$ are computed in the table below for $1 \le e \le 13$. The first two lines give the minimal solution of the Pell-type equations $\mathscr{P}_e(1)$ and $\mathscr{P}_{4e}(5)$ (when they exists). A $\star$ means that e is a perfect square. The last two lines indicate the "slope" ν of the "other" extremal ray of the cone (that is, the ray is generated by $L_2 - \nu\delta$). The sign $=$ means that the two cones are equal.

e	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathscr{P}_e(1)$	$\star$	$(3,2)$	$(2,1)$	$\star$	$(9,4)$	$(5,2)$	$(8,3)$	$(3,1)$	$\star$	$(19,6)$	$(10,3)$	$(7,2)$	$(649,180)$
$\mathscr{P}_{4e}(5)$	$\star$	–	–	$\star$	$(5,1)$	–	–	–	$\star$	–	$(7,1)$	–	–
$\mathrm{Mov}(S^{[2]})$	1	$\frac{4}{3}$	$\frac{3}{2}$	2	$\frac{20}{9}$	$\frac{12}{5}$	$\frac{21}{8}$	$\frac{8}{3}$	3	$\frac{60}{19}$	$\frac{33}{10}$	$\frac{24}{7}$	$\frac{2340}{649}$
$\mathrm{Nef}(S^{[2]})$	=	=	=	=	2	=	=	=	=	=	$\frac{22}{7}$	=	=

In the next table, we restrict ourselves to integers $1 < e \le 71$ for which the equation $\mathscr{P}_{4e}(5)$ is solvable (in other words, to those e for which the nef and movable cones are distinct). We indicate all solutions of that equation that give rise to rays that are in the movable cone (that is, with slope smaller than that of the "other" ray of the movable cone): they are the walls for the decomposition (10). As expected, there are two rays if and only if

[25]There is a typo in [BM2, Lemma 13.3(b)]: one should replace d with $2d$. Also, the general theory developed in [BM2] implies that this ray does lie inside the movable cone; this is also a consequence of the elementary inequalities (31) and (32).

b_1 is even and $5 \nmid e$.

e	5	11	19	29	31	41	55	71
$\mathscr{P}_e(1)$	$(9,4)$	$(10,3)$	$(170,39)$	$(9801,1820)$	$(1520,273)$	$(2049,320)$	$(89,12)$	$(3480,413)$
$\mathscr{P}_{4e}(5)$	$(5,1)$	$(7,1)$	$(9,1)$	$(11,1),(2251,209)$	$(657,59)$	$(13,1),(397,31)$	$(15,1)$	$(17,1)$
$\mathrm{Mov}(S^{[2]})$	$\frac{20}{9}$	$\frac{33}{10}$	$\frac{741}{170}$	$\frac{52780}{9801}$	$\frac{8463}{1520}$	$\frac{13120}{2049}$	$\frac{660}{89}$	$\frac{29323}{3480}$
Walls for (10)	2	$\frac{22}{7}$	$\frac{38}{9}$	$\frac{58}{11}, \frac{12122}{2251}$	$\frac{3658}{657}$	$\frac{82}{13}, \frac{2542}{397}$	$\frac{22}{3}$	$\frac{142}{17}$

Example 3.17 (Nef and movable cones of $S^{[m]}$)**.** A complete description of the cone $\mathrm{Mov}(S^{[m]})$ is given in [BM2, Proposition 13.1].[26] One extremal ray is spanned by L_m and generators of the other extremal ray are given as follows:

- if $e(m-1)$ is a perfect square, a generator is $(m-1)L_m - \sqrt{e(m-1)}\delta$ (with square 0);
- if $e(m-1)$ is not a perfect square and the equation $(m-1)a^2 - eb^2 = 1$ has a minimal solution (a_1, b_1), a generator is $(m-1)a_1 L_m - eb_1\delta$ (with square $2e(m-1)$);
- otherwise, a generator is $a'_1 L_m - eb'_1\delta$ (with square $2e$), where (a'_1, b'_1) is the minimal positive solution of the equation $a^2 - e(m-1)b^2 = 1$ that satisfies $a'_1 \equiv \pm 1 \pmod{m-1}$.[27]

A complete description of the cone $\mathrm{Nef}(S^{[m]})$ theoretically follows from [BHT, Theorem 1], although it is not as simple as for the movable cone. Here are a couple of facts:

- when $m \geq \frac{e+3}{2}$, the "other" extremal ray of $\mathrm{Nef}(S^{[m]})$ is spanned by $(m+e)L_m - 2e\delta$ ([BM1, Proposition 10.3]) and the movable and nef cones are different, except when $m = e+2$;[28]
- when $e = (m-1)b^2$, where b is a positive integer, the "other" extremal ray of $\mathrm{Nef}(S^{[m]})$ is spanned by $L_m - b\delta$ ([BM1, Theorem 10.6]) and the movable cone and the nef cone are equal.

[26] There is a typo in [BM2, (33)] hence also in statement (c) of that proposition.

[27] If (a, b) is a solution to that equation, its "square" $(a^2 + e(m-1)b^2, 2ab)$ is also a solution and $a^2 + e(m-1)b^2 \equiv a^2 \equiv 1 \pmod{m-1}$. So there always exist solutions with the required property.

Also, we always have the inequalities $\frac{eb_1}{a_1(m-1)} \leq \frac{eb'_1}{a'_1} < \sqrt{\frac{e}{m-1}}$ between slopes and the first inequality is strict when $m \geq 3$ (when the equation $(m-1)a^2 - eb^2 = 1$ has a minimal positive solution (a_1, b_1) and $m \geq 3$, the minimal solution of the equation $\mathscr{P}_{e(m-1)}(1)$ is $(2(m-1)a_1^2 - 1, 2a_1b_1)$ by Lemma A.2 and this is (a'_1, b'_1); hence $\frac{eb'_1}{a'_1} > \frac{eb_1}{a_1(m-1)}$).

[28] If they are equal, we are in one of the following cases described in Example 3.17 (we set $g := \gcd(m+e, 2e)$)

- either $(m+e)L_m - 2e\delta$ has square 0 and $2e(m+e)^2 = 4e^2(2m-2)$, which implies $(m+e)^2 = 2e(2m-2)$, hence $(m-e)^2 = -4e$, absurd;
- or $m + e = g(m-1)a_1$ and $2e = geb_1$, which implies $b_1 \leq 2$ and
$$a_1^2 \frac{e+1}{2} \leq a_1^2(m-1) = 1 + eb_1^2 + 1 \leq 1 + 4e + 1$$
hence either $a_1 = b_1 = 2$, absurd, or $a_1 = 1$, and $m+e = g(m-1)$ implies $g = 2$, $b_1 = 1$, and $m = e+2$, the only case when the cones are equal;
- or $m + e = ga'_1$ and $2e = geb'_1$, which implies $g \leq 2$ and
$$(m+e)^2 = g^2 a'^2_1 = g^2(e(m-1)b'^2_1 + 1) = 4e(m-1) + g^2$$
hence $(m-e)^2 = -4e + g^2 \leq -4e + 4 \leq 0$ and $e = m = 1$, absurd.

3.7.3. *Nef and movable cones for some other hyperkähler fourfolds with Picard number 2.* We now give an example of cones with irrational slopes. Let n and e' be a positive integers such that $n \equiv -1 \pmod 4$. Let (X, H) be a polarized hyperkähler fourfold of $\mathrm{K3}^{[2]}$-type with H of divisibility 2 and $\mathrm{Pic}(X) = \mathbf{Z}H \oplus \mathbf{Z}L$, with intersection matrix $\left(\begin{smallmatrix} 2n & 0 \\ 0 & -2e' \end{smallmatrix}\right)$.[29] By Theorem 3.14, cones of divisors on X can be described as follows.[30]

(a) The extremal rays of the movable cone $\mathrm{Mov}(X)$ are spanned by $H - \mu_{n,e'}L$ and $H + \mu_{n,e'}L$, where
- if the equation $\mathscr{P}_{n,e'}(-1)$ is not solvable, $\mu_{n,e} = \sqrt{n/e'}$;
- if the equation $\mathscr{P}_{n,e'}(-1)$ is solvable and (a_{-1}, b_{-1}) is its minimal solution, $\mu_{n,e'} = \frac{na_{-1}}{e'b_{-1}}$.

(b) The extremal rays of the nef cone $\mathrm{Nef}(X)$ are spanned by $H - \nu_{n,e'}L$ and $H + \nu_{n,e'}L$, where
- if the equation $\mathscr{P}_{n,4e'}(-5)$ is not solvable, $\nu_{n,e'} = \mu_{n,e'}$;
- if the equation $\mathscr{P}_{n,4e'}(-5)$ is solvable and (a_{-5}, b_{-5}) is its minimal solution, $\nu_{n,e'} = \frac{na_{-5}}{2e'b_{-5}}$.

The new element here is that we may have cones with irrational slopes (when $e'n$ is not a perfect square). As in Section 3.7.2, the walls of the chamber decomposition (10) correspond to the rays determined by the solutions to the equation $\mathscr{P}_{n,4e'}(-5)$ that sit inside the movable cone. This is only interesting when the nef and movable cones are different, so we assume that the equation $\mathscr{P}_{n,4e'}(-5)$ is solvable, with minimal solution (a_{-5}, b_{-5}), and the extremal rays of the nef cone have rational slopes $\pm\frac{na_{-5}}{2e'b_{-5}}$. This implies that ne' is not a perfect square[31] and the equation $\mathscr{P}_{n,4e'}(-5)$ has infinitely many solutions. By Lemma A.1, these solutions form two conjugate classes if $5 \nmid e'$, and one class if $5 \mid e'$.

This means that all the solutions (a, b) to the equation $\mathscr{P}_{n,4e'}(-5)$ are given by

$$na + b\sqrt{4ne'} = \pm(na_{-5} \pm b_{-5}\sqrt{4ne'})x_1^m\ , \quad m \in \mathbf{Z}, \tag{12}$$

where $x_1 = a_1 + b_1\sqrt{4ne'}$ corresponds to the minimal solution (a_1, b_1) to the equation $\mathscr{P}_{4ne'}(1)$. There are two cases:

a) either the equation $\mathscr{P}_{n,e'}(-1)$ is not solvable and the extremal rays of the movable cone have slopes $\pm\sqrt{n/e'}$, which is irrational by footnote 31;
b) or the equation $\mathscr{P}_{n,e'}(-1)$ is solvable, with minimal solution (a_{-1}, b_{-1}), and $x_1 = na_{-1}^2 + e'b_{-1}^2 + a_{-1}b_{-1}\sqrt{4ne'}$ by Lemma A.2.

In case a), all positive solutions (a, b) of the equation $\mathscr{P}_{n,4e'}(-5)$ satisfy

$$\frac{na}{2e'b} < \frac{n}{2e'}\sqrt{\frac{4e'}{n}} = \sqrt{\frac{n}{e'}}.$$

[29] In the notation of Section 3.10, these are fourfolds whose period point is very general in one component of the hypersurface $\mathscr{D}^{(2)}_{2n,2e'n}$ and we will prove in Theorem 3.27 that they exist if and only if $n > 0$ and $e' > 1$.

[30] Before the general results of [BHT] were available, the case $n = 3$ and $e' = 2$ had been worked out in [HT3, Proposition 7.2] using a beautiful geometric argument.

[31] Since the equation $\mathscr{P}_{n,4e'}(-5)$ is solvable, $d := \gcd(n, e') \in \{1, 5\}$. If ne' is a perfect square, we can write $n = du^2$ and $e' = dv^2$. This is easily checked to be incompatible with the equality $(ua+2vb)(ua-2vb) = -5/d$.

Therefore, there are infinitely many walls in the chamber decomposition (10): they correspond to the (infinitely many) solutions (a,b) of the equation $\mathscr{P}_{n,4e'}(-5)$ with $a>0$.

In case (b), the interior walls of the chamber decomposition (10) correspond to the solutions (a,b) of the equation $\mathscr{P}_{n,4e'}(-5)$ (as given in (12)) with $a>0$ that land in the interior of the movable cone, that is, that satisfy $\frac{na}{2e'|b|}<\frac{na_{-1}}{e'b_{-1}}$. We saw in (36) that there are only two such solutions, (a_{-5},b_{-5}) and $(a_{-5},-b_{-5})$, hence three chambers.

3.8. **The Torelli theorem.** A general version of the Torelli theorem for hyperkähler manifolds was proven by Verbitsky ([V]). It was later reformulated by Markman in terms of the *monodromy group* of a hyperkähler manifold X ([M2]): briefly, this is the subgroup $\mathrm{Mon}^2(X)$ of $O(H^2(X,\mathbf{Z}),q_X)$ generated by monodromy operators associated with all smooth deformations of X.

When X is of $\mathrm{K3}^{[m]}$-type, Markman showed that $\mathrm{Mon}^2(X)$ is the subgroup of the group $O(H^2(X,\mathbf{Z}),q_X)$ generated by reflections about (-2)-classes and the negative of reflections about $(+2)$-classes ([M2, Theorem 9.1]). Combined with a result of Kneser, he obtains that $\mathrm{Mon}^2(X)$ is the subgroup $\widehat{O}^+(H^2(X,\mathbf{Z}),q_X)$ of elements of $O^+(H^2(X,\mathbf{Z}),q_X)$ that act as $\pm\,\mathrm{Id}$ on the discriminant group $D(H^2(X,\mathbf{Z}))$ ([M2, Lemma 9.2]).[32]

This implies that the index of $\mathrm{Mon}^2(X)$ in $O^+(H^2(X,\mathbf{Z}),q_X)$ is $2^{\max\{\rho(m-1)-1,0\}}$, where $\rho(d)$ is the number of distinct prime divisors of an integer d;[33] in particular, these two groups are equal if and only if $m-1$ is a prime power (including 1). For the sake of simplicity, we will only state the first version of the Torelli theorem in that case ([M2, Theorem 1.3]); compare with Theorem 2.8).

Theorem 3.18 (Verbitsky, Markman; Torelli theorem, first version)**.** *Let m be a positive integer such that $m-1$ is a prime power. Let (X,H) and (X',H') be polarized hyperkähler manifolds of* $\mathrm{K3}^{[m]}$*-type. If there exists an isometry of lattices*

$$\varphi\colon (H^2(X',\mathbf{Z}),q_{X'}) \xrightarrow{\sim} (H^2(X,\mathbf{Z}),q_X)$$

such that $\varphi(H')=H$ and $\varphi_{\mathbf{C}}(H^{2,0}(X'))=H^{2,0}(X)$, there exists an isomorphism $\sigma\colon X\xrightarrow{\sim}X'$ such that $\varphi=\sigma^*$.

We will state a version of this theorem valid for all m in the next section (Theorem 3.19). In this present form, when $m-1$ is not a prime power, the isometry φ needs to satisfy additional conditions for it to be induced by an isomorphism between X and X'. For example, when $X=X'$, the isometry φ needs to be in the group $\widehat{O}(H^2(X,\mathbf{Z}),q_X)$ defined above.

3.9. **The period map.** Let ${}^m\mathscr{M}_\tau$ be the (not necessarily irreducible!) 20-dimensional quasi-projective moduli space of polarized hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type with fixed polarization type τ, that is, τ is the $O(\Lambda_{\mathrm{K3}^{[m]}})$-orbit of a primitive element h_τ with positive square (see

[32]As in footnote 9, $O^+(H^2(X,\mathbf{Z}),q_X)$ is the index-2 subgroup of $O(H^2(X,\mathbf{Z}),q_X)$ of isometries that preserve the positive cone $\mathrm{Pos}(X)$.

[33]One has $D(H^2(X,\mathbf{Z}))\simeq\mathbf{Z}/(2m-2)\mathbf{Z}$ and $O(D(H^2(X,\mathbf{Z})),\bar q_X)\simeq\{x \pmod{2m-2}\mid x^2\equiv 1 \pmod{4t}\}\simeq(\mathbf{Z}/2\mathbf{Z})^{\rho(m-1)}$ ([GHS2, Corollary 3.7]). One then uses the surjectivity of the canonical map $O^+(H^2(X,\mathbf{Z}),q_X)\to O(D(H^2(X,\mathbf{Z})),\bar q_X)$ (see Section 2.7).

Section 3.5). Fix such an element $h_\tau \in \Lambda_{\mathrm{K3}^{[m]}}$ and define as in Section 2.8 a 20-dimensional (non-connected) complex manifold

$$\begin{aligned}\Omega_{h_\tau} &:= \{[x] \in \mathbf{P}(h_\tau^\perp \otimes \mathbf{C}) \mid x \cdot x = 0,\ x \cdot \overline{x} > 0\} \\ &= \{[x] \in \mathbf{P}(\Lambda_{\mathrm{K3}^{[m]}} \otimes \mathbf{C}) \mid x \cdot h_\tau = x \cdot x = 0,\ x \cdot \overline{x} > 0\}.\end{aligned}$$

Instead of taking the quotient by the action of the group $O(\Lambda_{\mathrm{K3}^{[m]}}, h_\tau) := \{\varphi \in O(\Lambda_{\mathrm{K3}^{[m]}}) \mid \varphi(h_\tau) = h_\tau\}$ as we did in the K3 surface case, we consider, according to the discussion in Section 3.8, the (sometimes) smaller group

$$\widehat{O}(\Lambda_{\mathrm{K3}^{[m]}}, h_\tau) := \{\varphi \in O(\Lambda_{\mathrm{K3}^{[m]}}) \mid \varphi(h_\tau) = h_\tau \text{ and } \varphi \text{ acts as } \pm\mathrm{Id} \text{ on } D(\Lambda_{\mathrm{K3}^{[m]}})\}.$$

The quotient

$$\mathscr{P}_\tau := \widehat{O}(\Lambda_{\mathrm{K3}^{[m]}}, h_\tau)\backslash\Omega_{h_\tau}$$

is again an irreducible[34] quasi-projective normal variety of dimension 20 and one can define an algebraic period map

$$\wp_\tau \colon {}^m\mathscr{M}_\tau \longrightarrow \mathscr{P}_\tau.$$

When the polarization type is determined by its degree $2n$ and its divisibility γ, we will also write ${}^m\mathscr{P}_{2n}^{(\gamma)}$ instead of $\mathscr{P}_\tau$ and ${}^m\wp_{2n}^{(\gamma)}$ instead of $\wp_\tau$.

The Torelli theorem now takes the following form ([GHS3, Theorem 3.14], [M2, Theorem 8.4]).

Theorem 3.19 (Verbitsky, Markman; Torelli theorem, second version)**.** *Let m be an integer with $m \geq 2$ and let τ be a polarization type of $\Lambda_{\mathrm{K3}^{[m]}}$. The restriction of the period map $\wp_\tau$ to any irreducible component of ${}^m\mathscr{M}_\tau$ is an open embedding.*

As we did in Section 2.8 for K3 surfaces (Proposition 2.11), we will determine the image of the period map in Section 2.10 (at least when $m = 2$).

Remark 3.20. Recall the following facts about the irreducible components of ${}^m\mathscr{M}_\tau$:

- when the divisibility γ satisfies $\gamma = 2$ or $\gcd(\frac{2n}{\gamma}, \frac{2m-2}{\gamma}, \gamma) = 1$, the polarization type τ is determined by its square $2n$ and γ (in other words, ${}^m\mathscr{M}_\tau = {}^m\mathscr{M}_{2n}^{(\gamma)}$);
- when moreover $\gamma = 1$, or $\gamma = 2$ and $n + m \equiv 1 \pmod 4$, the spaces ${}^m\mathscr{M}_\tau = {}^m\mathscr{M}_{2n}^{(\gamma)}$ are irreducible (Theorem 3.5; see also footnote 15 for more cases where this happens).

Remark 3.21 (Strange duality)**.** [35] We assume here $\gamma = 1$, or $\gamma = 2$ and $n+m \equiv 1 \pmod 4$, so that the moduli spaces ${}^m\mathscr{M}_{2n}^{(\gamma)}$ are irreducible. Recall from Remark 3.6 the isomorphism of lattices $\Lambda^{(\gamma)}_{\mathrm{K3}^{[m]},2n} \simeq \Lambda^{(\gamma)}_{\mathrm{K3}^{[n+1]},2m-2}$. It translates[36] into an isomorphism

$$(13) \qquad {}^m\mathscr{P}_{2n}^{(\gamma)} \xrightarrow{\ \sim\ } {}^{n+1}\mathscr{P}_{2m-2}^{(\gamma)}$$

hence into a birational isomorphism

$$(14) \qquad {}^m\mathscr{M}_{2n}^{(\gamma)} \overset{\sim}{\dashrightarrow} {}^{n+1}\mathscr{M}_{2m-2}^{(\gamma)}.$$

[34] As in footnote 9, one usually chooses one component $\Omega^+_{h_\tau}$ of Ω_{h_τ}, so that $\mathscr{P}_\tau = \widehat{O}^+(\Lambda_{\mathrm{K3}^{[m]}}, h_\tau)\backslash\Omega^+_{h_\tau}$ (see [GHS3, Theorem 3.14]).

[35] This strange duality was already noticed in [Ap2, Proposition 3.2].

[36] The compatibility of the group actions were checked by J. Song; see also [Ap2].

So there is a way to associate with a general polarized hyperkähler manifold (X,H) of dimension $2m$, degree $2n$, and divisibility $\gamma\in\{1,2\}$ another polarized hyperkähler manifold (X',H') of dimension $2n+2$, degree $2m-2$, and same divisibility. Note that by formula (5), we have $h^0(X,H)=h^0(X',H')$. Of course, a true strange duality in the sense of Le Potier would require a canonical isomorphism $H^0(X,H)\xrightarrow{\sim}H^0(X',H')^\vee$.

There is a very nice interpretation of the birational isomorphism (14) when $m=2$, $n=3$, and $\gamma=2$. Recall from Section 3.6.1 that a general element of ${}^2\mathscr{M}_6^{(2)}$ is the variety $F(W)\hookrightarrow \mathsf{Gr}(2,6)\subset\mathbf{P}^{14}$ of lines contained in a smooth cubic hypersurface $W\subset\mathbf{P}^5$. In Section 3.6.3, we explained the construction of an element $X(W)\xrightarrow{72:1}\mathsf{Gr}(4,6)\subset\mathbf{P}^{14}$ of ${}^4\mathscr{M}_2^{(2)}$ associated with W; this is the strange duality correspondence.[37] Geometrically, one can recover W (hence also $F(W)$) from $X(W)$ as one of the components of the fixed locus of the canonical involution on $X(W)$ (footnote 43). There is also a canonical rational map

$$F(W)\times F(W)\dashrightarrow X(W) \tag{15}$$

of degree 6 described geometrically in [Vo1, Proposition 4.8] and the canonical involution on $X(W)$ is induced by the involution of $F(W)\times F(W)$ that interchanges the two factors.

Similarly, a Debarre–Voisin fourfold $X\subset\mathsf{Gr}(6,10)$ (a general element of ${}^2\mathscr{M}_{22}^{(2)}$; see Section 3.6.1) has an associated hyperkähler manifold $(X',H')\in{}^{12}\mathscr{M}_2^{(2)}$ of dimension 24 with $(H')^{24}=2^{12}\frac{24!}{12!2^{12}}=\frac{24!}{12!}$. By analogy with the construction above, could it be that there is a dominant rational map $X'\dashrightarrow\mathsf{Gr}(4,10)$?

Finally, a general element (X,H) of ${}^2\mathscr{M}_{38}^{(2)}$ (geometrically described in Section 3.6.1) should have an associated hyperkähler manifold $(X',H')\in{}^{20}\mathscr{M}_2^{(2)}$ of dimension 40, but I have no idea how to construct it geometrically.

Remark 3.22. There is a chain of subgroups[38] of finite index

$$\widetilde{O}(h_\tau^\perp)\xhookrightarrow{\iota_1}\widetilde{O}(\Lambda_{\mathrm{K3}^{[m]}},h_\tau)\xhookrightarrow{\iota_2}\widehat{O}(\Lambda_{\mathrm{K3}^{[m]}},h_\tau)\xhookrightarrow{\iota_3}O(\Lambda_{\mathrm{K3}^{[m]}},h_\tau)\xhookrightarrow{\iota_4}O(h_\tau^\perp)$$

(the lattices $h_\tau^\perp$ are described in [GHS2, Proposition 3.6.(iv)]) hence a further finite morphism

$$\mathscr{P}_\tau=\widehat{O}(\Lambda_{\mathrm{K3}^{[m]}},h_\tau)\backslash\Omega_{h_\tau}\longrightarrow O(h_\tau^\perp)\backslash\Omega_{h_\tau}$$

which is sometimes non-trivial.[39]

[37]The fact that the period points of $F(W)$ and $X(W)$ are identified via the isomorphism (14) is proved in [LPZ, Proposition 1.3].

[38]Let G be the subgroup $\Lambda_{\mathrm{K3}^{[m]}}/(\mathbf{Z}h_\tau\oplus h_\tau^\perp)$ of $D(\mathbf{Z}h_\tau)\times D(h_\tau^\perp)$. An element of $O(h_\tau^\perp)$ is in $O(\Lambda_{\mathrm{K3}^{[m]}},h_\tau)$ if and only if it induces the identity on $p_2(G)\subset D(h_\tau^\perp)$. This is certainly the case if it is in $\widetilde{O}(h_\tau^\perp)$ and the lift is then in $\widetilde{O}(\Lambda_{\mathrm{K3}^{[m]}},h_\tau)$ since it induces the identity on $D(\mathbf{Z}h_\tau)\times D(h_\tau^\perp)$, hence on its subquotient $D(\Lambda_{\mathrm{K3}^{[m]}})$.

[39]The following holds:
- the inclusion ι_1 is an equality if $\gcd(\frac{2n}{\gamma},\frac{2m-2}{\gamma},\gamma)=1$ ([GHS2, Proposition 3.12(i)]);
- the inclusion ι_2 is an equality if $m=2$, and has index 2 if $m>2$ ([GHS3, Remark 3.15]);
- the inclusion ι_3 defines a normal subgroup and is an equality if $m-1$ is a prime power (Section 3.8);
- the inclusion $\iota_3\iota_2$ defines a normal subgroup and, if $\gcd(\frac{2n}{\gamma},\frac{2m-2}{\gamma},\gamma)=1$, the corresponding quotient is the group $(\mathbf{Z}/2\mathbf{Z})^\alpha$, where $\alpha=\rho\big(\frac{m-1}{\gamma}\big)$ when γ is odd, and $\alpha=\rho\big(\frac{2m-2}{\gamma}\big)+\varepsilon$ when γ is even, with $\varepsilon=1$ if $\frac{2m-2}{\gamma}\equiv 0\pmod 8$, $\varepsilon=-1$ if $\frac{2m-2}{\gamma}\equiv 2\pmod 4$, and $\varepsilon=0$ otherwise ([GHS2, Proposition 3.12(ii)]).

Assume $\gamma \in \{1,2\}$. When $n = m-1$ (and n even when $\gamma = 2$), (13) gives an involution of ${}^m\mathscr{P}^{(\gamma)}_{2m-2}$ which corresponds to the element of $O(h_\tau^\perp)$ that switches the two factors of $D(h_\tau^\perp)$. This isometry is not in $O(\Lambda_{\mathrm{K3}^{[m]}}, h_\tau)$ (hence the involution of ${}^m\mathscr{P}^{(\gamma)}_{2m-2}$ is non-trivial), except when $n = \gamma = 2$, where the involution is trivial.

Assume now $m = 2$. The inclusions ι_1, ι_2, and ι_3 are then equalities and the group $O(h_\tau^\perp)/\widetilde{O}(h_\tau^\perp) \simeq O(D(h_\tau^\perp))$ acts on $\mathscr{P}_\tau$ (where $-\mathrm{Id}$ acts trivially). Here are some examples:

- if $\gamma = 1$ and n is odd, we obtain a generically free action of the group $(\mathbf{Z}/2\mathbf{Z})^{\rho(n)-1}$ when $n \equiv -1 \pmod 4$, and of the group $(\mathbf{Z}/2\mathbf{Z})^{\max\{\rho(n),1\}}$ when $n \equiv 1 \pmod 4$.[40]
- if $\gamma = 2$ and $n \equiv -1 \pmod 4$, we obtain a generically free action of the group $(\mathbf{Z}/2\mathbf{Z})^{\rho(n)-1}$.

These free actions translate into the existence of non-trivial (birational) involutions on the dense open subset ${}^2\mathscr{M}^{(\gamma)}_{2n}$ of ${}^2\mathscr{P}^{(\gamma)}_{2n}$. When $\gamma = n = 1$, O'Grady gave in [O3] a geometric description of the corresponding involution (general elements of ${}^2\mathscr{M}^{(1)}_{2}$ are double EPW sextics). When $\gamma = 2$, all cases where we have a geometric description of general elements of ${}^2\mathscr{M}^{(2)}_{2n}$ have n prime (Section 3.6.1), so there are no involutions.

The quotient of ${}^2\mathscr{P}^{(1)}_{2}$ by its non-trivial involution is isomorphic to ${}^3\mathscr{P}^{(2)}_{4}$. The authors of [IKKR1] speculate that given a Lagrangian A, the image of the period point of the double EPW sextic $\widetilde{Y}_A \in {}^2\mathscr{P}^{(1)}_{2}$ is the period point of the EPW cube associated with A (private discussion).

3.10. **The Noether–Lefschetz locus.** Let (X,H) be a polarized hyperkähler manifold of $\mathrm{K3}^{[m]}$-type with period $p(X,H) \in \Omega_{h_\tau}$. As in the case of K3 surfaces, the Picard group of X can be identified with the subgroup of $\Lambda_{\mathrm{K3}^{[m]}}$ generated by

$$p(X,H)^\perp \cap \Lambda_{\mathrm{K3}^{[m]}}$$

(which contains h_τ). This means that if the period of (X,H) is outside the countable union

$$\bigcup_K \mathbf{P}(K^\perp \otimes \mathbf{C}) \subset \Omega_{h_\tau}$$

of hypersurfaces, where K runs over the countable set of primitive, rank-2, signature-$(1,1)$ sublattices of $\Lambda_{\mathrm{K3}^{[m]}}$ containing h_τ, the group $\mathrm{Pic}(X)$ is generated by H. We let $\mathscr{D}_{\tau,K} \subset \mathscr{P}_\tau$ be the image of $\mathbf{P}(K^\perp \otimes \mathbf{C})$ and set, for each positive integer d,

$$\mathscr{D}_{\tau,d} := \bigcup_{K,\ \mathrm{disc}(K^\perp)=-d} \mathscr{D}_{\tau,K} \subset \mathscr{P}_\tau. \tag{16}$$

The inverse image in ${}^m\mathscr{M}_\tau$ of $\bigcup_d \mathscr{D}_{\tau,d}$ by the period map is called the *Noether–Lefschetz locus.* It consists of (isomorphism classes of) hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type with a polarization of type τ whose Picard group has rank at least 2.

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[40] By (6), we have $D(h_\tau^\perp) \simeq \mathbf{Z}/2n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, with $q(1,0) = -\frac{1}{2n}$ and $q(0,1) = -\frac{1}{2}$ in $\mathbf{Q}/2\mathbf{Z}$. When $n \equiv -1 \pmod 4$, one checks that any isometry must leave each factor of $D(h_\tau^\perp)$ invariant, so that $O(D(h_\tau^\perp)) \simeq O(\mathbf{Z}/2n\mathbf{Z})$. When $n \equiv 1 \pmod 4$, there are extra isometries $(0,1) \mapsto (n,0)$, $(1,0) \mapsto (a,1)$, where $a^2 + n \equiv 1 \pmod{4n}$.

Lemma 3.23. *Fix a polarization type τ and a positive integer d. The subset $\mathscr{D}_{\tau,d}$ is a finite union of (algebraic) hypersurfaces of $\mathscr{P}_\tau$.*

Proof. Let K be a primitive, rank-2, signature-$(1,1)$ sublattice of $\Lambda_{\mathrm{K3}^{[m]}}$ containing h_τ and let κ be a generator of $K\cap h_\tau^\perp$. Since K has signature $(1,1)$, we have $\kappa^2<0$ and the formula from [GHS3, Lemma 7.5] reads

$$d=\big|\operatorname{disc}(K^\perp)\big|=\left|\frac{\kappa^2\operatorname{disc}(h_\tau^\perp)}{s^2}\right|,$$

where s is the divisibility of κ in $h_\tau^\perp$. Since κ is primitive in $h_\tau^\perp$, the integer s is also the order of the element κ_* in the discriminant group $D(h_\tau^\perp)$ (Section 2.7). In particular, we have $s\le\operatorname{disc}(h_\tau^\perp)$, hence

$$|\kappa^2|=ds^2/\operatorname{disc}(h_\tau^\perp)\le d\operatorname{disc}(h_\tau^\perp).$$

Since τ is fixed, so is the (isomorphism class of the) lattice $h_\tau^\perp$, hence κ^2 can only take finitely many values. Since the element κ_* of the (finite) discriminant group $D(h_\tau^\perp)$ can only take finitely many values, Eichler's criterion (Section 2.7) implies that κ belongs to finitely many $\widetilde{O}(h_\tau^\perp)$-orbits, hence to finitely many $\widehat{O}(\Lambda_{\mathrm{K3}^{[m]}},h_\tau)$-orbits. Therefore, the images in $\mathscr{P}_\tau$ of the hypersurfaces $\mathscr{D}_{\tau,K}$ form finitely many hypersurfaces. $\square$

Describing the irreducible components of the loci $\mathscr{D}_{\tau,d}$ is a lattice-theoretic question (which we answer below in the case $m=2$). When the divisiblity γ is 1 or 2, the polarization type only depends on γ and the integer $h_\tau^2=:2n$ (Section 3.5). We use the notation ${}^m\mathscr{D}^{(\gamma)}_{2n,d}$ instead of $\mathscr{D}_{\tau,d}$ (and ${}^m\mathscr{P}^{(\gamma)}_{2n}$ instead of $\mathscr{P}_\tau$).

Proposition 3.24 (Debarre–Macrì)**.** *Let n and d be a positive integers and let $\gamma\in\{1,2\}$. If the locus ${}^2\mathscr{D}^{(\gamma)}_{2n,d}$ is non-empty, the integer d is even; we set $e:=d/2$.*

(1) (a) *The locus ${}^2\mathscr{D}^{(1)}_{2n,2e}$ is non-empty if and only if either e or $e-n$ is a square modulo $4n$.*

(b) *If n is* square-free *and e is divisible by n and satisfies the conditions in* (a)*, the locus ${}^2\mathscr{D}^{(1)}_{2n,2e}$ is an irreducible hypersurface, except when*

- *either $n\equiv 1\pmod 4$ and $e\equiv n\pmod{4n}$,*
- *or $n\equiv -1\pmod 4$ and $e\equiv 0\pmod{4n}$,*

in which cases ${}^2\mathscr{D}^{(1)}_{2n,2e}$ has two irreducible components.

(c) *If n is* prime *and e satisfies the conditions in* (a)*, ${}^2\mathscr{D}^{(1)}_{2n,2e}$ is an irreducible hypersurface, except when $n\equiv 1\pmod 4$ and $e\equiv 1\pmod 4$, or when $n\equiv -1\pmod 4$ and $e\equiv 0\pmod 4$, in which cases ${}^2\mathscr{D}^{(1)}_{2n,2e}$ has two irreducible components.*

(2) *Assume moreover $n\equiv -1\pmod 4$.*

(a) *The locus ${}^2\mathscr{D}^{(2)}_{2n,2e}$ is non-empty if and only if e is a square modulo n.*

(b) *If n is* square-free and $n\mid e$*, the locus ${}^2\mathscr{D}^{(2)}_{2n,2e}$ is an irreducible hypersurface.*

(c) *If n is* prime *and e satisfies the conditions in* (a)*, ${}^2\mathscr{D}^{(2)}_{2n,2e}$ is an irreducible hypersurface.*

In cases (1)(b) and (1)(c), when ${}^2\mathscr{D}^{(1)}_{2n,d}$ is reducible, its the two components are exchanged by one of the involutions of the period space $\mathscr{P}_\tau$ described at the end of Section 3.9 when $n \equiv 1 \pmod 4$, but not when $n \equiv -1 \pmod 4$ (in that case, these involutions are in fact trivial when n is prime).

Proof of the proposition. **Case $\gamma = 1$.** Let (u, v) be a standard basis for a hyperbolic plane U contained in $\Lambda_{\mathrm{K3}^{[2]}}$ and let ℓ be a basis for the $I_1(-2)$ factor. We may take $h_\tau := u + nv$ (it has the correct square and divisibility), in which case $h_\tau^\perp = \mathbf{Z}(u - nv) \oplus \mathbf{Z}\ell \oplus M$, where $M := \{u, v, \ell\}^\perp = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$ is unimodular. The discriminant group $D(h_\tau^\perp) \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$ is generated by $\ell_* = \ell/2$ and $(u - nv)_* = (u - nv)/2n$, with $\bar{q}(\ell_*) = -1/2$ and $\bar{q}((u - nv)_*) = -1/2n$.

Let κ be a generator of $K \cap h_\tau^\perp$. We write

$$\kappa = a(u - nv) + b\ell + cw,$$

where $w \in M$ is primitive. Since K has signature $(1, 1)$, we have $\kappa^2 < 0$ and the formula from [GHS3, Lemma 7.5] reads

$$d = \left|\operatorname{disc}(K^\perp)\right| = \left|\frac{\kappa^2 \operatorname{disc}(h_\tau^\perp)}{s^2}\right| = \frac{8n(na^2 + b^2 + rc^2)}{s^2} \equiv \frac{8n(na^2 + b^2)}{s^2} \pmod{8n}, \tag{17}$$

where $r := -\frac{1}{2}w^2$ and $s := \gcd(2na, 2b, c)$ is the divisibility of κ in $h_\tau^\perp$. If $s \mid b$, we obtain $d \equiv 2\left(\frac{2na}{s}\right)^2 \pmod{8n}$, which is the first case of the conclusion: d is even and $e := d/2$ is a square modulo $4n$. Assume $s \nmid b$ and, for any non-zero integer x, write $x = 2^{v_2(x)}x_{\mathrm{odd}}$, where x_{odd} is odd. One has then $\nu_2(s) = \nu_2(b) + 1$ and

$$d \equiv 2\Big(\frac{2na}{s}\Big)^2 + 2n\Big(\frac{b_{\mathrm{odd}}}{s_{\mathrm{odd}}}\Big)^2 \equiv 2\Big(\frac{2na}{s}\Big)^2 + 2n \pmod{8n},$$

which is the second case of the conclusion: d is even and $d/2 - n$ is a square modulo $4n$. It is then easy, taking suitable integers a, b, c, and vector w, to construct examples that show that these necessary conditions on d are also sufficient.

We now prove (b) and (c).

Given a lattice K containing h_τ with $\operatorname{disc}(K^\perp) = -2e$, we let as above κ be a generator of $K \cap h_\tau^\perp$. By Eichler's criterion (Theorem 2.9), the group $\widetilde{O}(h_\tau^\perp)$ acts transitively on the set of primitive vectors $\kappa \in h_\tau^\perp$ of given square and fixed $\kappa_* \in D(h_\tau^\perp)$. Since κ and $-\kappa$ give rise to the same lattice K (obtained as the saturation of $\mathbf{Z}h_\tau \oplus \mathbf{Z}\kappa$), the locus $\mathscr{D}^{(1)}_{2n,2e}$ will be irreducible (when non-empty) if we show that the integer e determines κ^2, and κ_* up to sign.

We write as above $\kappa = a(u - nv) + b\ell + cw \in h_\tau^\perp$, with $\gcd(a, b, c) = 1$ and $s = \operatorname{div}_{h_\tau^\perp}(\kappa) = \gcd(2na, 2b, c)$. From (17), we get

$$\kappa^2 = -es^2/2n = -2(na^2 + b^2 + rc^2) \quad \text{and} \quad \kappa_* = (2na/s, 2b/s) \in \mathbf{Z}/2n\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}. \tag{18}$$

If $s = 1$, we have $e \equiv 0 \pmod{4n}$ and $\kappa_* = 0$.

If $s=2$, the integer c is even and a and b cannot be both even (because κ is primitive). We have $e=n(na^2+b^2+rc^2)$ and

$$\begin{cases} e\equiv n^2 \pmod{4n} \quad \text{and } \kappa_*=(n,0) & \text{if } b \text{ is even (and } a \text{ is odd);}\\ e\equiv n \pmod{4n} \quad \text{and } \kappa_*=(0,1) & \text{if } b \text{ is odd and } a \text{ is even;}\\ e\equiv n(n+1) \pmod{4n} \quad \text{and } \kappa_*=(n,1) & \text{if } b \text{ and } a \text{ are odd.}\end{cases}$$

Assume now that n is square-free and $n\mid e$. From (17), we get $n \mid \left(\frac{2na}{s}\right)^2$, hence $s^2\mid 4na^2$, and $s\mid 2a$ because n is square-free. This implies $s=\gcd(2a,2b,c)\in\{1,2\}$.

When n is even (that is, $n\equiv 2 \pmod 4$), we see from the discussion above that both s (hence also κ^2) and κ_* are determined by e, so the corresponding hypersurfaces $\mathscr{D}^{(1)}_{2n,2e}$ are irreducible.

If n is odd, there are coincidences:

- when $n\equiv 1 \pmod 4$, we have $n\equiv n^2 \pmod{4n}$, hence $\mathscr{D}^{(1)}_{2n,2e}$ is irreducible when $e\equiv 0$ or $2n \pmod{4n}$, has two irreducible components (corresponding to $\kappa_*=(n,0)$ and $\kappa_*=(0,1)$) when $e\equiv n \pmod{4n}$, and is empty otherwise;
- when $n\equiv -1 \pmod 4$, we have $n(n+1)\equiv 0 \pmod{4n}$, hence $\mathscr{D}^{(1)}_{2n,2e}$ is irreducible when $e\equiv -n$ or $n \pmod{4n}$, has two irreducible components (corresponding to $\kappa_*=0$ and $\kappa_*=(n,1)$) when $e\equiv 0 \pmod{4n}$, and is empty otherwise.

This proves (b).

We now assume that n is prime and prove (c). Since $s\mid 2n$, we have $s\in\{1,2,n,2n\}$; the cases $s=1$ and $s=2$ were explained above. If $s=n$ (and n is odd), we have $n\mid b$, $n\mid c$, $n\nmid a$, and

$$e\equiv 4a^2 \pmod{4n} \quad \text{and} \quad \kappa_*=(2a,0).$$

If $s=2n$, the integer c is even, a and b cannot be both even, $n\mid b$, and $n\nmid a$. We have

$$\begin{cases} e\equiv a^2 \pmod{4n} \quad \text{and } \kappa_*=(a,0) & \text{if } 2n\mid b \text{ (hence } a \text{ is odd);}\\ e\equiv a^2+n \pmod{4n} \quad \text{and } \kappa_*=(a,1) & \text{if } b \text{ is odd (and } n \text{ is odd);}\\ e\equiv a^2+2 \pmod{8} \quad \text{and } \kappa_*=(a,1) & \text{if } 4\nmid b \text{ is odd and } n=2.\end{cases}$$

When $n=2$, one checks that the class of e modulo 8 (which is in $\{0,1,2,3,4,6\}$) completely determines s, and κ_* up to sign. The corresponding divisors $\mathscr{D}^{(1)}_{4,2e}$ are therefore all irreducible.

When $n\equiv 1 \pmod 4$, we have $n\equiv n^2 \pmod{4n}$ and $a^2\equiv (n-a)^2+n \pmod{4n}$ when a is odd (in which case $a^2\equiv 1 \pmod 4$). When $n\equiv -1 \pmod 4$, we have $n(n+1)\equiv 0 \pmod{4n}$ and $a^2\equiv (n-a)^2+n \pmod{4n}$ when a is even (in which case $a^2\equiv 0 \pmod 4$). Together with changing a into $-a$ (which does not change the lattice K), these are the only coincidences: the corresponding divisors $\mathscr{D}^{(1)}_{2n,2e}$ therefore have two components and the others are irreducible. This proves (c).

Case $\gamma=2$ (hence $n\equiv -1 \pmod 4$). We may take $h_\tau := 2\big(u+\frac{n+1}{4}v\big)+\ell$, in which case $h_\tau^\perp=\mathbf{Z}w_1\oplus\mathbf{Z}w_2\oplus M$, with $w_1:=v+\ell$ and $w_2:=-u+\frac{n+1}{4}v$. The intersection form on

$\mathbf{Z}w_1 \oplus \mathbf{Z}w_2$ has matrix $\begin{pmatrix} -2 & -1 \\ -1 & -\frac{n+1}{2} \end{pmatrix}$ as in (7) and the discriminant group $D(h_\tau^\perp) \simeq \mathbf{Z}/n\mathbf{Z}$ is generated by $(w_1 - 2w_2)_* = (w_1 - 2w_2)/n$, with $\bar{q}((w_1 - 2w_2)_*) = -2/n$.

Let (h_τ, κ') be a basis for K, so that $\operatorname{disc}(K) = 2n\kappa'^2 - (h_\tau \cdot \kappa')^2$. Since $\operatorname{div}(h_\tau) = \gamma = 2$, the integer $h_\tau \cdot \kappa'$ is even and since κ'^2 is also even (because $\Lambda_{\mathrm{K3}^{[2]}}$ is an even lattice), we have $4 \mid \operatorname{disc}(K)$ and $-\operatorname{disc}(K)/4$ is a square modulo n. Since the discriminant of $\Lambda_{\mathrm{K3}^{[2]}}$ is 2, the integer $d = |\operatorname{disc}(K^\perp)|$ is either $2\,|\operatorname{disc}(K)|$ or $\frac{1}{2}\,|\operatorname{disc}(K)|$, hence it is even and $e = d/2$ is a square modulo n, as desired.

Conversely, it is easy to construct examples that show that these necessary conditions on d are also sufficient. This proves (a).

We now prove (b) and (c). To prove that the loci $\mathscr{D}^{(2)}_{2n,2e}$ are irreducible (when non-empty), we need to show that e determines κ^2, and κ_* up to sign (where κ is a generator of $K \cap h_\tau^\perp$).

With the notation above, we have $\kappa = ((h_\tau \cdot \kappa')h_\tau - 2n\kappa')/t$, where $t := \gcd(h_\tau \cdot \kappa', 2n)$ is even and $\kappa^2 = \frac{2n}{t^2}\operatorname{disc}(K)$. Formula (17) then gives

$$2e = \left|\operatorname{disc}(K^\perp)\right| = \left|\frac{\kappa^2 \operatorname{disc}(h_\tau^\perp)}{\operatorname{div}_{h_\tau^\perp}(\kappa)^2}\right| = \left|\frac{2n^2 \operatorname{disc}(K)}{t^2 \operatorname{div}_{h_\tau^\perp}(\kappa)^2}\right|.$$

Since n is odd and t is even, and, as we saw above, $\operatorname{disc}(K) \in \{-e, -4e\}$, the only possibility is $\operatorname{disc}(K) = -4e$ and $t \operatorname{div}_{h_\tau^\perp}(\kappa) = 2n$.

Assume that n is square-free and $n \mid e$. Since $-4e = \operatorname{disc}(K) = 2n\kappa'^2 - (h_\tau \cdot \kappa')^2$, we get $2n \mid (h_\tau \cdot \kappa')^2$ hence, since n is square-free and odd, $2n \mid h_\tau \cdot \kappa'$. This implies $t = 2n$ and $\operatorname{div}_{h_\tau^\perp}(\kappa) = 1$; in particular, $\kappa_* = 0$ and $\kappa^2 = -2e/n$ are uniquely determined. This proves (b).

We now assume that n is prime. Since $t \operatorname{div}_{h_\tau^\perp}(\kappa) = 2n$ and t is even,

- either $(t, \operatorname{div}_{h_\tau^\perp}(\kappa), \kappa^2) = (2n, 1, -2e/n)$ and $n \mid e$;
- or $(t, \operatorname{div}_{h_\tau^\perp}(\kappa), \kappa^2) = (2, n, -2ne)$ and $n \nmid e$ (because $n \nmid h_\tau \cdot \kappa'$ and $d = -\frac{1}{2}\operatorname{disc}(K) \equiv \frac{1}{2}(h_\tau \cdot \kappa')^2 \pmod{n}$).

Given $e = a^2 + nn'$, the integer κ^2 is therefore uniquely determined by e:

- either $n \mid a$, $\kappa^2 = -2e/n$, and $\kappa_* = 0$;
- or $n \nmid a$, $\kappa^2 = -2ne$, $\kappa_* = \kappa/n$, and $\bar{q}(\kappa_*) = -2a^2/n \pmod{2\mathbf{Z}}$.

In the second case, $\kappa_* = \pm a(w_1 - 2w_2)_*$; it follows that in all cases, κ_* is also uniquely defined, up to sign, by e. This proves (c). ☐

Example 3.25 (Case $n = \gamma = 1$). The moduli space ${}^2\mathscr{M}_2^{(1)}$ is irreducible and contains a dense open subset $\mathscr{U}_2^{(1)}$ whose points correspond to double EPW sextics (Section 3.6.1). The complement ${}^2\mathscr{M}_2^{(1)} \smallsetminus \mathscr{U}_2^{(1)}$ contains the irreducible hypersurface $\mathscr{H}_2^{(1)}$ whose general points correspond to pairs $(S^{[2]}, L_2 - \delta)$, where (S, L) is a polarized K3 surface of degree 4 (Example 3.10).

The loci ${}^2\mathscr{C}^{(1)}_{2,d}$ (for $d > 0$) were studied in [DIM]. Assuming $d \equiv 0, 2,$ or $4 \pmod{8}$ (which is, according to Proposition 3.24, the condition for ${}^2\mathscr{D}^{(1)}_{2,d}$ to be non-empty), they were shown to be non-empty if and only if $d \notin \{2,8\}$ (see Remark 3.28 for a more precise statement). The locus ${}^2\mathscr{C}^{(1)}_{2,4}$ is the hypersurface $\mathscr{H}^{(1)}_2$.

There is a way to associate a double EPW sextic with any smooth *Gushel–Mukai fourfold* (these are by definition complete intersections of $\mathsf{Gr}(2,V_5) \subset \mathbf{P}(\bigwedge^2 V_5) = \mathbf{P}^9$ with a hyperplane and a quadric hypersurface; see [DK]). Double EPW sextics with small discriminant d correspond to Gushel–Mukai fourfolds with special geometric features ([DIM]).

Example 3.26 (Case $n = 3$ and $\gamma = 2$)**.** The moduli space ${}^2\mathscr{M}^{(2)}_6$ is irreducible and contains a dense open subset $\mathscr{U}^{(2)}_6$ whose points correspond to the varieties of lines $X := F(W)$ contained in a cubic fourfold $W \subset \mathbf{P}^5$ (Section 3.6.1). The complement ${}^2\mathscr{M}^{(2)}_6 \smallsetminus \mathscr{U}^{(2)}_6$ contains the irreducible hypersurface $\mathscr{H}^{(2)}_6$ whose general points correspond to pairs $(S^{[2]}, 2L_2 - \delta)$, where (S, L) is a polarized K3 surface of degree 2.

The loci ${}^2\mathscr{C}^{(2)}_{6,d}$ (for $d > 0$) were originally introduced and studied by Hassett in [H]. Assuming $d \equiv 0$ or $2 \pmod 6$ (which, according to Proposition 3.24, is the condition for ${}^2\mathscr{D}^{(2)}_{6,d}$ to be non-empty), he showed that they are non-empty if and only if $d \neq 6$ (see also Remark 3.28). The locus ${}^2\mathscr{C}^{(2)}_{6,2}$ is the hypersurface $\mathscr{H}^{(2)}_6$ (see [vD] for a very nice description of the geometry involved).

Small d correspond to cubics W with special geometric features: W contains a plane ($d = 8$), W contains a cubic scroll ($d = 12$), W is a Pfaffian cubic ($d = 14$), W contains a Veronese surface ($d = 20$).

3.11. **The image of the period map.** In this section, we study the image of the period map

$$\wp_\tau \colon {}^m\mathscr{M}_\tau \longrightarrow \mathscr{P}_\tau$$

defined in Section 3.9 (here τ is a polarization type for polarized hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type). For K3 surfaces ($m = 1$), we saw in Section 2.8 that this image is precisely the complement of the (images in $\mathscr{P}_\tau$ of the) hypersurfaces $y^\perp$, for all $y \in h_\tau^\perp$ with $y^2 = -2$.

It follows from the results of [AV1] that for all $m \geq 2$ (which we assume for now on), the image of the period map is the complement of the union of finitely many Heegner divisors (that is, of the type studied in Section 3.10). We will explain how to determine explicitly (in principle) these divisors. We will only do here the case $m = 2$; the general case will be treated in Appendix B.

Theorem 3.27. *Let n be a positive integer and let $\gamma \in \{1,2\}$. The image of the period map*

$${}^2\wp^{(\gamma)}_{2n} \colon {}^2\mathscr{M}^{(\gamma)}_{2n} \longrightarrow O(\Lambda_{\mathrm{K3}^{[2]}}, h_\tau)\backslash\Omega_\tau$$

is exactly the complement of the union of finitely many explicit Heegner divisors. More precisely, these Heegner divisors are

- *if $\gamma = 1$,*
  - *some irreducible components of the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,2n}$ (two components if $n \equiv 0$ or $1 \pmod 4$, one component if $n \equiv 2$ or $3 \pmod 4$);*
  - *one irreducible component of the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,8n}$;*
  - *one irreducible component of the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,10n}$;*

– *and, if* $n = 5^{2\alpha+1}n''$*, with* $\alpha \geq 0$ *and* $n'' \equiv \pm 1 \pmod 5$*, some irreducible components of the hypersurface* ${}^2\mathscr{D}^{(1)}_{2n,2n/5}$*;*

- *if* $\gamma = 2$ *(and* $n \equiv -1 \pmod 4$*), one irreducible component of the hypersurface* ${}^2\mathscr{D}^{(2)}_{2n,2n}$.

Remark 3.28. We proved in Proposition 3.24 that when n is square-free (so in particular $n \not\equiv 0 \pmod 4$),

- the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,2n}$ has two components if $n \equiv 1 \pmod 4$, one component otherwise;
- the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,8n}$ has two components if $n \equiv -1 \pmod 4$, one component otherwise;
- the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,10n}$ has two components if $n \equiv 1 \pmod 4$, one component otherwise;
- the hypersurface ${}^2\mathscr{D}^{(2)}_{2n,2n}$ is irreducible (when $n \equiv -1 \pmod 4$).

The complement of the image of ${}^2\wp^{(1)}_2$ is therefore the union of ${}^2\mathscr{D}^{(1)}_{2,2}$, ${}^2\mathscr{D}^{(1)}_{2,8}$, and one of the two components of ${}^2\mathscr{D}^{(1)}_{2,10}$ (compare with Example 3.25).

The complement of the image of ${}^2\wp^{(2)}_6$ is ${}^2\mathscr{D}^{(2)}_{6,6}$ (compare with Example 3.26).

Proof of Theorem 3.27. Take a point $x \in {}^2\mathscr{P}^{(\gamma)}_{2n}$. Since the period map for smooth compact (not necessarily projective) hyperkähler fourfolds is surjective ([Hu2, Theorem 8.1]), there exists a compact hyperkähler fourfold X' with the given period point x. Since the class h_τ is algebraic and has positive square, X' is projective by [Hu2, Theorem 3.11]. Moreover, the class h_τ corresponds to the class of an integral divisor H in the positive cone of X'. By Remark 3.15(a), we can let an element in the group W_{Exc} act and assume that the pair (X', H), representing the period point x and the class h_τ, is such that H is in $\overline{\mathrm{Mov}(X')} \cap \mathrm{Pos}(X')$. By Remark 3.15(b), we can find a projective hyperkähler fourfold X which is birational to X' (and so still has period x), such that the divisor H, with class h_τ, is nef and big on X and has divisibility γ. By [Hu2, Theorem 4.6], the fourfold X' is deformation equivalent to X, hence still of type $\mathrm{K3}^{[2]}$.

To summarize, the point x is in the image of the period map ${}^2\wp^{(\gamma)}_{2n}$ if and only if H is actually ample on X. We now use Theorem 3.14: H is ample if and only if it is not orthogonal to any algebraic class either with square -2, or with square -10 and divisibility 2.

If H is orthogonal to an algebraic class v with square -2, the Picard group of X contains a rank-2 lattice K with intersection matrix $\begin{pmatrix} 2n & 0 \\ 0 & -2 \end{pmatrix}$; the fourfold X is therefore special of discriminant $2e := -\operatorname{disc}(K^\perp)$ (its period point is in the hypersurface ${}^2\mathscr{D}^{(\gamma)}_{2n,K}$). In the notation of the proof of Proposition 3.24, v is the class κ.

If $\gamma = 1$, the divisibility $s := \operatorname{div}_{K^\perp}(\kappa)$ is either 1 or 2. By (18), we have $es^2 = -2n\kappa^2 = 4n$, hence

- either $s = 1$, $e = 4n$, and $\kappa_* = 0$: the period point is then in one irreducible component of the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,8n}$;
- or $s = 2$, $e = n$, and

– either $\kappa_* = (0,1)$;
– or $\kappa_* = (n,0)$ and $n \equiv 1 \pmod 4$;
– or $\kappa_* = (n,1)$ and $n \equiv 0 \pmod 4$.

The period point x is in one irreducible component of the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,2n}$ if $n \equiv 2$ or $3 \pmod 4$, or in the union of two such components otherwise.

If $\gamma = 2$, we have $e = -\operatorname{disc}(K)/4 = n$, $t = \sqrt{2n\operatorname{disc}(K)/\kappa^2} = 2n$, and $\operatorname{div}(\kappa) = 2n/t = 1$, hence $\kappa_* = 0$: the period point x is in one irreducible component of the hypersurface ${}^2\mathscr{D}^{(2)}_{2n,2n}$.

If H is orthogonal to an algebraic class with square -10 and divisibility 2, the Picard group of X contains a rank-2 lattice K with intersection matrix $\begin{pmatrix} 2n & 0 \\ 0 & -10\end{pmatrix}$, hence X is special of discriminant $2e := -\operatorname{disc}(K^\perp)$. Again, we distinguish two cases, keeping the same notation.

If $\gamma = 1$, the divisibility $s := \operatorname{div}_{h_\tau^\perp}(\kappa)$ is even (because the divisibility in $H^2(X,\mathbf{Z})$ is 2) and divides $\kappa^2 = -10$, hence it is either 2 or 10. Moreover, $es^2 = -2n\kappa^2 = 20n$, hence

- either $s = 2$, $e = 5n$, and $\kappa_* = (0,1)$: the period point x is then in one irreducible component of the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,10n}$;
- or $s = 10$ and $e = n/5$: the period point is then in the hypersurface ${}^2\mathscr{D}^{(1)}_{2n,n/5}$.

In the second case, since the divisibility of κ in $H^2(X,\mathbf{Z})$ is 2, a and c are even, so that b is odd and $\kappa_* = (a,1)$. Set $a' := a/2$ and $n' := n/5$; we have $e \equiv a^2 + n \pmod{4n}$, hence $a'^2 \equiv -n' \pmod{5n'}$. Write $a' = 5^\alpha a''$ and $n' = 5^\beta n''$, with a'' and n'' prime to 5. This congruence then reads $5^{2\alpha}a''^2 \equiv -5^\beta n'' \pmod{5^{\beta+1}n''}$, which implies $\beta = 2\alpha$ and $a''^2 \equiv -n'' \pmod{5n''}$. Finally, this last congruence is equivalent to $a''^2 \equiv 0 \pmod{n''}$ and $a''^2 \equiv -n'' \pmod 5$; these congruences are solvable (in a'') if and only if $n'' \equiv \pm 1 \pmod 5$.

In general, there are many possibilities for $a = 2\cdot 5^\alpha a''$ (modulo $2n$). However, if n'' is square-free, we have $a'' \equiv 0 \pmod{n''}$ and $\pm a$ (hence also $\pm\kappa_*$) is well determined (modulo $2n$), so we have a single component of ${}^2\mathscr{D}^{(1)}_{2n,n/5}$.

If $\gamma = 2$, we have $e = -\operatorname{disc}(K)/4 = 5n$ and $t^2 = 2n\operatorname{disc}(K)/\kappa^2 = n^2/10$, which is impossible.

Conversely, in each case described above, it is easy to construct a class κ with the required square and divisibility which is orthogonal to H. □

Example 3.29 (Case $n = \gamma = 1$)**.** We keep the notation of Example 3.25. O'Grady proved that the image of $\mathscr{U}_2^{(1)}$ in the period space does not meet $\mathscr{D}^{(1)}_{2,2}$, $\mathscr{D}^{(1)}_{2,4}$, $\mathscr{D}^{(1)}_{2,8}$, and one component of $\mathscr{D}^{(1)}_{2,10}$ ([O4, Theorem 1.3][41]); moreover, by [DIM, Theorem 8.1], this image does meet all the other components of the non-empty hypersurfaces $\mathscr{D}^{(1)}_{2,d}$. The hypersurface $\mathscr{H}_2^{(1)}$ maps to $\mathscr{D}^{(1)}_{2,4}$.

These results agree with Theorem 3.27 and Remark 3.28, which say that the image of $\mathscr{M}_2^{(1)}$ in the period space is the complement of the union of $\mathscr{D}^{(1)}_{2,2}$, $\mathscr{D}^{(1)}_{2,8}$, and one of the two components of $\mathscr{D}^{(1)}_{2,10}$. However, our theorem says nothing about the image of $\mathscr{U}_2^{(1)}$.

[41] O'Grady's hypersurfaces $\mathbb{S}'_2 \cup \mathbb{S}''_2$, $\mathbb{S}^\star_2$, $\mathbb{S}_4$ are our $\mathscr{D}^{(1)}_{2,2}$, $\mathscr{D}^{(1)}_{2,4}$, $\mathscr{D}^{(1)}_{2,8}$.

O'Grady conjectures that it is the complement of the hypersurfaces $\mathscr{D}^{(1)}_{2,2}$, $\mathscr{D}^{(1)}_{2,4}$, $\mathscr{D}^{(1)}_{2,8}$, and one component of $\mathscr{D}^{(1)}_{2,10}$; this would follow if one could prove $\mathscr{M}^{(1)}_2 = \mathscr{U}^{(1)}_2 \cup \mathscr{H}^{(1)}_2$.

Example 3.30 (Case $n = 3$ and $\gamma = 2$)**.** We keep the notation of Example 3.26. Theorem 3.27 and Remark 3.28 say that the image of $\mathscr{M}^{(2)}_6$ in the period space is the complement of the irreducible hypersurface $\mathscr{D}^{(2)}_{6,6}$. This (and much more) was first proved by Laza in [Laz, Theorem 1.1], together with the fact that $\mathscr{M}^{(2)}_6 = \mathscr{U}^{(2)}_6 \cup \mathscr{H}^{(2)}_6$; since $\mathscr{H}^{(2)}_6$ maps onto $\mathscr{D}^{(2)}_{6,2}$, the image of $\mathscr{U}^{(2)}_6$ is the complement of $\mathscr{D}^{(2)}_{6,2} \cup \mathscr{D}^{(2)}_{6,6}$.

4. Automorphisms of hyperkähler manifolds

We determine the group $\mathrm{Aut}(X)$ of biregular automorphisms and the group $\mathrm{Bir}(X)$ of birational automorphisms for some hyperkähler manifolds X of $\mathrm{K3}^{[m]}$-type with Picard number 1 or 2.

4.1. The orthogonal representations of the automorphism groups.

Let X be a hyperkähler manifold. As in the case of K3 surfaces, we have $H^0(X, T_X) \simeq H^0(X, \Omega^1_X) = 0$ and the group $\mathrm{Aut}(X)$ of biholomorphic automorphisms of X is discrete. We introduce the two representations

$$\Psi^A_X\colon \mathrm{Aut}(X) \longrightarrow O(H^2(X, \mathbf{Z}), q_X) \quad \text{and} \quad \overline{\Psi}^A_X\colon \mathrm{Aut}(X) \longrightarrow O(\mathrm{Pic}(X), q_X). \tag{19}$$

By Proposition 3.13, the group $\mathrm{Bir}(X)$ of birational automorphisms of X also acts on $O(H^2(X, \mathbf{Z}), q_X)$. We may therefore define two other representations

$$\Psi^B_X\colon \mathrm{Bir}(X) \longrightarrow O(H^2(X, \mathbf{Z}), q_X) \quad \text{and} \quad \overline{\Psi}^B_X\colon \mathrm{Bir}(X) \longrightarrow O(\mathrm{Pic}(X), q_X). \tag{20}$$

We have of course $\mathrm{Ker}(\Psi^A_X) \subset \mathrm{Ker}(\overline{\Psi}^A_X)$ and $\mathrm{Ker}(\Psi^B_X) \subset \mathrm{Ker}(\overline{\Psi}^B_X)$.

Proposition 4.1 (Huybrechts, Beauville)**.** *Let X be a hyperkähler manifold. The kernels of Ψ^A_X and Ψ^B_X are equal and finite and, if X is projective, the kernels of $\overline{\Psi}^B_X$ and $\overline{\Psi}^A_X$ are equal and finite.*

If X is of $\mathrm{K3}^{[m]}$ type, Ψ^A_X and Ψ^B_X are injective.

Sketch of proof. If X is projective and H is an ample line bundle on X, and if σ is a birational automorphism of X that acts trivially on $\mathrm{Pic}(X)$, we have $\sigma^*(H) = H$ and this implies that σ is an automorphism (Proposition 3.13). We have therefore $\mathrm{Ker}(\overline{\Psi}^B_X) = \mathrm{Ker}(\overline{\Psi}^A_X)$ (and $\mathrm{Ker}(\Psi^B_X) = \mathrm{Ker}(\Psi^A_X)$) and the latter group is a discrete subgroup of a general linear group hence is finite.

For the proof in the general case (when X is only assumed to be Kähler), we refer to [Hu2, Proposition 9.1] (one replaces H with a Kähler class).

When X is a punctual Hilbert scheme of a K3 surface, Beauville proved that Ψ^A_X is injective ([B2, Proposition 10]). It was then proved in [HT2, Theorem 2.1] that the kernel of Ψ^A_X is invariant by smooth deformations. □

Via the representation $\overline{\Psi}_X^A$, any automorphism of X preserves the cone $\mathrm{Nef}(X)$ and via the representation $\overline{\Psi}_X^B$, any birational automorphism of X preserves the cone $\mathrm{Mov}(X)$. The Torelli theorem (Theorem 3.18) implies that any element of $\widehat{O}(H^2(X,\mathbf{Z}),q_X)$ which is a Hodge isometry and preserves an ample class is induced by an automorphism of X.

Remark 4.2. Let X be a hyperkähler manifold. Oguiso proved in [Og4] and [Og5] that *when X is not projective,* both groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ are almost abelian finitely generated. When X is projective, the group $\mathrm{Bir}(X)$ is still finitely generated ([BS]) but whether the group $\mathrm{Aut}(X)$ is finitely generated is unknown.

4.2. **Automorphisms of very general polarized hyperkähler manifolds.** The Torelli theorem allows us to "read" biregular automorphisms of a hyperkähler manifold on its second cohomology lattice. For birational automorphisms, this is more complicated but there are still necessary conditions: a birational automorphism must preserve the movable cone. The next proposition describes the automorphism groups, both biregular and birational, for a very general polarized hyperkähler manifold of $\mathrm{K3}^{[}m]$-type; at best, we get groups of order 2 (as in the case of polarized K3 surfaces; see Proposition 2.15).

Proposition 4.3. *Let (X,H) be a polarized hyperkähler manifold corresponding to a very general point in any component of a moduli space ${}^m\mathscr{M}_{2n}^{(\gamma)}$ (with $m\geq 2$). The group $\mathrm{Bir}(X)$ of birational automorphisms of X is trivial, unless*

- $n=1$;
- $n=\gamma=m-1$ *and* -1 *is a square modulo* $m-1$.

In each of these cases, $\mathrm{Aut}(X)=\mathrm{Bir}(X)\simeq\mathbf{Z}/2\mathbf{Z}$ *and the corresponding involution of X is antisymplectic.*

Proof. As we saw in Section 3.10, the Picard group of X is generated by the class H. Any birational automorphism leaves this class fixed, hence is in particular biregular of finite order. Let σ be a non-trivial automorphism of X. Since σ extends to small deformations of (X,H), the restriction of σ^* to $H^\perp$ is a homothety[42] whose ratio is, by [B1, Proposition 7], a root of unity; since it is real and non-trivial (because Ψ_X^A is injective by Proposition 4.1), it must be -1. We will study under which conditions such an isometry of $\mathbf{Z}H\oplus H^\perp$ extends to an isometry $\varphi\in\widehat{O}(H^2(X,\mathbf{Z}),q_X)$.

Choose an identification $H^2(X,\mathbf{Z})\simeq\Lambda_{\mathrm{K3}^{[m]}}$ and write, as in Section 3.5, $H=ax+b\delta$, where $a,b\in\mathbf{Z}$ are relatively prime, x is primitive in Λ_{K3}, and $\gamma:=\mathrm{div}(H)=\gcd(a,2m-2)$. Using Eichler's criterion, we may write $x=u+cv$, where (u,v) is a standard basis for a hyperbolic plane in Λ_{K3} and $c=\frac{1}{2}x^2\in\mathbf{Z}$. We then have $\varphi(a(u+cv)+b\delta)=a(u+cv)+b\delta$ and

$$\varphi(u-cv) = -u+cv, \tag{21}$$
$$\varphi(b(2m-2)v+a\delta) = -b(2m-2)v-a\delta \tag{22}$$

[42] The argument is classical: let X be a hyperkähler manifold, let ω be a symplectic form on X, let σ be an automorphism of X, and write $\sigma^*\omega=\xi\omega$, where $\xi\in\mathbf{C}^\star$. Assume that (X,σ) deforms along a subvariety of the moduli space; the image of this subvariety by the period map consists of period points which are eigenvectors for the action of σ^* on $H^2(X,\mathbf{C})$ and the eigenvalue is necessarily ξ. Dans notre cas, the span of the image by the period map is $H^\perp$, which is therefore contained in the eigenspace $H^2(X,\mathbf{C})_\xi$.

(these two vectors are in $H^\perp$). From these identities, we obtain

$$(2a^2c - b^2(2m-2))\varphi(v) = 2a^2u + b^2(2m-2)v + 2ab\delta. \tag{23}$$

This implies that $2a^2c - b^2(2m-2)$, which is equal to $H^2 = 2n$, divides $\gcd(2a^2, b^2(2m-2)v, 2ab) = 2\gcd(a, m-1)$. Since H^2 is obviously divisible by $2\gcd(a, m-1)$, we obtain $n = \gcd(a, m-1) \mid \gamma$ hence, since $\gamma \mid 2n$,

$$n \mid m-1 \qquad \text{and} \qquad \gamma \in \{n, 2n\}.$$

One checks that conversely, if these conditions are satisfied, one can define an involution φ of $\Lambda_{\mathrm{K3}^{[m]}}$ that has the desired properties $\varphi(H) = H$ and $\varphi|_{H^\perp} = -\operatorname{Id}_{H^\perp}$: use (23) to define $\varphi(v)$, then (21) to define $\varphi(u)$, and finally (22) to define

$$\varphi(\delta) = -\frac{ab(2m-2)}{n}u - \frac{abc(2m-2)}{n}v - \frac{b^2(2m-2)+n}{n}\delta.$$

The involution φ therefore acts on $D(H^2(X, \mathbf{Z})) = \mathbf{Z}/(2m-2)\mathbf{Z} = \langle \delta_* \rangle$ by multiplication by $-\frac{b^2(2m-2)+n}{n}$. For φ to be in $\widehat{O}(H^2(X, \mathbf{Z}), q_X)$, we need to have

- either $-\frac{b^2(2m-2)+n}{n} \equiv -1 \pmod{2m-2}$, i.e., $n \mid b^2$; since $n \mid a$ and $\gcd(a, b) = 1$, this is possible only when $n = 1$;
- or $-\frac{b^2(2m-2)+n}{n} \equiv 1 \pmod{2m-2}$, which implies $m-1 \mid n$, hence $n = m-1$.

When the conditions

$$n \in \{1, m-1\} \qquad \text{and} \qquad \gamma \in \{n, 2n\} \tag{24}$$

are realized, φ acts on $D(H^2(X, \mathbf{Z}))$ as $-\operatorname{Id}$ in the first case and as Id in the second case (the relations $m-1 = n = a^2c - b^2(m-1)$ and $n \mid a$ imply $b^2 \equiv -1 \pmod{m-1}$). By the Torelli theorem, φ lifts to an involution of X.

Finally, we use the numerical conditions on m, γ, and n given in [Ap1, Proposition 3.1] to show that when (24) holds, the moduli space ${}^m\mathscr{M}_{2n}^{(\gamma)}$ is empty except in the cases given in the proposition. □

Remark 4.4. When $m = 2$ and $n = 1$, the involution is the canonical involution on double EPW sextics and the morphism $\varphi_H \colon X \to \mathbf{P}^5$ factors through the quotient of X by this involution followed by an embedding (see Section 3.6.1).

When $m = 4$ and $n = 1$, the involution again has a geometric description[43] and the map $\varphi_H \colon X \dashrightarrow \mathbf{P}^{14}$ also factors through the quotient of X by the involution, but it is not a morphism and has degree 72 onto its image (see Section 3.6.3).

When $m = 3$ and $n = \gamma = 2$ (second case of Proposition 4.3), the involution has a geometric description and $\varphi_H \colon X \to \mathbf{P}^{19}$ is a morphism that factors through the quotient of X by this involution followed by an embedding (see Section 3.6.2).

[43]Given a twisted cubic C contained in the cubic fourfold W, with span $\langle C \rangle \simeq \mathbf{P}^3$, take any quadric $Q \subset \langle C \rangle$ containing C; the intersection $Q \cap W$ is a curve of degree 6 which is the union of C and another twisted cubic C'. The curve C' depends on the choice of Q, but not its image in X (thanks to Ch. Lehn for this description). The map $[C] \mapsto [C']$ defines an involution ι on X; the fact that it is biregular is not at all clear from this description, but follows from the fact that $\iota^*(H) = (H)$. The fixed locus of ι has two (Lagrangian) components and one is the image of the canonical embedding $W \hookrightarrow X$ ([LLSvS, Theorem B(2)]).

The second example above shows that, for a general polarized hyperkähler manifold X of $\mathrm{K3}^{[m]}$-type with a polarization H of degree 2, and contrary to what the L conjecture from [O5, Conjecture 1.2] predicted, the map φ_H is not always a morphism of degree 2 onto its image. The question remains however of whether φ_H factors through the involution of X constructed in Proposition 4.3.

Remark 4.5. In Proposition 4.3, the assumption that X be very general in its moduli space is necessary and it is not enough in general to assume only that X have Picard number 1. The proof of [BCS, Theorem 3.1] implies that $\mathrm{Bir}(X)$ is trivial when $\rho(X) = 1$, unless $n \in \{1, 3, 23\}$. These three cases are actual exceptions: all fourfolds corresponding to points of ${}^2\mathscr{M}_2^{(1)}$ carry a non-trivial involution; there is a 10-dimensional subfamily of ${}^2\mathscr{M}_6^{(2)}$ whose elements consists of fourfolds that have an automorphism of order 3 and whose very general elements have Picard number 1 ([BCS, Section 7.1]); there is a (unique) fourfold in ${}^2\mathscr{M}_{46}^{(2)}$ with Picard number 1 and an automorphism of order 23 ([BCMS, Theorem 1.1]).

4.3. **Automorphisms of projective hyperkähler manifolds with Picard number 2.** When the Picard number is 2, the very simple structure of the nef and movable cones of a hyperkähler manifold X (they only have 2 extremal rays) can be used to describe the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$. We get potentially more interesting groups.

Theorem 4.6 (Oguiso)**.** *Let X be a projective hyperkähler manifold with Picard number 2. Exactly one of the following possibilities occurs:*

- *the extremal rays of the cones* $\mathrm{Nef}(X)$ *and* $\mathrm{Mov}(X)$ *are all rational and the groups* $\mathrm{Aut}(X)$ *and* $\mathrm{Bir}(X)$ *are both finite;*
- *both extremal rays of* $\mathrm{Nef}(X)$ *are rational and the group* $\mathrm{Aut}(X)$ *is finite, both extremal rays of* $\mathrm{Mov}(X)$ *are irrational and the group* $\mathrm{Bir}(X)$ *is infinite;*
- *the cones* $\mathrm{Nef}(X)$ *and* $\mathrm{Mov}(X)$ *are equal with irrational extremal rays and the groups* $\mathrm{Aut}(X)$ *and* $\mathrm{Bir}(X)$ *are equal and infinite.*

Moreover, when X is of $\mathrm{K3}^{[m]}$*-type, the group* $\mathrm{Aut}(X)$ *(resp.* $\mathrm{Bir}(X)$*), when finite, is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^r$, with $r \le 2$.*

We will see later that all three cases already occur when $\dim(X) = 4$.

Proof. One of the main ingredients of the proof in [Og2] is a result of Markman's ([M2, Theorem 6.25]): there exists a rational polyhedral closed cone $\Delta \subset \mathrm{Mov}(X)$ such that

$$\mathrm{Mov}(X) = \overline{\bigcup_{\sigma \in \mathrm{Bir}(X)} \sigma^* \Delta} \tag{25}$$

(where $\sigma^* = \overline{\Psi}_X^B(\sigma)$) and $\sigma^*(\mathring{\Delta}) \cap \mathring{\Delta} = \varnothing$ unless $\sigma^* = \mathrm{Id}$. This decomposition implies that if $\mathrm{Bir}(X)$ is finite, the cone $\mathrm{Mov}(X)$ is rational (i.e., both its extremal rays are rational).

Let x_1 and x_2 be generators of the two extremal rays of $\mathrm{Mov}(X)$. Let $\sigma \in \mathrm{Bir}(X)$; these extremal rays are either left stable by σ^* or exchanged. We can therefore write $\sigma^{*2}(x_i) = \alpha_i x_i$, with $\alpha_i > 0$; moreover, $\alpha_1 + \alpha_2 \in \mathbf{Z}$ and $\alpha_1\alpha_2 = \det(\sigma^{*2}) = 1$, since σ^* is defined over $\mathbf{Z}$.

If the ray $\mathbf{R}_{\ge 0}x_1$ is rational, we may choose x_1 integral primitive, hence α_1 is a (positive) integer. Since $\alpha_1 + \alpha_2 \in \mathbf{Z}$, so is α_2 and, since $\alpha_1\alpha_2 = \det(\sigma^{*2}) = 1$, we obtain $\alpha_1 = \alpha_2 = 1$

and $\sigma^{*2} = \mathrm{Id}$. Thus, every element of $\overline{\Psi}_X^B(\mathrm{Bir}(X))$ has order 1 or 2. In the orthogonal group of a real plane, the only elements order 2 are $-\mathrm{Id}$, which cannot be in $\overline{\Psi}_X^B(\mathrm{Bir}(X))$ since it does not fix the movable cone, and isometries with determinant -1. It follows that $\overline{\Psi}_X^B(\mathrm{Bir}(X))$ has order 1 or 2. Proposition 4.1 implies that $\mathrm{Bir}(X)$ is finite, hence $\mathrm{Mov}(X)$ is rational.

The same argument shows that if one extremal ray $\mathbf{R}_{\geq 0}y_1$ of the cone $\mathrm{Nef}(X)$ is rational, the group $\mathrm{Aut}(X)$ is finite and $\overline{\Psi}_X^A(\mathrm{Aut}(X))$ has order 1 or 2. To show that the other ray $\mathbf{R}_{\geq 0}y_2$ is also rational, we use a result of Kawamata (see [Og2, Theorem 2.3]) that implies that if $q_X(y_2) > 0$, the ray $\mathbf{R}_{\geq 0}y_2$ is rational. But if $q_X(y_2) = 0$, the ray $\mathbf{R}_{\geq 0}y_2$ is also, by (8), an extremal ray for $\mathrm{Mov}(X)$. The decomposition (25) then implies that for infinitely many distinct $\sigma_k \in \mathrm{Bir}(X)$, we have $\sigma_k^* \Delta \subset \mathrm{Nef}(X)$. Fix an integral class $x \in \mathring{\Delta}$; the integral class $a_k := \sigma_k^*(x)$ is ample, and $\sigma_0^* \sigma_k^{*-1}(a_k) = a_0$. This implies $\sigma_k^{-1}\sigma_0 \in \mathrm{Aut}(X)$, which is absurd since $\mathrm{Aut}(X)$ is finite.

If one extremal ray $\mathbf{R}_{\geq 0}y_1$ of the cone $\mathrm{Nef}(X)$ is irrational, the argument above implies that the other extremal ray $\mathbf{R}_{\geq 0}y_2$ is also irrational, and $q_X(y_1) = q_X(y_2) = 0$. The chain of inclusions (8) implies $\mathrm{Nef}(X) = \mathrm{Mov}(X)$, hence $\mathrm{Aut}(X) = \mathrm{Bir}(X)$.

To prove the last assertion, we use the classical (and rather elementary, see [Hu1, Lemma 3.1]) fact that the transcendental lattice $\mathrm{Pic}(X)^\perp \subset H^2(X, \mathbf{Z})$ carries a simple rational Hodge structure. Since its rank, 21, is odd, ± 1 is, for any $\sigma \in \mathrm{Bir}(X)$, an eigenvalue of the isometry σ^* and the corresponding eigenspace is a sub-Hodge structure of $\mathrm{Pic}(X)^\perp$. This implies $\sigma^*|_{\mathrm{Pic}(X)^\perp} = \pm \mathrm{Id}$. When X is of $\mathrm{K3}^{[m]}$-type, the morphisms Ψ_X^A and Ψ_X^B are injective and we just described their possible images when they are finite. This ends the proof of the theorem. ☐

One can go further when the nef and movable cones are explicitly known and obtain necessary conditions for non-trivial (birational or biregular) automorphisms to exist on a projective hyperkähler manifold X. To actually construct a biregular automorphism is easy if the nef cone is known: the Torelli theorem (see Section 3.8) implies that any isometry in $\widehat{O}(H^2(X, \mathbf{Z}), q_X)$ that preserves the nef cone is induced by an automorphism of X. Given an isometry $\varphi \in \widehat{O}(H^2(X, \mathbf{Z}), q_X)$ that preserves the movable cone, it is harder in general to construct a birational automorphism that induces that isometry. One way to do it is to identify, inside the movable cone of X, the nef cone of a birational model X' of X (see (10)) which is preserved by φ, and construct the birational automorphism of X as a biregular automorphism of X' using Torelli. This is however not always possible and requires a good knowledge of the chamber decomposition (10). We will describe some situations when this can be done without too much effort, using the results of Sections 3.7.2 and 3.7.3.

4.3.1. *Automorphisms of punctual Hilbert schemes of K3 surfaces.* [44] Let (S, L) be a polarized K3 surface such that $\mathrm{Pic}(S) = \mathbf{Z}L$ and let $m \geq 2$. The Picard group of $S^{[m]}$ is isomorphic to $\mathbf{Z}L_m \oplus \mathbf{Z}\delta$. The ray spanned by the integral class $L_m \in \mathrm{Pic}(S^{[m]})$ is extremal for both cones $\mathrm{Nef}(S^{[m]})$ and $\mathrm{Mov}(S^{[m]})$ (described in Example 3.17). We are therefore in the first case of Theorem 4.6 and both groups $\mathrm{Bir}(S^{[m]})$ and $\mathrm{Aut}(S^{[m]})$ are 2-groups of order at most 4. More precisely, if $e := \frac{1}{2}L^2$,

[44] Many of the results of this section can also be found in [Ca].

- either $e = 1$, the canonical involution on S induces a canonical involution on $S^{[m]}$ which generates the kernel of
$$\overline{\Psi}^B_{S^{[m]}}\colon \operatorname{Bir}(S^{[m]}) \longrightarrow O(\operatorname{Pic}(S^{[m]}), q_{S^{[m]}})$$
and the groups $\operatorname{Bir}(S^{[m]})$ and $\operatorname{Aut}(S^{[m]})$ are 2-groups of order 2 or 4;
- or $e > 1$, the morphism $\overline{\Psi}^B_{S^{[m]}}$ is injective, and the groups $\operatorname{Bir}(S^{[m]})$ and $\operatorname{Aut}(S^{[m]})$ have order 1 or 2.

As we explained earlier, it is hard to determine precisely the group $\operatorname{Bir}(S^{[m]})$, but we can at least formulate necessary conditions: when $e > 1$ and $m \geq 3$ (the case $m = 2$ will be dealt with in Proposition 4.11 below), one needs all the following properties to hold for $\operatorname{Bir}(S^{[m]})$ to be non-trivial:[45]

(a) $e(m-1)$ is not a perfect square;
(b) e and $m-1$ are relatively prime;
(c) the equation $(m-1)a^2 - eb^2 = 1$ is not solvable;
(d) if (a_1, b_1) is the minimal positive solution of the equation $a^2 - e(m-1)b^2 = 1$ that satisfies $a_1 \equiv \pm 1 \pmod{m-1}$, one has $a_1 \equiv \pm 1 \pmod{2e}$ and b_1 even.

The group $\operatorname{Bir}(S^{[m]})$ is therefore trivial in the following cases:

- $m = e + 1$, because item (a) does not hold;
- $m = e + 2$, because item (c) does not hold;
- $m = e + 3$, because item (d) does not hold ($a_1 = e + 1$ and $b_1 = 1$);
- $m = e - 1$, because item (d) does not hold ($a_1 = e - 1$ and $b_1 = 1$).

When $m = e \geq 3$, we will see in Example 4.8 that the group $\operatorname{Bir}(S^{[m]})$ has order 2: items (a) and (b) hold, so does (d) ($a_1 = 2e - 1$ and $b_1 = 2$), but I was unable to check item (c) directly!

In some cases, we can be more precise.

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[45]Since $e > 1$, the morphism $\overline{\Psi}^B_{S^{[m]}}$ is injective and any non-trivial birational involution σ of $S^{[m]}$ induces a non-trivial involution σ^* of $\operatorname{Pic}(S^{[m]})$ that preserves the movable cone; in particular, primitive generators of its two extremal rays need to have the same lengths. Since $m \geq 3$, this implies (Example 3.17) that items (a) and (c) hold. The other extremal ray of $\operatorname{Mov}(S^{[m]})$ is then spanned by $a_1 L_m - eb_1\delta$, where (a_1, b_1) is the minimal positive solution of the equation $a^2 - e(m-1)b^2 = 1$ that satisfies $a_1 \equiv \pm 1 \pmod{m-1}$.

The lattice $\operatorname{Pic}(S^{[m]}) = \mathbf{Z}L_m \oplus \mathbf{Z}\delta$ has intersection matrix $\left(\begin{smallmatrix} 2e & 0 \\ 0 & -(2m-2)\end{smallmatrix}\right)$ and its orthogonal group is
$$O(\operatorname{Pic}(S^{[m]})) = \left\{ \begin{pmatrix} a & \alpha m' b \\ e' b & \alpha a \end{pmatrix} \,\middle|\, a, b \in \mathbf{Z},\ a^2 - e'm'b^2 = 1,\ \alpha = \pm 1 \right\},$$
where $g := \gcd(e, m-1)$ and we write $e = ge'$ and $m - 1 = gm'$. Since $\sigma^*(L_m) = a_1 L_m - eb_1\delta$, we must therefore have $g = \gcd(e, m-1) = 1$.

The transcendental lattice $\operatorname{Pic}(S^{[2]})^\perp \subset H^2(S^{[2]}, \mathbf{Z})$ carries a simple rational Hodge structure (this is a classical fact found for example in [Hu1, Lemma 3.1]). Since the eigenspaces of the involution σ^* of $H^2(S^{[2]}, \mathbf{Z})$ are sub-Hodge structures, the restriction of σ^* to $\operatorname{Pic}(S^{[2]})^\perp$ is $\varepsilon \operatorname{Id}$, with $\varepsilon \in \{\pm 1\}$. As in the proof of Proposition 4.3 and with its notation, we write $L_m = u + ev$; we have
$$\begin{aligned} \sigma^*(u + ev) &= a_1(u + ev) - eb_1\delta \\ \sigma^*(u - ev) &= \varepsilon(u - ev), \end{aligned}$$
which implies $2e\sigma^*(v) = (a_1 - \varepsilon)u + e(a_1 + \varepsilon)v - eb_1\delta$, hence $2e \mid a_1 - \varepsilon$ and $2 \mid b_1$.

Example 4.7 (Case $m \geq \frac{e+3}{2}$)**.** Let (S, L) be a polarized K3 surface such that $\mathrm{Pic}(S) = \mathbf{Z}L$ and $L^2 =: 2e \geq 4$. When $m \geq \frac{e+3}{2}$, *the automorphism group of $S^{[m]}$ is trivial.* Indeed, we saw in Example 3.17 that the "other" extremal ray of $\mathrm{Nef}(S^{[m]})$ is spanned by the primitive vector $((m+e)L_m - 2e\delta)/g$, where $g := \gcd(m+e, 2e)$. Any non-trivial automorphism of $S^{[m]}$ acts non-trivially on $\mathrm{Pic}(S^{[m]})$ and exchanges the two extremal rays. The primitive generator L_m of the first ray (with square $2e$) is therefore sent to $((m+e)L_m - 2e\delta)/g$, with square $(2e(m+e)^2 - 4e^2(2m-2))/g^2$, hence

$$g^2 = (m+e)^2 - 2e(2m-2) = (m-e)^2 + 4e.$$

Since $g \mid m - e$, this implies $g^2 \mid 4e$; but it also implies $g^2 \geq 4e$, hence $g^2 = 4e$ and $m = e$. We obtain $g = \gcd(2e, 2e) = 2e$ and $e = 1$, which contradicts our hypothesis.

Example 4.8 (The Beauville involution (case $m = e$))**.** Let $S \subset \mathbf{P}^{e+1}$ be a K3 surface of degree $2e \geq 4$. By sending a general point $Z \in S^{[e]}$ to the residual intersection $(\langle Z \rangle \cap S) \smallsetminus Z$, one defines a birational involution σ of $S^{[e]}$; it is biregular if and only if $e = 2$ and S contains no lines ([B2, Section 6]).

Assume $\mathrm{Pic}(S) = \mathbf{Z}L$ (so that S contains no lines). The isometry σ^* of $\mathrm{Pic}(S^{[e]})$ must then exchange the two extremal rays of the movable cone; in particular, a primitive generator of the "other" extremal ray of $\mathrm{Mov}(S^{[e]})$ must have square $2e$. We must therefore be in the third case of Example 3.17 and $\sigma^*(L_e) = (2e-1)L_e - 2e\delta$ and $\sigma^*((2e-1)L_e - 2e\delta) = L_e$, so that $\sigma^*(L_e - \delta) = L_e - \delta$: the axis of the reflection σ^* is spanned by $L_e - \delta$.

When $e = 2$, the nef and movable cones are equal (see Example 3.16) and σ is biregular. When $e \geq 3$, the line bundle $L_e - \delta$ is still base-point-free by Corollary 3.8, hence nef. Since, by the Beauville result mentioned above (or by Example 4.7), σ is not biregular, the class $L_e - \delta$ cannot be ample. Its span is therefore the "other" extremal ray of the nef cone (as we already saw in Example 4.7) and the chamber decomposition (10) is

$$\mathrm{Mov}(S^{[e]}) = \mathrm{Nef}(S^{[e]}) \cup \sigma^* \mathrm{Nef}(S^{[e]}).$$

In particular, $S^{[e]}$ has no other hyperkähler birational model than itself and $\mathrm{Bir}(S^{[e]}) = \{\mathrm{Id}, \sigma\}$.

We end this section with a detailed study of the groups $\mathrm{Aut}(S^{[2]})$ (see [BCNS, Proposition 5.1]) and $\mathrm{Bir}(S^{[2]})$.

Proposition 4.9 (Biregular automorphisms of $S^{[2]}$)**.** *Let (S, L) be a polarized K3 surface of degree $2e$ with Picard group $\mathbf{Z}L$. The group $\mathrm{Aut}(S^{[2]})$ is trivial except when the equation $\mathscr{P}_e(-1)$ is solvable and the equation $\mathscr{P}_{4e}(5)$ is not, or when $e = 1$.*

If this is the case and $e > 1$, the only non-trivial automorphism of $S^{[2]}$ is an antisymplectic involution σ, the fourfold $S^{[2]}$ defines an element of ${}^2\mathscr{M}_2^{(1)}$ (hence generically a double EPW sextic) and σ is its canonical involution.

Proof. When $e = 1$, the canonical involution of S induces an involution on $S^{[2]}$. So we assume $e > 1$ and $\mathrm{Aut}(S^{[2]})$ non-trivial. The morphism $\overline{\Psi}^B_{S^{[m]}}$ is then injective, so we look for non-trivial involutions σ^* of $\mathrm{Pic}(S^{[2]})$ that preserve the nef cone.

If the equation $\mathscr{P}_{4e}(5)$ has a minimal solution (a_5, b_5), the extremal rays of $\mathrm{Nef}(S^{[2]})$ are spanned by L_2 and $a_5 L_2 - 2eb_5\delta$ (see Example 3.16), hence $\sigma^*(L_2) = (a_5L_2 - 2eb_5\delta)/g$, where $g = \gcd(a_5, 2eb_5)$. Taking lengths, we get $2eg^2 = 10e$, which is absurd. Therefore, the equation $\mathscr{P}_{4e}(5)$ is not solvable, the other ray of the nef cone is spanned by $a_1L_2 - eb_1\delta$, where (a_1, b_1) is the minimal solution to the equation $\mathscr{P}_e(1)$, and $\sigma^*(L_2) = a_1L_2 - eb_1\delta$. As in footnote 45 and with its notation, we have $2e \mid a_1 - \varepsilon$ and $2 \mid b_1$, with $\varepsilon \in \{-1, 1\}$. Setting $a_1 = 2ea + \varepsilon$ and $b_1 = 2b$, we rewrite the equation $\mathscr{P}_e(1)$ as

$$a(ea + \varepsilon) = b^2.$$

Since a and $ea+\varepsilon$ are relatively prime, there exists positive integers r and s such that $a = s^2$, $ea + \varepsilon = r^2$, and $b = rs$. The pair (r, s) satisfies the equation $r^2 = ea + \varepsilon = es^2 + \varepsilon$, i.e., $\mathscr{P}_e(\varepsilon)$. Since (a_1, b_1) is the minimal solution to the equation $\mathscr{P}_e(1)$, we must have $\varepsilon = -1$.

Conversely, if (r, s) is the minimal solution to the equation $\mathscr{P}_e(-1)$, the class $H := sL_2 - r\delta$ has square 2 and is ample (because it is proportional to $L_2 + (a_1L_2 - eb_1\delta)$). The pair $(S^{[2]}, H)$ is an element of ${}^2\mathscr{M}_2^{(1)}$ and as such, has an antisymplectic involution by Proposition 4.3. □

Remark 4.10. When $e = 2$, the equation $\mathscr{P}_2(-1)$ is solvable and the involution σ of $S^{[2]}$ is the (regular) Beauville involution described in Example 4.8. There is a dominant morphism $S^{[2]} \to \mathsf{Gr}(2, 4)$ of degree 6 that sends a length-2 subscheme of S to the line that it spans and it factors through the quotient $S^{[2]}/\sigma$. In that case, $S^{[2]}/\sigma$ is not an EPW sextic (although one could say that it is three times the smooth quadric $\mathsf{Gr}(2, 4) \subset \mathbf{P}^5$, which is a degenerate EPW sextic!) but $(S^{[2]}, L_2 - \delta)$ is still an element of ${}^2\mathscr{M}_2^{(1)}$ (see Example 3.10 and Section 3.6.1).

Proposition 4.11 (Automorphisms of $S^{[2]}$)**.** *Let (S, L) be a polarized K3 surface of degree $2e$ with Picard group $\mathbf{Z}L$. The group $\mathrm{Bir}(S^{[2]})$ is trivial except in the following cases:*

- *$e = 1$, or the equation $\mathscr{P}_e(-1)$ is solvable and the equation $\mathscr{P}_{4e}(5)$ is not, in which cases $\mathrm{Aut}(S^{[2]}) = \mathrm{Bir}(S^{[2]}) \simeq \mathbf{Z}/2\mathbf{Z}$;*
- *$e > 1$, and $e = 5$ or $5 \nmid e$, and both equations $\mathscr{P}_e(-1)$ and $\mathscr{P}_{4e}(5)$ are solvable, in which case $\mathrm{Aut}(S^{[2]}) = \{\mathrm{Id}\}$ and $\mathrm{Bir}(S^{[2]}) \simeq \mathbf{Z}/2\mathbf{Z}$.*

Proof. If $\sigma \in \mathrm{Bir}(S^{[2]})$ is not biregular, σ^* is a reflection that acts on the movable cone $\mathrm{Mov}(S^{[2]})$ in such a way that $\sigma^*(\mathrm{Amp}(S^{[2]})) \cap \mathrm{Amp}(S^{[2]}) = \varnothing$ (if the pull-back by σ of an ample class were ample, σ would be regular). This implies $\mathrm{Mov}(S^{[2]}) \neq \mathrm{Nef}(S^{[2]})$ hence (see Example 3.16) $e > 1$ and the equation $\mathscr{P}_{4e}(5)$ has a minimal solution (a_5, b_5) (which implies that e is not a perfect square). As in the proof of Proposition 4.9, we have $\sigma^*(L_2) = a_1L_2 - eb_1\delta$, $\sigma^*|_{\mathrm{Pic}(S^{[2]})^\perp} = -\mathrm{Id}$, $a_1 = 2eb_{-1}^2 - 1$, and $b_1 = 2a_{-1}b_{-1}$, where (a_{-1}, b_{-1}) is the minimal solution to the equation $\mathscr{P}_e(-1)$.

The chamber decomposition of the movable cone, which is preserved by σ^*, was determined in Example 3.16: since b_1 is even,

- either $5 \mid e$, there are two chambers, and the middle wall is spanned by $a_5L_2 - 2eb_5\delta$;
- or $5 \nmid e$, there are three chambers, and the middle walls are spanned by $a_5L_2 - 2eb_5\delta$ and $(a_1a_5 - 2eb_1b_5)L_2 - e(a_5b_1 - 2a_1b_5)\delta$ respectively.

In the first case, the two chambers are $\mathrm{Nef}(S^{[2]})$ and $\sigma^*(\mathrm{Nef}(S^{[2]}))$ hence the wall must be the axis of the reflection σ^*. It follows that there is an integer c such that $a_5 = cb_{-1}$ and $2eb_5 = ca_{-1}$; substituting these values in the equation $\mathscr{P}_{4e}(5)$, we get $5 = c^2b_{-1}^2 - c^2a_{-1}^2/e =$

c^2/e, hence $e = c = 5$. In that case, one can construct geometrically a non-trivial birational involution on $S^{[2]}$ (Example 4.12).

In the second case, since the reflection σ^* respects the chamber decomposition, the square-2 class $H := b_{-1}L_2 - a_{-1}\delta$ is in the interior of the "middle" chamber, which is the nef cone of a birational model X of $S^{[2]}$. It is therefore ample on X and the pair (X, H) is an element of ${}^2\mathscr{M}_2^{(1)}$; as such, it has an antisymplectic involution by Proposition 4.3, which induces a birational involution of $S^{[2]}$. □

Example 4.12 (The O'Grady involution)**.** We saw in Section 2.3 that a general polarized K3 surface S of degree 10 is the transverse intersection of the Grassmannian $\mathsf{Gr}(2, \mathbf{C}^5) \subset \mathbf{P}(\bigwedge^2 \mathbf{C}^5) = \mathbf{P}^9$, a quadric $Q \subset \mathbf{P}^9$, and a $\mathbf{P}^6 \subset \mathbf{P}^9$. A general point of $S^{[2]}$ corresponds to $V_2, W_2 \subset V_5$. Then

$$\mathsf{Gr}(2, V_2 \oplus W_2) \cap S = \mathsf{Gr}(2, V_2 \oplus W_2) \cap Q \cap \mathbf{P}^6 \cap \textstyle\bigwedge^2(V_2 \oplus W_2)) \subset \mathbf{P}^2$$

is the intersection of two general conics in $\mathbf{P}^2$ hence consists of 4 points. The (birational) O'Grady involution $S^{[2]} \dashrightarrow S^{[2]}$ takes the pair of points $([V_2], [W_2])$ to the residual two points of this intersection.

Remark 4.13. In the second case of the proposition, when $5 \nmid e$, the fourfold $S^{[2]}$ is birationally isomorphic to a double EPW sextic, whose canonical involution induces the only non-trivial birational automorphism of $S^{[2]}$.

Remark 4.14. There are cases where both equations $\mathscr{P}_e(-1)$ and $\mathscr{P}_{4e}(5)$ are solvable and $5 \nmid e$: when $e = m^2 + m - 1$, so that $(2m+1, 1)$ is the minimal solution of the equation $\mathscr{P}_{4e}(5)$, and $m \not\equiv 2 \pmod 5$, this happens for $m \in \{5, 6, 9, 10, 13, 21\}$. I do not know whether this happens for infinitely many integers m.

4.3.2. *Automorphisms of some other hyperkähler fourfolds with Picard number 2.* We determine the automorphism groups of the hyperkähler fourfolds of K3$^{[2]}$-type studied in Section 3.7.3 and find more interesting groups. The polarized hyperkähler fourfolds (X, H) under study are those for which H has square $2n$ and divisibility 2 (so that $n \equiv -1 \pmod 4$), and $\mathrm{Pic}(X) = \mathbf{Z}H \oplus \mathbf{Z}L$, with intersection matrix $\left(\begin{smallmatrix} 2n & 0 \\ 0 & -2e' \end{smallmatrix}\right)$, where $e' > 1$. If n is square-free, a very general element of the irreducible hypersurface ${}^2\mathscr{C}^{(2)}_{2n,2e'n} \subset {}^2\mathscr{M}^{(2)}_{2n}$ is of this type (Proposition 3.24(2)(b)).

Proposition 4.15. *Let (X, H) be a polarized hyperkähler fourfold as above.*

(a) *If both equations $\mathscr{P}_{n,e'}(-1)$ and $\mathscr{P}_{n,4e'}(-5)$ are not solvable and ne' is not a perfect square, the groups* $\mathrm{Aut}(F)$ *and* $\mathrm{Bir}(F)$ *are equal. They are infinite cyclic, except when the equation $\mathscr{P}_{n,e'}(1)$ is solvable, in which case these groups are isomorphic to the infinite dihedral group.*[46]
(b) *If the equation $\mathscr{P}_{n,e'}(-1)$ is not solvable but the equation $\mathscr{P}_{n,4e'}(-5)$ is, the group* $\mathrm{Aut}(X)$ *is trivial and the group* $\mathrm{Bir}(X)$ *is infinite cyclic, except when the equation $\mathscr{P}_{n,e'}(1)$ is solvable, in which case it is infinite dihedral.*
(c) *If the equation $\mathscr{P}_{n,e'}(-1)$ is solvable or if ne' is a perfect square, the group* $\mathrm{Bir}(X)$ *is trivial.*

[46]This is the group $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$, also isomorphic to the free product $\mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z}$.

When n is square-free, we can consider equivalently the equation $\mathscr{P}_{ne'}(\pm n)$ instead of $\mathscr{P}_{n,e'}(\pm 1)$ and the equation $\mathscr{P}_{4ne'}(-5n)$ instead of $\mathscr{P}_{n,4e'}(-5)$.

Proof. The map $\Psi_X^A\colon \operatorname{Aut}(X) \to O(H^2(X,\mathbf{Z}),q_X)$ is injective (Proposition 4.1). Its image consists of isometries which preserve $\operatorname{Pic}(X)$ and the ample cone and, since $b_2(X)-\rho(X)$ is odd, restrict to $\pm\operatorname{Id}$ on $\operatorname{Pic}(X)^\perp$ ([Og3, proof of Lemma 4.1]. Conversely, by Theorem 2.8, any isometry with these properties is in the image of Ψ_X^A. We begin with some general remarks on the group G of isometries of $H^2(X,\mathbf{Z})$ which preserve $\operatorname{Pic}(X)$ and the components of the positive cone, and restrict to $\varepsilon\operatorname{Id}$ on $\operatorname{Pic}(X)^\perp$, with $\varepsilon\in\{-1,1\}$.

As we saw in footnote 45, we have

$$O(\operatorname{Pic}(X),q_X)=\left\{\begin{pmatrix} a & \alpha e''b\\ n'b & \alpha a\end{pmatrix} \,\Big|\, a,b\in\mathbf{Z},\ a^2-n'e''b^2=1,\ \alpha\in\{-1,1\}\right\},$$

where $g:=\gcd(n,e')$, $n'=n/g$, and $e''=e'/g$. Note that α is the determinant of the isometry and

- such an isometry preserves the components of the positive cone if and only if $a>0$; we denote the corresponding subgroup by $O^+(\operatorname{Pic}(X),q_X)$;
- when ne' is not a perfect square, the group $SO^+(\operatorname{Pic}(X),q_X)$ is infinite cyclic, generated by the isometry R corresponding to the minimal solution of the equation $\mathscr{P}_{n'e''}(1)$ and the group $O^+(\operatorname{Pic}(X),q_X)$ is infinite dihedral;
- when ne' is a perfect square, so is $n'e''=ne'/g^2$, and $O^+(\operatorname{Pic}(X),q_X)=\{\operatorname{Id},\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\}$.

By Eichler's criterion, there exist standard bases (u_1,v_1) and (u_2,v_2) for two orthogonal hyperbolic planes in $\Lambda_{\mathrm{K3}^{[2]}}$, a generator ℓ for the $I_1(-2)$ factor, and an isometric identification $H^2(X,\mathbf{Z})\xrightarrow{\sim}\Lambda_{\mathrm{K3}^{[2]}}$ such that

$$H=2u_1+\frac{n+1}{2}v_1+\ell \quad\text{and}\quad L=u_2-e'v_2.$$

The elements Φ of G must then satisfy $a>0$ and

$$\begin{aligned}
\Phi(2u_1+\tfrac{n+1}{2}v_1+\ell) &= a(2u_1+\tfrac{n+1}{2}v_1+\ell)+n'b(u_2-e'v_2)\\
\Phi(u_2-e'v_2) &= \alpha e''b(2u_1+\tfrac{n+1}{2}v_1+\ell)+\alpha a(u_2-e'v_2)\\
\Phi(v_1+\ell) &= \varepsilon(v_1+\ell)\\
\Phi(u_1-\tfrac{n+1}{4}v_1) &= \varepsilon(u_1-\tfrac{n+1}{4}v_1)\\
\Phi(u_2+e'v_2) &= \varepsilon(u_2+e'v_2)
\end{aligned}$$

(the last three lines correspond to elements of $\operatorname{Pic}(X)^\perp$). From this, we deduce

$$\begin{aligned}
n\Phi(v_1) &= 2(a-\varepsilon)u_1+\big((a+\varepsilon)\tfrac{n+1}{2}-\varepsilon\big)v_1+(a-\varepsilon)\ell+n'b(u_2-e'v_2)\\
2\Phi(u_2) &= 2\alpha e''bu_1+\alpha e''b\tfrac{n+1}{2}v_1+\alpha e''b\ell+(\varepsilon+\alpha a)u_2+e'(\varepsilon-\alpha a)v_2\\
2e'\Phi(v_2) &= -2\alpha e''bu_1-\alpha e''b\tfrac{n+1}{2}v_1-\alpha e''b\ell+(\varepsilon-\alpha a)u_2+e'(\varepsilon+\alpha a)v_2.
\end{aligned}$$

From the first equation, we get $g\mid b$ and $a\equiv\varepsilon \pmod n$; from the second equation, we deduce that $e''b$ and $\varepsilon+\alpha a$ are even; from the third equation, we get $2g\mid b$ and $a\equiv\alpha\varepsilon$ (mod $2e'$). All this is equivalent to $a>0$ and

$$2g\mid b \quad,\quad a\equiv\varepsilon \pmod n \quad,\quad a\equiv\alpha\varepsilon \pmod{2e'}. \tag{26}$$

Conversely, if these conditions are realized, one may define Φ uniquely on $\mathbf{Z}u_1 \oplus \mathbf{Z}v_1 \oplus \mathbf{Z}u_2 \oplus \mathbf{Z}v_2 \oplus \mathbf{Z}\ell$ using the formulas above, and extend it by $\varepsilon\,\mathrm{Id}$ on the orthogonal of this lattice in $\Lambda_{\mathrm{K3}^{[2]}}$ to obtain an element of G.

The first congruence in (26) tells us that the identity on $\mathrm{Pic}(X)$ extended by $-\mathrm{Id}$ on its orthogonal does not lift to an isometry of $H^2(X,\mathbf{Z})$. This means that the restriction $G \to O^+(\mathrm{Pic}(X), q_X)$ is injective. Moreover, the two congruences in (26) imply $a \equiv \varepsilon \equiv \alpha\varepsilon \pmod g$. If $g > 1$, since n, hence also g, is odd, we get $\alpha = 1$, hence the image of G is contained in $SO^+(\mathrm{Pic}(X), q_X)$.

Assume $\alpha = 1$. The relations (26) imply that $a - \varepsilon$ is divisible by n and $2e'$, hence by their least common multiple $2gn'e''$. We write $b = 2gb'$ and $a = 2gn'e''a' + \varepsilon$ and obtain from the equality $a^2 - n'e''b^2 = 1$ the relation

$$4g^2n'^2e''^2a'^2 + 4\varepsilon gn'e''a' = 4g^2n'e''b'^2,$$

hence

$$gn'e''a'^2 + \varepsilon a' = gb'^2.$$

In particular, $a'' := a'/g$ is an integer and $b'^2 = a''(ne'a'' + \varepsilon)$.

Since $a > 0$ and a'' and $ne'a'' + \varepsilon$ are coprime, both are perfect squares and there exist coprime integers r and s, with $r > 0$, such that

$$a'' = s^2 \quad , \quad ne'a'' + \varepsilon = r^2 \quad , \quad b' = rs.$$

Since -1 is not a square modulo n, we obtain $\varepsilon = 1$; the pair (r, s) satisfies the Pell equation $r^2 - ne's^2 = 1$, and $a = 2ne's^2 + 1$ and $b = 2grs$. In particular, either ne' is not a perfect square and there are always infinitely many solutions, or ne' is a perfect square and we get $r = 1$ and $s = 0$, so that $\Phi = \mathrm{Id}$.

Assume $\alpha = -1$. As observed before, we have $g = 1$, i.e., n and e' are coprime. Using (26), we may write $b = 2b'$ and $a = 2a'e' - \varepsilon$. Since $2 \nmid n$ and $a \equiv \varepsilon \pmod n$, we deduce $\gcd(a', n) = 1$. Substituting into the equation $a^2 - ne'b^2 = 1$, we obtain

$$a'(e'a' - \varepsilon) = nb'^2,$$

hence there exist coprime integers r and s, with $r \geq 0$, such that $b' = rs$, $a' = s^2$, and $e'a' - \varepsilon = nr^2$. The pair (r, s) satisfies the equation $nr^2 - e's^2 = -\varepsilon$, and $a = 2e's^2 - \varepsilon$ and $b = 2rs$. In particular, one of the two equations $\mathscr{P}_{n,e'}(\pm 1)$ is solvable. Note that at most one of these equations may be solvable: if $\mathscr{P}_{n,e'}(\varepsilon)$ is solvable, $-\varepsilon e'$ is a square modulo n, while -1 is not. These isometries are all involutions and, since $n \geq 2$ and $e' \geq 2$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is not one of them. In particular, if ne' is a perfect square, $G = \{\mathrm{Id}\}$.

We now go back to the proof of the proposition. We proved that the composition $\mathrm{Aut}(X) \to G \to O^+(\mathrm{Pic}(X), q_X)$ is injective and, by the discussion in Section 4.1, so is the morphism $\mathrm{Bir}(X) \to G \to O^+(\mathrm{Pic}(X), q_X)$ (any element of its kernel is in $\mathrm{Aut}(X)$).

Under the hypotheses of (a), both slopes of the nef cone are irrational (Section 3.7.3), hence the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ are equal and infinite (Theorem 4.6). The calculations above allow us to be more precise: in this case, the ample cone is just one component of the positive cone and the groups $\mathrm{Aut}(X)$ and G are isomorphic. The conclusion follows from the discussions above.

Under the hypotheses of (c), the slopes of the extremal rays of the nef and movable cones are rational (Section 3.7.3) hence, by Theorem 4.6 again, $\mathrm{Bir}(X)$ is a finite group. By [Og2, Proposition 3.1(2)], any non-trivial element Φ of its image in $O^+(\mathrm{Pic}(X))$ is an involution which satisfies $\Phi(\mathrm{Mov}(X)) = \mathrm{Mov}(X)$, hence switches the two extremal rays of this cone. This means $\Phi(H \pm \mu L) = H \mp \mu L$, hence $\Phi(H) = H$, so that $\Phi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Since we saw that this is impossible, the group $\mathrm{Bir}(X)$ is trivial.

Under the hypotheses of (b), the slopes of the extremal rays of the nef cone are both rational and the slopes of the extremal rays of the movable cone are both irrational (Section 3.7.3). By Theorem 4.6 again, $\mathrm{Aut}(X)$ is a finite group and $\mathrm{Bir}(X)$ is infinite. The same reasoning as in case (c) shows that the group $\mathrm{Aut}(X)$ is in fact trivial; moreover, the group $\mathrm{Bir}(X)$ is a subgroup of $\mathbf{Z}$, except when the equation $\mathscr{P}_{n,e'}(1)$ is solvable, where it is a subgroup of $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$.

In the latter case, such an infinite subgroup is isomorphic either to $\mathbf{Z}$ or to $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$ and we exclude the first case by showing that there is indeed a regular involution on a birational model of X.

As observed in Appendix A, the positive solutions (a, b) to the equation $\mathscr{P}_{n,4e'}(-5)$ determine an infinite sequence of rays $\mathbf{R}_{\geq 0}(2e'bH \pm naL)$ in $\mathrm{Mov}(X)$. The nef cones of hyperkähler fourfolds birational to X can be identified with the chambers with respect to this collection of rays. In order to apply Lemma A.1 and show that the equation $\mathscr{P}_{n,4e'}(-5)$ has two classes of solutions, we need to check that 5 divides neither n nor e'. We will use quadratic reciprocity and, given integers r and s, we denote by $\left(\frac{r}{s}\right)$ their Jacobi symbol.

Assume first $5 \mid e'$. Since the equation $\mathscr{P}_{n,e'}(1)$ is solvable, we have $\left(\frac{n}{5}\right) = 1$; moreover, since $n \equiv -1 \pmod 4$, we have $\left(\frac{e'}{n}\right) = -1$. The solvability of the equation $\mathscr{P}_{n,4e'}(-5)$ implies $\left(\frac{5}{n}\right) = \left(\frac{e'}{n}\right)$; putting all that together contradicts quadratic reciprocity.

Assume now $5 \mid n$ and set $n' := n/5$. Since the equation $\mathscr{P}_{n,e'}(1)$ is solvable, we have $\left(\frac{e'}{5}\right) = 1$; moreover, since $n' \equiv -1 \pmod 4$, we have $\left(\frac{e'}{n'}\right) = -1$. Since $5 \nmid e'$, the equation $\mathscr{P}_{n',20e'}(-1)$ is solvable, hence $\left(\frac{5e'}{n'}\right) = 1$; again, this contradicts quadratic reciprocity.

The assumptions of Lemma A.1 are therefore satisfied and the equation $\mathscr{P}_{n,4e'}(-5)$ has two (conjugate) classes of solutions. We can reinterpret this as follows. Let (r, s) be the minimal solution to the equation $\mathscr{P}_{n,e'}(1)$; by Lemma A.2, the minimal solution to the equation $\mathscr{P}_{ne'}(1)$ is $(a, b) := (nr^2 + e's^2, 2rs)$ and it corresponds to the generator $R = \left(\begin{smallmatrix} a & e'b \\ nb & a \end{smallmatrix}\right)$ of $SO^+(\mathrm{Pic}(X), q_X)$ previously defined (we are in the case $\gcd(n, e') = 1$).

The two extremal rays of the nef cone of X are spanned by $\mathbf{x}_0 := 2e'b_{-5}H - na_{-5}L$ and $\mathbf{x}_1 := 2e'b_{-5}H + na_{-5}L$, where (a_{-5}, b_{-5}) is the minimal solution to the equation $\mathscr{P}_{n,4e'}(-5)$. If we set $\mathbf{x}_{i+2} := R(\mathbf{x}_i)$, the fact the $\mathscr{P}_{n,4e'}(-5)$ has two classes of solutions means exactly that the ray $\mathbf{R}_{\geq 0}\mathbf{x}_2$ is "above" the ray $\mathbf{R}_{\geq 0}\mathbf{x}_1$; in other words, we get an "increasing" infinite sequence of rays

$$\cdots < \mathbf{R}_{\geq 0}\mathbf{x}_{-1} < \mathbf{R}_{\geq 0}\mathbf{x}_0 < \mathbf{R}_{\geq 0}\mathbf{x}_1 < \mathbf{R}_{\geq 0}\mathbf{x}_2 < \cdots .$$

It follows from the discussion above that the involution $R\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ belongs to the group G and preserves the nef cone of the birational model X' of X whose nef cone is generated by $\mathbf{x}_1$ and $\mathbf{x}_2$. It is therefore induced by a biregular involution of X' which defines a birational involution of X. This concludes the proof of the proposition. $\square$

Remark 4.16. The case $n = 3$ and $e' = 2$ where ($\mathrm{Aut}(X) = \{\mathrm{Id}\}$ and $\mathrm{Bir}(X) \simeq \mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$) was treated geometrically by Hassett and Tschinkel in [HT3]. Here is a table for the groups $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ when $n = 3$ and $2 \le e' \le 11$.

e'	2	3	4	5	6	7	8	9	10	11
$\mathrm{Aut}(X)$	Id	Id	Id	Id	$\mathbf{Z}$	Id	Id	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$
$\mathrm{Bir}(X)$	$\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$	Id	Id	$\mathbf{Z}$	$\mathbf{Z}$	Id	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$

When $e' = 3a^2 - 1$, the pair $(a, 1)$ is a solution of $\mathscr{P}_{3,e'}(1)$, but neither $\mathscr{P}_{3,e'}(-1)$ nor, when $a \not\equiv \pm 1 \pmod 5$, $\mathscr{P}_{3,4e'}(-5)$ are solvable (reduce modulo 3 and 5). Therefore, we have $\mathrm{Aut}(X) = \mathrm{Bir}(X) \simeq \mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$.

5. Unexpected isomorphisms between special hyperkähler fourfolds and Hilbert squares of K3 surfaces

The nef cone of the Hilbert square of a polarized K3 surface (S, L) of degree $2e$ such that $\mathrm{Pic}(S) = \mathbf{Z}L$ was described in Example 3.16: its extremal rays are spanned by L_2 and $L_2 - \nu_e \delta$, where ν_e is a positive rational number that can be computed from the minimal solutions to the equations $\mathscr{P}_e(1)$ or $\mathscr{P}_{4e}(5)$.

We can use this result to parametrize some of the special divisors ${}^2\mathscr{C}^{(\gamma)}_{2n,2e} \subset {}^2\mathscr{M}^{(\gamma)}_{2n}$. We let $\mathscr{K}_{2e}$ be the quasi-projective 19-dimensional coarse moduli space of polarized K3 surfaces of degree $2e$ (Theorem 2.7).

Proposition 5.1. *Let n and e be positive integers and assume that the equation $\mathscr{P}_e(-n)$ has a positive solution (a, b) that satisfies the conditions*

$$\frac{a}{b} < \nu_e \qquad \text{and} \qquad \gcd(a, b) = 1. \tag{27}$$

The rational map

$$\begin{aligned} \varpi\colon \mathscr{K}_{2e} &\dashrightarrow {}^2\mathscr{M}^{(\gamma)}_{2n} \\ (S, L) &\longmapsto (S^{[2]}, bL_2 - a\delta), \end{aligned}$$

where the divisibility γ is 2 if b is even and 1 if b is odd, induces a birational isomorphism onto an irreducible component of ${}^2\mathscr{C}^{(\gamma)}_{2n,2e}$. In particular, if n is prime and b is even, it induces a birational isomorphism

$$\mathscr{K}_{2e} \overset{\sim}{\dashrightarrow} {}^2\mathscr{C}^{(2)}_{2n,2e}.$$

The case $n = 3$ and $e = m^2 + m + 1$ (one suitable solution is then $(a, b) = (2m + 1, 2)$) is a result of Hassett ([H]; see also [A]).

Proof. If (S, L) is a polarized K3 surface of degree $2e$ and $K := \mathbf{Z}L_2 \oplus \mathbf{Z}\delta \subset H^2(S^{[2]}, \mathbf{Z})$, the lattice $K^\perp$ is the orthogonal in $H^2(S, \mathbf{Z})$ of the class L. Since the lattice $H^2(S, \mathbf{Z})$ is unimodular, $K^\perp$ has discriminant $-2e$, hence $S^{[2]}$ is special of discriminant $2e$.

The class $H = bL_2 - a\delta$ has divisibility γ and square $2n$. It is primitive, because $\gcd(a, b) = 1$, and, if S is very general, ample on $S^{[2]}$ because of the inequality in (27). Therefore, the pair $(S^{[2]}, H)$ corresponds to a point of ${}^2\mathscr{C}^{(\gamma)}_{2n,2e}$.

The map ϖ therefore sends a very general point of $\mathscr{K}_{2e}$ to $\mathscr{C}^{(\gamma)}_{2n,2e}$. To prove that ϖ is generically injective, we assume to the contrary that there is an isomorphism $\varphi\colon S^{[2]} \xrightarrow{\sim} S'^{[2]}$ such that $\varphi^*(bL'_2 - a\delta') = bL_2 - a\delta$, although (S, L) and (S', L') are not isomorphic. It is straightforward to check that this implies $\varphi^*\delta' \neq \delta$ and that the extremal rays of the nef cone of $S^{[2]}$ are spanned by the primitive classes L_2 and $\varphi^*L'_2$. Comparing this with the description of the nef cone given in Example 3.16, we see that e is not a perfect square, $\varphi^*L'_2 = a_1L_2 - eb_1\delta$ and $\varphi^*(a_1L'_2 - eb_1\delta') = L_2$, where (a_1, b_1) is the minimal solution to the Pell equation $\mathscr{P}_e(1)$. The same proof as that of Proposition 4.9 implies $e > 1$, the equation $\mathscr{P}_e(-1)$ is solvable and the equation $\mathscr{P}_{4e}(5)$ is not.

By Proposition 4.9 again, $S^{[2]}$ has a non-trivial involution σ and $(\varphi \circ \sigma)^*(L'_2) = L_2$ and $(\varphi \circ \sigma)^*(\delta') = \delta$. This implies that $\varphi \circ \sigma$ is induced by an isomorphism $(S, L) \xrightarrow{\sim} (S', L')$, which contradicts our hypothesis. The map ϖ is therefore generically injective and since $\mathscr{K}_{2e}$ is irreducible of dimension 19, its image is a component of $\mathscr{C}^{(\gamma)}_{2n,2e}$. When n is prime and b is even, the conclusion follows from the irreducibility of ${}^2\mathscr{C}^{(2)}_{2n,2e}$ (Proposition 3.24). □

Example 5.2. Assume $n = 3$. For $e = 7$, one computes $\nu_7 = \frac{21}{8}$. The only positive solution to the equation $\mathscr{P}_7(-3)$ with b even that satisfy (27) is $(5, 2)$. A general element of ${}^2\mathscr{C}^{(2)}_{6,14}$ is therefore isomorphic to the Hilbert square of a polarized K3 surface of degree 14. This is explained geometrically by the Beauville–Donagi construction ([BD]).

Example 5.3. Assume $n = 3$. For $e = 13$, one computes $\nu_{13} = \frac{2340}{649}$. The only positive solutions to the equation $\mathscr{P}_{13}(-3)$ with b even that satisfy (27) are $(7, 2)$ and $(137, 38)$. A general element (X, H) of ${}^2\mathscr{C}^{(2)}_{6,26}$ is therefore isomorphic to the Hilbert square of a polarized K3 surface (S, L) of degree 26 in two different ways: H is mapped either to $2L_2 - 7\delta$ or to $38L_2 - 137\delta$. This can be explained as follows: the equation $\mathscr{P}_{13}(-1)$ has a minimal solution $(18, 5)$ and the equation $\mathscr{P}_{52}(5)$ is not solvable (reduce modulo 5), hence $S^{[2]}$ has a non-trivial involution σ (Proposition 4.9); one isomorphism $S^{[2]} \xrightarrow{\sim} X$ is obtained from the other by composing it with σ (and indeed, $\sigma^*(2L_2 - 7\delta) = 38L_2 - 137\delta$). This is a case where X is the variety of lines on a cubic fourfold, the Hilbert square of a K3 surfaces, and a double EPW sextic! There is no geometric explanation for this remarkable fact (which also happens for $e \in \{73, 157\}$).

Remark 5.4. The varieties $\mathscr{K}_{2e}$ are known to be of general type for $e > 61$ ([GHS1]). Proposition 5.1 implies that for any prime number n satisfying its hypotheses, the Noether–Lefschetz divisors ${}^2\mathscr{C}^{(2)}_{2n,2e}$, which are irreducible by Proposition 3.24(2), are also of general type. More precise results on the geometry of the varieties ${}^2\mathscr{C}^{(2)}_{6,2e}$ can be found in [Nu, TV, La] (they are known to be of general type for $e > 96$ ([TV]) and unirational for $e \leq 19$ ([Nu])).

Corollary 5.5. *Let n be a positive integer.*

(1) *Inside the moduli space ${}^2\mathscr{M}^{(1)}_{2n}$, the general points of some component of each of the infinitely many distinct hypersurfaces ${}^2\mathscr{C}^{(1)}_{2n,2(a^2+n)}$, where a describes the set of all positive integers such that $(n, a) \neq (1, 2)$, correspond to Hilbert squares of K3 surfaces.*

(2) *Assume moreover $n \equiv -1 \pmod 4$. Inside the moduli space ${}^2\mathscr{M}^{(2)}_{2n}$, the general points of some component of each of the infinitely many distinct hypersurfaces ${}^2\mathscr{C}^{(2)}_{2n,2\left(a^2+a+\frac{n+1}{4}\right)}$,*

where a describes the set of all nonegative integers such that $(n,a)\neq(3,1)$, correspond to Hilbert squares of K3 surfaces.

In both cases, the union of these hypersurfaces is dense in the moduli space ${}^2\mathscr{M}_{2n}^{(\gamma)}$ for the euclidean topology.

Proof. For (1), the pair $(a,1)$ is a solution of the equation $\mathscr{P}_e(-n)$, with $e=a^2+n$. We need to check that the inequality $a<\nu_e$ in (27) holds.

If e is a perfect square, we have $\nu_e=\sqrt{e}$ and (27) obviously holds.

If e is not a perfect square and the equation $\mathscr{P}_{4e}(5)$ is not solvable, we have $\nu_e=e\frac{b_1}{a_1}$ (Example 3.16). If the inequality (27) fails, since $\nu_e^2=e^2\frac{b_1^2}{a_1^2}=e-\frac{e}{a_1^2}=a^2+n-\frac{e}{a_1^2}$, we have $a_1^2\leq e/n$. Since $a_1^2=eb_1^2+1\geq e+1$, this is absurd and (27) holds in this case.

If the equation $\mathscr{P}_{4e}(5)$ has a minimal solution (a_5,b_5), we have $\nu_e=2e\frac{b_5}{a_5}$ (Example 3.16). If the inequality (27) fails, we have again $\nu_e^2=4e^2\frac{b_5^2}{a_5^2}=a^2+n-\frac{5e}{a_5^2}\leq a^2$, hence $a_5^2\leq 5e/n$. Since $a_5^2=4eb_5^2+5$, this is possible only if $n=b_5=1$, in which case $a_5^2=4e+5=4a^2+9$. This implies $a_5>2a$, hence $a_5^2\geq 4a^2+4a+1$ and $a\leq 2$. If $a=1$, the integer $4a^2+9$ is not a perfect square. If $a=2$, we have $a_5=e=5$ and $a=\nu_e$, but this is a case that we have excluded. The inequality (27) therefore holds in this case.

For (2), the pair $(2a+1,2)$ is a solution of the equation $\mathscr{P}_e(-n)$, with $e=a^2+a+\frac{n+1}{4}$ and we need to check that the inequality $a+\frac{1}{2}<\nu_e$ in (27) holds.

If e is a perfect square, we have $\nu_e=\sqrt{e}$ and (27) holds.

If e is not a perfect square and the equation $\mathscr{P}_{4e}(5)$ is not solvable, we have $\nu_e=e\frac{b_1}{a_1}$. If the inequality (27) fails, since $\nu_e^2=e^2\frac{b_1^2}{a_1^2}=e-\frac{e}{a_1^2}=a^2+a+\frac{n+1}{4}-\frac{e}{a_1^2}$, we have $a_1^2\leq 4e/n$. Since $a_1^2=eb_1^2+1$ and $n\equiv -1 \pmod 4$, this is possible only if $n=3$ and $b_1=1$, in which case $a_1^2=a^2+a+2$. This implies $a_1>a$, hence $a_1^2\geq a^2+2a+1$ and $a\leq 1$. If $a=0$, the integer a^2+a+2 is not a perfect square. If $a=1$, we have $a_1=2$, $e=3$, and $a=\nu_e$, but this is a case that we have excluded (and indeed, $\mathscr{C}_{6,6}^{(2)}$ is empty as noted in Example 3.26). The inequality (27) therefore holds in this case.

If the equation $\mathscr{P}_{4e}(5)$ has a solution (a_5,b_5), we have $\nu_e=2e\frac{b_5}{a_5}$. If the inequality (27) fails, we have again $\nu_e^2=4e^2\frac{b_5^2}{a_5^2}=a^2+a+\frac{n+1}{4}-\frac{5e}{a_5^2}\leq\left(a+\frac{1}{2}\right)^2$, hence $a_5^2\leq 20e/n$. Since $a_5^2=4eb_5^2+5$, this is possible only if $n=3$ and $b_5=1$, in which case $a_5^2=4e+5=4a^2+4a+9$. This implies $a_5>2a+1$, hence $a_5^2\geq 4a^2+8a+4$ and $a\leq 1$. If $a=1$, the integer $4a^2+4a+9$ is not a perfect square. If $a=0$, we have $a_5=3$, $e=1$, and $a=\nu_e=\frac{2}{3}>a+\frac{1}{2}$, so the inequality (27) always holds.

Finally, the density of the union of the special hypersurfaces in the moduli space follows from a powerful result of Clozel and Ullmo (Theorem 5.6 below). ☐

Theorem 5.6 (Clozel–Ullmo)**.** *The union of infinitely many Heegner divisors in any moduli space ${}^m\mathscr{M}_{2n}^{(\gamma)}$ is dense for the euclidean topology.*

Proof. This follows from the main result of [CU]: the space ${}^m\mathscr{M}_{2n}^{(\gamma)}$ is a (union of components of a) Shimura variety and each Heegner divisor $\mathscr{D}_x$ is a "strongly special" subvariety, hence is endowed with a canonical probability measure $\mu_{\mathscr{D}_x}$;. Given any infinite family $(\mathscr{D}_{x_a})_{a\in\mathbf{N}}$ of Heegner divisors, there exists a subsequence $(a_k)_{k\in\mathbf{N}}$, a strongly special subvariety $Z\subset {}^m\mathscr{M}_{2n}^{(\gamma)}$ which contains $\mathscr{D}_{x_{a_k}}$ for all $k\gg 0$ such that $(\mu_{\mathscr{D}_{x_{a_k}}})_{k\in\mathbf{N}}$ converges weakly to μ_Z ([CU, th. 1.2]). For dimensional reasons, we have $Z={}^m\mathscr{M}_{2n}^{(\gamma)}$; this implies that $\bigcup_a \mathscr{D}_{x_a}$ is dense in ${}^m\mathscr{M}_{2n}^{(\gamma)}$. □

Remark 5.7. It was proved in [MM] that Hilbert schemes of projective K3 surfaces are dense in the coarse moduli space of all (possibly non-algebraic) hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type.

Appendix A. Pell-type equations

We state or prove a few elementary results on some diophantine equations.

Given non-zero integers e and t with $e>0$, we denote by $\mathscr{P}_e(t)$ the Pell-type equation

$$a^2-eb^2=t, \tag{28}$$

where a and b are integers (the usual Pell equation is the case $t=1$). A solution (a,b) of this equation is called positive if $a>0$ and $b>0$. If e is not a perfect square, (a,b) is a solution if and only if the norm of $a+b\sqrt{e}$ in the quadratic number field $\mathbf{Q}(\sqrt{e})$ is t.

A positive solution with minimal a is called the minimal solution; it is also the positive solution (a,b) for which the "slope" $b/a=\sqrt{\frac{1}{e}-\frac{t}{ea^2}}$ is minimal when $t>0$, maximal when $t<0$. Since the function $x\mapsto x+\frac{t}{x}$ is increasing on the interval $(\sqrt{|t|},+\infty)$, the minimal solution is also the one for which the real number $a+b\sqrt{e}$ is $>\sqrt{|t|}$ and minimal.

Assume that e is not a perfect square. There is always a minimal solution (a_1,b_1) to the Pell equation $\mathscr{P}_e(1)$ and if $x_1:=a_1+b_1\sqrt{e}$, all the solutions of the equation $\mathscr{P}_e(1)$ correspond to the "powers" $\pm x_1^n$, for $n\in\mathbf{Z}$, in $\mathbf{Z}[\sqrt{e}]$.

If an equation $\mathscr{P}_e(t)$ has a solution (a,b), the elements $\pm(a+b\sqrt{e})x_1^n$ of $\mathbf{Z}[\sqrt{e}]$, for $n\in\mathbf{Z}$, all give rise to solutions of $\mathscr{P}_e(t)$ which are said to be *associated with* (a,b). The set of all solutions of $\mathscr{P}_e(t)$ associated with each other form a *class of solutions.* A class and its conjugate (generated by $(a,-b)$) may be distinct or equal.

Assume that t is positive but not a perfect square. Let (a,b) be a solution to the equation $\mathscr{P}_e(t)$ and set $x:=a+b\sqrt{e}$. If $x=\sqrt{t}$, we have $\bar{x}=t/x=x$, hence $b=0$; this contradicts our hypothesis that t is not a perfect square, hence $x\neq\sqrt{t}$.

A class of solutions to the equation $\mathscr{P}_e(t)$ and its conjugate give rise to real numbers which are ordered as follows[47]

$$\cdots<x_tx_1^{-2}\le\bar{x}_tx_1^{-1}<x_tx_1^{-1}\le\bar{x}_t<\sqrt{t}<x_t\le\bar{x}_tx_1<x_tx_1\le\bar{x}_tx_1^2<x_tx_1^2<\cdots \tag{29}$$

where $x_t=a_t+b_t\sqrt{e}$ corresponds to a solution which is minimal and its class and its conjugate. We have $x_t=\bar{x}_tx_1$ if and only if the class of the solution (a_t,b_t) is associated with

[47]Since $0<x_1<1$, we have $0<x_tx_1^{-1}<x_t$. Since x_t corresponds to a minimal solution, this implies $x_tx_1^{-1}<\sqrt{t}$, hence $\bar{x}_tx_1>\sqrt{t}$. By minimality of x_t again, we get $\bar{x}_tx_1\ge x_t$.

its conjugate. The inequality $x_t \le \bar{x}_t x_1$ implies $a_t \le a_t a_1 - e b_t b_1$, hence

$$\frac{b_t}{a_t} \le \frac{a_1 - 1}{e b_1} = \frac{a_1^2 - a_1}{e a_1 b_1} = \frac{b_1}{a_1} - \frac{a_1 - 1}{e a_1 b_1} < \frac{b_1}{a_1}. \tag{30}$$

This inequality between slopes also holds for the solution $\bar{x}_t x_1$ in the conjugate class (because $(\bar{x}_t x_1) x_1^{-1} = \bar{x}_t < \sqrt{t}$) but for no other positive solutions in these two classes.

We will need the following variation on this theme. We still assume that t is positive and not a perfect square. Let (a'_t, b'_t) be the minimal positive solution to the equation $\mathscr{P}_{4e}(t)$ (if it exists) and set $x'_t := a'_t + b'_t\sqrt{4e}$.[48]

If b_1 is even, $(a_1, b_1/2)$ is the minimal solution to the equation $\mathscr{P}_{4e}(1)$ and we obtain from (30) the inequality

$$\frac{b'_t}{a'_t} < \frac{b_1}{2a_1}. \tag{31}$$

As above, the only other solution (among the class of (a'_t, b'_t) and its conjugate) for which this inequality between slopes also holds is the "next" solution, which corresponds to $\bar{x}'_t x_1$.

If b_1 is odd, $(a'_1, b'_1) = (2eb_1^2 + 1, a_1 b_1)$ is the minimal solution to the equation $\mathscr{P}_{4e}(1)$, so that $x'_1 = x_1^2$. The solutions associated with (a'_t, b'_t) correspond to the $\pm x'_t x_1^{2n}$, $n \in \mathbf{Z}$. We still get from (30) the inequality

$$\frac{b'_t}{a'_t} \le \frac{a'_1 - 1}{4eb'_1} = \frac{(2eb_1^2 + 1) - 1}{4e a_1 b_1} = \frac{b_1}{2a_1} \tag{32}$$

which is in fact strict.[49] No other solution associated with (a'_t, b'_t) or its conjugate satisfies this inequality between slopes.

The following criterion ([N, Theorem 110]) ensures that in some cases, any two classes of solutions are conjugate, so that the discussion above applies to all the solutions.

Lemma A.1. *Let u be a positive integer which is either prime or equal to 1, let e be a positive integer which is not a perfect square, and let $\varepsilon \in \{-1, 1\}$. If the equation $\mathscr{P}_e(\varepsilon u)$ is solvable, it has one or two classes of solutions according to whether u divides $2e$ or not; if there are two classes, they are conjugate.*

We now extend slightly the class of equations that we are considering: if e_1 and e_2 are positive integers, we denote by $\mathscr{P}_{e_1,e_2}(t)$ the equation

$$e_1 a^2 - e_2 b^2 = t.$$

[48] If b_t is even, we have $a'_t = a_t$, $b'_t = b_t/2$, and $x'_t = x_t$. If t is even, we have $4 \mid t$, $a'_t = 2a_{t/4}$, $b'_t = b_{t/4}$, and $x'_t = 2x_{t/4}$. If b_t and t are odd,

- either b_1 is even (and a_1 is odd) and the second argument of any solution of $\mathscr{P}_e(t)$ associated with (a_t, b_t) or its conjugate remains odd, so that the solution $(a'_t, 2b'_t)$ of $\mathscr{P}_e(t)$ is in neither of these two classes;
- or b_1 is odd, we have $a_t b_1 - a_1 b_t \equiv a_t + a_1 \equiv t + e + 1 + e \equiv 0 \pmod 2$, hence $x'_t = \bar{x}_t x_1$.

[49] Since a_1 and b_1 are relatively prime, the equality $2a_1 b'_t = b_1 a'_t$ implies that there exists a positive integer c such that $2b'_t = cb_1$ and $a'_t = ca_1$; plugging these values into the equation $\mathscr{P}_{4e}(t)$, we get $t = c^2$, which contradicts our hypothesis that t is not a perfect square.

Given an integral solution (a,b) to $\mathscr{P}_{e_1,e_2}(t)$, we obtain a solution (e_1a,b) to $\mathscr{P}_{e_1e_2}(e_1t)$. If e_1 is square-free, all the solutions to $\mathscr{P}_{e_1e_2}(e_1t)$ arise in this way; in general, all the solutions whose first argument is divisible by e_1 arise. A positive solution (a,b) to $\mathscr{P}_{e_1,e_2}(t)$ is called minimal if a is minimal. If e_1e_2 is not a perfect square, we say that the solutions (a,b) and (a',b') of $\mathscr{P}_{e_1,e_2}(t)$ are associated if (e_1a,b) and (e_1a',b') are associated solutions of $\mathscr{P}_{e_1e_2}(e_1t)$.[50]

Let $\varepsilon \in \{-1,1\}$; assume that the equation $\mathscr{P}_{e_1,e_2}(\varepsilon)$ has a solution $(a_\varepsilon,b_\varepsilon)$ and set $x_\varepsilon := e_1a_\varepsilon + b_\varepsilon\sqrt{e_1e_2}$. Let $(a,b)\in \mathbf{Z}^2$ and set $x := e_1a+b\sqrt{e_1e_2}$. We have

$$xx_\varepsilon = e_1(e_1aa_\varepsilon + e_2bb_\varepsilon + (a_\varepsilon b + ab_\varepsilon)\sqrt{e_1e_2}) =: e_1y,$$

where $x\bar{x} = e_1\varepsilon y\bar{y}$. In particular, x is a solution to the equation $\mathscr{P}_{e_1e_2}(e_1t)$ if and only if y is a solution to the equation $\mathscr{P}_{e_1e_2}(\varepsilon t)$. This defines a bijection between the set of solutions to the equation $\mathscr{P}_{e_1,e_2}(t)$ and the set of solutions to the equation $\mathscr{P}_{e_1e_2}(\varepsilon t)$ (the inverse bijection is given by $y\mapsto x = \varepsilon\bar{x}_\varepsilon y$).

The proof of the following lemma is left to the reader.

Lemma A.2. *Let e_1 and e_2 be positive integers. Assume that for some $\varepsilon\in\{-1,1\}$, the equation $\mathscr{P}_{e_1,e_2}(\varepsilon)$ is solvable and let $(a_\varepsilon,b_\varepsilon)$ be its minimal solution. Then, e_1e_2 is not a perfect square and the minimal solution of the equation $\mathscr{P}_{e_1e_2}(1)$ is $(e_1a_\varepsilon^2+e_2b_\varepsilon^2, 2a_\varepsilon b_\varepsilon)$, unless $e_1=\varepsilon=1$ or $e_2=-\varepsilon=1$,*

We now assume $t<0$ (the discussion is entirely analogous when $t>0$ and leads to the reverse inequality in (35)) and $-t$ is not a perfect square. Let (a_t,b_t) be the minimal solution to the equation $\mathscr{P}_{e_1,e_2}(t)$ and set $x_t := e_1a_t + b_t\sqrt{e_1e_2}$, solution to the equation $\mathscr{P}_{e_1e_2}(e_1t)$ which is minimal among all solutions whose first argument is divisible by e_1. We have as in (30) the inequalities

$$\cdots < x_tx_1^{-1} \le -\bar{x}_t < \sqrt{-e_1t} < x_t \le -\bar{x}_tx_1 < x_tx_1 \le \cdots \tag{33}$$

with $x_1 := \frac{1}{e_1}x_\varepsilon^2$ by Lemma A.2. The increasing correspondence with the solutions to the equation $\mathscr{P}_{e_1e_2}(\varepsilon t)$ that we described above maps $x_tx_1^{-1}$ to $\frac{1}{e_1}x_tx_1^{-1}x_\varepsilon = x_tx_\varepsilon^{-1}$ and $-\bar{x}_t$ to $-\frac{1}{e_1}\bar{x}_tx_\varepsilon$. Since the product of these positive numbers is $-t$, we have

$$\cdots \le x_tx_\varepsilon^{-1} < \sqrt{-t} < -\frac{1}{e_1}\bar{x}_tx_\varepsilon \le \cdots \tag{34}$$

In particular, $-\frac{1}{e_1}\bar{x}_tx_\varepsilon$ corresponds to a positive solution, hence

$$\frac{a_t}{b_t} < \frac{a_\varepsilon}{b_\varepsilon}. \tag{35}$$

Moreover, (a_t,b_t) is the only positive solution to the equation $\mathscr{P}_{e_1,e_2}(t)$ among those appearing in (33) that satisfies this inequality. When $|t|$ is prime, Lemma A.1 implies that any two classes of solutions are conjugate, hence all positive solutions appear in (33), and (35) holds for only one positive solution to the equation $\mathscr{P}_{e_1,e_2}(t)$.

Finally, one small variation: we assume that $(a_\varepsilon,b_\varepsilon)$ is the minimal solution to the equation $\mathscr{P}_{e_1,e_2}(\varepsilon)$ but that (a_t,b_t) is the minimal solution to the equation $\mathscr{P}_{e_1,4e_2}(t)$. Then

[50] If (a,b) is a solution to $\mathscr{P}_{e_1,e_2}(t)$ and (a_1,b_1) is a solution to $\mathscr{P}_{e_1e_2}(1)$, and if we set $x_1 := a_1+b_1\sqrt{e_1e_2}$, then $(e_1a+b\sqrt{e_1e_2})x_1 =: e_1a'+b'\sqrt{e_1e_2}$, where (a',b') is again a solution to $\mathscr{P}_{e_1,e_2}(t)$.

we have

$$\frac{a_t}{2b_t} < \frac{a_\varepsilon}{b_\varepsilon} \tag{36}$$

and, when $|t|$ is prime, (a_t, b_t) is the only solution that satisfies that inequality. The proof is exactly the same: since $2a_\varepsilon b_\varepsilon$ is even, x_1 still corresponds to the minimal solution to the equation $\mathscr{P}_{4e_1e_2}(1)$; in (33), we have solutions to the equation $\mathscr{P}_{4e_1e_2}(e_1t)$ and in (34), we have solutions to the equation $\mathscr{P}_{e_1e_2}(\varepsilon t)$, but this does not change the reasoning.

Appendix B. The image of the period map (with E. Macrì)

In this second appendix, we generalize Theorem 3.27 in all dimensions. Recall the set up: a polarization type τ is the $O(\Lambda_{\mathrm{K3}^{[m]}})$-orbit of a primitive element h_τ of $\Lambda_{\mathrm{K3}^{[m]}}$ with positive square and ${}^m\mathscr{M}_\tau$ is the moduli space for hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type with a polarization of type τ. There is a period map

$$\wp_\tau \colon {}^m\mathscr{M}_\tau \longrightarrow \mathscr{P}_\tau = \widehat{O}(\Lambda_{\mathrm{K3}^{[m]}}, h_\tau)\backslash\Omega_{h_\tau}.$$

The goal of this appendix is to prove the following result.

Theorem B.1 (Bayer, Debarre–Macrì, Amerik–Verbitsky)**.** *Assume $m \geq 2$. Let τ be a polarization type. The image of the restriction of the period map $\wp_\tau$ to any component of the moduli space ${}^m\mathscr{M}_\tau$ is the complement of a finite union of explicit Heegner divisors.*

The procedure for listing these divisors will be explained in Remark B.5. The proof of the theorem is based on the description of the nef cone for hyperkähler manifolds in [BHT, Mo2] (another proof follows from [AV2]). We start by revisiting these results. Recall (see (11)) that the (unimodular) extended K3 lattice is

$$\widetilde{\Lambda}_{\mathrm{K3}} := U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}.$$

Let X be a projective hyperkähler manifold of $K3^{[m]}$-type. By [M2, Corollary 9.5], there is a canonical $O(\widetilde{\Lambda}_{\mathrm{K3}})$-orbit $[\theta_X]$ of primitive isometric embeddings

$$\theta_X \colon H^2(X, \mathbf{Z}) \hookrightarrow \widetilde{\Lambda}_{\mathrm{K3}}.$$

We denote by $\mathbf{v}_X$ a generator of the orthogonal of $\theta_X(H^2(X, \mathbf{Z}))$ in $\widetilde{\Lambda}_{\mathrm{K3}}$. It satisfies $\mathbf{v}_X^2 = 2m-2$. We endow $\widetilde{\Lambda}_{\mathrm{K3}}$ with the weight-2 Hodge structure $\widetilde{\Lambda}_X$ for which θ_X is a morphism of Hodge structures and $\mathbf{v}_X$ is of type $(1,1)$, and we set

$$\widetilde{\Lambda}_{\mathrm{alg},X} := \widetilde{\Lambda}_X \cap \widetilde{\Lambda}_X^{1,1},$$

so that $\mathrm{NS}(X) = \theta_X^{-1}(\widetilde{\Lambda}_{\mathrm{alg},X})$. Finally, we set

$$\mathscr{S}_X := \{\mathbf{s} \in \widetilde{\Lambda}_{\mathrm{alg},X} \mid \mathbf{s}^2 \geq -2 \text{ and } 0 \leq \mathbf{s}\cdot\mathbf{v}_X \leq \mathbf{v}_X^2/2\}.$$

The hyperplanes $\theta_X^{-1}(\mathbf{s}^\perp) \subset \mathrm{NS}(X) \otimes \mathbf{R}$, for $\mathbf{s} \in \mathscr{S}_X$, are locally finite in the positive cone $\mathrm{Pos}(X)$. The dual statement of [BHT, Theorem 1] is then the following (see also [BM2, Theorem 12.1 and Theorem 12.2]).

Theorem B.2. *Let X be a projective hyperkähler manifold of $K3^{[m]}$-type. The ample cone of X is the connected component of*

$$\mathrm{Pos}(X) \smallsetminus \bigcup_{\mathbf{s}\in\mathscr{S}_X} \theta_X^{-1}(\mathbf{s}^\perp) \tag{37}$$

that contains the class of an ample divisor.

Note that changing $\mathbf{v}_X$ into $-\mathbf{v}_X$ changes $\mathscr{S}_X$ into $-\mathscr{S}_X$, but the set in (37) remains the same.

We rewrite Theorem B.2 in terms of the existence of certain rank-2 lattices in the Néron-Severi group as follows.

Proposition B.3. *Let $m \geq 2$. Let X be a projective hyperkähler manifold of $K3^{[m]}$-type and let $h_X \in \mathrm{NS}(X)$ be a primitive class such that $h_X^2 > 0$. The following conditions are equivalent:*

(i) *there exist a projective hyperkähler manifold Y of $K3^{[m]}$-type, an ample primitive class $h_Y \in \mathrm{NS}(Y)$, and a Hodge isometry $g\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(Y,\mathbf{Z})$ such that $[\theta_X] = [\theta_Y \circ g]$ and $g(h_X) = h_Y$;*

(ii) *there are no rank-2 sublattices $L_X \subset \mathrm{NS}(X)$ such that*
- *$h_X \in L_X$, and*
- *there exist integers $0 \leq k \leq m-1$ and $a \geq -1$, and $\kappa_X \in L_X$, such that*

$$\kappa_X^2 = 2(m-1)(4(m-1)a - k^2), \qquad \kappa_X \cdot h_X = 0, \qquad \frac{\theta_X(\kappa_X) + k\mathbf{v}_X}{2(m-1)} \in \widetilde{\Lambda}_{\mathrm{K3}}.$$

Again, if one changes $\mathbf{v}_X$ into $-\mathbf{v}_X$, one needs to take $-\kappa_X$ instead of κ_X.

Proof. Assume that a pair (Y, h_Y) satisfies (i) but that there is a lattice L_X as in (ii). The rank-2 sublattice $L_Y := g(L_X) \subset H^2(Y,\mathbf{Z})$ is contained in $\mathrm{NS}(Y)$. We also let $\kappa_Y := g(\kappa_X) \in L_Y$ and choose the generator $\mathbf{v}_Y := \mathbf{v}_X \in \widetilde{\Lambda}_{\mathrm{K3}}$ of $H^2(Y,\mathbf{Z})^\perp$. We then have

$$\kappa_Y^2 = 2(m-1)(4(m-1)a - k^2), \qquad \kappa_Y \cdot h_Y = 0, \qquad \mathbf{s}_Y := \frac{\theta_Y(\kappa_Y) + k\mathbf{v}_Y}{2(m-1)} \in \widetilde{\Lambda}_{\mathrm{K3}}$$

and

$$\mathbf{s}_Y^2 = 2a \geq -2, \qquad 0 \leq \mathbf{s}_Y \cdot \mathbf{v}_Y = k \leq m-1 = \frac{\mathbf{v}_Y^2}{2}, \qquad \mathbf{s}_Y \cdot \theta_Y(h_Y) = 0.$$

Moreover, we have $\mathbf{s}_Y \in \widetilde{\Lambda}_{\mathrm{alg},Y}$. By Theorem B.2, h_Y cannot be ample on Y, a contradiction.

Conversely, assume that there exist no lattices L_X as in (ii). By [M2, Lemma 6.22], there exists a Hodge isometry $g'\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(X,\mathbf{Z})$ such that $[\theta_X \circ g'] = [\theta_X]$ and $g'(h_X) \in \overline{\mathrm{Mov}}(X)$. Since $g'(h_X) \in \mathrm{NS}(X)$ and $g'(h_X)^2 > 0$, by [M2, Theorem 6.17 and Lemma 6.22] and [HT1, Theorem 7], there exist a projective hyperkähler manifold Y of $K3^{[m]}$-type, a nef divisor class $h_Y \in \mathrm{Nef}(Y)$, and a Hodge isometry $g''\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(Y,\mathbf{Z})$ such that $[\theta_X] = [\theta_Y \circ g'']$ and $g''(g'(h_X)) = h_Y$. Assume that h_Y is not ample. Since $h_Y^2 > 0$ and h_Y is nef, there exists by Theorem B.2 a class $\mathbf{s}_Y \in \mathscr{S}_Y$ such that $\mathbf{s}_Y \cdot \theta_Y(h_Y) = 0$. We set $a := \frac{1}{2}\mathbf{s}_Y^2$, $k := \mathbf{s}_Y \cdot \mathbf{v}_Y$, and

$$\kappa_Y := \theta_Y^{-1}\left(2(m-1)\mathbf{s}_Y - k\mathbf{v}_Y\right) \in \theta_Y^{-1}(\widetilde{\Lambda}_{\mathrm{alg},Y}) = \mathrm{NS}(Y).$$

Let L_Y be the sublattice of $H^2(Y,\mathbf{Z})$ generated by h_Y and κ_Y. Then $L_X := g^{-1}(L_Y)$ satisfies the conditions in (ii), a contradiction. □

When $m-1$ is a prime number, this can be written only in terms of $H^2(X,\mathbf{Z})$.

Proposition B.4. *Assume that $p := m-1$ is either 1 or a prime number. Let X be a projective hyperkähler manifold of $K3^{[m]}$-type and let $h_X \in \mathrm{NS}(X)$ be a primitive class such that $h_X^2 > 0$. The following conditions are equivalent:*

- (i) *there exist a projective hyperkähler manifold Y of $K3^{[m]}$-type, an ample primitive class $h_Y \in \mathrm{NS}(Y)$, and a Hodge isometry $g\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(Y,\mathbf{Z})$ such that $g(h_X) = h_Y$;*
- (ii) *there are no rank-2 sublattices $L_X \subset \mathrm{NS}(X)$ such that*
  - *$h_X \in L_X$, and*
  - *there exist integers $0 \le k \le p$ and $a \ge -1$, and $\kappa_X \in L_X$, such that*

$$\kappa_X^2 = 2p(4pa - k^2), \qquad \kappa_X \cdot h_X = 0, \qquad 2p \mid \mathrm{div}_{H^2(X,\mathbf{Z})}(\kappa_X).$$

Proof. As explained in [M2, Sections 9.1.1 and 9.1.2], under our assumption on m, there is a unique $O(\widetilde{\Lambda}_{\mathrm{K3}})$-orbit of primitive isometric embeddings $\Lambda_{\mathrm{K3}^{[m]}} \hookrightarrow \widetilde{\Lambda}_{\mathrm{K3}}$. This implies that any Hodge isometry $H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(Y,\mathbf{Z})$ commutes with the orbits $[\theta_X]$ and $[\theta_Y]$.

We can choose the embedding $\theta\colon \Lambda_{\mathrm{K3}^{[m]}} \hookrightarrow \widetilde{\Lambda}_{\mathrm{K3}}$ as follows. Let us fix a canonical basis (u,v) of a hyperbolic plane U in $\widetilde{\Lambda}_{\mathrm{K3}}$. We define θ by mapping the generator ℓ of $I_1(-2p)$ to $u - pv$. We also choose $\mathbf{v} := u + pv$ as the generator of $\Lambda_{\mathrm{K3}^{[m]}}^{\perp}$.

By Proposition B.3, we only have to show that given a class $\kappa \in \Lambda_{\mathrm{K3}^{[m]}}$ with divisibility $2p$ and such that $\kappa^2 = 2p(4pa - k^2)$, either $\theta(\kappa) + k\mathbf{v}$ or $\theta(-\kappa) + k\mathbf{v}$ is divisible by $2p$ in $\widetilde{\Lambda}_{\mathrm{K3}}$. Since $\mathrm{div}_{H^2(X,\mathbf{Z})}(\kappa)$ is divisible by $2p$, we can write $\kappa = 2pw + r\ell$, with $w \in U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ and $r \in \mathbf{Z}$. By computing κ^2, we obtain the equality

$$r^2 - k^2 = 2p(w^2 - 2a).$$

In particular, $r^2 - k^2$ is even, hence so are $r + k$ and $-r + k$. Moreover, p divides $r^2 - k^2$, hence also $\varepsilon r + k$, for some $\varepsilon \in \{-1, 1\}$. Then

$$\theta(\varepsilon\kappa) + k\mathbf{v} = \varepsilon 2p\theta(w) + (\varepsilon r + k)u + (-\varepsilon r + k)pv$$

is divisible by $2p$, as we wanted. □

Before proving the theorem, we briefly review polarized marked hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type, their moduli spaces, and their periods, following the presentation in [M2, Section 7] (see also [Ap1, Section 1]).

Let ${}^m\mathfrak{M}$ be the (smooth, non-Hausdorff, 21-dimensional) coarse moduli space of marked hyperkähler manifolds of $\mathrm{K3}^{[m]}$-type (see [Hu2, Section 6.3.3]), consisting of isomorphisms classes of pairs (X,η) such that X is a compact complex (not necessarily projective) hyperkähler manifold of $\mathrm{K3}^{[m]}$-type and $\eta\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}^{[m]}}$ is an isometry. We let $({}^m\mathfrak{M}^t)_{t\in\mathfrak{t}}$ be the family of connected components of ${}^m\mathfrak{M}$. By [M2, Lemma 7.5], the set $\mathfrak{t}$ is finite and is acted on transitively by the group $O(\Lambda_{\mathrm{K3}^{[m]}})$.

Let $h \in \Lambda_{\mathrm{K3}^{[m]}}$ be a class with $h^2 > 0$ and let $t \in \mathfrak{t}$. Let ${}^m\mathfrak{M}^{t,+}_{h^\perp} \subset {}^m\mathfrak{M}^t$ be the subset parametrizing the pairs (X,η) for which the class $\eta^{-1}(h)$ is of Hodge type $(1,1)$ and belongs to the positive cone of X, and let ${}^m\mathfrak{M}^{t,a}_{h^\perp} \subset {}^m\mathfrak{M}^{t,+}_{h^\perp}$ be the open subset where $\eta^{-1}(h)$ is ample on X. By [M2, Corollary 7.3], ${}^m\mathfrak{M}^{t,a}_{h^\perp}$ is connected, Hausdorff, and 20-dimensional.

The relation with our moduli spaces ${}^m\mathscr{M}_\tau$ is as follows. Let τ be a polarization type and let M be an irreducible component of ${}^m\mathscr{M}_\tau$. Pick a point (X_0, H_0) of M and choose a marking $\eta_0\colon H^2(X_0,\mathbf{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}^{[m]}}$. If $h_0 := \eta_0(H_0)$, the $O(\Lambda_{\mathrm{K3}^{[m]}})$-orbit of h_0 is the polarization type τ, and the pair (X_0, η_0) is in ${}^m\mathfrak{M}^{t_0,a}_{h_0^\perp}$, for some $t_0 \in \mathfrak{t}$. As in [M2, (7.4)], we consider the disjoint union

$$ {}^m\mathfrak{M}^a_{\overline{h}_0} := \coprod_{(h,t)\in O(\Lambda_{\mathrm{K3}^{[m]}})\cdot(h_0,t_0)} {}^m\mathfrak{M}^{t,a}_{h^\perp}. $$

It is acted on by $O(\Lambda_{\mathrm{K3}^{[m]}})$ and, by [M2, Lemma 8.3], there is an analytic bijection

$$ M \xrightarrow{\ \sim\ } {}^m\mathfrak{M}^a_{\overline{h}_0}/O(\Lambda_{\mathrm{K3}^{[m]}}). \tag{38} $$

Proof of Theorem B.1. Let M be a irreducible component of ${}^m\mathscr{M}_\tau$. Pick a point (X_0, H_0) of M and choose a marking $\eta_0\colon H^2(X_0,\mathbf{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}^{[m]}}$. As above, set $h_0 := \eta_0(H_0) \in \Lambda_{\mathrm{K3}^{[m]}}$ and let $t_0 \in \mathfrak{t}$ be such that the pair (X_0, η_0) is in ${}^m\mathfrak{M}^{t_0,a}_{h_0^\perp}$. Given another point (X, H) of M, by using the bijection (38), we can always find a marking $\eta\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}^{[m]}}$ for which (X, η) is in ${}^m\mathfrak{M}^{t_0,a}_{h_0^\perp}$ and $H = \eta^{-1}(h_0)$.

By [M2, Corollary 9.10], for all $(X, \eta) \in {}^m\mathfrak{M}^{t_0}$, the primitive embeddings

$$ \theta_X \circ \eta^{-1}\colon \Lambda_{\mathrm{K3}^{[m]}} \hookrightarrow \widetilde{\Lambda}_{\mathrm{K3}} $$

are all in the same $O(\widetilde{\Lambda}_{\mathrm{K3}})$-orbit. We fix one such embedding θ and a generator $\mathbf{v} \in \widetilde{\Lambda}_{\mathrm{K3}}$ of $\theta(\Lambda_{\mathrm{K3}^{[m]}})^\perp$. As in Proposition B.3, we consider the Heegner divisors $\mathscr{D}_{\tau,K}$ in $\mathscr{P}_\tau := \widehat{O}(\Lambda_{\mathrm{K3}^{[m]}}, h_0)\backslash\Omega_\tau$, where K is a primitive, rank-2, signature-$(1,1)$, sublattice of $\Lambda_{\mathrm{K3}^{[m]}}$ such that $h_0 \in K$ and there exist integers $0 \le k \le m-1$ and $a \ge -1$, and $\kappa \in K$ with

$$ \kappa^2 = 2(m-1)(4(m-1)a - k^2), \qquad \kappa\cdot h_0 = 0, \qquad \frac{\theta(\kappa) + k\mathbf{v}}{2(m-1)} \in \widetilde{\Lambda}_{\mathrm{K3}}. \tag{39} $$

There are finitely many such divisors. Indeed, since the signature of K is $(1,1)$, we have $\kappa^2 < 0$. Moreover, we have $\kappa^2 \ge 2(m-1)(-4(m-1) - (m-1)^2)$, hence κ^2 may take only finitely many values. As explained in the proof of Lemma 3.23, this implies that there only finitely many Heegner divisors of the above form.

We claim that the image of the period map

$$ \wp_\tau\colon M \longrightarrow \mathscr{P}_\tau $$

coincides with the complement of the union of these Heegner divisors. We first show the image does not meet these divisors. Let $(X, H) \in M$ and, as explained above, choose a marking η such that $(X, \eta) \in {}^m\mathfrak{M}^{t_0,a}_{h_0^\perp}$, where $h_0 := \eta(H)$. If $(X, H) \in \mathscr{D}_{\tau,K}$, for K as above, the lattice $K \subset \mathrm{NS}(X)$ satisfies condition (ii) of Proposition B.3, which is impossible.

Conversely, take a point $x \in \mathscr{P}_\tau$. The refined period map defined in [M2, (7.3)] is surjective (this is a consequence of [Hu2, Theorem 8.1]). Hence there exists $(X, \eta_X) \in {}^m\mathfrak{M}^{t_0,+}_{h_0^\perp}$ such that $H_X := \eta_X^{-1}(h_0)$ is an algebraic class in the positive cone of X and (X, H_X) has period point x. By [Hu2, Theorem 3.11], X is projective. We can now apply Proposition B.3:

if x is outside the union of the Heegner divisors described above, there exist a projective hyperkähler manifold Y of $K3^{[m]}$-type, an ample primitive class H_Y, and a Hodge isometry $g\colon H^2(X,\mathbf{Z}) \xrightarrow{\sim} H^2(Y,\mathbf{Z})$ such that $[\theta_X]=[\theta_Y\circ g]$ and $g(H_X)=H_Y$. By [M2, Theorem 9.8], g is a *parallel transport operator;* by [M2, Definition 1.1(1)], this means that if $\eta_Y := \eta\circ g^{-1}$, the pair (Y,η_Y) belongs to the same connected component ${}^m\mathfrak{M}^{t_0}$ of ${}^m\mathfrak{M}$. Moreover, (Y,η_Y) is in ${}^m\mathfrak{M}^{t_0,+}_{h_0^\perp}$ and since $\eta_Y^{-1}(h_0)=H_Y$ is ample, it is even in ${}^m\mathfrak{M}^{t_0,a}_{h_0^\perp}$. By (38), it defines a point of M which still has period point x. This means that x is in the image of $\mathscr{P}_\tau$, which is what we wanted. □

Remark B.5. Let us explain how to list the Heegner divisors referred to in the statement of Theorem B.1. Fix a representative $h_\tau\in\Lambda_{\mathrm{K3}^{[m]}}$ of the polarization τ. For each $O(\widetilde{\Lambda}_{\mathrm{K3}})$-equivalence class of primitive embeddings $\Lambda_{\mathrm{K3}^{[m]}}\hookrightarrow\widetilde{\Lambda}_{\mathrm{K3}}$,[51] pick a representative θ and a generator $\mathbf{v}$ of $\theta(\Lambda_{\mathrm{K3}^{[m]}})^\perp$. Let T be the saturation in $\widetilde{\Lambda}_{\mathrm{K3}}$ of the sublattice generated by $\theta(h_\tau)$ and $\mathbf{v}$. By [Ap1, Theorem 2.1], the abstract isometry class of the pair $(T,\theta(h_\tau))$ determines a component M of ${}^m\mathscr{M}_\tau$ (and all the components are obtained in this fashion). Now list, for all integers $0\le k\le m-1$ and $a\ge -1$, all $\widehat{O}(\Lambda_{\mathrm{K3}^{[m]}},h_\tau)$-orbits of primitive, rank-2, signature-$(1,1)$, sublattices K of $\Lambda_{\mathrm{K3}^{[m]}}$ such that $h_\tau\in K$ and there exist $\kappa\in K$ satisfying (39).

The image $\wp_\tau(M)$ is then the complement in $\mathscr{P}_\tau$ of the union of the corresponding Heegner divisors $\mathscr{D}_{\tau,K}$. The whole procedure is worked out in a particular case in Example B.6 below (there are more examples in Section 3.11 in the case $m=2$).

A priori, the image of $\wp_\tau$ may be different when restricted to different components M of ${}^m\mathscr{M}_\tau$.

Example B.6. The moduli space ${}^4\mathscr{M}_2^{(2)}$ for hyperkähler manifolds of $\mathrm{K3}^{[4]}$-type with a polarization of square 2 and divisibility 2 is irreducible (Theorem 3.5). Let us show that the image of the period map ${}^4\wp_2^{(2)}$ is the complement of the three irreducible Heegner divisors ${}^4\mathscr{D}^{(2)}_{2,2}$, ${}^4\mathscr{D}^{(2)}_{2,6}$, and ${}^4\mathscr{D}^{(2)}_{2,8}$.

We follow the recipe given in Remark B.5. The irreducibility of ${}^4\mathscr{M}_2^{(2)}$ means that we can fix the embedding $\theta\colon\Lambda_{\mathrm{K3}^{[4]}}\hookrightarrow\widetilde{\Lambda}_{\mathrm{K3}}$ and the class $h=h_\tau\in\Lambda_{\mathrm{K3}^{[4]}}$ as we like. Let us fix bases (u_i,v_i), for $i\in\{1,\dots,4\}$, for each of the four copies of U in $\widetilde{\Lambda}_{\mathrm{K3}}$. We choose the embedding

$$\Lambda_{\mathrm{K3}^{[4]}}=U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}\oplus I_1(-6)\hookrightarrow U^{\oplus 4}\oplus E_8(-1)^{\oplus 2}=\widetilde{\Lambda}_{\mathrm{K3}}$$

given by mapping the generator ℓ of $I_1(-6)$ to u_1-3v_1. We also set $\mathbf{v}:=u_1+3v_1$ (so that $\mathbf{v}^\perp=\Lambda_{\mathrm{K3}^{[4]}}$) and $h:=2(u_2+v_2)+\ell$.

As explained in Remark B.5 (and using Proposition B.4), the image of the period map is the complement of the Heegner divisors of the form $\mathscr{D}_{\tau,K}$, where K is a primitive, rank-2, sublattice of $\Lambda_{\mathrm{K3}^{[4]}}$ such that $h\in K$ and there exist integers $0\le k\le 3$ and $a\ge -1$, and $\kappa\in K$ with

$$\kappa^2=6(12a-k^2)<0,\qquad \kappa\cdot h=0,\qquad 6\mid \mathrm{div}_{\Lambda_{\mathrm{K3}^{[4]}}}(\kappa).$$

[51]When $m-1$ is prime or equal to 1, there is a unique equivalence class; in general, the number of equivalence classes is the index of $\widehat{O}(\Lambda_{\mathrm{K3}^{[m]}})$ in $O^+(\Lambda_{\mathrm{K3}^{[m]}})$, that is $2^{\max\{\rho(m-1)-1,0\}}$ (Section 3.8 or [M2, Lemma 9.4]).

The last relation implies that we can write $\kappa = 6w + b\ell$, with $w \cdot \ell = 0$, and the equality $\kappa \cdot h = 0$ implies

$$0 = 12w \cdot (u_2 + v_2) - 6b,$$

so that b is even. In particular, $\kappa^2 = 36w^2 - 6b^2$ is divisible by 4 and k is even, so that $k \in \{0, 2\}$. The only possible pairs (a, k) are therefore $(0, 2)$, $(-1, 0)$, and $(-1, 2)$ and a straightforward computation shows that the corresponding classes κ can be taken to be

$$\kappa = 2(3v_2 - (u_1 - 3v_1)), \qquad \kappa = 6(u_2 - v_2), \qquad \kappa = 2(3u_3 + 6u_2 + 2(u_1 - 3v_1)),$$

The associated primitive classes $\kappa_{\rm prim}$ satisfy

$$\kappa_{\rm prim}^2 = -6, \qquad \kappa_{\rm prim}^2 = -2, \qquad \kappa_{\rm prim}^2 = -24$$

and the primitive lattices $K = \mathbf{Z}h \oplus \mathbf{Z}\kappa_{\rm prim}$ have intersection matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 \\ 0 & -24 \end{pmatrix}.$$

The discriminant of $K^\perp$ is equal to -2, -6, and -8, respectively. One checks that the divisors ${}^4\mathscr{D}^{(2)}_{2,2}$, ${}^4\mathscr{D}^{(2)}_{2,6}$, and ${}^4\mathscr{D}^{(2)}_{2,8}$ are irreducible (alternatively, one can use the strange duality isomorphism ${}^4\mathscr{P}^{(2)}_2 \xrightarrow{\sim} {}^2\mathscr{P}^{(2)}_6$ between period domains and note that it sends the Heegner divisors ${}^4\mathscr{D}^{(2)}_{2,2}$, ${}^4\mathscr{D}^{(2)}_{2,6}$, and ${}^4\mathscr{D}^{(2)}_{2,8}$ to the Heegner divisors ${}^2\mathscr{D}^{(2)}_{6,2}$, ${}^2\mathscr{D}^{(2)}_{6,6}$, and ${}^2\mathscr{D}^{(2)}_{6,8}$, which were proved in [H] to be irreducible). This concludes the proof.

Remark B.7. The ample cone of a projective hyperkähler fourfold of $K3^{[2]}$-type was already described in Theorem 3.14. This description involved only classes $\mathbf{s} \in \mathscr{S}_X$ such that $\mathbf{s}^2 = -2$ and $\mathbf{s} \cdot \mathbf{v}_X \in \{0, 1\}$. We still find these classes in higher dimensions ([Mo1, Corollary 2.9]):

- classes $\mathbf{s}$ with $\mathbf{s}^2 = -2$ and $\mathbf{s} \cdot \mathbf{v}_X = 0$; in the notation of Proposition B.3, we have $a = -1$, $k = 0$, and $\theta_X(\kappa_X) = 2(m-1)\mathbf{s}$, and the lattice $\mathbf{Z}h \oplus \mathbf{Z}\frac{1}{2(m-1)}\kappa_X$ has intersection matrix $\begin{pmatrix} h^2 & 0 \\ 0 & -2 \end{pmatrix}$ but may or may not be primitive;
- classes $\mathbf{s}$ with $\mathbf{s}^2 = -2$ and $\mathbf{s} \cdot \mathbf{v}_X = 1$; in the notation of Proposition B.3, we have $a = -1$, $k = 1$, and $\theta_X(\kappa_X) = 2(m-1)\mathbf{s} - \mathbf{v}_X$; the lattice $\mathbf{Z}h \oplus \mathbf{Z}\kappa_X$ has intersection matrix $\begin{pmatrix} h^2 & 0 \\ 0 & -2(m-1)(4m-3) \end{pmatrix}$ and may or may not be primitive.

When $m \in \{3, 4\}$, complete lists of possible pairs $(\kappa^2, \operatorname{div}(\kappa))$ are given in [Mo1, Sections 2.2 and 2.3]:

- when $m = 3$, we have
$$(\kappa^2, \operatorname{div}(\kappa)) \in \{(-2, 1), (-4, 2), (-4, 4), (-12, 2), (-36, 4)\};$$
- when $m = 4$, we have
$$(\kappa^2, \operatorname{div}(\kappa)) \in \{(-2, 1), (-6, 2), (-6, 3), (-6, 6), (-14, 2), (-24, 3), (-78, 6)\}.$$

Depending on m and on the polarization, not all pairs in these list might occur: in Example B.6 (where $m = 4$), only the pairs $(-6, 3)$, $(-2, 1)$, and $(-24, 3)$ do occur.

Whenever $m - 1$ is a power of a prime number, the pair $(\kappa^2, \operatorname{div}(\kappa)) = (-2m - 6, 2)$ is also in the list ([Mo1, Corollary 2.9]): in our notation, it corresponds to $\mathbf{s}^2 = 2a = -2$ and $\mathbf{s} \cdot \mathbf{v}_X = k = m - 1$.

When the divisibility γ of the polarization h is 1 or 2, we list some of the Heegner divisors that are avoided by the period map.

Proposition B.8. *Consider the period map*

$$ {}^m\wp_{2n}^{(\gamma)} \colon {}^m\mathscr{M}_{2n}^{(\gamma)} \longrightarrow O(L_{\mathrm{K3}^{[m]}}, h_\tau)\backslash\Omega_\tau $$

When $\gamma = 1$, the image of ${}^m\wp_{2n}^{(1)}$ does not meet

- *ν of the components of the hypersurface ${}^m\mathscr{D}_{2n,2n(m-1)}^{(1)}$, where $\nu \in \{0,1,2\}$ is the number of the following congruences that hold: $n+m \equiv 2 \pmod 4$, $m \equiv 2 \pmod 4$, $n \equiv 1 \pmod 4$;*
- *one of the components of the hypersurface ${}^m\mathscr{D}_{2n,8n(m-1)}^{(1)}$;*
- *when $m-1$ is a prime power, one component of the hypersurface ${}^m\mathscr{D}_{2n,2(m-1)(m+3)n}^{(1)}$.*

When $\gamma = 2$ (so that $n+m \equiv 1 \pmod 4$), the image of ${}^m\wp_{2n}^{(2)}$ does not meet

- *one of the components of the hypersurface ${}^m\mathscr{D}_{2n,2n(m-1)}^{(2)}$;*
- *when $m-1$ is a power of 2, one component of the hypersurface ${}^m\mathscr{D}_{2n,(m-1)(m+3)n/2}^{(2)}$.*

Proof. No algebraic class with square -2 can be orthogonal to h_τ (see Remark B.7). As in the proof of Theorem 3.27, there is such a class κ if and only if the period point belongs to the hypersurface ${}^m\mathscr{D}_{2n,K}^{(\gamma)}$, where K is the rank-2 lattice with intersection matrix $\left(\begin{smallmatrix} 2n & 0 \\ 0 & -2 \end{smallmatrix}\right)$. This hypersurface is a component of ${}^m\mathscr{D}_{2n,d}^{(\gamma)}$, where $d := |\operatorname{disc}(K^\perp)|$ can be computed using the formula

$$ d = \left| \frac{\kappa^2 \operatorname{disc}(h_\tau^\perp)}{s^2} \right| $$

from [GHS3, Lemma 7.5] (see (17)), where $s := \operatorname{div}_{h_\tau^\perp}(\kappa) \in \{1,2\}$.

Assume $\gamma = 1$, so that $D(h_\tau^\perp) \simeq \mathbf{Z}/(2m-2)\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$ (see (6)). As in the proof of Proposition 3.24, we write $\kappa = a(u-nv) + b\ell + cw$, so that $1 = -\frac{1}{2}\kappa^2 = na^2 + (m-1)b^2 + (\frac{1}{2}w^2)c^2$, and $s = \gcd(2na, 2(m-1)b, c)$ is 1 if c is odd, and 2 otherwise.

If $s = 1$, we have $d = 8n(m-1)$ and $\kappa_* = 0$, and, by Eichler's criterion, this defines an irreducible component of ${}^2\mathscr{D}_{2n,8n(m-1)}^{(1)}$ (to prove that it is non-empty, just take $a = b = 0$, $c = 1$, and $w^2 = -2$).

If $s = 2$, we have $d = 2n(m-1)$ and κ_* has order 2. More precisely,

$$ \kappa_* = \begin{cases} (0,n) & \text{if } a \text{ is odd and } b \text{ is even;} \\ (m-1,0) & \text{if } a \text{ is even and } b \text{ is odd;} \\ (m-1,n) & \text{if } a \text{ and } b \text{ are odd.} \end{cases} $$

Conversely, to obtain such a class κ with square -2, one needs to solve the equation $na^2 + (m-1)b^2 + rc^2 = 1$, with c even and r any integer. It is equivalent to solve $na^2 + (m-1)b^2 \equiv 1 \pmod 4$ and one checks that this gives the first item of the proposition.

Assume $\gamma = 2$, so that $\operatorname{disc}(h_\tau^\perp) = n(m-1)$ (see (7)) and $d = \frac{2}{s^2}n(m-1) \in \{2n(m-1), \frac{n(m-1)}{2}\}$. As in the proof of Proposition 3.24, we may take $h_\tau^\perp = \mathbf{Z}w_1 \oplus \mathbf{Z}w_2 \oplus M$, with $w_1 := (m-1)v + \ell$ and $w_2 := -u + \frac{n+m-1}{4}v$ and write $\kappa = aw_1 + bw_2 + cw$. The equality

$\kappa^2 = -2$ now reads $1 = a(a+b)(m-1) + b^2\left(\frac{n+m-1}{4}\right) - c^2\left(\frac{1}{2}w^2\right)$, and again, $s = 1$ if c is odd, $s = 2$ if c is even.

Again, the case $s = 1$ gives rise to a single component of ${}^2\mathscr{D}^{(1)}_{2n,2n(m-1)}$, since it corresponds to $\kappa_* = 0$ (take $a = b = 0$ and $c = 1$).[52]

Assume now that $m-1$ is a prime power. In that case, no algebraic class with square $-2m-6$ and divisibility 2 can be orthogonal to h_τ (see Remark B.7). Assume that there is such a class κ.

When $\gamma = 1$, keeping the same notation as above, we need to solve $na^2 + (m-1)b^2 + (\frac{1}{2}w^2)c^2 = m+3$, with a and c even. Just taking $c = 2$, $a = 0$, and $b = 1$ works, with $d = \frac{8(m+3)n(m-1)}{\gcd(2na,2(m-1)b,c)^2} = 2(m-1)(m+3)n$. In the notation of Example B.6, we may take $\mathbf{v} = u_1 + (m-1)v_1$, $\ell = u_1 - (m-1)v_1$, $h = u_2 + nv_2$, $\kappa = (m-1)(2(u_3 - v_3) + \ell)$, and $\mathbf{s} = u_3 - v_3 + u_1$.

When $\gamma = 2$, we need to solve $a(a+b)(m-1) + b^2\left(\frac{n+m-1}{4}\right) - c^2\left(\frac{1}{2}w^2\right) = m+3$, with $a(m-1)$ and c even, and $d = \frac{2(m+3)n(m-1)}{\gcd(2na,2(m-1)b,c)^2} \mid (m-1)(m+3)n/2$, so that both n and $m-1$ are even. In that case, take $c = 2$, $b = 0$, and $a = 1$. In the notation of Example B.6, we may take $\mathbf{v} = u_1 + (m-1)v_1$, $\ell = u_1 - (m-1)v_1$, $h = 2(u_2 + \frac{n+m-1}{4}v_2) + \ell$, $\kappa = (m-1)((m-1)v_2 + 2(u_3 - v_3) + \ell)$, and $\mathbf{s} = \frac{m-1}{2}v_2 + u_3 - v_3 + u_1$. □

B.1. **The period map for cubic fourfolds.** Let $\mathscr{M}_{\mathrm{cub}}$ be the moduli space of smooth cubic hypersurfaces in $\mathbf{P}^5$. With any cubic fourfold W, we can associate its period, given by the weight-2 Hodge structure on $H^4_{\mathrm{prim}}(W,\mathbf{Z})$. The Torelli Theorem for cubic fourfolds ([Vo2]) says that the period map is an open embedding. By [BD, Proposition 6], there is a Hodge isometry $H^4_{\mathrm{prim}}(W,\mathbf{Z})(-1) \simeq H^2_{\mathrm{prim}}(F(W),\mathbf{Z})$. Hence, we can equivalently consider the period map for (smooth) cubic fourfolds as the period map of their (smooth) Fano varieties of lines, which are hyperkähler fourfolds of $\mathrm{K3}^{[2]}$-type with a polarization of square 6 and divisibility 2. The latter form a dense open subset in the moduli space ${}^2\mathscr{M}^{(2)}_6$.

By [DM, Theorem 8.1], the image of the corresponding period map ${}^2\mathscr{M}^{(2)}_6 \to {}^2\mathscr{P}^{(2)}_6$ is exactly the complement of the Heegner divisor ${}^2\mathscr{D}^{(2)}_{6,6}$. The description of the image of the period map for cubic fourfolds, which is smaller, is part of a celebrated result of Laza and Looijenga ([Laz, Lo]).

Theorem B.9 (Laza, Looijenga)**.** *The image of the period map for cubic fourfolds is the complement of the divisor ${}^2\mathscr{D}^{(2)}_{6,2} \cup {}^2\mathscr{D}^{(2)}_{6,6}$.*

It had been known for some time (see [H]) that the image of the period map for cubic fourfolds *is contained in* the complement of ${}^2\mathscr{D}^{(2)}_{6,2} \cup {}^2\mathscr{D}^{(2)}_{6,6}$.

We will present a new simple proof (due to Bayer and Mongardi), based on Theorem B.1, of a weaker form of Theorem B.9.

[52]If $s = 2$, we have $d = n(m-1)/2$, so that both n and $m-1$ are even, and we need to solve $a(a+b)(m-1) + b^2\left(\frac{n+m-1}{4}\right) \equiv 1 \pmod 4$. The reader is welcome to determine the number of solutions of this equation and see how many different elements of order 2 in the group $D(h_\tau^\perp)$ (which was determined in footnote 17) are obtained in this way in each case. This will produce more hypersurfaces avoided by the image of the period map.

Proposition B.10 (Bayer, Mongardi). *The image of the period map for cubic fourfolds contains the complement of* ${}^2\mathscr{D}^{(2)}_{6,2} \cup {}^2\mathscr{D}^{(2)}_{6,6} \cup {}^2\mathscr{D}^{(2)}_{6,8}$.

Before giving a brief idea of the proof of the proposition, we recall a construction from [LLSvS]. Let W be a smooth cubic fourfold that contains no planes. The moduli space of twisted cubic curves on W gives rise (after contraction) to a hyperkähler eightfold $X(W) \in {}^4\mathscr{M}^{(2)}_2$ (see Section 3.6.3). There is an anti-symplectic involution on $X(W)$ whose fixed locus has exactly two (smooth and 4-dimensional) connected components, and one of them is the cubic W itself. The varieties $F(W)$ and $X(W)$ are "strange duals" in the sense of Remark 3.21.

This follows from a more general "strange duality" statement as follows. Consider the *Kuznetsov component* $\mathscr{K}u(W)$ of the derived category $\mathrm{D^b}(W)$ ([Ku]). As explained in [AT, Section 2], with such category, one can associate a weight-2 Hodge structure on the lattice $\widetilde{\Lambda}_{\mathrm{K3}}$; we denote it by $\widetilde{\Lambda}_W$. Moreover, there is a natural primitive sublattice $A_2 \hookrightarrow \widetilde{\Lambda}_{\mathrm{alg},W}$; we denote its canonical basis by (u_1, u_2): it satisfies $u_1^2 = u_2^2 = 2$ and $u_1 \cdot u_2 = -1$.

Let $\sigma_0 = (\mathscr{A}_0, Z_0)$ be the Bridgeland stability condition constructed in [BLMS, Theorem 1.2]. Given a Mukai vector $\mathbf{v} \in \widetilde{\Lambda}_{\mathrm{alg},W}$, we denote by $M_{\sigma_0}(\mathbf{v})$ the moduli spaces of σ_0-semistable objects in $\mathscr{A}_0$ with Mukai vector $\mathbf{v}$. If there are no properly σ_0-semistable objects, $M_{\sigma_0}(\mathbf{v})$ is, by [BLMNPS], a smooth projective hyperkähler manifold of dimension $\mathbf{v}^2+2$ and there is a natural Hodge isometry $H^2(M_{\sigma_0}(\mathbf{v}), \mathbf{Z}) \simeq \mathbf{v}^\perp$ such that the embedding $\theta_{M_{\sigma_0}(\mathbf{v})}$ can be identified with $\theta_{\mathbf{v}} \colon \mathbf{v}^\perp \hookrightarrow \widetilde{\Lambda}_{\mathrm{K3}}$. Moreover, there is by [BM1] a natural ample class $\ell_{\sigma_0}(\mathbf{v})$ on $M_{\sigma_0}(\mathbf{v})$.

By [LPZ], there are isomorphisms $F(W) \simeq M_{\sigma_0}(u_1)$ and $X(W) \simeq M_{\sigma_0}(u_1 + 2u_2)$. Moreover, possibly after multiplying by a positive constant, the class $\ell_{\sigma_0}(u_1)$ on $M_{\sigma_0}(u_1)$ (respectively, the class $\ell_{\sigma_0}(u_1+2\lambda_2)$ on $M_{\sigma_0}(u_1+2u_2)$) corresponds to the Plücker polarization on $F(W)$ (respectively, to the degree-2 polarization on $X(W)$). Finally, an easy computation shows $\ell_{\sigma_0}(u_1) = \theta_{u_1}(u_1 + 2u_2)$ and $\ell_{\sigma_0}(u_1 + 2u_2) = \theta_{u_1+2u_2}(u_1)$. The strange duality statement follows directly from this: the periods of both polarized varieties are identified with the orthogonal of the A_2 sublattice. We notice that, in this example, this gives a precise formulation of the strange duality between polarized hyperkähler manifolds in terms of Le Potier's strange duality ([Le]) between the two moduli spaces (on a non-commutative K3 surface).

Sketch of proof of Proposition B.10. By Proposition 4.3, any eightfold X in the moduli space ${}^4\mathscr{M}^{(2)}_2$ has an anti-symplectic regular involution. Upon varying the eightfold in the moduli space, the fixed loci form a smooth family. In particular, since, when $X = X(W)$, the fixed locus contains a copy of the cubic fourfold W, and any smooth deformation of a cubic fourfold is a cubic fourfold as well (this is due to Fujita; see [IP, Theorem 3.2.5]), there exists for any $X \in {}^4\mathscr{M}^{(2)}_2$ a smooth cubic fourfold $W_X \subset X$ fixed by the involution, and the periods of W_X and X are compatible because they are strange duals, as explained above.

By Example B.6, the image of the period map for ${}^4\mathscr{M}^{(2)}_2$ is exactly the complement of the union of the Heegner divisors ${}^4\mathscr{D}^{(2)}_{2,2}$, ${}^4\mathscr{D}^{(2)}_{2,6}$, ${}^4\mathscr{D}^{(2)}_{2,8}$, and these divisors correspond by strange duality to the (irreducible) Heegner divisors ${}^2\mathscr{D}^{(2)}_{6,2}$, ${}^2\mathscr{D}^{(2)}_{6,6}$, and ${}^2\mathscr{D}^{(2)}_{6,8}$ in the period domain of cubic fourfolds. □

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Université Paris-Diderot, PSL Research University, CNRS, École normale supérieure, Département de Mathématiques et Applications, 45 rue d'Ulm, 75230 Paris cedex 05, France

E-mail address: `olivier.debarre@ens.fr`